CHAPTER I

INTRODUCTION

I.1. INTRODUCTION

During the last decade Quantum Chromodynamics (QCD) has emerged as one of the most promising candidate for a dynamical theory of strong interaction. It is basically a non-abelian gauge field theory in which coloured quarks interact via exchange of coloured vector gluons. One of the most spectacular feature of QCD is that, unlike in QED, the effective coupling constant becomes small at large momentum transfers, so that quarks behave almost like free particles at very small seperations. This concept, known as asymptotic freedom[1,2] is consistent with the recent deep-inelastic scattering data[3]. On the other hand, there is a general expectation that the effective coupling constant may become very large at the infrared asymptotic region[4-6]. Quarks feel, therefore, an increasingly strong restoring force at larger interquark distances. Though the existence of a confining phase has not yet been established conclusively, the confinement hypothesis of quarks seems to be an attractive concept. According to this hypothesis hadrons are supposed to be colour singlet states of an exact SU(3) colour group. The present picture allows considerable freedom in the choice of quark masses. Earlier, non-availability of free quarks led one to assume that quarks were extremely massive objects and they formed low-lying hadronic states through an extremely high binding force. The confinement hypothesis as propounded in QCD does not, however, require a very large mass for a quark, since even light quarks will not be detectable in free states, being trapped in a potential which presumably increases with the interquark distance.

The recent discovery of heavy mesons (Ψ and Υ families) with new hidden flavours, charm (c)[7,8] and bottom[9,10] (b) has also contributed to our understanding of quark dynamics within the framework of QCD. It is assumed that the new quarks c and b are more massive than the u , d and s quarks. The colour singlet states, Ψ , Ψ', Ψ''... and Υ , Υ', ... etc. are quark-antiquark ($q\,\bar{q}$) bound states, which can be described simply by a Schrodinger equation[11]. The relativistic correction to their masses will be small, since the quark mass (assumed to be roughly half of the meson mass) is assumed to be large compared to the inter-quark potential, which is also taken to be flavour independent. It is expected that an understanding of the phenomenology of the new heavy mesons within the framework of a Schrodinger equation will help to clarify the more fundamental aspects of quark dynamics in general.

In non-relativistic potential model calculations, one usually assumes the static interquark potential to be of the form

$$V(r) = -\frac{\Sigma}{r} + V_c(r) \qquad (1.1)$$

The coulomb-like term involving a running coupling constant may be generated by one-gluon exchange diagrams. $V_c(r)$, on the other hand, is assumed to be responsible for quark confinement and is generally undetermined. Eichten et al[12], in their extensive investigations of the phenomenology of the Ψ-particles used an idea (originally obtained from lattice gauge models) which states that the effective quark potential should grow at large distances as the inter-quark distance. In this approximation, one may take $V_c(r) = \alpha r$, with a suitable choice of the confining coupling constant α.

The success of this simple model is not, however, conclusive in the sense that a large class of potentials is found to give more or less acceptable fit to the experimental data for Ψ as well as Υ families. This makes it difficult to select a potential uniquely. It is, therefore, more useful to obtain rigorous constraints on energy levels and the wave function, particularly at the origin, for general classes of potentials which may be compared directly with the experimental data for the new heavy meson families. This approach is followed independently by Quigg, Rosner and Thacker[13–15] and by Martin and his collaborators[16–23]. The former group has obtained some interesting results for specific potential models by using simple scaling laws and also semi-classical arguments. They have also proposed an inverse scattering method for reconstructing the quarkonium potential directly from the experimental data and to use later on this potential to make predictions. Martin et al, on the other hand, have tried to find out general features of classes of potentials to discriminate models which are allowed or forbidden by the data. Their study reveals, for example, that the cannnical level ordering $E(1S) < E(1P) < E(2S) < E(1D)$, which has been observed experimentally, favours a special class of potentials which includes a general superposition of power potentials, viz:

$$V(r) = \int_{-1}^{2} d\alpha \, \epsilon(\alpha) \, r^{\alpha} \rho(\alpha) \, , \ \rho \geqslant 0 \, , \ \epsilon = \frac{\alpha}{|\alpha|} \, . \qquad (1.2)$$

Further constraints on the potentials may be obtained by examining the mass and angular momentum dependence of the corresponding energy levels and the wave function at the origin[21-23].

The study of general properties of Schrodinger energy levels for confining potential is, therefore, a subject of considerable interest. The aim of this thesis is to study this problem analytically for some interesting classes of quark confining potentials. In particular, we investigate[24-26] the analyticity of the energy levels for the class of potentials

$$V(r) = -\frac{\zeta}{r} + \alpha r + \beta r^2 \quad , \qquad (1.3)$$

in the ζ, β coupling constant planes. Potentials (1.3) is a special class of the allowed quarkonium potentials (1.2) and can be obtained by replacing $\rho(\alpha')$ by $\zeta\,\delta(\alpha'+1) + \alpha\,\delta(\alpha'-1) + \beta\,\delta(\alpha'-2)$.

