

CHAPTER –II
FINITE DIFFERENCE METHOD

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2.1 ABSTRACT

The biharmonic equation is used to model the deflections arising in two dimensional rectangular orthotropic symmetric plates. The plate can be subjected to external perpendicular loading resulting deflections under various boundary conditions. The edges can be simply supported, clamped or free. In this chapter an autonomic procedure is defined which produces the difference equations for different problems under investigation. The procedure is illustrated using MATLAB, with numerical results for problems with and without analytical solutions.

2.2 INTRODUCTION

Since Lagrange obtained the equation of bending of thin elastic plates:

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = q(x,y)/D \quad \text{-----}(2.1)$$

in 1811, a lot of work has been done on the plates.

The solution of plate problems via the analysis method is limited in that the boundary conditions were restricted, e.g., simple supports, etc ([14]) . If these conditions are more complex, the analysis becomes increasingly tedious and even impossible. In such a case numerical and approximate method are the approaches that can be employed. Among the numerical techniques presently available, the finite difference method is one of the most general. In applying finite difference method, the derivatives in the differential equation under consideration are placed by difference quantities at some selected points, referred to as nodes or pivotal points, or simply as points of division. The numerical solution is thus obtained from differential equation, which are applicable to the actual continuous structure.

In the last part of this century, with development of composite material, many scientists and engineers in mechanical, civil, aircraft and aerospace, and

other fields work on the composites. Of composite, orthotropic symmetric laminated plates are widely used in engineering. Its bending equation is

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = q(x,y) \quad \text{-----}(2.2)$$

where

$w(x,y)$ is deflection along the z direction.

$q(x,y)$ is the intensity of transverse distributed load per unit area acting on the thin plate.

D_{11} , D_{12} , D_{66} , D_{22} , are the flexural coefficients of orthotropic symmetric laminated plates.

Because of the great success of finite difference method for common material plate, noticing the similarity of equations (2.1) and (2.2), we considered employing finite difference method to solve the bending problem of orthotropic symmetric laminated plates.

First we built the patterns of the finite difference coefficients of orthotropic symmetric laminated plate with simply supported edge condition. Then we develop a program (GB1.M) using MATLAB to define an autonomic procedure. To check the efficiency of our program based on finite difference method, we choose a few problems with and without analytical solution. The results by our method is in good agreement with the analytical solution.

Results tell us that the finite difference patterns of orthotropic symmetric laminate plate developed in this chapter is useful to solve equations and display 3D pictures of MATLAB language. To use the procedure to automatically generate the finite difference approximations of orthotropic symmetric laminate plates, what a user needs to do is input the number of mesh and the loads, then the output is a 3-D picture describing the deflection of plate acted on given load.

2.3 THEORY

For the symmetric laminated thin plates with loading q .

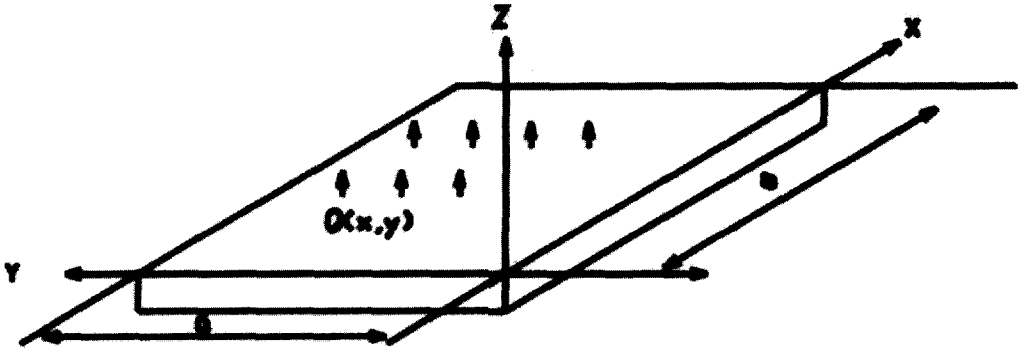


Fig.- 2

Its bending equation in terms of displacements is() [57]

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = Q(x, y) \quad \text{-----(2.3)}$$

where

$w(x, y)$ is the deflection along the z direction.

