INTRODUCTION

The theory of wave propagation in elastic solids possesses a distinguished history of nearly hundred years. The eminent mathematician Galileo Galilei paid his attention to the vibrations of pendulums, the resonance phenomenon and the vibrations of strings. The first stage of the investigation on wave propagation associated with the names of **NAVIER, CAUCHY, HOOKE, POISSON, STOKES, RAYLEIGH, KELVIN, GREEN, LAME** and **CLEBSCH** is characterized by development of the extensive theory of elasticity to the problem of wave propagation and vibrating bodies in elastic material.

During the first quarter of this century the subject lost much of its glamour and interest, perhaps because of a gap between the advancement of theoratical and experimental work, as there was no practical methods available in laboratory for observing the passage of stress waves in elastic materials. But in the later part of the century the interest in the study of elastic waves has been growing rapidly because of the application of the theory in Seismology, Geophysics and in Engineering science. During the last three decades there has been a remarkable revival of interest in this subject.

Most of the experimental works carried out on the wave propagation of elasticity are concerned with studying propagation in specimens of comparatively simple geometrical shape, the results of this experiment could be compared directly with the exact or approximate theoratical predictions. With increasing confidence in the experimental techniques and in the interpretation of observations, it is now possible to study more complicated problems of elastodynamics.

All the elastic bodies may be divided roughly into two categories

(i)homogeneous and non-homogeneous

(ii) isotropic and anisotropic

A homogeneous body is the body whose elastic properties are the same at different points and a non-homogeneous body has different elastic properties at different points. If the elastic moduli vary from point to point in a continuous manner, the non-homogeneity may also be termed continuous. If, however, the elastic moduli undergo discontinuties in passing from point to point, for example change abruptly, the non-homogeneity is said to be discontinuous or discrete.

An isotropic body , with regard to its elastic properties , is one in which these properties are the same for all directions drawn through a given point. An anisotropic body has, in general, different elastic properties for different directions drawn through a given point. A body may be isotropic or anisotropic and at the same time homogeneous or non-homogeneous depending on its own structure.

In an unbounded homogeneous isotropic solid, two types of elastic waves may be propagated with two different velocities. These are dilatational wave or longitudinal wave and distortional wave or shear wave. Obviously longitudinal waves arrive earlier than shear waves. In the case of the deformation of elastic body, both longitudinal and distortional waves will normally be produced and when a wave of either type impinges on the boundary of the solid, two types of waves are generated. In addition to the existence of these two types of waves of the body, a third type of wave may exist whose effects are confined closely along the surface of the body; this type of waves are known as Rayleigh-wave. Their effects decrease exponentially with depth and their velocity of propagation is smaller than the other types of elastic waves. They are of great importance in seismic phenomena. Bullein (1963), Ewing et. al. (1957), Cagniard (1962) and Pilant (1979) have discussed about seismic waves in their books.

Some important equations governing the motion of a homogeneous, isotropic, linearly isotropic elastic solid are listed below: Consider a rectangular cartesian co-ordinate system with reference to the co-ordinate axes x_i (i= 1,2,3) and assume u_i to be the components of the displacement vector field. The system of displacement equations of motion in indicial notation may be expressed as

$$\mu\, u_{i,jj} + (\lambda+\mu)\, u_{j,ji} = \rho\, \ddot{u}_i \qquad (1)$$

where λ , μ are Lame's elastic constants and ρ is the mass density.

A plane displacement wave propagating in an arbitrary direction in an unbounded medium is represented by

$$u_i = f(x_k p_k - ct)\, d_i \tag{2}$$

where d_i and p_i are the components of unit vectors in the directions of motion and of propagation and x_i are the components of the position vector, $p_i x_i$ = constant represents a plane normal to the unit vector with components p_i. Equation (2) represents a plane wave whose planes of constant phase propagate with velocity c. Substitution of equation (2) into the displacement equation of motion yields

$$(\mu - \rho c^2) d_i + (\lambda+\mu)(p_j d_j) p_i = 0 \tag{3}$$

Since p_i and d_i denote two different unit vectors, equation (3) may be satisfied in only two ways:

(i) $p_i = \pm d_i$, consequently $p_j d_j = \pm 1$ and equation (3) yields

$$c^2 = c_L^2 = (\lambda+2\mu)/\rho \tag{4}$$

(ii) If $p_i \neq d_i$, both terms in equation (4) have to vanish independently yielding

$$c^2 = c_T^2 = \mu/\rho \qquad \text{and} \qquad p_j d_j = 0 \tag{5}$$

The displacement corresponding to transverse wave whose velocity of propagation is given by (5) can have any direction in a plane normal to the direction of propagation but usually the x_1-x_2 plane is chosen to contain the vector $\vec{p}$ and transverse motions are considered in the x_1-x_2 plane or normal to the x_1-x_2 plane. These are called "vertically" and "horizontally" polarised transverse waves, respectively.

A convenient representation of the displacement components is

$$u_i = \phi,_j + e_{ijk}\,\psi_{k,j}\,, \qquad \psi_{k,k} = 0 \tag{6}$$

where e_{ijk} is the alternating tensor. Substitution of equation (6) in equation (1) shows that this representation satisfies the displacement equations of motion, provided that

$$\phi,_{ii} = \frac{1}{c_L^2}\,\ddot{\phi} \tag{7.a}$$

and

$$\psi_{k,jj} = \frac{1}{c_T^2}\,\ddot{\psi}_k \tag{7.b}$$

where c_L and c_T are given by equations (4) and (5) respectively and equations (7.a-b) are uncoupled wave equations.

When the field variables are independent of one of the cartesian co-ordinates say x_3, wave motions uncouple into anti-plane and in-plane motions. A displacement distribution defined by $u_3(x_1,x_2,t)$ describes anti-plane strain and $u_1(x_1,x_2,t)$, $u_2(x_1,x_2,t)$ defines a state of plane strain.

Since the x,y,z co-ordinate system is more convenient to refer the motions, so the displacement components are denoted by u(x,y,t), v(x,y,t) and w(x,y,t) respectively and the components of the stress tensor by $\tau_x(x,y,t)$, $\tau_{xy}(x,y,t)$ etc.

The two dimensional anti-plane motion governing the displacement component w(x,y,t) is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c_T^2}\,\frac{\partial^2 w}{\partial t^2} \tag{8}$$

The non-vanishing stress components are

$$\tau_{xz} = \mu \frac{\partial w}{\partial x} \quad \text{and} \quad \tau_{yz} = \mu \frac{\partial w}{\partial y} \tag{9}$$

For the case of plane strain it is expedient to employ the decomposition (6) and reduce the displacement components to

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \tag{10}$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \tag{11}$$

The functions ϕ and ψ satisfy the following two dimensional wave equations

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c_L^2} \frac{\partial^2 \phi}{\partial t^2} \tag{12}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{c_T^2} \frac{\partial^2 \psi}{\partial t^2} \tag{13}$$

where c_L and c_T are defined by equations (4) and (5) respectively. The corresponding components of the stress tensor are

$$\tau_{xx} = \lambda \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) + 2\mu \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right) \tag{14}$$

$$\tau_{yy} = \lambda \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) + 2\mu \left(\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} \right) \tag{15}$$

$$\tau_{xy} = \mu \left(2\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \tag{16}$$

However, natural or artificial materials in our surroundings are generally inhomogeneous and anisotropic.

