CHAPTER 1

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Let f be an entire function and $M(r) \equiv M(r, f) = \max_{|z|=r} |f(z)|$ be the maximum modulus function of f on $|z| = r$. The function $M(r)$ plays a vital role in many situations. By Liouville's theorem it is known that a bounded entire function is constant, which implies that for nonconstant f, the maximum modulus function $M(r)$ is unbounded. The following theorem is due to Cauchy.

Theorem 1.0.1 **{*Theorem 1, p.5,[45]*}.** *The maximum of the modulus of a function f, which is regular in a closed connected region D, bounded by one or more curves C, is attained on the boundary.*

This theorem implies that when f is an entire function, $M(r)$ is a nondecreasing function of r for all values of r. Using the uniform continuity of f in any closed region and the above theorem , i.e. , the value $M(r)$ is attained by f on $|z| = r$, it follows that $M(r)$ is a continuous function of r. Also $M(r)$ is differentiable in adjacent intervals { Theorem 10, p.27, [45]}. In view of Hadamard's theorem {Theorem9, p.20,[45]} we know that $\log M(r)$ is a continuous, convex and ultimately increasing function of $\log r$.

Let f be an entire function and $M(r)$ be its maximum modulus function on $|z| = r$. f is said to be of finite order if there exists a positive number k such that $\log M(r) < r^k$ for all sufficiently large values of r. If there exists no such $k(> 0)$, f is called a function of infinite order. Let $\rho = \inf\{k : k > 0, \log M(r) < r^k$ for all sufficiently large values of $r\}$.

The number $\rho\,(\geq 0)$ is called the order of f. It can be easily verified that the order ρ of f has the following alternative definition

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

The number λ, defined by $\lambda = \liminf_{r\to\infty} \frac{\log\log M(r)}{\log r}$, is called the lower order of f. Clearly $\lambda \leq \rho$. If in particular, $\lambda = \rho$ for an entire function f, it is called of regular growth. For example, a polynomial or the function e^z is of regular growth.

Extending this notion, Sato **[34]** defined the generalised order and generalised lower order of an entire function as follows :

Definition 1.0.1 *The generalised order $\rho_f^{[l]}$ and the generalised lower order $\lambda_f^{[l]}$ of an entire function f are defined as :*

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M(r)}{\log r} \quad \textit{and} \quad \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M(r)}{\log r} \quad \textit{where}$$

$$log^{[l]}x = \log\left(\log^{[l-1]} x\right) \ \textit{for } l = 1, 2, 3, \ldots \ \textit{and } log^{[0]}x = x.$$

Juneja, Kapoor and Bajpai **[20]** gave a more generalised concept of Definition 1.0.1 which may be given in the following way :

Definition 1.0.2 *The (p, q) th order $\rho_f(p, q)$ and the (p, q) th lower order $\lambda_f(p, q)$ of an entire function f are defined respectively as follows :*

$$\rho_f(p, q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r)}{\log^{[q]} r} \quad \textit{and} \quad \lambda_f(p, q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r)}{\log^{[q]} r},$$

where p, q are positive integers with $p > q$.

If f is an entire function of positive finite order ρ, the number τ given by $\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho}$ is called the type of f. The quantities ρ, λ and τ are extensively used to the study of growth properties of f. It is well- known that the order and type of an entire function f is equal to those of its derivative f' {Theorem 2.4.1, p. 13, **[1]**}.

Let f be an entire function of finite order ρ.When more precise specification of the rate of growth of f is desired, one can use the proximate order, a function $\rho(r)$ with the following properties {p.64[**45**]},

I. $\rho(r)$ is continuous for $r > r_0$, say,

II. $\rho(r)$ is differentiable in adjacent intervals,

III. $\limsup_{r\to\infty} \rho(r) = \rho$ and $\liminf_{r\to\infty} \rho(r) \geq \beta$, where $0 \leq \beta \leq \rho$,

IV. $\lim_{r\to\infty} r\rho'(r)\log r = 0$,

V. $\limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho_f(r)}} = 1$.

Using some results of Blumenthal, Valiron {pp.64-67, [**45**]} proved the existence of a proximate order for an entire function of finite order. Shah [**33**] introduced the notion of lower proximate order for an entire function in the following way and proved its existence.