$Q(x, y)$ is the intensity of transverse distributed load per unit area acting on the thin plate. D_{11} , D_{16} , D_{12} , D_{66} , D_{26} , D_{22} are the flexural rigidity coefficients of the laminated plate.

For specially orthotropic laminates ($D_{16} = D_{26} = 0$), the governing differential equation becomes

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = q(x, y) \quad \text{-----(2.4)}$$

To solve the differential equation numerically by finite differences we generally require to replace the derivation of a function by difference of the function at the nodes.

2.3.1 REPRESENTATION OF DERIVATIVE BY FINITE DIFFERENCE

Figure 3 represents a function $y=f(x)$. The derivative (or the slope) of the curve at point

x_{i-1} and x_{i+1} can be approximated by

$$f'(x)_i = \left(\frac{dy}{dx} \right)_i = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)_i \approx \frac{y_{i+1} - y_{i-1}}{2\lambda} \quad (2.5)$$

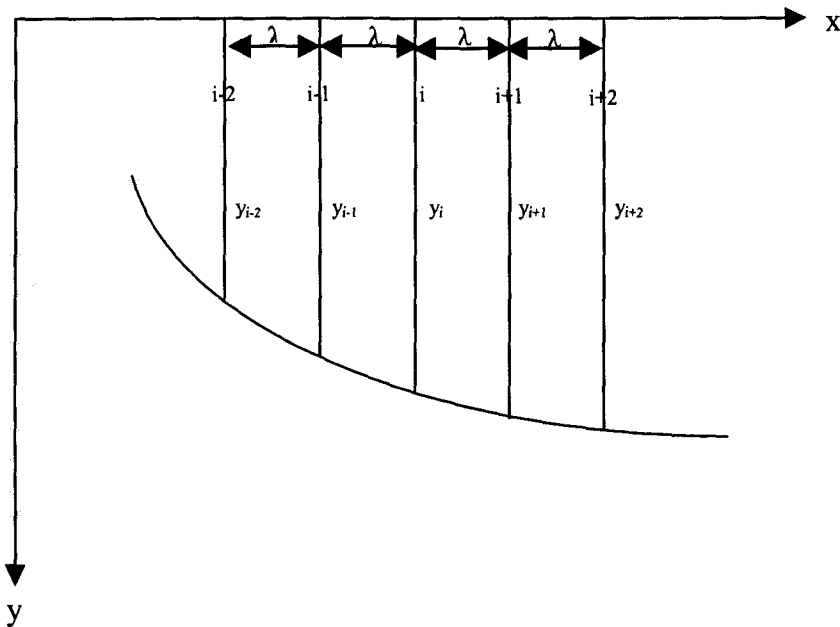


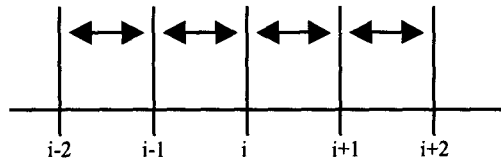
Fig..3

We can also write

$$\left(\frac{d^2 y}{dx^2} \right)_i \approx \frac{1}{\lambda^2} (y_{i+1} - 2y_i + y_{i-1}) \quad \text{-----(2.6)}$$

In the above expressions we have used central differences because the derivative of the function in each case was expressed in terms of the values of the function at points located symmetrically with respect to the point considered.