Since 1950 the theory of elasticity for anisotropic bodies has been continually developed and enriched with new investigations of both serious problems as a general nature and individual aspects of these problems. Thus the general theory has been placed on a rigorous scientific basis and a number of laws have been established with the result that this theory, first worked out by B.de Saint Venant and P.V.Bekhterev (1925),has been revived.

Of great importance is the development and construction of many entirely new anisotropic materials which possesses a number of advantages over those previously known (for example,glass-fibre reinforced plastics). Thus, over two or three decades this branch of science has made great progress, both in a theoratical and a purely practical way, i.e. in constructing new anisotropic materials.

Now we recapitulate the fundamental principles of the theory of elasticity and the general equations which will be used in what follows for the construction of solutions to specific problems of the theory of elasticity for anisotropic bodies.

In studying the states of stress and strain in anisotropic bodies produced by an external load, we make a number of assumptions imposing certain restrictions. The most important of these assumptions reduce to the following:

(1) A body is solid (a continuous medium), the stresses on any plane within the body and on its surface are forces per unit area. In otherwords, the couple stresses are neglected, as is done in the classical theory of elasticity.

(2) The relation between the components of strain and the projections of displacement and their first derivatives with respect to the co-ordinates is linear.

(3) The stress-strain relations are linear, i.e., the material follows the generalized Hooke's law, the co-efficients in these linear relations may be either constant (homogeneous body) or variable, i.e. functions of position, continuous or discontinuous (in the case of a non-homogeneous body).

(4) The initial stresses i.e. those existing without any external load, including the thermal stresses are disgarded; specific problems of dynamics are not considered.

Thus, the theory of anisotropic elastic bodies can be studied from the classical linear theory of homogeneous or non-homogeneous elastic bodies.

The stresses acting on planes normal to the co-ordinate directions are each resolved into three components: one normal(normal stress) and two tangential (shearing stresses).

The deformation of a body in the neighbourhood of a point is characterized by the components of strain, viz three extensions and three shearing strains.

The components of the displacement of a point on the axes of cartesian co-ordinates (x,y,z) are denoted as u,v,w. Let ε_x and ε_y are the extensions of segments of unit length originally parallel to x and y, γ_{xy} is the change in angle between segments whose original directions are x and y.

The strain components ε_i ($i = 1,2,3$) and $\frac{1}{2}\gamma_{ij}$($j \neq i$, $j = 1,2,3$) constitute a symmetrical tensor of rank two. For a cartesion system x,y,z, it can be written as in the matrix form

$$\begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{vmatrix}$$

The relation between the components of displacement and strain in cartesian system are given below:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{17}$$

If the strains are not small, the extensions and shears,

$\varepsilon_i (i=1,2,3)$, $\frac{1}{2}\ \gamma_{ij} (i \neq j,\ j=1,2,3)$ are related to the displacements by non-linear equation which are given as follows:

$$\varepsilon_x = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1$$

$$\varepsilon_y = \sqrt{1 + 2\frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} - 1 \qquad (18)$$

$$\sin \gamma_{xy} = \frac{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}}{(1 + \varepsilon_x)(1 + \varepsilon_y)}$$

The other three components ε_z, γ_{yz}, γ_{xz} are found from (18) by cyclic permutation of the subscripts.

The corresponding stress components are in the matrix form

$$\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$$

In the general case of anisotropy each strain component is a linear function of all six components. For a homogeneous body having anisotropy of the most general kind, the equations expressing the generalized Hooke's law for this system are

$$\begin{aligned}
\varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{xz} + a_{16}\tau_{xy} \\
\varepsilon_y &= a_{21}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{24}\tau_{yz} + a_{25}\tau_{xz} + a_{26}\tau_{xy} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\gamma_{xy} &= a_{61}\sigma_x + a_{62}\sigma_y + a_{63}\sigma_z + a_{64}\tau_{yz} + a_{65}\tau_{xz} + a_{66}\tau_{xy}
\end{aligned} \tag{19}$$

In the general case equations (19) contain 36 co-efficients a_{ij}, but actually they are always fewer;

Suppose that the 6th order determinent of the co-efficients a_{ij}, written down successively, is not zero, amd hence equation (19) for σ and τ are solvable. The generalized Hooke's law equations for the general case are thus obtained in an alternative equivalent form:

$$\begin{aligned}
\sigma_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_y + A_{13}\varepsilon_z + A_{14}\gamma_{yz} + A_{15}\gamma_{xz} + A_{16}\gamma_{xy} \\
\sigma_y &= A_{21}\varepsilon_x + A_{22}\varepsilon_y + A_{23}\varepsilon_z + A_{24}\gamma_{yz} + A_{25}\gamma_{xz} + A_{26}\gamma_{xy} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\tau_{xy} &= A_{61}\varepsilon_x + A_{62}\varepsilon_y + A_{63}\varepsilon_z + A_{64}\gamma_{yz} + A_{65}\gamma_{xz} + A_{66}\gamma_{xy}
\end{aligned} \tag{20}$$

However if strain energy function $\bar{V}$ exists such that

$$\sigma_x = \frac{\partial \bar{V}}{\partial \varepsilon_x}, \qquad \sigma_y = \frac{\partial \bar{V}}{\partial \varepsilon_y}, \qquad , \tau_{xy} = \frac{\partial \bar{V}}{\partial \gamma_{xy}}$$

then differentiation of the stress components with respect to the strain components yield

$$\frac{\partial \sigma_x}{\partial \varepsilon_y} = \frac{\partial \sigma_y}{\partial \varepsilon_x}, \qquad \frac{\partial \sigma_x}{\partial \gamma_{xy}} = \frac{\partial \tau_{xy}}{\partial \varepsilon_x} \quad \text{etc.} \tag{21}$$

It follows from the equalities of (21) and (20) that

$$A_{21} = A_{12}, \quad A_{31} = A_{13}, \ldots\ldots\ldots\ldots, A_{65} = A_{56}$$

and, in general,

$$A_{ij} = A_{ji} \quad (i, j = 1, 2, \ldots, 6) \tag{22}$$

Solving equation (19), we obtain six expressions for ε and γ in which the co-efficients on the right hand sides are also symmetrical:

$$a_{ij} = a_{ji} \quad (i, j = 1, 2, \ldots, 6)$$

So we can write the generalized Hooke's law equations in the general case as

$$\begin{aligned}
\varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + \ldots\ldots\ldots\ldots + a_{16}\tau_{xy} \\
\varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + \ldots\ldots\ldots\ldots + a_{26}\tau_{xy} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + \ldots\ldots\ldots\ldots + a_{66}\tau_{xy}
\end{aligned} \tag{23}$$

or

$$\begin{aligned}
\sigma_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_y + \ldots\ldots\ldots\ldots + A_{16}\gamma_{xy} \\
\sigma_y &= A_{12}\varepsilon_x + A_{22}\varepsilon_y + \ldots\ldots\ldots\ldots + A_{26}\gamma_{xy} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\tau_{xy} &= A_{16}\varepsilon_x + A_{26}\varepsilon_y + \ldots\ldots\ldots\ldots + A_{66}\gamma_{xy}
\end{aligned} \tag{24}$$

In general case of anisotropy the number of elastic constants A_{ij}, a_{ij} is 21, but among these the independent constants are fewer.