Definition 1.0.3 *Let f be an entire function of finite lower order λ. The function $\lambda(r)$ is called a lower proximate order of f if it satisfies the following properties:*

I $\lambda(r)$ is a non-negative continuous function of r for $r > r_0$, say,

II. $\lambda(r)$ is differentiable for $r > r_0$ except at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist,

III. $\lim_{r\to\infty} r\lambda'(r)\log r = 0$,

IV. $\lim_{r\to\infty} \lambda(r) = \lambda$ and

V. $\liminf_{r\to\infty} \frac{\log M(r)}{r^{\lambda(r)}} = 1$.

Using the notion of proximate order and lower proximate order it is some times possible to make sharper estimation of the number of zeros of an entire function f within the circle $|z| = r$. If $n(r)$ denotes the number of zeros of f within $|z| = r$, counted with multiplicities, then the following two inequalities hold

I $n(r) < kr^{\rho(r)}$ for all sufficiently large values of r and $k > 0$ {p.68,[**45**]},

II $n(r) \leq kr^{\lambda(r)}$ for a sequence of values of r tending to infinity and $k > 0$ {[**45**]}.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then the sequence $|a_0|, |a_1|r, |a_2|r^2, \ldots |a_n|r^n$ tends to zero for each value of $|z| = r$. For each value of r there is a term of this sequence which is greater than or equal to the rest of the terms. This term (or more) is called the maximum term for the given value of r and is denoted by $\mu(r, f)$, the term of the highest rank is to be taken as the maximum term and this rank of the maximum term is denoted by $\nu(r)$.

It is possible to estimate in terms of proximate order and lower proximate order the rank of the maximum term for $|z| = r$ of an entire infinite series which is given in the following theorem of Shah [**33**].

Theorem 1.0.2 *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is of finite order,*
I $\nu(r) \le kr^{\lambda(r)}$ *for a sequence of values of* r *tending to infinity,* $k > 0$,
II $\nu(r) \ge \frac{kr^{\rho(r)}}{\log r}$, *for a sequence of values of* r *tending to infinity,* $k > 0$.

D. Somasundaram and R. Thamizharasi [**41**] considered a positive continuous function $L(r)$ which increases slowly i.e. $L(ar) \sim L(r)$ as $r \to \infty$ and for every positive constant a . The collection of all such functions are denoted by $\mathcal{L}$.

They [**41**] introduced the definition of L-order and L-type of an entire function f denoting respectively by ρ_L and T_L as follows:

$$\text{For } L(r) \in \mathcal{L}$$
$$\rho_L = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log [rL(r)]}$$

and

$$T_L = \limsup_{r \to \infty} \frac{\log M(r)}{[rL(r)]^{\rho_L}}, \quad 0 < \rho_L < \infty.$$

Using the definition of L-order of an entire function they [**41**] proposed that if f_1 and f_2 are entire functions of L-orders ρ_1 and ρ_2 respectively and if $\rho_1 < \rho_2$ then the L-orders of $f_1 + f_2$ is ρ_2. They [**41**] hinted that the proposition fails if $\rho_1 = \rho_2$.

Another proposition [**41**] relating ρ_L and T_L states the following:

Theorem 1.0.3 *If* f *is of positive L-order* ρ_L *and finite L-type* T_L *then*

$$L = \limsup_{r \to \infty} \left\{ \frac{n(r) \exp(-\rho_L \log r)}{\exp(r \log L(r))} \right\} \le e\rho_L T_L$$

and

$$l = \liminf_{r \to \infty} \left\{ \frac{n(r) \exp(-\rho_L \log r)}{\exp(rL(r))} \right\} \le \rho_L T_L,$$

where $n(r)$ *denotes the number of zeros in* $|z| \le r$.

A. P. Singh [**42**] proved a theorem to establish a relation between $M(r,f)$ and $\mu(r,f)$.

Theorem 1.0.4 *{ Theorem A,[42]}. For* $0 \leq r < R$,

$$\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f).$$

In [42] the following two results are also proved.

