# CHAPTER - I.

## Bending of uniformly compressed circular plates of variable thickness.*

### PAPER - I.

**Nomenclature :**

The following nomenclature are used in this paper.

$a$ = radius of the plate,

$b$ = radius of the inner boundary,

$h$ = thickness of the plate at a distance $r$ from the centre,

$D$ = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,

$E$ = Young's modulus,

$\sigma$ = Poisson's ratio.

**Introduction :**

Holzer ( 1918 ) discussed first the problem of symmetrical bending of circular plates of variable thickness. Since then, many authors have investigated the problem, outstanding of which are the investigations of Pichler ( 1928 ) and Olsson ( 1937 ). The last named author ( 1939 ) has also solved the problem of unsymmetrical bending of circular plates. Conway ( 1948 ) investigated the problem of symmetrically loaded circular plates of variable thickness with various types of thickness variations.

---

*Published in Indian Journal of Mechanics and Mathematics, vol. V. no. 1.

Basuli ( 1961 ) solved the problem of bending of uniformly compressed annular plates, thickness of which varies linearly from the centre.

In this paper an attempt has been made to solve the same problem with the thickness varying inversely as the distance from the centre.

Theory :

Let $T =$ uniform pressure per unit length and thickness of the section of the deflection surface bounded by two concentric cylindrical surfaces of radii $r$ and $r + dr$ and two radial planes including a small angle $d\theta$ at the centre of the plate, at a distance $r$; $M_r, M_\theta =$ bending moments per unit length of the section perpendicular to radius and tangent, $Q_r =$ shearing forces per unit length, acting normally to the middle plane, $\phi =$ slope at a distance $r = -\frac{dw}{dr}$, $w$ being the corresponding displacement.

Considering the equilibrium of the element and taking moments we have the following differential equation,

$$D\frac{d}{dr}\left(\frac{d\phi}{dr} + \frac{\phi}{r}\right) + \frac{dD}{dr}\left(\frac{d\phi}{dr} + \sigma\frac{\phi}{r}\right) + Th\phi = -Q_r \qquad \ldots (1)$$

where the flexural rigidity $D$ is a variable quantity.

[ Basuli. S ( 1961 ) ]

**Problem :**

(a) <u>Outer boundary clamped and supported, inner boundary clamped. Line load along the inner boundary.</u>

Let us consider an annular plate whose thickness at a distance $r$ is given by

$$h = h_0 r^{-1} \qquad \ldots(2)$$

subjected to a total normal load $P$ distributed uniformly round the radius of the hole.

Then the differential equation ( 1 ) will take the form

$$r^2 \frac{d^2\phi}{dr^2} - 2r\frac{d\phi}{dr} + \left(\alpha^2 r^4 - 3\sigma - 1\right)\phi = -\frac{\alpha^2 r^4 P}{2\pi h_0 T} \qquad \ldots(3)$$

where $\alpha^2 = \dfrac{12(1-\sigma^2)T}{E h_0^2}$ , $Q_r = \dfrac{P}{2\pi r}$.

The complementary function for the equation ( 3 ) can be put in the form

$$r^{3/2}\left[A J_\mu\left(\frac{\alpha r^2}{2}\right) + B Y_\mu\left(\frac{\alpha r^2}{2}\right)\right]$$

where $\mu^2 = \dfrac{13+12\sigma}{16}$ ; $A$, $B$ being constants and $J_\mu\left(\frac{\alpha r^2}{2}\right)$, $Y_\mu\left(\frac{\alpha r^2}{2}\right)$ being the Bessel functions of 1st and 2nd kind of order $\mu$.

[Forsyth, A.R. (1929)]

The particular integral is $-\dfrac{P\alpha^{3/4} r^{3/2} S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{\pi h_0 T\cdot 2^{7/4}}$

where $S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right) = \sum_{m=0}^{\infty} \dfrac{(-1)^m \left(\frac{\alpha r^2}{2}\right)^{1/4+1+2m}}{\left\{(1/4+1)^2-\mu^2\right\}\cdots\left\{(1/4+1+2m)^2-\mu^2\right\}}$

is the Lommel's function.

[Erdelyi, A. (1953)]

Hence the general solution is

$$\phi = r^{3/2}\left[A J_\mu\left(\frac{\alpha r^2}{2}\right) + B Y_\mu\left(\frac{\alpha r^2}{2}\right) - \frac{P\alpha^{3/4}\, S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{7/4}\pi T h_0}\right] \quad \ldots (4)$$

If the outer boundary be clamped and supported and the inner boundary be clamped

boundary conditions are $\phi = 0$ when $r = a$, $r = b$, $W = 0$ at

$$r = a \quad \ldots (5)$$

Considering equations ( 4 ) and ( 5 ) and solving for the constants, we get

$$A = \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\left[\frac{S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha a^2}{2}\right)}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - J_\mu\left(\frac{\alpha b^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha a^2}{2}\right)}\right] \quad \ldots (6)$$

$$B = \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\left[\frac{S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right) J_\mu\left(\frac{\alpha b^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)\cdot J_\mu\left(\frac{\alpha a^2}{2}\right)}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - J_\mu\left(\frac{\alpha b^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha a^2}{2}\right)}\right] \quad \ldots (7)$$

Using equations ( 4 ), ( 6 ) and ( 7 ) we get $\phi$ in the form

$$\begin{aligned}\phi = \frac{P r^{3/2}\alpha^{3/4}}{2^{7/4}\pi T h_0}\Bigg[ & J_\mu\left(\frac{\alpha r^2}{2}\right)\frac{S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right) Y_\mu\left(\frac{\alpha b^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha a^2}{2}\right)}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)} \\ & + Y_\mu\left(\frac{\alpha r^2}{2}\right)\frac{S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)\cdot J_\mu\left(\frac{\alpha a^2}{2}\right)}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)} \\ & - S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)\Bigg]\end{aligned} \quad \ldots (8)$$

Considering equation ( 4 ) we get

$$\begin{aligned}M_r = D\left[\frac{d\phi}{dr} + \sigma\frac{\phi}{r}\right] = \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\cdot D\Bigg[ & A\left\{\frac{3}{2}r^{1/2} J_\mu\left(\frac{\alpha r^2}{2}\right) + r^{5/2}\alpha\cdot J'_\mu\left(\frac{\alpha r^2}{2}\right)\right\} + B\left\{\frac{3}{2}r^{1/2} Y_\mu\left(\frac{\alpha r^2}{2}\right) + r^{5/2}\alpha\cdot Y'_\mu\left(\frac{\alpha r^2}{2}\right)\right\} \\ & + \sigma r^{1/2}\left\{A J_\mu\left(\frac{\alpha r^2}{2}\right) + B Y_\mu\left(\frac{\alpha r^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)\right\} \\ & - \alpha r\, S'_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)\Bigg]\end{aligned} \quad \ldots (9)$$