NOVOZHILOV (1958) states that, all co-ordinate systems are equivalent in Geometry, neverthless as regards the elastic and in general, physical properties symmetry may be observed even in the most general case. Consequently, even in the most general case the number of independent elastic constants is not 21, but fewer, namely 18.

By changing the notation for the elastic constants and stress components, we may write the generalized Hooke's law equations in an extremely simple form. Let the elastic constants be denoted by "a" with four subscripts and setting

(1) $a_{ij} = a_{mnkl}$ if $i, j = 1,2,3$ (all possible cases where $j=i$ are included)

(2) $a_{ij} = 2a_{mnkl}$ if either of the two subscripts, i or j is 4,5,6.

(3) $a_{ij} = 4a_{mnkl}$ if both subscripts $i, j = 4,5,6$.

The six equations (23) are then written as a single one:

$$\varepsilon_{ij} = a_{ijkl}\sigma_{kl} \qquad (i, j, k, l = 1,2,3) \qquad (25)$$

The number of all constants a_{ijkl} with four subscripts is 81, but, when grouped, they reduce to 21 (of these 18 constants are independent) elastic constants.

The generalized Hooke's law equations, solved for the stress components, are of the form

$$\sigma_{ij} = A_{ijkl}\varepsilon_{kl} \qquad (26)$$

The notation for the elastic constants "a" and "A" with four subscripts has been used by Malmeister, Tamuzh and Terters (1972) in their book.

If the structure of an anisotropic body has some kind of symmetry, the elastic properties also exhibit symmetry. The elastic symmetry is expressed in the fact that at each point there are symmetrical directions equivalent as regards the elastic properties.

F.Neumann (1885) established the relationship between the structural symmetry and the elastic symmetry for crystals, which may be stated as follows:

With respect to its physical properties (including the elastic properties), a material exhibits the same kind of symmetry as its crystallographic form or more perfect symmetry. The principle is also extended to bodies that are not crystals, but have structural symmetry (wood, plywood, glass fibre reinforced plastics).

If there is symmetry of the elastic properties (elasticsymmetry) in an anisotropic body, the generalized Hooke's law equations for it are simplified since some of the co-efficients a_{ij} are zero, while among others there are linear relations.

The following four cases of elastic symmetry are the most important, and these will now be discussed below:

(1) PLANE OF ELASTIC SYMMETRY:—

Suppose a plane passing through each point of a body possesses the following property:

Every two symmetric directions with respect to this plane are equivalent as regards the elastic properties. A direction normal to the plane of elastic symmetry will be termed the principal direction of elasticity. In this case only one principal direction passes through a point of the body.

If the z-axis is taken normal to the plane of elastic symmetry and the other two axes lie in this plane, we conclude that 8 elastic constants must be zero, namely

$$a_{14} = a_{24} = a_{34} = a_{46} = a_{15} = a_{25} = a_{35} = a_{56} = 0$$

and the number of elastic constants a_{ij} reduces to 13 which are given below

$$\begin{matrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} \\ & a_{22} & a_{23} & 0 & 0 & a_{26} \\ & & a_{33} & 0 & 0 & a_{36} \\ & & & a_{44} & a_{45} & 0 \\ & & & & a_{55} & 0 \\ & & & & & a_{66} \end{matrix} \qquad (27)$$

For arbitrary directions of the axes, these equations contain 13 strain co-efficients a_{ij}, not explicitly related in any way.

(2) THREE PLANES OF ELASTIC SYMMETRY (ORTHOGONAL BODY)

If through each point of a body there pass three mutually perpendicular (orthogonal) planes of elastic symmetry and the like planes of elastic symmetry are parallel at each point, then, taking the co-ordinate axes normal to the planes of elastic symmetry (along the principal directions) we find, in addition to 8-elastic constants of the preceeding case, there are 4-more constants equal to zero:

$$a_{16} = a_{26} = a_{36} = a_{45} = 0$$

The generalized Hooke's law equations and the Schematic expression for the elastic potential in terms of the constants a_{ij} take the form

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z$$

$$\varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z$$

$$\varepsilon_z = a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z \qquad (28)$$

$$\gamma_{yz} = a_{44}\tau_{yz}, \quad \gamma_{xz} = a_{55}\tau_{xz}, \quad \gamma_{xy} = a_{66}\tau_{xy}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ & a_{22} & a_{23} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & & & & a_{55} & 0 \\ & & & & & a_{66} \end{matrix} \qquad (29)$$

Introducing the engineering constants E_i, G_{ij}, ν_{ij} equation (28) can be rewritten in the following form

$$\varepsilon_x = \frac{1}{E_1}\sigma_x - \frac{\nu_{21}}{E_2}\sigma_y - \frac{\nu_{31}}{E_3}\sigma_z$$

$$\varepsilon_y = -\frac{\nu_{12}}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y - \frac{\nu_{32}}{E_3}\sigma_z$$

$$\varepsilon_z = -\frac{\nu_{13}}{E_1}\sigma_x - \frac{\nu_{23}}{E_2}\sigma_y + \frac{1}{E_3}\sigma_z \qquad (30)$$

$$\gamma_{yz} = \frac{1}{G_{23}}\tau_{yz}, \qquad \gamma_{xz} = \frac{1}{G_{13}}\tau_{xz}, \qquad \gamma_{xy} = \frac{1}{G_{12}}\tau_{xy}$$

A body having three orthogonal planes of elastic symmetry at each point is said to be orthogonally anisotropic or in short, orthotropic. The principal directions at a given point may not be equivalent. Of the 12 elastic constants entering into equations (30) only 9 constants are independent. By virtue of the symmetry of the matrix of the right hand side of the equations expressing the generalized Hooke's law, we always have

$$E_1\nu_{21} = E_2\nu_{12}, \quad E_2\nu_{32} = E_3\nu_{23}, \quad E_3\nu_{13} = E_1\nu_{31} \tag{31}$$

It is important to note that no further reduction of the elastic constants is possible here since, in contrast to the case of a plane symmetry, a_{ij} from equations (28) or E_i, G_{ij}, ν_{ij} from equations (30) are invariant constants themselves. They are alternatively called the principal constants.

(3) PLANE OF ISOTROPY (AXIS OF ROTATIONAL SYMMETRY)

(3a) TRANSVERSELY ISOTROPIC BODY.

Suppose a body possesses the properties that through all points there pass parallel planes of elastic symmetry in which all directions are elastically equivalent (planes of isotropy). A body with such properties is said to be transversely isotropic.

Considering the z-axis to be taken normal to a plane of isotropy, with the x and y axes arbitrarily in this plane, the generalized Hooke's law equations with the 5 independent elastic constants are then written as

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z$$

$$\varepsilon_y = a_{12}\sigma_x + a_{11}\sigma_y + a_{13}\sigma_z$$

$$\varepsilon_z = a_{13}(\sigma_x + \sigma_y) + a_{33}\sigma_z \tag{32}$$

$$\gamma_{yz} = a_{44}\tau_{yz}, \quad \gamma_{xz} = a_{44}\tau_{xz}, \quad \gamma_{xy} = 2(a_{11} - a_{12})\tau_{xy}$$

In some cases a transversely isotropic material is called, in short, transtropic.

