CHAPTER - I

SEMI-INFINITE CRACKS PROPAGATING IN AN ELASTIC MEDIUM

PAPER 1 : Steady state propagation of a series of parallel cracks in antiplane state of strain in an inhomogeneous elastic medium.

PAPER 2 : Scattering of antiplane shear wave by a propagating crack at the interface of two dissimilar elastic media.

media when the crack moves in the direction of the modulus variation. Steady state crack propagation due to shear waves in a medium of monoclinic type has recently been studied by Chattopadhyay and Bandyopadhyay (1988).

In our paper, we have considered the steady state propagation of a series of semi-infinite, rectilinear parallel and uniformly spaced cracks in an infinite inhomogeneous medium. Cracks are assumed to move steadily in the direction of modulus variation, it being assumed that the moduli vary exponentially. We further assume that the medium possesses constant elastic wave speeds. These assumptions are necessary for the steady state solution to exist. We assume that the loading is such that Mode III conditions prevail. Mode III is the simplest mode to analyze mathematically. Nevertheless, it can be expected that the results for the stress intensity factor obtained here will be qualitatively similar to other modes, even though the specific structure of the stress variation near the crack tip will differ in each case. Following Atkinson and List (1978), we have also assumed in our paper that the edges of the cracks are loaded on their entire length by constant strain.

2. FORMULATION OF THE PROBLEM

Consider an infinite elastic medium with spatially varying density and elastic moduli divided partially by an infinite number of semi-infinite, rectilinear, parallel and uniformly spaced cracks. The semi-infinite cracks are situated parallel to the negative x_1-axis at 2h distances apart and move along positive x_1-direction at a constant velocity $c < c_2$.

The cracks are assumed to propagate steadily in the direction of modulus variation. We assume that the elastic moduli and density both vary exponentially in the same manner; so that the medium may have constant elastic wave speeds.

Owing to the symmetry of the problem, it is reduced to the problem of an infinite elastic strip of thickness 2h weakened in the middle plane $x_2=0$ by a semi-infinite crack $x_1<0$, the surfaces $x_2 = \pm h$ of the strip being rigidly clamped.

The displacement $\vec{U}$ in the anti-plane state of strain in a rectangular co-ordinate system (x_1, x_2, x_3) is in the form

$$\vec{U} = [0, 0, w(x_1, x_2, t)] \tag{1}$$

The non-vanishing components of this state of strain are given by the following relations:-

$$e_{13} = \frac{\partial w}{\partial x_1}, \qquad e_{23} = \frac{\partial w}{\partial x_2}$$

$$\begin{aligned} \sigma_{13} &= \mu \frac{\partial w}{\partial x_1} , & \sigma_{23} &= \mu \frac{\partial w}{\partial x_2} \\ &= \mu_0 e^{2\alpha x_1} \frac{\partial w}{\partial x_1} & &= \mu_0 e^{2\alpha x_1} \frac{\partial w}{\partial x_2} \end{aligned} \tag{2}$$

where the shear modulous $\mu(x_1) == \mu_0 e^{2\alpha x_1}$, μ_0 and α are constants.

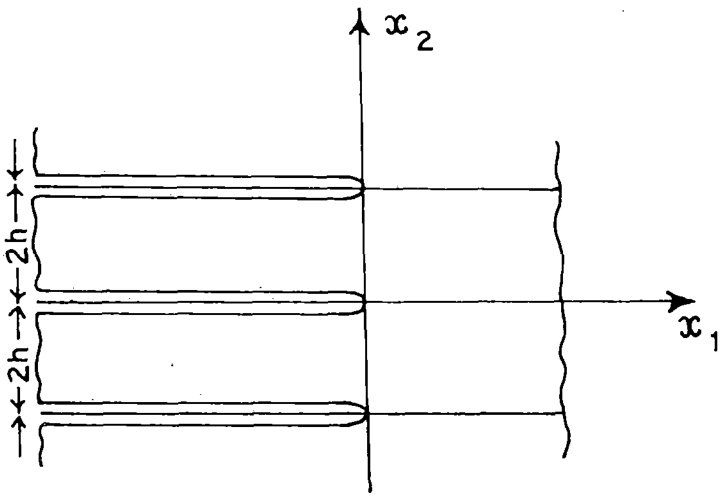


Fig. 1. Geometry of the problem .

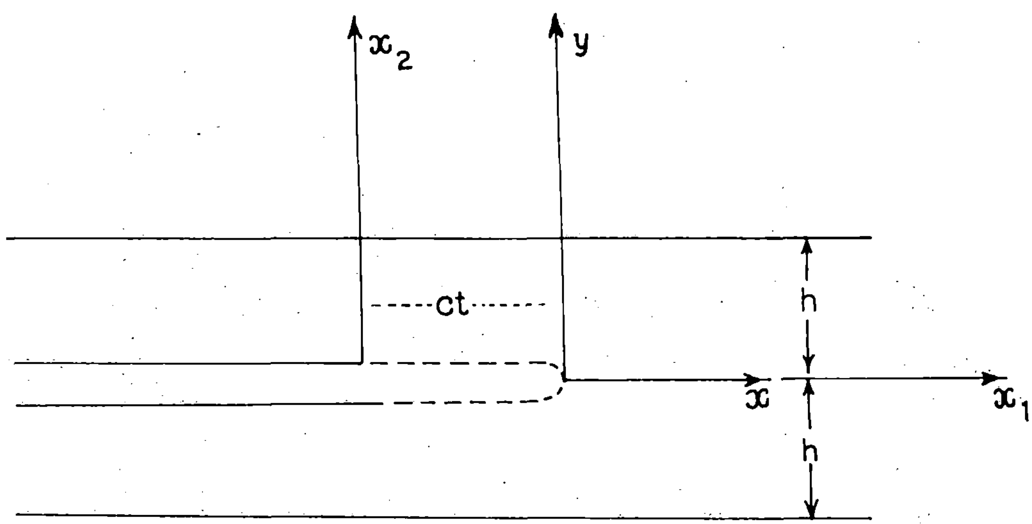


Fig. 2. Crack propagating in the strip.

Using relation (2), the equation of motion of SH-waves is

$$\frac{\partial}{\partial x_1}\left[\mu(x_1)\frac{\partial w}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[\mu(x_1)\frac{\partial w}{\partial x_2}\right] = \rho(x_1)\frac{\partial^2 w}{\partial t^2}$$

or,
$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + 2\alpha\frac{\partial w}{\partial x_1} = c_2^{-2}\frac{\partial^2 w}{\partial t^2} \qquad (3)$$

where $\rho(x_1)=\rho_o e^{2\alpha x_1}$; so $c_2 = \sqrt{\mu(x_1)/\rho(x_1)} = \sqrt{\mu_o/\rho_o}$ is the shear-wave velocity.

The fixed co-ordinate system may be repalced by the conventional system (x,y,z) moving with the crack tip,

$$x_1 = x + ct, \quad x_2 = y, \quad x_3 = z \qquad (4)$$

Using relation (4) , equation (3) becomes

$$\left(1-\frac{c^2}{c_2^2}\right)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 2\alpha\frac{\partial w}{\partial x} = 0 \qquad (5)$$

Applying complex Fourier transform in x, equation (5) becomes

$$\frac{d^2\overline{W}}{dy^2} - \beta^2\overline{W} = 0 \qquad (6)$$

where
$$\beta^2 = \left(1-\frac{c^2}{c_2^2}\right)\zeta^2 + 2i\alpha\zeta \qquad (7.1)$$

and

$$\overline{W}(\zeta,y) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} w(x,y)\, e^{i\zeta x}\, dx \qquad (7.2)$$

The solution of equation (6) becomes

$$\overline{W}(\zeta,y) = A\sinh(\beta y) + B\cosh(\beta y) \qquad (8)$$

where the constants A and B are to be determined.

3. SOLUTION OF THE PROBLEM FOR CONSTANT STRAIN $\frac{\partial w}{\partial y} = P$ OF THE CRACK EDGES $x<0$

We now consider the problem when the constant strain given by

$$\frac{\partial w}{\partial y} = P \tag{9}$$

is applied to the crack faces $y = 0$, $x < 0$.

We shall therefore consider the steady state crack propagation under the boundary conditions

$$\frac{\partial w}{\partial y} = P, \qquad \text{for} \quad x<0, \; y=0 \tag{10.1}$$

$$w(x,y) = 0, \qquad \text{for} \quad x>0, \; y=0 \tag{10.2}$$

$$w(x,y) = 0, \qquad \text{for} \quad |x|<\infty, \; y=h. \tag{10.3}$$

Now we can write

$$\frac{\partial w}{\partial y} = P, \qquad \text{for} \quad x<0, \; y=0$$

$$= e(x) \qquad \text{for} \quad x>0, \; y=0$$

where $e(x)$ is the unknown function which is to be determined.

In our case

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} \frac{\partial w}{\partial y}\, e^{i\zeta x}\, dx = (2\pi)^{-1/2}\int_{-\infty}^{0} \frac{\partial w}{\partial y}\, e^{i\zeta x}\, dx + (2\pi)^{-1/2}\int_{0}^{\infty} \frac{\partial w}{\partial y}\, e^{i\zeta x}\, dx$$

$$\frac{\partial \overline{W}(\zeta,0)}{\partial y} = (2\pi)^{-1/2}\int_{-\infty}^{0} P\, e^{i\zeta x}\, dx + (2\pi)^{-1/2}\int_{0}^{\infty} e(x)\, e^{i\zeta x}\, dx \tag{11}$$

Therefore using (8) and writing $(2\pi)^{-1/2}\int_0^{\infty} e(x)\, e^{i\zeta x}\, dx = E_+(\zeta)$,

$$\beta A = (2\pi)^{-1/2}\frac{P}{i\zeta} + E_+(\zeta) \quad \text{for } -k < \operatorname{Im}\zeta < 0 \qquad (12)$$

$$\text{if } e(x) \sim O(e^{-kx}) \text{ as } x \to \infty.$$

Using the conditions (10.2) and(10.3), it can be easily shown that

$$A = -\frac{\overline{W}_-(\zeta,0)}{\tanh(\beta h)} \qquad (13)$$

where $\overline{W}_-(\zeta,0) = (2\pi)^{-1/2}\int_{-\infty}^{0} w(x,0)\, e^{i\zeta x}\, dx$ is analytic in the lower half-plane $\operatorname{Im}\zeta<k_1$, if we assume $w(x,0)$ $O(e^{k_1 x})$ as $x \to -\infty$.

