ON DIFFERENT TOPOLOGY-LIKE STRUCTURES

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By

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To

MY WELL-WISHERS

DECLARATION

I declare that the thesis entitled "**ON DIFFERENT TOPOLOGY-LIKE STRUCTURES**", has been prepared by me under the guidance of Dr. S. De Sarkar, Associate Professor of Department of Mathematics, University of North Bengal. No part of this thesis has formed the basis for the award of any degree or fellowship previously.

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ABSTRACT

The 20^{th} century saw the rise of Point Set Topology, Generalized Topology and the present century brought the concepts of Minimal Structure as well as Weak Structure. These arose from the effort to establish a solid base for Analysis and the Hausdorff's book "Grundz*ü*ge der Mengenlehre " indicates that the development of such topology and topology-like structures are intimately linked to the progress in Set Theory. This thesis extends the fields by changing the domain of thoughts and using the properties of different operators on a nonvoid set endowed with aforementioned topological, minimal or weak structures and developing the theory of those spaces in new settings. The richness of the theory presented in the current thesis is shown by the fundamental fact that a nonvoid set may be equipped with countably many p-structures, which are connected to each other by some rules, where $p \in$ {topological, minimal, weak}. A careful study of most important topological notions and concepts like density, closeness, separation axioms, various operators and their components has been made in the new settings.

This thesis consists of seven chapters and among them the first is preparatory or, may be called, introductory. The essence of the first chapter of the thesis is to warm-up topologically. Some basic definitions and results that are commonly used directly or indirectly in different chapters of the thesis are also accumulated here to provide source of ready references.

In the second chapter, notions of $K\Omega$-closure operators, relative closure operators, and their components are introduced. Furthermore, relation among themselves are investigated, additivity and productivity behavior of such operators are observed and the sequential topologies (ST) induced by them are studied. Two dimensional expansion, in some sense, of a sequential set by the aforementioned operators plays the main and important role in this chapter. K. Kuratowski introduced the concept of an operator (classically known under the name "Kuratowski's closure operator") $K : P(X) \to P(X)$, $X \neq \phi$ from the power set of X into itself which generates a topology on X and the closure operator on this topological space coincides with the operator K. This chapter starts with a change in the domain and codomain of the Kuratowski's closure operator. We consider the domain and codomain to be $(P(X))^{\mathbb{N}}$ instead of $P(X)$. It is observed that such a change leads to remarkable consequences in the theory producing a topology-like structure which we call sequential topology (briefly ST). Theory of $K\Omega$-interior operators has been developed analogously. Main results of this chapter are published in [**M. Singha** and S. De Sarkar, *On* $K\Omega$ *and Relative Closure Operators in* $(P(X))^{\mathbb{N}}$, J. Adv. Stud. Topol. **3** (1) (2012), 72 – 80.]

In the last few decades, closure operators, interior operators and various compositions of them have been studied that produced different classes of subsets of a topological space. A large number of papers is devoted to the study of such classes, possessing properties more or less similar to those of open or closed sets of the underlying topological space. It is seen that different open and

closed sets correspond to separation axioms of different forms producing extremely interesting theories. Composition of $K\Omega$-closure and $K\Omega$-interior operators in various permutations yields theory of different types of open and closed sequential sets in STSs. In the third chapter, considering the theory of semi open and semi closed sets introduced by N. Levine as a model, we develop a theory of semi open and semi closed sequential sets in newly introduced $K\Omega$-spaces by using the composition 'C∘I'of $K\Omega$-closure and $K\Omega$-interior operators 'C' and 'I' respectively. Some results of this chapter are the generalization of the results published in [S. Das, **M. Singha** and S. De Sarkar, *Semi Open and Weakly Semi Open Sequential Sets in Sequential Topological Spaces*, Vesnik, BSPU, **9** 2(19) (2009), 40 – 52.]

In the fourth chapter, development of the study of richness of sequential topological spaces (STSs) has been considered. In doing so it is noticed that the existing theory related to separation axioms in STS suffer from major deficiency because of the ambiguity in defining "distinct points" in such spaces. We get rid of such obstacles by providing appropriate definitions and then develop the axioms up to T_4. Results of this chapter are published in [N. Tamang, **M. Singha** and S. De Sarkar, *Separation Axioms in Sequential Topological Spaces in the Light of Reduced and Augmented Bases*, Int. J. Contemp. Math. Sci. **6** (23) (2011), 1137 – 1150.]

It is surprising to note that every composition of closure and interior operators in a topological space are monotonic. This means if 'comp' is a composition of closure and interior operators in any permutation then $A \subset B \Rightarrow \text{Comp}(A) \subset \text{Comp}(B)$. Taking the behavior of these compositions into account Á. Császár introduced operator $\gamma : P(X) \to P(X)$, possessing the property of monotony to define γ-open and γ-closed sets. The collection of all these γ-open sets provides a structure on X known as generalized topology (briefly GT) which drew remarkable attention of researchers in this area. Once such an operator γ is defined it is natural to ask "What happens if both of domain and codomain of γ are taken as $(P(X))^{\mathbb{N}}$?". One may wish to ask more specifically: "Does the new operator resemble generalized sequential topology (GST) in the same way as the γ-operator does GT ? ". In the fifth chapter we answer these and similar questions in the affirmative sense and obtain generalized sequential topology which provides many interesting results. Notions of monotonic sequential operators and their components (on and off) are introduced and studied sequential as well as generalized sequential topologies induced by them. Some properties of such operators and their components have been derived using the concept of operators introduced by Á. Császár and investigated the relations among themselves. A generalized concept of connectors of GTs is obtained and a version of Teitze's extension theorem in GTS has been established. Expansion or contraction of a component of sequential set depends upon some other components of that sequential set; this kind of dependence plays the main role and yields plenty of results in this chapter. The paper [**M. Singha** and S. De Sarkar, *On Monotonic Sequential Operators*, Accepted for its publication in Southeast Asian Bull. Math.] is totally based on this chapter.

We then turn our attention to some other lines of development of the theory; namely, "theories of minimal and weak structures". Considerable development has been made in these directions during the last twelve years. V. Popa and T. Noiri introduced 'minimal structure' while 'weak structure' was introduced by *Á*. Cs*á*sz*á*r. In both cases some condition in the definition of topology were relaxed to obtain these new definitions. Currently many researchers are engaged in developing theories with these new concepts and studies are being made towards defining different types of open and closed sets and related separation axioms. In this thesis, we take up some problems related to separation axioms and continuity (e. g., Urysohn's lemma) in spaces endowed with minimal structure and weak structure.

With necessary modifications and corrections of some results in the existing literature on minimal structures we propose, in the sixth chapter, a development in the related theory that enables us, along with other important results, to establish a generalization of Urysohn's lemma in this new setting. In doing so, various separation axioms and different kinds of continuity related to Urysohn's lemma in minimal structures have been studied. Main results of this chapter are included in the paper [**M. Singha** and S. De Sarkar, *Towards Urysohn's Lemma in Minimal Structures*, Int. J. Pure Appl. Math. **85** (2) (2013), 255 – 264.]

What happens to various separation axioms and different kinds of continuity in topological spaces, GTS and minimal structure when the spaces are replaced by weak structures? We consider this question in the last chapter and it is seen that among many results, a variant of Urysohn's lemma is achievable in the later case. All the results of this chapter are published in [**M. Singha**, *Urysohn's Lemma in Weak Structures*, Bull. Cal. Math. Soc. **104** (6) (2012), 547 – 552.]

It is hoped that our investigations in the present thesis have thrown much light on the issues of several open questions in point set topology, generalized topology, minimal structures, weak structures, sequential topology etc. and paved the way for further research in this direction.

PREFACE

The idea to work on different topology-like structures, and more importantly, the desire to do so, is a direct consequence of searching possible identity in the differences among point set topology, sequential topology, generalized topology, minimal structure and weak structure. During this work many new mathematical notions were observed and studied, that yielded plenty of results on which this thesis is based.

The completion of the thesis would not have been possible without the support of a number of people to whom I owe many thanks. First and foremost, I thank God and my parents who only know all the ups and downs in my life I passed through to reach at the present situation where I am and to whom I share all my happiness and sorrows; I am highly proud to be their son. I am indebted to my supervisor Dr. Suparna De Sarkar for sparking my interest in the aforementioned area of research. Whenever I faced difficulties, my respected guide encouraged me not to be upset and advised me to confront the problems with new energies. This helped me immensely in developing the entire content of the present thesis. I would like to take this opportunity to thank my colleagues (present and past) for their cooperation during entire period of investigation. Finally, I am deeply grateful to Debasmita, Mukti, my grandparents, my friends and other relatives especially Mrityunjoy uncle, Sudhir uncle, Dhananjoy uncle and Swapna anti, without their support I never able to be in a position of writing a thesis.

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(2) N. Tamang, **M. Singha** and S. De Sarkar, *Separation Axioms in Sequential Topological Spaces in the Light of Reduced and Augmented Bases*, Int. J. Contemp. Math. Sci. **6** (23) (2011), 1137 – 1150.

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(5) **M. Singha**, *Urysohn's Lemma in Weak Structures*, Bull. Cal. Math. Soc. **104** (6) (2012), 547 – 552.

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Chapter I

Topological Warm-up

1.1. Symbols and their meanings/definitions

$$\begin{aligned}
\Phi &\Leftrightarrow \text{void set.}\\
x \in A &\Leftrightarrow x \text{ belongs to } A\\
x \notin A &\Leftrightarrow \text{just negation of } x \in A\\
A - B &\Leftrightarrow \text{complement of B in A}\\
A^c &\Leftrightarrow \text{complement of A in whole set}\\
\{A_\lambda; \lambda \in \Lambda\} &\Leftrightarrow \text{family of sets.}\\
\cup\{A_\lambda; \lambda \in \Lambda\} &\Leftrightarrow \text{union of a family of sets.}\\
\cap\{A_\lambda; \lambda \in \Lambda\} &\Leftrightarrow \text{intersection of a family of sets.}\\
A \subset B &\Leftrightarrow B \supset A\\
&= A \text{ consists of some or all elements of } B\\
&\quad\ A \text{ is called subset of } B \text{ and}\\
&\quad\ B \text{ is called superset of } A.\\
A \not\subset B &\Leftrightarrow \text{negation of } A \subset B.\\
A = B &\Leftrightarrow \text{equality of sets } A \text{ and } B.\\
\mathcal{P}(X) &\Leftrightarrow \text{power set of } X.\\
\mathbb{N}, \mathbb{Z}, \mathbb{Q} \text{ and } \mathbb{R} &\Leftrightarrow \text{set of natural numbers, integers, rational}\\
&\quad\ \text{numbers and real numbers respectively}\\
Y^X &\Leftrightarrow \text{collection of all functions from } X \text{ to } \mathbb{R}.
\end{aligned}$$

1.2. Prerequisites

An amazing feature of present century mathematics has been its recognition of the power of the topological approach. This abstract approach has given rise to a plenty of new results and problems and has, probably, led us to open whole new areas of mathematics whose very existence had not even been suspected. In this thesis, some such new areas e. g., Sequential Topology, Generalized Topology, Minimal Structure, Weak Structure have been cultivated and various kind of operators and their components has been introduced for such investigations.

Open sets have played an extremely important role in much of mathematics. They are the main tools to study Analysis. Considering the collection of such open sets as a model the definition of Topology has been set up. Let X be a nonvoid set and τ be a collection of some subsets of X such that, it contains two extreme subsets Φ and X, it is closed under arbitrary union and it is also closed under finite intersection. Then the collection τ is called a topology on X and the ordered pair (X, τ) is called a topological space. The members of τ are called τ-open sets or simply, open sets (if no confusion arises) in the underlying set X and their complements are called closed sets. It is immediate that the collection of all closed sets in a topological space contains the two extreme subsets of the space and is closed under arbitrary intersection as well as under finite union. Different operators has been introduced in terms of open and closed sets. Two important among them are closure and interior operators in topological spaces. Let (X, τ) be a topological space and $A \subset X$. Then the intersection $\overline{A}$ or $cl(A)$ or $c(A)$ of all closed sets containing A and the union $\overset{\circ}{A}$ or $Int(A)$ or $i(A)$ of all open sets contained in A are called closure and interior of the set A respectively. The closure and interior operators in a topological space possess several properties. Four independent properties of closure operator in a topological space are as follows; it is expansive, it has ϕ as a fixed point, it commutes with finite union and it is idempotent. The interior operator in a topological space is contractive, X is a fixed point of it, it commutes with finite intersection and it is also idempotent. An operator $K : P(X) \to P(X)$, $X \neq \Phi$ having all the four properties of aforementioned closure operator in a topological space is called Kuratowski's closure operator. The collection of complements of all fixed points of a Kuratowski's closure operator $K : P(X) \to P(X)$ forms a topology on the underlying set X and the closure operator therein coincides with the operator K. Similarly, a mapping $I : P(X) \to P(X)$ satisfying all the four independent properties of an interior operator in a topological space is called an interior operator. The collection of all fixed points of I forms a topology on X and the interior operator in that topological space is I. A Kuratowski's closure operator $K : P(X) \to P(X)$ and an interior operator $I : P(X) \to P(X)$ both generate same topology on the underlying set X iff they are related by the equation $(K(A))^c = I(A^c)$,

$\forall\ A \in P(X)$ or any equivalent form of it. As said before, different operators on a topological space have been defined and studied in terms of open and closed sets to characterize openness, closedness of a set in that topological space and continuity of a function. Three natural operations on a subset of a space are interior, closure and compliment. It is surprising and interesting fact that the number of distinct sets one may produce using these operations on a subset of a topological space is limited. In 1922, K. Kuratowski proved that for any given subset A of a topological space there are, at most, 14 distinct sets that may be constructed by applying composition of the interior, closure and complement operations on A in different permutations. It is also observed that there exists a subset $(A = [0,1]\bigcup(2,3)\bigcup[(4,5)\bigcap\mathbb{Q}]\bigcup[(6,8)-\{7\}]\bigcup\{9\})$ of the real line, with the standard topology, which yields 14 distinct sets when operated on by every permutation of the interior, closure and compliment operations. A number of mathematicians have used various combinations of the axioms what Kuratowski assumed to define closure operators. In 1963 N. Levin used the composition $coInt$, where 'c' denotes closure and 'Int' denotes interior operators in a topological space and introduced the concept of semi-open sets. He defined that a subset A of a topological space X is semi-open if $A \subset coInt(A)$ and shown that a set A is semi-open iff $\exists$ an open set O so that $O \subset A \subset c(O)$. Complement of a semi-open set is called semi-closed set. In the literature it has been seen that mathematicians used various compositions of closure, interior and complements to define different type of sets in a topological space. Some of them are as follows: A subset A of a topological space (X, τ) is said to be regular open if $A = Int(c(A))$, regular closed if $A = c(Int(A))$, δ-open if for each $x \in A$ there exists a regular open set G such that $x \in G \subset A$, δ-closed if its complement is δ-open, α-open if $A \subset Int(c(Int(A)))$, pre-open or nearly open if $A \subset Int(c(A))$, β-open or semi-pre-open if $A \subset c(Int(c(A)))$, b-open or sp-open if $A \subset c(Int(A)) \bigcup Int(c(A))$, feebly open if there exists an open set O so that $O \subset A \subset scl(A)$, where $scl(A) =$ semi-closure of $A =$ intersection of all semi-closed set containing A, θ-open if for each $x \in A$ there exists an open set U such that $x \in U \subset c(A) \subset A$ and complement of θ-open set is called θ-closed set. Various separation axioms in topological spaces have been developed in literature using different open and closed sets mentioned above. A topological space (X, τ) is called T_0 space if for any two distinct points there is an open set that contains one of them but does not contain the other (i. e., if any pair of distinct points can be separated by an open sets). If a topological space (X, τ) is not T_0 then there would exists a pair of distinct points x and y in X such that every open set in (X, τ) contains either both of them or none of them and so, any topological statement about one of them will imply a corresponding statement about the other and vice versa. A singleton subset of a T_0 space may not be closed. A topological space is called T_1 space if for any two distinct points $x, y \in X$ $\exists$ open sets U, V in (X, τ) such that $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$

(i. e., if any pair of distinct points can be separated by a pair of open sets). A topological space is T_1 if and only if every finite subset of it is closed. A T_1 space is T_0 but the converse is not true in general. If for any two distinct points there are disjoint open sets in (X, τ) containing them (i. e., if any pair of distinct points can be separated by a pair of disjoint open sets) then the space is called T_2 or Hausdorff space. Every T_2 space is T_1 but the converse may not be true. In a T_2 space limit of a convergent sequence is unique which may not be true in T_1 space. But uniqueness of limit of every convergent sequence in a topological space (X, τ) does not imply that the space is T_2. A topological space is called regular if for any point and any closed set not containing that point $\exists$ two disjoint open sets containing them (i. e., if any point and any closed set not containing that point can be separated by a pair of disjoint open sets). A topological space is regular if and only if for any point x and any open set U containing x, $\exists$ an open set V so that, $x \in V \subset \overline{V} \subset U$. In a regular space singleton subsets may not be closed and so a regular space may not be T_1 and so, it may not be T_2. A topological space which is regular as well as T_1 is called T_3 space. A T_3 space is of course T_2 but the converse need not be true. If in a topological space it is possible to separate any pair of disjoint closed sets by a pair of disjoint open sets then the space is called normal. A topological space is normal if and only if for any closed set C and any open set U containing C, $\exists$ an open set V such that, $C \subset V \subset \overline{V} \subset U$. Singletons in a normal space may not be closed and so normal space need not be T_1. A normal space which is also T_1 is called T_4 space. Though T_1 space may not be T_4 but every T_4 space is T_1. P. Urysohn characterized normality of a space by the help of continuous functions. He shown that a topological space (X, τ) is normal if and only if for any pair of nonvoid disjoint closed sets C and D $\exists$ a continuous function $f : (X, \tau) \to (\mathbb{R}, \mathcal{U})$, where $\mathcal{U}$ is the usual or standard topology on $\mathbb{R}$, so that, $f(C) = \{0\}$ and $f(D) = \{1\}$. A point to be noted that, in the last result, 0 and 1 can be replaced by any two distinct real numbers a and b. Using this result H. Tietze characterized normal topological spaces by the help of extensibility of real valued continuous functions on nonvoid closed subsets to the whole of the space as follows: a topological space (X, τ) is normal if and only if for every closed $C \subset X$, every continuous function $f : C \to \mathbb{R}$ has a continuous extension $F : X \to \mathbb{R}$.

A directed set $\mathcal{J}$ is a set with a partial order $\leq$ such that for each pair α, β of elements of $\mathcal{J}$, there exists an element γ of $\mathcal{J}$ having the property that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A subset $\mathcal{K}$ of $\mathcal{J}$ is said to be cofinal in $\mathcal{J}$ if for each $\alpha \in \mathcal{J}$, there exists $\beta \in \mathcal{K}$ so that $\alpha \leq \beta$. Obviously, every cofinal subset of a directed set is derected. Let X be a nonvoid set. A net in X is a function f from a directed set $\mathcal{J}$ into X. If $\alpha \in \mathcal{J}$, we usually denote $f(\alpha)$ by x_α. We denote the net f itself by the symbol $(x_\alpha)_{\alpha \in \mathcal{J}}$ or merely by (x_α) if the index set is understood. A net $(x_\alpha)_{\alpha \in \mathcal{J}}$ in a topological space X is said to be convergent if there is a point $x \in X$ so that for any open set

U containing x there is a $\gamma \in \mathcal{J}$ such that $\gamma \leq \alpha$ implies $x_\alpha \in U$; point x is called a limit of the net. Let $A \subset X$. Then a point $x \in \overline{A}$ if and only if there is a net in A converging to x. A topological space is Hausdorff iff every convergent net in it has unique limit.

As in [6, 57, 60, 62] any sequence $A(s) = \{A_n\}$, where $A_n \subset X$ for all $n \in \mathbb{N}$ is called a sequential set in X and A_n is called n^{th} component of $A(s)$; we write $P_n(A(s)) = A_n$. Thus sequential sets in X are precisely the members of $(P(X))^{\mathbb{N}}$. If each $A_n = X$, $n \in \mathbb{N}$, then the corresponding sequential set is denoted by $X(s)$ and $\Phi(s)$ denotes the sequential set having each term equal to Φ. If $F(s) \in (P(X))^{\mathbb{N}}$ and $A \in P(X)$ then ${}_nF_A(s)$ denotes the sequential set obtained from $F(s)$ replacing n^{th} component of it by A. Let $A(s) = \{A_n\}$ be a sequential set so that, $A_n \neq \phi$ iff $n \in P \subset \mathbb{N}$ then P is called the base of $A(s)$. A sequential set $A(s) = \{A_n\}$ in X is called a sequential point if $A_n = \{x\}$ for $n \in P \subset \mathbb{N}$, $P \neq \Phi$ and $A_n = \Phi$ for $n \in \mathbb{N} - P$ and it is denoted by $p = (x\ ,\ P)$, if further $P = \{n\}$ then the sequential point is called a simple sequential point and write $p = (x\ ,\ n)$ rather than $p = (x\ ,\ \{n\})$. A sequential point $p = (x\ ,\ P)$ is called a reduced sequential point of $q = (x\ ,\ Q)$ if $P \subset Q$ and in this case $q = (x\ ,\ Q)$ is called an augmented sequential point of $p = (x\ ,\ P)$. A sequential set $A(s) = \{A_n\}$ is said to be contained in a sequential set $B(s)$ if $A_n \subset B_n$ for all $n \in \mathbb{N}$ and it is denoted by $A(s) \subset B(s)$ or $B(s) \supset A(s)$. If $A(s) \subset B(s)$ and $B(s) \subset A(s)$ then $A(s)$ and $B(s)$ are said to be identical and write $A(s) = B(s)$. So, two sequential points $(x\ ,\ P)$

and $(y\ ,\ Q)$ are identical if $x = y$ and $P = Q$. Two sequential points $(x\ ,\ P)$ and $(y\ ,\ Q)$ are said to be distinct if they are not identical. $p = (x,\ P) \in$ (resp. $\in_w$)$A(s) = \{A_n\}$ means that $x \in A_n$ for all n(resp. for some n) $\in P$. The union and intersection of the sequential sets $A(s) = \{A_n\}$ and $B(s) = \{B_n\}$ in X are defined as $A(s) \cup B(s) = \{A_n \cup B_n\}$ and $A(s) \cap B(s) = \{A_n \cap B_n\}$ respectively. The complement of $B(s)$ in $A(s)$ is denoted by $A(s) - B(s)$ and is defined by $A(s) - B(s) = \{A_n - B_n\}$. $X(s) - A(s)$ is called the complement of $A(s)$ and it is denoted by $A^c(s)$. Let X be a nonvoid set. A subset τ of $(P(X))^{\mathbb{N}}$ is called a sequential topology (briefly ST) on X if

i. τ contains $X(s)$ and $\Phi(s)$,

ii. τ is closed under arbitrary union,

iii. τ is closed under finite intersection

and ordered pair (X, τ) is called a sequential topological space(STS). The members of τ are called $\tau-$open sequential sets or open sequential sets simply, if no confusion arises. A sequential set $A(s)$ is said to be closed if its complement is open. Let (X, τ) be a sequential topological space and Y be a nonvoid subset of X then the sequential topological space $(Y, \tau_Y(s))$, where

$$\tau_Y(s) = \{A(s) \cap Y(s); A(s) \in \tau\}$$

is called a sequential subspace of the sequential topological space (X, τ). For any topology D on X, $D^{\mathbb{N}}$ forms an ST on X, called the ST generated by D and is denoted by $\tau < D >$. In an STS

(X, τ) the collection $D_n(\tau)$ of the n^{th} components of members of τ forms a topology on X and is called the n^{th} component topology of τ on X. If $A(s) = \{A_n\} \in \tau$, then $A_n \in D_n(\tau)$ for each $n \in \mathbb{N}$, but the converse is not necessarily true. A sequential set $A(s)$ is called a neighborhood(resp. weak neighborhood) of a sequential point p if there is an open sequential set $G(s)$ such that $p \in$ (resp.$\in_w$)$G(s) \subset A(s)$. A sequential point $p = (x\ ,\ P)$ is called a limit point of a sequential set $A(s) = \{A_n\}$ if for any weak neighborhood $H(s) = \{H_n\}$ of p, either $x \in H_n \cap A_n$ for some $n \notin P$ or $y \in H_n \cap A_n$ for some $n \in \mathbb{N}$ and $y \neq x$. Any reduced sequential point of a limit point of a sequential set is also a limit point of that sequential set, but the converse is not necessarily true. The union of all limit points of $A(s)$ is called the derived sequential set of $A(s)$ and it is written as $A^{'}(s)$. The closure (resp. interior) of a sequential set $A(s)$, denoted by $\overline{A(s)}$ or $Cl_\tau(A(s))$ (resp. $A(s)^\circ$ or $Int_\tau(A(s))$), is defined to be the intersection (resp. union) of all closed (resp. open) sequential sets containing (resp. contained in) $A(s)$. A sequential point $\alpha = (x, P)$ is said to be an interior point of a sequential set A(s) in the sequential topological space (X, τ) if there exists a $\tau-$open sequential set $B_\alpha(s)$ such that $\alpha = (x, P) \in B_\alpha(s) \subset A(s)$. If $\alpha = (x, P)$ is an interior point of A(s) then the simple sequential point $\beta = (x, n)$ is also an interior point of A(s) for each $n \in P$. The union of all interior points of a sequential set A(s) is the largest open sequential set contained in A(s). In an STS (X, τ) the following hold: (1). $A(s) \subset \overline{A(s),}$ (2). $\overline{X(s)} = X(s)$ and $\overline{\Phi(s)} = \Phi(s)$, (3).

$\overline{\overline{A(s)}} = \overline{A(s)}$, (4). $\overline{A(s) \cup B(s)} = \overline{A(s)} \cup \overline{B(s)}$, (5). $\overline{A(s) \cap B(s)} \subseteq \overline{A(s)} \cap \overline{B(s)}$, (6). A sequential point $p = (x\ ,\ P) \in \overline{A(s)}$ if and only if every weak neighborhood of p and $A(s)$ are intersecting, (7). For any sequential set $A(s)$, $\overline{A(s)} = A(s) \cup A^{'}(s)$, (8). $A(s)^{\circ} \subset A(s)$, (9). $\Phi(s)^{\circ} = \Phi(s)$ and $X(s)^{\circ} = X(s)$ (10). $(A(s)^{\circ})^{\circ} = A(s)^{\circ}$ (11). $(A(s) \cap B(s))^{\circ} = A(s)^{\circ} \cup B(s)^{\circ}$ (12). $A(s)^{\circ} \cup B(s)^{\circ} \subset (A(s) \cup B(s))^{\circ}$ (13). $\overline{A(s)} = X(s) - (X(s) - A(s))^{\circ}$. An STS (X, τ) is called a T_0-space if for every pair of distinct sequential points p and q there exists an open sequential set $U(s)$ containing weakly one of p and q but not the other. An STS (X, τ) is a T_0-space if and only if for every pair of distinct sequential points p and q either is does not belong to the closure of q or q does not belong to the closure of p. Thus an STS is T_0 if and only if distinct sequential points have distinct closures. A topological space $(X,\ D)$ is T_0 if and only if the generated STS $(X,\ \tau(D))$ is T_0. Component spaces of a T_0 STS are T_0. But the converse is not true. An STS is T_1 if for every pair of distinct sequential points p and q there exist two open sequential sets $U(s)$ and $V(s)$ such that $p \in_w U(s)$, $q \in_w V(s)\, p \notin_w V(s)$ and $q \notin_w U(s)$. Every component of a T_1 STS is T_1. A topological space is T_1 if and only if the corresponding generated STS is T_1. An STS is T_1 if and only if every sequential point therein is closed. An STS is said to be a Hausdorff space or a T_2-space if for any two distinct sequential points p and q there exists two open sequential sets $U(s)$ and $V(s)$ such that $p \in_w U(s)$, $q \in_w V(s)\, p \notin_w \overline{V(s)}$ and $q \notin_w \overline{U(s)}$. An STS is said to be a weak

Hausdorff or (w)Hausdorff if for any two distinct sequential points there exists two open sequential sets $U(s)$ and $V(s)$ such that $p \in_w U(s)$, $q \in_w V(s)$ and $U(s) \cap V(s) = \Phi(s)$. An STS is Hausdorff if and only if for any two distinct sequential points p and q there exist two open sequential sets $G(s)$ and $H(s)$ such that $p \in G(s)$, $q \in_w H(s)$, $G(s) \cap H(s) = \Phi(s)$ and there exist two open sequential sets $D(s)$ and $E(s)$ such that $p \in_w D(s)$, $q \in E(s)$, $D(s) \cap E(s) = \Phi(s)$. Every Hausdorff space is (w)Hausdorff but the converse is not true in general. Components of a (w)Hausdorff space and hence that of Hausdorff space are Hausdorff.

CHAPTER II

KΩ Closure and interior operators in $(P(X))^{\mathbb{N}}$

Polish mathematician Kazimierz Kuratowski (1896-1980) introduced the concept of an operator $K : P(X) \to P(X)$, $X \neq \Phi$ which generates a topology on X and the closure operator on this topological space coincides with the operator K. Then a number of mathematicians have used various combinations of the axioms what Kuratowski assumed to define closure operators. In this chapter, those concepts to study different operators from $(P(X))^{\mathbb{N}}$ to itself and their components are invited. It is also observed that what do the operators and their components produce and what are the relations among themselves.
Throughout the chapter, by $i = m(1)n$, we mean i assumes the values from m to n step by step, increasing 1 at every step, where m and n are positive integers with $m \leq n$.

2.1. KΩ-closure, KΩ-interior and relative closure operators

DEFINITION 2.1.1. Let X be a nonvoid set. An operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$, subject to the conditions:
i. $A(s) \subset C(A(s))$ for all $A(s) \in (P(X))^{\mathbb{N}}$,that is, C is expansive.
ii. $C(\Phi(s)) = \Phi(s)$, that is, $\Phi(s)$ is a fixed point of C.
iii. $C(C(A(s))) = C(A(s))$ for all $A(s) \in (P(X))^{\mathbb{N}}$,
that is, C is idempotent.
iv. $C(\underset{i=1(1)n}{\cup} A_i(s)) = \underset{i=1(1)n}{\cup} C(A_i(s))$, where $A_i(s) \in (P(X))^{\mathbb{N}}$,

$i = 1(1)n$, that is, C commutes with finite union,
is called a $K\Omega$-closure operator on $(P(X))^{\mathbb{N}}$.

EXAMPLE 2.1.1. The operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by

$$\begin{aligned} C(A(s)) &= A(s) \cup F(s) \text{ whenever } A(s) \neq \Phi(s), \\ &= \Phi(s) \text{ whenever } A(s) = \Phi(s) \end{aligned}$$

where $F(s)$ is a fixed sequential set in X, is a $K\Omega$-closure operator on $(P(X))^{\mathbb{N}}$.

EXAMPLE 2.1.2. The operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $C(A(s)) = \{\bigcup\limits_{k=1}^{n} A_k\}_n$ for all $A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$ is a $K\Omega$-closure operator on $(P(X))^{\mathbb{N}}$, where X is any nonvoid set.

THEOREM 2.1.1. *Let $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be a $K\Omega$-closure operator. Then $\tau_C = \{A(s)^c\ ;\ A(s) \in (P(X))^{\mathbb{N}}$ and $C(A(s)) = A(s)\} = \{(C(A(s)))^c\ ;\ A(s) \in (P(X))^{\mathbb{N}}\}$ forms an ST on X and $C(A(s)) = \overline{A(s)}$ for all $A(s) \in (P(X))^{\mathbb{N}}$ where $\overline{A(s)}$ is the closure of $A(s)$ in $(X\ ,\tau_C)$.*

PROOF. Proof is omitted. ■

DEFINITION 2.1.2. $\tau_C = \{A(s)^c\, ; A(s) \in (P(X))^{\mathbb{N}}$ and $C(A(s)) = A(s)\}$ is called the ST generated or induced by the $K\Omega-$closure operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$.

DEFINITION 2.1.3. Let A be a nonvoid subset of a nonvoid set X. A sequential set $A(s) = \{A_n\}$, where $A_n = \Phi$ for all $n \neq p \in \mathbb{N}$ and $A_p = A$ is called a simple sequential set in X and it is denoted by $(A\ ,\, p)$.

EXAMPLE 2.1.3. Let $X = \{a, b, c\}$ and $A = \{b, c\}$. Sequential set $(A\ , 1)$ in X whose all but first terms are Φ and the first term is A is a simple sequential set, that is $(A, 1) = A, \Phi, \Phi, \Phi, \Phi, \Phi, \Phi, \Phi$.............so on.

In Example 2.1.1 n^{th} component of the closure of a sequential set in $(X\ , \tau_C)$ is equal to the closure of the n^{th} component of that sequential set in $(X\ , D_n(\tau_C))$ but in Example 2.1.2 they are not equal whenever $n \neq 1$. This kindles the idea of components of a $K\Omega-$closure operator as follows.

DEFINITION 2.1.4. Let X be a nonvoid set and $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be a $K\Omega-$closure operator. A function $C_n : P(X) \to P(X)$ defined by $C_n(A) = n^{th}$ term of $C((A, n))$ if $A \neq \Phi$ and $C_n(\Phi) = \Phi$ is called n^{th} component of C, $n \in \mathbb{N}$.

EXAMPLE 2.1.4. n^{th} component of the $K\Omega-$closure operator in Example 2.1.1 is $C_n : P(X) \to P(X)$ defined by

$$\begin{aligned} C_n(A) &= A \cup F_n \text{ whenever } A \neq \Phi, \\ &= \Phi \text{ whenever } A = \Phi \end{aligned}$$

where F_n is the n^{th} component of $F(s)$, $n \in \mathbb{N}$.

EXAMPLE 2.1.5. n^{th} component of the $K\Omega-$closure operator in Example 2.1.2 is $C_n : P(X) \to P(X)$ defined by $C_n(A) = A$ for all $A \in P(X)$, $n \in \mathbb{N}$.