The same equation also determines

$$M_\theta = D\left[\frac{\phi}{r} + \sigma\frac{d\phi}{dr}\right] \qquad \ldots (10)$$

To get the deflection we know that $\quad \phi = -\dfrac{dw}{dr} \qquad \ldots (11)$

On integrating equation (11) we get,

$$\begin{aligned} W = & -(2/\alpha)^{1/4}\Big[A\Big\{(\mu-3/4)\,J_\mu\left(\tfrac{\alpha r^2}{2}\right) S_{-3/4,\mu-1}\left(\tfrac{\alpha r^2}{2}\right) \\ & - J_{\mu-1}\left(\tfrac{\alpha r^2}{2}\right) S_{1/4,\mu}\left(\tfrac{\alpha r^2}{2}\right)\Big\}\tfrac{1}{2}r^2 \\ & + B\Big\{(\mu-3/4)\,Y_\mu\left(\tfrac{\alpha r^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha r^2}{2}\right) \\ & - Y_{\mu-1}\left(\tfrac{\alpha r^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha r^2}{2}\right)\Big\}\tfrac{1}{2}r^2 \\ & - \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m\cdot r^{5+4m}\cdot\left(\frac{\alpha}{2}\right)^{5/4+2m}}{(5+4m)\{(5/4)^2-\mu^2\}\cdots\{(5/4+2m)^2-\mu^2\}}\right\}\Big] \\ & + K_1 \text{ ( constant )} \end{aligned} \qquad \ldots (12)$$

The boundary condition is $W=0$ at $r = a \qquad \ldots (13)$

Hence

$$\begin{aligned} K_1 = & (2/\alpha)^{1/4}\Big[\frac{Aa^2}{2}\Big\{(\mu-3/4)\,J_\mu\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha a^2}{2}\right) \\ & - J_{\mu-1}\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha a^2}{2}\right)\Big\} + \frac{Ba^2}{2}\Big\{(\mu-3/4)\,Y_\mu\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha a^2}{2}\right) \\ & - Y_{\mu-1}\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha a^2}{2}\right)\Big\} \\ & - \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\cdot a^{5+4m}\cdot\left(\frac{\alpha}{2}\right)^{5/4+2m}}{(5+4m)\{(5/4)^2-\mu^2\}\cdots\{(5/4+2m)^2-\mu^2\}}\Big] \end{aligned} \qquad \ldots (14)$$

Therefore $W$ is determined.

(b) Outer boundary clamped and supported, inner boundary clamped. Load uniform.

For the same plate if the load be uniformly distributed with intensity $q$ then

$$Q_r = \frac{1}{2\pi r}\int_b^r q \cdot 2\pi r\, dr = \frac{q}{2r}\left(r^2 - b^2\right) \qquad \ldots (15)$$

With this value of $Q_r$ the differential equation ( 1 ) takes the form

$$r^2 \frac{d^2\phi}{dr^2} - 2r\frac{d\phi}{dr} + \left(\alpha^2 r^4 - 3\sigma - 1\right)\phi = \frac{q\alpha^2}{2Th_0}\left(b^2 r^4 - r^6\right) \qquad \ldots (16)$$

Solution of equation (16) can be put in the form

$$\phi = r^{3/2}\left[C\, J_\mu\left(\frac{\alpha r^2}{2}\right) + D\, Y_\mu\left(\frac{\alpha r^2}{2}\right) + \frac{\alpha^{3/4} b^2 S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right) q}{2^{7/4}\, Th_0} - \frac{q\, S_{5/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{3/4}\alpha^{1/4}\, Th_0}\right] \qquad \ldots (17)$$

Boundary conditions are $\phi = 0$ when $r = a$, $r = b$ ... (18)

Considering equations (17) and (18) and solving for the constants, we get

$$C = \frac{Y_\mu\left(\frac{\alpha b^2}{2}\right)\frac{q}{Th_0}\left[\frac{S_{5/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2 S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right] - Y_\mu\left(\frac{\alpha a^2}{2}\right)\frac{q}{Th_0}\left[\frac{S_{5/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2 S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right]}{J_\mu\left(\frac{\alpha a^2}{2}\right) Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right) J_\mu\left(\frac{\alpha b^2}{2}\right)} \qquad \ldots (19)$$

$$D = \frac{J_\mu\left(\frac{\alpha a^2}{2}\right)\frac{q}{Th_0}\left[\frac{S_{5/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2 S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right] - J_\mu\left(\frac{\alpha b^2}{2}\right)\frac{q}{Th_0}\left[\frac{S_{5/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2 S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right]}{J_\mu\left(\frac{\alpha a^2}{2}\right) Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right) J_\mu\left(\frac{\alpha b^2}{2}\right)} \qquad \ldots (20)$$

Combining equations (17), (19) and (20), $\phi$ is determined in the form

$$\phi = r^{3/2}\Bigg[J_\mu\left(\frac{\alpha r^2}{2}\right)\Bigg\{Y_\mu\left(\frac{\alpha b^2}{2}\right)\frac{q}{Th_0}\left(\frac{S_{5/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right)$$

$$-Y_\mu\left(\frac{\alpha a^2}{2}\right)\frac{q}{Th_0}\left(\frac{S_{5/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right)\Bigg\}$$

$$\Big/\left\{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)\right\}$$

$$+ Y_\mu\left(\frac{\alpha r^2}{2}\right)\Bigg\{J_\mu\left(\frac{\alpha a^2}{2}\right)\frac{q}{Th_0}\left(\frac{S_{5/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right)$$

$$- J_\mu\left(\frac{\alpha b^2}{2}\right)\frac{q}{Th_0}\left(\frac{S_{5/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right)\Bigg\}$$

$$\Big/\left\{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)\right\}$$

$$+ \frac{\alpha^{3/4}\cdot q\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{7/4}\cdot Th_0}$$

$$- \frac{q\cdot S_{5/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{3/4}\alpha^{1/4}\,Th_0}\Bigg] \qquad \ldots(21)$$

To get the deflection $W$ we know that $\phi = -\frac{dw}{dr}$

Hence

$$\frac{dw}{dr} = -r^{3/2}\Bigg[C J_\mu\left(\frac{\alpha r^2}{2}\right) + D Y_\mu\left(\frac{\alpha r^2}{2}\right) + \frac{\alpha^{3/4}\cdot q\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{7/4}\cdot T h_0}$$

$$- \frac{q\cdot S_{5/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}\cdot Th_0}\Bigg] \qquad \ldots(22)$$