Eliminating A by equations (12) and (13)

$$-\beta\frac{\overline{W}_-(\zeta,0)}{\tanh(\beta h)} = -\frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta} + E_+(\zeta) \qquad (14)$$

Let $K(\zeta) = \beta \coth(\beta h) = \frac{1}{h}\beta h\frac{\cosh(\beta h)}{\sinh(\beta h)} = \frac{1}{h}\prod_{n=1}^{\infty}\left\{\frac{1-\left(\frac{i\beta h}{\pi(n-1/2)}\right)^2}{1-\left(\frac{i\beta h}{n\pi}\right)^2}\right\}$ (15)

[cf. Noble (1958) eqns (3.96a) and (3.96b), p.123]

Now consider

$$1-\left(\frac{i\beta h}{n\pi}\right)^2 = 1+\left(\frac{\beta h}{n\pi}\right)^2 = \left(\frac{h}{n\pi}\right)^2\left[\nu^2\zeta^2 + 2i\alpha\zeta + \left(\frac{n\pi}{h}\right)^2\right]$$

$$= \left(\frac{\nu h}{n\pi}\right)^2\left[\zeta^2 + \frac{2i\alpha\zeta}{\nu^2} + \left(\frac{n\pi}{\nu h}\right)^2\right] \qquad (16)$$

where $\nu^2 = 1 - c^2/c_2^2$.

So equation (16) can be written as

$$1 - \left(\frac{i\beta h}{n\pi}\right)^2 = \left(\frac{\nu h}{n\pi}\right)^2 (\zeta + i\eta_n^+)(\zeta + i\eta_n^-)$$

where

$$\eta_n^{\pm} = \frac{\alpha}{\nu^2} \pm \left[\frac{\alpha^2}{\nu^4} + \left(\frac{n\pi}{\nu h}\right)^2\right]^{1/2}$$

Similarly, $$1 - \left[\frac{i\beta h}{(n-1/2)\pi}\right]^2 = \left(\frac{\nu h}{n\pi}\right)^2 (\zeta + i\eta^+_{n-1/2})(\zeta + i\eta^-_{n-1/2})$$

It may be noted that η_n^- and $\eta^-_{n-1/2}$ are negative real quantities. So equation (15) becomes

$$K(\zeta) = \frac{1}{h}\prod_{n=1}^{\infty} \frac{(\zeta + i\eta^-_{n-1/2})(\zeta + i\eta^+_{n-1/2})\, n^2}{(\zeta + i\eta_n^-)(\zeta + i\eta_n^+)\ (n-1/2)^2}$$

$$= \frac{1}{h}\prod_{n=1}^{\infty} \frac{(\zeta + i\eta^-_{n-1/2})}{(\zeta + i\eta_n^-)} \frac{n}{(n-1/2)} \cdot \prod_{n=1}^{\infty} \frac{(\zeta + i\eta^+_{n-1/2})}{(\zeta + i\eta_n^+)} \frac{n}{(n-1/2)}$$

$$= K^-(\zeta).K^+(\zeta) \quad \text{(say)} \qquad (17)$$

where $K^-(\zeta)$ is analytic in the lower half-plane given by $\mathrm{Im}\ \zeta < -\eta^-_{1/2}$ whereas $K^+(\zeta)$ is analytic in the upper half plane given by $\mathrm{Im}\ \zeta > -\eta^+_{1/2}$.

Now

$$K^+(\zeta) = \prod_{n=1}^{\infty} \frac{(\zeta + i\eta^+_{n-1/2})}{(\zeta + i\eta_n^+)} \frac{(n-0)}{(n-1/2)}$$

$$= \prod_{n=1}^{\infty} \frac{\left[\zeta + i\left\{\frac{\alpha}{\nu^2} + \left(\frac{\alpha^2}{\nu^4} + \frac{(n-1/2)^2\pi^2}{\nu^2 h^2}\right)^{1/2}\right\}\right](n-0)}{\left[\zeta + i\left\{\frac{\alpha}{\nu^2} + \left(\frac{\alpha^2}{\nu^4} + \frac{n^2\pi^2}{\nu^2 h^2}\right)^{1/2}\right\}\right](n-1/2)}$$

$$= \prod_{n=1}^{\infty} \frac{\left[\frac{\zeta\nu h}{\pi} + i\left\{\frac{\alpha h}{\nu\pi} + \left\{\frac{\alpha^2 h^2}{\nu^2\pi^2} + (n-1/2)^2\right\}^{1/2}\right\}\right](n-0)}{\left[\frac{\zeta\nu h}{\pi} + i\left\{\frac{\alpha h}{\nu\pi} + \left\{\frac{\alpha^2 h^2}{\nu^2\pi^2} + n^2\right\}\right\}^{1/2}\right](n-1/2)}$$

Now elastic moduli and density are assumed to be varying slowly with x_1 so that αh may be assumed to be small.

So neglecting $\alpha^2 h^2$ we get

$$K^+(\zeta) = \prod_{n=1}^{\infty} \frac{\left[\frac{\zeta\nu h}{\pi} + i\left\{\frac{\alpha h}{\nu\pi} + (n-1/2)\right\}\right](n-0)}{\left[\frac{\zeta\nu h}{\pi} + i\left(\frac{\alpha h}{\nu\pi} + n\right)\right](n-1/2)}$$

$$= \prod_{n=1}^{\infty} \frac{\left[n - \left(\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right)\right](n-0)}{\left[n - \left(\frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right)\right](n-1/2)} \tag{18}$$

Next using the formula

$$\prod_{n=1}^{\infty}\left\{\frac{(n-a_1)\ldots\ldots\ldots\ldots(n-a_k)}{(n-b_1)\ldots\ldots\ldots\ldots(n-b_k)}\right\} = \prod_{m=1}^{k} \frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}$$

which expresses the general infinite product in terms of the Gamma function (cf. Whittaker and Watson ,1969, p.239) we obtain from

(18)

$$K^{+}(\zeta) = \frac{\Gamma(\frac{1}{2})\ \Gamma\ [\ 1 - (\ \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\ )]}{\Gamma(1)\ \Gamma\ [\frac{1}{2} - (\frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\ )]} \tag{19}$$

Similarly, for small values of αh, neglecting $\alpha^2 h^2$, it can be easily shown that

$$K^{-}(\zeta) = \frac{\sqrt{\pi}}{h}\frac{\Gamma\ [\ 1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\ ]}{\Gamma\ [\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\ ]} \tag{20}$$

Now writing $\beta\ \coth(\beta h) = K(\zeta) = K^{+}(\zeta)K^{-}(\zeta)$, equation (14) becomes

$$- K^{+}(\zeta)K^{-}(\zeta)\ \overline{W}_{-}(\zeta,0) = - \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta} + E_{+}(\zeta)$$

so,

$$- K^{-}(\zeta)\ \overline{W}_{-}(\zeta,0) = - \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(\zeta)} + \frac{E_{+}(\zeta)}{K^{+}(\zeta)}$$

$$= - \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\left[\frac{1}{K^{+}(\zeta)} - \frac{1}{K^{+}(0)}\right] - \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(0)} + \frac{E_{+}(\zeta)}{K^{+}(\zeta)}$$

Therefore,

$$- K^{-}(\zeta)\ \overline{W}_{-}(\zeta,0) + \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\frac{1}{K^{+}(0)} = \frac{E_{+}(\zeta)}{K^{+}(\zeta)} - \frac{iP}{\sqrt{2\pi}}\frac{1}{\zeta}\left[\frac{1}{K^{+}(\zeta)} - \frac{1}{K^{+}(0)}\right]. \tag{21}$$

The expression on the left hand side of equation (21) is regular in the half-plane $\mathrm{Im}\ \zeta < 0$ whereas R.H.S. is regular in $\mathrm{Im}\ \zeta > - K_1$

where $K_1 = \min(k, \eta^+_{1/2})$. The equation (21) holds in the strip $-K_1 < \mathrm{Im}\ \zeta < 0$ and therefore using analytic continuation and Liouville's theorem we can write

$$\overline{W}_-(\zeta,0) = \frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \frac{1}{K^+(0)K^-(\zeta)} \tag{22}$$

and

$$E_+(\zeta) = \frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \left[1 - \frac{K^+(\zeta)}{K^+(0)} \right] \tag{23}$$

Therefore, by help of (11) and (23), we obtain

$$\frac{\partial \overline{W}(\zeta,0)}{\partial y} = - \frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \frac{K^+(\zeta)}{K^+(0)} .$$

So,

$$\frac{\partial w}{\partial y} = - \frac{iP}{\sqrt{2\pi}} \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{1}{\zeta} \frac{K^+(\zeta)}{K^+(0)} e^{-i\zeta x}\, d\zeta \quad \text{where } -K_1 < \varepsilon < 0. \tag{24}$$