THEOREM 2.1.2. *Let X be a nonvoid set. If $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is a $K\Omega-$closure operator then each component $C_n : P(X)$*

$\to P(X)$, $n \in \mathbb{N}$ *is a Kuratowski closure operator(closure operator* [13, 42, 63]*). Also* $D_n(\tau_C) = {}_{C_n}\tau$*, where* τ_C *is the ST on* X *induced by the* $K\Omega-$*closure operator* $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ *and* ${}_{C_n}\tau$ *is the topology on* X *induced by the component* $C_n : P(X) \to P(X)$ *of* C*,* $n \in \mathbb{N}$*.*

Proof. Let $A \in P(X)$ then $(A, n) \subset C((A, n)) \Rightarrow A \subset C_n(A)$. By definition, Φ is a fixed point of C_n. Now $(A, n) \subset (C_n(A), n)$ therefore $C((A, n)) \subset C((C_n(A), n)) \Rightarrow C_n(A) \subset C_n(C_n(A))$.
Also $C(C((A, n))) = C((A, n)) \Rightarrow C((C_n(A), n)) \subset C((A, n)) \Rightarrow C_n(C_n(A)) \subset C_n(A)$ hence $C_n(C_n(A)) = C_n(A)$. Let $A_i \in P(X)$, $i = 1(1)n$. $C(\overset{i=n}{\underset{i=1}{\cup}}(A_i\ ,\ n)) = \overset{i=n}{\underset{i=1}{\cup}} C(A_i\ ,\ n) \Rightarrow C(\overset{i=n}{\underset{i=1}{\cup}} A_i, n) = \overset{i=n}{\underset{i=1}{\cup}} C(A_i, n) \Rightarrow C_n(\overset{i=n}{\underset{i=1}{\cup}} A_i) = \overset{i=n}{\underset{i=1}{\cup}} C_n(A_i)$. For the next part, let $A \in D_n(\tau_C)$ therefore $X - A$ is a closed set in $(X, D_n(\tau_C))$. This implies there exists a closed sequential set $B(s) = \{B_m\}$ in $(X\ ,\ \tau_C)$ such that $B_n = X - A$. Now $(X - A, n) \subset B(s) \Rightarrow C((X - A, n)) \subset C(B(s)) = B(s) \Rightarrow C_n(X - A) \subset B_n = X - A$. So $A \in_{C_n} \tau$. Again let $A \in {}_{C_n}\tau$, therefore $C_n(X - A) = X - A$. Let $B(s) = C((X - A, n))$ then $B(s)$ is a closed sequential set and its n^{th} component is $X - A$. Therefore $A \in D_n(\tau_C)$ and the proof is done. ∎

Theorem 2.1.3. *Let* $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ *be a* $K\Omega$ *closure operator and* $A \subset X$*. Then* $C_A : (P(A))^{\mathbb{N}} \to (P(A))^{\mathbb{N}}$ *defined by* $C_A(B(s)) = \{A\}^{\mathbb{N}} \cap C(B(s))$ *is a* $K\Omega$ *closure operator and* $(C_A)_n(B) = A \cap C_n(B)$ *for all* $B \in P(A)$*.*

PROOF. It is straightforward that $C_A : (P(A))^{\mathbb{N}} \to (P(A))^{\mathbb{N}}$ satisfies all but the following condition to be a $K\Omega$ closure operator and $C_A(C_A(B(s)) = C_A(\{A\}^{\mathbb{N}} \cap C(B(s))) = \{A\}^{\mathbb{N}} \cap C(\{A\}^{\mathbb{N}} \cap C(B(s))) \subset \{A\}^{\mathbb{N}} \cap C(\{A\}^{\mathbb{N}}) \cap C(C(B(s))) = \{A\}^{\mathbb{N}} \cap C(B(s)) = C_A(B(s) \Rightarrow C_A : (P(A))^{\mathbb{N}} \to (P(A))^{\mathbb{N}}$ is a $K\Omega$ closure operator. And $(C_A)_n(B) = n^{\text{th}}$ term of $\{A\}^{\mathbb{N}} \cap C((B\ ,\ n)) = A \cap n^{\text{th}}$ term of $C((B\ ,\ n)) = A \cap C_n(B)$. ■

THEOREM 2.1.4. *Let $\{C_\lambda : (P(X_\lambda))^{\mathbb{N}} \to (P(X_\lambda))^{\mathbb{N}}\ ;\ \lambda \in \Lambda\}$ be a family of $K\Omega-$closure operators, where $X_\lambda \cap X_\mu = \Phi$ for any two distinct $\lambda,\ \mu \in \Lambda$. If $X = \underset{\lambda\in\Lambda}{\cup} X_\lambda$, then $SC_\lambda : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $SC_\lambda(A(s)) = \underset{\lambda\in\Lambda}{\cup} C_\lambda(X_\lambda(s) \cap A(s))$ is a $K\Omega$-closure operator and $(SC_\lambda)_n = \oplus(C_\lambda)_n$* [42]

PROOF. For $A(s) \in (P(X))^{\mathbb{N}}$, $SC_\lambda(SC_\lambda(A(s))) = SC_\lambda(\underset{\lambda\in\Lambda}{\cup} C_\lambda(X_\lambda(s) \cap A(s))) = \underset{\lambda\in\Lambda}{\cup} C_\lambda(X_\lambda(s) \cap (\underset{\lambda\in\Lambda}{\cup} C_\lambda(X_\lambda(s) \cap A(s)))) = \underset{\lambda\in\Lambda}{\cup} C_\lambda(X_\lambda(s) \cap C_\lambda(X_\lambda(s) \cap A(s))) = \underset{\lambda\in\Lambda}{\cup} C_\lambda(C_\lambda(X_\lambda(s) \cap A(s))) = \underset{\lambda\in\Lambda}{\cup} C_\lambda(C_\lambda(X_\lambda(s) \cap A(s))) = \underset{\lambda\in\Lambda}{\cup} C_\lambda(X_\lambda(s) \cap A(s)) = SC_\lambda(A(s))$, other conditions to be a $K\Omega-$closure operator fallow from the definition of SC_λ and the proof of Theorem **3.3** [42]. Also $\oplus(C_\lambda)_n(A) = \underset{\lambda\in\Lambda}{\cup} (C_\lambda)_n(X_\lambda \cap A) = n^{\text{th}}$ term of $\underset{\lambda\in\Lambda}{\cup} C_\lambda(X_\lambda \cap A\ ,\ n) = (SC_\lambda)_n(A)$ for all $A \in P(X)$. This proves the theorem . ■

DEFINITION 2.1.5. Let $\{X_\lambda;\ \lambda \in \Lambda\}$ be a nonvoid family of nonvoid sets, $X = \underset{\lambda\in\Lambda}{\Pi} X_\lambda$ and $SP(A)$ be the collection of sequential points in any nonvoid set A. Then the map $P_\lambda : SP(X) \to SP(X_\lambda)$ defined by $P_\lambda((x\ ,\ P)) = (\pi_\lambda(x)\ ,\ P)$ for all $(x\ ,\ P) \in SP(X)$, where π_λ is the projection map of X into its λ^{th} factor is called $S_\lambda-$projection map of $SP(X)$ into $SP(X_\lambda)$.

THEOREM 2.1.5. *Let* $\{C_\lambda : (P(X_\lambda))^{\mathbb{N}} \to (P(X_\lambda))^{\mathbb{N}}, \lambda \in \Lambda\}$ *be a family of* $K\Omega$*-closure operators, then the operator* $PC_\lambda : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$*, where* $X = \prod_{\lambda\in\Lambda} X_\lambda$*, defined as follows, for* $A(s) \in (P(X))^{\mathbb{N}}$*, a sequential point* $(x\ ,\ P) \in PC_\lambda(A(s))$ *if and only if for any finite cover* $\{A_i(s); i = 1(1)n\}$ *of* $A(s)$ *there exists* i_k*,* $1 \leq i_k \leq n$ *and* $k \in P$ *such that* $P_\lambda((x\ ,\ P)) \in C_\lambda(P_\lambda(\underset{k\in P}{\cup} A_{i_k}(s)))$ *for all* $\lambda \in \Lambda$ *is a* $K\Omega$*-closure operator and* $(PC_\lambda)_n = \otimes(C_\lambda)_n$ *[42].*

PROOF. Since only finite cover of $\Phi(s)$ is $\{\Phi(s)\}$, $PC_\lambda(\Phi(s)) = \Phi(s)$. Let $A(s) \in (P(X))^{\mathbb{N}}$ and $\{A_i(s),\ i = 1(1)n\}$ be any finite cover of $A(s)$. Therefore $(x\ ,\ P) \in A(s) \Rightarrow (x\ ,\ k) \in A(s)$ for all $k \in P \Rightarrow (x\ ,\ k) \in A_{i_k}(s)$ for some i_k, $1 \leq i_k \leq n \Rightarrow P_\lambda((x\ ,\ k)) \in P_\lambda(A_{i_k}(s)) \subset C_\lambda(P_\lambda(A_{i_k}(s))$ for all $\lambda \in \Lambda \Rightarrow P_\lambda((x\ ,\ P)) = \underset{k\in P}{\cup} P_\lambda((x\ ,\ k)) \in \underset{k\in P}{\cup} C_\lambda(P_\lambda(A_{i_k}(s)) = C_\lambda(\underset{k\in P}{\cup} P_\lambda(A_{i_k}(s)) = C_\lambda(P_\lambda(\underset{k\in P}{\cup} A_{i_k}(s)))$ for all $\lambda \in \Lambda \Rightarrow (x\ ,\ P) \in PC_\lambda(A(s))$. A simple sequential point $(x\ ,\ k) \notin PC_\lambda(A(s)) \cup PC_\lambda(B(s))$, where $A(s)$, $B(s) \in (P(X))^{\mathbb{N}} \Rightarrow$ there exist covering $\{D_i(s)\ ,\ i = 1(1)m\}$ of $A(s)$, $\{D_i(s)\ ,\ i = (m+1)(1)n\}$ of $B(s)$ and indices λ_i, $i = 1(1)n$ such that $P_{\lambda_i}((x\ ,\ k)) \notin C_{\lambda_i}(P_{\lambda_i}(D_i(s))$ for $i = 1(1)n. \Rightarrow (x\ ,\ k) \notin PC_\lambda(A(s) \cup B(s))$. If a simple sequential point $(x\ ,\ k) \notin PC_\lambda(A(s))$, where $A(s) \in (P(X))^{\mathbb{N}}$, then, there exists covering $\{A_i(s)\ ,\ i = 1(1)n\}$ of $A(s)$ together with indices λ_i, $i = 1(1)n$, such that $P_{\lambda_i}((x\ ,\ k)) \notin C_{\lambda_i}(P_{\lambda_i}(A_i(s))$ for $i = 1(1)n$. If possible let $(x\ ,\ k) \in PC_\lambda(PC_\lambda(A(s)))$, since $\{PC_\lambda(A_i(s))\ ,\ i = 1(1)n\}$ is a covering of $PC_\lambda(A(s))$ there exists i, $1 < i < n$ so that $P_\lambda((x\ ,\ k)) \in C_\lambda(P_\lambda(PC_\lambda(A_i(s))) \subset C_\lambda(C_\lambda(P_\lambda(A_i(s))) = C_\lambda(P_\lambda(A_i(s))$ for all $\lambda \in \Lambda$ which is a contradiction. Therefore $PC_\lambda : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is a $K\Omega$-closure operator. At the end $x \in (PC_\lambda)_n(A) \Rightarrow$for any finite covering $\{A_i\ ,\ i = 1(1)k\}$ of A there exists i, $1 < i < k$ such that $P_\lambda((x\ ,\ n)) \in C_\lambda(P_\lambda((A_i\ ,\ n)))$ for all $\lambda \in \Lambda \Rightarrow \pi_\lambda(x) \in C_\lambda((\pi_\lambda(A_i)\ ,\ n))$ for all $\lambda \in \Lambda \Rightarrow \pi_\lambda(x) \in (C_\lambda)_n(\pi_\lambda(A_i)$ for all $\lambda \in \Lambda \Rightarrow x \in \otimes(C_\lambda)_n(A)$ and conversely. ■

Definition 2.1.6. Let X be a nonvoid set, $F(s)$ be a sequential set in X and $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be a $K\Omega-$closure operator. A function $C_n^{F(s)} : P(X) \to P(X)$ defined by $C_n^{F(s)}(A) = n^{th}$ term of $C({}_nF_A(s))$, where ${}_nF_A(s)$ is the sequential set in X obtained from $F(s)$ replacing n^{th} term of it by A, is called n^{th} relative closure operator of C with respect to $F(s)$.

If $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is a $K\Omega$-closure operator then obviously $C_n^{\Phi(s)} = C_n$ and consequently ${}_{C_n^{\Phi(s)}}\tau = {}_{C_n}\tau$.

Theorem 2.1.6. *Let X be a nonvoid set and $C_n^{F(s)} : P(X) \to P(X)$ be a relative closure operator of the $K\Omega$-closure operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ with respect to a sequential set $F(s)$. Then*

(i) $C_n^{F(s)}$ is expansive,

(ii) $C_n^{F(s)}$ is idempotent,

(iii) $C_n^{F(s)}$ commutes with finite union,

(iv) Φ need not be a fixed point of $C_n^{F(s)}$.

PROOF. (*i*) Let $A \in P(X)$ then ${}_nF_A(s) \subset C({}_nF_A(s)) \Rightarrow A \subset C_n^{F(s)}(A)$, so $C_n^{F(s)}$ is expansive.

(*ii*) Let $A \in P(X)$ then, $A \subset C_n^{F(s)}(A)$. Therefore

$$\begin{array}{rl} & {}_nF_A(s) \subset {}_nF_{C_n^{F(s)}(A)}(s) \\ \Rightarrow & C({}_nF_A(s)) \subset C({}_nF_{C_n^{F(s)}(A)}(s)) \\ \Rightarrow & C_n^{F(s)}(A) \subset C_n^{F(s)}(C_n^{F(s)}(A)). \\ \text{Also} & C(C({}_nF_A(s))) = C({}_nF_A(s)) \\ \Rightarrow & C({}_nF_{C_n^{F(s)}(A)}(s)) \subset C({}_nF_A(s)) \\ \Rightarrow & C_n^{F(s)}(C_n^{F(s)}(A)) \subset C_n^{F(s)}(A). \end{array}$$

Hence $C_n^{F(s)}$ is idempotent.

(*iii*) Let $A_i \in P(X)$, $i = 1(1)m$. Now $C(\bigcup_{i=1}^{i=m} {}_nF_{A_i}(s)) = \bigcup_{i=1}^{i=m} C({}_nF_{A_i}(s)) \Rightarrow C({}_nF_{\bigcup_{i=1}^{i=m} A_i}(s)) = \bigcup_{i=1}^{i=m} C({}_nF_{A_i}(s)) \Rightarrow C_n^{F(s)}(\bigcup_{i=1}^{i=m} A_i) = \bigcup_{i=1}^{i=m} C_n^{F(s)}(A_i)$. Therefore $C_n^{F(s)}$ commutes with finite union.

(*iv*) Let X be a nonvoid set. Define a function $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ by

$$\begin{aligned} C(A(s)) &= X(s) \text{ if } A(s) \neq \Phi(s), \\ &= \Phi(s) \text{ if } A(s) = \Phi(s). \end{aligned}$$

Then for any sequential set $F(s) \neq \Phi(s)$ in X, $C_n^{F(s)}(\Phi) = X$ for all $n \in \mathbb{N}$ so that ${}_nF_\phi(s) \neq \phi(s)$. ■

THEOREM 2.1.7. *Let X be a nonvoid set and $C_n^{F(s)} : P(X) \to P(X)$ be a relative closure operator of a KΩ-closure operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ with respect to a sequential set $F(s)$. Then ${}_{C_n^{F(s)}}\tau = \{X,\ A^c\ ;\ A \in P(X)$ and $C_n^{F(s)}(A) = A\}$ forms a topology* *on X and on $P(X) - \{\Phi\}$, the closure operator in the topological space $(X\ ,\ _{C_n^{F(s)}}\tau)$ and $C_n^{F(s)}$ are identical.*

Proof. Proof is omitted. ■

Definition 2.1.7. The topology $_{C_n^{F(s)}}\tau = \{X\ ,\ A\ ;\ A \in P(X)$ and $C_n^{F(s)}(A) = A\}$ induced by the relative closure operator $C_n^{F(s)} : P(X) \to P(X)$ is called the n^{th} relative topology induced by the $K\Omega$-closure operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ with respect to a sequential set $F(s)$.

Theorem 2.1.8. *Let $A(s) = \{A_n\}$ be a sequential set in a non-void set X, $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be a $K\Omega$-closure operator and C_n, $n \in \mathbb{N}$ be the n^{th} component of C. Then (i) $C(A(s)) \supset \{C_n(A_n)\}$ and the equality holds if $A(s)$ is a closed sequential set in (X, τ_C). (ii) If $C(A(s)) = \{C_n(A_n)\}$ and A_n is closed in $(X\ ,\ _{C_n}\tau)$ for each $n \in \mathbb{N}$ then $A(s)$ is closed in $(X\ ,\ \tau_C)$. (iii) $C(A(s)) = \{C_n^{A(s)}(A_n)\}$.*

Proof. Proof is omitted. ■

In an STS $(X\ ,\ \tau)$, if $A(s) = \{A_n\}$ is closed then A_n is closed in $(X\ ,\ D_n(\tau))$ for each $n \in \mathbb{N}$ but the converse is not true [6]. Theorem 2.1.8 provides a pair of if and only if conditions for a sequential set $A(s)$ to be closed in an STS. We list up these conditions in the following

Corollary 2.1.1. *In an STS $(X\ ,\ \tau)$ a sequential set $A(s) = \{A_n\}$ is closed*

(i) if and only if $\overline{A(s)} = \{L_n\}$ *and* A_n *is closed in* $(X\ ,\ D_n(\tau))$ *for each* $n \in \mathbb{N}$*, where* $L_n = n^{th}$ *component of* $\overline{(A, n)}$*.*
(ii) if and only if A_n *is closed in* $(X\ ,\ {}_{R_n}\tau)$ *for each* $n \in \mathbb{N}$*, where* R_n *is the* n^{th} *relative closure operator of the closure operator in* $(X\ ,\ \tau)$ *with respect to* $A(s)$*.*

THEOREM 2.1.9. *If* $\{F_\lambda(s)\ ;\ \lambda \in \Lambda\}$ *is a chain of sequential sets in* $((P(X))^{\mathbb{N}}\ ,\ \subset)$ *then* $\{{}_{C_n^{F_\lambda(s)}}\tau\ ,\ \lambda \in \Lambda\}$ *is a chain of topologies on* X *for each* $n \in \mathbb{N}$*, where* $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ *is a* $K\Omega$*-closure operator.*

PROOF. It is sufficient to show that $F(s) \subset G(s) \Rightarrow {}_{C_n^{G(s)}}\tau \subset {}_{C_n^{F(s)}}\tau$. Let $A \in {}_{C_n^{G(s)}}\tau \Rightarrow C_n^{G(s)}(X - A) = X - A \Rightarrow n^{th}$ term of $C({}_nG_{X-A}(s)) = X - A$. Therefore n^{th} term of $C({}_nF_{X-A}(s)) \subset X - A \Rightarrow C_n^{F(s)}(X - A) \subset X - A$. So $A \in {}_{C_n^{F(s)}}\tau$. ■

COROLLARY 2.1.2. *If* $\{{}_mF_{A_\lambda}(s)\ ;\ \lambda \in \Lambda\ \}$*, where* $\{\ A_\lambda\ ;\ \lambda \in \Lambda\ \}$ *is a chain in* $(X, \subset)$ *then* $\{{}_{C_n^{{}_mF_{A_\lambda}(s)}}\tau\ ,\ \lambda \in \Lambda\}$ *is a chain of topologies on* X *for each* $n \in \mathbb{N} - \{m\}$*.*

DEFINITION 2.1.8. Each member except possibly X of ${}_{C_n^{F(s)}}\tau$ is a subset of $X - C_n^{F(s)}(\Phi)$ and so ${}_{C_n^{F(s)}}\tau$ is called $(X - C_n^{F(s)}(\Phi))$-cut of ${}_{C_n}\tau$.

THEOREM 2.1.10. *Let* $\{K_n : P(X) \to P(X)\}$ *be a sequence of Kuratowski closure operators. Then the operator* $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ *defined by* $C(A(s)) = \{K_n(A_n)\}$ *for all* $A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$ *is a* $K\Omega$*-closure operator.*

PROOF. Proof is straightforward. ■

DEFINITION 2.1.9. Let X be a nonvoid set. An operator $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ subject to the conditions:

(*i*) $I(A(s)) \subset A(s)$ for all $A(s) \in (P(X))^{\mathbb{N}}$ i. e., I is contractive,

(*ii*) $I(X(s)) = X(s)$ i. e., X(s) is a fixed point of I,

(*iii*) $I(I(A(s))) = I(A(s))$ for all $A(s) \in (P(X))^{\mathbb{N}}$
i. e., I is idempotent,

(*iv*) $I(\bigcap_{i=1}^{n} A_i(s)) = \bigcap_{i=1}^{n} I(A_i(s))$, where $A_i(s) \in (P(X))^{\mathbb{N}}$, $i = 1(1)n$ i. e., I commutes with finite intersection,

is called a $K\Omega$-interior operator on $(P(X))^{\mathbb{N}}$.

EXAMPLE 2.1.6. The operator $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by

$$I(A(s)) = A(s) \text{ for all } A(s) \in (P(X))^{\mathbb{N}}$$

is a $K\Omega$-interior operator.

EXAMPLE 2.1.7. The operator $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $I(A(s)) = \{\bigcap\limits_{i=n}^{\infty} A_i\}_{n=1}^{\infty}$ is a $K\Omega$-interior operator.

For, (i) Let $A(s) = \{A_n\}_n \in (P(X))^{\mathbb{N}}$. Then, $P_n(I(A(s))) = \bigcap\limits_{i=n}^{\infty} A_i \subset A_n$ for all $n \in \mathbb{N}$. Therefore, $I(A(s)) \subset A(s)$.

(ii) Clearly, I(X(s))=X(s).

(iii) $I(I(A(s))) = I(\{\bigcap\limits_{i=n}^{\infty} A_i\}_{n=1}^{\infty}) = \{\bigcap\limits_{i=n}^{\infty} A_i\}_{n=1}^{\infty} = I(A(s))$.

(iv) Let $A_i(s) \in (P(X))^{\mathbb{N}}$, $i = 1(1)n$.
We claim that, $I(\bigcap\limits_{i=1}^{n} A_i(s)) = \bigcap\limits_{i=1}^{n} I(A_i(s))$.

For,

$$\begin{aligned} A_1(s) &= {A_1}^1 , {A_2}^1 , {A_3}^1 , ... \\ A_2(s) &= {A_1}^2 , {A_2}^2 , {A_3}^2 , ... \\ A_3(s) &= {A_1}^3 , {A_2}^3 , {A_3}^3 , ... \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_n(s) &= {A_1}^n , {A_2}^n , {A_3}^n , ... \\ \bigcap_{k=1}^{n} A_k(s) &= \bigcap_{k=1}^{n} {A_1}^k , \bigcap_{k=1}^{n} {A_2}^k , \bigcap_{k=1}^{n} {A_3}^k , ... \\ \text{Hence, } I(\bigcap_{k=1}^{n} A_k(s)) &= \{\bigcap_{j=m}^{\infty}(\bigcap_{k=1}^{n} {A_j}^k)\}_m. \end{aligned}$$

On the other hand,

$$\begin{aligned} I(A_1(s)) &= \bigcap_{j=1}^{\infty} {A_j}^1 , \bigcap_{j=2}^{\infty} {A_j}^1 , \bigcap_{j=3}^{\infty} {A_j}^1 , ... \\ I(A_2(s)) &= \bigcap_{j=1}^{\infty} {A_j}^2 , \bigcap_{j=2}^{\infty} {A_j}^2 , \bigcap_{j=3}^{\infty} {A_j}^2 , ... \\ I(A_3(s)) &= \bigcap_{j=1}^{\infty} {A_j}^3 , \bigcap_{j=2}^{\infty} {A_j}^3 , \bigcap_{j=3}^{\infty} {A_j}^3 , ... \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ I(A_n(s)) &= \bigcap_{j=1}^{\infty} {A_j}^n , \bigcap_{j=2}^{\infty} {A_j}^n , \bigcap_{j=3}^{\infty} {A_j}^n , ... \end{aligned}$$

$$\begin{aligned} \text{So, } \bigcap_{k=1}^{n} I(A_k(s)) &= \bigcap_{k=1}^{n}(\bigcap_{j=1}^{\infty} {A_j}^k) , \bigcap_{k=1}^{n}(\bigcap_{j=2}^{\infty} {A_j}^k) , \bigcap_{k=1}^{n}(\bigcap_{j=3}^{\infty} {A_j}^k)....... \\ &= \{\bigcap_{k=1}^{n}(\bigcap_{j=m}^{\infty} {A_j}^k)\}_m = \{\bigcap_{j=m}^{\infty}(\bigcap_{k=1}^{n} {A_j}^k)\}_m. \end{aligned}$$

THEOREM 2.1.11. *Let* $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ *be a* $K\Omega$*-interior operator. Then* $\tau_I = \{A(s) : A(s) \in (P(X))^{\mathbb{N}},\ I(A(s)) = A(s)\}$ *forms an ST on X and* $I(A(s)) = A(s)^\circ$*, for all* $A(s) \in (P(X))^{\mathbb{N}}$*, where* $A(s)^\circ$ *is the interior of A(s) in* (X, τ_I)*.*

PROOF. (i) By definition of $K\Omega$-interior operator,

$$\begin{aligned} & I(\phi(s)) \subset \phi(s) \\ \Rightarrow\quad & I(\phi(s)) = \phi(s) \text{ [Since, } \phi(s) \subset I(\phi(s))] \\ \Rightarrow\quad & \phi(s) \in \tau_I. \\ \text{Also}\quad & X(s) \in \tau_I. \text{ [Since I(X(s))=X(s)]} \end{aligned}$$

(ii) Let $A(s) \subset B(s)$ then

$$\begin{aligned} & A(s) = A(s) \cap B(s) \\ \Rightarrow\quad & I(A(s)) = I(A(s) \cap B(s)) \\ \Rightarrow\quad & I(A(s)) = I(A(s)) \cap I(B(s)) \\ \Rightarrow\quad & I(A(s)) \subset I(B(s)) \longrightarrow (1) \end{aligned}$$

Let $A_\lambda(s) \in \tau_I, \lambda \in \Lambda$. Then $I(A_\lambda(s)) = A_\lambda(s)$ for all $\lambda \in \Lambda$.
Let $A(s) = \underset{\lambda\in\Lambda}{\cup} A_\lambda(s)$. Therefore,

$$\begin{aligned} & A_\lambda(s) \subset A(s) \text{ for all } \lambda \in \Lambda \\ \Rightarrow\quad & I(A_\lambda(s)) \subset I(A(s)) \text{ for all } \lambda \in \Lambda \text{ [by (1)]} \\ \Rightarrow\quad & A_\lambda(s) \subset I(A(s)) \text{ for all } \lambda \in \Lambda. \end{aligned}$$

Therefore, $\underset{\lambda\in\Lambda}{\cup} A_\lambda(s) \subset I(A(s))$. Thus, $A(s) \subset I(A(s))$. Again since I is contractive, $I(A(s)) \subset A(s)$.
Therefore, $I(A(s)) = A(s) \Rightarrow A(s) \in \tau_I$. Thus, $\underset{\lambda\in\Lambda}{\cup} A_\lambda(s) \in \tau_I$.
(iii) Again let $A_i(s) \in \tau_I$, $i = 1(1)n$ and $B(s) = \bigcap_{i=1}^{n} A_i(s)$. Now,

$$I(B(s)) = I(\bigcap_{i=1}^{n} A_i(s)) = \bigcap_{i=1}^{n} I(A_i(s)) = \bigcap_{i=1}^{n} A_i(s) = B(s)$$

$$\Rightarrow \quad B(s) = \bigcap_{i=1}^{n} A_i(s) \in \tau_I.$$

Therefore, (X, τ_I) is a sequential topological space. At the end,

$$\begin{aligned} A(s)^{\circ} &= \cup\{B(s) : B(s) \subset A(s), B(s) \in \tau_I\} \\ &= \cup\{B(s) : B(s) \subset A(s), I(B(s)) = B(s)\} \end{aligned}$$

Now, $B(s) \subset A(s) \Rightarrow I(B(s)) \subset I(A(s)) \Rightarrow B(s) \subset I(A(s))$ and $\{B(s) : B(s) \subset A(s),\ I(B(s)) = B(s)\}$ contains I(A(s)). Thus, $A(s)^{\circ} = I(A(s))$. ■

Definition 2.1.10. $\tau_I = \{A(s) : A(s) \in (P(X))^{\mathbb{N}},\ I(A(s) = A(s)\}$ is called the sequential topology (ST) generated or induced by the $K\Omega$-interior operator $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$.

Definition 2.1.11. Let X be a nonvoid set and $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be a $K\Omega$-interior operator. A function $I_n : P(X) \to P(X)$ defined by $I_n(A) = n^{th}$ term of $I({}_nX_A(s))$, if $A \neq X$ and $I_n(X) = X$ is called the n^{th} component of I, $n \in \mathbb{N}$.

In Example 2.1.6 the n^{th} component of the interior of a sequential set in (X, τ_I) is equal to the interior of the n^{th} component of that sequential set in $(X, D_n(\tau_I))$ but, in Example 2.1.7 they are not equal.

EXAMPLE 2.1.8. n^{th} component $I_n : P(X) \to P(X)$ of both the $K\Omega$ operators in Example 2.1.6 and Example 2.1.7 is defined by

$$I_n(A) = A, \text{ for all } A \subset X.$$

THEOREM 2.1.12. *Let X be a nonvoid set. If $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is a $K\Omega$- interior operator then each component $I_n : P(X) \to P(X)$, $n \in \mathbb{N}$ is an interior operator. Also $D_n(\tau_I) =_{I_n} \tau$, where τ_I is the ST on X induced by the $K\Omega$-interior operator $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ and ${}_{I_n}\tau$ is the topology on X induced by the component $I_n : P(X) \to P(X)$ of I, $n \in \mathbb{N}$.*

PROOF. Let $A \in P(X)$. Then, $I({}_nX_A(s)) \subset {}_nX_A(s) \Rightarrow I_n(A) \subset A$. By definition we have X is fixed point of I_n. Now,

$$\begin{aligned} & {}_nX_{I_n(A)}(s) \subset {}_nX_A(s) \\ \Rightarrow \quad & I({}_nX_{I_n(A)}(s)) \subset I({}_nX_A(s)) \\ \Rightarrow \quad & I_n(I_n(A)) \subset I_n(A). \end{aligned}$$

Also, $I(I({}_nX_A(s))) = I({}_nX_A(s))$.

Clearly, $I({}_nX_A(s)) \subset {}_nX_{I_n(A)}(s) \Rightarrow I(I({}_nX_A(s))) \subset I({}_nX_{I_n(A)}(s))$.

Hence, $I({}_nX_A(s)_) \subset I({}_nX_{I_n(A)}(s)) \Rightarrow I_n(A) \subset I_n(I_n(A))$.

Therefore we have, $I_n(I_n(A)) = I_n(A)$.

Let $A_i \in (P(X))$, $i = 1(1)n$. Now,

$$\begin{aligned} & I(\bigcap_{i=1}^{n} {}_nX_{A_i}(s)) = \bigcap_{i=1}^{n} I({}_nX_{A_i}(s)) \\ \Rightarrow \quad & I({}_nX_{\bigcap_{i=1}^{n} A_i}(s)) = \bigcap_{i=1}^{n} I({}_nX_{A_i}(s)) \\ \Rightarrow \quad & I_n(\bigcap_{i=1}^{n} A_i) = \bigcap_{i=1}^{n} I_n(A_i). \end{aligned}$$

Hence I_n is an interior operator.
For the last part, let $A \in D_n(\tau_I)$. Thus A is an open set and hence there is an open sequential set $B(s) = \{B_n\}$ in (X, τ_I) such that $B_n = A$. Now,

$$B(s) \subset {}_nX_A(s) \Rightarrow I(B(s)) \subset I({}_nX_A(s))$$
$$\Rightarrow B(s) \subset I({}_nX_A(s)) \Rightarrow A \subset I_n(A) \subset A$$
$$\Rightarrow I_n(A) = A \in {}_{I_n}\tau.$$

Again let $A \in {}_{I_n}\tau$. Therefore, $I_n(A) = A$.
Let $B(s) = I({}_nX_A(s))$ then B(s) is an open sequential set and its n^{th} component is $I_n(A) = A$ [Since, $A \in {}_{I_n}\tau$]. Therefore, $A \in D_n(\tau_I)$ and the proof is done. ■

NOTE 2.1.1. The $K\Omega$-interior operator in Example 2.1.7 generates an ST on X but it is not discrete though every component of this operator induces discrete topology on X.

DEFINITION 2.1.12. Let X be a nonvoid set, F(s) be a sequential set in X and $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be a $K\Omega$-interior operator. A function ${I_n}^{F(s)} : P(X) \to P(X)$ defined by ${I_n}^{F(s)}(A) = n^{th}$ term of $I({}_nF_A(s))$ is called n^{th} relative interior operator of I with respect to F(s).

NOTE 2.1.2. If $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is a $K\Omega$-interior operator then obviously ${I_n}^{X(s)} = I_n$ and consequently ${}_{{I_n}^{X(s)}}\tau = {}_{I_n}\tau$.

THEOREM 2.1.13. *Let X be a nonvoid set and $I_n^{F(s)} : P(X) \to P(X)$ be a relative interior operator of the $K\Omega$-interior operator*

$I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ *with respect to sequential set F(s). Then :*

(i) $I_n^{F(s)}$ *is contractive,*

(ii) $I_n^{F(s)}$ *is idempotent,*

(iii) $I_n^{F(s)}$ *commutes with finite intersection,*

(iv) *X need not be a fixed point of* $I_n^{F(s)}$.

Proof. (i) Let $A \in P(X)$. Then

$$\begin{aligned} & I({}_nF_A(s)) \subset {}_nF_A(s) \\ \Rightarrow \quad & I_n^{F(s)}(A) \subset A. \end{aligned}$$

Therefore $I_n^{F(s)}$ is contractive.

(ii) Let $A \in P(X)$ then $I_n^{F(s)}(A) \subset A$. Therefore

$$\begin{aligned} & {}_nF_{I_n^{F(s)}(A)}(s) \subset {}_nF_A(s) \\ \Rightarrow \quad & I({}_nF_{I_n^{F(s)}(A)}(s)) \subset I({}_nF_A(s)) \\ \Rightarrow \quad & I_n^{F(s)}(I_n^{F(s)}(A)) \subset I_n^{F(s)}(A). \longrightarrow (1) \end{aligned}$$

Also, $I(I({}_nF_A(s))) = I({}_nF_A(s))$. Since

$$\begin{aligned} & I({}_nF_A(s)) \subset {}_nF_{I_n^{F(s)}(A)}(s) \\ \Rightarrow \quad & I(I({}_nF_A(s))) \subset I({}_nF_{I_n^{F(s)}(A)}(s)) \\ \Rightarrow \quad & I({}_nF_A(s)) \subset I({}_nF_{I_n^{F(s)}(A)}(s)) \\ \Rightarrow \quad & I_n^{F(s)}(A) \subset I_n^{F(s)}(I_n^{F(s)}(A)). \longrightarrow (2) \end{aligned}$$

Combining (1) and (2), we get $I_n^{F(s)}(A) = I_n^{F(s)}(I_n^{F(s)}(A))$. Hence $I_n^{F(s)}$ is idempotent.

(iii) Let $A_i \in P(X)$, i=1(1)m. Now

$$I(\bigcap_{i=1}^{m} {}_nF_{A_i}(s)) = \bigcap_{i=1}^{m} I({}_nF_{A_i}(s))$$

$$\Rightarrow \quad I({}_nF_{\bigcap_{i=1}^{m} A_i}(s)) = \bigcap_{i=1}^{m} I({}_nF_{A_i}(s))$$
$$\Rightarrow \quad I_n^{F(s)}(\bigcap_{i=1}^{m} A_i) = \bigcap_{i=1}^{m} I_n^{F(s)}(A_i).$$

That is, $I_n^{F(s)}$ commutes with finite intersection.

(iv) Let X be a nonvoid set. Define a function $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ by

$$\begin{aligned} I(A(s)) &= \phi(s), \text{ whenever } A(s) \neq X(s) \\ &= X(s), \text{ whenever } A(s) = X(s). \end{aligned}$$

It is $K\Omega$-interior operator and for any sequential set $F(s) \neq X(s)$ in X, $I_n^{F(s)}(X) = \phi$ for all those $n \in \mathbb{N}$ so that ${}_nF_X(s) \neq X(s)$ ■

2.2. Connector and its Applications

In this section we make a tool called connector that connects two topologies in such a manner that a sequence of connectors connecting a sequence of topologies on a nonvoid set provides a sequential topology on the underlying set.

DEFINITION 2.2.1. The operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $C(A(s)) = \{K_n(A_n)\}$ for all $A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$ is called a $K\Omega$-closure operator induced by a sequence $\{K_n : P(X) \to P(X)\}$ of Kuratowski closure operators.

DEFINITION 2.2.2. Let D and E be topologies on X. A subset K of E^D such that (*i*) $O_\lambda \in D$ and $f_\lambda \in K$, $\lambda \in \Lambda \Rightarrow$there exists $f \in K$ so that $f(\underset{\lambda\in\Lambda}{\cup} O_\lambda) = \underset{\lambda\in\Lambda}{\cup} f_\lambda(O_\lambda)$,

(*ii*) $O_i \in D$ and $f_i \in K$, $i = 1(1)n \Rightarrow$there exists $f \in K$ so that $f(\overset{n}{\underset{i=1}{\cap}} O_i) = \overset{n}{\underset{i=1}{\cap}} f_i(O_i)$, and

(*iii*) $E = \underset{f\in K}{\cup} f(D)$ is called a connector of D to E.