Theorem 1.0.5 *[42]. Let f and g be two entire functions.Then for every $\alpha > 1$ and $0 < r < R$*

$$\mu(r, f \circ g) \leq \frac{\alpha}{\alpha - 1}\mu\left(\frac{\alpha R}{R-r}\mu(R,g), f\right).$$

In particular taking $\alpha = 2$ and $R = 2r$,

$$\mu(r, f \circ g) \leq 2\mu(4\mu(2r,g), f).$$

Theorem 1.0.6 *[42]. Let f and g be two entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Also let $0 < \delta < 1$ then*

$$\mu(r, f \circ g) \geq (1-\delta)\mu(c(\alpha)\mu(\alpha\delta r, g), f).$$

If g is any entire function, then for $\alpha = \delta = \frac{1}{2}$ and for all sufficiently large values of r,

$$\mu(r, f \circ g) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right).$$

Now we state the following theorem due to Clunie [**6**].

Theorem 1.0.7 . *Let f and g be two entire functions with $g(0) = 0$. Let α satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Then for $r > 0$*

$$M(r, f \circ g) \geq M(c(\alpha) M(\alpha r, g), f).$$

Further if g is any entire function, then for $\alpha = \frac{1}{2}$ and for all sufficiently large values of r,

$$M(r, f \circ g) \geq M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right).$$

Also $M(r, f \circ g) \leq M(M(r,g), f)$ is an immediate consequence of the definition.

Let f be a meromorphic function in the finite complex plane and let $n(r,a;f) \equiv n(r,a)$ which is a non-negative integer for each r, denote the number of a- points of f in $|z| \leq r$, counted with proper multiplicities, for a complex number 'a', finite or infinite. Obviously $n(r,\infty) \equiv n(r,f)$ represents the number of poles of f in $|z| \leq r$ counted with proper multiplicities. The definition of the function $N(r,a)$ is as follows:

$$N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} dt + n(0,a) \log r$$

and $N(r,\infty) \equiv N(r,f)$.

Next let us define

$$\begin{aligned} log^+ x &= \log x \text{ if } x \geq 1 \\ &= 0 \text{ if } 0 \leq x < 1. \end{aligned}$$

The following properties are then obvious.

(i) $\log^+ x \geq 0$ if $x \geq 0$

(ii) $\log^+ x \geq \log x$ if $x > 0$

(iii) $\log^+ x \geq \log^+ y$ if $x > y$

(iv) $\log x = \log^+ x - \log^+ \frac{1}{x}$ if $x > 0$.

The proximity function $m(r,f)$ of f is defined as follows {p.4,[**19**]}.

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta.$$

The term $m(r,f)$ is a sort of average magnitude of $\log |f(z)|$ on arcs of $|z| = r$ where $|f(z)|$ is large.

We write $T(r,f) = m(r,f) + N(r,f)$. The function $T(r,f)$ is called Nevanlinna's Characteristic function of f {p.4,[**19**]} and it plays an important role in the theory of meromorphic functions as the function $M(r,f)$ plays in the theory of entire functions.

Now we express Poisson-Jensen formula {p.1,[**19**]} in the form of the following theorem :

Theorem 1.0.8 . *Suppose that f is meromorphic in $|z| \leq R\,(0 < R < \infty)$ and that a_μ $(\mu = 1, 2, ..., M)$ are the zeros and b_ν $(\nu = 1, 2, ..., N)$ are the poles of f in $|z| < R$. Then if $z = re^{i\theta}$ $(0 < r < R)$ and if $f(z) \neq 0, \infty$ we have*

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\phi}\right)\right| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$

$$+ \sum_{\mu=1}^{M} \log\left|\frac{R(z - a_\mu)}{R^2 - \bar{a}_\mu z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z - b_\nu)}{R^2 - \bar{b}_\nu z}\right|.$$

The theorem holds good also when f has zeros and poles on $|z| = R$. When $z = 0$, we obtain Jensen's formula

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\phi}\right)\right| d\phi + \sum_{\mu=1}^{M} \log\frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log\frac{|b_\nu|}{R},$$

provided that $f(0) \neq 0, \infty$.