For $x<0$, considering a semi-circular contour in the upper half ζ-plane it can easily be verified that

$$\frac{\partial w}{\partial y} = P$$

Now for $x>0$, substituting the values of $K^+(\zeta)$ and $K^+(0)$ from (19) and (24) we obtain

$$\frac{\partial w}{\partial y} = - \frac{iP}{2\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{1}{\zeta} \frac{\Gamma\left[1 - \frac{i\zeta\nu h}{\pi} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[\frac{1}{2} - \frac{i\zeta\nu h}{\pi} + \frac{\alpha h}{\nu\pi}\right]} e^{-i\zeta x}\, d\zeta$$

$(x>0)$

$$= -\frac{iP}{2\pi}\frac{\Gamma[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]}{\Gamma[1+\frac{\alpha h}{\nu\pi}]} e^{-\frac{\pi x}{\nu h}(\frac{1}{2}+\frac{\alpha h}{\nu\pi})}\int_{-i\infty+s}^{i\infty+s}\frac{1}{(\frac{1}{2}-p+\frac{\alpha h}{\nu\pi})}\times$$

$$\times\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} e^{\frac{\pi x}{\nu h}p} dp$$

$$\text{where} \quad s = \frac{1}{2}+\frac{\alpha h}{\nu\pi}-\frac{\nu h\varepsilon}{\pi}$$

$$= \frac{iP}{2\pi}\frac{\Gamma[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]}{\Gamma[1+\frac{\alpha h}{\nu\pi}]} e^{-\frac{\pi x}{\nu h}(\frac{1}{2}+\frac{\alpha h}{\nu\pi})}\int_{-i\infty+s}^{i\infty+s}\frac{\Gamma(p-\frac{1}{2}-\frac{\alpha h}{\nu\pi})}{\Gamma(p+\frac{1}{2}-\frac{\alpha h}{\nu\pi})}\times$$

$$\times\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)} e^{\frac{\pi x}{\nu h}p} dp$$

$$= -P\frac{\Gamma[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]}{\Gamma[1+\frac{\alpha h}{\nu\pi}]} e^{-\frac{\pi x}{\nu h}(\frac{1}{2}+\frac{\alpha h}{\nu\pi})}\frac{e^{\frac{\pi x}{\nu h}(\frac{1}{2}+\frac{\alpha h}{\nu\pi})}(1-e^{-\frac{\pi x}{\nu h}})^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}$$

$$\times {}_2F_1(-\tfrac{1}{2};-\tfrac{\alpha h}{\nu\pi};\tfrac{1}{2};1-e^{-\frac{\pi x}{\nu h}})$$

[cf. Erdelyi et al. (1954) formula no. 7 . p.262]

$$= -\frac{P}{\sqrt{\pi}}\frac{\Gamma[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]}{\Gamma[1+\frac{\alpha h}{\nu\pi}]}\frac{1}{\sqrt{1-\exp(-\pi x/\nu h)}}\,{}_2F_1(-\tfrac{1}{2};-\tfrac{\alpha h}{\nu\pi};\tfrac{1}{2};1-e^{-\frac{\pi x}{\nu h}})$$

where ${}_2F_1(-\frac{1}{2};-\frac{\alpha h}{\nu\pi};\frac{1}{2};1-e^{-\frac{\pi x}{\nu h}})$ is the hypergeometric function.

It is known that the Hypergeometric series

$${}_2F_1(a,b,c,z) = 1+\frac{a.b}{1.c}z+\frac{a(a+1)\,b(b+1)}{1.2.c(c+1)}z^2+$$

$$+ \frac{a(a+1)(a+2)\ b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)} z^3 + \ldots\ldots\ldots\ldots\ldots$$

therefore neglecting $\left(\frac{\alpha h}{\nu\pi}\right)^2$ and higher powers of $\frac{\alpha h}{\nu\pi}$,

$${}_2F_1\left(-\frac{1}{2};\ -\frac{\alpha h}{\nu\pi};\ \frac{1}{2};\ 1 - e^{-\frac{\pi x}{\nu h}}\right) = 1 + \frac{\alpha h}{\nu\pi}\left(\frac{z}{1} + \frac{z^2}{2.3} + \frac{z^3}{3.4} + \frac{z^4}{4.7} + \ldots\ldots\right)$$

where $z = 1 - e^{-\frac{\pi x}{\nu h}}$;

After a little algebraic simplification it can be shown that for small $\frac{\alpha h}{\nu\pi}$

$${}_2F_1\left(-\frac{1}{2};\ -\frac{\alpha h}{\nu\pi};\ \frac{1}{2};\ z\right) = 1 + \frac{\alpha h}{\nu\pi}\Big[(1+\sqrt{z})\log(1+\sqrt{z})+(1-\sqrt{z})\log(1-\sqrt{z})\Big]$$

Therefore

$$\frac{\partial w}{\partial y} = -\frac{P}{\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1 - \exp(-\pi x/\nu h)}} \times$$

$$\times\left\{1 + \frac{\alpha h}{\nu\pi}\Big[(1+\sqrt{z})\log(1+\sqrt{z})+(1-\sqrt{z})\log(1-\sqrt{z})\Big]\right\} \quad (x>0) \qquad (25)$$

Next in order to determine the crack opening displacement consider equation (22) viz.

$$\bar{W}_-(\zeta,0) = \frac{iP}{\sqrt{2\pi}} \frac{1}{\zeta} \frac{1}{K^+(0)K^-(\zeta)}$$

which by help of equations (19) and (20) becomes

$$\bar{W}_-(\zeta,0) = \frac{iP}{\sqrt{2\pi}} \frac{h}{\pi} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\zeta}$$

Therefore

$$w(x,0) = \frac{ihP}{\pi}\frac{1}{2\pi}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\zeta} e^{-i\zeta x}\, d\zeta$$

Obviously for $x > 0$, $w(x,0) \equiv 0$. In order to find $w(x,0)$ for $x < 0$, we firstly evaluate $\frac{dw(x,0)}{dx}$ which is given by

$$\frac{dw}{dx} = \frac{hP}{\pi}\frac{1}{2\pi}\frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\Gamma\left[\frac{1}{2} + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{i\zeta\nu h}{\pi} - \frac{\alpha h}{\nu\pi}\right]} e^{-i\zeta x}\, d\zeta$$

$$\text{so,}\quad \frac{dw}{dx} = \frac{P\, e^{\frac{\pi x}{\nu h}\left(\frac{1}{2} - \frac{\alpha h}{\nu\pi}\right)}}{\nu\sqrt{\pi}} \frac{1}{2\pi i} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \int_{s-i\infty}^{s+i\infty} \frac{\Gamma(\frac{1}{2})\,\Gamma(p)}{\Gamma(p+\frac{1}{2})} e^{-\frac{\pi x}{\nu h}p}\, dp$$

where $p = \frac{1}{2} + \frac{i\nu\zeta h}{\pi} - \frac{\alpha h}{\nu\pi}$ and $s = \frac{1}{2} - \frac{\alpha h}{\nu\pi} + \frac{\nu h\varepsilon}{\pi}$.

Using the table of inverse Laplace transform (1954), we find

$$\frac{dw}{dx} = \frac{P\, e^{\frac{\pi x}{\nu h}\left(\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right)}}{\nu\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \frac{1}{\sqrt{1 - \exp(\pi x/\nu h)}}$$

Integrating w.r.t. x we obtain

$$w(x,0) = \frac{P}{\nu\sqrt{\pi}} \frac{\Gamma\left[\frac{1}{2} + \frac{\alpha h}{\nu\pi}\right]}{\Gamma\left[1 + \frac{\alpha h}{\nu\pi}\right]} \int_0^x e^{-\frac{\alpha x}{\nu^2}} \frac{e^{\frac{\pi x}{2\nu h}}}{\sqrt{1 - \exp(\pi x/\nu h)}}\, dx \qquad (\text{for } x<0)$$

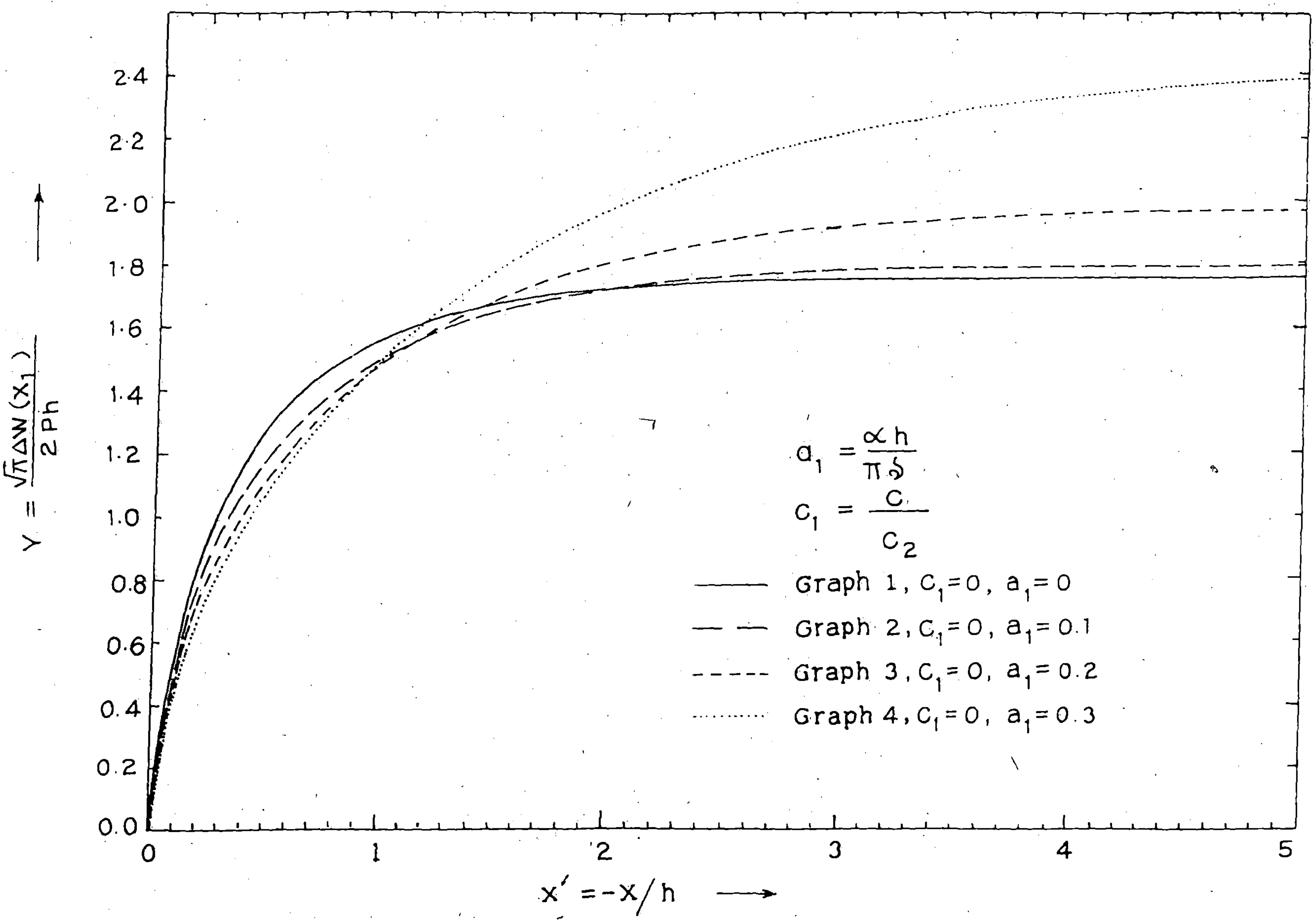
Making $x \longrightarrow -\infty$, it can easily be shown that