Example 2.2.1. Let D and E be two topologies on a nonvoid set X. A function $f : D \to E$ defined by $f(O) = U$ for all $O \in D$, where U is a fixed element of E is called a constant function from D into E. Now let K be the collection of all such constant functions from D into E. If $O_\lambda \in D$ and $f_\lambda \in K$, $\lambda \in \Lambda$ then $f_\lambda(O_\lambda) \in E$ for all $\lambda \in \Lambda \Rightarrow \underset{\lambda\in\Lambda}{\cup} f_\lambda(O_\lambda) \in E$. Consider the function $f \in K$ which maps each element of D to $\underset{\lambda\in\Lambda}{\cup} f_\lambda(O_\lambda)$. Since $\underset{\lambda\in\Lambda}{\cup} O_\lambda \in D$ we have $f(\underset{\lambda\in\Lambda}{\cup} O_\lambda) = \underset{\lambda\in\Lambda}{\cup} f_\lambda(O_\lambda)$. Again if $O_i \in D$ and $f_i \in K$, $i = 1(1)n$ then $f \in K$ defined by $f(O) = \overset{i=n}{\underset{i=1}{\cap}} f_i(O_i)$ for all $O \in D$, is the desired function to satisfy the second condition to be a connector. Also for each $U \in E$ define a function $f_U : D \to E$ by $f_U(O) = U$ for all $O \in D$. Then $K = \{f_U : D \to E, U \in E\}$ and $f_U(D) = \{U\} \Rightarrow \underset{U\in E}{\cup} f_U(D) = E \Rightarrow \underset{f\in K}{\cup} f(D) = E$. So, K is a connector of D to E.

Definition 2.2.3. Let D and E be two topologies on a nonvoid set X. Then the collection of all constant functions from D into E forms a connector of D to E. This is called the discrete connector of D to E.

One can connect a sequence $\{\tau_n\}$ of topologies on a nonvoid set X by any sequence $\{F_n\}$ of connectors and each connection provides a unique ST on X Theorem 2.2.1 which is denoted by $\tau < \{\tau_n\}$, $\{F_n\} >$ such that $D_n(\tau < \{\tau_n\}, \{F_n\} >) = \tau_n$ for $n \in \mathbb{N}$ and it is called the ST generated by $\{\tau_n\}$ and $\{F_n\}$. If further each F_n is the discrete connector of τ_n to τ_{n+1} then the ST is said to be generated by $\{\tau_n\}$ and is denoted by $\tau < \{\tau_n\} >$.

THEOREM 2.2.1. *Let $\{K_n : P(X) \to P(X)\}$ be a sequence of Kuratowski closure operators. Then for any sequence $\{F_n\}$ of connectors such that F_n connects ${}_{k_n}\tau$ to ${}_{k_{n+1}}\tau$ for all $n \in \mathbb{N}$ there is a unique sequential topology $\tau < \{{}_{K_n}\tau\}, \{F_n\} >$ on X such that $D_n(\tau < \{{}_{K_n}\tau\}$, $\{F_n\} >) = {}_{k_n}\tau$ and the components of the closure operator on $(X, \tau < \{{}_{K_n}\tau\}$, $\{F_n\} >)$ are K_n, $n \in \mathbb{N}$. Also for any STS $(X$, $\tau)$ there is a sequence $\{F_n\}$ of connectors such that F_n connects $D_n(\tau)$ to $D_{n+1}(\tau)$, $n \in \mathbb{N}$ and $\tau = \tau < \{D_n(\tau)\}$, $\{F_n\} >$.*

PROOF. Let $F = \prod_{n=1}^{\infty} F_n$, $l = \{l_n\} \in F$ and $O \in {}_{k_1}\tau$. Define $A_1 = O$, and $A_n = l_{n-1}l_{n-2}.....l_2l_1O$, $n > 1$ assuming $l_nO = l_n(O)$ for all $n \in \mathbb{N}$. Let $A^l_O(S) = \{A_n\} \in (P(X))^{\mathbb{N}}$ and consider $\tau = \{X(s), \Phi(S)\} \cup \{A^l_O(S);\ l \in F$ and $O \in {}_{k_1}\tau\}$. Consider $A_\lambda(S) = A^{l_\lambda}_{O_\lambda}(S) \in \tau$, $\lambda \in \Lambda$, where Λ is an index set and $O = \bigcup_{\lambda\in\Lambda} O_\lambda \in {}_{k_1}\tau$. For $l_{\lambda 1} \in F_1$ and $O \in {}_{k_1}\tau$ there exists $l_1 \in F_1$ such that $l_1O = \bigcup_{\lambda\in\Lambda} l_{\lambda 1}O_\lambda$; $l_{\lambda n} \in F_n$ and $l_{n-1}l_{n-2}.....l_2l_1O \in {}_{k_n}\tau$ there exists $l_n \in F_n$ such that $l_nl_{n-1}.....l_2l_1O = \bigcup_{\lambda\in\Lambda} l_{\lambda n}l_{\lambda n-1}.....l_{\lambda 2}l_{\lambda 1}O_\lambda$. Obviously $\bigcup_{\lambda\in\Lambda} A_\lambda(S) = \bigcup_{\lambda\in\Lambda} A^{l_\lambda}_{O_\lambda}(S) = A^l_O(S) \in \tau$, where $l = \{l_n\}$. Arguing in the same way it clearly can be shown that τ is closed under finite intersection. Therefore, $(X,\ \tau)$ is a sequential topological space.

The third condition to be a connectors ensures that $D_n(\tau) = {}_{k_n}\tau$ for all $n \in \mathbb{N}$. At the end, for any $A \in P(X)$, $\overline{(A, n)}$ is the smallest closed set in $(X\ ,\ \tau)$ having n^{th} term $K_n(A)$. For the next part, for each $n \in \mathbb{N}$ define a relation $\rho^{n,\, n+1}$ on τ by ${}_{A(s)=\{A_n\}}\rho^{n,\, n+1}_{B(s)=\{B_n\}}$ if and only if $A_n = B_n$, then, $\rho^{n,\, n+1}$ defines a partition of τ say $\{C(A(s))\ ;\ A(s) \in \tau^{n,\, n+1} \subset \tau\}$, where $\tau^{n,\, n+1}$ is a family of open sequential sets taking exactly one from each part of the partition of τ defined by $\rho^{n,\, n+1}$ and let $F^{n,\, n+1} = \underset{A(s)\in\tau^{n,\, n+1}}{\Pi} C(A(s))$. Then each $l \in F^{n,\, n+1}$ defines a function $f_l : D_n(\tau) \to D_{n+1}(\tau)$ by $f_l(U) = U_{n+1}$, where $U \in D_n(\tau)$ and $U(s) = \{U_n\} \in l$ with $U_n = U$. Then for every $n \in \mathbb{N}$ $F_n = \{f_l;\ l \in \tau^{n,\, n+1}\}$ is a connector connecting $D_n(\tau)$ to $D_{n+1}(\tau)$ and properties of connector ensure that $\tau = \tau < \{D_n(\tau)\}\ ,\ \{F_n\} >$. ■

Corollary 2.2.1. *If $\{\tau_n\}$ is a sequence of topologies on X such that $\tau_n = D$ for all $n \in \mathbb{N}$ then $\tau < \{\tau_n\} >= \tau < D >$.*

Corollary 2.2.2. *If $\{K_n : P(X) \to P(X)\}$ is a sequence of Kuratowski closure operators and*
C is the $K\Omega$-closure operator induced by $\{K_n\}$ then $\tau_C = \tau < \{\tau_n\} >$, where τ_n is the topology on X induced by K_n , $n \in \mathbb{N}$.

Chapter III

Density Property and Semi Open Sequential Sets in $K\Omega$-Spaces

Semi open and weakly semi open sequential sets in sequential topological spaces are introduced and studied in [22]. [22, 15, 17] kindle the idea to think about Kuratowski space and $K\Omega$-space.

3.1. Density Property of Sequential Sets in $K\Omega$-Spaces

We have seen that, in the second chapter of the thesis, $K\Omega-$closure operator expands a sequential set and $K\Omega-$interior operator contracts. This section mainly deals with two kinds of sequential sets, one is extremely expansive and other is extremely contractive under some $K\Omega-$closure and $K\Omega-$ interior operators.

Definition 3.1.1. Let X be a nonvoid set. Suppose $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be a $K\Omega$-closure operator and $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be a $K\Omega$-interior operator, then (X, C, I) is called $K\Omega$-space.

Definition 3.1.2. Let (X, C, I) be a $K\Omega$-space. The closure operator $C_I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ induced by the $K\Omega$-interior operator $I : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is defined by $C_I(A(s)) = X(s) - I(X(s) - A(s))$ for all $A(s) = \{A_n\}_{n=1}^{\infty} \in P(X)^{\mathbb{N}}$.

Definition 3.1.3. Let (X, C, I) be a $K\Omega$-space. The interior operator $I_C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ induced by the $K\Omega$-closure operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is defined by $I_C(A(s)) = X(s) - C(X(s) - A(s))$ for all $A(s) = \{A_n\}_{n=1}^{\infty} \in (P(X))^{\mathbb{N}}$.

Definition 3.1.4. A sequential point $\alpha = (x, P)$ in a KΩ-space (X, C, I) is said to be a C-limit point of a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ in X if for any sequential set B(s) such that $\alpha \in_{\omega} I_C(B(s)) = \{H_n\}_{n=1}^{\infty}$, either

$$(a)\ x \in H_n \cap A_n \text{ for some } n \notin P$$

$$or \quad (b)\ y \in H_n \cap A_n \text{ for some } n \in \mathbb{N} \text{ and } y \neq x.$$

Definition 3.1.5. A sequential point $\alpha = (x, P)$ in a $K\Omega$-space (X, C, I) is called I-limit point of a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$in X if for any sequential set B(s) such that $\alpha \in_{\omega} I(B(s)) = \{H_n\}_{n=1}^{\infty}$, either

$$(a)\ x \in H_n \cap A_n \text{ for some } n \notin P$$

$$or \quad (b)\ y \in H_n \cap A_n \text{ for some } n \in \mathbb{N} \text{ and } y \neq x.$$

Definition 3.1.6. In a $K\Omega$-space (X, C, I), a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ is said to be I-open if $I(A(s)) = A(s)$ and C-open if $I_C(A(s)) = A(s)$.

Note 3.1.1. Let D be a topology on a nonvoid set X. Then $C_D : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $C_D(A(s)) = \{cl(A_n)\}_{n=1}^{\infty}$ for all $A(s) = \{A_n\}_{n=1}^{\infty} \in (P(X))^{\mathbb{N}}$ is a $K\Omega$-closure operator, where '*cl*' denotes the closure operator in (X, D).

NOTE 3.1.2. Let D be a topology on a nonvoid set X. Then $I_D : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is a $K\Omega$-interior operator defined by $I_D(A(s)) = \{int(A_n)\}_{n=1}^{\infty}$ for all $A(s) = \{A_n\}_{n=1}^{\infty} \in (P(X))^{\mathbb{N}}$, where '$int$' denotes the interior operator in (X, D).

DEFINITION 3.1.7. A sequential set $A(s)$ in $K\Omega-$space (X, C, I) is said to be $(\omega)C-$dense if $C(A(s)) = X(s)$. If further $C_n(A_n) = X$ for all $n \in \mathbb{N}$ then $A(s)$ is called $C-$dense in (X, C, I).

DEFINITION 3.1.8. A sequential set $A(s)$ in $K\Omega-$space (X, C, I) is said to be $(\omega)I-$dense if $C_I(A(s)) = X(s)$. If further $C_{I_n}(A_n) = X$ for all $n \in \mathbb{N}$, then $A(s)$ is called $I-$dense in (X, C, I).

NOTE 3.1.3. Let C and I be the closure and interior operators in an STS (X, τ). Since $\tau_C = \tau_I = \tau$, the space (X, τ) can be identified with $K\Omega$-space (X, C, I)

EXAMPLE 3.1.1. Let $X = \{1, 2, 3, 4\}$ and $\tau =\{\phi(s)$, A(s), B(s), C(s), D(s), E(s), F(s), G(s), H(s), M(s), N(s), P(s), Q(s), R(s), S(s), T(s), X(s)$\}$, where

$$\begin{aligned} A(s) = \{A_n\}_{n=1}^{\infty};\ A_n &= \{1\};\ \text{whenever n is an odd,} \\ &= \phi;\ \text{whenever n is an even.} \\ B(s) = \{B_n\}_{n=1}^{\infty};\ B_n &= \{3\};\ \text{whenever n is an odd,} \\ &= \phi;\ \text{whenever n is an even.} \\ C(s) = \{C_n\}_{n=1}^{\infty};\ C_n &= \phi;\ \text{whenever n is an odd,} \\ &= \{1\};\ \text{whenever n is an even.} \end{aligned}$$

$$
\begin{aligned}
D(s) = \{D_n\}_{n=1}^{\infty};\ D_n &= \phi;\ \text{whenever n is an odd,}\\
&= \{2\};\ \text{whenever n is an even.}\\
E(s) = \{E_n\}_{n=1}^{\infty};\ E_n &= \{1,3\};\ \text{whenever n is an odd,}\\
&= \phi;\ \text{whenever n is an even.}\\
F(s) = \{F_n\}_{n=1}^{\infty};\ F_n &= \{1\};\ \text{for all } n \in \mathbb{N}.\\
G(s) = \{G_n\}_{n=1}^{\infty};\ G_n &= \{1\};\ \text{whenever n is an odd,}\\
&= \{2\};\ \text{whenever n is an even.}\\
H(s) = \{H_n\}_{n=1}^{\infty};\ H_n &= \{3\};\ \text{whenever n is an odd,}\\
&= \{1\};\ \text{whenever n is an even.}\\
M(s) = \{M_n\}_{n=1}^{\infty};\ M_n &= \{3\};\ \text{whenever n is an odd,}\\
&= \{2\};\ \text{whenever n is an even.}\\
N(s) = \{N_n\}_{n=1}^{\infty};\ N_n &= \phi;\ \text{whenever n is an odd,}\\
&= \{1,2\};\ \text{whenever n is an even.}\\
P(s) = \{P_n\}_{n=1}^{\infty};\ P_n &= \{1,3\};\ \text{whenever n is an odd,}\\
&= \{1\};\ \text{whenever n is an even.}\\
Q(s) = \{Q_n\}_{n=1}^{\infty};\ Q_n &= \{1,3\};\ \text{whenever n is an odd,}\\
&= \{2\};\ \text{whenever n is an even.}\\
R(s) = \{R_n\}_{n=1}^{\infty};\ R_n &= \{1,3\};\ \text{whenever n is an odd,}\\
&= \{1,2\};\ \text{whenever n is an even.}\\
S(s) = \{S_n\}_{n=1}^{\infty};\ S_n &= \{1\};\ \text{whenever n is an odd,}\\
&= \{1,2\};\ \text{whenever n is an even.}\\
T(s) = \{T_n\}_{n=1}^{\infty};\ T_n &= \{3\};\ \text{whenever n is an odd,}\\
&= \{1,2\};\ \text{whenever n is an even.}
\end{aligned}
$$

The sequential set $U(s) = \{U_n\}_{n=1}^{\infty}$, where

$$\begin{aligned} U_n &= \{1,2,3\};\ \text{if n is an odd,} \\ &= \{1,2,4\};\ \text{otherwise} \end{aligned}$$

is $C-$dense in the $K\Omega-$space (X, C, I), where C and I are the closure and interior operators in the STS (X, τ). We reason as follows: The closed sequential sets therein are

$A^c(S) = \{\{2,3,4\}, X, \{2,3,4\}, X, ...\}$,

$B^c(S) = \{\{1,2,4\}, X, \{1,2,4\}, X, ...\}$,

$C^c(S) = \{X, \{2,3,4\}, X, \{2,3,4\}...\}$,

$D^c(S) = \{X, \{1,3,4\}, X, \{1,3,4\}...\}$,

$E^c(S) = \{\{2,4\}, X, \{2,4\}, X, ...\}$,

$F^c(S) = \{\{2,3,4\}, \{2,3,4\}, \{2,3,4\}, \{2,3,4\}, ...\}$,

$G^c(S) = \{\{2,3,4\}, \{1,3,4\}, \{2,3,4\}...\}$,

$H^c(S) = \{\{1,2,4\}, \{2,3,4\}, \{1,2,4\}, \{2,3,4\}...\}$,

$M^c(S) = \{\{1,2,4\}, \{1,3,4\}, \{1,2,4\}, \{1,3,4\}...\}$,

$N^c(S) = \{X, \{3,4\}, X, \{3,4\}...\}$,

$P^c(S) = \{\{2,4\}, \{2,3,4\}, \{2,4\}, \{2,3,4\}...\}$,

$Q^c(S) = \{\{2,4\}, \{1,3,4\}, \{2,4\}, \{1,3,4\}...\}$,

$R^c(S) = \{\{2,4\}, \{3,4\}, \{2,4\}, \{3,4\}...\}$,

$S^c(S) = \{\{2,3,4\}, \{3,4\}, \{2,3,4\}, \{3,4\}...\}$,

$T^c(S) = \{\{1,2,4\}, \{3,4\}, \{1,2,4\}, \{3,4\}...\}$,

$\phi(s) = \{\phi, \phi, \phi, \phi, ...\}$ and

$X(s) = \{X, X, X, X...\}$.

Here only closed sequential set containing $U(s)$ is $X(s)$, i. e., $C(U(s)) = X(s)$. Again, X is the only closed set in $(X, {}_{C_n}\tau)$

containing U_n, for all $n \in \mathbb{N}$. Hence $C(U(s)) = X(s)$ with the property $C_n(U_n) = X$ for each $n \in \mathbb{N}$. Therefore, $U(s)$ is $C-$dense in the $K\Omega-$space (X, C, I).

THEOREM 3.1.1. *Let D be a topology on X. In the $K\Omega-$space $(X,\ C_D,\ I_D)$ if a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ is $(\omega)C-$dense then it is $C-$dense.*

PROOF. Since A(s) is $(\omega)C-$dense, then $C_D(A(s)) = X(s)$. Again from the definition of C_D we have $C_D(A(s)) = \{cl(A_n)\}_{n=1}^{\infty}$. Thus $X(s) = \{cl(A_n)\}_{n=1}^{\infty} \Rightarrow X = cl(A_n)$. Hence the proof. ■

NOTE 3.1.4. In the $K\Omega-$space $(X,\ C_D,\ I_D)$, $C-$density and $I-$density are same. For,

$$\begin{aligned} C_{I_D}(A(s)) &= X(s) - I_D(X(s) - A(s)) \\ &= X(s) - I_D\{X - A_n\} \\ &= X(s) - \{int(X - A_n)\} \\ &= \{X - int(X - A_n)\} \\ &= \{cl(A_n)\} \\ &= C_D(A(s)). \end{aligned}$$

NOTE 3.1.5. Let $K\Omega-$closure operator C and $K\Omega-$interior operator I be connected by $C(A(s)) = X(s) - I(X(s) - A(s))$, for any sequential set $A(s)$ in the underlying set X. Then $C-$density and $I-$density in the corresponding $K\Omega-$space $(X,\ C,\ I)$ coincide.

DEFINITION 3.1.9. A sequential set $A(s)$ in $K\Omega-$space (X, C, I) is said to be $(\omega)CI-$nowhere dense if $I(C(A(s))) = \phi(s)$. If further $I_n(C_n(A_n)) = \phi$ then $A(s)$ is called $CI-$nowhere dense.

EXAMPLE 3.1.2. In Example 3.1.1 the sequential set $V(s) = \{V_n\}_{n=1}^{\infty}$, where

$$\begin{aligned} V_n &= \{2\};\ \text{when n is an odd} \\ &= \{3\};\ \text{otherwise} \end{aligned}$$

is $CI-$nowhere dense. Because, $C(V(s)) = \{2,4\}$, $\{3,4\}$, $\{2,4\}$, $\{3,4\}$ .... and so, $IC(V(s)) = \phi(s)$ along with $I_n(C_n(V_n)) = \phi$.

NOTE 3.1.6. From the Definitions 3.1.7 and 3.1.9 we observe that $C-$dense and $CI-$nowhere dense sequential sets are also (ω) $C-$dense and $(\omega)CI-$ nowhere dense sequential sets respectively. But the converses need not be true as seen from the following example

EXAMPLE 3.1.3. Consider X={1,2,3,4} and $\tau = \{\phi(s), A(s), B(s), C(s), D(s), X(s)\}$, where

$$\begin{aligned} A(s) = \{A_n\}_{n=1}^{\infty};\ A_n &= \{1\}; \text{n is an odd}, \\ &= \phi; \text{n is an even}. \\ B(s) = \{B_n\}_{n=1}^{\infty};\ B_n &= \{1,3\}; \text{n is an odd}, \\ &= X; \text{n is an even}. \\ C(s) = \{C_n\}_{n=1}^{\infty};\ C_n &= \{3\}; \text{n is an odd}, \\ &= \phi; \text{n is an even}. \end{aligned}$$

$$D(s) = \{D_n\}_{n=1}^{\infty};\ D_n = \{1,3\}; \text{n is an odd,}$$
$$= \phi; \text{n is an even.}$$

Then the sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ defined by,

$$U_n = \{1,3\}; \text{n is an odd,}$$
$$= \phi; \text{n is an even.}$$

and

$$V_n = \{2\}; \text{n is an odd,}$$
$$= X; \text{n is an even.}$$

are $(\omega)C-$dense and $(\omega)CI$-nowhere dense respectively in the $K\Omega$-space (X, C, I), where C and I are the closure and interior operators in the STS (X, τ). But neither $U(s)$ is $C-$dense nor $V(s)$ is $CI-$nowhere dense in (X, τ) as shown below:

The nontrivial open sequential sets are

$$A^c(s) = \{\{1\}, \phi, \{1\}, \phi, ...\}$$
$$B^c(s) = \{\{1,3\}, X, \{1,3\}, X, ...\}$$
$$C^c(s) = \{\{3\}, \phi, \{3\}, \phi, ...\}$$
$$D^c(s) = \{\{1,3\}, \phi, \{1,3\}, \phi, \ldots\}.$$

Therefore nontrivial closed sequential sets are

$$A^c(s) = \{\{2,3,4\}, X, \{2,3,4\}, X, ...\}$$
$$B^c(s) = \{\{2,4\}, \phi, \{2,4\}, \phi, ...\}$$
$$C^c(s) = \{\{1,2,4\}, X, \{1,2,4\}, X, ...\}$$
$$D^c(s) = \{\{2,4\}, \{X\}, \{2,4\}, \{X\}, \ldots\}.$$

Since there is no closed sequential set containing $U(s)$ except $X(s)$, $C(U(s)) = X(s)$, i. e., $U(s)$ is $(\omega)C-$dense.

$$\begin{aligned} \text{But } {}_{C_n}\tau &= \{\{1\},\{1,3\},\{3\},\phi,X\}; \text{ if n is odd,} \\ &= \{\phi,\ X\}; \text{ when n is even.} \\ \text{Therefore, } C_n(U_n) &= X; \text{ whenever n is odd,} \\ &= \phi; \text{ if n is even.} \end{aligned}$$

$\Longrightarrow$ U(s) is not C-dense.

Similarly, $C(V(s)) = A^c(s) \cap C^c(s) = \{\{2,4\},X,\{2,4\},X...\}$

But $I(C(V(s)))=$ largest open set contained in $C(V(s)) = \phi(s)$.

$$\begin{aligned} \text{Also, } C_n(V_n) &= \{2,4\};\ \text{if n is odd,} \\ &= X; \text{ for else} \\ \text{Now, } I_n(C_n(V_n)) &= \phi; \text{ for odd } n \\ &= X; \text{ when n is even.} \end{aligned}$$

i.e, $V(s)$ is $(\omega)CI-$nowhere dense, but not $CI-$nowhere dense in the $K\Omega-$space $(X,\ C,\ I)$.

THEOREM 3.1.2. *Let D be a topology on X. In the $K\Omega-$space $(X,\ C_D,\ I_D)$ if a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ is $(\omega)CI$-nowhere dense then it is CI-dense.*

PROOF. Proof is omitted. ■

THEOREM 3.1.3. *Let A(s) be a sequential set in a $K\Omega$-space (X, C, I) so that $C(I(A(s))) \subset C_I(I(A(s)))$ and $I_C(I(A(s))) = I(A(s))$. Then $C(I(A(s))) - I(A(s))$ is $(\omega)CI-$nowhere dense.*

PROOF. Assume that $I(C(C(I(A(s))) - I(A(s)))) \neq \phi(s)$. So there exist at least one sequential point $\alpha = (x, P)$ such that $\alpha = (x, P) \in I(C(C(I(A(s))) - I(A(s))))$. As $I(C(C(I(A(s))) - I(A(s))))$ is $I-$open, there exists a $I-$open sequential set $G(s) = \{G_n\}_{n=1}^{\infty}$ such that,

$$\begin{aligned} \alpha \in G(s) &\subset I(C(C(I(A(s))) - I(A(s)))) \\ &\subset C(C(I(A(s)))) - I(A(s))[\text{since I is contractive}] \\ &= C(I(A(s))) - I(A(s))[\text{since I(A(s)) is C-open}] \\ &\subset C_I(I(A(s)))) - I(A(s)). \end{aligned}$$

We claim that none of the sequential point belonging to $I(A(s))$ is an $I-$limit point of $C_I(I(A(s))) - I(A(s))$. If $\beta = (y, Q) \in I(A(s))$, then there exists an $I-$open sequential set $G_1(s)$ such that $\beta = (y, Q) \in G_1(s) \subset I(A(s))$. Since $G_1(s)$ is any $I-$nbd of β, then it is also a $I - weak$ nbd of β with $G_1(s) \cap (C_I(I(A(s))) - I(A(s))) = \phi(s)$. So $\beta \notin_\omega C_I(I(A(s))) - I(A(s))$. Since $G(s) \subset C_I(I(A(s))) - I(A(s)) \Rightarrow \beta \notin_\omega G(s)$. Therefore we have $I(A(s)) \subset G^c(s)$, this implies $C_I(I(A(s))) \subset C_I(G^c(s)) = G^c(s)$. Thus we get $G(s) \subset C_I(I(A(s))) - I(A(s)) \subset G^c(s) - I(A(s))$, which is a contradiction. Therefore, $C(I(A(s))) - I(A(s))$ is $(\omega)CI-$nowhere dense. ■

REMARK. A similar discussion can be made in (X, c, i) what we call a Kuratowski space, where $c : (P(X)) \to (P(X))$ is a Kuratowski closure operator and $i : (P(X)) \to (P(X))$ is an interior operator.

3.2. Semi-open and weakly semi-open sequential sets in $K\Omega$-Spaces

Compositions of closure and interior operators in different permutations in a topological space generate a class of subsets and several mathematicians contributed to present some intellectually stimulating studies with those subsets. N. Levin [36] introduced the concept of semi open sets in point set topological spaces using one of such composition. He studied semi open sets in a topological space by using the closure and interior operators of the underlying space. We study a new kind of semi open set defined by using any Kuratowski closure and interior operators. This section also introduces $CI-$semi-open and $(\omega)CI-$ semi-open sequential sets using the composition $C \circ I$ of some $K\Omega$ closure operator C and $K\Omega$ interior operator I.

Definition 3.2.1. Let X be a nonvoid set. Suppose $c : P(X) \to P(X)$ be a Kuratowski closure operator and $i : P(X) \to P(X)$ be an interior operator, then $(X,\, c,\, i)$ is called a Kuratowski space.

Definition 3.2.2. Let $(X,\, c,\, i)$ be a Kuratowski space. The interior operator $i_c : P(X) \to P(X)$ induced by the Kuratowski closure operator $c : P(X) \to P(X)$ is defined by $i_c(A) = X - c(X - A)$ for all $A \subset X$.

Definition 3.2.3. Let $(X,\, c,\, i)$ be a Kuratowski space. The Kuratowski closure operator $c_i : P(X) \to P(X)$ induced by the interior operator $i : P(X)\to P(X)$ is defined by $c_i(A) = X - i(X - A)$ for all $A \subset X$.

DEFINITION 3.2.4. A point α in a kuratowski space (X, c, i) is called a c-limit point of $A \subset X$ if for any $B \subset X$ with $\alpha \in i_c(B)$, $i_c(B) \cap A$ contains a point other than α.

DEFINITION 3.2.5. A point α in a kuratowski space (X, c, i) is called an i-limit point of $A \subset X$ if for any $B \subset X$ with $\alpha \in i(B)$, $i(B) \cap A$ contains a point other than α.

DEFINITION 3.2.6. Let (X, c, i) be a Kuratowski space. $A \subset X$ is said to be *ci*-semi open if there exists a $B \subset X$ such that

$$i(B) \subset A \subset c(i(B)).$$

THEOREM 3.2.1. *Let X be a non empty set. $A \subset X$ is ci-semi open in a Kuratowski space (X, c, i) if and only if $A \subset c(i(A))$.*

PROOF. Let us first suppose that A be ci-semi open. Then there exists $B \subset X$ such that $i(B) \subset A \subset c(i(B))$. Now

$$\begin{aligned} & i(B) \subset A \\ & \Rightarrow i(B) \subset i(A) \\ & \Rightarrow c(i(B)) \subset c(i(A)) \\ & \Rightarrow A \subset c(i(A)). \end{aligned}$$

Conversely, let $A \subset c(i(A))$. We know that $i(A) \subset A$. Therefore $i(A) \subset A \subset c(i(A))$. Hence A is ci-semi open. ■

THEOREM 3.2.2. *Let X be a nonempty set and $A \subset X$. Then $i(A)$ is ci-semi open in (X, c, i).*

PROOF. Since $i(A) \subset i(A) \subset c(i(A))$, it follows that $i(A)$ is ci-semi open. ■

THEOREM 3.2.3. *Arbitrary union of ci-semi open sets is ci-semi open in Kuratowski space (X, c, i).*

PROOF. Let $\{A_\lambda \subset X, \lambda \in \Lambda\}$ be a family of ci-semi open set in (X, c, i). Then there exists $B_\lambda \subset X, \lambda \in \Lambda$ such that

$$i(B_\lambda) \subset A_\lambda \subset c(i(B_\lambda)), \lambda \in \Lambda.$$

$$\bigcup_{\lambda\in\Lambda} i(B_\lambda) \subset \bigcup_{\lambda\in\Lambda} A_\lambda \subset \bigcup_{\lambda\in\Lambda} c(i(B_\lambda)) \subset c(\bigcup_{\lambda\in\Lambda}(i(B_\lambda)).$$

Therefore $i(\bigcup_{\lambda\in\Lambda} i(B_\lambda)) \subset \bigcup_{\lambda\in\Lambda} A_\lambda \subset c(i(\bigcup_{\lambda\in\Lambda}(i(B_\lambda))).$
Hence $\bigcup_{\lambda\in\Lambda} A_\lambda$ is ci-semi open in (X, c, i). ■

THEOREM 3.2.4. *Let X be a nonvoid set and (X, c, i) be a Kuratowski space. Let A be a set in Kuratowski space (X, c, i) such that $i_c(A)$=$i(A)$ and B be a ci-semi open in (X, c, i) such that $i_c(B) = i(B)$. Then $i(A) \cap B$ is ci-semi open in (X, c, i).*

PROOF. Since B is ci-semi open in (X, c, i), then $B \subset c(i(B))$. Now

$$B \subset c(i(B))$$
$$\Rightarrow i(A) \cap B \subset i(A) \cap c(i(B)). \longrightarrow (*)$$

We claim that, $i(A) \cap c(i(B)) \subset c(i(A \cap B))$. If $i(A) \cap c(i(B)) = \phi$, then the proof is complete.
If $i(A) \cap c(i(B)) \neq \phi$, then there are two possibilities :

(i) $\alpha \in i(A)$ and $\alpha \in i(B)$.

(ii) $\alpha \in i(A)$ and $\alpha \in (c(i(B)) - i(B))$.

In the first case, $\alpha \in i(A) \cap i(B) = i(A \cap B) \subset c(i(A \cap B))$ and the proof is done.
In the later case, $\alpha \in (c(i(B)) - i(B))$, it comes out α is a c-limit point of i(B).
Let $i_c(G)$ be any set containing α. Then $\alpha \in i(A) \cap i_c(G) = i_c(A \cap G)$ [as $i(A) = i_c(A)$].
As α is a c-limit point of i(B),

$$i(B) \cap i_c(A \cap G) \neq \phi$$
$$\Rightarrow i_c(B) \cap i_c(A \cap G) \neq \phi$$
$$\Rightarrow i_c(B \cap A) \cap i_c(G) \neq \phi.$$

It follows that, α is c-limit point of $i_c(B \cap A)$. So $\alpha \in c(i_c(B \cap A)) = c(i_c(B) \cap i_c(A)) = c(i(B) \cap i(A)) = c(i(A \cap B))$. [Since $i_c(A) = i(A)$ and $i_c(B) = i(B)$.]
This implies $i(A) \cap c(i(B)) \subset c(i(A \cap B))$.
Consequently $i(A) \cap B \subset c(i(A \cap B))$ [by $(*)$].
That is, $i(A) \cap B \subset c(i(i(A) \cap B))$. Therefore $i(A) \cap B$ is ci-semi open in (X, c, i). ■

Definition 3.2.7. Let X be a nonvoid set and (X, C, I) be a $K\Omega$-space. A sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ is said to be

(ω) $CI-$semi open if there exists a sequential set $B(s)$ such that $I(B(s)) \subset A(s) \subset C(I(B(s)))$.

THEOREM 3.2.5. *A sequential set* $A(s) = \{A_n\}_{n=1}^{\infty}$ *in a* $K\Omega$*-space (X, C, I) is* $(\omega)CI-$*semi open if and only if* $A(s) \subset C(I(A(s)))$.

PROOF. Since A(s) is (ω) $CI-$semi open, there exists a sequential set B(s) such that $I(B(s)) \subset A(s) \subset C(I(B(s)))$. Since I is idempotent,

$$I(B(s)) \subset I(A(s)) \; [\because I(B(s)) \subset A(s)]$$

$$\Rightarrow C(I(B(s)) \subset C(I(A(s))).$$

Therefore we get $A(s) \subset C(I(A(s)))$.
Conversely, $I(A(s)) \subset A(s) \subset C(I(A(s))$. So A(s) is (ω) CI-semi open. ■

THEOREM 3.2.6. *Let* $A(s) = \{A_n\}_{n=1}^{\infty}$ *be a* (ω) *CI-semi open sequential set in the* $K\Omega$*-space (X, C, I) with* $A(s) \subset B(s) \subset C(A(s))$. *Then B(s) is* (ω) *CI-semi open.*

PROOF. Since A(s) is (ω) CI-semi open sequential set in the $K\Omega$-space (X, C, I), then there exists a sequential set D(s) such that $I(D(s)) \subset A(s) \subset C(I(D(s))$. Now

$$A(s) \subset C(I(D(s))$$

$$\Rightarrow C(A(s)) \subset C(I(D(s))). \longrightarrow (i)$$

On the other hand $A(s) \subset B(s) \subset C(A(s))$ and $I(D(s)) \subset A(s)$. Combining these two and (i) we get

$$I(D(s)) \subset B(s) \subset C(I(D(s))).$$

Hence B(s) is (ω) CI-semi open. ■

THEOREM 3.2.7. *Arbitrary union of $(\omega)CI-$semi open set is $(\omega)CI-$semi open in $K\Omega$ space (X, C, I).*

PROOF. Proof is omitted. ■

EXAMPLE 3.2.1. Consider the $K\Omega$-space (X, C, I) as in Example 3.1.3 and the sequential set $S(s) = \{S_n\}_{n=1}^{\infty}$ defined by

$$\begin{aligned} S_n &= \{1,4\}; \text{ when n is odd} \\ &= \{2\}; \text{ if n is even} \end{aligned}$$

Here we see that,

$$I(A(s)) = A(s) \subset S(s) \subset C(I(A(s))) = C(A(s)) = C^c(s).$$

Thus $S(s)$ is $(\omega)CI$-semi open. Now,

$$C_n(I_n(B)) = \phi, \text{ whenever n is even; for any } B \subset X$$

Thus for no $B \subset X$ and no even n, S_n is contained in $C_n(I_n(B))$. Hence S_n is not C_nI_n-semi open, whenever n is even.