The process can be repeated to calculate higher derivatives , in which case the values of y at a greater of equally spaced points are required. The finite-difference pattern of coefficients is shown in Fig.4.



$$\left(\frac{dy}{dx}\right)_i = \frac{1}{2\lambda} \begin{bmatrix} -1 & & +1 & & \end{bmatrix}$$

$$\left(\frac{d^2y}{dx^2}\right)_i = \frac{1}{\lambda^2} \begin{bmatrix} +1 & -2 & +1 & & \end{bmatrix}$$

$$\left(\frac{d^3y}{dx^3}\right)_i = \frac{1}{2\lambda^3} \begin{bmatrix} +1 & +2 & & -2 & +1 \end{bmatrix}$$

$$\left(\frac{d^4y}{dx^4}\right)_i = \frac{1}{\lambda^4} \begin{bmatrix} +1 & -4 & +6 & -4 & +1 \end{bmatrix}$$

fig -4

Finite difference expressions can also be obtained by considering forward or backward differences, in which the derivatives at any point is expressed in terms of the value of the function at points. In ascending or descending order with respect to the point under consideration. As the central differences are more accurate than either forward or backward difference, they will be used here.

2.3.2 ERRORS IN FINITE DIFFERENCE EQUATIONS

It is apparent from the foregoing that the finite-difference approach involves errors compared with the continuous function , and it is useful to be aware of the

magnitude of error involved. The value of a function $y(x)$ at a point x_{i+1} can be expressed in terms of $y(x_i)$ and its derivatives by Taylor's expression

$$y(x_{i+1}) = y(x_i) + \frac{\lambda}{1!} y'(x_i) + \frac{\lambda^2}{2!} y''(x_i) + \frac{\lambda^3}{3!} y'''(x_i) + \dots \quad (2.7)$$

Similarly,

$$y(x_{i-1}) = y(x_i) - \frac{\lambda}{1!} y'(x_i) + \frac{\lambda^2}{2!} y''(x_i) - \frac{\lambda^3}{3!} y'''(x_i) + \dots \quad (2.8)$$

Subtracting (2.8) from (2.7), we obtain

$$y(x_i) = \frac{1}{2\lambda} [y(x_{i+1}) - y(x_{i-1})] - \frac{\lambda^3}{3!} y'''(x_i) - \frac{\lambda^5}{5!} y^{(5)}(x_i) - \dots \quad (2.9)$$

Comparing (2.9) with the finite-difference expression given in the Fig.4., we can see that the error in the finite-difference expression for the first derivative $\left(\frac{dy}{dx}\right)_i$,

$$\varepsilon_1 = -\frac{\lambda^2}{3!} y'''(x_i) - \frac{\lambda^4}{5!} y^{(5)}(x_i) - \dots \quad (2.10)$$

If we add (2.7) and (2.8), it can be shown that

$$y(x)_i = \frac{1}{\lambda^2} (y_{i-1} - 2y_i + y_{i+1}) - \frac{2\lambda^2}{4!} y''''(x_i) - \frac{2\lambda^4}{6!} y^{(6)}(x_i) - \dots \quad (2.11)$$

The error in the finite-difference expression for the second derivative $\left(\frac{d^2y}{dx^2}\right)_i$ is therefore

$$\varepsilon_2 = -\frac{2\lambda^2}{4!} y''''(x_i) - \frac{2\lambda^4}{6!} y^{(6)}(x_i) - \dots \quad (2.12)$$

Similarly, it can be shown that the error in the third derivative given in Fig.4. is

$$\varepsilon_3 = -\frac{\lambda}{4} y''''(x_i) - \dots \quad (2.13)$$

and the error in the fourth derivative is

$$\varepsilon_4 = -\frac{\lambda^4}{6} y^{(6)}(x_i) - \dots \quad (2.14)$$

From the above equations it is evident that the first error term is of the order of λ^2 . The accuracy can be improved by reducing λ .

2.3.3 REPRESENTATION OF PARTIAL DERIVATIVES BY FINITE DIFFERENCE

Thin plates subjected to transverse(normal) loading are subjected to bending and are therefore referred to as plates in bending. They undergo transverse deflections which are small compared with the dimensions of the plate. As a result , the stretching of the middle plane of the middle plane is negligible , and the in-plane displacement of point on the middle plane is assumed to be zero. Thus the displacements of points on the middle plane (or surface)of the plate, say x-y plane can be defined by a translation in the z direction and two rotations about x and y axes. The plate-bending problem involves the solution of a partial differential equation, for which the finite-difference method will be used.