Integrating equation (22) we get

$$W = -(2/\alpha)^{1/4}\Big[C\Big\{(\mu-3/4)J_\mu\left(\tfrac{\alpha r^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha r^2}{2}\right)$$

$$-J_{\mu-1}\left(\tfrac{\alpha r^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha r^2}{2}\right)\Big\}\tfrac{1}{2}r^2$$

$$+D\Big\{(\mu-3/4)Y_\mu\left(\tfrac{\alpha r^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha r^2}{2}\right)$$

$$-Y_{\mu-1}\left(\tfrac{\alpha r^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha r^2}{2}\right)\Big\}\tfrac{1}{2}r^2$$

$$-\frac{\alpha^{3/4}\,q\,b^2}{2^{7/4}\,T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\alpha}{2}\right)^{5/4+2m}\cdot r^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$+\frac{q}{2^{3/4}\alpha^{1/4}T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{\alpha}{2}\right)^{9/4+2m}\cdot r^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}}\Big]$$

$$+K_2 \qquad \ldots (23)$$

Boundary condition is $W=0$ at $r=a$

Therefore the constant $K_2$ is given by

$$K_2 = (2/\alpha)^{1/4}\Big[C\Big\{(\mu-3/4)J_\mu\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha a^2}{2}\right)$$

$$-J_{\mu-1}\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha a^2}{2}\right)\Big\}\tfrac{1}{2}a^2$$

$$+D\Big\{(\mu-3/4)Y_\mu\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha a^2}{2}\right)$$

$$-Y_{\mu-1}\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha a^2}{2}\right)\Big\}\tfrac{1}{2}a^2$$

$$- \frac{\alpha^{3/4} \cdot q \cdot b^2}{2^{7/4} \cdot T h_o} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot a^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$+ \frac{q}{2^{3/4} \cdot \alpha^{1/4} \cdot T h_o} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot a^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}} \Bigg] \qquad \ldots (24)$$

**Substituting the value of $K_2$ from equation (24) in (23) $W$ is determined.**

**(c) <u>Outer boundary clamped, inner boundary clamped and supported, Load uniform.</u>**

**For the same plate if the inner boundary be clamped and supported and the outer boundary be clamped, boundary conditions are $\phi = 0$ at $r = a$, $r = b$ and $W = 0$ at $r = b$.**

**If the load be uniformly distributed with intensity $q$, equation (23) reduces to**

$$W = -(2/\alpha)^{1/4} \Bigg[ C\left\{ (\mu - 3/4) J_\mu\left(\frac{\alpha r^2}{2}\right) \cdot S_{-3/4,\mu-1}\left(\frac{\alpha r^2}{2}\right) - J_{\mu-1}\left(\frac{\alpha r^2}{2}\right) \cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right) \right\} \frac{1}{2} r^2 + D\left\{ (\mu - 3/4) Y_\mu\left(\frac{\alpha r^2}{2}\right) \cdot S_{-3/4,\mu-1}\left(\frac{\alpha r^2}{2}\right) - Y_{\mu-1}\left(\frac{\alpha r^2}{2}\right) \cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right) \right\} \frac{1}{2} r^2$$

$$- \frac{\alpha^{3/4} \cdot q \cdot b^2}{2^{7/4} \cdot T h_o} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot r^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$+ \frac{q}{2^{3/4} \cdot \alpha^{1/4} \, T h_o} \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot r^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}} \Bigg]$$

$$+ (2/\alpha)^{1/4}\Big[\frac{Cb^2}{2}\Big\{(\mu-3/4)\,J_\mu\Big(\frac{\alpha b^2}{2}\Big)\cdot S_{-3/4,\mu-1}\Big(\frac{\alpha b^2}{2}\Big) - J_{\mu-1}\Big(\frac{\alpha b^2}{2}\Big)\cdot S_{1/4,\mu}\Big(\frac{\alpha b^2}{2}\Big)\Big\}$$

$$+ \frac{Db^2}{2}\Big\{(\mu-3/4)\,Y_\mu\Big(\frac{\alpha b^2}{2}\Big)\cdot S_{-3/4,\mu-1}\Big(\frac{\alpha b^2}{2}\Big) - Y_{\mu-1}\Big(\frac{\alpha b^2}{2}\Big)\cdot S_{1/4,\mu}\Big(\frac{\alpha b^2}{2}\Big)\Big\}$$

$$- \frac{\alpha^{3/4}\, q\, b^2}{2^{7/4}\, T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\,(\alpha/2)^{5/4+2m}\cdot b^{5+4m}}{(5+4m)\{(5/4)^2-\mu^2\}\cdots\{(5/4+2m)^2-\mu^2\}}$$

$$+ \frac{q}{2^{3/4}\alpha^{1/4}\, T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\,(\alpha/2)^{9/4+2m}\cdot b^{7+4m}}{(7+4m)\{(9/4)^2-\mu^2\}\cdots\{(9/4+2m)^2-\mu^2\}}\Big] \quad \ldots (25)$$

which is the deflection under uniform load at a distance $r$.

Deflection will be maximum at the outer boundary $r=a$.

Thus we get,

$$(W)_{max} = (2/\alpha)^{1/4}\Big[\frac{Cb^2}{2}\Big\{(\mu-3/4)\,J_\mu\Big(\frac{\alpha b^2}{2}\Big)\cdot S_{-3/4,\mu-1}\Big(\frac{\alpha b^2}{2}\Big)$$

$$- J_{\mu-1}\Big(\frac{\alpha b^2}{2}\Big)\cdot S_{1/4,\mu}\Big(\frac{\alpha b^2}{2}\Big)\Big\} + \frac{Db^2}{2}\Big\{(\mu-3/4)\,Y_\mu\Big(\frac{\alpha b^2}{2}\Big)\cdot S_{-3/4,\mu-1}\Big(\frac{\alpha b^2}{2}\Big)$$

$$- Y_{\mu-1}\Big(\frac{\alpha b^2}{2}\Big)\cdot S_{1/4,\mu}\Big(\frac{\alpha b^2}{2}\Big)\Big\}$$

$$- \frac{\alpha^{3/4}\, q\, b^2}{2^{7/4}\, T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\,(\alpha/2)^{5/4+2m}\cdot b^{5+4m}}{(5+4m)\{(5/4)^2-\mu^2\}\cdots\{(5/4+2m)^2-\mu^2\}}$$

$$+ \frac{q}{2^{3/4}\alpha^{1/4}\, T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\,(\alpha/2)^{9/4+2m}\cdot b^{7+4m}}{(7+4m)\{(9/4)^2-\mu^2\}\cdots\{(9/4+2m)^2-\mu^2\}}\Big]$$