$$w(x,0) \longrightarrow -\frac{Ph}{\pi}\frac{\Gamma\,[\frac{1}{2}+\frac{\alpha h}{\nu\pi}]\;\Gamma\,[\frac{1}{2}-\frac{\alpha h}{\nu\pi}]}{\Gamma\,[1+\frac{\alpha h}{\nu\pi}]\;\Gamma\,[1-\frac{\alpha h}{\nu\pi}]} \tag{26}$$

Putting $\alpha = 0$ in (25) and (26) expressions for $\frac{\partial w(x,0)}{\partial y}$ and $w(x,0)$ for homogeneous medium can be derived and they are found to be identical with the results given by Matczynski (1973).

Crack opening displacement is obviously $\Delta w = 2w(x,0)$ where $w(x,0)$ is given by (26). In Figs. 3-5 dimensionless values of the crack opening displacement given by $Y = \frac{\pi\Delta w}{2ph}$ have been plotted against the dimensionless distance $x' = -\frac{x}{h}$ along the length of the crack for different values of $\alpha_1 = \frac{\alpha h}{\nu\pi}$ and $c_1 = c/c_2$.

It is interesting to note that for a fixed value of c_1, crack opening displacement increases with the increase in the values of the inhomogeneity parameter α_1 for large values of x' whereas for small values of $x'(x' \neq 0)$, the result is just the opposite. Further it may be noted that for any given value of the inhomogeneity parameter α_1, crack opening displacement Y at any point x' increases with the increase in the value of the crack propagation velocity.



2.4
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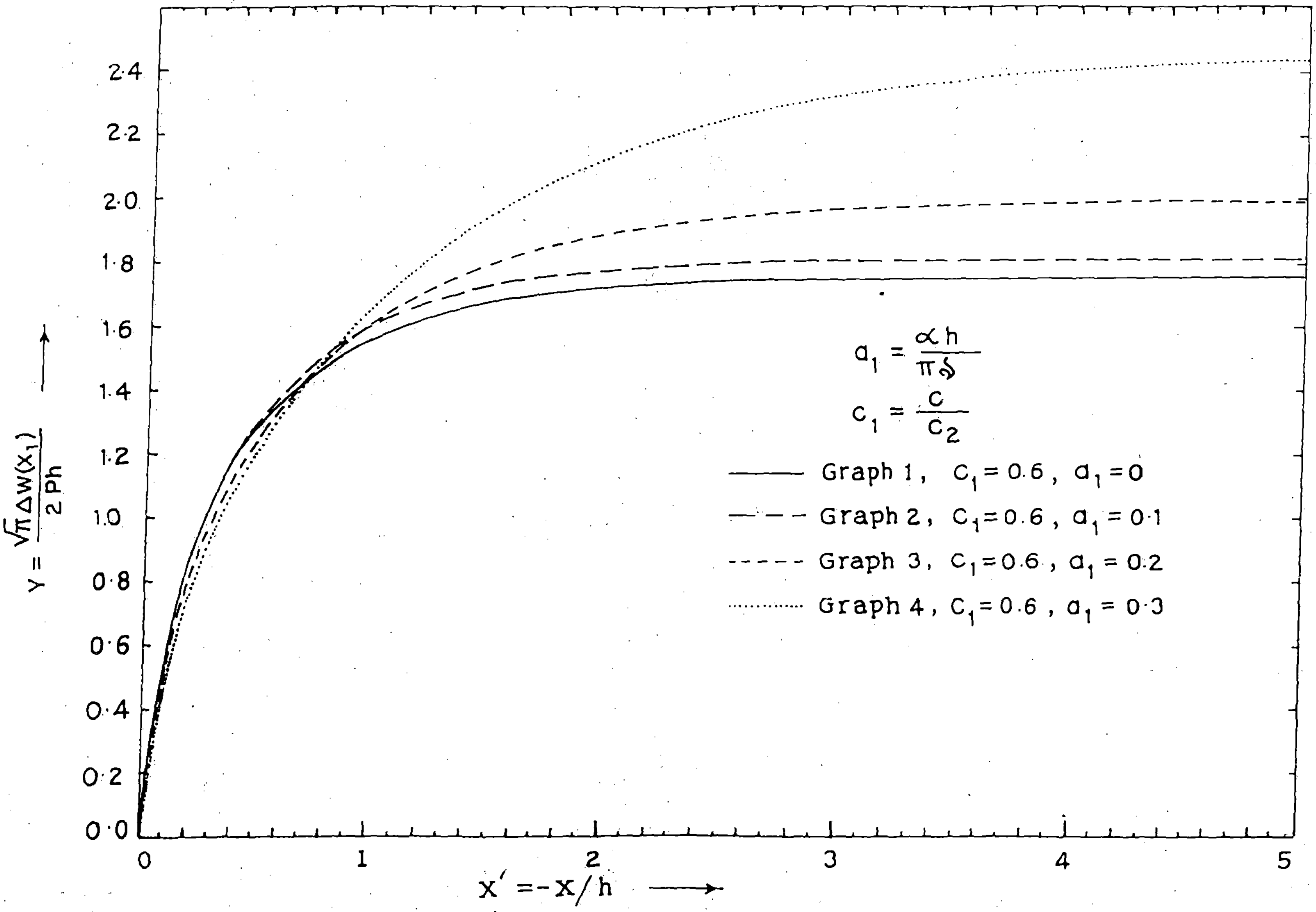
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
$Y = \frac{\sqrt{\pi}\Delta W(x_1)}{2Ph}$
$x' = -x/h$
$a_1 = \frac{\alpha h}{\pi \delta}$
$c_1 = \frac{c}{c_2}$
Graph 1, $C_1 = 0$, $a_1 = 0$
Graph 2, $C_1 = 0$, $a_1 = 0.1$
Graph 3, $C_1 = 0$, $a_1 = 0.2$
Graph 4, $C_1 = 0$, $a_1 = 0.3$

Fig. 3 Y vs x'



$Y = \frac{\sqrt{\pi}\Delta W(x_1)}{2Ph}$
$x' = -x/h$
0
1
2
3
4
5
0·0
0·2
0·4
0·6
0·8
1.0
1·2
1·4
1·6
1·8
2·0
2·2
2·4
$a_1 = \frac{\alpha h}{\pi \delta}$
$c_1 = \frac{c}{c_2}$
Graph 1, $C_1 = 0.6$, $a_1 = 0$
Graph 2, $C_1 = 0.6$, $a_1 = 0{\cdot}1$
Graph 3, $C_1 = 0.6$, $a_1 = 0{\cdot}2$
Graph 4, $C_1 = 0.6$, $a_1 = 0{\cdot}3$

Fig. 4   Y vs. x'