EXAMPLE 3.2.2. Consider $X = \{1, 2, 3, 4\}$ and τ ={$\phi(s)$,A(s), B(s),C(s),X(s)}, where $A(s) = \{A_n\}_{n=1}^{\infty}$, $B(s) = \{B_n\}_{n=1}^{\infty}$, $C(s) = \{C_n\}_{n=1}^{\infty}$ are defined as $A_n = \{1\}$, $B_n = \{2, 3\}$, $C_n = \{1, 2, 3\}$, for all $n \in \mathbb{N}$. Let C and I be the closure and interior operators in (X, τ). For the sequential set $E(s) = \{E_n\}_{n=1}^{\infty}$, where $E_n = \{1\}$ for odd n and $E_n = \{2, 3\}$ for even n, each component E_n of $E(s)$ is C_nI_n-semi open in (X, C_n, I_n), but $E(s)$ is not $CI-$semi open in (X, τ) as shown below. Here, $E(s) = \{\{1\}, \{2, 3\}, \{1\}, \{2, 3\}, ...\}$

and the closed sequential sets are

$$\begin{aligned} A^c(S) &= \{\{2,3,4\},\{2,3,4\},\{2,3,4\},...\} \\ B^c(S) &= \{\{1,4\},\{1,4\},\{1,4\},...\} \\ C^c(S) &= \{\{4\},\{4\},\{4\},...\} \\ \phi(s) &= \{\phi,\phi,\phi,...\} \\ X(s) &= \{X,X,X,...\} \end{aligned}$$

Here only $\phi(s) \subset E(s)$ and $C(I(\phi(s))) = \phi(s)$, i. e., $E(s)$ is not $(\omega)CI-$semi open in (X, C, I). Now $A_n \subset E_n$ when n is odd and ${B_n}^c$ and X are only closed sets in $(X, D_n(\tau))$ containing A_n, for odd n. Therefore $\overline{A_n} = {B_n}^c \cap X = {B_n}^c = \{1,4\}$. Thus we have $A_n \subset E_n \subset \overline{A_n}$ when n is odd. Again we see that $B_n \subset E_n$ when n is even and ${A_n}^c$ and X are only closed sets in $(X, D_n(\tau))$ containing B_n for even n. Therefore $\overline{B_n} = {A_n}^c \cap X = {A_n}^c = \{2,3,4\}$. Thus we have $B_n \subset E_n \subset \overline{B_n}$ when n is even. Therefore each component of $E(s)$, i. e., E_n is C_nI_n-semi open in (X, C_n, I_n) for all $n \in \mathbb{N}$.

Above Example 3.2.1 and Example 3.2.2 force to make the following

Remark. There is no relation between the $(\omega)CI$-semi openness of a sequential set and the C_nI_n-semi openness of its components in the sense as follows:

A sequential set $A(s)$ is $(\omega)CI$-semi open $\not\Leftrightarrow$ n^{th} component of $A(s)$ is C_nI_n-semi open.

Definition 3.2.8. A sequential set $A(s)$ in $K\Omega - space$ (X, C, I) is said to be $CI-$semi open, if there exists a sequential set

$B(s)$ such that, $I(B(s)) \subset A(s) \subset C(I(B(s)))$ with the property $I_n(D_n) \subset A_n \subset C_nI_n(D_n)$ for each $n \in \mathbb{N}$ and for some $D_n \subset X$ depending on n.

Every $CI-$semi open sequential set in a $K\Omega$-space (X, C, I) is $(\omega)CI$-semi open but the converse is not true in general as seen from the Example 3.2.2

THEOREM 3.2.8. *Let D be a topology on X. A sequential set A(s) is C_DI_D-semi open in $(X,\ C_D,\ I_D)$ if and only if each component A_n of A(s) is $C_{D_n}I_{D_n}-$semi open in $(X,\ C_{D_n},\ I_{D_n})$.*

PROOF. First suppose that A(s) is C_DI_D-semi open in $(X,\ C_D,\ I_D)$. Then there exists a sequential set B(s) such that $I_D(B(s)) \subset A(s) \subset C_D(I_D(B(s)))$ with the property $I_{D_n}(E_n) \subset A_n \subset C_{D_n}I_{D_n}(E_n)$ for each $n \in \mathbb{N}$ and for some $E_n \subset X$ depending on n. Since $I_{D_n}(E_n) \subset A_n \subset C_{D_n}I_{D_n}(E_n)$ for some $E_n \subset X$, it follows that each component A_n is $C_{D_n}I_{D_n}-$semi open in $(X,\ C_{D_n},\ I_{D_n})$.
Conversely, let each component of $A(s) = \{A_n\}_{n=1}^{\infty}$ is $C_{D_n}I_{D_n}-$semi open in $(X,\ C_{D_n},\ I_{D_n})$, then there exists some $E_n \subset X$ such that

$$I_{D_n}(E_n) \subset A_n \subset C_{D_n}I_{D_n}(E_n), \text{ for all } n \in \mathbb{N}$$
$$\Rightarrow \{I_{D_n}(E_n)\} \subset \{A_n\} \subset \{C_{D_n}I_{D_n}(E_n)\}$$
$$\Rightarrow I_D(E(s)) \subset A(s) \subset C_D(I_D(E(s))).$$

Hence $A(s)$ is C_DI_D-semi open in $(X,\ C_D,\ I_D)$. ■

THEOREM 3.2.9. *If $A(s)$ is $CI-$semi open in the $K\Omega-$space $(X,\ C,\ I)$, then each component A_n of $A(s)$ is C_nI_n-semi open in Kuratowski space $(X,\ C_n,\ I_n)$.*

PROOF. Proof is straightforward. ■

Example 3.2.2 shows that the converse of the Theorem 3.2.9 is not true in general.

THEOREM 3.2.10. *If A(s), B(s) be sequential sets in $K\Omega-$space (X, C, I) such that $I(A(s)) \cap I(B(s)) = \phi(s)$ and D(s) is sequential set with $I(B(s)) \subset D(s) \subset C(I(B(s)))$. Then $I(A(s)) \cap D(s) = \phi(s)$ if $I_C(I(A(s))) = I(A(s))$.*

PROOF. Given $I(A(s)) \cap I(B(s)) = \phi(s) \Rightarrow I(B(s)) \subset X(s) - I(A(s)) \Rightarrow C(I(B(s))) \subset C(X(s) - I(A(s))) = X(s) - I(A(s))$[since $I_C(I(A(s))) = I(A(s))$] i. e., $C(I(B(s))) \subset X(s) - I(A(s))$.
Also D(S) is a sequential set with $I(B(s)) \subset D(s) \subset C(I(B(s)))$
$\Rightarrow I(A(s)) \cap I(B(s)) \subset I(A(s)) \cap D(s) \subset I(A(s)) \cap C(I(B(s)))$
$\subset I(A(s)) \cap (X(s) - I(A(s)))$[Since $C(I(B(s))) \subset X(s) - I(A(s))$]
$\Rightarrow \phi(s) \subset I(A(s)) \cap D(s) \subset \phi(s)$.
Thus we have $I(A(s)) \cap D(s) = \phi(s)$. Hence the proof. ■

THEOREM 3.2.11. *Arbitrary union of $CI-$semi open sequential sets is $CI-$semi open.*

PROOF. Let $\{A^\lambda(s)\}_{\lambda\in\Lambda}$ be an arbitrary collection of CI-semi open sequential sets in $K\Omega-$space (X, C, I). For every $\lambda \in \Lambda$, there exists a sequential set $B^\lambda(s)$ such that $I(B^\lambda(s)) \subset A^\lambda(s) \subset C(I(B^\lambda(s)))$ with the property $I_n(D_n^\lambda) \subset A_n^\lambda \subset C_n(I_n(D_n^\lambda))$ for each

$n \in \mathbb{N}$ and for some $D_n^\lambda \subset X$ depending on n. Now,

$$I(B^\lambda(s)) \subset A^\lambda(s) \subset C(I(B^\lambda(s)))$$

$$\Rightarrow \quad \underset{\lambda\in\Lambda}{\cup} I(B^\lambda(s)) \subset \underset{\lambda\in\Lambda}{\cup} A^\lambda(s) \subset \underset{\lambda\in\Lambda}{\cup} C(I(B^\lambda(s))) \subset C(\underset{\lambda\in\Lambda}{\cup} I(B^\lambda(s)))$$

$$\Rightarrow \quad I(\underset{\lambda\in\Lambda}{\cup} I(B^\lambda(s))) \subset \underset{\lambda\in\Lambda}{\cup} A^\lambda(s) \subset C(I(\underset{\lambda\in\Lambda}{\cup} I(B^\lambda(s))).$$

Similarly it can be shown that,

$I_n(\underset{\lambda\in\Lambda}{\cup} I_n(D_n^\lambda)) \subset \underset{\lambda\in\Lambda}{\cup} A_n^\lambda \subset C_n(I_n(\underset{\lambda\in\Lambda}{\cup} I_n(D_n^\lambda)))$, for each $n \in \mathbb{N}$

Hence the theorem. ■

The following Example 3.2.3 shows that, complement of a $CI-$ or $(\omega)CI-$semi open sequential set may not be $(\omega)CI-$semi open and intersection of two $CI-$ or $(\omega)CI-$semi open sequential sets may fail to be $(\omega)CI-$semi open.

EXAMPLE 3.2.3. In the $K\Omega-$space (X, C, I) as considered in Example 3.1.1 the sequential sets $K(s) = \{K_n\}_{n=1}^\infty$, $L(s) = \{L_n\}_{n=1}^\infty$ and $W(s) = \{W_n\}_{n=1}^\infty$ defined by

$$\begin{aligned} K_n &= \{1,4\}; \text{ for odd n} \\ &= \{2,3\}; \text{ for even n} \\ L_n &= \{3,4\}; \text{ for odd n} \\ &= \{1,3\}; \text{ for even n} \\ &\quad \text{and} \\ W_n &= \{1,3,4\}; \text{ when n is odd} \\ &= \{1,3\}; \text{ otherwise} \end{aligned}$$

are $CI-$semi open but neither $K(s) \cap L(s)$ nor $W^c(s)$ is $CI-$semi open. Let's explain the facts below:

Here $G(s) \subset K(s)$ and $B^c(s)$, $C^c(s)$, $H^c(s)$ and $X(s)$ are the only closed sequential sets containing $G(s)$ in (X, τ). So, $C(G(s)) = B^c(s) \cap C^c(s) \cap H^c(s) \cap X(s) = H^c(s)$. Also $K(s) \subset C(G(s))$. Thus we have $I(G(s)) \subset K(s) \subset C(I(G(s)))$. Also $C_n(G_n) = H^c{}_n$ =the n^{th} component of $H^c(s)$ and $K_n \subset C_n(G_n)$ for each $n \in \mathbb{N}$. Therefore $K(s)$ is $CI-$semi open in (X, C, I). Now $B(s)$, $C(s)$ and $H(s)$ are the open sequential sets in (X, τ) that contained in $L(s)$. $C(B(s)) = A^c(s) \cap C^c(s) D^c(s) \cap F^c(s) \cap G^c(s) \cap N^c(s) \cap S^c(s) = S^c(s)$. But $L(s)$ is not contained in $C(B(s))$. Now $C(C(s)) = A^c(s) \cap B^c(s) \cap D^c(s) \cap E^c(s) \cap G^c(s) \cap M^c(s) \cap Q^c(s) = Q^c(s)$. But $L(s)$ is not contained in $Q^c(S)$. Finally, $C(H(s)) = A^c(s) \cap D^c(s) \cap G^c(s) = G^c(s)$ and $L(s) \subset G^c(s) = C(H(s))$. Therefore $I(H(s)) \subset L(s) \subset C(I(H(s)))$. Also $C_n(H_n) = {G_n}^c$ and $L_n \subset C_n(H_n)$ for each $n \in \mathbb{N}$. Therefore $L(s)$ is $CI-$semi open in (X, C, I). Now $K(s) \cap L(s) = \{\{4\}, \{3\}, ...\}$. Here $\phi(s)$ is the only open sequential set contained in $K(s) \cap L(s)$. Thus $K(s) \cap L(s)$ is not $CI-$semi open.

On the other hand, $R(s) \subset W(s) \subset C(R(s)) = X(s)$ and so $W(s)$ is $CI-$semi open but since $\phi(s)$ is the only sequential open set contained in $W^c(s)$, $W^c(s)$ can not be $CI-$semi open.

THEOREM 3.2.12. *A sequential set $A(s)$ in $K\Omega-$space (X, C, I) is CI-semi open if and only if $A(s) \subset C(I(A(s)))$ with the property $A_n \subset C_n I_n(A_n)$ for each $n \in \mathbb{N}$.*

PROOF. First we suppose that A(s) is CI-semi open. Then there exists a sequential set B(s) such that, $I(B(s)) \subset A(s) \subset C(I(B(s)))$ with the property $I_n(D_n) \subset A_n \subset C_n I_n(D_n)$ for each

$n \in \mathbb{N}$ and for some $D_n \subset X$ depending on n. Now, $I(B(s)) \subset A(s) \Rightarrow I(B(s)) \subset I(A(s)) \Rightarrow C(I(B(s))) \subset C(I(A(s)))$. Therefore $A(s) \subset C(I(A(s)))$. Similarly it can be shown that $A_n \subset C_nI_n(A_n)$ for each $n \in \mathbb{N}$.

Conversely, let $A(s) \subset C(I(A(s)))$ and $A_n \subset C_nI_n(A_n)$ for each $n \in \mathbb{N}$. Therefore $I(A(s)) \subset A(s) \subset C(I(A(s)))$ and $I_n(A_n) \subset A_n \subset C_nI_n(A_n)$ for each $n \in \mathbb{N}$.
Hence A(s) is CI-semi open. ■

Theorem 3.2.13. *Let A(s) be sequential set in* $K\Omega-$*space* (X, C, I) *such that* $I_C(A(s)) = I(A(s))$. *Let B(s) be a* $(\omega)CI-$*semi open sequential set such that* $I_C(B(s)) = I(B(s))$. *Then* $I(A(s)) \cap B(s)$ *is* (ω)*CI-semi open. If further* $I(A(s)) = \{I_n(A_n)\}_{n=1}^{\infty}$ *and* $A_n \cap B_n$ *is* C_nI_n-*semi open for each* $n \in \mathbb{N}$, *then* $I(A(s)) \cap B(s)$ *is CI-semi open.*

Proof. Since B(s) is $(\omega)CI-$semi open sequential set, then $B(s) \subset C(I(B(s)))$.
Now, $B(s) \subset C(I(B(s))) \Rightarrow I(A(s)) \cap B(s) \subset I(A(s)) \cap C(I(B(s)))$.
We claim that, $I(A(s)) \cap C(I(B(s))) \subset C(I(A(s)) \cap (B(s)))$.
If $I(A(s)) \cap C(I(B(s))) = \phi(s)$, then proof is done.
Let $I(A(s)) \cap C(I(B(s))) \neq \phi(s)$ and $\alpha = (x, p) \in I(A(s)) \cap C(I(B(s)))$. Now there are two possibilities:
(i) $\alpha \in I(A(s))$ and $\alpha \in I(B(s))$
(ii) $\alpha \in I(A(s))$ and $\alpha \in C(I(B(s))) - I(B(s))$.
In the first case the proof is complete. In the later case, $\alpha \in$

$C(I(B(s))) - I(B(s))$, and it follows that α is a C-limit point of I(B(s)). Let $I_C(G(s))$ be any set containing α. Then, $\alpha \in I(A(s)) \cap I_C(G(s)) = I_C(A(s) \cap G(s))$[since $I_C(A(s)) = I(A(s))$].
As α is a C-limit point of I(B(s)), thus
$I(B(s)) \cap I_C(A(s) \cap G(s)) \neq \phi(s)$
$\Rightarrow I_C(B(s)) \cap I_C(A(s) \cap G(s)) \neq \phi(s)$
$\Rightarrow I_C(B(s) \cap A(s)) \cap I_C G(s) \neq \phi(s)$.
Therefore α is a C-limit point of $I_C(B(s) \cap A(s))$, so $\alpha \in C(I_C(B(s) \cap A(s))) = C(I_C(B(s)) \cap I_C(A(s))) = C(I(B(s) \cap A(s)))$[since $I(A(s)) = I_C(A(s))$ and $I(B(s)) = I_C(B(s))$].
Hence $I(A(s)) \cap B(s)$ is (ω)CI-semi open.
Moreover, if $I(A(s)) = \{I_n(A_n)\}_{n=1}^{\infty}$ then $P_n(I(A(s))) \cap B_n = I_n(A_n) \cap B_n$. Since $A_n \cap B_n$ is $C_n I_n-$semi open for each $n \in \mathbb{N}$, we have $A_n \cap B_n \subset C_n I_n(A_n \cap B_n)$
$\Rightarrow I_n(A_n) \cap B_n \subset A_n \cap B_n \subset C_n I_n(A_n \cap B_n)$
$\Rightarrow I_n(A_n) \cap B_n \subset C_n I_n(A_n \cap B_n)$.
Hence $I(A(s)) \cap B(s)$ is semi open. ■ The following Example 3.2.4 shows that analogue of Theorem 3.2.6 for $CI-$semi open sets may not be true in general.

Example 3.2.4. Consider a set X={1,2,3,4} and τ ={$\phi(s)$,A(s), B(s),C(s),D(s),E(s),F(s),G(s),H(s),M(s),N(s),X(s)}, where

$$A(s) = \{A_n\}_{n=1}^{\infty};\ A_n = \{1\}; \text{n is an odd}$$
$$= \phi; \text{n is an even}$$

$$
\begin{aligned}
B(s) = \{B_n\}_{n=1}^{\infty};\ B_n &= \{1,3\};\text{n is an odd}\\
&= X;\text{n is an even}\\
C(s) = \{C_n\}_{n=1}^{\infty};\ C_n &= \{3\};\text{n is an odd}\\
&= \{2\};\text{n is an even}\\
D(s) = \{D_n\}_{n=1}^{\infty};\ D_n &= \{1,3\};\text{n is an odd}\\
&= \phi;\text{n is an even}\\
E(s) = \{E_n\}_{n=1}^{\infty};\ E_n &= \{3\};\text{n is an odd}\\
&= \phi;\text{n is an even}\\
F(s) = \{F_n\}_{n=1}^{\infty};\ F_n &= \{1,3\};\text{n is an odd}\\
&= \{2\};\text{n is an even}\\
G(s) = \{G_n\}_{n=1}^{\infty};\ G_n &= \{3\};\text{n is an odd}\\
&= \{1\};\text{n is an even}\\
H(s) = \{H_n\}_{n=1}^{\infty};\ H_n &= \{3\};\text{n is an odd}\\
&= \{1,2\};\text{n is an even}\\
M(s) = \{M_n\}_{n=1}^{\infty};\ M_n &= \{1,3\};\text{n is an odd}\\
&= \{1\};\text{n is an even}\\
N(s) = \{N_n\}_{n=1}^{\infty};\ N_n &= \{1,3\};\text{n is an odd}\\
&= \{1,2\};\text{n is an even.}
\end{aligned}
$$

Also let C and I be the closure and interior operators in (X, τ). Consider two sequential sets

$$
\begin{aligned}
S(s) = \{S_n\}_{n=1}^{\infty};\ S_n &= \{1,4\};\text{n is an odd}\\
&= \phi;\text{n is an even}
\end{aligned}
$$

$$\text{and } T(s) = \{T_n\}_{n=1}^{\infty};\ T_n = \{1,2,4\}; \text{n is an odd}$$
$$= \{4\}; \text{n is an even}$$

Here A(s) is the only τ open set contained in S(s). Now, $C(A(s)) = C^c(s) \cap E^c(s) \cap G^c(s) \cap H^c(s) \cap X^c(s) = H^c(s)$. Also, $S(s) \subset H^s(s) = C(A(s))$. Again

$$C_n(A_n) = \{1,2,4\}; \text{n is an odd}$$
$$= \phi; \text{n is an even}$$

and $S_n \subset C_n(A_n)$ for each $n \in \mathbb{N}$. This shows that $S(s)$ is $CI-$semi open sequential set. Also $C(S(s)) = C^c(s) \cap E^c(s) \cap G^c(s) \cap H^c(s) \cap X^c(s) = H^c(s)$ and $S(s) \subset T(s) \subset H^c(s) = S^c(s)$. Again $A(s) \subset T(s) \subset C(A(s))$, then $T(s)$ is $(\omega)CI-$ semi open. But $T_n \nsubseteq C_n I_n(T_n) = \phi$ for every even n. Thus $T(s)$ is not $CI-$semi open. Now $I(T(s)) = A(s) \cup \phi(s) = A(s)$. Therefore $C(I(T(s)) = C(A(s)) = H^c(s)$. We have $T(s) \subset H^c(s) = C(I(T(s)))$. Also $I(S(s)) = A(s) \cup \phi(s) = A(s)$, therefore $C(I(S(s))) = C(A(s)) = H^c(s)$. We also have $S(s) \subset T(s) \subset C(I(S(s)))$. Hence $S(s)$ is $CI-$semi open with the property $S(s) \subset T(s) \subset C(S(s))$. Moreover, $T(s) \subset C(I(T(s)))$ and $S(s) \subset T(s) \subset C(I(S(s))$.

Chapter IV

Separation Axioms in Sequential Topological Spaces

Separation axioms up to regularity in sequential topological spaces are studied by Bose and Lahiri in the year 2002 [6]; some definitions and results regarding sequential topological spaces, that will be used to explore the study, are given in chapter I of this thesis. As in [6] two sequential points (x, P) and (y, Q) are said to be distinct if either $x \neq y$ or none of P and Q is a subset of the other. But according to this definition two sequential points may be neither distinct nor identical as seen in the following example: sequential points $(x, 2\mathbb{N})$ and $(x, 4\mathbb{N})$ are not distinct. Also these two sequential points are not identical since they do not satisfy condition of equality of sequential sets. We propose to present an appropriate definition as follows:

Two sequential points $p = (x, P)$ and $q = (y, Q)$ are said to be identical if $x = y$ and $P = Q$; otherwise they are distinct. Let P and Q be respectively the bases of the sequential sets $A(s) = \{A_n\}_{n=1}^{\infty}$ and $B(s) = \{B_n\}_{n=1}^{\infty}$. Then $B(s)$ is said to be a reduced sequential set of $A(s)$ if $A_n \subset B_n \; \forall \; n \in Q \subset P$. In this case $A(s)$ is said to be an augmented sequential set of $B(s)$. A sequential topological space (X, τ) is said to be T_o space if for any two distinct sequential points $p = (x, P)$ and $q = (y, Q)$ $\exists$ an open sequential set $U(s)$ in (X, τ) such that $p \in_w^{P-Q} U(s)$ and $q \notin_w U(s)$, whenever q is a reduced sequential point of the sequential point p; otherwise $\exists$ an open sequential set $U(s)$ in (X,τ) such that $p \in_w U(s)$ and $q \notin_w U(s)$. A sequential topological space (X,τ) is said to be T_1 space if for any two distinct sequential points $p = (x,P)$ and $q = (y,Q)$ $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X,τ) such that $p \in_w^{P-Q} U(s)$, $q \in_w V(s)$, $p \notin_w^{P-Q} V(s)$ and $q \notin_w U(s)$, whenever q is a reduced sequential point of the sequential point p; otherwise $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X,τ) such that $p \in_w U(s)$, $q \in_w V(s)$, $p \notin_w V(s)$ and $q \notin_w U(s)$. A sequential topological space (X,τ) is said to be Hausdorff or T_2 space if for any two distinct sequential points $p = (x,P)$ and $q = (y,Q)$ $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X,τ) such that $p \in_w^{P-Q} U(s)$, $q \in_w V(s)$, $p \notin_w^{P-Q} \overline{V(s)}$ and $q \notin_w \overline{U(s)}$, whenever q is a reduced sequential point of the sequential point p; otherwise $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X,τ) such that $p \in_w U(s)$, $q \in_w V(s)$, $p \notin_w \overline{V(s)}$ and $q \notin_w \overline{U(s)}$. A sequential topological space (X,τ) is said to be weak Hausdorff or (w) Hausdorff if for any two distinct sequential points $p = (x,P)$ and $q = (y,Q)$ $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X,τ) such that $p \in_w^{P-Q} U(s)$, $q \in_w V(s)$, $U(s) \cap V(s) = \phi(s)$, whenever q is a reduced sequential point of the sequential point p; otherwise $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X,τ) such that $p \in_w U(s)$, $q \in_w V(s)$, $U(s) \cap V(s) = \phi(s)$.

The above definitions of T_0, T_1 and T_2 spaces are different from the corresponding definitions given in [6], but in spite of the differences, all the results related to these spaces given in [6] remain unchanged. For this reason we exclude those portions of separation axioms in this study.

We present an example which reveals the fact that how the changes in our definitions affect the further study and what consequences we are going to encounter. Consider $X = \{a, b\}$ and $\mathbf{D} = \{\phi, \{a\}, \{b\}, X\}$, then $(X, \mathbf{D})$ is a regular topological space. Also consider the sequential topology $\tau < \mathbf{D} >$ generated by $\mathbf{D}$ on X. Then $(X, \tau < \mathbf{D} >)$ is not a regular sequential topological space according to the definition of regularity in [6], since for the closed sequential set $F(s) = \{F_n\}_{n=1}^{\infty}$, where

$$\begin{aligned} F_n &= \{a\} \text{ for } n = 1, 2 \\ &= \phi \text{ for } n = 3, 4, 5... \end{aligned}$$

and the sequential point $p = (a, \{1, 2, 3\}) \notin F(s)$, $\nexists$ open sequential sets $U(s)$ and $V(s)$ in $(X, \tau < \mathbf{D} >)$ such that $p \in_w U(s)$, $F(s) \subset_w V(s), p \notin_w \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s)$. Thus with definitions in [6] regularity of (X,D) does not imply that of $(X, \tau < \mathbf{D} >)$ (cf.Theorem:15, [6]). Here we define regularity in an appropriate manner and establish the truth of this result in our setting. Also an attempt of defining T_3 -spaces according to definition of regularity in [6] leads to simple contradictions. In this study we develop the concept of T_3 -spaces, normal sequential topological spaces and T_4 -spaces and obtain some important results. [28, 35, 7, 5, 62] provide some basic clues towards establishing such a theory.

4.1. Regularity in the Light of Reduced and Augmented Bases

DEFINITION 4.1.1. A sequential topological space (X, τ) is said to be regular if for any sequential point $p = (x, P)$ and any closed sequential set $F(s)$ with $p \notin F(s)$, $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in_w^{P-Q} U(s), F(s) \subset_w V(s), p \notin_w^{P-Q} \overline{V}(s), F(s) \subset X(s) - \overline{U}(s)$, whenever $F(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in_w U(s), F(s) \subset_w V(s), p \notin_w \overline{V}(s), F(s) \subset X(s) - \overline{U}(s)$.

DEFINITION 4.1.2. A sequential topological space (X, τ) is said to be weakly regular or (w) regular if for any sequential point $p = (x, P)$ and any closed sequential set $F(s)$ with $p \notin F(s)$, $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in_w^{P-Q} U(s), F(s) \subset_w V(s), U(s) \cap V(s) = \phi(s)$, whenever $F(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in_w U(s), F(s) \subset_w V(s), U(s) \cap V(s) = \phi(s)$.

DEFINITION 4.1.3. A sequential topological space (X, τ) is said to be a T_3 space if it is regular and T_1.

REMARK. A T_3 space is a T_2. That the converse may not be true is shown by Example 4.1.3.

Remark. Example 4.1.4 shows that a regular sequential topological space may not be a T_1.

Definition 4.1.4. A sequential topological space (X, τ) is said to be a weakly T_3 space or (w) T_3 if it is (w) regular and T_1.

Theorem 4.1.1. *A sequential topological space (X, τ) is regular if and only if for any sequential point $p = (x, P)$ and for any closed sequential set $F(s)$ with $p \notin F(s)$ $\exists$ open sequential sets $G(s)$ and $H(s)$ in (X, τ) such that $p \in^{P-Q} G(s), F(s) \subset_w H(s), G(s) \cap H(s) = \phi(s)$ and $\exists$ open sequential sets $D(s)$ and $E(s)$ in (X, τ) such that $p \in_w^{P-Q} D(s), F(s) \subset E(s), D(s) \cap E(s) = \phi(s)$, whenever $F(s)$ is a reduced sequential set with base Q of the sequential point p; otherwise $\exists$ open sequential sets $G(s)$ and $H(s)$ in (X, τ) such that $p \in G(s), F(s) \subset_w H(s), G(s) \cap H(s) = \phi(s)$ and $\exists$ open sequential sets $D(s)$ and $E(s)$ in (X, τ) such that $p \in_w D(s), F(s) \subset E(s), D(s) \cap E(s) = \phi(s)$.*

Proof. Suppose (X, τ) is regular. Let $p = (x, P)$ be any sequential point and $F(s)$ be any closed sequential set such that $p \notin F(s)$. Then $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in_w^{P-Q} U(s)$, $F(s) \subset_w V(s)$, $p \notin_w^{P-Q} \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s)$ whenever $F(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in_w U(s)$, $F(s) \subset_w V(s)$, $p \notin_w \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s)$. If we take $G(s) = X(s) - \overline{V}(s)$, $H(s) = V(s)$, $D(s) = U(s)$ and $E(s) = X(s) - \overline{U}(s)$, then we are done.

Conversely, suppose the given conditions are true. Let $p = (x, P)$ be any sequential point and $F(s)$ be any closed sequential set such that $p \notin F(s)$. Then $\exists$ open sequential sets $G(s)$ and $H(s)$ in (X, τ) such that $p \in^{P-Q} G(s)$, $F(s) \subset_w H(s)$, $G(s) \cap H(s) = \phi(s)$ and $\exists$ open sequential sets $D(s)$ and $E(s)$ in (X, τ) such that $p \in_w^{P-Q} D(s)$, $F(s) \subset E(s)$, $D(s) \cap E(s) = \phi(s)$, whenever $F(s)$ is a reduced sequential set with base Q of the sequential point p; otherwise $\exists$ open sequential sets $G(s)$ and $H(s)$ in (X, τ) such that $p \in G(s)$, $F(s) \subset_w H(s)$, $G(s) \cap H(s) = \phi(s)$ and $\exists$ open sequential sets $D(s)$ and $E(s)$ in (X, τ) such that $p \in_w D(s)$, $F(s) \subset E(s)$, $D(s) \cap E(s) = \phi(s)$ Taking $U(s) = G(s) \cap D(s)$ and $V(s) = H(s) \cap E(s)$ the theorem follows. ■

COROLLARY 4.1.1. *If (X, τ) is regular, then it is (w) regular.*

THEOREM 4.1.2. *A sequential topological space (X, τ) is regular if and only if for any sequential point $p = (x, P)$ and an open sequential set $G(s)$ with $p \in_w G(s)$, $\exists$ an open sequential set $H(s)$ in (X, τ) such that $p \in^{P-Q} H(s)$, $\overline{H}(s) \subset_w G(s)$ and $\exists$ an open sequential set $B(s)$ in (X, τ) such that $p \in_w^{P-Q} B(s)$, $\overline{B}(s) \subset G(s)$ whenever $X(s) - G(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise $\exists$ an open sequential set $H(s)$ in (X, τ) such that $p \in H(s)$, $\overline{H}(s) \subset_w G(s)$ and $\exists$ an open sequential set $B(s)$ in (X, τ) such that $p \in_w B(s)$, $\overline{B}(s) \subset G(s)$.*

PROOF. Suppose (X, τ) is a regular sequential topological space. Let $p = (x, P)$ be any sequential point and $G(s)$ be an open sequential set such that $p \in_w G(s)$ i.e $p \notin X(s) - G(s) = F(s)$ (say). Then $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in^{P-Q} U(s)$, $F(s) \subset_w V(s)$, $U(s) \cap V(s) = \phi(s)$ and $\exists$ open sequential sets $D(s)$ and $E(s)$ in (X, τ) such that $p \in_w^{P-Q} D(s)$, $F(s) \subset E(s)$ $D(s) \cap E(s) = \phi(s)$, whenever $X(s) - G(s) = F(s)$ is a reduced sequential set with base Q of the sequential point p; otherwise, $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $p \in U(s)$, $F(s) \subset_w V(s)$, $U(s) \cap V(s) = \phi(s)$ and $\exists$ open sequential sets $D(s)$ and $E(s)$ in (X, τ) such that $p \in_w D(s)$, $F(s) \subset E(s)$, $D(s) \cap E(s) = \phi(s)$. If we take $H(s) = U(s)$ and $B(s) = D(s)$ we are done.

Conversely, suppose given conditions are true. Let $p = (x, P)$ be any sequential point and $F(s)$ be any closed sequential set such that $p \notin F(s)$ i.e $p \in_w X(s) - F(s) = G(s)$ (say). Then $\exists$ an open sequential set $H(s)$ in (X, τ) such that $p \in^{P-Q} H(s)$, $\overline{H}(s) \subset_w G(s)$ and $\exists$ an open sequential set $B(s)$ in (X, τ) such that $p \in_w^{P-Q} B(s)$, $\overline{B}(s) \subset G(s)$ whenever $F(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise $\exists$ an open sequential set $H(s)$ in (X, τ) such that $p \in H(s)$, $\overline{H}(s) \subset_w G(s)$ and $\exists$ an open sequential set $B(s)$ in (X, τ) such that $p \in_w B(s)$, $\overline{B}(s) \subset G(s)$.
Considering $U(s) = H(s)$, $V(s) = X(s) - \overline{H}(s)$, $D(s) = B(s)$ and $E(s) = X(s) - \overline{B}(s)$ we are done. ■

THEOREM 4.1.3. *A sequential topological space (X, τ) is (w) regular if and only if for any sequential point $p = (x, P)$ and any open sequential set $G(s)$ with $p \in_w G(s)$, $\exists$ an open sequential set $H(s)$ in (X, τ) such that $p \in_w^{P-Q} H(s)$, $\overline{H}(s) \subset_w G(s)$ whenever $X(s) - G(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise $\exists$ an open sequential set $H(s)$ in (X, τ) such that $p \in_w H(s)$, $\overline{H}(s) \subset_w G(s)$.*

PROOF. Suppose (X, τ) is (w) regular. Let $p = (x, P)$ be any sequential point and $G(s)$ be an open sequential set with $p \in_w G(s)$ i.e $p \notin X(s) - G(s) = F(s)$ (say). Then $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that

$p \in_w^{P-Q} U(s)$, $F(s) \subset_w V(s)$, $U(s) \cap V(s) = \phi(s)$.
whenever $F(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise, $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that

$p \in_w U(s)$, $F(s) \subset_w V(s)$, $U(s) \cap V(s) = \phi(s)$. If we take $H(s) = U(s)$, then we are done.
Conversely, suppose the given conditions are true. Let $p = (x, P)$ be any sequential point and $F(s)$ be any closed sequential set such that $p \notin F(s)$ i.e $p \in_w X(s) - F(s) = G(s)$ (say). Then $\exists$ an open sequential set $H(s)$ in (X, τ) such that $p \in_w^{P-Q} H(s)$, $\overline{H}(s) \subset_w G(s)$ whenever $X(s) - G(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise, $\exists$ an open sequential set $H(s)$ in (X, τ) such that $p \in_w H(s)$, $\overline{H}(s) \subset_w G(s)$. If we take $U(s) = H(s)$ and $V(s) = X(s) - \overline{H}(s)$, then we are done. ■

THEOREM 4.1.4. *If the sequential topological space (X, τ) is regular, then for any closed sequential set $A(s)$ in X, $A(s) = \cap\{N(s) :$ $N(s)$ is a closed nbd of $A(s)\}$*————————> (1)

PROOF. Suppose (X, τ) is a regular sequential topological space. Let $A(s)$ be a closed sequential set in X. Let $p = (x, P)$ be a sequential point in X with $p \notin A(s)$. Then $p \in_w X(s) - A(s) = G(s)$

(say). Then $\exists$ an open sequential set $B(s)$ in (X, τ) such that $p \in_w^{P-Q} B(s)$, $\overline{B}(s) \subset G(s)$ whenever $A(s)$ is a reduced sequential set, with base Q, of the sequential point p; otherwise $\exists$ an open sequential set $B(s)$ in (X, τ) such that $p \in_w B(s)$, $\overline{B}(s) \subset G(s)$. This implies $A(s) \subset X(s) - \overline{B}(s) = H(s)$ (say). Again $p \notin X(s) - B(s)$ and hence $p \notin \overline{H}(s)$.
Thus (1) holds. Hence the theorem . ■

REMARK. Converse of Theorem 4.1.4 may not be true, which is shown by Example 4.1.1.

EXAMPLE 4.1.1. Let $\mathbf{U}$ be the usual topology on $\mathbb{R}$. Fix $a \in \mathbb{R}$. For every open set $G \in \mathbf{U}$, we consider sequential sets $U^G(s) = \{U_n^G\}_{n=1}^\infty$ and $V^G(s) = \{V_n^G\}_{n=1}^\infty$, where

$$\begin{aligned} U_n^G &= V_n^G = G \ \forall\ n \neq 2, \\ U_2^G &= \{a\} \text{ if } a \in G \\ &= \mathbb{R} - \{a\} \text{ if } a \notin G \\ V_2^G &= \mathbb{R} \text{ if } a \in G \\ &= \phi \text{ if } a \notin G. \end{aligned}$$

The collection τ_a of sequential sets $U^G(s)$ and $V^G(s)$ $\forall\ G \in \mathbf{U}$ forms a sequential topology on $\mathbb{R}$. Any closed sequential set $F(s)$ in $(\mathbb{R}, \tau_a)$ is the intersection of all closed nbds of $F(s)$ but $(\mathbb{R}, \tau_a)$ is not regular.