The study of the coupling constant analyticity of the Schrodinger energy levels was initiated by Bender and Wu[27] in connection with an anharmonic oscillator

$$H = p^2 + x^2 + \lambda x^4 \quad , \quad p = -i\frac{d}{dx} \quad . \qquad (1.4)$$

The motivation for this study came from the fact that the exact behaviour of the singular perturbation series in most of the cases, is not fully understood. It is only accepted on heuristic grounds that the series may be asymptotic near the weak coupling limit. The attention is, therefore, focussed on simpler cases of singular perturbations where the exact analytic properties of the theory may be studied. It is suggested that a systematic study of the simpler cases will help to understand the more complex features of the perturbation theory. Using an improved WKB technique, Bender and Wu investigated the λ -plane analyticity of the energy levels $\varepsilon_n(\lambda)$ for the Hamiltonian (1.4). The Bender-Wu results were subsequently established more rigorously by Simon[28]. Following the Kato-Rellich perturbation theory for linear operators in a Hilbert space, Simon developed a general method of handling singular perturbations and applied it to the anharmonic oscillator Hamiltonian (1.4). Later on the method was also used to obtain rigorous results for atomic Stark effects[29]. Graffi et al[30], in particular, investigated the analyticity of the Schrodinger energy levels $\varepsilon_n(\alpha)$ for the potential

$$V(x) = -\frac{\zeta}{x} + \alpha x \,, \quad x > 0 \,. \qquad (1.5)$$

The charmonium-like potential, eqn. (1.5), represents, in atomic physics, the one-dimensional Stark potential in hydrogen atom (α being the electric field parameter). We have used, in our calculations, the results of Simon[28] as well as of Graffi et al[30].

The class of potentials (1.3) studied by us has important applications also in other branches of physics. In atomic physics, this could be interpreted as the potential seen by an electron of an atom exposed to a suitable admixture of electric and magnetic fields. Recently Rau[31] has suggested a method to realize this type of potentials. According to the model, an electron at the surface ($z = 0$) of liquid helium experiences, in addition to the force due to the one dimensional image coulomb potential arising from polarization of the helium atoms, an electric field normal to the surface and a crossed magnetic field along the surface. The resulting potential should then assume a form similar to eqn. (1.3) i.e. $-\zeta/z + \alpha z + \beta z^2$. Since electrons are assumed to move freely in the xy plane, the wave equations describing the motion parallel and perpendicular to the surface are separable.

Hence the non-trivial factor of the wave function

$$\psi(r) = e^{i\vec{k}\cdot\vec{R}}\,\phi(z)$$

where $\vec{R}$ is a vector in the plane of the surface, satisfies the Schrodinger equation[32]

$$-\frac{\hbar^2}{2m^2}\frac{d^2\phi}{dz^2} + V(z)\,\phi(z) = E\,\phi(z)\,, \qquad (1.6)$$

with the boundary conditions $\phi(0) = 0$, $\phi(\infty) = 0$. Eqn. (1.6) is clearly similar to the S-wave radial equation for the potential (1.3). A systematic study of the analyticity of the energy levels for potentials (1.3) in the coupling constant planes, therefore, may be useful for many problems. The results of this study constitute a major part of the thesis. In the remaining part, we investigate[42,43] the angular momentum (ℓ) analyticity of the Schrodinger energy levels. As pointed out by Grosse and Martin[23], the study of this problem is relevant for the quarkonium physics. They initiated this study and proved the analyticity for $\mathrm{Re}\,\ell > -\frac{1}{2}$ for a single power law potential $[\,V(r) = r^{\alpha}\,,\ \alpha > 0\,]$. We have been able to use a different method, the linear operator theory of Kato and Rellich, to prove the analyticity, for $\mathrm{Re}\,\ell > -1/2$ for a class of superposition of power potentials, which includes, in particular, the conventional charmonium potential, eqn. (1.5).

I.2. SUMMARY OF THE WORK

The present work consists of the following parts:

(a) Analyticity by Linear Operator Technique

In chapter III we investigate the coupling constant analyticity of the energy level E_n for the potential (1.3) using the perturbation theory of linear operators in a Hilbert space. It is shown that the corresponding Schrodinger operator $H(\zeta, \alpha, \beta)$ defined on a dense domain D (H) of the square integrable functions, is a semi-bounded self-adjoint operator with compact resolvent for real values of ζ, α and $\beta > 0$. Corresponding energy levels $E_n(\zeta, \alpha, \beta)$ are, therefore, analytic simultaneously in the three real parameters ($\beta > 0$). For complex parameters, we make use of the standard results obtained by Simon[28] and Graffi et al[30]. It is shown that for complex ζ, α and β in the cut plane $|\arg \beta| < \pi$, $H(\zeta, \alpha, \beta)$ is a self-adjoint holomorphic family of type A in each parameter, when the other two remain fixed in the respective domains. We see, therefore, that each energy level E_n defined for real parameters, can be analytically continued for complex values of the parameters and the spectrum of the operator $H(\zeta, \alpha, \beta)$ with complex parameters consists precisely of these analytically continued E_n. We also see that the singularities for the function can only be branch points or natural boundaries. The Herglotz condition satisfied by E_n for each complex parameter guarantees the non-occurrence of an isolated pole or an isolated essential singularity in the finite portions of the complex plane.