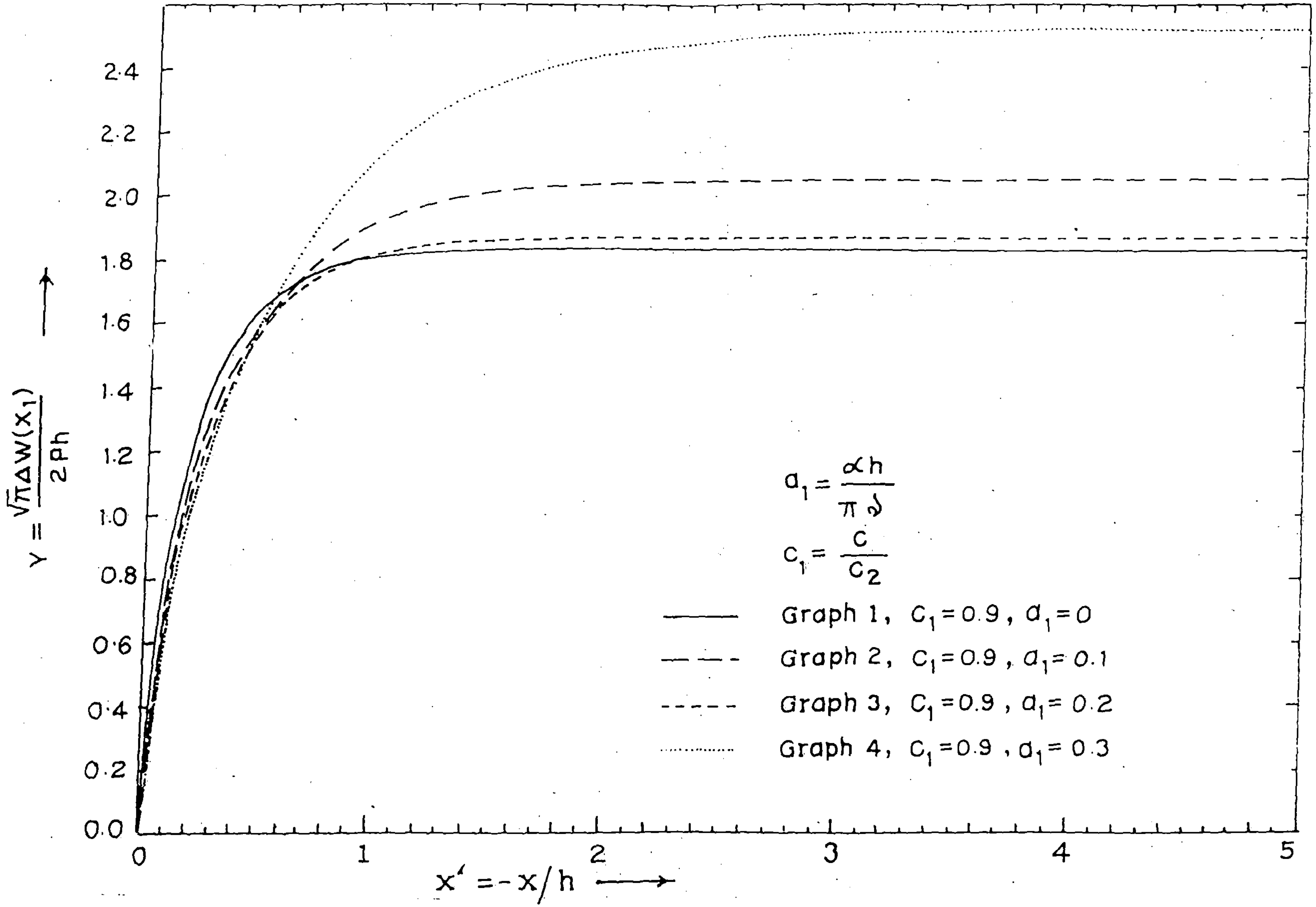

2·4
2·2
2·0
1·8
1·6
1·4
1·2
1·0
0·8
0·6
0·4
0·2
0.0
0
1
2
3
4
5
$Y = \frac{\sqrt{\pi}\Delta W(x_1)}{2Ph}$
$x' = -x/h$
$a_1 = \frac{\alpha h}{\pi \delta}$
$c_1 = \frac{c}{c_2}$
Graph 1, $c_1 = 0.9$, $a_1 = 0$
Graph 2, $c_1 = 0.9$, $a_1 = 0.1$
Graph 3, $c_1 = 0.9$, $a_1 = 0.2$
Graph 4, $c_1 = 0.9$, $a_1 = 0.3$

Fig. 5. Y vs. x'

SCATTERING OF ANTIPLANE SHEAR WAVE BY A PROPAGATING CRACK AT THE INTERFACE OF TWO DISSIMILAR ELASTIC MEDIA

1. INTRODUCTION

It is well known that the problems of diffraction of elastic wave by cracks or inclusions are of considerable importance in view of their application in seismology and geophysics. If the cracks or inclusions are located at the interface of layered media, the study becomes more relevant. The extensive use of composite materials in modern technology has also evoked interest in the wave propagation problems in layered media with interfacial discontinuities. Onder et al. (1975) studied the diffraction of monochromatic plane SH-waves obliquely incident on a rigid half-plane between the two different semi-infinite media.

In this paper we have considered the problem of the diffraction of a plane harmonic SH-wave by a semi-infinite crack running uniformly along the interface of two dissimilar semi-infinite elastic media. The problem of scattering of plane harmonic polarized shear wave by a half-plane crack in an infinite isotropic medium extending under antiplane strain was studied

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earlier by Jahanshahi (1967). Chen and Sih (1973, 1975) also solved the in-plane problem of the diffraction of stress waves by a running crack in an incident wave field in an infinite elastic medium. We have applied Fourier transform and Wiener-Hopf technique (1958) to solve the mixed boundary value problem. The resulting integrals have been evaluated asymptotically to obtain the displacement and stress field near about the crack tip. It is found that the stress intensity factor depends sensitively upon the speed of crack propagation, the angle of incidence of the incoming wave and on the material properties of the elastic media. Quantitative assessment of the effect of the aforementioned parameters on the stress intensity factor has been made by displaying the numerical results graphically for two pairs of different materials.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION

Let a plane crack move at a constant velocity V on the interface of two bonded dissimilar elastic semi-infinite medium due to the incidence of the plane harmonic SH-wave

$$v_1^o = V_1 \exp[-i\{\Lambda_1 (X\cos\Theta_1 + Y\sin\Theta_1) + \Omega T\}] \qquad (1)$$

in the medium where the co-efficient of rigidity, density and shear wave velocity respectively are given by μ_1, ρ_1 and C_1. The crack lies on the bimaterial interface along Y=0 with respect to the fixed rectangular co-ordinate system (X,Y,Z).

We assume that the displacement and stress due to the scattered field are

$$v_j = v_j(X,Y,T) \tag{2}$$

and $$\left(\tau_{xz}\right)_j = \mu_j \frac{\partial v_j}{\partial X}, \qquad \left(\tau_{yz}\right)_j = \mu_j \frac{\partial v_j}{\partial Y} \tag{3}$$

where the subscript j=1,2 refers to the upper and lower half-planes and T, the time.

The equations of SH-wave motion in either elastic half-space are given by

$$\frac{\partial^2 v_j}{\partial X^2} + \frac{\partial^2 v_j}{\partial Y^2} = \frac{1}{C_j^2}\frac{\partial^2 v_j}{\partial T^2} \quad (j=1,2) \tag{4}$$

where $C_j = (\mu_j/\rho_j)^{1/2}$ is the shear-wave velocity. Without any loss of generality, we further assume that $C_1 > C_2$.

Due to the incident wave given in (1), the reflected and transmitted waves in the absence of the crack may be written in the form

$$v_1^r(X,Y,T) = V_1^r \exp[-i\{\Lambda_1(X\cos\Theta_1 - Y\sin\Theta_1) + \Omega T\}]$$

and (5)

$$v_2^T(X,Y,T) = V_2^T \exp[-i\{\Lambda_2(X\cos\Theta_2 + Y\sin\Theta_2) + \Omega T\}]$$

where

$$V_1^r = \frac{\mu_1\Lambda_1\sin\Theta_1 - \mu_2\Lambda_2\sin\Theta_2}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_1^R V_1 \text{ (say)}$$

and (6)

$$V_2^T = \frac{2\mu_1\Lambda_1\sin\Theta_1}{\mu_1\Lambda_1\sin\Theta_1 + \mu_2\Lambda_2\sin\Theta_2} V_1 = A_2^T V_1 \text{ (say)}$$

with

$$\Lambda_1 \cos\Theta_1 = \Lambda_2 \cos\Theta_2 \ .$$

V_1, V_1^r and V_2^T are the incident, reflected and transmitted wave amplitude respectively, Λ_j the wave number, $\Omega = \Lambda_j C_j$ the circular frequency and Θ_1, Θ_2 the angles of incidence and refraction respectively.

Assume that the crack has been moving in the horizontal direction along the interface for a sufficiently long time and that a steady state has been reached in the neighbourhood of the crack.

A set of moving co-ordinate systems (x,y,z,t) attached to the crack tip moving at a constant velocity V is introduced in accordance with

$$x = X-Vt \ , \quad y_j = s_j Y \ , \quad z = Z \ , \quad t = T \tag{7}$$

where $s_j = (1-M_j^2)^{1/2}$ and $M_j = V/C_j$ is the Mach number.

In terms of the moving co-ordinate system (x,y,t), (4) becomes

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial y_j^2} + \frac{1}{C_j^2 \ s_j^2} \frac{\partial}{\partial t}\left[2M_j C_j \frac{\partial v_j}{\partial x} - \frac{\partial v_j}{\partial t}\right] = 0. \tag{8}$$

It is convenient to define an apparent circular frequency $\omega = \alpha \, \Omega$ and the angles of reflection ϕ_1 and refraction ϕ_2 are given by

$$\cos\phi_j = M_j + (\Lambda_j/\lambda_j)\cos\Theta_j \ , \quad \sin\phi_j = (s_j/\alpha)\sin\Theta_j \ ,$$

where (9)

$$\alpha = (1+M_j\cos\Theta_j) \quad \text{and} \quad \lambda_j = (\Lambda_j/s_j^2)\alpha.$$

Using these relations in a moving system, (1) and (5) take the form

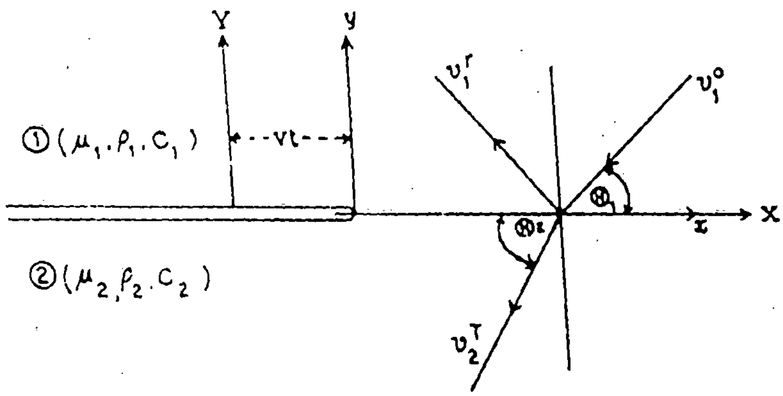

Figure 1. Geometry of the propagating crack.