If f has a zero of order λ or a pole of order -λ at $z = 0$ such that $f(z) = C_\lambda z^\lambda + ...$ then Jensen's formula takes the form

$$\log|C_\lambda| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\phi}\right)\right| d\phi + \sum_{\mu=1}^{M} \log\frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log\frac{|b_\nu|}{R} - \lambda\log R.$$

This complicated modification is one of the minor irritations of the theory.Generally we shall assume that our function behave in such a way that the terms in the Jensen's formula do not become infinite in our use of that formula knowing that the exceptional cases can be treated.

When f has no a-points (i.e. the roots of the equation $f = a$) at $z = 0$, then from Riemann-Stieltjes integral it follows that

$$\sum_{0<|a_\nu|\leq r} \log\frac{|a_\nu|}{r} = \int_0^r \frac{n(t,a)}{t} dt,$$

where a_ν's are a-points of f in $|z| \leq r$.

Again since $N(r, 0) = N\left(r, \frac{1}{f}\right)$, from Jensen's formula we get

$$\log|f(0)| = m(R, f) - m\left(R, \frac{1}{f}\right) + N(R, f) - N\left(R, \frac{1}{f}\right)$$

$$\text{or,} T(R,f) = T\left(R, \frac{1}{f}\right) + \log|f(0)|.$$

Now we express Nevanlinna's First Fundamental theorem as the following form by denoting $m(r,a)$ the function $m\left(r, \frac{1}{f-a}\right)$ for any finite complex number a and $m(r,\infty) = m(r,f)$.

Theorem 1.0.9 *{p.6,[19]}. If f is a meromorphic function in $|z| < \infty$ and 'a' is any complex number, finite or infinite, then*

$$m(r,a) + N(r,a) = T(r,f) + O(1).$$

This result shows the remarkable symmetry exhibited by a meromorphic function in its behaviour relative to different complex number 'a', finite or infinite. The sum $m(r,a) + N(r,a)$ for different values of 'a' maintains a total, given by the quantity $T(r,f)$ which is invariant up to a bounded additive term involving r.

One part of this invariant sum, the quantity $N(r,a)$ hints how densely the roots of the equation $f = a$ are distributed in the average in the disc $|z| < r$. The large the number of a-points the faster this counting function for a-points grows with r.

The first term $m(r,a)$ which is defined to be the mean value of $\log^+\left|\frac{1}{f-a}\right|$ (or $\log^+|f|$ if $a = \infty$) on the circle $|z| = r$, receives a remarkable contribution only from those arcs on the circle where the functional values differ very little from the given value 'a'. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle $|z| = r$ of the functional value f from the value 'a'.

If the a-points of a meromorphic function are relatively scarce for a certain $'a'$, this fact finds expression analytically in the relatively slow growth of the function $N(r,a)$ as $r \to \infty$. In the extreme case where 'a' is a Picard's exceptional value of the function (so that $f \neq a$ in $|z| < \infty$), $N(r,a)$ is identically zero. But this fact on 'a'-points finds a compensation: the function deviates in the mean slightly from the value $'a'$ in question; the corresponding proximity function $m(r,a)$ will be relatively large, so that the sum $m(r,a) + N(r,a)$ reaches the magnitude $T(r,f)$, characteristic function of the function f.

If f is an entire function, $N(r,f) = 0$ and $T(r,f) = m(r,f)$. For an entire function f the study of the comparative growth properties of $T(r,f)$

and $\log M(r,f)$ is a popular problem among the researchers. Now we express a fundamental inequality relating $T(r,f)$ and $\log M(r,f)$.

Theorem 1.0.10 *{p.18,[19]}. If f is regular for $|z| \le R$ then*

$$T(r,f) \le \log^{+} M(r,f) \le \frac{R+r}{R-r} T(R,f),\ 0 \le r < R.$$

At this stage we introduce the following definition.