THEOREM 4.1.5. *A topological space* $(X, \mathbf{D})$ *is regular if and only if the generated sequential topological space* $(X, \tau < \mathbf{D} >)$ *is regular.*

Proof. Suppose $(X, \mathbf{D})$ is regular. Let $p = (x, P)$ be any sequential point in X and $F(s)$ be any closed sequential set with the base Q, such that $p \notin F(s)$.

Case 1: Suppose $F(s) = \{F_n\}_{n=1}^{\infty}$ is a reduced sequential set of the sequential point p. Let $U \in \mathbf{D}$ such that $F_j \subset U$ for some $j \in Q$. Let $i \in P - Q$. Consider open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in $(X, \tau < \mathbf{D} >)$, where $U_i = U$, $U_n = \phi \ \forall \ n(\neq i)$, $V_j = U$, $V_n = \phi \ \forall \ n(\neq j)$.

Then $p \in_w^{P-Q} U(s)$, $F(s) \subset_w V(s)$, $p \notin_w^{P-Q} \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s)$.

Case 2: Suppose $F(s) = \{F_n\}_{n=1}^{\infty}$ such that $x \in F_n \ \forall n \in Q$ and $F_n = \phi \ \forall \ n \in \mathbb{N} - Q$ with $P \cap Q = \phi$. Let $U \in \mathbf{D}$ such that $F_j \subset U$ for some $j \in Q$. Let $i \in P$. Consider open sequential set $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in $(X, \tau < \mathbf{D} >)$, where $U_i = U$, $U_n = \phi \ \forall \ n(\neq i)$, $V_j = U$, $V_n = \phi \ \forall \ n(\neq j)$.

Then $p \in_w U(s)$, $F(s) \subset_w V(s)$, $p \notin_w \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s)$.

Case 3: In any other case different from case 1 and case 2, we proceed as follows,

$p \notin F(s) \implies \exists$ a closed set F in $(X, \mathbf{D})$ such that $x \notin F$. Since $(X, \mathbf{D})$ is regular, $\exists \ U, V \in \mathbf{D}$ such that $x \in U$, $F \subset V$, $U \cap V = \phi$. Now consider open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in $(X, \tau < \mathbf{D} >)$, where $U_n = U \ \forall n$, $V_n = V \ \forall n$.

Then $p \in_w U(s)$, $F(s) \subset_w V(s)$, $p \notin_w \overline{V}(s), F(s) \subset X(s) - \overline{U}(s)$.

Combining Case 1, Case 2 and Case 3, $(X, \tau < \mathbf{D} >)$ is regular.

Conversely, suppose $(X, \tau < \mathbf{D} >)$ is regular. Let x be any point in X and F be a closed set in $(X, \mathbf{D})$ such that $x \notin F$. Then

$p = (x, n)$ is a sequential point for any $n \in \mathbb{N}$ and $F(s) = \{F_n\}_{n=1}^{\infty}$, where $F_n = F \ \forall \ n$, is a closed sequential set in $(X, \tau < \mathbf{D} >)$ such that $p \notin F(s)$. Hence $\exists$ open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in $(X, \tau < \mathbf{D} >)$ such that $p \in_w U(s)$, $F(s) \subset_w V(s)$, $p \notin_w \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s)$.
Letting $U = U_n$ and $V = X - \overline{U_n}$, $\overline{U_n}$ being the n^{th} component of $\overline{U}(s)$ we are done. ■

Theorem 4.1.6. *For any topological space $(X, \mathbf{D})$, the generated sequential topological space $(X, \tau < \mathbf{D} >)$ is (w) regular.*

Proof. The proof is omitted. ■

Theorem 4.1.7. *If a space (X, τ) is regular its components $(X, D_n(\tau))$ are regular.*

Proof. The proof is omitted. ■

Remark. Converse of Theorem 4.1.7 may not be true, which is shown by Example 4.1.2.

Example 4.1.2. Let X be nonvoid set and $a \in X$. Then the sets ϕ, $\{a\}$, $X - \{a\}$, X forms a regular topology on X. Consider sequential sets $A^i(s) = \{A_n^i\}_{n=1}^{\infty}$, $B^i(s) = \{B_n^i\}_{n=1}^{\infty}$ and $C^i(s) = \{C_n^i\}_{n=1}^{\infty}$ $(i = 1, 2, 3.........)$, where $A_n^i = B_n^i = C_n^i = X - \{a\} \ \forall \ n \neq i$ and $A_i^i = \{a\}$, $B_i^i = \phi$, $C_i^i = X$. Then the collection S of all sequential sets $A^i(s)$, $B^i(s)$ and $C^i(s)$ $(i = 1, 2, 3......)$ and $\phi(s)$ forms a subbase of a sequential topology say τ on X; (X, τ) is not regular, though the component spaces are regular.

EXAMPLE 4.1.3. Let $\mathbb{R}$ be the set of real numbers. We define a topology $\mathbf{D}$ on $\mathbb{R}$ as follows - for any non zero point in $\mathbb{R}$, the $\mathbf{D}$-nbds are as the usual topology in $\mathbb{R}$. The $\mathbf{D}$- nbds of 0 are of the form $N - A$, where N is a nbd of 0 and $A = \{1, 1/2, 1/3......\}$. Then since $\mathbf{D}$ is finer than the usual topology $\mathbf{U}$ and $(\mathbb{R}, \mathbf{U})$ is $T_{2,}$, $(\mathbb{R}, \mathbf{D})$ is a T_2- space. Hence $(\mathbb{R}, \tau < \mathbf{D} >)$ is T_2-space but it is not regular.

EXAMPLE 4.1.4. Let $X = \{a, b, c\}$ and $\mathbf{D} = \{\phi, \{a\}, \{b, c\}, X\}$. Then $(X, \mathbf{D})$ is a regular topological space but it is not a T_1 space. Hence the sequential topological space $(X, \tau < \mathbf{D} >)$, where $\tau < \mathbf{D} >$ is the sequential topology generated by $\mathbf{D}$, is a regular sequential topological space but it is not $T_{1.}$

4.2. Normality in the Light of Reduced and Augmented Bases

DEFINITION 4.2.1. Two sequential sets $A(s) = \{A_n\}_{n=1}^{\infty}$ and $B(s) = \{B_n\}_{n=1}^{\infty}$ are said to be weakly disjoint if $A_n \cap B_n = \phi$ for some n, where at least one of A_n or B_n is nonvoid.

DEFINITION 4.2.2. A sequential topological space (X, τ) is said to be normal if for any two weakly disjoint closed sequential sets $A(s) = \{A_n\}_{n=1}^{\infty}$ and $B(s) = \{B_n\}_{n=1}^{\infty}$ with bases P and Q respectively i.e $A_m \cap B_m = \phi$ for some $m \in \mathbb{N}$, where either $A_m \neq \phi$ or $B_m \neq \phi$, $\exists$ open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset_w^{P-Q} U(s)$, $B(s) \subset_w V(s)$, $A(s) \subset^{P-Q} X(s) - \overline{V}(s)$, $B(s) \subset X(s) - \overline{U}(s)$, $A_m \subset U_m$, $B_m \subset V_m$

whenever $B(s)$ is a reduced sequential set of $A(s)$; otherwise $\exists$ open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset_w U(s)$, $B(s) \subset_w V(s)$, $A(s) \subset X(s) - \overline{V}(s)$, $B(s) \subset X(s) - \overline{U}(s)$, $A_m \subset U_m$, $B_m \subset V_m$.

Definition 4.2.3. A sequential topological space (X, τ) is said to be weakly normal or (w) normal if for any two weakly disjoint closed sequential sets $A(s) = \{A_n\}_{n=1}^{\infty}$ and $B(s) = \{B_n\}_{n=1}^{\infty}$ with bases P and Q respectively i.e $A_m \cap B_m = \phi$ for some $m \in \mathbb{N}$, where either $A_m \neq \phi$ or $B_m \neq \phi$, $\exists$ open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset_w^{P-Q} U(s)$, $B(s) \subset_w V(s), U(s) \cap V(s) = \phi(s)$, $A_m \subset U_m$, $B_m \subset V_m$ whenever $B(s)$ is a reduced sequential set of $A(s)$; otherwise $\exists$ open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset_w U(s)$, $B(s) \subset_w V(s)$, $U(s) \cap V(s) = \phi(s)$, $A_m \subset U_m$, $B_m \subset V_m$.

Definition 4.2.4. A sequential topological space (X, τ) is said to be a T_4 space if it is normal and T_1.

Remark. A normal sequential topological space may not be T_1 , which is shown by Example 4.2.1.

Example 4.2.1. Consider the Sierpinski space (X, τ), where $X = \{0, 1\}$, $\tau = \{\{0\}, X, \phi\}$. Let $A(s) = \{A_n\}_{n=1}^{\infty}$, where $A_n = \{0\}$ $\forall n \in \mathbb{N}$. let $\tau' = \{\phi(s), A(s), X(s)\}$. Then (X, τ') forms a normal sequential topological space but it is not T_1.

Definition 4.2.5. A sequential topological space (X, τ) is said to be weakly T_4 space or (w) T_4 if it is (w) normal and T_1.

THEOREM 4.2.1. *A sequential topological space* (X, τ) *is normal if and only if for any two weakly disjoint closed sequential sets* $A(s) = \{A_n\}_{n=1}^{\infty}$ *and* $B(s) = \{B_n\}_{n=1}^{\infty}$ *with bases* P *and* Q *respectively i.e* $A_m \cap B_m = \phi$ *for some* $m \in \mathbb{N}$*, where either* $A_m \neq \phi$ *or* $B_m \neq \phi$, $\exists$ *open sequential sets* $G(s) = \{G_n\}_{n=1}^{\infty}$ *and* $H(s) = \{H_n\}_{n=1}^{\infty}$ *in* (X, τ) *such that* $A(s) \subset^{P-Q} G(s)$, $B(s) \subset_w H(s)$, $G(s) \cap H(s) = \phi(s)$, $A_m \subset G_m$, $B_m \subset H_m$ *and* $\exists$ *open sequential sets* $D(s) = \{D_n\}_{n=1}^{\infty}$ *and* $E(s) = \{E_n\}_{n=1}^{\infty}$ *in* (X, τ) *such that* $A(s) \subset_w^{P-Q} D(s)$, $B(s) \subset E(s)$, $D(s) \cap E(s) = \phi(s)$, $A_m \subset D_m$, $B_m \subset E_m$ *whenever* $B(s)$ *is a reduced sequential set of* $A(s)$*; otherwise* $\exists$ *open sequential sets* $G(s) = \{G_n\}_{n=1}^{\infty}$ *and* $H(s) = \{H_n\}_{n=1}^{\infty}$ *in* (X, τ) *such that* $A(s) \subset G(s)$, $B(s) \subset_w H(s)$, $G(s) \cap H(s) = \phi(s)$, $A_m \subset G_m$, $B_m \subset H_m$ *and* $\exists$ *open sequential sets* $D(s) = \{D_n\}_{n=1}^{\infty}$ *and* $E(s) = \{E_n\}_{n=1}^{\infty}$ *in* (X, τ) *such that* $A(s) \subset_w D(s)$, $B(s) \subset E(s)$, $D(s) \cap E(s) = \phi(s)$, $A_m \subset D_m$, $B_m \subset E_m$.

PROOF. Suppose (X, τ) is a normal sequential topological space. Let $A(s) = \{A_n\}_{n=1}^{\infty}$ and $B(s) = \{B_n\}_{n=1}^{\infty}$ be two weakly disjoint closed sequential sets with bases P and Q respectively i.e $A_m \cap B_m = \phi$ for some $m \in \mathbb{N}$, where either $A_m \neq \phi$ or $B_m \neq \phi$. Then $\exists$ open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset_w^{P-Q} U(s)$, $B(s) \subset_w V(s)$, $A(s) \subset^{P-Q} X(s) - \overline{V}(s)$, $B(s) \subset X(s) - \overline{U}(s)$, $A_m \subset U_m$, $B_m \subset V_m$ whenever $B(s)$ is a reduced sequential set of $A(s)$; otherwise $\exists$ open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset_w$

$U(s)$, $B(s) \subset_w V(s)$, $A(s) \subset X(s) - \overline{V}(s)$, $B(s) \subset X(s) - \overline{U}(s)$, $A_m \subset U_m$, $B_m \subset V_m$. If we take $G(s) = X(s) - \overline{V}(s)$, $H(s) = V(s)$, $D(s) = U(s)$, $E(s) = X(s) - \overline{U}(s)$, then we are done.

Conversely, suppose the given conditions are true. Let $A(s) = \{A_n\}_{n=1}^{\infty}$ and $B(s) = \{B_n\}_{n=1}^{\infty}$ be two weakly disjoint closed sequential sets with bases P and Q respectively i.e $A_m \cap B_m = \phi$ for some $m \in \mathbb{N}$, where either $A_m \neq \phi$ or $B_m \neq \phi$. Then $\exists$ open sequential sets $G(s) = \{G_n\}_{n=1}^{\infty}$ and $H(s) = \{H_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset^{P-Q} G(s)$, $B(s) \subset_w H(s)$, $G(s) \cap H(s) = \phi(s)$, $A_m \subset G_m$, $B_m \subset H_m$ and $\exists$ open sequential sets $D(s) = \{D_n\}_{n=1}^{\infty}$ and $E(s) = \{E_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset_w^{P-Q} D(s)$, $B(s) \subset E(s)$, $D(s) \cap E(s) = \phi(s)$, $A_m \subset D_m$, $B_m \subset E_m$ whenever $B(s)$ is a reduced sequential set of $A(s)$; otherwise $\exists$ open sequential sets $G(s) = \{G_n\}_{n=1}^{\infty}$ and $H(s) = \{H_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset G(s)$, $B(s) \subset_w H(s)$, $G(s) \cap H(s) = \phi(s)$, $A_m \subset G_m$, $B_m \subset H_m$ and $\exists$ open sequential sets $D(s) = \{D_n\}_{n=1}^{\infty}$ and $E(s) = \{E_n\}_{n=1}^{\infty}$ in (X, τ) such that $A(s) \subset_w D(s)$, $B(s) \subset E(s)$, $D(s) \cap E(s) = \phi(s)$, $A_m \subset D_m$, $B_m \subset E_m$. If we take $U(s) = G(s) \cap D(s)$ and $V(s) = H(s) \cap E(s)$, then we are done. ■

COROLLARY 4.2.1. *A normal sequential topological space is (w) normal.*

REMARK. Converse of Corollary 4.2.1 is not true. This is shown by Example 4.2.2.

EXAMPLE 4.2.2. Let $X = \mathbb{R}_l^2$. Let $\mathbf{D}$ denotes the product topology on X. Now consider the sequential topology generated by $\mathbf{D}$ on X i.e $\tau < \mathbf{D} >$. Then $(X, \tau < \mathbf{D} >)$ is not normal.
Now let $A(s) = \{A_n\}_{n=1}^{\infty}$ and $B(s) = \{B_n\}_{n=1}^{\infty}$ be two weakly disjoint closed sequential sets with bases P and Q respectively i.e $A_m \cap B_m = \phi$ for some $m \in \mathbb{N}$, where either $A_m \neq \phi$ or $B_m \neq \phi$.
Case 1: Suppose one of $A(s)$ and $B(s)$, say $B(s)$ is a reduced sequential set of $A(s)$ i.e $A_n \subset B_n \; \forall n \in Q \subset P$. Now consider open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in $(X, \tau < \mathbf{D} >)$, where

$$\begin{aligned} U_n &= A_n \; \forall \; n \in P - Q, \\ &= \phi, \text{ otherwise.} \end{aligned}$$

and

$$\begin{aligned} V_n &= B_n \;\; \forall \; n \in Q, \\ &= \phi, \text{ otherwise.} \end{aligned}$$

Then $A(s) \subset_w^{P-Q} U(s)$, $B(s) \subset_w V(s)$, $U(s) \cap V(s) = \phi(s)$, $A_m \subset U_m$, $B_m \subset V_m$.
Case 2: Suppose none of $A(s)$ and $B(s)$ is reduced from the other. Since in $\mathbb{R}_l^2$ every set is clopen, so are A_n and B_n. Now consider open sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ in $(X, \tau < \mathbf{D} >)$, where $U_m = A_m$, $U_n = \phi \; \forall n \neq m$ and $V_m = B_m$, $V_n = \phi \; \forall n \neq m$. Then $A(s) \subset_w U(s)$, $B(s) \subset_w V(s)$, $U(s) \cap V(s) = \phi(s)$, $A_m \subset U_m$, $B_m \subset V_m$. Hence $(X, \tau < \mathbf{D} >)$ is (w) normal but not normal.

REMARK. A normal sequential topological space may not be regular, which is shown by Example 4.2.3.

EXAMPLE 4.2.3. Consider the Sierpinski space (X,τ), where $X = \{0.1\}$, $\tau = \{\{0\}, X, \phi\}$. Let $A(s) = \{A_n\}_{n=1}^{\infty}$, where $A_n = \{0\}$ $\forall n \in N$. Let $\tau' = \{\phi(s), A(s), X(s)\}$. Then (X,τ') forms a normal sequential topological space but it is not regular.

THEOREM 4.2.2. *A normal sequential topological space which is* T_1 *is a regular space i.e a* T_4 *space is a* T_3 *space.*

PROOF. Let (X,τ) be a normal sequential topological space which is T_1. Let $p = (x, P)$ be a sequential point in X and $F(s) = \{F_n\}_{n=1}^{\infty}$ be a closed sequential set in (X,τ) with $p \notin F(s)$. Since p and $F(s)$ are weakly disjoint closed sequential sets in (X,τ), $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X,τ) such that $p \in_w^{P-Q} U(s)$, $F(s) \subset_w V(s)$, $p \in^{P-Q} X(s) - \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s) \Longrightarrow p \in_w^{P-Q} U(s)$, $F(s) \subset_w V(s)$, $p \notin_w^{P-Q} \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s)$, whenever $F(s)$ is a reduced sequential set with base Q of sequential point p; otherwise $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X,τ) such that $p \in_w U(s)$, $F(s) \subset_w V(s)$, $p \in X(s) - \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s) \Longrightarrow p \in_w U(s)$, $F(s) \subset_w V(s)$, $p \notin_w \overline{V}(s)$, $F(s) \subset X(s) - \overline{U}(s)$.
Hence (X,τ) is regular. ■

REMARK. Example 4.2.4 shows that converse of Theorem 4.2.2 is not true.

EXAMPLE 4.2.4. Let $X = \mathbb{R}_l^2$ and let $\mathbf{D}$ denotes the product topology on X. Now consider the sequential topology $\tau < \mathbf{D} >$ generated by $\mathbf{D}$ on X. Then $(X, \tau < \mathbf{D} >)$ is not normal but it is regular.

THEOREM 4.2.3. *If (X, τ) is a regular sequential topological space, where X is finite, then it is (w) normal.*

PROOF. Let (X, τ) be a regular sequential topological space, where X is finite. Let $A(s) = \{A_n\}_{n=1}^{\infty}$ and $B(s) = \{B_n\}_{n=1}^{\infty}$ be two weakly disjoint closed sequential sets in (X, τ) with bases P and Q respectively i.e $A_m \cap B_m = \phi$ for some $m \in \mathbb{N}$, where either $A_m \neq \phi$ *or* $B_m \neq \phi$.
Case 1. Suppose one of $A(s)$ and $B(s)$, say $B(s)$ is a reduced sequential set of $A(s)$ i. e., $A_n \subset B_n$ $\forall n \in Q \subset P$.
Here $A_m \neq \phi$, $B_m = \phi$ and $m \in P - Q$. Let $x \in A_m$. Then $p = (x, m) \notin B(s)$. So $\exists$ open sequential sets $U^x(s)$ and $V^x(s)$ in (X, τ) such that $p \in_w^{P-Q} U^x(s)$, $B(s) \subset_w V^x(s)$, $p \notin_w^{P-Q} \overline{V^x}(s)$, $B(s) \subset X(s) - \overline{U^x}(s)$. Corresponding to each $x \in A_m$, we can find such $U^x(s)$ and since A_m is finite, so $\exists$ finitely many open sequential sets in (X, τ) say $U_1(s)$, $U_2(s)$, $U_3(s)$,........, $U_k(s)$ such that $p \in_w^{P-Q} U_i(s)$, $B(s) \subset X(s) - \overline{U_i}(s)$, $i = 1, 2, 3........k$.
Let $U(s) = \{U_n\}_{n=1}^{\infty} = \cup_{i=1}^{k} U_i(s)$, $V(s) = \{V\}_{n=1}^{\infty} = \cap_{i=1}^{k}(X(s) - \overline{U}_i(s)) = X(s) - \cup_{i=1}^{k}\overline{U}_i(s)$. Then $A(s) \subset_w^{P-Q} U(s)$ and $B(s) \subset V(s)$. Thus $U(s)$, $V(s) \in \tau$ such that $A(s) \subset_w^{P-Q} U(s)$, $B(s) \subset_w V(s)$ and $U(s) \cap V(s) = \phi(s)$, $A_m \subset U_m$, $B_m \subset V_m$.
Case 2. Suppose none of $A(s)$ and $B(s)$ is reduced from the other

and let $A_m \neq \phi$. Let $x \in A_m$. Then $p = (x, m) \notin B(s)$. So $\exists$ open sequential sets $U^x(s)$ and $V^x(s)$ in (X, τ) such that $p \in_w U^x(s)$, $B(s) \subset_w V^x(s)$, $p \notin_w \overline{V^x}(s)$, $B(s) \subset X(s) - \overline{U^x}(s)$. Now proceeding in the same way as in case 1 we can find open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $A(s) \subset_w U(s)$, $B(s) \subset_w V(s)$ and $U(s) \cap V(s) = \phi(s)$, $A_m \subset U_m$, $B_m \subset V_m$.
Hence (X, τ) is (w) normal. ■

Remark. That a regular sequential topological space (X, τ), where X is infinite, may not be (w) normal is shown by Example 4.2.5.

Example 4.2.5. Let $\mathbf{U}$ be the usual topology on $\mathbb{R}$ and let $a \in \mathbb{R}$. For any open set $G \in \mathbf{U}$, *let us consider sequential sets* $A^G(s) = \{A_n^G\}_{n=1}^{\infty}$, $B^G(s) = \{B_n^G\}_{n=1}^{\infty}$, $C^G(s) = \{C_n^G\}_{n=1}^{\infty}$, $D^G(s) = \{D_n^G\}_{n=1}^{\infty}$, where $A_n^G = B_n^G = C_n^G = D_n^G = G \; \forall n \neq 2$ and $A_2^G = \{a\}$, $B_2^G = \mathbb{R} - \{a\}$, $C_2^G = \phi$, $D_2^G = \mathbb{R}$. Then τ_a, the collection of all sequential sets of the form $A^G(s)$, $B^G(s)$, $C^G(s)$, $D^G(s)$ $\forall G \in \mathbf{U}$, forms a regular sequential topology on X. But (X, τ_a) is not (w) normal.

Remark. A (w) normal sequential topological space may not be regular, which is shown by Example 4.2.3.

Theorem 4.2.4. *Let (X, τ) be a normal sequential topological space, then $(X, D_n(\tau))$ is a normal space for each n.*

Proof. Let A and B be two disjoint nonvoid closed sets in $(X, D_n(\tau))$. Then $\exists$ closed sequential sets $A(s) = \{A_n\}_{n=1}^{\infty}$ and

$B(s) = \{B_n\}_{n=1}^{\infty}$ in (X, τ) such that $A_n = A$ and $B_n = B$. So $\exists$ open sequential sets $U(s)$ and $V(s)$ in (X, τ) such that $A(s) \subset_w U(s)$, $B(s) \subset_w V(s)$, $A(s) \subset X(s) - \overline{V}(s)$, $B(s) \subset X(s) - \overline{U}(s)$, $A \subset U_n$, $B \subset V_{n.}$. Taking $U = X - \overline{V_n}$and $V = V_n$, $\overline{V_n}$ being the n^{th} component of $\overline{V}(s)$, we have $A \subset U$, $B \subset V$, $U \cap V = \phi$. Thus $(X, D_n(\tau))$ is normal. ■

Remark. Converse of Theorem 4.2.4 is not true. This is shown by Example 4.2.6.

Example 4.2.6. Let X be a nonvoid set and $a \in X$. Let $\tau = \{\phi, \{a\}, X - \{a\}, X\}$. Then (X, τ) is a normal topological space. Now consider sequential sets $A^i(s) = \{A_n^i\}_{n=1}^{\infty}$, $B^i(s) = \{B_n^i\}_{n=1}^{\infty}$, $C^i(s) = \{C_n^i\}_{n=1}^{\infty}$, $i = 1, 2, 3...$, where $A_n^i = B_n^i = C_n^i = X - \{a\}$ $\forall n \neq i$ and $A_i^i = \{a\}$, $B_i^i = \phi$, $C_i^i = X$. Then S, the collection of all sequential sets $A^i(s)$, $B^i(s)$, $C^i(s)$ $(i = 1, 2, 3....)$ and $\phi(s)$ forms a subbase of a sequential topology say τ' on X . The sequential topological space (X, τ') is not normal though the component spaces are normal.

Chapter V

Generalized Topology, Monotonic Sequential Operators and Generalized Sequential Topology

5.1. Urysohn's Lemma and Tietze's Extension Theorem

The year 2002 brought the raise of generalized topology by *Á*. Cs*á*sz*á*r [16]. He started with any monotonic operators on power set of any nonvoid set into itself and collect all those subsets of the underlying set which are expanded after operation. Then he observed that such collection always contains the void set and is closed under arbitrary union. Considering this into account, he named any family of subsets that behaves like those collections as generalized topology (GT) and studied it. One such collection is the collection of semi-open sets introduced by N. Levine. Clearly every topology is a generalized topology. But a generalized topology may fail to be a topology. In this section we examine separation axioms, continuity, Urysohn's lemma and the extension theorem in the light of generalized topology. Very recent, generalized topology was studied by J. Li, W. K. Min, R. Shen, E. Ekiki, B. Roy, GE Xun, GE Ying, C. Cao, B. Wang, W. Wang etc. [11, 19, 20, 32, 38, 43, 56, 67]. They extend the field by introducing notions of various separation axioms, continuity and a version of Urysohn's lemma in this context but the Tietze's extension theorem is not extended to this field. We observe that a variant of Tietze's extension theorem in normal generalized topological spaces is achievable. At first we give some basic definitions and outline of the results relevant to this extension formally.

Definition 5.1.1. Let $X \neq \phi$. A family g of subsets of X is called a generalized topology (briefly GT) on X if it contains the void set and it is closed under arbitrary union; the ordered pair (X, g) is called a generalized topological space (in brief GTS). Members of a GT g are called g-open set and their complements are called g-closed sets.

Example 5.1.1. Let v be the collection of arbitrary union of members of $\beta = \{(-\infty, s), s \in \mathbb{R}\} \cup \{(t, \infty), t \in \mathbb{R}\}$ then v is a GT on $\mathbb{R}$ but not a topology on it.

In such cases we call v is a GT generated by the basis β.

Definition 5.1.2. As in general, a GTS is called normal if for any pair of disjoint g-closed sets $\exists$ a pair of disjoint g-open sets containing them.

The above GTS $(\mathbb{R}, v)$ shows that the class of normal GTSs is nonvoid. A GTS that does not contain the underlying set is, vacuously, a normal GTS.

Definition 5.1.3. A function $f : X \to \mathbb{R}$ from a GTS (X, g) to the set $\mathbb{R}$ of real numbers is said to be an upper(resp. lower) semi-continuous if for any upper(resp. lower) open ray R in $\mathbb{R}$ with usual ordering $f^{-1}(R) \in g$.

As in general case, an upper(resp. a lower) semi-continuous function may not be lower(resp. upper) semi-continuous but unlike in point set topology an upper as well as lower semi-continuous (upper-lower-semi-continuous) function from a GTS (X, g) to $\mathbb{R}$ with usual topology τ may not be (g, τ)-continuous as shown in the following

EXAMPLE 5.1.2. The identity function $I : (\mathbb{R}, \upsilon) \to (\mathbb{R}, \tau)$, where υ is the generalized topology generated by the base $\beta = \{(-\infty, s); s \in \mathbb{R}\} \cup \{(t, \infty); t \in \mathbb{R}\}$ and τ is the usual topology on $\mathbb{R}$, is upper-lower-semi-continuous but not (υ, τ)-continuous.

Characterization of Normality

THEOREM 5.1.1. *A GTS (X, g) is normal iff for any g-closed set C and any g-open set U containing C there is a g-open set V and a g-closed set V^* such that $C \subset V \subset V^* \subset U$.*

PROOF. Proof is omitted. ■

THEOREM 5.1.2. *A GTS (X, g) is normal iff for any pair of disjoint g-closed sets C and D there is an upper-lower-semi-continuous function $f : X \to [0, 1]$ so that $f(x) = 0 \ \forall \ x \in C$ and $f(x) = 1 \ \forall \ x \in D$.*

PROOF. Let C and D be any disjoint closed set in a normal GTS (X, g), $V_1 = X - D$ and $V_0{}^* = C$. Since the g-closed set $V_0{}^*$ is contained in the g-open set V_1, by using normality there is a

g-open set $V_{\frac{1}{2}}$ and a g-closed set $V_{\frac{1}{2}}{}^*$ so that $V_0{}^* \subset V_{\frac{1}{2}} \subset V_{\frac{1}{2}}{}^* \subset V_1$. Applying the hypothesis on (X, g) to each pair $V_0{}^*$, $V_{\frac{1}{2}}$ and $V_{\frac{1}{2}}{}^*$, V_1, we have g-open sets $V_{\frac{1}{4}}$, $V_{\frac{3}{4}}$ and g-closed sets $V_{\frac{1}{4}}{}^*$, $V_{\frac{3}{4}}{}^*$ so that $V_0{}^* \subset V_{\frac{1}{4}} \subset V_{\frac{1}{4}}{}^* \subset V_{\frac{1}{2}} \subset V_{\frac{1}{2}}{}^* \subset V_{\frac{3}{4}} \subset V_{\frac{3}{4}}{}^* \subset V_1$. Continuing this process one can define g-open sets V_s, V_t and g-closed sets $V_s{}^*$, $V_t{}^*$ for any dyadic rational s and t in $[0, 1]$ of the form $\frac{k}{2^n}$, $k = 1, 2, 3.......2^n - 1$ and $n \in \mathbb{N}$ so that $s < t \Rightarrow V_0{}^* \subset V_s \subset V_s{}^* \subset V_t \subset V_t{}^* \subset V_1$. If s is any other dyadic rational, let $V_s = \phi$ for $s \leq 0$, $V_s = X$ for $s > 1$, and $V_s{}^* = \phi$ for $s < 0$, $V_s{}^* = X$ for $s \geq 1$. Note that V_s for $s > 1$ may not be g-open and so $V_s{}^*$ for $s < 0$ may not be g-closed. Now consider a function $f : X \to [0, 1]$ defined by $f(x) = \inf\{s; x \in V_s\} = \inf\{s; x \in V_s{}^*\}\ \forall\ x \in X$. Then $f(x) = 0$, $\forall\ x \in C$ and $f(x) = 1$, $\forall\ x \in D$. Definition of the function f and construction of the sets V_s and $V_s{}^*$ for dyadic rational number s show that $x \in V_s{}^* \Rightarrow f(x) \leq s$ and $x \notin V_s \Rightarrow f(x) \geq s$. Now for the ray $[0, 1]$ (which is left as well as right open ray in $[0, 1]$) U is an g-open set containing x_0 so that $f(U) \subset [0, 1]$, where $U = V_1$ if $f(x_0) = 0$ and if $f(x_0) = 1$ then $U = X - V_0{}^*$. For any left open ray $[0, d)$ in $[0, 1]$ containing $f(x_0)$ choose a dyadic rational q such that $f(x_0) < q < d$ and consider the V_q. Since $0 < q < 1$, V_q is g-open set containing x_0 because otherwise $f(x_0) \geq q$. Also $f(V_q) \subset [0, d)$, since $x \in V_q \Rightarrow f(x) \leq q$. And for any right open ray $(c, 1]$ in $[0, 1]$ containing $f(x_0)$ select a dyadic rational p such that $c < p < f(x_0)$ then, $X - V_p{}^*$ is a g-open set containing x_0 so that $f(X - V_p{}^*) \subset (c, 1]$. Converse is trivial so we exclude it. ■

Some Other Properties of Normal Generalized Topological Spaces

THEOREM 5.1.3. *Inverse image of closed set C in $\mathbb{R}$ with usual topology under upper-lower-semi-continuous function on a GTS (X, g) is g-closed if C is of the form $(-\infty, a]$ or $[a, \infty)$ or $[a, b]$ or $\{a\}$, where $a, b \in \mathbb{R}$*

Sum and product of upper-lower-semi-continuous functions on GTSs need not be upper-lower-semi-continuous as seen in the following

EXAMPLE 5.1.3. Let us consider $\mathbb{R}$ with generalized topology v generated by the basis $\beta = \{(-\infty, s); s \in \mathbb{R}\} \cup \{(t, \infty); t \in \mathbb{R}\}$. Then the functions $f_i : (\mathbb{R}, v) \to \mathbb{R}, i = 1, 2$ defined by

$$f_1(x) = x \ \forall \ x \in \mathbb{R} \text{ and}$$

$$\begin{aligned} f_2(x) &= 0 \ \forall \ x \geq 0 \\ &= -2x \ \forall \ x < 0 \end{aligned}$$

both are upper-lower-semi-continuous but none of

$$(f_1 + f_2)(x) = |x| \ \ \forall \ x \in \mathbb{R} \text{ and}$$

$$(f_1.f_1)(x) = f_1(x).f_1(x) = x^2 \ \forall \ x \in \mathbb{R}$$

is so.

DEFINITION 5.1.4. Let (X, g) be a GTS; a function $f : (X, g) \to \mathbb{R}$ is called weak $M(g)$ upper (resp. lower) semi-continuous if for any $x \in X$ and any left (resp. right) open ray R containing $f(x)$ $\exists \ U_i \in g, i = 1, 2, 3, ...n$ such that $x \in \bigcap_{i=1}^{n} U_i$ and $f(\bigcap_{i=1}^{n} U_i) \subset R$. $f : (X, g) \to \mathbb{R}$ is said to be weak $M(g)$ upper-lower-semi-continuous if it is weak $M(g)$ upper as well as weak $M(g)$ lower semi-continuous.

Obviously in the above definition replacement of GT g by a topology reduces the weak $M(g)$ upper-lower-semi-continuity to continuity. It is also straightforward that upper (resp., lower) semi-continuity implies weak $M(g)$ upper (resp., lower) semi-continuity but the converse is not necessarily true as seen in the Example 5.1.3.

THEOREM 5.1.4. *Finite sum of upper-lower-semi-continuous functions on a GTS (X, g) is weak $M(g)$ upper-lower-semi-continuous.*

PROOF. Let f_1 and f_2 be upper-lower-semi-continuous functions on a GTS (X, g) and f be their sum. For any right open ray (r, ∞) in $\mathbb{R}$ such that $f(x) \in (r, \infty)$ choose real numbers $r_1, r_2 > 0$ so that $r < f_1(x) - r_1 + f_2(x) - r_2$. Since f_1 and f_2 are lower-semi-continuous, $U = f_1^{-1}(f_1(x) - r_1, \infty)$ and $V = f_2^{-1}(f_2(x) - r_2, \infty)$ both are g-open sets containing x. Now $f_1(U) \subset (f_1(x) - r_1, \infty) \Rightarrow f_1(U \cap V) \subset (f_1(x) - r_1, \infty)$ and similarly $f_2(U \cap V) \subset (f_2(x) - r_2, \infty)$. Hence $f(U \cap V) \subset (f_1(x) - r_1 + f_2(x) - r_2, \infty)$ and so f is weak $M(g)$ lower-semi-continuous. Arguing in the same way it can be shown that f is weak $M(g)$ upper-semi-continuous too. ∎

THEOREM 5.1.5. *Finite product of weak $M(g)$ upper-lower-semi-continuous functions on a GTS (X, g) is weak $M(g)$ upper-lower-semi-continuous.*

PROOF. Proof is omitted. ∎

Now it is straightforward that finite product of functions on a GTS (X, g) some of which are upper-lower-semi-continuous and others are weak $M(g)$ upper-lower-semi-continuous is weak $M(g)$ upper-lower-semi-continuous.