$$\begin{bmatrix} v_1^o \\ v_1^r \\ v_2^T \end{bmatrix} = \begin{bmatrix} w_1^o(x,y_1) \\ w_1^r(x,y_1) \\ w_2^T(x,y_2) \end{bmatrix} \exp\left\{i(M_1\lambda_1 x-\omega t)\right\} \tag{10}$$

where

$$w_1^o(x,y_1) = V_1 \exp\left\{-i\lambda_1(x\cos\phi_1 + y_1\sin\phi_1)\right\}$$

$$w_1^r(x,y_1) = A_1^R V_1 \exp\left\{-i\lambda_1(x\cos\phi_1 - y_1\sin\phi_1)\right\}$$

$$w_2^T(x,y_2) = A_2^T V_1 \exp\left[-i\left\{(\beta_2+\lambda_2\cos\phi_2)x+\lambda_2 y_2\sin\phi_2\right\}\right] \tag{11}$$

and

$$\beta_2 = M_1\lambda_1\left[1 - \frac{\lambda_2 C_1}{\lambda_1 C_2}\right] < 0 \quad \text{since} \quad C_1 > C_2.$$

Using (10), we assume the solution of the governing equation (8) as

$$v_j(x,y_j,t) = w_j(x,y_j)\exp[i(M_j\lambda_j x-\omega t)]. \tag{12}$$

Substitution of (12) in (8) yields the Helmhotz equation

$$\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y_j^2} + \lambda_j^2 w_j = 0 \qquad (j=1,2). \tag{13}$$

Applying Fourier transform, (13) can be solved and the result is

$$w_1(x,y_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(\xi)\exp\left\{-i\xi x-(\xi^2-\lambda_1^2)^{1/2}y_1\right\} d\xi, \quad (y_1>0)$$

and

$$w_2(x,y_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(\xi)\exp\left\{-i\xi x-(\xi^2-\lambda_2^2)^{1/2}y_2\right\} d\xi, \quad (y_2<0)$$

(14)

where $A_1(\xi)$ and $A_2(\xi)$ are the unknown functions to be determined.

From (12) and (14) we obtain the displacement components due to scattered field as

$$v_1 = \exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(u)\exp[-iux-\gamma_1 y_1]du\ , \quad (y_1>0)$$

and

$$v_2 = \exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty} A_2(u)\exp[-iux+\gamma_2 y_2]du\ , \quad (y_2<0)$$

(15)

where $\gamma_1 = (u^2-\lambda_1^2)^{1/2}$ and $\gamma_2 = [(u-\beta_2)^2-\lambda_2^2]^{1/2}$. (16)

Therefore, the expressions for the stresses are

$$\left(\tau_{xz}\right)_1 = -i\mu_1\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}(u-M_1\lambda_1)A_1(u)\exp[-iux-\gamma_1 y_1]du$$

$$\left(\tau_{xz}\right)_2 = -i\mu_2\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}(u-M_1\lambda_1)A_2(u)\exp[-iux+\gamma_2 y_2]du$$

and

$$\left(\tau_{y_1z}\right)_1 = -\mu_1 s_1\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_1 A_1(u)\exp[-iux-\gamma_1 y_1]du$$

$$\left(\tau_{y_2z}\right)_2 = \mu_2 s_2\exp[i(M_1\lambda_1 x-\omega t)]\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_2 A_2(u)\exp[-iux+\gamma_2 y_2]du.$$

(17)

The unknown functions $A_1(u)$ and $A_2(u)$ are to be determined from the following boundary conditions at the interface y=0

(i) $\quad v_1(x,0) = v_2(x,0)\ , \quad x>0$

(ii) $\quad \mu_1 s_1 \dfrac{\partial v_1}{\partial y_1} = \mu_2 s_2 \dfrac{\partial v_2}{\partial y_2}\ , \quad -\infty<x<\infty$

and

$$\text{(iii)} \qquad \frac{\partial v_1^o}{\partial y_1} + \frac{\partial v_1^r}{\partial y_1} + \frac{\partial v_1}{\partial y_1} = 0 , \quad x<0, \quad y \to 0+ .$$

From the boundary condition (ii) we obtain

$$- \mu_1 s_1 \gamma_1 A_1(u) + \mu_2 s_2 \gamma_2 A_2(u) = 0 \tag{18}$$

and from other two boundary conditions, we get

$$\int_{-\infty}^{\infty} B_1(u)\exp(-iux)du = 0 \quad (x>0)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} M(u)B_1(u)\exp(-iux)du = N \exp(-i\lambda_1 x\cos\phi_1) , \quad (x<0) \tag{19}$$

where

$$B_1(u) = \frac{\mu_1 s_1 \gamma_1 + \mu_2 s_2 \gamma_2}{\mu_2 s_2 \gamma_2} A_1(u)$$

$$M(u) = \gamma_1 \frac{\mu_2 s_2 \gamma_2}{(\mu_1 s_1 \gamma_1 + \mu_2 s_2 \gamma_2)} \tag{20}$$

and

$$N = - \frac{i\Lambda_1 v_1 \sin\Theta_1}{s_1} (1-A_1^R).$$

The solution of the dual integral equation may be obtained by a method based on the Wiener-Hopf technique. The first part of (19) can be satisfied if we choose

$$B_1(u) = L_-(u) \tag{21}$$

where $L_-(u)$ is a function of u, analytic in the lower half of the complex u-plane. The second part is satisfied if we take

$$M(u)B_1(u) = \frac{N}{i(u-\alpha_1)} \frac{U_+(u)}{U_+(\alpha_1)} \tag{22}$$

where $\alpha_1=\lambda_1\cos\phi_1$ and $U_+(u)$ is a function free from zeros and singularities in the upper half of the complex u-plane. Thus (22) is a solution of the second part of (19) can easily be shown by completing the path from $-\infty$ to ∞ by a semi-circle of infinite radius in the upper u-plane and then applying the residue theorem and Jordan's Lemma. The path of integration is chosen to avoid possible branch points and is indented below the pole $u=\alpha_1$.

Eliminiting $B_1(u)$ from (21) and (22) we obtain

$$\frac{L_-(u)}{U_+(u)} = \frac{N}{i(u-\alpha_1)M(u)} \frac{1}{U_+(\alpha_1)} \tag{23}$$

and

$$M(u) = \frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2} (u^2-\lambda_1^2)^{1/2} F(u) \tag{24}$$

where

$$F(u) = \frac{\gamma_2(\mu_1 s_1+\mu_2 s_2)}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)}$$

and

$$F(u) \to 1 \quad \text{as} \quad |u| \to \infty .$$

The function F(u) can be expressed as the product of two functions such that

$$F(u) = F_+(u)F_-(u) \tag{25}$$

where $F_+(u)$ and $F_-(u)$ are analytic in the upper and lower half of the complex u-plane respectively. The expressions for $F_+(u)$ and $F_-(u)$ have been derived in the appendix.

In view of (25), (24) assumes the form

$$\frac{U_+(u)}{(u+\lambda_1)^{1/2}F_+(u)} = \frac{L_-(u)}{\dfrac{N}{iU_+(\alpha_1)}\,\dfrac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}} \tag{26}$$

where

$$U_+(u) = (u+\lambda_1)^{1/2}\ F_+(u). \tag{27}$$

So

$$L_-(u) = \frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\ \frac{\mu_1 s_1+\mu_2 s_2}{\mu_2 s_2(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)} \tag{28}$$

Hence the functions $A_1(u)$ and $A_2(u)$ are

$$A_1(u) = \frac{N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\ \frac{\gamma_2(\mu_1 s_1+\mu_2 s_2)}{(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)}$$

and

$$A_2(u) = \frac{-N}{i(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\ \frac{\mu_1 s_1\gamma_1(\mu_1 s_1+\mu_2 s_2)}{\mu_2 s_2(\mu_1 s_1\gamma_1+\mu_2 s_2\gamma_2)(u-\alpha_1)(u-\lambda_1)^{1/2}F_-(u)} \tag{29}$$

The singular behaviour of the stress components for the scattered waves at the crack-tip is due to the divergence of the integrals around $x=y_j=0$ in (17). Making use of (29) and asymptotic expressions of the integrands of (17) for large values of u, we obtain near about the crack-tip,

$$\left(\tau_{xz}\right)_1 = \frac{B(1+i)}{s_1}\int_0^\infty u^{-1/2} \exp[-s_1 uY](\cos ux-\sin ux)du\ .$$

$$\left(\tau_{xz}\right)_2 = -\frac{B(1+i)}{s_2}\int_0^\infty u^{-1/2} \exp[-s_2 u|Y|](\cos ux-\sin ux)du$$

$$\left(\tau_{yz}\right)_1 = -B(1+i)\int_0^\infty u^{-1/2} \exp[-s_1 uY](\cos ux+\sin ux)du$$

$$\left(\tau_{yz}\right)_2 = -B(1+i)\int_0^\infty u^{-1/2} \exp[-s_2 u|Y|](\cos ux+\sin ux)du$$

(30)

where

$$B = -\frac{N\mu_1 s_1}{2\pi(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)}\ ; \quad y_j = s_j Y \quad (j=1,2).$$

Using the results

$$\int_0^\infty u^{-1/2} \exp[-s_1 uY]\cos ux\ dx = \sqrt{\frac{\pi}{2}}\left[\frac{(s_1^2y^2+x^2)^{1/2}+s_1 y}{s_1^2y^2+x^2}\right]^{1/2}$$

$$\int_0^\infty u^{-1/2} \exp[-s_1 uY]\sin ux\ dx = \sqrt{\frac{\pi}{2}}\left[\frac{(s_1^2y^2+x^2)^{1/2}-s_1 y}{s_1^2y^2+x^2}\right]^{1/2}$$

(31)

the stresses near about the crack tip given by (30) can be evaluated. The displacement near about the crack tip can be obtained from the crack tip stresses by integration.