$$- (2/\alpha)^{1/4}\Big[\frac{Ca^2}{2}\Big\{(\mu-3/4)\,J_\mu\Big(\frac{\alpha a^2}{2}\Big)\cdot S_{-3/4,\mu-1}\Big(\frac{\alpha a^2}{2}\Big)$$

$$- J_{\mu-1}\Big(\frac{\alpha a^2}{2}\Big)\cdot S_{1/4,\mu}\Big(\frac{\alpha a^2}{2}\Big)\Big\} + \frac{Da^2}{2}\Big\{(\mu-3/4)\,Y_\mu\Big(\frac{\alpha a^2}{2}\Big)\cdot S_{-3/4,\mu-1}\Big(\frac{\alpha a^2}{2}\Big)$$

$$- Y_{\mu-1}\Big(\frac{\alpha a^2}{2}\Big)\cdot S_{1/4,\mu}\Big(\frac{\alpha a^2}{2}\Big)\Big\}$$

$$- \frac{\alpha^{3/4} \cdot q \cdot b^2}{2^{7/4} \cdot Th_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot a^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$\left. + \frac{q}{2^{3/4} \alpha^{1/4} Th_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot a^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}} \right] \quad \ldots (26)$$

**In particular, if $\sigma = 0{\cdot}25$, the deflection and the maximum deflection can be obtained from equations (25) and (26) as given below.**

$$W = -(2/\alpha)^{1/4} \left[ C_1 r^2 \left\{ \frac{1}{8} J_1\left(\frac{\alpha r^2}{2}\right) \cdot S_{-3/4,0}\left(\frac{\alpha r^2}{2}\right) - \frac{1}{2} J_0\left(\frac{\alpha r^2}{2}\right) \cdot S_{1/4,1}\left(\frac{\alpha r^2}{2}\right) \right\} \right.$$

$$+ D_1 r^2 \left\{ \frac{1}{8} Y_1\left(\frac{\alpha r^2}{2}\right) \cdot S_{-3/4,0}\left(\frac{\alpha r^2}{2}\right) - \frac{1}{2} Y_0\left(\frac{\alpha r^2}{2}\right) \cdot S_{1/4,1}\left(\frac{\alpha r^2}{2}\right) \right\}$$

$$- \frac{\alpha^{3/4} q b^2}{2^{7/4} Th_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot r^{5+4m}}{(5+4m)\left\{(5/4)^2-1\right\}\cdots\left\{(5/4+2m)^2-1\right\}}$$

$$\left. + \frac{q}{2^{3/4} \alpha^{1/4} Th_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot r^{7+4m}}{(7+4m)\left\{(9/4)^2-1\right\}\cdots\left\{(9/4+2m)^2-1\right\}} \right]$$

$$+ (2/\alpha)^{1/4} \left[ C_1 b^2 \left\{ \frac{1}{8} J_1\left(\frac{\alpha b^2}{2}\right) \cdot S_{-3/4,0}\left(\frac{\alpha b^2}{2}\right) - \frac{1}{2} J_0\left(\frac{\alpha b^2}{2}\right) \cdot S_{1/4,1}\left(\frac{\alpha b^2}{2}\right) \right\} \right.$$

$$+ D_1 b^2 \left\{ \frac{1}{8} Y_1\left(\frac{\alpha b^2}{2}\right) \cdot S_{-3/4,0}\left(\frac{\alpha b^2}{2}\right) - \frac{1}{2} Y_0\left(\frac{\alpha b^2}{2}\right) \cdot S_{1/4,1}\left(\frac{\alpha b^2}{2}\right) \right\}$$

$$- \frac{\alpha^{3/4} q b^2}{2^{7/4} Th_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot b^{5+4m}}{(5+4m)\left\{(5/4)^2-1\right\}\cdots\left\{(5/4+2m)^2-1\right\}}$$

$$\left. + \frac{q}{2^{3/4} \alpha^{1/4} Th_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot b^{7+4m}}{(7+4m)\left\{(9/4)^2-1\right\}\cdots\left\{(9/4+2m)^2-1\right\}} \right] \quad \ldots (27)$$

where

$$C_1 = \frac{\frac{q}{Th_0}\cdot Y_1\left(\frac{\alpha b^2}{2}\right)\left[\frac{S_{5/4,1}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2 . S_{1/4,1}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right] - \frac{q}{Th_0}\cdot Y_1\left(\frac{\alpha a^2}{2}\right)\left[\frac{S_{5/4,1}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2 . S_{1/4,1}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right]}{J_1\left(\frac{\alpha a^2}{2}\right)\cdot Y_1\left(\frac{\alpha b^2}{2}\right) - Y_1\left(\frac{\alpha a^2}{2}\right)\cdot J_1\left(\frac{\alpha b^2}{2}\right)}$$

$$D_1 = \frac{\frac{q}{Th_0}\cdot J_1\left(\frac{\alpha a^2}{2}\right)\left[\frac{S_{5/4,1}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}.\alpha^{1/4}} - \frac{\alpha^{3/4} b^2 . S_{1/4,1}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right] - \frac{q}{Th_0}\cdot J_1\left(\frac{\alpha b^2}{2}\right)\left[\frac{S_{5/4,1}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2 . S_{1/4,1}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right]}{J_1\left(\frac{\alpha a^2}{2}\right)\cdot Y_1\left(\frac{\alpha b^2}{2}\right) - Y_1\left(\frac{\alpha a^2}{2}\right)\cdot J_1\left(\frac{\alpha b^2}{2}\right)}$$