(4) ISOTROPIC BODY:-

If all directions in a body are elastically equivalent and principal, then the generalized Hooke's law for an isotropic body of Young's modulus E, Poission's ratio ν, and shear modulus G, is

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu (\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E} [\sigma_y - \nu (\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_x + \sigma_y)] \qquad (33)$$

$$\gamma_{yz} = \frac{1}{G} \tau_{yz} , \qquad \gamma_{xz} = \frac{1}{G} \tau_{xz} , \qquad \gamma_{xy} = \frac{1}{G} \tau_{xy}$$

In recent years problems of diffraction of elastic waves by cracks or by inclusions have attracted considerable attention in view of their application in Seismology and Geophysics. Cracks or inclusions are present in essentially all structural materials either as natural defects or as a result of fabrication processes. Moreover, in many cases the cracks or inclusions are sufficiently small so that their presence does not significantly reduce the strength of the material. In other cases, however, the imperfections are large enough through fatigue, stress corosion cracking etc., so that they must be taken into account in determining the strength.

From the standpoint of engineering applications it has been the macroscopic theories based on the notions of continuum solid mechanics and classical thermodynamics which have provided the quantitative working tools for dealing with the fracture of structural materials. In the macroscopic-continuum approach to fracture it is implicitly assumed that the material contains some macroscopic flaw which may act as fracture nuclei and that the medium is a homogeneous continuum in the sense that the size of the macroscopic flaws is large in comparison with the characteristic microstructural dimension of the material. The problem is then to study the effects of the applied loads, the flaw geometry and the environmental conditions on the fracture process in the solid.

Fracture mechanics is concerned with the analysis of the stability of cracks. A fracture criterion can subsequently be employed to determine the conditions for crack propagation, both stable and unstable, and for crack arrest.

Fracture mechanics problems that have to be treated as dynamic problem may be classified in two types:

(1) Cracked bodies subjected to rapidly varying loads

(2) Bodies containing rapidly propagating cracks.

In both the cases the crack tip is an environment disturbed by wave motions.

Impact and vibration problems fall into the first type of dynamic problems. It is often found that at inhomogeneities in a body the dynamic stresses are higher than the stresses computed from the corresponding problem of static equillibrium in the analysis of this type of problem.

The second type of problem is equally important. There are several kinds of large engineering structures e.g., gas transmission pipelines, ship-hulls, aircraft fuselages and nuclear reactor components, in which rapid crack growth is a definite possibility. The study of earthquake mechanisms is the another area to which the analysis of rapidly propagating cracks is relevant.

Recently, there have been a number of comprehensive articles in the general area of fracture mechanics. Some references are those of Achenbach (1972,1976) , Freund (1975,1976,1990) and Kanninen (1978).

Engineering structures requiring protection against the possibility of large scale catastrophic crack propagation are, however, generally constructed of ductile, tough materials. Current progress in this area, and a starting point for the development of a dynamic plastic propagating crack tip analysis have recently been presented by Achenbach and Kanninen.

A problem of central importance in dynamic fracture mechanics is that of predicting the way in which a crack will grow in a deformable solid, given the geometrical configuration of the solid, a characterization of the material, the applied load distribution and suitable initial conditions. In the interpretation of laboratory data on rapid crack propagation, a problem of equal importance is that of determining the values of fracture characterizing parameters from measurments of the crack motion and applied load distribution.

In order to determine an equation of motion for a crack tip, two main ingradients are essential. The first of these is a crack propagation criterion which must be stated as a fundamental physical postulate, distinct from the postulates dealing with bulk material behaviour and momentum balance. Generally these later postulates can be satisfied for any motion of the crack tip. It is the role of the fracture criterion to select the motion of the crack tip from the class of all such dynamically admissible motions.

The only geometrical configuration for which exact solutions of the elastodynamic field equations, valid for nonuniform crack motion, have been found is a semi-infinite crack motion in an otherwise unbounded solid or configurations which can be shown to be equivalent to this by linear superposition arguments. The solution for anti -plane deformation was presentedby Kostrov (1964,1966) and Achenbach (1970) and for in-plane deformation by Freund (1973), Burridge (1976) and Kostrov (1975). Although these solutions have been of major importance in addressing certain fundamental questions on rapid crack propagation, they have been found to be inadequate for describing some dynamic fracture processes of practical importance.

The shape of the cracks which have been studied uptil now are as follows:

(i) Semi-infinite plane cracks

(ii) Finite Griffith cracks

(iii) Penny shaped and annular cracks

(iv) Non-planer cracks

A transient problem in which a semi-infinite crack appears suddenly in a stretched elastic sheet was solved by Maue (1954) and was also discussed by Ang (1958) as his dissertation. Baker (1962) solved the problem of a semi-infinite crack suddenly bearing and growing at a constant velocity in a stretched body. A steady state problem in which a semi-infinite crack extends at constant speed through an elastic sheet was solved by Craggs (1960). Using the method of matched asymptotic expansion the problem involving diffraction of plane elastic waves by a semi-infinite boundary of finite width was solved by Viswanathan and Sharma (1978) and by Viswanathan, Sharma and Datta (1982).

The diffraction problem of a semi-infinite crack has been solved by the Wiener-Hopf (1958) technique.

In 1921 Griffith considered the problem of a fracture of a glass containing crack like defects. Griffith's work presented a theory of fracture. Among other workers investigating crack problems are

Orowan (1948), Sack (1946), Irwin (1957) etc. A number of crack problems in the theory of classical elasticity can be found in the literature [e.g. Sneddon and Lowengrub (1969), Sih (1972)].
Yoffe (1951) considered the inplane problem of propagation of a finite Griffith crack of fixed length at a constant speed in an isotropic elastic solid of infinite extent. Other references treating elastodynamic problems involving a single finite Griffith crack are of Sato (1961), Williams (1957,1961), Karp and Karal (1962), Ang and Knopoff (1964), Loeber and Sih (1968), Sih and Loeber (1968,1969,1970), Willis (1967), Atkinson and Eshebly (1968), Mal (1970a,1970b,1972), Hilton and Sih (1971), Chang (1971), Thau and Lu (1971), Sih, Embley and Ravera (1972), Kanninen (1974), Chen (1978), Sih and Chen (1980), Takei, Shindo and Atsumi (1982), Ueda, Shindo and Atsumi (1983), Shindo (1985). Some other references are of Srivastava, Palaiya and Karaulia (1980a, 1980b), Srivastava, Gupta and Palaiya (1981), Erguven (1987).

Carrier (1946) studied the propagation of waves in orthotropic medium. Achenbach and Bazant (1975) considered the problem of elastodynamic near-tip stress and displacement fields for rapidly propagating cracks in orthotropic meterials. Kassir and Tse (1983) also solved the problem of moving a Griffith crack in an orthotropic material.