Now introducing the factor $\exp[i(M_1\lambda_1 x-\omega t)]$ and taking the real part, the stresses and displacements near about the moving crack-tip are found to be equal to

$$\begin{bmatrix} (\tau_{yz})_j \\ \\ (\tau_{xz})_j \\ \\ v_j \end{bmatrix} = \mathrm{Re}\begin{bmatrix} K_1\left[\dfrac{(s^2Y^2+x^2)^{1/2}+x}{s_j^2Y^2+x^2}\right]^{1/2} \\ \\ (-1)^j\,\dfrac{K_1}{s_1}\left[\dfrac{(s_j^2Y^2+x^2)^{1/2}-x}{s_j^2Y^2+x^2}\right]^{1/2} \\ \\ (-1)^{j+1}\,\dfrac{2K_1}{\mu_j s_j}\left[(x^2+s_j^2Y^2)^{1/2}-x\right]^{1/2} \end{bmatrix} \exp\left[i(M_1\lambda_1 x-\omega t-\frac{\pi}{4})\right] \tag{32}$$

where

$$K_1 = \sqrt{\frac{2}{\pi}}\,\frac{\mu_1\mu_2\Lambda_1\Lambda_2 V_1 \sin\Theta_1 \sin\Theta_2}{(\alpha_1+\lambda_1)^{1/2}F_+(\alpha_1)(\mu_1\Lambda_1\sin\Theta_1+\mu_2\Lambda_2\sin\Theta_2)}. \tag{33}$$

In the case of crack propagation in an isotropic elastic medium using the result $\mu_1=\mu_2$, $\rho_1=\rho_2$ and $F_+(\alpha_1)=1$, we obtain

$$K_1 = (1/\pi)^{1/2}\mu_1\Lambda_1^{1/2}V_1(1-M_1)^{1/2}\sin(\Theta_1/2). \tag{34}$$

Putting $r=(x^2+y^2)^{1/2}$, $\tan\phi=|Y|/x$, the expression of displacements and stresses given by (32) near about the moving crack-tip is found to be equal to

$$v_1 = \frac{2K_1}{\mu_1 s_1}\, r^{1/2}\left\{(1-M_1^2\sin^2\phi)^{1/2} - \cos\phi\right\}^{1/2}\cos(\omega t+\frac{\pi}{4}) + O(r^{3/2})$$

$$v_2 = -\frac{2K_1}{\mu_2 s_2}\, r^{1/2}\left\{(1-M_2^2\sin^2\phi)^{1/2} - \cos\phi\right\}^{1/2}\cos(\omega t+\frac{\pi}{4}) + O(r^{3/2})$$

$$\left[\tau_{yz}\right]_1 = \frac{K_1}{r^{1/2}}\left\{\frac{(1-M_1^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_1^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{yz}\right]_2 = \frac{K_1}{r^{1/2}}\left\{\frac{(1-M_2^2\sin^2\phi)^{1/2}-\cos\phi}{1-M_2^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{xz}\right]_1 = \frac{-K_1}{s_1 r^{1/2}}\left\{\frac{(1-M_1^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_1^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{xz}\right]_2 = \frac{K_1}{s_2 r^{1/2}}\left\{\frac{(1-M_2^2\sin^2\phi)^{1/2}+\cos\phi}{1-M_2^2\sin^2\phi}\right\}^{1/2}\cos(\omega t+\frac{\pi}{4}) + O(r^{1/2})$$

(35)

Taking the value of K_1 given by (34), the results given by (35) agree with the results of the crack propagation in an isotropic elastic medium as given by Jahanshahi (1967).

When the crack is stationary, the corresponding results of stresses and displacements near about the crack-tip can be derived by making M_1 and M_2 approach zero and are given by

$$\left[\tau_{yz}\right]_1 = K_1^*(2/r)^{1/2}\cos\tfrac{1}{2}\phi\cos(\Omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{yz}\right]_2 = K_1^*(2/r)^{1/2}\cos\tfrac{1}{2}\phi\cos(\Omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{xz}\right]_1 = -K_1^*(2/r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\frac{\pi}{4}) + O(r^{1/2})$$

$$\left[\tau_{xz}\right]_2 = K_1^*(2/r)^{1/2}\sin\tfrac{1}{2}\phi\cos(\Omega t+\frac{\pi}{4}) + O(r^{1/2}) \qquad (36)$$

and

$$v_1 = \frac{2\sqrt{2}K_1^*}{\mu_1} r^{1/2} \sin\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4}) + O(r^{3/2})$$

$$v_2 = -\frac{2\sqrt{2}K_1^*}{\mu_2} r^{1/2} \sin\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4}) + O(r^{3/2}) \qquad (37)$$

where

$$K_1^* = \sqrt{\frac{2}{\pi}} \frac{\mu_1\mu_2\Lambda_1\Lambda_2 V_1 \sin\Theta_1 \sin\Theta_2}{(\Lambda_1\cos\phi_1+\Lambda_1)^{1/2} F_+^*(\Lambda_1\cos\phi_1)(\mu_1\Lambda_1\sin\phi_1+\mu_2\Lambda_2\sin\phi_2)} \qquad (38)$$

and

$$F_+^*(\Lambda_1\cos\phi_1) = \exp\left[\frac{1}{\pi}\int_{\Lambda_1}^{\Lambda_2} \tan^{-1}\left\{\frac{\mu_1(s^2-\Lambda_1^2)^{1/2}}{\mu_2(\Lambda_2^2-s^2)^{1/2}}\right\} \frac{ds}{s+\Lambda_1\cos\phi_1}\right]. \qquad (39)$$

If we put $\mu_1=\mu_2$, $\rho_1=\rho_2$, the corresponding results of the stationary crack in an isotropic elastic medium are found to be

$$\left(\tau_{yz}\right)_{1,2} = V_1 \sin\tfrac{1}{2}\Theta_1 \cos\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4})\left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2} + O(r^{1/2})$$

$$\left(\tau_{xz}\right)_{1,2} = \mp V_1 \sin\tfrac{1}{2}\Theta_1 \cos\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4})\left[\frac{2\Lambda_1\mu_1^2}{\pi r}\right]^{1/2} + O(r^{1/2})$$

and

$$v_{1,2} = \pm V_1 \sin\tfrac{1}{2}\Theta_1 \sin\tfrac{1}{2}\phi \cos(\Omega t+\tfrac{\pi}{4})\left[\frac{8\Lambda_1 r}{\pi}\right]^{1/2} + O(r^{3/2}) \qquad (40)$$

which are same as given by Jahanshahi (1967).

3. RESULTS AND DISCUSSION

K_1 given by (33) is the dynamic stress intensity factor at the moving crack-tip and K_1^* given by (38) is the value of the corresponding quantity when the crack is stationary. The variation of K_1/K_1^* with the values of V/C_2 where V is the crack speed has been depicted graphically for the following two sets of materials.

First set :

Wrought iron	$\rho_1 = 7.8 g/cm^3$,	$\mu_1 = 7.7\times10^{11} dyn/cm^2$
Copper	$\rho_2 = 8.96 g/cm^3$,	$\mu_2 = 4.5\times10^{11} dyn/cm^2$

Second set :

Steel	$\rho_1 = 7.6 g/cm^3$,	$\mu_1 = 8.33\times10^{11} dyn/cm^2$
Aluminium	$\rho_2 = 2.7 g/cm^3$,	$\mu_2 = 2.63\times10^{11} dyn/cm^2$

It is found that in both the cases the stress intensity factor gradually decreases with an increase in the value of V/C_2 and approaches zero as $V/C_2 \to 1$; the decrease in the value of K_1/K_1^* for the second set being more rapid than for the first set. We also find that in both the cases for any fixed value of C_1/C_2, K_1/K_1^* decreases with decrease in the value of Θ.

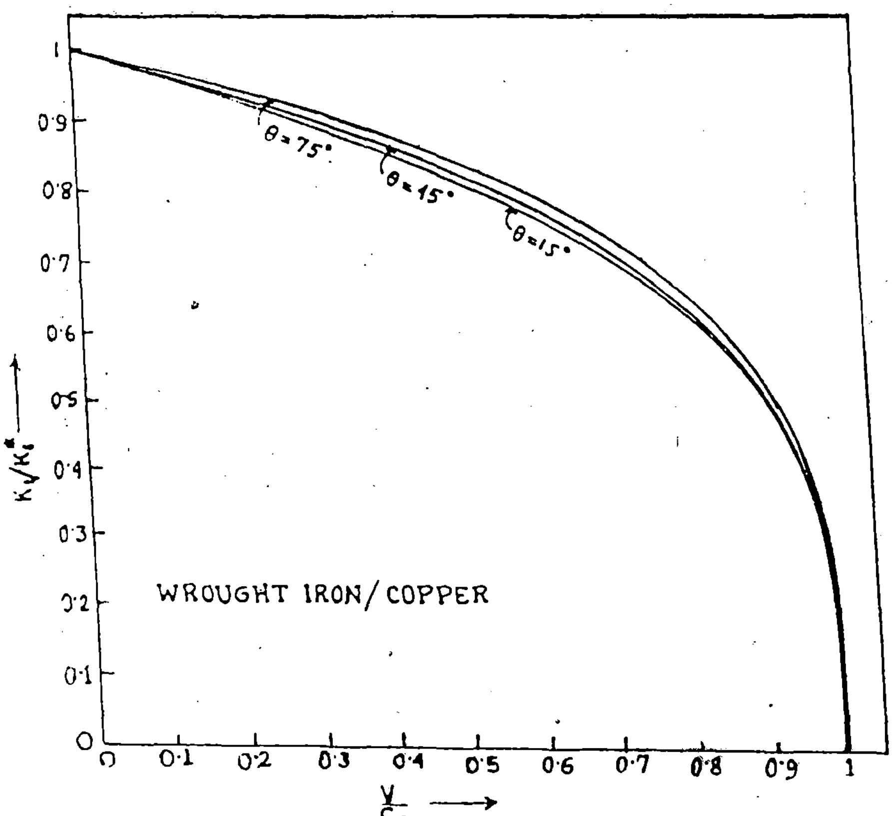

Figure 2. Stress intensity factor vs dimensionless crack speed (wrought iron/copper).