Figure 5. shows the meshes drawn on a surface representing a function $w=f(x,y)$, which can , for example , be the deflected surface of a plate in bending. The derivative of 'w' with respect to x or y can be expressed as a difference of the values of w at the nodes of the mesh in the ordinary finite differences considered in 3-3 . In the following expressions, central differences are used to express values at point 7.

The slope of the surface in the x direction is

$$\left(\frac{\partial w}{\partial x}\right)_7 \approx \frac{1}{\lambda_x} [-1 \ 1] \begin{Bmatrix} w_{11} \\ w_7 \\ w_3 \end{Bmatrix} \quad (2.15)$$

and the curvature in the x direction is

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_7 \approx \frac{1}{2\lambda_x^2} [1 \ -2 \ 1] \begin{Bmatrix} w_{11} \\ w_7 \\ w_3 \end{Bmatrix} \quad (2.16)$$

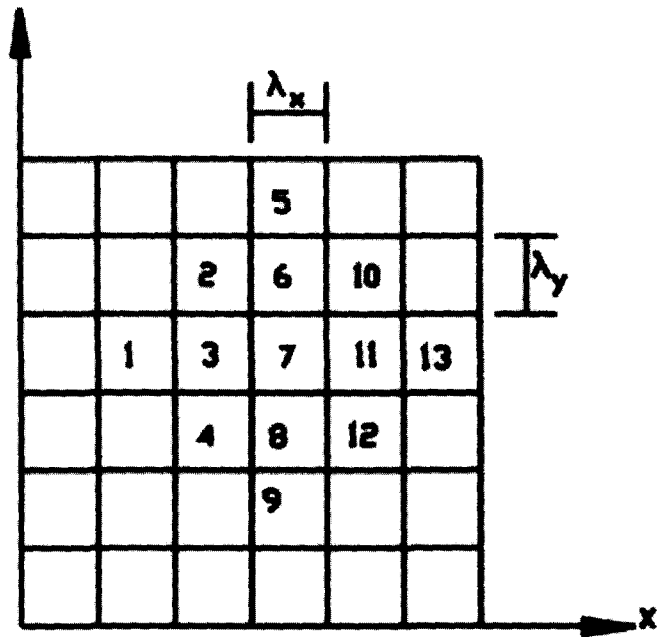


Fig :-5

Similar expression can be written for derivatives with respect to y.

The Laplacian operator in the x and y variables

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

applied at a general point 7 can be put in the finite-difference form.

$$(\nabla^2 w)_7 \approx \frac{1}{2\lambda_x^2} [1 \ -2 \ 1] \begin{Bmatrix} w_{11} \\ w_7 \\ w_3 \end{Bmatrix} + \frac{1}{2\lambda_y^2} [1 \ -2 \ 1] \begin{Bmatrix} w_6 \\ w_7 \\ w_8 \end{Bmatrix} \dots\dots\dots(2.17)$$

The mixed derivatives at point I, the center of the hatched rectangle in Fig5. is

$$\left(\frac{\partial^2 w}{\partial x \partial y} \right)_7 \approx \frac{1}{2\lambda_y} \left[\left(\frac{\partial w}{\partial x} \right)_8 - \left(\frac{\partial w}{\partial x} \right)_6 \right] \quad (2.18)$$

Whence

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)_i \approx \frac{1}{4\lambda_x \lambda_y} [1 \ -1 \ 1 \ -1] \begin{Bmatrix} w_4 \\ w_{12} \\ w_2 \\ w_{10} \end{Bmatrix} \quad (2.19)$$

The coefficients of w at the nodes in Eqs.(2.17) and (2.19) applied at general node 'i' are given in Fig6.(a) and (b). The coefficients of w for the derivatives $(\partial^2 w / \partial x \partial y)_i$ are given by Fig6.(c). We use central-difference to express the approximations of partial derivatives λ_x and λ_y are mesh widths in x and y directions.