THEOREM 5.1.6. *Let* $\{f_n\}$ *be a sequence of upper-lower-semi-continuous functions on a GTS* (X, g) *so that* $|f_n(x)| \leq M_n$ *for each* $n \in \mathbb{N}$, *where* $\sum_{n=1}^{\infty} M_n$ *is a convergent series of real numbers. Then* $f(x) = \sum_{n=1}^{\infty} f_n(x)$ *exists and is a weak* $M(g)$ *upper-lower-semi-continuous function* $f : (X, g) \to \mathbb{R}$.

PROOF. It can be shown that, as in general case, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ exists $\forall x \in X$. For the second part, let $x_0 \in X, \epsilon > 0$ then $|f(x) - s_{n_o}(x)| < \frac{\epsilon}{3} \forall x \in X$, where n_0 is a positive integer as large as required and in particular $|s_{n_o}(x_0) - f(x_0)| < \frac{\epsilon}{3}$. Now $s_{n_0} : (X, g) \to \mathbb{R}$ is weak $M(g)$ upper-lower-semi-continuous therefore for $(s_{n_0}(x_0) - \frac{\epsilon}{3}, \infty) \exists U_i \in g, i = 1, 2, 3, ...p$ (say) such that $s_{n_0}(\bigcap_{i=1}^{p} U_i) \subset (s_{n_0}(x_0) - \frac{\epsilon}{3}, \infty) \Rightarrow s_{n_0}(x) \in (s_{n_0}(x_0) - \frac{\epsilon}{3}, \infty) \forall x \in \bigcap_{i=1}^{p} U_i$. Thus for those $x \in \bigcap_{i=1}^{p} U_i$ $s_{n_0}(x_0) - \frac{\epsilon}{3} < s_{n_0}(x)$, $(s_{n_0}(x) - \frac{\epsilon}{3} < f(x)$ and $f(x_0) - \frac{\epsilon}{3} < s_{n_0}(x_0)$; adding all these three inequalities we have $f(x_0) - \epsilon < f(x)$ and so f is weak $M(g)$ lower-semi-continuous. Similarly one can show that f is weak $M(g)$ upper-semi-continuous.

■

THEOREM 5.1.7. *If* (X, g) *is a normal GTS then for every g-closed set* $C \subset X$, *each upper-lower-semi-continuous function* $f : (C, g|C) \to [-m, m]$ *has a weak* $M(g)$ *upper-lower-semi-continuous extension* $F : (X, g) \to [-m, m]$, *where* m *is any positive integer.*

Proof. First let $|f(c)| \leq m$ on C and construct a sequence $\{r_n\}$ of real numbers, where $r_n = \frac{m}{2}(\frac{2}{3})^n$, $n \in \mathbb{N}$. So, $2r_n = 3r_n$ $\forall\ n \in \mathbb{N}$; in particular $m = 3r_1 \therefore |f(c)| \leq 3r_1$ on C. Let $A_1 = \{c \in C; f(c) \leq -r_1\}$ and $B_1 = \{c \in C; f(c) \geq r_1\}$. Then $A_1 \cap B_1 = \phi$ and A_1, B_1 are $g|C$-closed; since C is g-closed, A_1 and B_1 are also g-closed [18]. Therefore $\exists$ an upper-lower-semi-continuous function $g_1 : (X, g) \to [-r_1, r_1]$ such that $g_1(A_1) = -r_1$ and $g_1(B_1) = r_1$ by virtu of Theorem 3.3 of [18]. Rename f by f_1 and set $f_2 = f_1 - g_1|C$, where $g_1|C$ is the restriction of g_1 on C. Then $|f_2(c)| \leq 2r_1 = 3r_2$ on C. Set $A_2 = \{c \in C; f_2(c) \leq -r_2\}$ and $B_2 = \{c \in C; f_2(c) \geq r_2\}$ then for the same reason $\exists$ an upper-lower-semi-continuous function $g_2 : (X, g) \to [-r_2, r_2]$ such that $g_2(A_2) = -r_2$ and $g_2(B_2) = r_2$. Set $f_3 = f_2 - g_2|C$ and continue this process indefinitely. These result two sequences $\{f_n\}$ and $\{g_n\}$ such that $f_{n+1} = f_n - g_n|C|$, $|f_n(c)| \leq 3r_n$ on C and $|g_n(x)| \leq r_n$ on X $\forall\ n \in \mathbb{N}$. Since $\sum\limits_{n=1}^{\infty} r_n = \frac{m}{2}\sum\limits_{n=1}^{\infty}(\frac{2}{3})^n < \infty$, by Theorem 5.1.6 and Theorem 5.1.4 it follows that $F : (X, g) \to \mathbb{R}$ defined by $F(x) = \sum\limits_{n=1}^{\infty} g_n(x)$ is weak $M(g)$ upper-lower-semi-continuous on X. Let $c \in C$. Then $F(c) = \sum\limits_{n=1}^{\infty} g_n(c) = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} g_i(c) = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}[f_i(c) - f_{i+1}(c)] = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}[f_1(c) - f_{n+1}(c)] = f_1(c) = f(c)$. Thus F is an extension of f. Also $|F(x)| \leq \frac{m}{2}\sum\limits_{n=1}^{\infty}(\frac{2}{3})^n = m$. ■

We conclude this section with the following

Problem : Semiopen sets in a topological space form a generalized topology. Does there exist a topological space so that semiopen sets therein forms a given generalized topology on the underlying set?

5.2. Monotonic Sequential Operators and Generalized Sequential Topology

In the last few decades, closure operators, interior operators and various compositions of them have been studied what produced different forms of open and closed sets, namely, $\hat{g}$-closed sets, $^{*}g$-closed sets, $^{\sharp}g$-semi-closed sets and $\tilde{g}$-closed sets [9], semi-open sets [36] (called $\beta-$sets in [47]), regular open sets [45], $\alpha-$open sets (called $\alpha-$sets in [47]), feebly open sets [39], preopen sets [41], $\beta-$ open sets [1] (called semi-preopen sets in [24]), locally closed sets [10], $\mathcal{A}-$sets [64] (called $A-$sets in [45]), $\mathcal{B}-$sets [65], δ-open sets and θ-open sets [66], δ-preopen sets [55], δ-semiopen sets [50], b-open sets [3], semi-θ-open sets [23], g-open sets [37], gs-open sets [4], gp-open sets and αg-open sets [51], δ-g-open sets [26], gb-open sets [33], θ-g-open sets [27], $g\delta s$-open sets [52], gsp-open sets [25], $g\delta p$-open sets [31], θgs-open sets [46], $g\gamma$-open sets [29], αg-open sets [40], $g\mu$-open sets [48] etc.. A common generalization of these classes has been the concept of γ-open sets introduced by *Á*. Cs*á*sz*á*r [14] which form generalized topology [16, 17, 19]. The key idea behind such generalization was the use of mappings $\gamma : \exp X \to \exp X$ from the power set $\exp X$ of the underlying set X into itself, possessing the property of monotony [14, 15, 16, 17, 19]. The purpose of the present work is to change the domain of thoughts and to exhibit the relations between *Á*. Cs*á*sz*á*r's and ours results [60]. A significant contribution to the theory of $K\Omega$ and relative closure operators in $(P(X))^{\mathbb{N}}$ and their components has been made in [57], where $P(X)$ is the power set of the underlying set X (denoted by $\exp X$ in [14]). Here we introduce different monotonic operators $M : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ and their components, investigate the relations among themselves and study some of their properties and observe what such operators produce.

Monotonic sequential operators and their dual

We consider the operators $M_i; i = 1, 2, 3, 4, 5, 6, 7$, and some properties enjoyed by them.

(1) $M_1 : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $M_1(A(s)) = A(s) \cup F(s)$, $\forall\ A(s) \in (P(X))^{\mathbb{N}}$, where $F(s)$ $(\neq \Phi(s))$ is a fixed sequential set in X.

M_1 is expansive; commute with finite union and it is idempotent [57].

(2) $M_2 : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $M_2(A(s)) = A(s) \cap F(s)$, $\forall\ A(s) \in (P(X))^{\mathbb{N}}$, where $F(s)$ $(\neq \Phi(s))$ is a fixed sequential set in X.

M_2 is contractive; commute with finite intersection and it is idempotent.

(3) $M_3 : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $M_3(A(s)) = \{\bigcup_{i=1}^{n} A_i\}$, $\forall$ $A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$.

M_3 is expansive; commute with finite union; $\Phi(s)$ is a fixed sequential set of M_3; and it is idempotent.

(4) $M_4 : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $M_4(A(s)) = \{\overset{n}{\underset{i=1}{\cap}} A_i\}$, $\forall$ $A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$.

M_4 is contractive; commute with finite intersection, $X(s)$ is a fixed sequential set of M_4; and it is idempotent.

(5) $M_5 : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $M_5(A(s)) = \{A_n \cup A_{n+1}\}$, $\forall$ $A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$.

M_5 is expansive; commute with finite union; $\Phi(s)$ is a fixed sequential set of M_5 but it is not idempotent.

(6) $M_6 : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $M_6(A(s)) = \{A_n \cap A_{n+1}\}$, $\forall$ $A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$.

M_6 is contractive, commute with finite intersection, $X(s)$ is a fixed sequential set of M_6 but it is not idempotent.

(7) $M_7 : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $M_7(A(s)) = B(s) = \{B_n\}$ $\forall A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$.

$$\begin{aligned} B_n &= \{A_n \cup A_{n+1}\}, \text{ when } n \text{ is odd,} \\ &= \{A_n \cap A_{n+1}\}, \text{ when } n \text{ is even.} \end{aligned}$$

It is neither expansive nor contractive.

Article [57] mainly deals with those operators (called $K\Omega-$closure operator) which satisfy the four properties of M_3 as mentioned above. The collection τ_C of all fixed sequential sets of any $K\Omega$-closure operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ forms a sequential topology on X called the sequential topology [6, 62] generated by $K\Omega$-closure operator $C : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$. Evidently different operator M_i, ($i = 1, 2, 3, 4, 5, 6$ and 7) possesses different class of properties, but interestingly, all these operators have monotonicity

$(A(s) \subset B(s) \Rightarrow M_i(A(s)) \subset M_i(B(s)))$ which is not listed above but can be obtained by simple derivation. Our motivation rests on the use of monotonicity of operators from $(P(X))^{\mathbb{N}}$ to $(P(X))^{\mathbb{N}}$ as the key to unlock the door of a new truth. Let $S((P(X))^{\mathbb{N}})$ be the collection of all monotonic sequential operators $M : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$; we write S if the omission of "$(P(X))^{\mathbb{N}}$" does not create a confusion. Here domain of the operator $M : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is the denumerable cartesian product of $P(X)$ with itself; $P(X)$ is called factor of the domain of M. If τ is a sequential topology on X then Cl_τ, $Int_\tau \in S((P(X))^{\mathbb{N}})$ we rename them as C and I respectively. An operator $M : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is said to satisfy the property $p = \Phi$, X, E, C, I, $\cup$, $\cap$ and C_2 if $\Phi(s)$ is a fixed sequential set of M, $X(s)$ is a fixed sequential set of M, M is expansive, M is contractive, M is idempotent, M commutes with finite union, M commutes with finite intersection and $M(M(A(s))) \subset M(A(s))$ $\forall\ A(s) \in (P(X))^{\mathbb{N}}$ resp.. By S_p we mean the subcollection of S, where the operators $M : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ satisfy the property p and for the sake of simplicity we write $S_{pq} = S_p \cap S_q$. The operators M_3, M_4, M_5, M_6, M_7 ensure that S_p for all those p mentioned above are nonvoid. The operator $M_n^{off} : P(X) \to P(X)$, associated with $M \in S((P(X))^{\mathbb{N}})$, defined by $M_n^{off}(A) = P_n(M({}_n\Phi_A(s)))$ (i. e., n^{th} component of $M({}_n\Phi_A(s))$) $\forall\ A \in P(X)$ is called the n^{th} off-component of M and the operator $M_n^{on} : P(X) \to P(X)$, associated with $M \in S((P(X))^{\mathbb{N}})$, defined by $M_n^{on}(A) = P_n(M({}_nX_A(s)))$ $\forall\ A \in P(X)$ is called the n^{th} on-component of M, we call $M_n^{off}(A)$

is the null action of M on A at n (without interference of all but n^{th} factor of the domain) and $M_n^{on}(A)$ is the full action of M on A at n (in the full interference of all factors of the domain). Obviously $M \in S \Rightarrow M_n^{on},\ M_n^{off} \in \Gamma$ [14], $M \in S_p$, $p = \Phi,\ X,\ E,\ C,\ I,\ C_2 \Rightarrow M_n^{on},\ M_n^{off} \in \Gamma_n$, $n = 0,\ 1,\ +,\ -,$ $2,\ -2$ resp. [14] and $M \in S_p$, $p = \cup, \cap \Rightarrow M_n^{on},\ M_n^{off} \in \Gamma_n$, $n = \cup, \cap$ resp., where $\Gamma_\cup$ and $\Gamma_\cap$ bear their similar meanings that $S_\cup$ and $S_\cap$ do mean resp.. We denote the generalized topologies (GTs) formed by $M_n^{on}$and $M_n^{off}-$open sets [16] by $g_{M_n^{on}}$ and $g_{M_n^{off}}$ resp.. Let $M \in S((P(X))^{\mathbb{N}})$, a sequential set $A(s)$ in X is called $M-$sequential open if $A(s) \subset M(A(s))$. $A(s)$ is $M-$sequential open $\Rightarrow A(s) \subset M(A(s)) \subset M({}_nX_{A_n}),\ \forall\ n \in \mathbb{N} \Rightarrow A_n \subset M_n^{on}(A_n)$, $\forall\ n \in \mathbb{N} \Rightarrow A_n$ is $M_n^{on}-$open, $n \in \mathbb{N}$; but if we let $X = \{a,\ b,$ $c\}$ in the definition of M_7 then $A(s) = \{A_n\}$, where $A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{a,\ b\}$ and all others A_n are Φ is M_7-sequential open, whereas A_2 is not $(M_7)_2^{off}-$open, this forces to conclude that n^{th} component of an $M-$sequential open set need not be $M_n^{off}-$open, $n \in \mathbb{N}$. If $M \in S_C$ then a sequential set $A(s)$ is $M-$sequential open iff $A(s) = M(A(s))$ and if $M \in S_E$ then every sequential set is $M-$sequential open and hence each component of $M-$sequential open set is $M_n^{on}-$ as well as $M_n^{off}-$open because for any $A \subset X$, ${}_n\Phi_A(s) \subset M({}_n\Phi_A(s))$ and this implies $A \subset M_n^{off}(A)$. Let $G_{\tau M}$ be the collection of all $M-$sequential open sets in X. Clearly $\Phi(s) \in G_{\tau M}$ and $A_\lambda(s) \in G_{\tau M},\ \lambda \in \Lambda \Rightarrow \forall\ \lambda \in \Lambda,\ A_\lambda(s) \subset M(A_\lambda(s)) \subset M(\underset{\lambda\in\Lambda}{\cup} A_\lambda(s)) \Rightarrow \underset{\lambda\in\Lambda}{\cup} A_\lambda(s) \subset$

$M(\underset{\lambda\in\Lambda}{\cup} A_\lambda(s)) \Rightarrow \underset{\lambda\in\Lambda}{\cup} A_\lambda(s) \in G_{\tau M}$ i. e., $G_{\tau M}$ is closed under arbitrary union. If we agree in saying that a sub-collection G_τ of $(P(X))^{\mathbb{N}}$ is a generalized sequential topology (GST) on X if $\Phi(s)$ is a member of G_τ and G_τ is closed under arbitrary union we can say that any $M \in S((P(X))^{\mathbb{N}})$ generates a GST $G_{\tau M}$ on X. Like sequential topology, n^{th} components, $n \in \mathbb{N}$, of members of G_τ form GT on X called n^{th} component generalized topology of G_τ and it is denoted by $D_n(G_\tau)$. Thus for any $M \in S((P(X))^{\mathbb{N}})$ and $n \in \mathbb{N}$ we have three GTs on X in hand namely, $D_n(G_{\tau M})$, $g_{M_n^{on}}$ and $g_{M_n^{off}}$. Since ${}_n\Phi_A(s) \subset_n X_A(s)\ \forall\ A \in P(X),\ n \in \mathbb{N},\ g_{M_n^{off}} \subset g_{M_n^{on}}\ \forall\ n \in \mathbb{N}$. Also, $A \in D_n(G_{\tau M}) \Rightarrow \exists$ a sequential set $A(s) \in G_{\tau M}$ whose n^{th} component is A, but then $A(s) \subset_n X_A(s)$ which implies $A(s) \subset M(A(s)) \subset M({}_nX_A(s)) \Rightarrow A \subset M_n^{on}(A) \Rightarrow A \in g_{M_n^{on}}$ and hence $D_n(G_{\tau M}) \subset g_{M_n^{on}}$. Now consider the operator in the

Example 5.2.1. $M_8 : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ defined by $M_8(A(s)) = B(s) = \{B_n\}$, where $X = \{a, b, c\}$ and

$$B_n = \left(\bigcup_{i=1}^{n} A_i\right) \cap \{b\}, \text{ when } n \text{ is odd,}$$
$$= \left(\bigcap_{i=1}^{n} A_i\right) \cup \{c\}, \text{ when } n \text{ is even.}$$

The definition of the operator M_8 shows that $\{a\} \in g_{(M_8)_n^{on}}\ \forall$ even $n \in \mathbb{N}$. We claim that $\{A_n\} = A(s) \subset M_8(A(s)) \Rightarrow$no A_n is $\{a\}$. $A_n \neq \{a\}$ for all odd $n \in \mathbb{N}$ is obvious , if possible let $A_n = \{a\}$ for some even $n = e$, then since $A(s) \subset M(A(s))$, $a \in A_n\ \forall\ n \leq e$ but then $A_n \nsubseteq P_n(M(A(s)))\ \forall$ odd $n < e$ which is a contradiction. Thus if $A(s) \in G_{\tau M_8}$ then $A_n \neq \{a\}$

$\forall$ even $n \in \mathbb{N} \Rightarrow \{a\} \notin D_n(G_{\tau M_8})$ $\forall$ even $n \in \mathbb{N}$ and therefore $g_{(M_8)_n^{on}} \nsubseteq D_n(G_{\tau M_8})$ $\forall$ even $n \in \mathbb{N}$. Again $A \in g_{M_n^{off}} \Rightarrow A \subset M_n^{off}(A) \Rightarrow_n \Phi_A(s) \subset M({}_n\Phi_A(s)) \Rightarrow_n \Phi_A(s) \in G_{\tau M} \Rightarrow A \in D_n(G_{\tau M})$ and so $g_{M_n^{off}} \subset D_n(G_{\tau M})$; the operator M_7 shows that $g_{M_n^{off}} \neq D_n(G_{\tau M})$ in general. Now like [16] let G_τ be any GST on X and $M : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ be defined by $M(A(s)) = \cup\{O(s);\ O(s) \in G_\tau$ and $O(s) \subset A(s)\}$ then $M \in S((P(X))^{\mathbb{N}})$, $M(A(s)) \subset A(s)$ and hence any $M-$sequential open set in X belongs to G_τ. Also $O(s) \in G_{\tau M} \Rightarrow M(O(s)) = O(s) \supset O(s) \Rightarrow O(s)$ is an $M-$sequential open set. Therefore any GST on a set X can be rediscovered by a monotonic sequential operator on $(P(X))^{\mathbb{N}}$. Associate with any $M \in S((P(X))^{\mathbb{N}})$ there is a sequential operator $I_M : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$, called $M-$interior sequential operator, defined by $I_M(A(s))$ = union of all $M-$ sequential open sets contained in $A(s)$; obviously $I_M(A(s)) \subset A(s)$ which implies $I_M(\Phi(s)) = \Phi(s)$, $A(s) \in G_{\tau M} \Rightarrow I_M(A(s)) = A(s)$, $I_M(A(s)) \in G_{\tau M}$ $\forall$ $A(s) \in (P(X))^{\mathbb{N}}$ which implies $I_M(I_M(A(s))) = I_M(A(s))$ $\forall$ $A(s) \in (P(X))^{\mathbb{N}}$, $(B(s) \in G_{\tau M}$ and $B(s) \subset A(s)) \Rightarrow B(s) \subset I_M(A(s))$ and $A(s) \subset B(s) \Rightarrow I_M(A(s)) \subset A(s) \subset B(s) \Rightarrow I_M(A(s)) \subset I_M(B(s))$, i. e., $I_M \in S_{CI\Phi}$. Also $M \in S_{CI\Phi} \Rightarrow M(A(s)) \in G_{\tau M}$ and $M(A(s)) \subset A(s)$ $\forall$ $A(s) \in (P(X))^{\mathbb{N}}$, therefore for any $B(s) \in G_{\tau M}$ with $B(s) \subset A(s)$ $B(s) \subset M(B(s)) \subset M(A(s)) \subset A(s) \Rightarrow I_M(A(s)) = M(A(s))$ $\forall$ $A(s) \in (P(X))^{\mathbb{N}} \Rightarrow I_M = M$ and hence $I_{I_M} = I_M$ for any $M \in S((P(X))^{\mathbb{N}})$. It is also observed that $I_M \in S_X$ iff $M \in S_X$ and a sequential set

$A(s)$ is $M-$sequential open iff $A(s) = I_M(A(s))$ which happens iff $A(s)$ is I_M-open. For any $A \in P(X)$, $i_{M_n^{off}}(A) = \cup\{B;\ B \subset A,\ B \subset M_n^{off}(B)\} = \cup\{n^{\text{th}}$ component of ${}_n\Phi_B(s)$; n^{th} component of ${}_n\Phi_B(s) \subset n^{\text{th}}$ component of ${}_n\Phi_A(s)$, n^{th} component of ${}_n\Phi_B(s) \subset n^{\text{th}}$ component of $M({}_n\Phi_B(s))\} = n^{\text{th}}$ component of $\cup\{{}_n\Phi_B(s);\ n^{\text{th}}$ component of ${}_n\Phi_B(s) \subset n^{\text{th}}$ component of ${}_n\Phi_A(s)$, n^{th} component of ${}_n\Phi_B(s) \subset n^{\text{th}}$ component of $M({}_n\Phi_B(s))\} = n^{\text{th}}$ component of $\cup\{{}_n\Phi_B(s);\ {}_n\Phi_B(s) \subset_n \Phi_A(s),\ {}_n\Phi_B(s) \subset M({}_n\Phi_B(s))\} = n^{\text{th}}$ component of $I_M({}_n\Phi_A(s))$ since any sequential set $B(s)$ that contained in ${}_n\Phi_A(s)$ is of the form ${}_n\Phi_B(s)$ for some $B \subset X$ and therefore $(I_M)_n^{off} = i_{M_n^{off}} = i_{(I_M)_n^{off}}$, where $i_{M_n^{off}}$ and $i_{(I_M)_n^{off}}$ are the M_n^{off} and $(I_M)_n^{off}-$interior of A [14], but $(I_M)_n^{on} \neq i_{M_n^{on}}$ for, see the Example 5.2.1, where $i_{M_n^{on}}(\{a\}) = \{a\}$ but $(I_M)_n^{on}(\{a\}) = \Phi$. It can be shown that the interior operators in the generalized topological spaces $(X\ ,\ g_{M_n^{on}})$, $(X\ ,\ D_n(G_{\tau M}))$ and $(X\ ,\ g_{M_n^{off}})$ are $i_{M_n^{on}}$, $(I_M)_n^{on}$ and $(I_M)_n^{off}$ resp.. Now $A(s) \in G_{\tau I_M} \Leftrightarrow A(s) \subset I_M(A(s)) \subset A(s) \Leftrightarrow I_M(A(s)) = A(s) \Leftrightarrow A(s) \in G_{\tau M}$ consequently $G_{\tau I_M} = G_{\tau M}$ and $D_n(G_{\tau I_M}) = D_n(G_{\tau M})\ \forall\ n \in \mathbb{N}$. Also $(I_M)_n^{off} = i_{M_n^{off}} \Rightarrow g_{M_n^{off}} = g_{(I_M)_n^{off}}\ \forall\ n \in \mathbb{N}$ but Example 5.2.1 shows that $g_{M_n^{on}}$ may not be $g_{(I_M)_n^{on}}\ \forall\ n \in \mathbb{N}$. If the change of the components of sequential set by some sequential operator $M \in S((P(X))^{\mathbb{N}})$ do not depend on the other components then $M(A(s)) = \{M_n^{off}(A_n)\} = \{M_n^{on}(A_n)\}$ $\forall\ A(s) = \{A_n\} \in (P(X))^{\mathbb{N}} \Rightarrow g_{M_n^{on}} = D_n(G_{\tau M}) = g_{M_n^{off}}\ \forall\ n \in \mathbb{N}$. As in [14] we say $A(s) \in (P(X))^{\mathbb{N}}$ is $M-$sequential closed for some $M \in S((P(X))^{\mathbb{N}})$ if $X(s) - A(s)$ is $M-$sequential open.

By $C_M(A(s))$ we mean the intersection of all $M-$sequential closed sets containing $A(s)$ called $M-$sequential closure of $A(s)$. It is straightforward that $(G_{\tau M})' = \{X(s) - A(s);\ A(s) \in G_{\tau M}\}$ is exactly the collection of all $M-$sequential closed sets. Some properties of $M-$sequential closed sets and $M-$sequential closure C_M are directly fallow from the last sentence which are enlisted in the Theorem 5.2.1. Considering the equation, what maintains the relation between interior and closure operators in point set topological space, as a model we define an operator $M^o : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ by $M^o(A(s)) = X(s) - M(X(s) - A(s))\ \forall\ A(s) \in (P(X))^{\mathbb{N}}$ associate with each $M \in S((P(X))^{\mathbb{N}})$ and call it the dual operator of M. For any $M \in S$, $M^o \in S$ follows from the fact that $A(s) \subset B(s) \Rightarrow X(s) - B(s) \subset X(s) - A(s) \Rightarrow M(X(s) - B(s)) \subset M(X(s) - A(s)) \Rightarrow X(s) - M(X(s) - A(s)) \subset X(s) - M(X(s) - A(s))$ and also $(M^o)^o(A(s)) = X(s) - M^o(X(s) - A(s)) \Rightarrow (M^o)^o(A(s)) = X(s) - (X(s) - M(A(s)) = M(A(s)) \Rightarrow (M^o)^o = M$. $M \in S_p$, $p = \Phi, X, E, I, C, \cup$ and $\cap \Leftrightarrow M^o \in S_q$, $q = X, \Phi, C, I, E, \cap$ and $\cup$ resp., also $(C_M)^o = I_M$, the proof is as simple as that in [14]. Now consider the operator M_7 with $X = \mathbb{R}$ and choose $A(s) = \{A_n\}$ in $\mathbb{R}$ in such a way that $A_{n+1} \subset A_n$ for odd n and otherwise A_{n+1}, A_n are incomparable; this sequential set is neither M_7-sequential open nor M_7-sequential closed but it is $(M_7)^\circ-$sequential closed set. Actually a sequential set $A(s)$ is $M^\circ-$sequential closed $\Leftrightarrow X(s) - A(s) \subset M^\circ(X(s) - A(s)) = X(s) - M(A(s)) \Leftrightarrow M(A(s)) \subset A(s)$. Since for each $M \in S$, $M^o \in S$, all the results for M

extracted from the above discussion are true for M^o too. Now $C_{I_M}(A(s)) = \cap\{B(s);\ B(s) \supset A(s),\ X(s) - B(s)$ is I_M-sequential open$\} = X(s) - \cup\{X(s) - B(s);\ X(s) - B(s) \subset X(s) - A(s)$, $X(s) - B(s)$ is I_M-sequential open$\} = X(s) - I_{I_M}(X(s) - A(s)) = (I_M)^\circ(A(s)) = C_M(A(s))\ \forall\ A(s) \in (P(X))^{\mathbb{N}} \Rightarrow C_{I_M} = C_M$ and it is straightforward that $I_{C_M} = E =$ the identity operator on $(P(X))^{\mathbb{N}} = E^\circ = (I_{C_M})^\circ = C_{C_M}$. Since $c_{M_n^{off}} = (i_{M_n^{off}})^\circ$, $c_{(I_M)_n^{off}} = (i_{(I_M)_n^{off}})^\circ$ [14] and $i_{M_n^{off}} = i_{(I_M)_n^{off}}$, $c_{M_n^{off}} = c_{(I_M)_n^{off}}$ but $c_{M_n^{off}} \neq c_{(I_M)_n^{off}}$ one of the witnesses for this is the Example 5.2.1. For any $A \in P(X)$, $(M^\circ)_n^{off}(A) = P_n(X(s) - M(X(s) -_n \Phi_A(s))) = X - P_n(M({}_nX_{(X-A)}(s))) = X - M_n^{on}(X - A) = (M_n^{on})^*(A) \Rightarrow (M_n^{on})^* = (M^\circ)_n^{off}$ and similarly $(M_n^{off})^* = (M^\circ)_n^{on}$. All other results related to dual operators listed in the Theorem 5.2.1 can be proved using these last results. Now for $M \in S((P(X))^{\mathbb{N}})$ define $\gamma : P(X) \to P(X)$ by $\gamma(A) = \overset{\infty}{\underset{n=1}{\cup}} P_n(I_M(\{A\}^{\mathbb{N}}))$, obviously $\gamma \in \Gamma(X)$. The γ defined in this way defines the generalized topology $g_\gamma = g_{\tau M} = \{A;\ A = \overset{\infty}{\underset{n=1}{\cup}} A_n$, where $A(s) = \{A_n\} \in G_{\tau M}\}$ on X for, let $A \in g_\gamma$ then $A \subset \gamma(A) \subset A \Rightarrow A = \gamma(A) = \overset{\infty}{\underset{n=1}{\cup}} P_n(I_M(\{A\}^{\mathbb{N}}))$ but $I_M(\{A\}^{\mathbb{N}}) \in G_{\tau M}$ therefore $A \in g_{\tau M}$ and $B \in g_{\tau M} \Rightarrow \exists\ B(s) = \{B_n\} = I_M(B(s))$ such that $B = \overset{\infty}{\underset{n=1}{\cup}} B_n = \overset{\infty}{\underset{n=1}{\cup}} P_n(I_M(B(s))) \subset \overset{\infty}{\underset{n=1}{\cup}} P_n(I_M(\{B\}^{\mathbb{N}})) \subset \gamma(B) \Rightarrow B \in g_\gamma$. We close this section by listing the results emerged from the above discussion in the Theorem 5.2.1 followed by a diagram representing interrelations between different classes of operators. In the diagram, a pair of lines indicates "equality", arrow mean "obtained from".

THEOREM 5.2.1. *Let* $M \in S((P(X))^{\mathbb{N}})$, $X \neq \Phi$ *then*

(1) $G_{\tau M}$ *forms a GST on* X.

(2) $M \in S_p((P(X))^{\mathbb{N}})$, $p = \Phi, X, E, C, I, \cup, \cap$ *and* $C_2 \Rightarrow M_n^{off}$, $M_n^{on} \in \Gamma_q(X)$, $q = 0, 1, +, -, 2, \cup, \cap$ *and* -2 *respectively.*

(3) $g_{M_n^{on}} \supset D_n(G_{\tau M}) \supset g_{M_n^{off}}\ \forall\ n \in \mathbb{N}$.

(4) *if the change of the components of sequential sets in* X *by* M *do not depend on the other components of the corresponding sequential set then* $g_{M_n^{on}} = D_n(G_{\tau M}) = g_{M_n^{off}}\ \forall\ n \in \mathbb{N}$.

(5) *any GST on* X *can be rediscovered by a monotonic sequential operator on* $(P(X))^{\mathbb{N}}$.

(6) $I_M \in S_{CI\Phi}((P(X))^{\mathbb{N}})$.

(7) *If* $M \in S_{CI\Phi}((P(X))^{\mathbb{N}})$ *then* $I_M = M$ *and hence* $I_{I_M} = I_M$ *for any* $M \in S((P(X))^{\mathbb{N}})$.

(8) $I_M \in S_X((P(X))^{\mathbb{N}})$ *iff* $M \in S_X((P(X))^{\mathbb{N}})$.

(9) *a sequential set* $A(s) \in G_{\tau M}$ *iff* $A(s) = I_M(A(s))$ *and* $I_M(B(s))$ *is the largest open set in* $G_{\tau M}$ *contained in* $B(s)$.

(10) $(I_M)_n^{off} = i_{M_n^{off}} = i_{(I_M)_n^{off}} = i_{i_{M_n^{off}}}$ *and* $(I_M)_n^{on} = i_{(I_M)_n^{on}} \neq i_{i_{M_n^{on}}} = i_{M_n^{on}}$

(11) $G_{\tau I_M} = G_{\tau M}$ *and so* $D_n(G_{\tau I_M}) = D_n(G_{\tau M})\ \forall\ n \in \mathbb{N}$.

(12) $g_{M_n^{off}} = g_{(I_M)_n^{off}}$ *and* $g_{M_n^{on}} \neq g_{(I_M)_n^{on}}$, $n \in \mathbb{N}$.

(13) $i_{g_{M_n^{off}}} = (I_M)_n^{off}$ *and* $c_{g_{M_n^{off}}} = (C_M)_n^{on}$.

(14) $i_{g_{M_n^{on}}} = i_{M_n^{on}}$ *and* $c_{g_{M_n^{on}}} = c_{M_n^{on}}$.

(15) $i_{D_n(G_{\tau M})} = (I_M)_n^{on}$ *and* $c_{D_n(G_{\tau M})} = (C_M)_n^{off}$.

(16) *intersection of* $M-$*sequential closed sets is* $M-$*sequential closed.*

(17) $C_M \in S_{EIX}((P(X))^{\mathbb{N}})$.

(18) $C_M \in S_\Phi((P(X))^{\mathbb{N}})$ *iff* $M \in S_\Phi((P(X))^{\mathbb{N}})$.

(19) *sequential set* $A(s)$ *in* X *is* $M-$*sequential closed*$\Leftrightarrow A(s) = C_M(A(s)) \Leftrightarrow A(s)$ *is* I_M-*sequential closed.*

(20) $C_M(A(s))$ *is the smallest closed sequential set in* $G_{\tau M}$ *containing* $A(s)$.

(21) $M^\circ \in (P(X))^{\mathbb{N}}$ *and* $(M^\circ)^\circ = M$.

(22) $M \in S_p((P(X))^{\mathbb{N}})$, $p = \Phi,\ X,\ E,\ I,\ C,\ \cup,$ *and* $\cap \Rightarrow M^\circ \in S_q((P(X))^{\mathbb{N}})$, $q = X,\ \Phi,\ C,\ I,\ E,\ \cap,$ *and* $\cup$ *respectively.*

(23) $(C_M)^\circ = I_M$ *hence* $(I_M)^\circ = C_M$.

(24) *a sequential set* $A(s)$ *in* X *is* $M^\circ-$*sequential closed iff* $M(A(s)) \subset A(s)$.

(25) $C_{I_M} = C_M$.

(26) $I_{C_M} = E = E^\circ = C_{C_M}$, *where* E *is the identity operator in* $(P(X))^{\mathbb{N}}$.

(27) $c_{M_n^{off}} = c_{(I_M)_n^{off}} = c_{i_{M_n^{off}}} = (C_M)_n^{on}$ *and* $c_{M_n^{on}} = c_{i_{M_n^{on}}} \neq c_{(I_M)_n^{on}} = (C_M)_n^{off}$.

(28) $(M_n^{off})^* = (M^\circ)_n^{on}$ *and* $(M_n^{on})^* = (M^\circ)_n^{off}$.