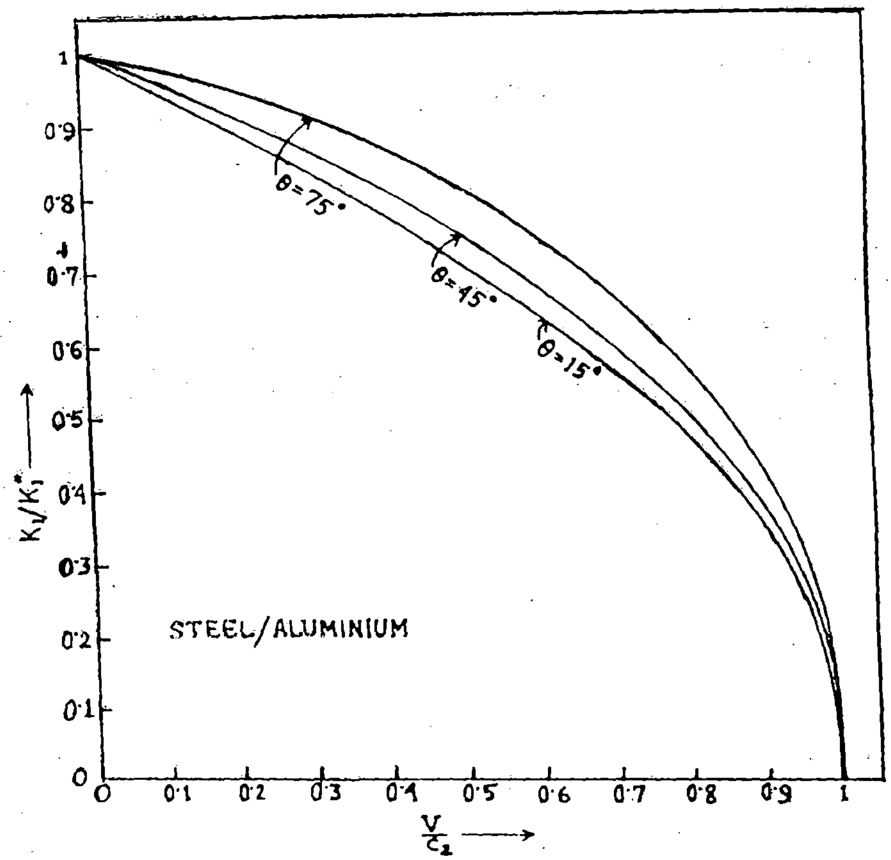

Figure 3. Stress intensity factor vs dimensionless crack speed (steel/aluminium).

APPENDIX

FACTORIZATION OF $F(\xi)$ INTO $F_+(\xi)$ AND $F_-(\xi)$:

Consider

$$F(\xi) = \frac{\gamma_2(\mu_1 s_1 + \mu_2 s_2)}{(\mu_1 s_1 \gamma_1 + \mu_2 s_2 \gamma_2)} \tag{A1}$$

The branch points of $F(\xi)$ are at $\xi = \lambda_1, -\lambda_1, \lambda_2+\beta_2, -(\lambda_2-\beta_2)$ where

$$-(\lambda_2-\beta_2) < -\lambda_1 < \lambda_1 < \lambda_2+\beta_2 \quad \text{since } C_2 < C_1.$$

Since $F(\xi) \to 1$ as $|\xi| \to \infty$, $F(\xi)$ possesses no singularity within the rectangular contour (shown in fig.4), by Cauchy's residue theorem we can write

$$\log F(\xi) = \frac{1}{2\pi i}\int_{c_+ + c_-} \frac{\log F(s)}{s-\xi}\,ds \tag{A2}$$

$$= \frac{1}{2\pi i}\int_{c_+} \frac{\log F(s)}{s-\xi}\,ds + \frac{1}{2\pi i}\int_{c_-} \frac{\log F(s)}{s-\xi}\,ds$$

$$\log F(\xi) = \log F_+(\xi) + \log F_-(\xi), \tag{A3}$$

where $F_+(\xi)$ and $F_-(\xi)$ are analytic in the upper and lower half of the complex ξ-plane respectively and can be expressed as

$$F_+(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c_+} \frac{\log F(s)}{s-\xi}\,ds\right]$$

and (A4)

$$F_-(\xi) = \exp\left[\frac{1}{2\pi i}\int_{c_-} \frac{\log F(s)}{s-\xi}\,ds\right]$$

In order to evaluate $F_-(\xi)$ the path of integration C_- can be deformed to the path C_1 round the branch cut through λ_1 and $\lambda_2+\beta_2$ as shown in fig.5.

After a little algebraic manipulation it can be shown that

$$F_-(\xi) = \exp\left[\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2} \frac{1}{s-\xi} \log\left\{1+i\,\frac{m_1}{m_2}\,\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\} ds\; -\right.$$

$$\left. -\frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2+\beta_2} \frac{1}{s-\xi} \log\left\{1-i\,\frac{m_1}{m_2}\,\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\} ds\right] \qquad (A5)$$

which after simplification becomes

$$F_-(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2+\beta_2} \frac{1}{s-\xi} \tan^{-1}\left\{\frac{m_1}{m_2}\,\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2-s)^2]^{1/2}}\right\} ds\right] \qquad (A6)$$

where

$$m_1 = \frac{\mu_1 s_1}{\mu_1 s_1+\mu_2 s_2} \quad \text{and} \quad m_2 = \frac{\mu_2 s_2}{\mu_1 s_1+\mu_2 s_2}\,. \qquad (A7)$$

Similarly it can be shown that

$$F_+(\xi) = \exp\left[\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2-\beta_2} \frac{1}{s+\xi} \tan^{-1}\left\{\frac{m_1}{m_2}\,\frac{(s^2-\lambda_1^2)^{1/2}}{[\lambda_2^2-(\beta_2+s)^2]^{1/2}}\right\} ds\right] \qquad (A8)$$

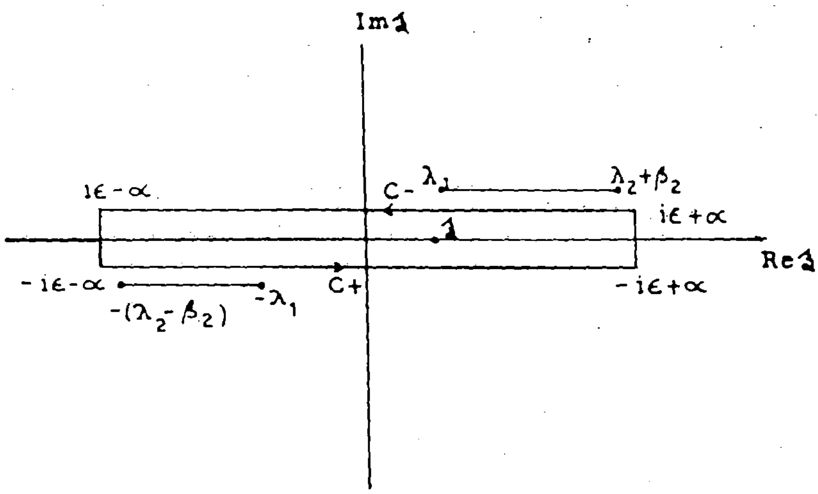

Figure 4. Rectangular contour in the complex ξ-plane.

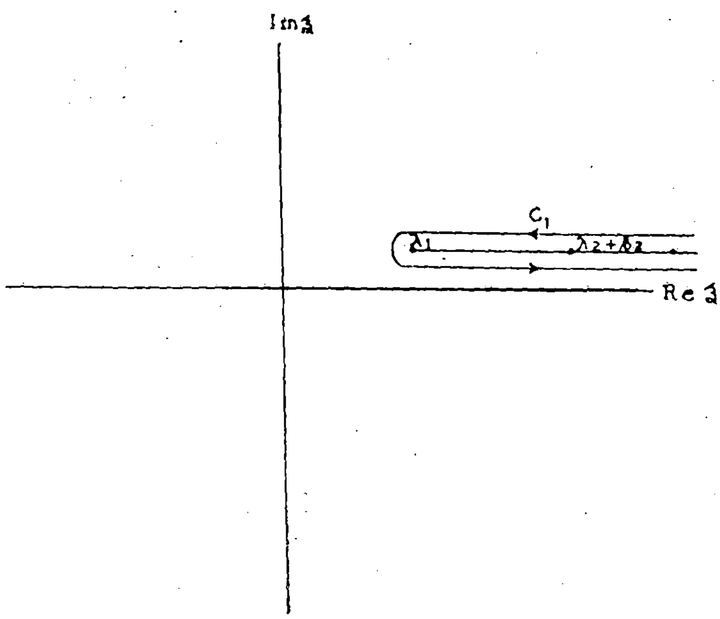

Figure 5. Path of integration C_1 round the branch cut.