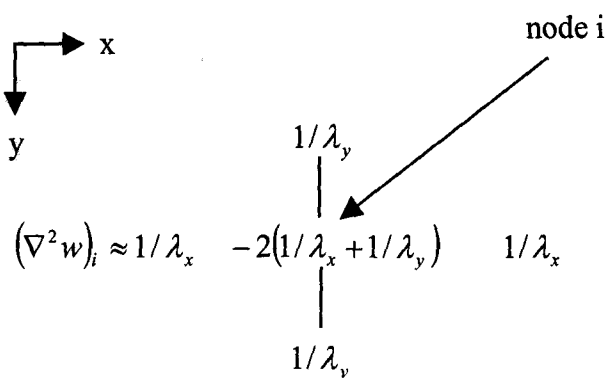


fig-6(a)

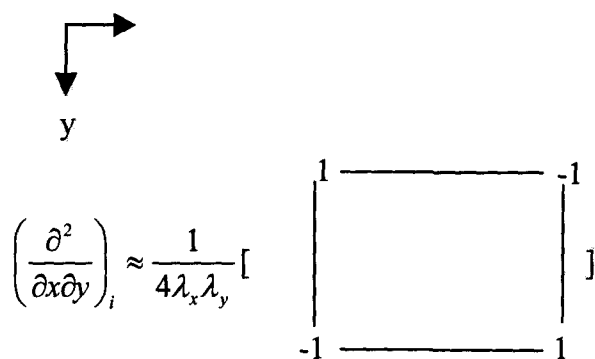


fig-6(b)

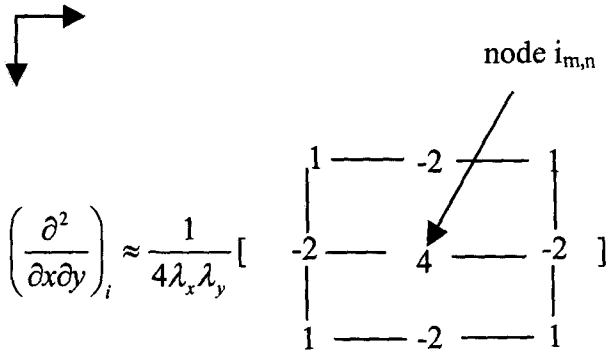


Fig.-6(c)

The finite difference expressions for the fourth-order derivatives of Eq. (2.1) can be written as

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_{m,n} \approx \frac{1}{\lambda^2} (w_{m+2,n} - 4w_{m+1,n} + 6w_{m,n} - 4w_{m-1,n} + w_{m,n-2}) \quad \dots(2.20)$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_{m,n} \approx \frac{1}{\lambda^2} (w_{m,n+2} - 4w_{m,n+1} + 6w_{m,n} - 4w_{m,n-1} + w_{m,n-2}) \quad \dots(2.21)$$

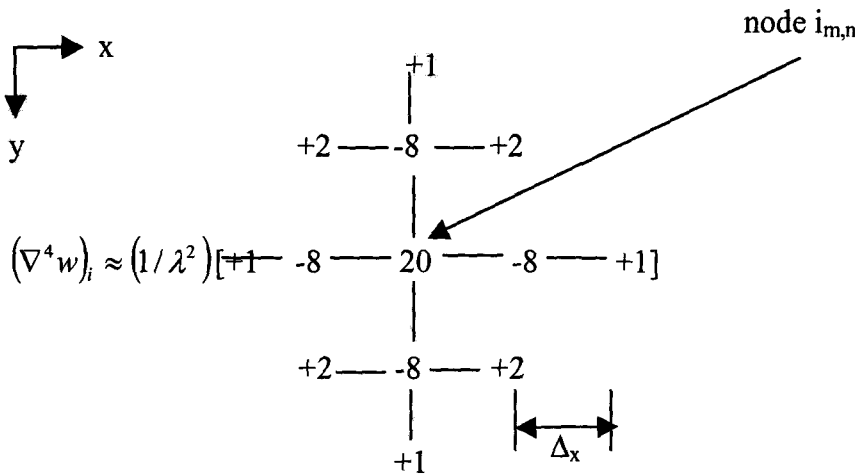


Fig.7.