$$\begin{aligned}
(W)_{max} = {} & (2/\alpha)^{1/4}\Big[C_1 b^2\Big\{\tfrac{1}{8} J_1\left(\tfrac{\alpha b^2}{2}\right).S_{-3/4,0}\left(\tfrac{\alpha b^2}{2}\right) - \tfrac{1}{2} J_0\left(\tfrac{\alpha b^2}{2}\right).S_{1/4,1}\left(\tfrac{\alpha b^2}{2}\right)\Big\} \\
& + D_1 b^2\Big\{\tfrac{1}{8} Y_1\left(\tfrac{\alpha b^2}{2}\right).S_{-3/4,0}\left(\tfrac{\alpha b^2}{2}\right) - \tfrac{1}{2} Y_0\left(\tfrac{\alpha b^2}{2}\right).S_{1/4,1}\left(\tfrac{\alpha b^2}{2}\right)\Big\} \\
& - \frac{\alpha^{3/4} q . b^2}{2^{7/4}\, Th_0}\sum_{m=0}^{\infty}\frac{(-1)^m . \left(\frac{\alpha}{2}\right)^{5/4+2m} . b^{5+4m}}{(5+4m)\left\{(5/4)^2-1\right\}\cdots\left\{(5/4+2m)^2-1\right\}} \\
& + \frac{q}{2^{3/4}\alpha^{1/4}\, Th_0}\sum_{m=0}^{\infty}\frac{(-1)^m . \left(\frac{\alpha}{2}\right)^{9/4+2m} . b^{7+4m}}{(7+4m)\left\{(9/4)^2-1\right\}\cdots\left\{(9/4+2m)^2-1\right\}}\Big] \\
& - (2/\alpha)^{1/4}\Big[C_1 a^2\Big\{\tfrac{1}{8} J_1\left(\tfrac{\alpha a^2}{2}\right).S_{-3/4,0}\left(\tfrac{\alpha a^2}{2}\right) - \tfrac{1}{2} J_0\left(\tfrac{\alpha a^2}{2}\right).S_{1/4,1}\left(\tfrac{\alpha a^2}{2}\right)\Big\} \\
& + D_1 a^2\Big\{\tfrac{1}{8} Y_1\left(\tfrac{\alpha a^2}{2}\right).S_{-3/4,0}\left(\tfrac{\alpha a^2}{2}\right) - \tfrac{1}{2} Y_0\left(\tfrac{\alpha a^2}{2}\right).S_{1/4,1}\left(\tfrac{\alpha a^2}{2}\right)\Big\} \\
& - \frac{\alpha^{3/4} q . b^2}{2^{7/4} . Th_0}\sum_{m=0}^{\infty}\frac{(-1)^m . \left(\frac{\alpha}{2}\right)^{5/4+2m} . a^{5+4m}}{(5+4m)\left\{(5/4)^2-1\right\}\cdots\left\{(5/4+2m)^2-1\right\}} \\
& + \frac{q}{2^{3/4}\alpha^{1/4}\, Th_0}\sum_{m=0}^{\infty}\frac{(-1)^m . \left(\frac{\alpha}{2}\right)^{9/4+2m} . a^{7+4m}}{(7+4m)\left\{(9/4)^2-1\right\}\cdots\left\{(9/4+2m)^2-1\right\}}\Big] \qquad \ldots (28)
\end{aligned}$$

**Numerical calculation :**

Let us assume $\alpha = 1,\ b = 5,\ a = 10$

Putting the above values of $\alpha$, $b$ and $a$ in (28) we have, $(W)_{max} = \frac{q}{Th_0} \times 0{\cdot}4135$

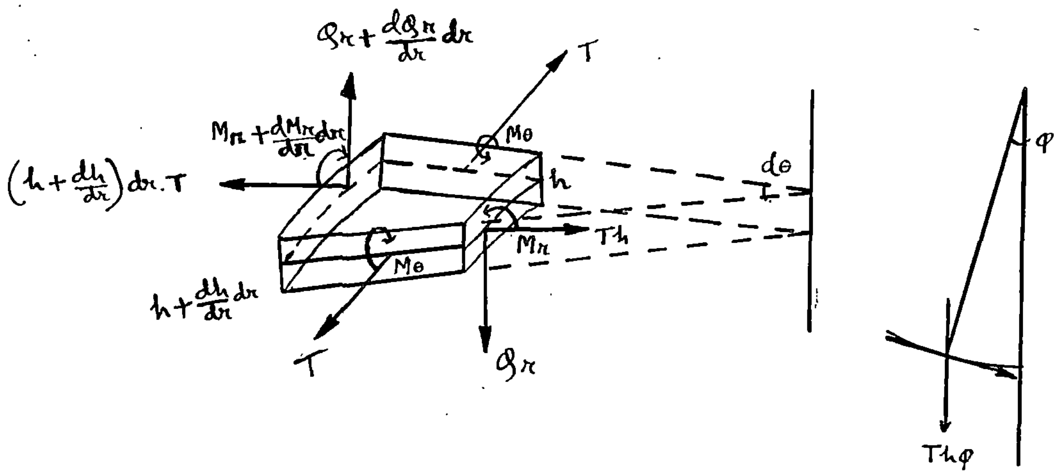


FIG. 1

Stresses and moments on an element bounded by two adjacent cylindrical surfaces and radial planes.

Note on the deflection of a square plate of variable thickness under a variable load and uniform tension in the middle plane of the plate.*

## PAPER - II

**Nomenclature :**

The following nomenclature are used in this paper.

$a$ = side of the plate,

$D$ = flexural rigidity of the plate $= \frac{Eh^3}{12(1-\sigma^2)}$,

$h$ = thickness at a distance $x$ $= h_0 e^{\lambda \frac{x}{a}}$ where $\lambda$ is a parameter, small in magnitude,

$E$ = Young's modulus,

$q$ = load,

$T$ = Uniform tension in the middle plane of the plate,

$\sigma$ = Poisson's ratio,

$w$ = deflection, normal to the plate.

**Introduction :**

Several problems of bending of rectangular plates of uniform thickness under the combined action of lateral loads and forces in the middle plane of the plate have been discussed by Timoshenko and Woinowsky - Krieger ( 1959 ), Conway H.D.(1949), Chang C.C. and Conway H.D.(1952). The

---

*Accepted for publication in the Journal of Theoretical Physics, Calcutta(Published in Vol. 15, No. 1, 1967).

object of this paper is to solve the problem of a square plate of variable thickness under the combined action of a variable load and uniform tension in the middle plane of the plate, thickness varying exponentially. It may be mentioned that the same problem without any compressive force was discussed by Favre H. and Glig B.(1952). The corresponding problem with linearly varying thickness was due to Basuli (1961).

**Theory :**

The figure ( 1 ) represents an element of the plate. Prejecting normal and shearing forces on the $z$-axis and considering the equation of equilibrium we obtain the following differential equation (Timoshenko and Woinowsky-Krieger, Theory of plates and shells. Page - 379):-

$$D\nabla\nabla W + 2\frac{dD}{dx}\cdot\frac{\partial}{\partial x}\cdot\nabla W + \frac{d^2D}{dx^2}\nabla W - (1-\sigma)\cdot\frac{d^2D}{dx^2}\cdot\frac{\partial^2 W}{\partial y^2} = q + Th\frac{\partial^2 W}{\partial x^2} \qquad \ldots(1)$$

**Problem :**

Let us consider a plate stretched in the direction of $x$- axis.

Hence $N_x = T$, $N_y = N_{xy} = N_{yx} = 0$. Let us assume load $q$ to be hydrostatic, represented by $q = q_0 \cdot \frac{x}{a}$.

Let $W = \sum_{m=0}^{\infty} W_m \lambda^m$ where $\lambda$ is a parameter defined earlier.