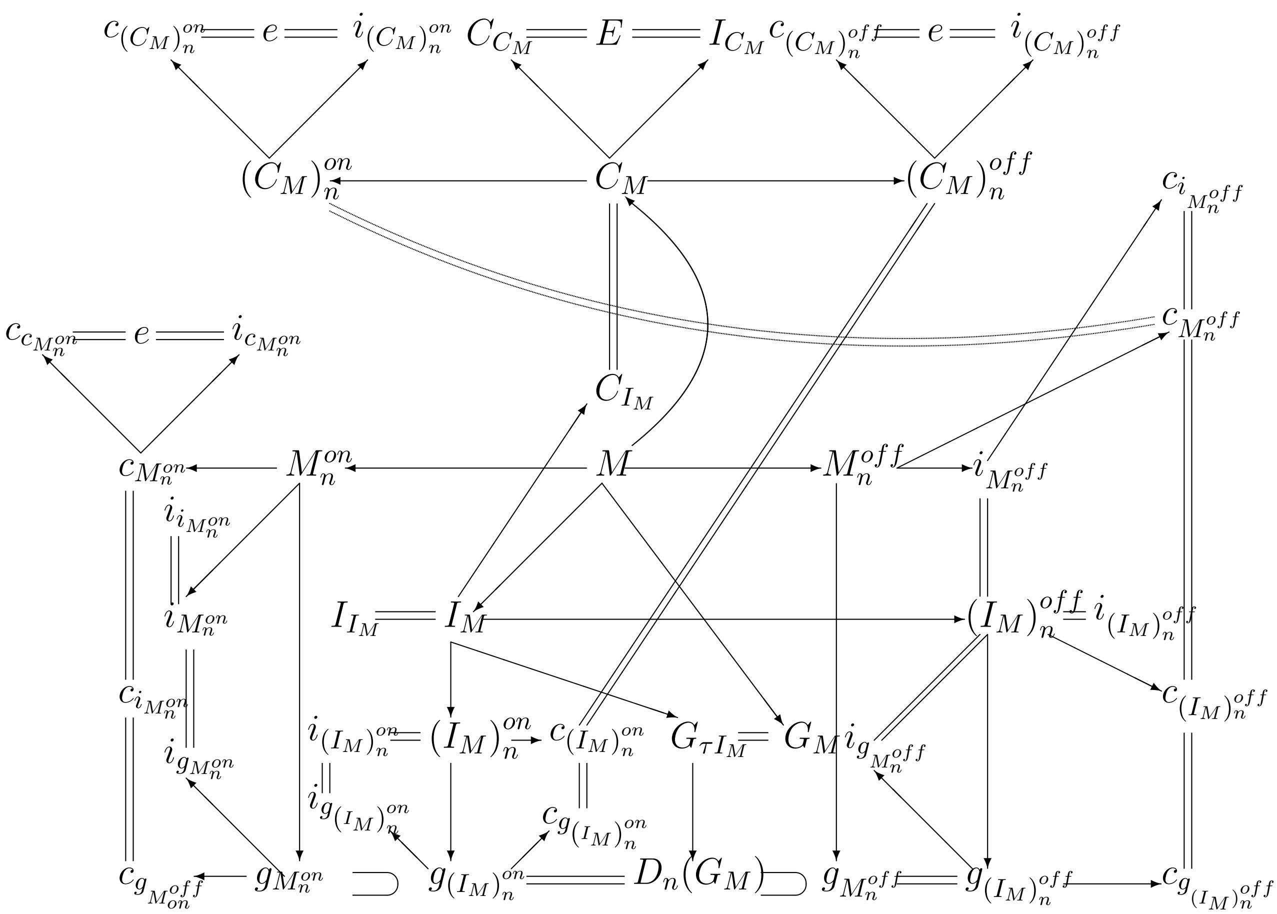
(29) $((I_M)_n^{off})^* = (C_M)_n^{on}$ *and* $((I_M)_n^{on})^* = (C_M)_n^{off}$.

(30) $((C_M)_n^{off})^* = (I_M)_n^{on}$ *and* $((C_M)_n^{on})^* = (I_M)_n^{off}$.

(31) $i_{(C_M)_n^{on}} = c_{(C_M)_n^{on}} = c_{(C_M)_n^{off}} = i_{(C_M)_n^{off}} = e = e^*$, *where* e *is the identity operator on* $P(X)$.

(32) $g_{\tau M} = \{A;\ A = \bigcup_{n=1}^{\infty} A_n,$ *where* $A(s) = \{A_n\} \in G_{\tau M}\}$ *forms a generalized topology on* X.

(33) $\gamma : P(X) \to P(X)$ *given by* $\gamma(A) = \bigcup_{n=1}^{\infty} P_n(I_M(\{A\}^{\mathbb{N}}))\ \forall A \in P(X)$ *defines the generalized topology* $g_\gamma = G_{\tau M}$.



Pictorial Representation of Relations among Different Monotonic Operators

Monotonic sequential operators in generalized sequential topological spaces

We recall that a subfamily G_τ of $(P(X))^{\mathbb{N}}$ is said to be a generalized sequential topology (GST) on X if it is closed under arbitrary union and contains $\Phi(s)$; the ordered pair $(X,\ G_\tau)$ is called generalized sequential topological space (GSTS). It is obvious that every ST is GST. But the converse is not true as follows from the

EXAMPLE 5.2.2. Let $X = \{a, b, c\}$, $A(s) = \{A_n\}$ and $B(s) = \{B_n\}$, where

$$A_{3n-2} = \{a, b\}, A_{3n-1} = \{b, c\} \text{ and } A_{3n} = \{c, a\},\ \forall n \in \mathbb{N}$$

$$B_{3n-2} = \{b, c\}, B_{3n-1} = \{c, a\} \text{ and } B_{3n} = \{a, b\} \forall n \in \mathbb{N}.$$

$G_\tau = \{\Phi(s),\ A(s),\ B(s)\}$ is a GST but not an ST on X.

The elements of a GST are called generalized sequential open (GSO) sets and their complements are called generalized sequential closed (GSC) sets. Let $D_n(G_\tau)$ be the collection of n^{th} components of all GSO sets of G_τ; since $\Phi(s) \in G_\tau$, $\Phi \in D_n(G_\tau)$. Also $O_\lambda \in D_n(G_\tau)$, $\lambda \in \Lambda$ implies there are $O_\lambda(s) = \{O_n^{(\lambda)}\} \in G_\tau$, $\forall\ \lambda \in \Lambda$ such that $O_n^{(\lambda)} = O_\lambda\ \forall\ \lambda \in \Lambda$ so $\underset{\lambda\in\Lambda}{\cup} O_n^{(\lambda)} = \underset{\lambda\in\Lambda}{\cup} O_\lambda \in D_n(G_\tau)$ because $\underset{\lambda\in\Lambda}{\cup} O_\lambda(s) \in G_\tau$. Thus $D_n(G_\tau)$ forms a generalized topology (GT) on X called n^{th} component GT of the GST G_τ on X. It is straightforward that if $A(s) = \{A_n\}$ is a GSO set in GSTS $(X\ ,\ G_\tau)$ then $A_n \in D_n(G_\tau)\ \forall\ n \in \mathbb{N}$ but the converse is not true as shown in the following

EXAMPLE 5.2.3. Consider the Example 5.2.2 and a sequential set $C(s) = \{C_n\}$, where

$$\begin{aligned} C_n &= A_n, \text{ when } n \text{ is odd and} \\ &= B_n, \text{ when } n \text{ is even.} \end{aligned}$$

Here $C_n \in D_n(G_\tau)\ \forall\ n \in \mathbb{N}$ but $C(s)$ is not a GSO set in the GSTS $(X\ ,\ G_\tau)$. Thus we have the following

THEOREM 5.2.2. (1) *Every ST is GST but the converse is not true.*
(2) *$D_n(G_\tau)$-the collection of n^{th} components of all GSO sets of G_τ forms GT on X.*
(3) *If $A(s) = \{A_n\}$ is a GSO set in GSTS $(X\ ,\ G_\tau)$ then $A_n \in D_n(G_\tau)\ \forall\ n \in \mathbb{N}$ but the converse is not true.*

The union of all GSO set in a GSTS $(X\ ,\ G_\tau)$ contained in a sequential set $A(s)$ in X is called generalized sequential interior (GSInt) of $A(s)$ in the GSTS $(X\ ,\ G_\tau)$ and it is denoted by $I_G(A(s))$ hence $I_G(A(s)) = \cup\{B(s);\ B(s) \subset A(s) \text{ and } B(s) \in G_\tau\}$. Generalized sequential closure (GSCl) of a sequential set $A(s)$ in a GSTS $(X\ ,\ G_\tau)$ is defined by the intersection of all GSC sets containing $A(s)$ and it is denoted by $C_G(A(s))$ that is, $C_G(A(s)) = \cap\{B(s);\ A(s) \subset B(s) \text{ and } X(s) - B(s) \in G_\tau\}$. Some results easily followed by the definitions of GSInt and GSCl operators are given in the following

THEOREM 5.2.3. *Let $A(s)$ and $B(s)$ be sequential sets in a GSTS $(X\ ,\ G_\tau)$ then*

(1) $A(s) \subset C_G(A(s))$ *and hence* $C_G(X(s)) = X(s)$.

(2) $A(s) \subset B(s) \Rightarrow C_G(A(s)) \subset C_G(B(s))$ *that is* C_G *is monotonic.*

(3) $C_G(\Phi(s)) \neq \Phi(s)$.

(4) $A(s)$ *is GSC iff* $C_G(A(s)) = A(s)$ *and hence* $C_G(C_G(A(s))) = C_G(A(s))$ *that is* C_G *is idempotent.*

(5) $C_G(A(s))$ *is the smallest GSC set containing* $A(s)$.

(6) $C_G(A(s) \cap B(s)) \subset C_G(A(s)) \cap C_G(B(s))$.

(7) $C_G(A(s) \cup B(s)) \supset C_G(A(s)) \cup C_G(B(s))$.

(8) $I_G(A(s)) \subset A(s)$ *therefore* $I_G(\Phi(s)) = \Phi(s)$.

(9) $A(s) \subset B(s) \Rightarrow I_G(A(s)) \subset I_G(B(s))$ *that is* I_G *is monotonic.*

(10) $I_G(X(s)) \neq X(s)$.

(11) $A(s)$ *is GSO iff* $I_G(A(s)) = A(s)$ *and so* $I_G(I_G(A(s))) = I_G(A(s))$ *that is* I_G *is idempotent.*

(12) $I_G(A(s))$ *is the largest GSO set contained in* $A(s)$.

(13) $I_G(A(s) \cup B(s)) \supset I_G(A(s)) \cup I_G(B(s))$.

(14) $I_G(A(s) \cap B(s)) \subset I_G(A(s)) \cap I_G(B(s))$.

(15) $C_G(A(s)) = X(s) - I_G(X(s) - A(s))$.

(16) $I_G(A(s)) = X(s) - C_G(X(s) - A(s))$.

We produce some examples to justify the relations (3), (7), (10) and (14) in Theorem 5.2.3. For (3) and (10)

Example 5.2.4. Consider $X = \{a, b, c\}$ and $G_\tau = \{\Phi(s), A(s) = \{A_n\}, B(s) = \{B_n\}, C(s) = \{C_n\}$, where

$$A_{3n-2} = \{c\},\ A_{3n-1} = \{a\} \text{ and } A_{3n} = \{b\}\ \forall\ n \in \mathbb{N},$$

$$B_{3n-2} = \{a\},\ B_{3n-1} = \{b\} \text{ and } B_{3n} = \{c\}\ \forall\ n \in \mathbb{N} \text{ and}$$

$C_{3n-2} = \{c, a\}$, $C_{3n-1} = \{a, b\}$ and $C_{3n} = \{b, c\}$ $\forall\, n \in \mathbb{N}$.

Here $I_G(X(s)) = C(s)$ and $C_G(\Phi(s)) = D(s) = \{D_n\}$ with $D_{3n-2} = \{b\}$, $D_{3n-1} = \{c\}$ and $D_{3n} = \{a\}$ $\forall\, n \in \mathbb{N}$.

For (7) and (14)

EXAMPLE 5.2.5. Consider $X = \{a, b, c\}$ and $G_\tau = \{X(s), \Phi(s), P(s) = \{P_n\}, Q(s) = \{Q_n\}, R(s) = \{R_n\}\}$, where

$$\begin{aligned} P_n &= a \text{ for } n = 1 \qquad Q_n = b \text{ for } n = 2 \qquad R_n = c \text{ for } n = 3 \\ &= X \text{ for } n \neq 1 \qquad\ \ = X \text{ for } n \neq 2 \qquad\ \ = X \text{ for } n \neq 3 \end{aligned}$$

Let $A(s) = \{A_n\}$ and $B(s) = \{B_n\}$, where

$$\begin{aligned} A_n &= \{b, c\} \text{ for } n = 1 \qquad B_n = \{a, c\} \text{ for } n = 2 \\ &= \Phi \text{ otherwise} \qquad\qquad\ = \Phi \text{ otherwise.} \end{aligned}$$

Then $C_G(A(s) \cup B(s)) = X(s) \nsubseteq C(s) = \{C_n\} = C_G(A(s)) \cup C_G(B(s))$ and $I_G(P(s) \cap Q(s)) = \Phi(s) \nsupseteq E(s) = \{E_n\} = I_G(P(s)) \cap I_G(Q(s))$ where

$$\begin{aligned} C_1 &= \{b, c\}, C_2 = \{c, a\} \text{ and } C_n = \Phi \text{ otherwise;} \\ E_1 &= \{a\}, E_2 = \{b\} \text{ and } E_n = X \text{ otherwise;} \end{aligned}$$

The n^{th} component of the GSInt $I_G({}_nX_A(s))$ (resp. GSInt $I_G({}_n\Phi_A(s))$) of the sequential set ${}_nX_A(s)$ (resp. ${}_n\Phi_A(s))$) in a GSTS $(X\ , G_\tau)$ is called the n^{th} on (resp. off) interior of A in the GSTS $(X\ , G_\tau)$ and it is denoted by $(I_G)_n^{on}(A)$ (resp. $(I_G)_n^{off}(A)$). Similarly n^{th} on (resp. off) closure of A in a GSTS $(X\ , G_\tau)$ which is denoted by $(C_G)_n^{on}(A)$ (resp. $(C_G)_n^{off}(A)$) is the n^{th} component of the GSCl

$C_G({}_nX_A(s))$ (resp. GSCl $C_G({}_n\Phi_A(s))$) of the sequential set ${}_nX_A(s)$ (resp. ${}_n\Phi_A(s)$)) in the GSTS $(X\ ,\ G_\tau)$. Some results concerning components of GSInt and GSCl operators in a GSTS which follow from the discussion in section 5.2 with a bit manipulations, are listed in the following

THEOREM 5.2.4. *If* $(X\ ,\ G_\tau)$ *is a GSTS then the following hold;*
(1) $(I_G)_n^{on} \in \Gamma_{02-}$ *and* $(I_G)_n^{on} \in \Gamma_1$ *iff* $X \in D_n(G_\tau)$.
(2) $(I_G)_n^{off} \in \Gamma_{02-}$ *and* $(I_G)_n^{off} \in \Gamma_1$ *iff* ${}_n\Phi_X(s) \in G_\tau$.
(3) $(C_G)_n^{off} \in \Gamma_{12+}$ *and* $(C_G)_n^{off} \in \Gamma_0$ *iff* $X \in D_n(G_\tau)$.
(4) $(C_G)_n^{on} \in \Gamma_{12+}$ *and* $(C_G)_n^{on} \in \Gamma_0$ *iff* ${}_n\Phi_X(s) \in G_\tau$.
(5) $g_{(I_G)_n^{off}} \subset g_{(I_G)_n^{on}} = D_n(G_\tau)$.
(6) $(I_G)_n^{on}$ *coincides with the interior operator and* $(C_G)_n^{off}$ *is the closure operator in the GTS* $(X\ ,\ D_n(G_\tau))$ *i. e.,* $i_{D_n(G_\tau)} = (I_G)_n^{on} = i_{(I_G)_n^{on}}$ *and* $c_{D_n(G_\tau)} = (C_G)_n^{off} = c_{(I_G)_n^{on}}$.
(7) $i_{g_{(I_G)_n^{off}}} = (I_G)_n^{off}$ *and* $c_{g_{(I_G)_n^{off}}} = (C_G)_n^{on}$.

Now let $(D\ ,\ \geq)$ be a directed set and $X \times \mathbb{N}$ be the collection of all simple sequential points in X, then any function $x : D \to X \times \mathbb{N}$ is called simple sequential net (SSN) in $X(s)$ and it is denoted by $\{x_j\}_{j\in D}$ or by $\{x_j\}$ if no confusion can result. In particular if D is replaced by the set $\mathbb{N}$ of natural numbers with usual ordering then x is called simple sequential sequence (SSS). If E is a cofinal subset of D then $\{x_j\}_{j\in E}$ is called a sub-SSN of $\{x_j\}_{j\in D}$. An SSN $\{x_j\}_{j\in D}$ in $X(s)$ is said to converge to a simple sequential point $(x\ ,\ n)$ in X with respect to a GST G_τ if for any $O(s) \in G_\tau$ containing $(x\ ,\ n)$ $\exists\ j_0 \in D$ so that $j_0 \leq j \in D \Rightarrow x_j \in O(s)$; $(x\ ,\ n)$ is called a limit of $\{x_j\}_{j \in D}$ and the SSN is called convergent. In a GSTS $(X\ ,\ G_\tau)$ it is observed that if an SSN $\{x_j\}_{j \in D}$ in a sequential set $A(s)$ (i. e., $x_j \in A(s)\ \forall\ j \in D$) converges to $(x\ ,\ n)$ then $(x\ ,\ n) \in C_G(A(s))$ and an SSN $\{x_j\}_{j \in D}$ in $X(s)$ converges to $(x\ ,\ n)$ iff every sub-SSN of it converges to $(x\ ,\ n)$. A GSTS $(X\ ,\ G_\tau)$ is said to posses SSN-closure property if for each sequential set $A(s)$ in X and $(x\ ,\ n) \in C_G(A(s))$ $\exists$ an SSN in $A(s)$ converging to $(x\ ,\ n)$. In a GTS $(X\ ,\ g)$ if for each $A \subset X$ and $x \in \overline{A}$ $\exists$ a net in A converging to x then the GTS is said to have net-closure property [61]. Now Consider the n^{th} component $(X\ ,\ D_n(G_\tau))$ of a GSTS $(X\ ,\ G_\tau)$ which possesses SSN-closure property. Let $A \subset X$ and $x \in \overline{A}$ (closure of A in $(X\ ,\ D_n(G_\tau))$). Then the simple sequential point $(x\ ,\ n) \in C_G({}_n\Phi_A(s))$, since $\overline{A} = (C_G)_n^{off}(A)$ by (5), (6) of Theorem 5.2.4. Therefore $\exists$ an SSN $\{x_j\}_{j \in D}$ in ${}_n\Phi_A(s)$ converging to $(x\ ,\ n)$. Let $x_j = (y_j\ ,\ n) \in_n \Phi_A(s)\ \forall\ j \in D$ then $\{y_j\}_{j \in D}$ is a net in A that converges to $x \in \overline{A}$. This proves that if a GSTS possesses SSN-closure property then every component of it does posses net closure property. The following example shows that the converse is not true in general.

Example 5.2.6. Let $X = \{a,\ b,\ c\}$ and $G_\tau = \{\Phi(s),\ X(s),\ A(s) = \{A_n\},\ B(s) = \{B_n\},\ C(s) = \{C_n\},\ D(s) = \{D_n\},\ E(s) = \{E_n\},\ F(s) = \{F_n\},\ G(s) = \{G_n\}\}$, where

$$A_{3n-2} = \{a\},\ \ A_{3n-1} = \{b\} \text{ and } A_{3n} = \{c\}\ \forall\ n \in \mathbb{N},$$

$$B_{3n-2} = \{b\}\ B_{3n-1} = \{c\} \text{ and } B_{3n} = \{a\}\ \forall\ n \in \mathbb{N},$$

$$C_{3n-2} = \{a,b\},\ \ C_{3n-1} = \{b,c\} \text{ and } C_{3n} = \{c,a\}\ \forall\ n \in \mathbb{N},$$

$$D_1 = \{a\} \text{ and } D_n = X, \ \forall\ n \in \mathbb{N}\ n \neq 1,$$

$$E_1 = \{a, b\} \text{ and } E_n = X, \ \forall\ n \in \mathbb{N}\ n \neq 1,$$

$$F_2 = \{c\} \text{ and } F_n = X \ \forall\ n \in \mathbb{N}\ n \neq 2,$$

$$G_2 = \{b, c\} \text{ and } G_n = X \ \forall\ n \in \mathbb{N}\ n \neq 2.$$

This GSTS $(X\ ,G_\tau)$ does not posses SSN-closure property but every component of it possesses net closure property. This example also leads to conclude that components of a GST which is not an ST may be topologies. We summaries the results obtained from the last fold of the discussion in

THEOREM 5.2.5. *In a GSTS* $(X\ ,\ G_\tau)$
(1) *if an SSN* $\{x_j\}_{j\in D}$ *in a sequential set* $A(s)$ *(i. e.,* $x_j \in A(s)\ \forall\ j \in D$*) converges to* $(x\ ,\ n)$ *then* $(x\ ,\ n) \in C_G(A(s))$*.*
(2) *an SSN* $\{x_j\}_{j\in D}$ *in* $X(s)$ *converges to* $(x\ ,\ n)$ *iff every sub-SSN of it converges to* $(x\ ,\ n)$*.*
(3) *if a GSTS possesses SSN-closure property then every component of it does posses net closure property but the converse is not true.*

We end this section by answering the question "How one can check whether a GST is an ST or not ?". Let $(X\ ,\ G_\tau)$ be a GSTS and consider $A(s),\ B(s) \in G_\tau$ then $C(s) = X(s) - A(s)$ and $D(s) = X(s) - B(s)$ are GSC sets in $(X\ ,\ G_\tau)$. If G_τ satisfies SSN-closure property and $(x\ ,\ n) \in C_G(C(s) \cup D(s))$ then $\exists$ an SSN $\{x_j\}_{j\in D}$ in $C(s) \cup D(s)$ converging to $(x\ ,\ n)$. $\{x_j\}_{j\in D}$ has a sub-SSN $\{x_j\}_{j\in E}$ either in $C(s)$ or in $D(s)$; let $\{x_j\}_{j\in E}$ be in $C(s)$ and by Theorem 5.2.5 $(x\ ,\ n) \in C_G(C(s)) = C(s) \subset C(s) \cup D(s)$.

Thus $C(s) \cup D(s)$ is GSC set and hence $A(s) \cap B(s)$ is GSO set. This proves the assertion of the following

THEOREM 5.2.6. *A GST G_τ on X containing $X(s)$ and possessing SSN-closure property is an ST.*

On generalized connectors

This subsection concludes the chapter by introducing generalized connector that connects $\gamma-$operators introduced by *Á*. Cs*á*sz*á*r. We call a family $K \subset (g_{\gamma_2})^{g_{\gamma_1}}$ a generalized connector of γ_1 to γ_2 or g_{γ_1} to g_{γ_2} where γ_1, $\gamma_2 \in \Gamma(X)$ [14, 15, 16, 17, 19] if
(1) $\exists\ f \in K$ so that $f(\Phi) = \Phi$,
(2) for $O_\lambda \subset \gamma_1(O_\lambda) \subset X$, that is $O_\lambda \in g_{\gamma_1}$ and $f_\lambda \in K$, $\lambda \in \Lambda$ $\exists$ $f \in K$ such that $f(\underset{\lambda\in\Lambda}{\cup} O_\lambda) = \underset{\lambda\in\Lambda}{\cup} f_\lambda(O_\lambda)$,
(3) $g_{\gamma_2} = \underset{f\in K}{\cup} f(g_{\gamma_1})$.
Now consider any $A(s) = \{A_n\} \in (P(X))^{\mathbb{N}}$; each $B \subset X$ so that $i_{\gamma_1}(B) \subset A_1$ and each sequence $\{f_n\}$, where $f_n \in K_n$ such that $f_n f_{n-1}.......f_1(i_{\gamma_1}(B)) \subset A_{n+1}$, $n \in \mathbb{N}$ generates a sequential set denoted by ${}_{\{f_n\}}B(s)_{A(s)} = \{B_n\}$, where $B_n = f_{n-1}f_{n-2}.......f_1 f_0(i_{\gamma_1}(B))$ $\forall\ n \in \mathbb{N}$ with the understanding that $f_0(i_{\gamma_1}(B)) = i_{\gamma_1}(B)$, we call ${}_{\{f_n\}}B(s)_{A(s)}$ a worm sequential set in $A(s)$ with foot B and sequential direction $\{f_n\}$. Define $M(A(s))$ = union of all such worm sequential sets in $A(s)$. Then $M : (P(X))^{\mathbb{N}} \to (P(X))^{\mathbb{N}}$ is a monotonic sequential operator, $I_M = M$ and hence $(I_M)^{on}_n = M^{on}_n = i_{D_n(G_{\tau M})}$. The third condition to be a generalized connector ensures that $D_n(G_{\tau M}) = g_{\gamma_n}$ that implies $i_{D_n(G_{\tau M})} = i_{g_{\gamma_n}} = i_{\gamma_n}$,

therefore $(I_M)_n^{on} = i_{\gamma_n}$. Now let M be any monotonic sequential operator. As in [57] for each $n \in \mathbb{N}$ define a relation $R^{n,n+1}$ on $G_{\tau M}$ by $A(s) = \{A_n\} R^{n,\, n+1} B(s) = \{B_n\}$ if and only if $A_n = B_n$, then, $R^{n,\, n+1}$ defines a partition of $G_{\tau M}$ say $\{C(A(s))\ ;\ A(s) \in G_{\tau M}^{n,\, n+1} \subset G_{\tau M}\}$, where $G_{\tau M}^{n,\, n+1}$ is a family of open sequential sets taking exactly one from each equivalence class of the partition of $G_{\tau M}$ defined by $R^{n,\, n+1}$ and let $F^{n,\, n+1} = \prod_{A(s) \in G_{\tau M}^{n,\, n+1}} C(A(s))$. Let $l \in F^{n,n+1}$; for any $U \in D_n(G_{\tau M})$ $\exists$ a unique $U(s) = \{U_k\} \in G_{\tau M}$ as a component of l so that $U_n = U$, this defines a function $f_l : D_n(G_{\tau M}) \to D_{n+1}(G_{\tau M})$ defined by $f_l(U) = U_{n+1}$. Then $F_n = \{f_l;\ l \in F^{n,\, n+1}\}$ is a generalized connector connecting $D_n(G_{\tau M})$ to $D_{n+1}(G_{\tau M})$. Also $i_{D_n(G_{\tau M})} = (I_M)_n^{on} = i_{(I_M)_n^{on}}$. Thus there is a relation between Á. Császár's $\gamma-$operator and our $M-$sequential operator what is exhibited in the

Theorem 5.2.7. *For any sequence $\{\gamma_n\}$ of Á. Császár's $\gamma-$operators and any sequence $\{K_n\}$ of generalized connectors so that K_n connects γ_n to γ_{n+1} $\exists$ a unique monotonic sequential operator M and conversely so that $(I_M)_n^{on} = i_{\gamma_n}$.*

Corollary 5.2.1. *Any sequence $\{g_n\}$ of generalized topologies and any sequence $\{K_n\}$ of generalized connectors so that, K_n connects g_n to g_{n+1}, together define a GST G_τ and conversely so that $D_n(G_\tau) = g_{\gamma_n}$.*

Remark. Separation axioms, continuity and other topological properties like compactness, connectedness, paracompactness etc.

can be studied in the influence of a sequence of topologies connected by a sequence of connectors. Particularly if one choose a sequence of copies of a topology (resp. generalized topology) and connects them by a sequence of connectors (resp. generalized connectors) then it will give rise to a new topological space (resp. generalized topological space), where the open sets are constrained, that is, they are connected by some rule.

Chapter VI

Minimal structures with special emphasis on separation axioms

Minimal structure, as the name suggests, involves least possible requirements for a class of subsets of a nonvoid set to have some "structure" from the view point of topological studies. This unique concept, emerged in 2000 (V. Popa and T. Noiri [53]), drew the attention of a number of researchers in the related field. Very recent, it was further studied by Carlos Carpintero, Ennis Rosas, Margot Salas [12], A. Pushpalatha, E. Subha [54], R. Parimelazhagan, K. Balachandran, N. Nagaveni [49], Sunisa Buadong, Chokchai Viriyapong, Chawalit Boonpok [8], Won Keun Min, Young Key Kim [44] etc.. We extend the field by introducing notions of various separation axioms, continuity and establish the Urysohn's lemma in this context. At first we give some basic definitions and outline of the results relevant to this extension. As in [53, 12, 54, 49, 8, 44] a family m of subsets of a nonvoid set X containing the void subset and the whole set is called a minimal structure and we call the ordered pair (X, m) a minimal structured space (briefly MSS). Elements of a minimal structure m on X are called m-open sets and their complements are called m-closed sets. For any $A \subset X$, $i_m(A)$ ($m-Int$ in [44]) denotes the union of all m-open sets contained in A and $c_m(A)$ ($m - Cl$ in [44]) denotes the intersection of all m-closed sets containing A; presence of the void set in m well-defines $i_m(A)$

and $c_m(A)$. i_m is contractive c_m is expansive, both are monotonic and idempotent and their relationship is $i_m(A) = X - c_m(X - A)$, for any $A \subset X$. $x \in i_m(A)$ iff there is an m-open set $B \subset A$ containing x and $x \in c_m(A)$ iff $B \cap A \neq \phi$ whenever $x \in B \in m$. All these are similar results to other topology-like structures (topology or generalized topology or weak structure) but unlike in topology or generalized topology $i_m(A)$ may not be m-open and $c_m(B)$ may not be m-closed, for $A, B \subset X$; though they will be so if A is m-open and B is m-closed. In the present work we have shown that how this dissimilarities propagate differences in separation axioms, continuity and Urysohn's lemma.

6.1. Correction of some results appeared in [44]

Let's begin with the following example.

Example 6.1.1. Let us consider the minimal structure $m_X = \{\phi, \{a,b\}, \{b,c\},\ X\}$ on $X = \{a,b,c\}$, $A = \{a,b\}$, $B = \{b,c\}$, $C = \{c\}$ and $D = \{a\}$ then $m - Int(A) = A$, $m - Int(B) = B$, $m - Int(A \cap B) = \phi$, $m - Cl(C) = C$, $m - Cl(D) = D$ and $m - Cl(C \cup D) = X$.

This shows that the statement in Theorem 2.5 (5) [44] is not true and it must be replaced by $m - Int(A \cap B) \subset m - Int(A) \cap m - Int(B)$ and $m - Cl(A) \cup m - Cl(B) \subset m - Cl(A \cup B)$.

In the proof of the Theorem 3.8 [44] and Theorem 4.20 [44] monotonicity of the operators I_m and Cl_m have been used but none of them is monotonic as shown in the following example;

EXAMPLE 6.1.2. Consider the minimal structure $m_X = \{\phi, \{a, b\}, \{b, c\}, \{c, d\}, X\}$ on $X = \{a, b, c, d\}$, $A = \{a, b, c\}$, $B = \{a, b\}$ and $C = \{a\}$ then $I_m(A) = \phi$, $I_m(B) = B$, $Cl_m(B) = B$ and $Cl_m(C) = X$.

The statement of the Theorem 3.8 [44] is wrong because(3) $\not\Leftrightarrow$ (4) there. The following examples are supporting witness.

EXAMPLE 6.1.3. Let $X = \{a, b, c\}$, $Y = \{1, 2\}$, $m_X = \{\phi, \{b\}, X\}$, $m_Y = \{\phi, \{2\}, Y\}$ and $f : X \to Y$ be a function defined by $f(a) = f(c) = 1, f(b) = 2$; f is M^*-continuous and Theorem 3.8 (3) [44] holds but $f(Cl_{m_X}(A)) = f(X) = \{1, 2\} \nsubseteq \{1\} = Cl_{m_Y}(\{1\}) = Cl_{m_Y}(f(A))$, where $A = \{c\}$, shows that Theorem 3.8 (4) [44] does not hold.

EXAMPLE 6.1.4. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $m_X = \{\phi, \{b\}, X\}$ and $m_Y = \{\phi, \{2\}, Y\}$. Then the function $f : X \to Y$ defined by $f(a) = f(b) = 1, f(c) = 2$ is not M^*-continuous and Theorem 3.8 (3) [44] is not true for this function but it can be verified that Theorem 3.8 (4) [44] holds.

The Theorem 4.20 [44] seems to be incorrect, we put it in the revised form in the following

THEOREM 6.1.1. *A bijection* $f : (X, m_X) \to (Y, m_Y)$*, where* m_X *and* m_Y *are minimal structures on* X *and* Y *respectively, is* M^**-open if and only if* $I_m(f^{-1}(B)) \subset f^{-1}(I_m(B))$ *for each* $B \subset Y$*.*

A corrected version of the Theorem 3.8 [44] is included in the Theorem 3.11 of the following Section 3.

6.2. Separation axioms, Continuity and a version of Urysohn's lemma

DEFINITION 6.2.1. A minimal structured space (X, m) is called $m - T_1$ if for any two distinct points $x, y \in X$ there are m-open sets U and V so that $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$.

EXAMPLE 6.2.1. Let $X = \{a, b, c\}$. On this X, $m_1 = \{\Phi, X, \{a\}, \{b\}, \{c\}\}$, $m_2 = \{\Phi, X, \{a, b\}, \{b, c\}, \{c, a\}\}$ are MSs and the corresponding MSSs are $m - T_1$.

The above MSSs in Example 6.2.1 show that a finite MSS may be $m - T_1$ without being $P(X)$.

DEFINITION 6.2.2. Let (X, m) be an MSS. An element A of $P(X)$ is called a fixed point of c_m if $c_m(A) = A$.

THEOREM 6.2.1. *A MSS* (X, m) *is* $m - T_1$ *iff every singleton is a fixed point of* c_m*.*

PROOF. If $x \neq y \in c_m(\{x\})$ then every m-open set containing y will contain x and (X, m) would not be $m - T_1$. Thus in a $m - T_1$

MSS (X, m) every singleton is a fixed point of c_m. Conversely for any two distinct points x and y in a MSS (X, m), where every singleton is a fixed point of c_m, $y \notin c_m(\{x\})$ and $x \notin c_m(\{y\})$; this implies there are m-open sets U and V containing x and y respectively such that $U \bigcap \{y\} = \Phi$ and $V \bigcap \{x\} = \Phi$, so (X, m) is $m - T_1$. ■

Note that those singletons may not be m-closed, (X, m_1) in Example 6.2.1 is one such wetness.

Definition 6.2.3. An MSS (X, m) is called $m - T_2$ if for any two distinct points $x, y \in X$ there exists disjoint m-open sets U and V so that $x \in U$, $y \in V$.

In Example 6.2.1 (X, m_1) is a $m - T_2$ but (X, m_2) is not $m - T_2$ though it is $m - T_1$. Obviously an $m - T_2$ MSS is $m - T_1$.

Definition 6.2.4. We say that a sequence $\{x_n\}$ in an MSS (X, m) converges to $x \in X$ if for any m-open set U there exists a positive integer N so that $x \in U$ for all $n \geq N$; in this case $\{x_n\}$ is called a convergent sequence and x is called limit of it.

Theorem 6.2.2. *In an $m - T_2$ MSS every convergent sequence has unique limit.*

Proof. If x, y are two distinct limits of a sequence $\{x_n\}$ in an $m - T_2$ MSS (X, m), then for any two m-open sets U and V containing x and y respectively there exists a positive integer N so that $x_n \in U \cap V$ for all $n \geq N$ and this contradicts the fact that the MSS is $m - T_2$. ■

DEFINITION 6.2.5. An MSS (X, m) is called m-regular if for any point $x \in X$ and any m-closed set C not containing x there exists disjoint m-open sets U and V so that $x \in U$ and $C \subset V$.