The problem of diffraction of finite Griffith crack along the interface of two dissimilar elastic media have been solved by Goldshtein (1966,1967), Brock and Achenbach (1973a,1973b,1974), Atkinson (1974), Matczynski (1974), Brock (1975), Srivastava, Gupta and Palaiya (1978), Neerhoff (1979), Srivastava, Palaiya and Karaulia (1980), Bostrom (1987). Bostrom (1987) solved the two dimensional scalar problem of scattering of elastic waves under anti-plane strain from an interfacecrackbetweentwoelastic half-spaces. The problem of interaction of anti-plane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half-spaces was considered by Srivastava, Palaiya and Karaulia (1980).

The transient stress and displacement fields arround an embedded crack in the shape of a circle were first investigated by Embley and Sih (1971) for extensional impact and by Sih and Embley (1971) for torsional impact. Their method of solution involves isolating the singular portion dynamic stresses in the Laplace transform domain such that the dynamic stress intensity factor can be obtained by direct application of the Laplace inversion theorem. Some other references are Mal (1968, 1970.a), Olesiak and Sneddon (1959), Pal and Sridharan (1980.a, 1980.b), Arwin and Erdogan (1971), Green (1949), Dhawan (1973), Krenk and Schimidt (1982). Robertson (1967) solved the problem of diffraction of a plane longitudinal wave by a penny-shaped crack.

We now discuss a certain type of mixed boundary value problems which are known as contact problem in the theory of elasticity. The contact problem is formulated as a problem about the influence of a rigid body or an elastic body.

Hertz investigated the punch problem in 1882. In his time many researchers followed his work. Chaplygin (1950) collected a number of punch problems worked out during the 19th century. Many authors such as Glagolev (1942), Mushkelishvilli (1953,1963), Mossakovski (1958), Ufliand (1956), Spence (1968,1975) investigated punch problems. In the literature [e.g. Gladwell (1980)] a variety of punch problems can be found.

The problem of diffraction of anti-plane shear wave by one or more finite rigid strip at the interface has been treated by Palaiya and Majumder (1981),Singh and Dhaliwal (1984), Tait and Moodie (1981), Mandal and Ghosh (1992a,1992b). Palaiya and Majumder (1981) considered the problem of diffraction of anti-plane shear wave by a finite rigid strip at the interface of two bonded dissimilar half spaces. The problem of diffraction of anti-plane shear wave by a pair of parallel rigid strips at the interface of two bonded dissimilar elastic media was solved by Mandal and Ghosh (1992a). De Sarkar (1985a,1985b) solved the punch problem on a micropolar elastic solid.

Different techniques have been adopted by many authors to solve these type of crack and inclusion problems. From these standpoint, these problems may be divided into two categories:

(i) One for low frequency oscillation of the source or long wave scattering or transmission and

(ii) the other for high frequency oscillation or short wave scattering or transmission in the medium.

The term long and short are used in comparison to the region of the source of disturbance or the size of the crack or strip etc. inside the medium to the wave length of disturbance. In case of low frequency oscillations Noble's (1963) method of solving dual integral equations, Tranter's (1968) technique for solving dual integral equations, Matched asymtotic expassion, and Variational principle are found to be very useful techniques.

NOBLE'S METHOD :

Suppose that a mixed boundary value problem is formulated by suitable integral transform so as to be governed by a set of dual integral equations of the form

$$\int_0^\infty x^{-1}\left[1+K(x)\right]S(x)J_\nu(rx)dx = f(r) \quad , \; 0\le r<a$$

$$\int_0^\infty S(x)J_\nu(rx)dx = g(r) \quad , \quad r>a$$

where the functions K(x), f(r) and g(r) are known.

According to Noble (1963), when $\nu>-\frac{1}{2}$.

$$S(x) = \sqrt{\frac{2x}{\pi}}\left\{\int_0^a t^{1/2}\theta(t)J_{\nu-1/2}(xt)dt + \int_a^\infty t^{\nu+1/2}G(t)J_{\nu-1/2}(xt)dt\right\}$$

where $\theta(t)$ satisfies the Fredholm integral equation

$$\theta(t) + \frac{1}{\pi}\int_0^a M(\tau,t)\theta(\tau)d\tau = t^{-\nu}F(t) - H(t) \qquad (0\le t<a) \qquad (34)$$

in which $$M(\tau,t) = \pi\sqrt{\tau t}\int_0^\infty xK(x)J_{\nu-1/2}(\tau x)J_{\nu-1/2}(tx)dx$$

$$F(t) = \frac{d}{dt}\int_0^t f(r)r^{\nu+1}\,(t^2-r^2)^{-1/2}\,dr$$

$$H(t) = t^{1/2}\int_0^\infty xK(x)J_{\nu-1/2}(xt)dx\int_a^\infty \xi^{\nu+1/2}G(\xi)J_{\nu-1/2}(x\xi)d\xi$$

$$G(\xi) = \int_\xi^\infty g(r)r^{-\nu+1}\,(r^2-\xi^2)^{-1/2}\,dr\ .$$

The integral equation (34) can be solved for $\theta(t)$ iteratively for low frequency and consequently $S(x)$ can be determined.

Singh and Dhaliwal (1984) solved the closed form solutions of dynamic punch problems by integral transform method. The mixed boundary value problem was reduced to a set of dual integral equations with trigonometrical kernels. The solutions were obtained by using Hilbert transform technique [Srivastava and Lowengrub (1968)]. We now discuss the Hilbert transform technique as follows.

HILBERT TRANSFORM TECHNIQUE :

Using the theorem (Tricomi, 1951), if $p \in L_2(a,b)$, then the equation

$$\frac{1}{\pi}\int_a^b \frac{h(x)}{x-y}\,dx = p(y)\quad ,\qquad y \in (a,b)$$

has the solution

$$h(x) = -\frac{1}{\pi}\left(\frac{x-a}{b-x}\right)^{1/2}\int_a^b \left(\frac{b-y}{y-a}\right)^{1/2}\frac{p(y)}{y-x}\,dy + \frac{C}{\sqrt{(x-a)(b-x)}}$$

where C is an arbitrary constant and the first term belongs to the class $L_2(a,b)$. Srivastava and Lowengrub (1968) found that the solution of the integral equation

$$\frac{1}{\pi}\int_a^b \frac{2th(t^2)}{t^2-y^2}\,dt = p(y) \quad , \quad y \in (a,b)$$

(provided that p satisfies the conditions of the above theorem) is given by

$$h(t^2) = -\frac{1}{\pi}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\int_a^b \left(\frac{b^2-y^2}{y^2-a^2}\right)^{1/2}\frac{2yp(y)}{y^2-t^2}\,dy + \frac{C}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$

where C is an arbitrary constant.

Using Hilbert transform technique problems involving pair of cracks or strips can easily be solved. Using Hilbert transform technique and also applying the modified Tricomi (1951) theorem of Singh (1973) Singh and Dhaliwal (1984) obtained a closed form solution of dynamic punch problem involving two moving punches.

All the axisymmetrical contact problems may be solved by using Hankel transforms and they then reduce to the solution of a number of sets (or pairs) of dual integral equations. To solve these dual integral equations there are various methods one of which is

Tranter's method. We discuss briefly the method of Tranter (1968) in solving axisymmetric problems.