(b) Perturbation Expansion for Energy Eigenvalues

General results of perturbation theory[28] can be applied to show that for non-negative ζ and α, the point $\beta = 0$ is an essential singularity, being the limit point of a sequence of branch points. This makes the Rayleigh-Schrodinger perturbation expansion of $E_n(\beta)$ in β totally divergent. The series is, however, asymptotic near $\beta = 0$ in an open domain. Moreover, following the arguments of Loeffel and Martin[39] it is shown that the first sheet of the β-cut plane is free from any singularity. Thus $E_n(\beta)$ is analytic in the entire cut plane $|\arg \beta| < \pi$. The asymptotic nature, the first-sheet analyticity together with the Symanzik scaling law thus assure the summability of the divergent series, in the Borel[40] and Pade[41] senses, to yield the correct energy level $E_n(\beta)$. Also, using Symanzik's scaling law and the analyticity of E_n in ζ near $\zeta = 0$, the existence of an expansion of

E_n in β^{-1} is proved for large values of β for the restricted class of potentials

$$V(r) = -\zeta/r + \beta^3 r + \beta^4 r^2 .$$

(c) Analyticity in the Complex Angular Momentum Plane

The angular momentum (ℓ) analyticity of $E_n(\ell)$ is a subject of considerable interest in many branches of physics. However, only in few cases the exact analyticity of the Regge trajectories $\ell(\sqrt{E})$ or $E(\ell)$ have been proved. It will be interesting to study the ℓ -plane analyticity of E_n for the class of potentials which may be relevant for the phenomenology of the quarkonia systems. Grosse and Martin[23] proved the analyticity of E_n for $Re\,\ell > -1/2$ for a pure power potential. In the first part of chapter IV we have modified their arguments to prove the analyticity for a class of potentials of the form

$$V(r) = \sum_{i=1}^{n} \kappa_i \, r^{\alpha_i} , \quad \kappa_i > 0 , \quad \alpha_1 > \alpha_2 > \cdots > \alpha_n . \tag{1.7}$$

Poles and isolated essential singulatities can be eliminated by a general argument. To discard branch points the approach is to start with the usual characterization of the energy level $E_n(\ell)$ by the number of nodes of the corresponding Schrodinger wave function $u_n(r)$ for real

$\ell > -1/2$ and to show that this characterization continues to be true even for complex ℓ with $Re\,\ell > -1/2$. The proof of this statement rests on showing that the number of zeros of the wave function $u_n(r)$ within a sector of the complex r-plane does not change as real ℓ is allowed to assume complex values in the half-plane $Re\,\ell > -1/2$. Thus starting from a point on the real ℓ axis and returning to it along a complex path in the half-plane $Re\,\ell > -1/2$ one gets back the same wave function. So no level crossing is observed in $Re\,\ell > -1/2$. Natural boundaries can also be discarded following the steps of Grosse-Martin.

In the second part the Rellich-Kato perturbation theory is applied to prove the analyticity for a larger class of potentials

$$V(r) = \begin{cases} r^{\alpha} + \sum k_i r^{\alpha_i}, & k_i \text{ real}, \ 0<\alpha_i<\alpha, \alpha \neq 1 \\ r + \sum k_i r^{\alpha_i}, & k_i \text{ real}, \ -2<\alpha_i<1 \end{cases} \quad (1.8)$$

Firstly, it is shown that the energy level E_n for the potential

$$V(r) = r^{\alpha} + k\beta r^{\beta}, \quad \beta < \alpha \qquad \ldots (1.9)$$

is analytic in β for (i) $-2 < \beta < 1$, $\alpha = 1$,

(ii) $0 < \beta < \alpha, \alpha \neq 1$ and also that every level is non-degenerate for β in the interval. Next, an argument is devised which shows that $E_n(\ell)$ is analytic in $R_e \ell > -1/2$ for the potential (1.9) with $\beta \neq 0$ once it is known to be true for $\beta = 0$. Finally, it has been shown that the arguments valid for the potential (1.9) can also be generalized for potentials (1.8) and hence the analyticity of the corresponding energy levels $E_n(\ell)$ follows.