Definition 1.0.4 *{p.16,[19]}. Let S be a real and non-negative function increasing for $r_0 \le r < \infty$, $r_0 > 0$. The order k and lower order λ of the function $S(r)$ are defined as*

$$k = \limsup_{r\to\infty} \frac{\log S(r)}{\log r} \text{ and } \lambda = \liminf_{r\to\infty} \frac{\log S(r)}{\log r}.$$

Moreover if $0 < k < \infty$, we set $c = \limsup_{r\to\infty} \frac{\log S(r)}{r^k}$ and distinguish the following possibilities:
(a) $S(r)$ has maximal type if $c = +\infty$;
(b) $S(r)$ has mean type if $0 < c < +\infty$;
(c) $S(r)$ has minimal type if $c = 0$ and
(d) $S(r)$ has convergence class if $\int_{r_0}^{\infty} \frac{S(t)}{t^{k+1}} dt$ converges.

From the above theorem the following theorem can be proved easily.

Theorem 1.0.11 *{p.18,[19]}. If f is an entire function then the order k of the function $S_1(r) = \log^{+} M(r,f)$ and $S_2(r) = T(r,f)$ is the same. Further if $0 < k < \infty$, $S_1(r)$ and $S_2(r)$ belong to the same classes (a), (b), (c) or (d).*

Here we note that $S_1(r)$ and $S_2(r)$ have the same lower order.

A function f meromorphic in the plane is said to have order ρ, lower order λ and maximal, minimal, mean type or convergence class if the function $T(r,f)$ has this property. For entire functions these coincide by the above theorem with the corresponding definition in terms of $M(r,f)$ which is classical. The type of a meromorphic function f is defined by

$$\tau = \limsup_{r\to\infty} \frac{\log T(r,f)}{r^{\rho}}, \quad 0 < \rho < \infty.$$

As we know that the order of an entire function f and its derivative are equal, the same result holds for a meromorphic function also.

After revealing the important symmetry property of a meromorphic function f, which is expressed in the first fundamental theorem through the invariance of the sum $m(r,a) + N(r,a)$, it is natural to attempt for a more careful investigation of the relative strength of two terms in the sum, of the proximity component $m(r,a)$ and of the counting component $N(r,a)$ .Individual results have been obtained in this direction {p.234,[**19**]} :

1. Picard's theorem shows that the counting function for a nonconstant meromorphic function in the finite complex plane can vanish for at most two values of a.

2.For a meromorphic function of finite non integral order there is atmost one Picard's exceptional value.

3. That the counting function $N(r,a)$ is in general i.e., for the great majority of the values of 'a', large in comparison with the proximity function.

We now state Nevanlinna's Second Fundamental theorem.

Theorem 1.0.12 *{p.31,[19]}. Suppose that f is a non-constant meromorphic function in $|z| \leq r$. Let $a_1, a_2, ...a_q$ where $q \geq 2$, be distinct finite complex numbers, $\delta > 0$ and suppose that*

$$|a_\mu - a_\nu| \geq \delta \text{ for } 1 \leq \mu < \nu \leq q.$$

Then

$$m(r,\infty) + \sum_{\nu=1}^{q} m(r,a_\nu) \leq 2T(r,f) - N_1(r) + S(r),$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N\left(r,\frac{1}{f'}\right) + 2N(r,f) - N\left(r,f'\right)$$

$$\text{and } S(r) = m\left(r,\frac{f'}{f}\right) + m\left\{r,\sum_{\nu=1}^{q}\frac{f'}{(f-a_\nu)}\right\} + q\log^{+}\frac{3q}{\delta} + \log 2 + \log\frac{1}{|f'(0)|},$$

with modifications if $f(0) = 0$ or ∞ and $f'(0) = 0$.

The quantity $S(r)$ will in general play the role of an unimportant error term. The combination of this fact with the above theorem yields the second fundamental theorem.

The following theorem gives an estimation of $S(r)$.

Theorem 1.0.13 *{p.34,[19]}. Suppose that f is a meromorphic function and not constant in $|z| < R_0 \leq \infty$ and that $S(r) \equiv S(r,f)$ is defined as in the above theorem. Then we have*

(i) If $R_0 = +\infty$, $S(r,f) = O\{\log T(r,f)\} + O(\log r)$, as $r \to \infty$ through all values if f has finite order and as $r \to \infty$ outside a set E of finite linear measure otherwise

(ii) If $0 < R_0 < +\infty$, $S(r,f) = O\left\{\log^+ T(r,f) + \log \frac{1}{R_0 - r}\right\}$ as $r \to R_0$ outside a set E such that $\int\limits_E \frac{dr}{R_0 - r} < \infty$.