Substituting this in ( 1 ) and equating the coefficients of successive powers of $\lambda$ we have the following sequence of differential equations

$$D_0 \nabla\nabla W_0 - Th_0 \frac{\partial^2 W_0}{\partial x^2} - q = 0 \qquad \ldots(2\cdot 0)$$

$$D_0 3\frac{x}{a} \nabla\nabla W_0 + D_0 \nabla\nabla W_1 + \frac{6D_0}{a} \cdot \frac{\partial}{\partial x} \nabla W_0 - Th_0 \frac{x}{a} \frac{\partial^2 W_0}{\partial x^2} - Th_0 \frac{\partial^2 W_1}{\partial x^2} = 0 \qquad \ldots(2\cdot 1)$$

$$\frac{9}{2} D_0 \frac{x^2}{a^2} \nabla\nabla W_0 + D_0 \nabla\nabla W_2 + \frac{6}{a} D_0 \frac{\partial}{\partial x} \nabla W_1 + 18 D_0 \frac{x}{a^2} \frac{\partial}{\partial x} \nabla W_0 +$$

$$+ \frac{9D_0}{a^2} \left\{ \nabla W_0 - (1-\sigma) \frac{\partial^2 W_0}{\partial y^2} \right\} - Th_0 \frac{x^2}{2a^2} \frac{\partial^2 W_0}{\partial x^2} - Th_0 \frac{\partial^2 W_2}{\partial x^2} - q = 0 \qquad \ldots(2\cdot 2)$$

and so on.

For a simply supported plate using the method of M. Levy we take the solution in the form

$$W_0 = \sum_{n=1,3,\ldots}^{\infty} X_{0n} \sin \frac{n\pi y}{a} \qquad \ldots(3\cdot 0)$$

$$W_1 = \sum_{n=1,3,\ldots}^{\infty} X_{1n} \sin \frac{n\pi y}{a} \qquad \ldots(3\cdot 1)$$

. . . . . . . . . . . . . . . . . . . . . . . . .

$$W_m = \sum_{n=1,3,\ldots}^{\infty} X_{mn} \sin \frac{n\pi y}{a} \qquad \ldots(3\cdot m)$$

( $m = 0, 1, 2, \ldots$ ) $X_{mn}$ being some functions of $x$ only.

We can finally represent the load $q$ by

$$q = \frac{4q_0 x}{\pi a} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi y}{a} \qquad \ldots(4)$$

**Considering equations (2·0), (3·0) and (4) we have the following differential equations of 4th order for $X_{on}$.**

$$\frac{d^4X_{on}}{dx^4} - \frac{d^2X_{on}}{dx^2}\left(\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2}\right) + X_{on}\frac{n^4\pi^4}{a^4} = \frac{4q_0 x}{\pi . a . n . D_0} \quad \ldots(5)$$

**Solving we get**

$$X_{on} = A_{01}e^{m_1x} + A_{02}e^{m_2x} + A_{03}e^{m_3x} + A_{04}e^{m_4x} + \frac{4q_0 x a^3}{n^5\pi^5 D_0} \quad \ldots(6)$$

**where** $A_{01}, A_{02}, A_{03}, A_{04}$ **are arbitrary constants.**

**Using boundary conditions**

$X_{on} = 0$ at $x = 0$ **and** $x = a$

$X''_{on} = 0$ at $x = 0$ **and** $x = a$

**one gets**

$$A_{01} = \frac{a^4 . 4q_0}{n^5\pi^5 D_0}\left[\frac{P + Q + R}{P(e^{m_2a} - e^{m_1a}) + Q(e^{m_3a} - e^{m_1a}) + R(e^{m_4a} - e^{m_1a})}\right] \quad \ldots(7{\cdot}1)$$

$$A_{02} = -\frac{4q_0 P a^4}{n^5\pi^5 D_0}\left[\frac{1}{P(e^{m_2a} - e^{m_1a}) + Q(e^{m_3a} - e^{m_1a}) + R(e^{m_4a} - e^{m_1a})}\right] \quad \ldots(7{\cdot}2)$$

$$A_{03} = -\frac{4q_0 Q a^4}{n^5\pi^5 D_0}\left[\frac{1}{P(e^{m_2a} - e^{m_1a}) + Q(e^{m_3a} - e^{m_1a}) + R(e^{m_4a} - e^{m_1a})}\right] \quad \ldots(7{\cdot}3)$$

$$A_{04} = -\frac{4q_0 R a^4}{n^5\pi^5 D_0}\left[\frac{1}{P(e^{m_2a} - e^{m_1a}) + Q(e^{m_3a} - e^{m_1a}) + R(e^{m_4a} - e^{m_1a})}\right] \quad \ldots(7{\cdot}4)$$

**where**

$$m_1 = +\sqrt{\frac{\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2} + \sqrt{\frac{T^2h_0^2}{D_0^2} + \frac{4n^2\pi^2}{a^2}\cdot\frac{Th_0}{D_0}}}{2}}$$

$$m_2 = +\sqrt{\frac{\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2} - \sqrt{\frac{T^2h_0^2}{D_0^2} + \frac{4n^2\pi^2}{a^2}\cdot\frac{Th_0}{D_0}}}{2}}$$

$$m_3 = -\sqrt{\frac{\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2} + \sqrt{\frac{T^2h_0^2}{D_0^2} + \frac{4n^2\pi^2}{a^2}\cdot\frac{Th_0}{D_0}}}{2}}$$

$$m_4 = -\sqrt{\frac{\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2} - \sqrt{\frac{T^2h_0^2}{D_0^2} + \frac{4n^2\pi^2}{a^2}\cdot\frac{Th_0}{D_0}}}{2}}$$

**and**

$$P = m_4^2(m_3^2 - m_1^2)(e^{m_4a} - e^{m_1a}) - (m_4^2 - m_1^2)\cdot m_3^2(e^{m_3a} - e^{m_1a})$$

$$Q = m_2^2(m_4^2 - m_1^2)(e^{m_2a} - e^{m_1a}) - m_4^2(e^{m_4a} - e^{m_1a})(m_2^2 - m_1^2)$$

$$R = m_3^2(m_2^2 - m_1^2)(e^{m_3a} - e^{m_1a}) - m_2^2(m_3^2 - m_1^2)(e^{m_2a} - e^{m_1a})$$