TRANTER'S METHOD :

The solution of certain physical problems involving axisymmetric geometry can be reduced to the determination of F(p) from so called dual integral equations of the form

$$\int_0^\infty G(p)F(p)J_\nu(rp)dp = f(r) \quad , \quad 0<r<1$$

$$\int_0^\infty pF(p)J_\nu(rp)dp = 0 \quad , \quad 1<r<\infty \qquad (35)$$

where G(p) and f(r) are known functions.

A solution F(p) of the above integral equations as a series of Bessel functions can be found by setting

$$F(p) = p^{-k} \sum_{m=0}^{\infty} a_m J_{\nu+2m+k}(p) \qquad (36)$$

where k is at present an arbitrary parameter, and proceeding as follows.

Substituting from (36) in the second equation of (35) and changing the order of integration and summation, one gets

$$\int_0^\infty pF(p)J_\nu(rp)dp = \sum_{m=0}^{\infty} a_m \int_0^\infty p^{1-k} J_\nu(rp)J_{\nu+2m+k}(p)dp \qquad (37)$$

Provided $\nu>-1$ and $k>0$, the formula

$$I(\nu,\mu,\lambda,a,b) = \int_0^\infty \frac{J_\nu(at)J_\mu(bt)}{t^\lambda}\,dt = \frac{b^\mu \Gamma(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2})}{2^\lambda a^{\mu-\lambda+1}\Gamma(\mu+1)\Gamma(\frac{\lambda}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2})}$$

$$\times\ {}_2F_1\left(\frac{\nu+\mu-\lambda+1}{2},\ \frac{\mu-\lambda-\nu+1}{2}\ ;\ \mu+1\ ;\ \frac{b^2}{a^2}\right)$$

shows that all the integrals on the right hand side of (37) vanish when $r>1$ (because of the factor $\Gamma(-m)$ in the denominator of the term multiplying the hypergeometric function) and hence the series in (36) automatically satisfies the second of the dual equations (35). The coefficients a_m have now to be chosen so that the series in (36) satisfies the first of the dual equations (35). For this purpose we need the result

$$p^{-k} J_{\nu+2n+k}(p) = \frac{\Gamma(\nu+n+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(n+k)} \int_0^1 r^{\nu+1}(1-r^2)^{k-1}F_n(k+\nu,\nu+1,r^2) \times$$

$$\times\ J_\nu(pr)dr \qquad (38)$$

where n is a positive integer or zero and

$$F_n(\alpha,\gamma,x) = {}_2F_1(-n,\alpha+n;\gamma;x) \qquad (39)$$

is Jacobi's polynomial.

Substituting from (36) in the first of (35), multiplication by

$$r^{\nu+1}(1-r^2)^{k-1}F_n(k+\nu,\nu+1,r^2),$$

integration with respect to r between 0 and 1, interchange of the order of integrations and use of (38) give

$$\sum_{m=0}^{\infty} a_m \int_0^\infty G(p)p^{-2k}\ J_{\nu+2m+k}(p)J_{\nu+2n+k}(p)dp = E(\nu,n,k) \qquad (40)$$

where

$$E(\nu,n,k) = \frac{\Gamma(\nu+n+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(n+k)} \int_0^1 f(r)r^{\nu+1}(1-r^2)^{k-1}F_n(k+\nu,\nu+1,r^2)dr \tag{41}$$

Equation (40) with n=0,1,2,3,....... gives a set of simultaneous equations for the determination of the coefficients a_m. These simultaneous equations can be rewritten in a more convenient form by making use of the formula

$$\int_0^\infty p^{-1} J_{\nu+2m+k}(p)J_{\nu+2n+k}(p)dp = \begin{cases} 0 \ , & m\neq n \\ (2\nu+4n+2k)^{-1} \ , & m=n \end{cases} \tag{42}$$

this being the form taken by equation

$$\int_0^\infty \frac{J_\nu(at)J_\mu(at)}{t} dt = \frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2})}{2\Gamma(1+\frac{\nu}{2}-\frac{\mu}{2})\Gamma(1+\frac{\nu}{2}+\frac{\mu}{2})\Gamma(1-\frac{\nu}{2}+\frac{\mu}{2})}$$

$$= \frac{2}{\pi}\frac{\sin\frac{1}{2}(\mu-\nu)\pi}{\mu^2-\nu^2} \tag{43}$$

when μ and ν are replaced respectively by $\nu+2n+k$, $\nu+2m+k$ and when 'at' is replaced by p. We find in this way

$$a_n + \sum_{m=0}^{\infty} L_{m,n}\, a_m = (2\nu+4n+2k)E(\nu,n,k) \tag{44}$$

where

$$L_{m,n} = (2\nu+4n+2k)\int_0^\infty \left[G(p)p^{1-2k} - 1\right]p^{-1} J_{\nu+2m+k}(p)J_{\nu+2n+k}(p)dp \tag{45}$$

The iterative solution of the simultaneous equations (44) is

$$a_n = E_n - E_n' + E_n'' - \ldots\ldots \tag{46}$$

where

$$E_n = (2\nu+4n+2k)E(\nu,n,k)$$

$$E'_n = \sum_{m=0}^{\infty} L_{m,n}\, E_m \quad , \quad E''_n \doteq \sum_{m=0}^{\infty} L_{m,n}\, E'_m \qquad (47)$$

and so on.

Equations (36), (46), (47), (45) and (42) provide a theoratical solution of the dual integral equations. For a practical solution it is necessary to be able to choose the parameter k so that the expression $\left[G(p)p^{1-2k} - 1\right]$, which occurs in the formula (45) for $L_{m,n}$, is fairly small.

Low frequency diffraction due to disc, cone and rigid cylinder have been studied by Asvestas and Kleinman (1970), Senior (1971), Datta (1974), Roy (1982a,1982b,1982c), Sleeman (1967), Roy and Sabina (1982). Datta (1970) considered the problem of diffraction of a plane compressional elastic wave by a rigid circular disc.

The problem of diffraction of elastic waves by two or more co-planar Griffith cracks are very few in number. As regards the dynamic crack problem, research has been restricted mainly to the case of a single crack because of the severe mathematical complexity encountered in finding solutions for two or more cracks.

Itou (1978) solved the problem of dynamic stress concentration around two co-planar Griffith cracks in an infinite elastic medium. Itou (1980a,1980b) also considered two different problems involving two finite cracks. The problem of determining the transient stress distribution in an infinite elastic medium weakened by two coplanar Griffith cracks has been reduced to the following equation

$$\sum_{n=1}^{\infty} c_n(s)\left[-\frac{4c_L^3}{k^2 s^2 b}\int_0^{\infty} g(s,\xi)\sin\left(\frac{a+b}{2}\xi - \frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\xi\right)\cos(\xi x)d\xi\right]$$

$$= - Pc_L(bs) \quad , \quad a<x<b \tag{48}$$

with

$$g(s,\xi) = \frac{[\xi^2+k^2s^2/(2c_L^2)]^2 - \xi^2\gamma_1\gamma_2}{\xi\gamma_1} \tag{49}$$

where locations of the cracks are $a\le|x|\le b$, $|y|<\infty$, $z = 0$,

$c_L = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$, $c_T = \left(\frac{\mu}{\rho}\right)^{1/2}$, $k = c_L/c_T$ and $c_n(s)$ are the unknown coefficients.