Further there is a point r outside E for which $\rho < r < \rho'$ provided that $0 < R - \rho' < e^{-2}(R - \rho)$.

Consequently we get the following theorem.

Theorem 1.0.14 *{p.41,[19]}. Let f be meromorphic and nonconstant in $|z| < R_0$. Then $\frac{S(r,f)}{T(r,f)} \to 0$ (*) as $r \to R_0$ with the following provisions:*

(a) (*) holds without restrictions if $R_0 = +\infty$ and f is of finite order in the plane.

(b) If f has infinite order in the plane, (*) still holds as $r \to \infty$ outside a certain exceptional set E_0 of finite length. Here E_0 depends only on f.

(c) If $R_0 < +\infty$ and $\limsup\limits_{r \to \infty} \frac{T(r,f)}{\log\left\{\frac{1}{(R_0 - r)}\right\}} = +\infty$, then (*) holds as $r \to R_0$ through a suitable sequence r_n, which depends on f only.

This theorem points out why $S(r)$ plays the role of an unimportant error term.

Let f be meromorphic and not constant in the plane We shall call an error term and denote by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set r of finite linear measure. Also we shall denote by $a(z), a_0(z), a_1(z)$ etc. functions meromorphic in the plane and satisfying $T\{r, a(z)\} = S(r,f)$ as $r \to \infty$. Now we introduce Milloux's theorem which is important in studying the properties of the derivatives of meromorphic functions.

Theorem 1.0.15 *{p.55,[19]}. Let $\imath$ be a positive integer and $\psi = \sum_{\nu=0}^{\imath} a_\nu f^{(\nu)}$. Then $m\left(r, \frac{\psi}{f}\right) = S(r,f)$ and $T(r,\psi) \leq (\imath+1)\,T(r,f) + S(r,f)$.*

Milloux showed that in the second fundamental theorem we can replace the counting functions for certain roots of $f = a$ by roots of the equation $\psi = b$, where ψ is given as in the above theorem. In this connection we state the following theorem.

Theorem 1.0.16 *{p.57,[19]}. Let f be meromorphic and nonconstant in the plane and $\psi = \sum_{\nu=0}^{\imath} a_\nu f^{(\nu)}$, where $\imath$ is a positive integer, be nonconstant. Then*

$$T(r,f) < \bar{N}(r,f) + N\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{\psi - 1}\right) - N_0\left(r, \frac{1}{\psi'}\right) + S(r,f),$$

where in $N_0\left(r, \frac{1}{\psi'}\right)$ only zeros of ψ' not corresponding to the repeated roots of $\psi = 1$ are to be considered.

Here we note that this result reduces to second fundamental theorem if $\psi = f$ and $q = 3$.

Now we set

$$\begin{aligned}\delta(a) = \delta(a;f) &= \liminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} \\ &= 1 - \limsup_{r\to\infty} \frac{N(r,a)}{T(r,f)},\end{aligned}$$

$$\Theta(a) = \Theta(a;f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r,a)}{T(r,f)},$$

where $\bar{N}(r,a;f) \equiv \bar{N}(r,a)$ is the counting function for distinct $a-$points,

$$\theta(a) = \theta(a;f) = \liminf_{r\to\infty} \frac{N(r,a) - \bar{N}(r,a)}{T(r,f)}.$$

Evidently, given $\varepsilon\,(> 0)$, we have for sufficiently large values of r,

$$N(r,a) - \bar{N}(r,a) > \{\theta(a) - \varepsilon\} T(r,f),$$
$$N(r,a) < \{1 - \delta(a) + \varepsilon\} T(r,f) \text{ and hence}$$
$$\bar{N}(r,a) < \{1 - \delta(a) - \theta(a) + 2\varepsilon\} T(r,f) \text{ so that}$$
$$\Theta(a) \geq \delta(a) + \theta(a).$$

The quantity $\delta(a)$ is called the deficiency of the value 'a' and $\theta(a)$ is called the index of multiplicity. Evidently $\delta(a)$ is positive only if there are relatively few roots of the equation $f = a$, while $\theta(a)$ is positive if there are relatively many multiple roots.