**With these values of $A_{01}, A_{02}, A_{03}, A_{04}$ and combining equations (6) and (3·0) $W_0$ is obtained.**

**Considering equations (2·1), (3·1) and (4) we have**

$$\frac{d^4X_{1n}}{dx^4} - \frac{d^2X_{1n}}{dx^2}\left(\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2}\right) + X_{1n}\frac{n^4\pi^4}{a^4}$$

$$= -\Bigg[2\frac{x}{a}\cdot\frac{Th_0}{D_0}\left\{A_{01}m_1^2e^{m_1x} + A_{02}m_2^2e^{m_2x} + A_{03}\cdot m_3^2e^{m_3x} + A_{04}\cdot m_4^2e^{m_4x}\right\} +$$

$$+ 12\frac{x^2}{a^2}\cdot\frac{q_0}{D_0}\cdot\frac{1}{\pi n} + \frac{6}{a}\Big\{A_{01}e^{m_1x}\left(m_1^3 - m_1\frac{n^2\pi^2}{a^2}\right) + A_{02}e^{m_2x}\left(m_2^3 - m_2\frac{n^2\pi^2}{a^2}\right) +$$

$$+ A_{03}e^{m_3x}\left(m_3^3 - m_3\frac{n^2\pi^2}{a^2}\right) +$$

$$+ A_{04}e^{m_4x}\left(m_4^3 - m_4\frac{n^2\pi^2}{a^2}\right)\Big\}\Bigg] \quad \text{...(8)}$$

Solution of equation (8) can be put in the form

$$X_{1n} = A_{11}e^{m_1x} + A_{12}e^{m_2x} + A_{13}e^{m_3x} + A_{14}e^{m_4x}$$

$$- \left[ 2\frac{Th_0}{D_0 a} \left\{ \frac{A_{01}m_1e^{m_1x}}{4m_1^2-2\lambda_1^2}\left(\frac{x^2}{2} - \frac{6m_1^2-\lambda_1^2}{4m_1^3-2\lambda_1^2m_1}\cdot x\right) \right.\right.$$

$$+ \frac{A_{02}m_2e^{m_2x}}{4m_2^2-2\lambda_1^2}\left(\frac{x^2}{2} - \frac{6m_2^2-\lambda_1^2}{4m_2^3-2\lambda_1^2m_2}\cdot x\right)$$

$$+ \frac{A_{03}m_3e^{m_3x}}{4m_3^2-2\lambda_1^2}\left(\frac{x^2}{2} - \frac{6m_3^2-\lambda_1^2}{4m_3^3-2\lambda_1^2m_3}\cdot x\right)$$

$$\left. + \frac{A_{04}m_4e^{m_4x}}{4m_4^2-2\lambda_1^2}\left(\frac{x^2}{2} - \frac{6m_4^2-\lambda_1^2}{4m_4^3-2\lambda_1^2m_4}\cdot x\right)\right\}$$

$$+ \frac{12 q_0 a^4}{D_0 n^5\pi^5}\left\{\frac{x^2}{a^2} + \frac{\left(\frac{2Th_0}{D_0} + \frac{4n^2\pi^2}{a^2}\right)a^2}{n^4\pi^4}\right\} + \frac{6}{a}\left\{\frac{A_{01}e^{m_1x}}{4m_1^2-2\lambda_1^2}\left(m_1^2 - \frac{n^2\pi^2}{a^2}\right)x\right.$$

$$+ \frac{A_{02}e^{m_2x}}{4m_2^2-2\lambda_1^2}\left(m_2^2 - \frac{n^2\pi^2}{a^2}\right)x + \frac{A_{03}e^{m_3x}}{4m_3^2-2\lambda_1^2}\left(m_3^2 - \frac{n^2\pi^2}{a^2}\right)x$$

$$\left.\left. + \frac{A_{04}e^{m_4x}}{4m_4^2-2\lambda_1^2}\left(m_4^2 - \frac{n^2\pi^2}{a^2}\right)x\right\}\right]$$

$$= A_{11}e^{m_1x} + A_{12}e^{m_2x} + A_{13}e^{m_3x} + A_{14}e^{m_4x} + F(x) \qquad \ldots(9)$$

where

$$\lambda_1^2 = \frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2}$$

Using boundary conditions and solving for the constants, we have

$$A_{11} = F(0) - \frac{F''(0) - m_1^2 F(0)}{m_2^2 - m_1^2} - \left(\frac{m_3^2 - m_1^2}{m_2^2 - m_1^2} + 1\right)\left(\frac{LM_2 - L_2M}{L_1M_2 - L_2M_1}\right)$$
$$- \left(\frac{m_4^2 - m_1^2}{m_2^2 - m_1^2} + 1\right)\left(\frac{LM_1 - L_1M}{L_2M_1 - L_1M_2}\right)$$

$$A_{12} = \frac{F''(0) - m_1^2 F(0) - (m_3^2 - m_1^2)\left\{\frac{LM_2 - L_2M}{L_1M_2 - L_2M_1}\right\} - (m_4^2 - m_1^2)\left\{\frac{LM_1 - L_1M}{L_2M_1 - L_1M_2}\right\}}{m_2^2 - m_1^2}$$

$$A_{13} = \frac{LM_2 - L_2M}{L_1M_2 - L_2M_1}, \qquad A_{14} = \frac{LM_1 - L_1M}{L_2M_1 - L_1M_2}$$

where

$$L = F''(a) - m_1^2 F(a) - F''(0)e^{m_2a} + m_1^2 e^{m_2a} F(0)$$

$$M = F''(a)(m_2^2 - m_1^2) - e^{m_1a}(m_2^2 - m_1^2)F''(0) - m_2^2(e^{m_2a} - e^{m_1a})F''(0)$$
$$+ m_1^2 m_2^2 (e^{m_2a} - e^{m_1a}) F(0)$$

$$L_1 = (m_3^2 - m_1^2)(e^{m_3a} - e^{m_2a})$$
$$M_1 = m_3^2(m_2^2 - m_1^2)(e^{m_3a} - e^{m_1a}) - m_2^2(m_3^2 - m_1^2)(e^{m_2a} - e^{m_1a})$$
$$L_2 = (m_4^2 - m_1^2)(e^{m_4a} - e^{m_2a})$$
$$M_2 = m_4^2(m_2^2 - m_1^2)(e^{m_4a} - e^{m_1a}) - m_2^2(m_4^2 - m_1^2)(e^{m_2a} - e^{m_1a})$$

Thus $W_1$ is determined completely.

By similar procedure we can have $W_2, W_3, \dots W_m$. Hence $W$ is obtained.

A graph is plotted showing $\frac{W D_0}{q_0 a^4}$ against $\frac{x}{a}$ of the section $\frac{y}{a}$ with $\lambda = 0.1$; $\sigma = 0.25$, and $a^2 \frac{T_{h0}}{D_0} = 1$.