To determine the coefficients $c_n(s)$ by Schmidt's method (1958) equation (48) can be rewritten as

$$\sum_{n=1}^{\infty} c_n(s)E_n(s,x) = - u(s,x) \quad , \quad a<|x|<b \tag{50}$$

where $E_n(s,x)$ and $u(s,x)$ are known functions and the coefficients $c_n(s)$ are unknown.

A set of functions $P_n(s,x)$ which satisfy the orthogonality condition

$$\int_o^b P_m(s,x)P_n(s,x)dx = N_n\ \delta_{mn} \quad , \qquad N_n = \int_a^b P_n^2(s,x)dx \qquad (51)$$

can be constructed from the function, $E_n(s,x)$, such that

$$P_n(s,x) = \sum_{n=1}^{\infty} \frac{M_{in}}{M_{nn}} E_i(s,x) \qquad (52)$$

where M_{in} is the cofactor of the element d_{in} of D_n , which is defined as

$$D_n = \begin{vmatrix} d_{11} & d_{12} & \cdots\cdots & d_{1n} \\ d_{21} & & & \\ \vdots & & & \\ d_{n1} & \cdots\cdots & & d_{nn} \end{vmatrix} \qquad (53)$$

$$d_{in} = \int_a^b E_i(s,x)E_n(s,x)\ dx\ .$$

Using equations (50) and (51) one can obtain

$$c_n(s) = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \qquad (54)$$

with $$q_j = -\frac{1}{N_j}\int_a^b u(s,x)P_j(s,x)\ dx \qquad (55)$$

In case of high frequency oscillation Wiener-Hopf (Noble, 1958) technique and Keller's (1958) geometrical theory are found to be most suitable. We now briefly discuss these methods.

WIENER- HOPF METHOD:

The typical problem obtained by applying Fourier transforms to partial differential equations is the following. One shall have to find unknown functions $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ satisfying

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\Psi_-(\alpha) + C(\alpha) = 0 \tag{56}$$

where this equation holds in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ of the complex α-plane, $\Phi_+(\alpha)$ is regular in the half-plane $\tau > \tau_-$, $\Psi_-(\alpha)$ is regular in $\tau < \tau_+$, and certain information which will be specified later is available regarding the behaviour of these functions as α tends to infinity in appropriate half-planes. The functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ are given function of α, regular in the strip. For simplicity let us assume that A, B are also non-zero in the strip.

The fundamental step in the Wiener-Hopf procedure for solution of this equaion is to find $K_+(\alpha)$ regular and non-zero in $\tau > \tau_-$, $K_-(\alpha)$ regular and non-zero in $\tau < \tau_+$, such that

$$A(\alpha)/B(\alpha) = K_+(\alpha)/K_-(\alpha) \tag{57}$$

Use (57) to rearrange (56) as

$$K_+(\alpha)\Phi_+(\alpha) + K_-(\alpha)\Psi_-(\alpha) + K_-(\alpha)C(\alpha)/B(\alpha) = 0 \tag{58}$$

Decompose $K_-(\alpha)C(\alpha)/B(\alpha)$ in the form

$$K_-(\alpha)C(\alpha)/B(\alpha) = C_+(\alpha) + C_-(\alpha) \quad (59)$$

where $C_+(\alpha)$ is regular in $\tau>\tau_-$, $C_-(\alpha)$ is regular in $\tau<\tau_+$.
With the help of (59) rearrange (58) so as to define a function $J(\alpha)$ by

$$J(\alpha) = K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha) = - K_-(\alpha)\Psi_-(\alpha) - C_-(\alpha) \quad (60)$$

So far this equation defines $J(\alpha)$ only in the strip $\tau_- < \tau < \tau_+$. But the second part of the equation is defined and is regular in $\tau > \tau_-$, and the third part is defined and is regular in $\tau < \tau_+$. Hence by analytic continuation $J(\alpha)$ must be regular in the whole α-plane. Then by the extended form of Liouville's theorem $J(\alpha)$ is a polynomial $p(\alpha)$

$$K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha) = p(\alpha)$$

$$(61)$$

$$K_-(\alpha)\Psi_-(\alpha) + C_-(\alpha) = - p(\alpha)$$

These equations determine $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ to within the arbitrary polynomial $p(\alpha)$, i.e. to within a finite number of arbitrary constants which must be determined otherwise.

KELLER'S GEOMETRICAL METHOD :

Keller's theory of geometrical diffraction applied to elastodynamics states that the two conical surfaces of diffracted rays are generated when an incident ray strikes an edge. The surface of the inner cone consists of rays of longitudinal motion,

while the surface of the outer cone is composed of rays of transverse motion. The half-angles of the cones are related by Snell's law. Fig.1 shows the cones generated by an incident longitudinal ray. For this case the diffracted longitudinal rays make the same angle ϕ_L with the tangent to the edge as the incident ray, and the diffracted rays of transverse motion are under an angle ϕ_T with the edge, where $C_L \cos\phi_T = C_T \cos\phi_L$.
For a straight diffracting edge, and an incident longitudinal ray, the diffracted displacement fields are related quantitatively to the incident field by

$$\vec{u}_d^L = e^{i\omega s_1/C_L} \left[S_1(1+S_1/R_i)\right]^{-1/2} D_L \hat{i}_L^d A e^{i\omega(s_o/C_L-t)}$$

$$\vec{u}_d^T = e^{i\omega s_2/C_T} \left[S_2(1+S_2/R_d)\right]^{-1/2} D_T \hat{i}_T^d A e^{i\omega(s_o/C_L-t)}$$

Here $A\exp[i\omega(S_o/C_L-t)]$ defines the amplitude and the phase of the incident field at the point of diffraction, and D_L and D_T are diffraction coefficents which relate the diffracted field to the incident field. Also S_1 and S_2 are the smaller of the principal radii of curvature of the diffracted wave front, or equivalently the distances along the diffracted rays from the points of diffraction to the observation point. The unit vectors $\hat{i}_L^d$ and $\hat{i}_T^d$ relate the directions of displacement of the diffracted field to the direction of displacement of the incident field. For a straight diffracting edge R_i is the radius of curvature at the point of diffraction of the curve formed by the intersection of the incident wave front and the plane which contains the incident

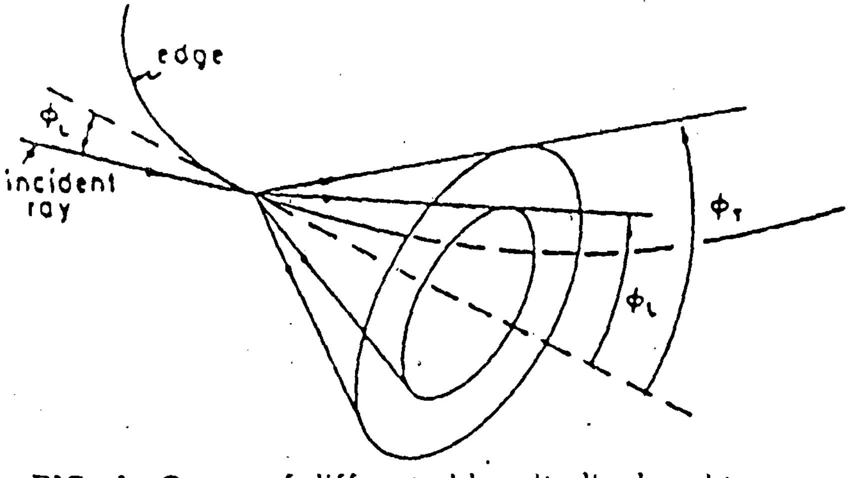

FIG. 1. Cones of diffracted longitudinal and transverse rays for an incident longitudinal ray.

