Abstract

The value distribution theory of entire and meromorphic functions is a prominent branch of complex analysis. For a non-constant entire or meromorphic function f, it is necessary in certain problem to study the behavior of roots of the equation $f - a = 0$, where a is a complex number.
The value distribution theory of entire and meromorphic functions which was developed by famous mathematician Rolf Nevanlinna is the back-bone of the thesis. The uniqueness theory of entire and meromorphic functions has become an active subfield of the value distribution theory. Mostly the uniqueness theory studies how an entire and meromorphic function on a domain can be determined uniquely under certain conditions. The thesis mainly focuses on some uniqueness problems of entire and meromorphic functions and their derivatives.

The thesis consists of eight chapters.

In chapter 1, we give some definitions, notations and some basic as well as significant results of the value distribution theory.

The following theorem plays an important role in Nevanlinna value distribution Theory.

Theorem 1. (**The Poisson-Jensen's Formula**) Suppose $f(\zeta)$ is meromorphic in $|\zeta| \leq R(0 < R < \infty)$ and that a_μ $(\mu = 1, 2, ..., m)$ and b_ν $(\nu = 1, 2, ..., n)$ are the zeros and poles of $f(\zeta)$ in $|\zeta| \leq R$ respectively. If $z = re^{i\theta}$ is a point in $|\zeta| < R$, distinct from a_μ and b_ν, then

$$\log |f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\varphi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2} d\varphi$$
$$+ \sum_{\mu=1}^{m} log \left| \frac{R(z - a_\mu)}{R^2 - \overline{a}_\mu z} \right| - \sum_{\nu=1}^{n} log \left| \frac{R(z - b_\nu)}{R^2 - \overline{b}_\nu z} \right|.$$

Definition 1. Let f be a non-constant meromorphic function in the open complex plane and $a \in \mathbb{C} \cup \{\infty\}$. Then

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r,$$

is called the integrated counting function of the a-points of f, where $n(r, a; f)$ is the number of a-points of f in $|z| \leq r$ counted according to multiplicities.

Definition 2. For $x \geq 0$, we define

$$\log^+ x = \max(\log x, 0) = \begin{cases} \log x, & \text{if } x \geq 1 \\ 0, & \text{if } 0 \leq x < 1. \end{cases}$$

Then
$(i)\log^+ x \geq 0$, for all $x \geq 0$,
$(ii)\log^+ x \geq \log^+ y$, for $x \geq y$,
$(iii)\log x = \log^+ x - \log^+ \frac{1}{x}$, for $x > 0$.

Definition 3. Let f be a non-constant meromorphic function on $|z| \leq R$ $(0 < R < \infty)$. The function

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\psi})|d\psi,\ 0 < r < R,$$

is called the proximity function of f. Every so often $m(r, f)$ is expressed as $m(r, \infty; f)$. Also for $a \neq \infty$

$$m(r, a; f) = m(r, \frac{1}{f-a}) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\psi}) - a|}d\psi.$$

Definition 4. The Nevanlinna characteristic function of a meromorphic function f is

$$T(r, f) = m(r, f) + N(r, f).$$

Clearly $T(r, f)$ is non-negative. The quantity $T(r, f)$ plays an important role in the Nevanlinna's value distribution theory of meromorphic functions.

Using Nevanlinna characteristic function Jensen's formula can be written as

$$T(r, f) = T(r, \frac{1}{f}) + \log |f(0)|,$$

where $f(0) \neq 0,\ \infty$. This shows that the characteristic functions of f and $\frac{1}{f}$ differ only by a constant.

Definition 6. Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E(a, f)$ the set $E(a, f) = \{z \in \mathbb{C} : f(z) - a = 0\}$, where each zero with multiplicity m is counted m times and by zeros of $f - \infty$ we mean poles of f. If we ignore multiplicities, then the set is denoted by $\overline{E}(a, f)$. We say that f and g share a IM (CM) provided that $\overline{E}(a, f) = \overline{E}(a, g)$ $(E(a, f) = E(a, g))$. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (CM), then we say that f and g share ∞ IM (CM).

Considering the problem of value sharing Nevanlinna proved the remarkable result known as Nevanlinna's five-value theorem which state that if two non-constant meromorphic functions in the complex plane share five distinct values IM, then they are identical.

In chapter 2, we have studied the uniqueness of meromorphic functions whose differential polynomials share a non-constant polynomial.

In chapter 3 deals with the uniqueness result of meromorphic functions when the differential polynomial $(f^n)^{(k)}$ share one small function with differential polynomial $P[f^q]$ of meromorphic function f^q.

In chapter 4, we have investigated the uniqueness problem of meromorphic functions when their differential polynomials share a small function. Here we have proved either $P[f] \equiv P[g]$ or $P[f]P[g] \equiv a^2$ when $P[f]$ and $P[g]$ share (a, l).

In chapter 5, we have investigated the uniqueness of meromorphic functions when they share a set of roots of unity.

In chapter 6 and 7, using the notion of weakly weighted-sharing, we have studied the uniqueness of non-constant homogeneous differential polynomials $P[f]$ and $P[g]$ generated by meromorphic functions f and g respectively.

In chapter 8 deals with the uniqueness problem for higher order derivatives of meromorphic functions on annuli.

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