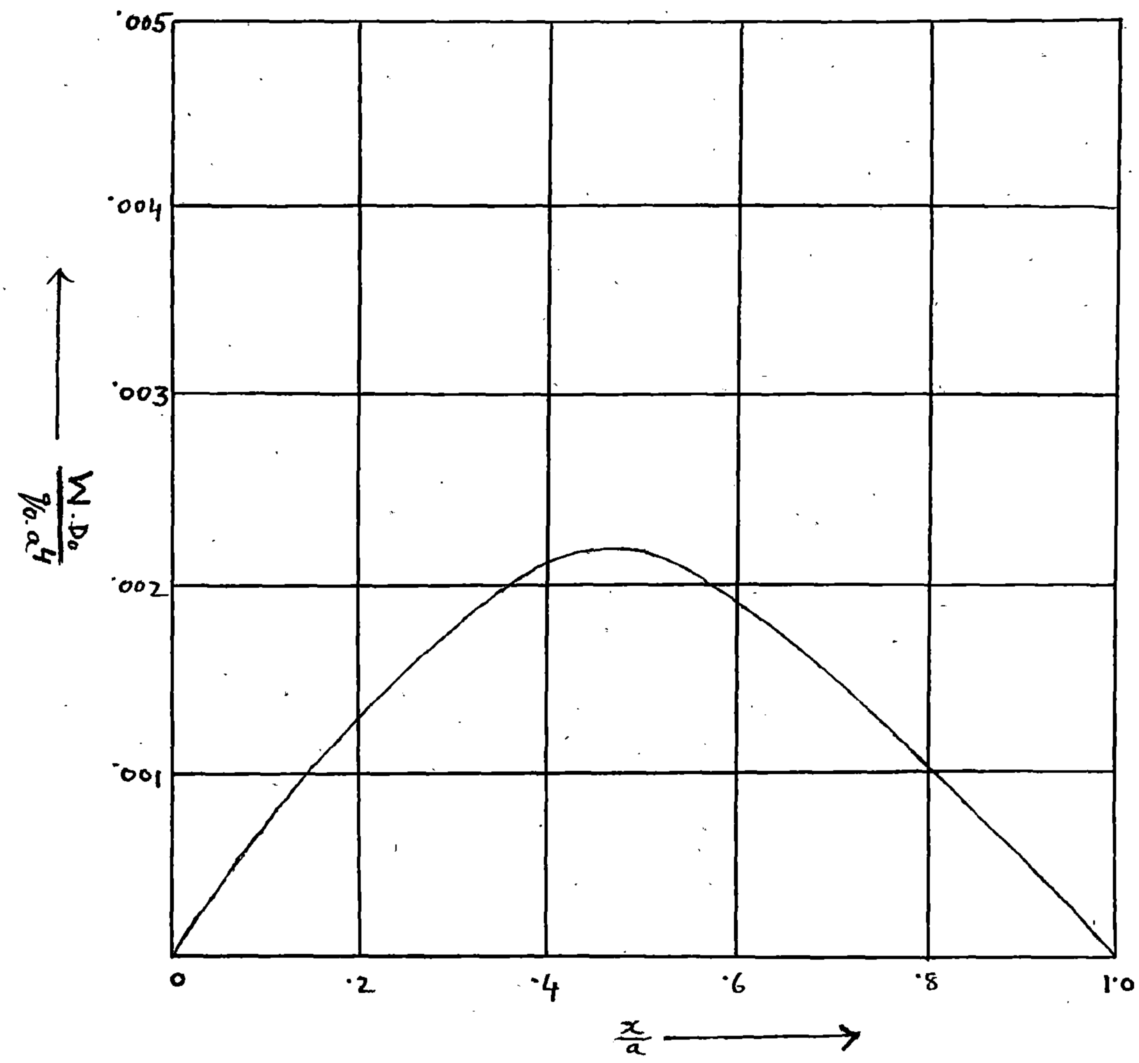


FIG. 2

Deflection of the section $y = \frac{a}{2}$

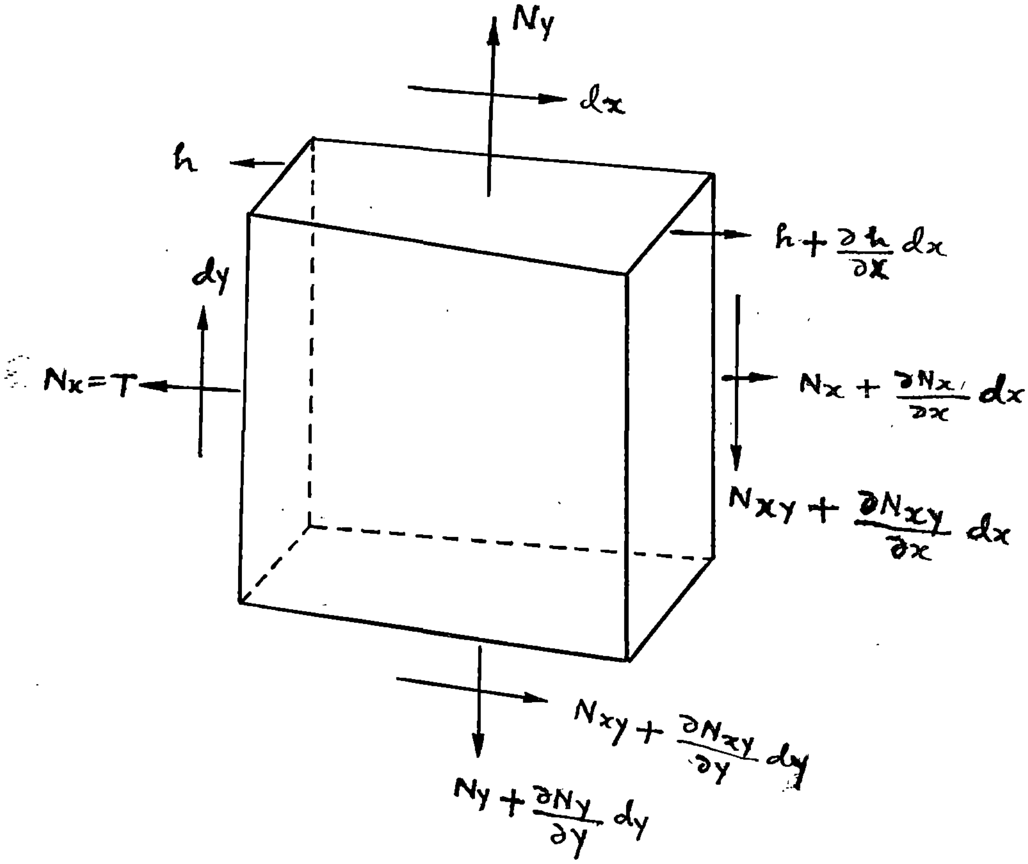


FIG. 3

Stresses on an element of the plate.