THEOREM 6.2.3. *If (X, m) is m-regular MSS then for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c_m(V) \subset U$.*

PROOF. Let (X, m) be an m-regular MSS and U be any m-open set, then for any point $x \in U$ the m-closed set $X - U$ does not contain x, so there exists m-open sets V and E such that $x \in V$, $X - U \subset E$ and $V \cap E = \phi$. This implies $V \subset X - E$ and hence $c_m(V) \subset X - E \subset U$, since $X - E$ is m-closed. Thus for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c_m(V) \subset U$. ■

But the converse of the Theorem 6.2.3 is not true as seen in the MSS (X, m_1) of the Example 6.2.1 which is true in regular topological spaces.
We call the family $M(x) = \{U; x \in U \in m\}$, where m is a minimal structure on X, a minimal-star at x and by the help of the family $\{M(x); x \in X\}$ of minimal-stars we define $i^*{}_m(A) = \{x \in A; A \in M(x)\}(= I_m(A)$ in [44]) and $c^*{}_m(A) = \{x \in X; X - A \notin M(x)\}(= Cl_m(A)$ in [44]). Now $i^*{}_m$ agrees with i_m in m and $c^*{}_m$ agrees with c_m in $m^c = \{X - U; U \in m\}$; also if $A \notin m$ then $i^*{}_m(A) = \Phi$ and $c^*{}_m(X - A) = X$. So, $i^*{}_m(A) \subset i_m(A)$, $c_m(A) \subset c^*{}_m(A)$. Let

$X = \{a, b, c\}$ and $m = \{\Phi, X, \{a\}, \{b, c\}\}$ then $i^*{}_m(\{a\}) = \{a\} \supset \Phi = i^*{}_m(\{a, b\})$ and $c^*{}_m(\{b, c\}) = \{b, c\} \subset X = c^*{}_m(\{b\})$, hence i^* and c^* are not monotonic on the power set $P(X)$ of the underlying set X. Now let $x \notin X - i^*{}_m(X - A) \Leftrightarrow x \in i^*{}_m(X - A) \Leftrightarrow x \in X - A \in m \Leftrightarrow X - A \in M(x) \Leftrightarrow x \notin c^*{}_m(A)$. Thus $c^*{}_m(A) = X - i^*{}_m(X - A)$ and similarly $i^*{}_m(A) = X - c^*{}_m(X - A)$ for all $A \subset X$. The purpose of the definitions of $i^*{}_m$ and $c^*{}_m$ is the fact that $i^*{}_m(A) = A \Rightarrow A \in m$ and $c^*{}_m(A) = A \Rightarrow X - A \in m$. Thus we have

THEOREM 6.2.4. *Let m be a minimal structure on X. Then*

1. $i^*{}_m(A)=i_m(A)\ \forall\ A \in m$
2. $c^*{}_m(X - A)=c_m(X - A)\ \forall\ A \in m$
3. $i^*{}_m(A) = \Phi$ *and* $c^*{}_m(X - A) = X$ *whenever* $A \in P(X) - m$.
4. $i^*{}_m(A) \subset i_m(A)$, *and* $c_m(A) \subset c^*{}_m(A)\ \forall\ A \subset X$.
5. $i^*{}_m$ *and* $c^*{}_m$ *are monotonic on* m *but not on the power set* $P(X)$ *(if* $m \neq P(X)$*) of the underlying set* X.
6. $c^*{}_m(A) = X - i^*{}_m(X - A)\ \forall\ A \subset X$.
7. $i^*{}_m(A) = A \Rightarrow A \in m$ *and* $c^*{}_m(A) = A \Rightarrow X - A \in m$.

THEOREM 6.2.5. *An MSS (X, m) is m-regular if for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c^*{}_m(V) \subset U$.*

PROOF. Now in an MSS (X, m), let for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c^*{}_m(V) \subset U$. Let x be any point of X and C be any m-closed set not containing x, so $x \in X - C$ and $X - C$ is an m-open set and therefore by the hypothesis there exists m-open set V such that $x \in V \subset {c^*}_m(V) \subset X - C \Rightarrow x \in V$ and $C \subset X - {c^*}_m(V) = T$ (say). Then since $V \cap T = \phi$, the MSS (X, m) is m-regular. Thus an MSS (X, m) is m-regular if for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset {c^*}_m(V) \subset U$ ∎

Example 6.2.2. The MSS $(\mathbb{R}, m)$, where m is the collection of all right and left open rays including ϕ and $\mathbb{R}$ shows that the condition in Theorem 6.2.5 is not necessary.

Theorem 6.2.6. *If an MSS (X, m) is m-regular then for any $x \in U \in m$ there exists $V \in m$ so that $x \in V \subset c_m(V) \subset U$.*

The MSS (X, m_1) in the Example 6.2.1 ensures that this condition is not sufficient. We search a necessary as well as sufficient condition for an MSS (X, m) to be m-regular.

Theorem 6.2.7. *An MSS (X, m) is m-regular if and only if for any point $x \in X$ and any m-open set U containing x there exist m-open set V and m-closed set V^* so that $x \in V \subset V^* \subset U$.*

Proof. Let an MSS (X, m) be m-regular, x be any point in X and U be any m-open set containing x. Then $X - U$ is m-closed set not containing x. So, there are m-open sets E and F such that $x \in E$, $(X - U) \subset F$ and $E \cap F = \phi$. Hence $E \subset (X - F) \subset U$. Taking $V = E$ and $V^* = (X - F)$ we have V is m-open, V^* is m-closed and $x \in V \subset V^* \subset U$. Conversely, let for any point $x \in X$ and any m-open set U containing x there exists m-open set V and m-closed set V^* so that $x \in V \subset V^* \subset U$. Let $x \in X$ and

E be any m-closed set not containing x. Then $x \in (X - E)$ and $(X - E)$ is m-open. Hence by the hypothesis there is an m-open set V and an m-closed set V^* so that $x \in V \subset V^* \subset (X - E)$. Therefore $E \subset (X - V^*)$, also $V \cap (X - V^*) \subset V \cap (X - V) = \phi$ hence $V \cap (X - V^*) = \phi$ which implies (X, m) is m-regular. ■

DEFINITION 6.2.6. An MSS (X, m) is called m-normal if for any two disjoint m-closed sets C and D there exists two disjoint m-open sets U and V so that $C \subset U$ and $D \subset V$.

THEOREM 6.2.8. *Let C be any m-closed set and U be any m-open set containing C. Then there exists an m-open set V so that $C \subset V \subset {c^*}_m(V) \subset U \Rightarrow$ MSS (X, m) is m-normal $\Rightarrow$ there exists an m-open set V so that $C \subset V \subset c_m(V) \subset U$.*

THEOREM 6.2.9. *A necessary and sufficient condition for an MSS (X, m) is m-normal if for any m-closed set C and any m-open set U containing C there is an m-open set V and an m-closed set V^* such that $C \subset V \subset V^* \subset U$.*

DEFINITION 6.2.7. A function $f : X \to Y$ from an MSS (X, m_X) into an MSS (Y, m_Y) is said to be M-continuous at a point $x \in X$ if for any $V \in M(f(x))$, there is $U \in M(x)$ such that $f(U) \subset V$; f is called M-continuous if it is so at each point of its domain.

THEOREM 6.2.10. *For any function $f : X \to Y$ from an MSS (X, m_X) into an MSS (Y, m_Y), as in general case, it is observed* *that the following are equivalent:*

(1) f is M-continuous.

(2) $f(c_{m_X}(A)) \subset c_{m_Y}(f(A))$ for $A \subset X$

(3) $c_{m_X}(f^{-1}(B)) \subset f^{-1}(c_{m_Y}(B))$ for $B \subset Y$.

(4) $f^{-1}(i_{m_Y}(B)) \subset i_{m_X}(f^{-1}(B))$ for $B \subset Y$.

Definition 6.2.8. A function $f : X \to Y$ from an MSS (X, m_X) into an MSS (Y, m_Y) is said to be M^*-continuous if for every $B \in m_Y$, $f^{-1}(B) \in m_X$.

Obviously M^*-continuity implies M-continuity but the converse is not true as seen in the following Example 6.2.3.

Example 6.2.3. Let $X = Y = \{a, b, c\}$, $m_X = \{\phi, X, \{a\}, \{b\}\}$, and $m_Y = \{\phi, X, \{a, b\}\}$ then the identity function is M-continuous but not M^*-continuous.

If $x \in i^*_m(A)$ then there is a $U \in M(x)$ contained in A but $x \in {c^*}_m(A)$ does not mean that every $V \in M(x)$ intersects A as shown in the following Example 6.2.4.

Example 6.2.4. Let $X = \{a, b, c\}$, $m = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}\}$ and $A = \{b\}$ then, $a \in {c^*}_m(A) = X$ and $V = \{a\} \in M(a)$ but $V \cap A = \phi$.

This fact together with the non-monotonicity of the operators i^* and c^* cause the difference between M-continuity and M^*-continuity.

THEOREM 6.2.11. *Let $f : X \to Y$ be a function from an MSS (X, m_X) into an MSS (Y, m_Y), then the following are equivalent:*
(i) f is M^*-continuous.
(ii) $f^{-1}(i^*_{m_Y}(B)) \subset i^*_{m_X}(f^{-1}(B))$ for $B \subset Y$.
(iii) $c^*_{m_X}(f^{-1}(B)) \subset f^{-1}(c^*_{m_Y}(B))$ for $B \subset Y$.
If further f is a bijection, the necessity of it is given below, then all these three statements are equivalent to
(iv) $f(c^*_{m_X}(A)) \subset c^*_{m_Y}(f(A))$ for $A \subset X$.

EXAMPLE 6.2.5. Let $X = \{a, b, c\}, Y = \{1, 2\}, m_X = \{\phi, X, \{b\}\}$, $m_Y = \{\phi, Y, \{2\}\}$ and $f : X \to Y$ be a function defined by $f(a) = f(c) = 1, f(b) = 2$; f is M^*-continuous hence (iii) holds but $f(c^*_{m_X}(A)) = f(X) = \{1, 2\} \nsubseteq \{1\} = c^*_{m_Y}(\{1\}) = c^*_{m_Y}(f(A))$, where $A = \{c\}$ shows that (iv) does not hold. Also let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $m_X = \{\phi, X, \{b\}\}$ and $m_Y = \{\phi, Y, \{2\}\}$. Then the function $f : X \to Y$ defined by $f(a) = f(b) = 1, f(c) = 2$ is not M^*-continuous, so (iii) can not be true but it can be verified that (iv) holds.

DEFINITION 6.2.9. Intersection of any right open ray (a, ∞) in the linearly ordered set of real numbers with $[0, 1]$ is called a right open ray in $[0, 1]$; similarly define left open ray in $[0, 1]$. A function $f : X \to [0, 1]$ from an MSS (X, m_X) into the subspace $[0, 1]$ of $\mathbb{R}$ with usual topology is said to be m_X-upper-semi-continuous (resp. m_X-lower-semi-continuous) [36] at a point $x \in X$ if for any left (resp. right) open ray R in $[0, 1]$ containing $f(x)$, there is $U \in M(x)$

such that $f(U) \subset R$; f is called m_X-upper-semi-continuous (resp. m_X-lower-semi-continuous) if it is so at each point of its domain.

EXAMPLE 6.2.6. Let m_l and m_r be the collections of all left-open rays and right-open rays respectively in $[0, 1]$ together with ϕ and $[0, 1]$. Then the identity mapping e on $[0, 1]$ is m_l-upper-semi-continuous, m_r-lower-semi-continuous and $m_l \cup m_r$-upper-lower-semi-continuous but neither it is m_l-lower-semi-continuous nor m_r-upper-semi-continuous even $e : ([0, 1], m_l \cup m_r) \to [0, 1]$ is not M-continuous.

We end this chapter with the following generalization of Urysohn's lemma.

THEOREM 6.2.12. *For any pair of disjoint m-closed sets C and D in an m-normal MSS (X, m) there is a m-upper-lower-semi-continuous function $f : X \to [0, 1]$ so that $f(x) = 0\ \forall\ x \in C$ and $f(x) = 1\ \forall\ x \in D$.*

PROOF. Let C and D be any disjoint m-closed sets in an m-normal MSS (X, m), $V_1 = X - D$ and ${V_0}^* = C$. Since the m-closed set ${V_0}^*$ is contained in the m-open set V_1, by using m-normality there is an m-open set $V_{\frac{1}{2}}$ and an m-closed set ${V_{\frac{1}{2}}}^*$ so that ${V_0}^* \subset V_{\frac{1}{2}} \subset {V_{\frac{1}{2}}}^* \subset V_1$. Applying the hypothesis on (X, m) to each pair ${V_0}^*$, $V_{\frac{1}{2}}$ and ${V_{\frac{1}{2}}}^*$, V_1, we have m-open sets $V_{\frac{1}{4}}$, $V_{\frac{3}{4}}$ and m-closed sets ${V_{\frac{1}{4}}}^*$, ${V_{\frac{3}{4}}}^*$ so that ${V_0}^* \subset V_{\frac{1}{4}} \subset {V_{\frac{1}{4}}}^* \subset V_{\frac{1}{2}} \subset {V_{\frac{1}{2}}}^* \subset V_{\frac{3}{4}} \subset {V_{\frac{3}{4}}}^* \subset V_1$. Continuing this process one can define m-open sets V_s, V_t and m-closed sets ${V_s}^*$, ${V_t}^*$ for any dyadic rational s and t in $[0, 1]$ of the form $\frac{k}{2^n}$, $k = 1, 2, 3.......2^n - 1$ and $n \in \mathbb{N}$ so that $s < t \Rightarrow {V_0}^* \subset V_s \subset {V_s}^* \subset V_t \subset V_t^* \subset V_1$. If s is any other dyadic rational, let $V_s = \phi$ for $s \leq 0$, $V_s = X$ for $s > 1$, and ${V_s}^* = \phi$ for $s < 0$, ${V_s}^* = X$ for $s \geq 1$. Now consider a function $f : X \to [0, 1]$ defined by $f(x) = \inf\{s; x \in V_s\} = \inf\{s; x \in {V_s}^*\}\ \forall\ x \in X$. Then $f(x) = 0$, $\forall\ x \in C$ and $f(x) = 1$, $\forall\ x \in D$. Definition of the function f and construction of the sets V_s and ${V_s}^*$ for dyadic rational number s show that $x \in {V_s}^* \Rightarrow f(x) \leq s$ and $x \notin V_s \Rightarrow f(x) \geq s$. Now for the ray $[0, 1]$ (which is left as well as right open ray in $[0, 1]$) U is an m-open set containing x_0 so that $f(U) \subset [0, 1]$, where $U = V_1$ if $f(x_0) = 0$ and if $f(x_0) = 1$ then $U = X - {V_0}^*$. For any left open ray $[0, d)$ in $[0, 1]$ containing $f(x_0)$ choose a dyadic rational q such that $f(x_0) < q < d$ and consider the V_q. Since $0 < q < 1$, V_q is m-open set containing x_0 because otherwise $f(x_0) \geq q$. Also $f(V_q) \subset [0, d)$ since $x \in V_q \Rightarrow f(x) \leq q$. And for any right open ray $(c, 1]$ in $[0, 1]$ containing $f(x_0)$ select a dyadic rational p such that $c < p < f(x_0)$ then, $X - {V_p}^*$ is an m-open set containing x_0 so that $f(X - {V_p}^*) \subset (c, 1]$. ■

Remark. In Theorem 6.2.12, if the minimal structure m is a topology then C and D become disjoint closed sets in the corresponding normal topological space, m-upper-lower-semi-continuity corresponds to continuity and Urysohn's lemma of point set topology will follow. Thus Theorem 6.2.12 is not a translation of Urysohn's lemma, it is a generalization of Urysohn's lemma.

Chapter VII

Towards a variant of Urysohn's lemma in weak structures

The year 2011 brought the raise of weak structure by Á. Császár [21]. Very recent, it was further studied by E. Ekici [30], A. Al-Omari, T. Noiri [2], A. M. Zahran, Kamal El-Saady and A. Ghareab [68], A. Güldürdek [34], myself [58] etc.. We extend the field by introducing notions of various separation axioms, continuity and established the Urysohn's lemma in this context. At first we give some basic definitions and outline of the results relevant to this extension. As in [21, 30, 2] a family w of subsets of a nonvoid set X containing the void subset is called a weak structure and we call the ordered pair (X, w) a weak structured space (briefly WSS). Elements of a weak structure w on X are called w-open sets and their complements are called w-closed sets. For any $A \subset X$, $i_w(A)$ denotes the union of all w-open sets contained in A and $c_w(A)$ denotes the intersection of all w-closed sets containing A; presence of the void set in w ensures that i_w and c_w are well-defined. It is to note that i_w is contractive, c_w is expansive, both are monotonic and idempotent and they are connected by the relation $i_w(A) = X - c_w(X - A)$, for any $A \subset X$. $x \in i_w(A)$ iff there is a w-open set $B \subset A$ containing x and $x \in c_w(A)$ iff $B \cap A \neq \phi$ whenever $x \in B \in w$. Unlike in topology or generalized topology $i_w(A)$ may not be w-open and $c_w(B)$ may not be w-closed, for $A, B \subset X$ unless A is w-open and B is w-closed. In this chapter we have shown, translating some results of [59] in the light of weak structures, that how these dissimilarities exhibit differences in separation axioms, continuity and Urysohn's lemma.

7.1. Star-interior, Star-closure and Continuity

This is a preparatory section to reach our goal. Let's begin with the family $W(x) = \{U; x \in U \in w\}$, where w is a weak structure on X and call it weak-star at x and by the help of the family $\{W(x); x \in X\}$ of weak-stars we define star-interior i^* and star-closure c^* operators as follows: $i^*{}_w(A) = \{x \in A; A \in W(x)\}$ and $c^*{}_w(A) = \{x \in X; X - A \notin W(x)\}$. In this setting it is observed that $i^*{}_w$ agrees with i_w in w and $c^*{}_w$ agrees with c_w in $w^c = \{X - U; U \in w\}$; also if $A \notin w$ then $i^*{}_w(A) = \Phi$ and $c^*{}_w(X - A) = X$. So, $i^*{}_w(A) \subset i_w(A)$, $c_w(A) \subset c^*{}_w(A)$. Let $X = \{a, b, c\}$ and $w = \{\Phi, \{a\}, \{b, c\}\}$ then $i^*{}_w(\{a\}) = \{a\} \supset \Phi = i^*{}_w(\{a, b\})$ and $c^*{}_w(\{b, c\}) = \{b, c\} \subset X = c^*{}_w(\{b\})$, hence i^* and c^* are not monotonic on the power set $P(X)$ of the underlying set X. Now let $x \notin X - i^*{}_w(X - A) \Leftrightarrow x \in i^*{}_w(X - A) \Leftrightarrow x \in X - A \in w \Leftrightarrow X - A \in W(x) \Leftrightarrow x \notin c^*{}_w(A)$. Thus $c^*{}_w(A) = X - i^*{}_w(X - A)$ and similarly $i^*{}_w(A) = X - c^*{}_w(X - A)$ for all $A \subset X$. The purpose of the definitions of $i^*{}_w$ and $c^*{}_w$ is the fact that $i^*{}_w(A) = A \Rightarrow A \in w$ and $c^*{}_w(A) = A \Rightarrow X - A \in w$. Some results emerged from the above discussion are accumulated in the following

Theorem 7.1.1. *Let w be a weak structure on X. Then*

1. $i^*{}_w(A){=}i_w(A)\ \forall\ A \in w$

2. $c^*{}_w(X-A){=}c_w(X-A)\ \forall\ A \in w$

3. $i^*{}_w(A) = \Phi$ *and* $c^*{}_w(X-A) = X$ *whenever* $A \in P(X) - w$.

4. $i^*{}_w(A) \subset i_w(A)$, *and* $c_w(A) \subset c^*{}_w(A)\ \forall\ A \subset X$.

5. $i^*{}_w$ *and* $c^*{}_w$ *are monotonic on* w *but not on the power set* $P(X)$ *(if* $w \neq P(X)$*) of the underlying set* X.

6. $c^*{}_w(A) = X - i^*{}_w(X-A)\ \forall\ A \subset X$.

7. $i^*{}_w(A) = A \Rightarrow A \in w$ *and* $c^*{}_w(A) = A \Rightarrow X - A \in w$.

A function $f : X \to Y$ from a WSS (X, w_X) into a WSS (Y, w_Y) is said to be W-continuous at a feasible point $x \in X$ if for any $V \in W(f(x))$, there is $U \in W(x)$ such that $f(U) \subset V$; f is called W-continuous if it is so at each feasible point of its domain. Three equivalent definitions for a function $f : X \to Y$ from a WSS (X, w_X) into a WSS (Y, w_Y) to be W-continuous are given by the following

THEOREM 7.1.2. *For any function* $f : X \to Y$ *from a WSS* (X, w_X) *into a WSS* (Y, w_Y)*, as in general case, it is observed that the following are equivalent:*

(1) f *is W-continuous.*

(2) $f(c_{w_X}(A)) \subset c_{w_Y}(f(A))$ *for* $A \subset X$

(3) $c_{w_X}(f^{-1}(B)) \subset f^{-1}(c_{w_Y}(B))$ *for* $B \subset Y$.

(4) $f^{-1}(i_{w_Y}(B)) \subset i_{w_X}(f^{-1}(B))$ *for* $B \subset Y$.

PROOF. As in point set topology. ■

A function $f : X \to Y$ from a WSS (X, w_X) into a WSS (Y, w_Y) is said to be W^*-continuous if for every $B \in w_Y$, $f^{-1}(B) \in w_X$. Obviously W^*-continuity implies W-continuity but the converse is not true as seen in the following Example 7.1.1.

EXAMPLE 7.1.1. Let $X = Y = \{a, b, c\}$, $w_X = \{\phi, \{a\}, \{b\}\}$,and $w_Y = \{\phi, \{a, b\}\}$ then the identity function is W-continuous but not W^*-continuous.

If $x \in i^*_w(A)$ then there is a $U \in W(x)$ contained in A but $x \in c^*_w(A)$ does not mean that every $V \in W(x)$ intersects A as shown in the following Example 7.1.2.

EXAMPLE 7.1.2. Let $X = \{a, b, c\}$, $w = \{\phi, \{a\}, \{a, b\}, \{b, c\}\}$ and $A = \{b\}$ then, $a \in c^*{}_w(A) = X$ and $V = \{a\} \in W(a)$ but $V \cap A = \phi$.

This fact together with the non-monotonicity of the operators i^* and c^* cause the difference between W-continuity and W^*-continuity. A set of necessary and sufficient conditions for a function $f : X \to Y$ from a WSS (X, w_X) into a WSS (Y, w_Y) to be W^*-continuous are given in the following

THEOREM 7.1.3. *Let $f : X \to Y$ be a function from a WSS (X, w_X) into a WSS (Y, w_Y), then the following are equivalent:*
(i) f is W^*-continuous.
(ii) $f^{-1}(i^*_{w_Y}(B)) \subset i^*_{w_X}(f^{-1}(B))$ for $B \subset Y$.
(iii) $c^*_{w_X}(f^{-1}(B)) \subset f^{-1}(c^*_{w_Y}(B))$ for $B \subset Y$.

If further f is a bijection, the necessity of it is given below, then all these three statements are equivalent to
(iv) $f(c^*_{w_X}(A)) \subset c^*_{w_Y}(f(A))$ for $A \subset X$.

Example 7.1.3. Let $X = \{a, b, c\}$, $Y = \{1, 2\}$, $w_X = \{\phi, \{b\}\}$, $w_Y = \{\phi, \{2\}\}$ and $f : X \to Y$ be a function defined by $f(a) = f(c) = 1, f(b) = 2$; f is W^*-continuous hence (iii) holds but $f(c^*_{w_X}(A)) = f(X) = \{1, 2\} \not\subseteq \{1\} = c^*_{w_Y}(\{1\}) = c^*_{w_Y}(f(A))$, where $A = \{c\}$ shows that (iv) does not hold. Also let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $w_X = \{\phi, \{b\}\}$ and $w_Y = \{\phi, \{2\}\}$. Then the function $f : X \to Y$ defined by $f(a) = f(b) = 1, f(c) = 2$ is not W^*-continuous, so (iii) can not be true but it can be verified that (iv) holds.

Intersection of any right open ray (a, ∞) in the linearly ordered set of real numbers with $[0, 1]$ is called a right open ray in $[0, 1]$; similarly define left open ray in $[0, 1]$. A function $f : X \to [0, 1]$ from a WSS (X, w_X) into the subspace $[0, 1]$ of $\mathbb{R}$ with usual topology is said to be w_X-upper-semi-continuous (resp. w_X-lower-semi-continuous) [36] at a feasible point $x \in X$ if for any left (resp. right) open ray R in $[0, 1]$ containing $f(x)$, there is $U \in W(x)$ such that $f(U) \subset R$; f is called w_X-upper-semi-continuous (resp. w_X-lower-semi-continuous) if it is so at each feasible point of its domain.

Example 7.1.4. Let w_l and w_r be the collections of all left-open rays and right-open rays respectively in $[0, 1]$ together with ϕ. Then the identity mapping e on $[0, 1]$ is w_l-upper-semi-continuous, w_r-lower-semi-continuous and $w_l \cup w_r$-upper-lower-semi-continuous but neither it is w_l-lower-semi-continuous nor w_r-upper-semi-continuous even $e : ([0, 1], w_l \cup w_r) \to [0, 1]$ is not W-continuous.

7.2. Richness and a variant of Urysohn's Lemma

In a WSS (X, w) there may be some points that are members of no w-open set; we call the collection of all such points as the dark set in (X, w) and denote it by $D(X, w)$. Elements of $D(X, w)$ are called dark points and others are called feasible points.

EXAMPLE 7.2.1. Consider the WSs $w_0 = \{\phi, \{a\}, \{b\}\}$, $w_1 = \{\phi, \{a\}, \{b\}, \{c\}\}$ and $w_2 = \{\phi, \{a, b\}, \{b, c\}, \{c, a\}\}$ on $X = \{a, b, c\}$ then $\{c\}$ is the dark set and c is a dark point of (X, w_0) but (X, w_1) and (X, w_2) has no dark point.

Obviously dark set $D(X, w)$ is subset of any closed set in (X, w) and consequently it is a subset of $c_w(A)$ and $c^*{}_w(A)$ for any $A \subset X$. Therefore neither closed sets nor $c_w(A)$ or $c^*{}_w(A)$ for $A \subset X$ can be contained in an w-open set. These break the usual path for studying separation axioms in weak structures.

A weak structured space (X, w) is called $w - T_1$ if for any two distinct feasible points $x, y \in X$ there are w-open sets U and V so that $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$. The above WSSs in Example 7.2.1 show that a finite WSS may be $w - T_1$ without being $P(X)$.

THEOREM 7.2.1. *A WSS (X, w) is $w - T_1$ iff $c_w(\{x\}) = \{x\} \cup D(X, w)$, $\forall x \in X$.*

PROOF. If $x \in D(X,w)$ then $c_w(\{x\}) = D(X,w) = \{x\} \cup D(X,w)$. If $x \notin D(X,w)$ then $y \in c_w(\{x\})$ for some feasible $y \neq x$ implies every w-open set containing y contains x and so (X,w) is not $w-T_1$. Since $x \in c_w(\{x\}) \supset D(X,w)$ $c_w(\{x\}) = \{x\} \cup D(X,w)$. Conversely for any two distinct feasible points x and y in a WSS (X,w), by the hypothesis, $c_w(\{x\}) = \{x\} \cup D(X,w)$ and $c_w(\{y\}) = \{y\} \cup D(X,w)$. Clearly $y \notin c_w(\{x\})$ and $x \notin c_w(\{y\})$; this implies there are w-open sets U and V containing x and y respectively such that $U \bigcap \{y\} = \Phi$ and $V \bigcap \{x\} = \Phi$, so (X,w) is $w - T_1$. ■

Note that those singletons may not be w-closed, (X,w_1) in Example 7.2.1 is one such wetness. Thus in a $w - T_1$ WSS singletons may not be w-closed.

A WSS (X,w) is called $w - T_2$ if for any two distinct feasible points $x, y \in X$ there exists disjoint w-open sets containing them. In Example 7.2.1 (X,w_1) is a $w - T_2$ but (X,w_2) is not $w - T_2$ though it is $w - T_1$. Obviously a $w - T_2$ WSS is $w - T_1$. A sequence $\{x_n\}$ in a WSS (X,w) is said to be convergent if it is not eventually infeasible and $\exists$ a feasible point $x \in X$ such that for any w-open set $U \ni x$ there exists a positive integer N so that $x_n \in U$ for all $n \geq N$; in this case $\{x_n\}$ is called a convergent sequence and x is called limit of it.

THEOREM 7.2.2. *In a $w - T_2$ WSS every convergent sequence has unique limit.*

PROOF. If x, y are two distinct limits of a sequence $\{x_n\}$ in a

$w - T_2$ WSS (X, w), then for any two w-open sets U and V containing x and y respectively there exists a positive integer N so that $x_n \in U \cap V$ for all $n \geq N$ and this contradicts the fact that the WSS is $w - T_2$. ∎

A WSS (X, w) is called w-regular if for any point $x \in X$ and any w-closed set C not containing x there exists disjoint w-open sets U and V so that $x \in U$ and $v(C) \subset V$, where $v(A) = A - D(X, w)$ called feasible part of $A \subset X$.

THEOREM 7.2.3. *If (X, w) is w-regular WSS then for any $x \in U \in w$ there exists $V \in w$ so that $x \in V \subset v(c_w(V)) \subset U$.*

PROOF. Let (X, w) be a w-regular WSS and U be any w-open set, then for any point $x \in U$ the w-closed set $X - U$ does not contain x, so there exists w-open sets V and E such that $x \in V$, $v(X - U) \subset E$ and $V \cap E = \phi$. This implies $X - E \subset U \cup D(X, w)$ and $V \subset X - E$ hence $c_w(V) \subset X - E \subset U \cup D(X, w)$. Thus for any $x \in U \in w$ there exists $V \in w$ so that $x \in V \subset v(c_w(V)) \subset U$. ∎

But the converse of the Theorem 7.2.3 is not true as seen in the WSS (X, w_1) of the Example 7.2.1 which is true in regular topological spaces.

Let in a WSS (X, w) for any $x \in U \in w$ there exists $V \in w$ so that $x \in V \subset v({c^*}_w(V)) \subset U$. Let x be any point of X and C be any w-closed set not containing x, so $x \in X - C$ and $X - C$ is a w-open set and therefore by the hypothesis there exists w-open set V such that $x \in V \subset v(c^*{}_w(V)) \subset X - C \Rightarrow x \in V$ and $v(C) = C - D(X,w) \subset X - c^*{}_w(V) = T$ (say). If T is nonvoid then the conditions for the WSS (X,w) to be w-regular will follow and it is not possible unless V is w-clopen (w-closed as well as w-open). Thus we obtain the following

THEOREM 7.2.4. *A WSS (X,w) is w-regular if for any $x \in U \in w$ there exists a w-clopen set V so that $x \in V \subset U$.*

Obviously the Theorem 7.2.4 will not be used to check w-regularity of a WSS having dark points.

EXAMPLE 7.2.2. The WSS $(\mathbb{R}, w)$, where w is the collection of all right and left open rays including ϕ shows that the condition in Theorem 7.2.4 is not necessary.

Theorem 7.2.3 and Theorem 7.2.4 give clue to obtain a necessary and sufficient condition for a WSS (X,w) to be w-regular. The conditions are listed in the following

THEOREM 7.2.5. *A WSS (X,w) is w-regular if and only if for any point $x \in X$ and any w-open set U containing x there exist w-open set V and w-closed set V^* so that $x \in V \subset v(V^*) \subset U$.*

PROOF. Let a WSS (X,w) be w-regular, x be any point in X and U be any w-open set containing x. Then $X - U$ is w-closed set not containing x. So, there are w-open sets E and F such that $x \in E$, $v(X - U) \subset F$ and $E \cap F = \phi$. Hence $v(X - F) \subset U$

and $E \subset (X - F) \subset U$, so $v(E) = E \subset v(X - F)$. Taking $V = E$ and $V^* = (X - F)$ we have V is w-open, V^* is w-closed and $x \in V \subset v(V^*) \subset U$. Conversely, let for any point $x \in X$ and any w-open set U containing x there exists w-open set V and w-closed set V^* so that $x \in V \subset V^* \subset U$. Let $x \in X$ and E be any w-closed set not containing x. Then $x \in (X - E)$ and $(X - E)$ is w-open. Hence by the hypothesis there is a w-open set V and a w-closed set V^* so that $x \in V \subset v(V^*) \subset (X - E)$. Therefore $V^* - D(X, w) \subset X - E$ and so $v(E) \subset (X - V^*)$, also $V \cap (X - V^*) \subset V \cap (X - V) = \phi$ hence $V \cap (X - V^*) = \phi$ which implies (X, w) is w-regular. ■

A WSS (X, w) is called w-normal if for any two w-closed sets C and D with $v(C) \cap v(D) = \phi$ there exists two disjoint w-open sets U and V so that $v(C) \subset U$ and $v(D) \subset V$. Some results related to w-normality of a WSS (X, w) without proof, because of similarity with regularity, are listed below

THEOREM 7.2.6. *Let C be any w-closed set and U be any w-open set containing C. Then there exists a w-open set V so that $v(C) \subset V \subset v({c^*}_w(V)) \subset U \Rightarrow$ WSS (X, w) is w-normal $\Rightarrow$ there exists a w-open set V so that $v(C) \subset V \subset v(c_w(V)) \subset U$.*

THEOREM 7.2.7. *A necessary and sufficient condition for a WSS (X, w) is w-normal if for any w-closed set C and any w-open set U with $v(C) \subset U$ there is a w-open set V and a w-closed set V^* such that $v(C) \subset V \subset v(V^*) \subset U$.*

We conclude with the following generalization of Urysohn's lemma.

THEOREM 7.2.8. *For any pair of w-closed sets C and D in a w-normal WSS (X, w) with $v(C) \cap v(D) = \phi, v(C) \neq \phi$ and $v(D) \neq \phi$ there is a w-upper-lower-semi-continuous function $f : X \to [0, 1]$ so that $f(x) = 0 \ \forall \ x \in v(C)$ and $f(x) = 1 \ \forall \ x \in v(D)$.*

PROOF. Let C and D be any closed set in a w-normal WSS (X, w) with $v(C) \cap v(D) = \phi$, $V_1 = X - D$ and ${V_0}^* = C$. Since the feasible part of w-closed set ${V_0}^*$ is contained in the w-open set V_1, by using w-normality there is a w-open set $V_{\frac{1}{2}}$ and a w-closed set ${V_{\frac{1}{2}}}^*$ so that $v({V_0}^*) \subset V_{\frac{1}{2}} \subset v({V_{\frac{1}{2}}}^*) \subset V_1$. Applying the hypothesis on (X, w) to each pair ${V_0}^*$, $V_{\frac{1}{2}}$ and ${V_{\frac{1}{2}}}^*$, V_1, we have w-open sets $V_{\frac{1}{4}}$, $V_{\frac{3}{4}}$ and w-closed sets ${V_{\frac{1}{4}}}^*$, ${V_{\frac{3}{4}}}^*$ so that $v({V_0}^*) \subset V_{\frac{1}{4}} \subset v({V_{\frac{1}{4}}}^*) \subset V_{\frac{1}{2}} \subset v({V_{\frac{1}{2}}}^*) \subset V_{\frac{3}{4}} \subset v({V_{\frac{3}{4}}}^*) \subset V_1$. Continuing this process one can define w-open sets V_s, V_t and w-closed sets ${V_s}^*$, ${V_t}^*$ for any dyadic rational s and t in $[0, 1]$ of the form $\frac{k}{2^n}$, $k = 1, 2, 3.......2^n - 1$ and $n \in \mathbb{N}$ so that $s < t \Rightarrow v({V_0}^*) \subset V_s \subset v({V_s}^*) \subset V_t \subset v({V_t}^*) \subset V_1$. If s is any other dyadic rational, let $V_s = \phi$ for $s \leq 0$, $V_s = X$ for $s > 1$, and ${V_s}^* = \phi$ for $s < 0$, ${V_s}^* = X$ for $s \geq 1$. Note that V_s for $s > 1$ may not be w-open and so ${V_s}^*$ for $s < 0$ may not be w-closed. Now consider a function $f : X \to [0, 1]$ defined by $f(x) = \inf\{s; x \in V_s\} = \inf\{s; x \in {V_s}^*\} \ \forall \ x \in X$. Then $f(x) = 0$, $\forall \ x \in C$ and $f(x) = 1$, $\forall \ x \in D$. Definition of the function f and construction of the sets V_s and ${V_s}^*$ for dyadic rational number s show that $x \in {V_s}^* \Rightarrow f(x) \leq s$ and $x \notin V_s \Rightarrow f(x) \geq s$. Let x_0 be any feasible point in (X, w). Now for $[0, 1]$ (which is left as well as right open ray in $[0,1]$) U is an w-open set containing x_0 so that $f(U) \subset [0,1]$, where $U = V_1$ if $f(x_0) = 0$ and $U = X - {V_0}^*$ if $f(x_0) = 1$. For any left open ray $[0,d)$ in $[0,1]$ containing $f(x_0)$ choose a dyadic rational q such that $f(x_0) < q < d$ and consider the V_q. Since $0 < q < 1$, V_q is w-open set containing x_0 because otherwise $f(x_0) \geq q$. Also $f(V_q) \subset [0,d)$ since $x \in V_q \Rightarrow f(x) \leq q$. And for any right open ray $(c,1]$ in $[0,1]$ containing $f(x_0)$ select a dyadic rational p such that $c < p < f(x_0)$ then, $X - {V_p}^*$ is a w-open set containing x_0 so that $f(X - {V_p}^*) \subset (c,1]$; this completes the proof. ■

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