2.3.4 Finite Difference Equation at Interior node of a plate in bending

Let us restrict our derivative to equally spaced square mesh, Assuming $\Delta x = \Delta y = \lambda$ mesh width, the finite difference expressions for the fourth order derivatives of equation (2.1).

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_{m,n} = \frac{1}{\lambda^2} (w_{m+2,n} - 4w_{m+1,n} + 6w_{m,n} - 4w_{m-1,n} + w_{m-2,n}) \quad \dots\dots (2.20)$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_{m,n} = \frac{1}{\lambda^2} (w_{m,n+2} - 4w_{m,n+1} + 6w_{m,n} - 4w_{m,n-1} + w_{m,n-2}) \quad \dots\dots (2.21)$$

Similarly the finite difference expression of the mixed fourth derivative (using central-difference to express the approximations of partial derivatives taking λ_x and λ_y as mesh widths in x and y directions shown in fig. 6) can be written as

$$\left(\frac{\partial^4 w}{\partial x^2 \partial y^2}\right)_{m,n} = \frac{1}{\lambda^2} (4w_{m,n} - 2(w_{m+1,n} + w_{m-1,n} + w_{m,n+1} + w_{m,n-1}) + (w_{m-1,n+1} + w_{m+1,n-1} + w_{m-1,n+1} + w_{m-1,n-1})) \quad \dots\dots (2.22)$$

Therefore, the finite difference representation of (2.1) at pivotal point m,n is

$$\begin{aligned} [D\nabla^4 W]_{m,n} &= \frac{D}{\lambda^2} [20w_{m,n} - 8(w_{m+1,n} + w_{m-1,n} + w_{m,n+1} + w_{m,n-1}) + 2(w_{m+1,n+1} + w_{m-1,n+1} + w_{m+1,n-1} + \\ &\quad w_{m-1,n-1}) + w_{m+2,n} + w_{m-2,n} + w_{m,n-2} + w_{m,n-2}] + O(\lambda^2) \\ &= q(x, y) \quad \dots\dots\dots(2.23) \end{aligned}$$

where $O(\lambda^2)$ the error term describing the discrepancy between the exact expression of the biharmonic operator (∇^2 operating on w) and its finite difference representation.

$$\text{So, [co-efficients] } \{w\} = q/D \quad \dots\dots (2.24)$$

Employing above difference expressions and boundary conditioned (used to eliminate the deflections of fictitious points), we can get equations to determine the

deflection of plate. Further, we can determine the internal forces and moments, which are the foundation of stress analysis. This is tedious. We will not describe more.

Here, we will give a case to show how to use finite – difference method to find the maximum deflection of uniformity loaded square plate with fixed boundary conditions. We select the same finite difference mesh ($\lambda=a/4$). In numbering the mesh points the apparent double symmetry of the deflected middle surface is considered. From the clamped boundary conditions it follows the fictitious points outside of the plate are the same as the corresponding points within the plate.

TABLE -12

No. of mesh	PRESENT	% ERROR
6 X 6	.0015344	21.78
8 X 8	.0014244	13.05
10 X 10	.0013697	8.706
14 X 14	.0013198	4.746
18 X 18	.0012986	3.0635
22 X 22	.0012877	2.1984
26 X 26	.0012814	1.6984
30 X 30	.0012774	1.381
34 X 34	.0012748	1.1746
38 X 38	.0012729	1.0238
42 X 42	.0012715	0.9127