Let us now state a fundamental theorem called Nevanlinna's theorem on deficient values.

Theorem 1.0.17 *{p.43,[19]}. Let f be a non constant meromorphic function defined on the plane. Then the set of values 'a' for which $\Theta(a) > 0$ is countable and we have, on summing over all such values 'a',*

$$\sum_a \{\delta(a) + \theta(a)\} \leq \sum_a \Theta(a) \leq 2.$$

The magnitude of the deficiency $\delta(a)$ lies in the closed unit interval $[0,1]$ and it gives us a very accurate measure for the relative density of the points where the function f assumes the value $'a'$ in question. The larger the deficiency is, the more rare are latter points.The deficiency reaches its maximum value 1 when the latter have been very sparsely distributed, as for example, in the extreme case where the value $'a'$ is a Picard exceptional value i.e., a complex number which is not assumed by the function f. We shall call every value of vanishing deficiency $\delta(a)$, a normal value in contrast to the deficient values for which $\delta(a)$ is positive.

It is known from Picard's theorem that a meromorphic function can have atmost two Picard exceptional values. This theorem follows easily from Nevanlinna's theorem on deficient values because as we have stated before that for a Picard exceptional value $'a'$, $\delta(a) = 1$.

The quantity $\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a)}{T(r,f)}$ gives another measure of deficiency and is called the Valiron deficiency. Clearly $0 \leq \delta(a;f) \leq \Delta(a;f) \leq 1$.

Apart from **Chapter 1** the thesis consists of eight chapters.

- In **Chapter 2** we compare the relative Valiron defect with the relative Nevanlinna defect of a meromorphic function of finite order. The results of this chapter have been published in **Journal of Mathematics,** see [8].

- In **Chapter 3** we compare the relative Valiron defect with the relative Nevanlinna defect of differential polynomials generated by a meromorphic function . The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences,** see [10].

- In **Chapter 4** we consider several meromorphic functions having common roots and find some relations involving their relative proximate defects. The results of this chapter have been published in **International Journal of Pure and Applied Mathematics,** see [13].

- In **Chapter 5** we wish to introduce an alternative definition of zero order (zero lower order) of a meromorphic function f and establish the equivalence of this definition with the classical one. In this chapter we also study the comparative growth properties of composite entire and meromorphic functions considering left factor or right factor to be of order zero. The results of this chapter have been published in **International Mathematical Forum,** see [15] and **International Journal of Mathematical Analysis,** see [16].

- In **Chapter 6** we study the comparative growth properties of composite entire functions on the basis of relative order, relative L-order and relative L*-order where $L = L(r)$ is a slowly changing function . The results of this chapter have been published in **International Journal of Pure and Applied Mathematics,** see [12].

- In **Chapter 7** we discuss about the comparative growth of composite entire or meromorphic functions and differential polynomials generated by one of the factors . Also we study the relationship between the $L-(p,q)$ th order of a transcendental meromorphic function and that of a special type of linear differential polynomial viz. the wronskian generated by it (a transcendental meromorphic function) where p, q are positive integers and $p > q$. Further we intend to establish a few theorems related to a result of W. Bergweiler [4] , I.Lahiri and D. K. Sharma [**25**]. The results of this chapter have been published in **Wesleyan Journal of Research, see** [**14**] and in **International Journal of Pure and Applied Mathematics, see** [**9**] ,[**11**].

- In **Chapter 8** we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of (p, q) th order ((p, q) th lower order) improving some earlier results where p, q are positive integers and $p > q$. .

From **Chapter 2** onwards when we write **Theorem** $a.b.c$ (or **Corollary** $a.b.c$ etc.) where a, b and c are positive integers, we mean the c-**th theorem** (or c-**th corollary** etc.) of the b-**th section** in the a-**th chapter**. Also by **equation number** $(a.b)$ we mean the b-**th equation** in the a-**th chapter** for positive integers a and b. Individual chapters have been presented in such a manner that they are almost independent of the other chapters. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

The author of the thesis is thankful to the authors of various papers and books which have been consulted during the preparation of the entire thesis.

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