ray and the edge, and

$$R_d = R_i \frac{\sin \phi_T}{\sin \phi_L} \frac{\tan \phi_T}{\tan \phi_L}.$$

In the thesis presented here we have studied some mixed boundary value problems in elastodynamics involving punches, inclusions and cracks. The work has been presented in three chapters. The first chapter deals with the diffraction problems in elastic media by propagating semi-infinite cracks. The second chapter deals with the diffraction problems in isotropic media involving finite width Griffith cracks when the boundaries are present in the medium. The last chapter i.e. chapter III deals with diffraction problem of elastic waves in an infinite orthotropic elastic medium in presence of strips or cracks of finite width. Here we give the summary of the thesis chapterwise.

In the first paper of the chapter-I, we have considered the problem of a series of semi-infinite, parallel and equally spaced cracks subjected to identic loads satisfying the conditions of anti-plane state of strain and steadily propagating in an infinite inhomogeneous medium. Cracks are assumed to move steadily in the direction of modulus variation, it being assumed that the moduli vary exponentially. We further assume that the medium possesses constant elastic wave speeds. These assumptions are necessary for the steady state solution to exist. We assume that the loading is such that Mode III conditions prevail. Mode III is the simplest mode to analyze mathematically. This problem has been solved by the application of Wiener-Hopf technique. We have solved the problem for two types of anti-plane loading. Firstly, the case when the crack edges are loaded at fixed distance from the crack-edge by a concentrated force of constant magnitude has been solved. Secondly, the crack propagation in the case of constant strain on the crack edges has been treated. In both the cases expressions of the stress intensity factor and the crack opening displacement have been derived in closed form and the effect of inhomogeneity of the medium has been shown by means of graphs.

In paper-2, we have considered the problem of the diffraction of a plane harmonic SH-wave by a semi-infinite crack running uniformly along the interface of two dissimilar semi-infinite elastic media. We have applied Fourier transform and Wiener-Hopf technique (1958) to solve the mixed boundary value problem. The resulting integrals have been evaluated asymptotically to obtain the displacement and stress-field near about the crack-tip. The effect of different values of the material parameter, the angle of incidence of incident wave and the crack propagation velocity on the stress intensity factor have been illustrated graphically.

In the second chapter of the thesis, we have considered three problems involving the diffraction of elastic waves by finite width Griffith cracks in a strip and also a problem of diffraction of elastic waves by a series of periodically placed Griffith cracks in an isotropic elastic medium.

First problem of chapter II deals with the diffraction of elastic SH-waves by a Griffith crack in an infinitely long inhomogeneous elastic strip. The shear modulus (μ) and density (ρ) of the material have been assumed to vary in the vertical direction. Applying the Fourier transform, the mixed boundary value problem has been reduced to the solution of the dual integral equations. The dual integral equations have been finally reduced to a Fredholm integral equation of second kind by applying the Abel transform. The numerical values of stress intensity factor and crack opening displacement have been illustrated graphically to show the effect of inhomogeneity of the material.

In the second paper of this chapter we have studied the two dimensional problems of diffraction of longitudinal waves by a series of periodically spaced co-planar Griffith cracks in an infinite, isotropic elastic medium. Due to the periodicity of the geometry, the problem can be reduced to the problem with a single crack in a strip with boundaries such that shear stress and normal displacement are zero on them. On use of Fourier transform the mixed boundary value problem for a typical strip has been reduced to the solution of dual integral equations and finally to that of a Fredholm integral equation of the second kind by applying Abel's transform. Expressions for the stress intensity factor and crack opening displacement have been derived in closed form. Numerical values of stress intensity factor and the crack opening displacement have been plotted graphically.

Paper 3 deals with the dynamic antiplane problem of determining

stress and displacement due to three coplanar Griffith cracks moving steadily at a subsonic speed in an infinite elastic strip. Employing Fourier integral transform, the problem when the rigidly clamped edges on the strip are pulled apart in opposite directions has been reduced to solving a set of four integral equations. These integral equations have been solved using the finite Hilbert transform technique and Cook's result (1970) to obtain the exact form of crack opening displacement and stress intensity factors. Making the length of the inner crack tend to zero, the diffraction problem for two cracks have been obtained.
Again, letting the distance between the edges of the inner and outer cracks tend to zero, the diffraction problem for a single crack has also been derived. Numerical results of stress intensity factors are presented in the form of graphs.

In the last problem, i.e. in paper 4 of chapter II, we investigated the problem of determining the antiplane dynamic stress distributions around four coplanar finite length Griffith cracks moving steadily with constant velocity in an infinitely long finite width strip. The two-dimensional Fourier transform has been used to reduce the mixed boundary value problem to the solution of five integral equations. These integral equations have been solved using the finite Hilbert transform technique to obtain the analytical form of crack opening displacement and stress intensity factors. Numerical results for the stress intensity factors at the crack tips have also been depicted graphically. Letting the distance between the inner cracks tend to zero, the corresponding solution of diffraction problem in the presence of three cracks has been derived.

Chapter III deals with some contact problems and crack problems in elastodynamics in an orthotropic elastic medium.

In the first problem of chapter III, the elastodynamic response of a pair of parallel rigid strips embedded in an infinite orthotropic medium due to elastic waves incident normally on the strips has been investigated. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Iterative solution valid for low frequency has been obtained. From the solution of the integral equation, we have found out the normal stress and vertical displacement at points in the plane of the strips. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal (1972.b) Numerical values of the stress intensity factors at inner and outer edges of the strips for several orthotropic materials have been calculated and plotted graphically to show the effect of material orthotropy.

Problem 2 of this chapter deals with the problem of diffraction of normally incident elastic waves by two coplanar Griffith cracks situated in an infinite orthotropic medium. Fourier and Hilbert transform techniques have been used to solve this mixed boundary value problem. The resulting triple integral equation has been reduced to the solution of an integro-differential equation and approximate solution has been obtained. These solutions have been used to obtain approximate analytical results for stress intensity factors and crack opening displacements when the wave lengths are large compared to the crack length. Making the distance between two cracks zero, the corresponding results for a single crack has been presented. Finally, choosing the engineering elastic constants of the orthotropic material suitably the results for isotropic material have been deduced and compared with the results obtained by Jain and Kanwal (1972a) To display the influence of the material orthotropy numerical values of stress intensity factors and crack opening displacements have been plotted for several orthotropic materials.

In paper 3, we have considered the problem of dynamic response of three coplanar Griffith cracks in an infinite orthotropic medium due to elastic waves incident normally on the cracks. Fourier transform technique has been used to reduce the elastodynamic problem to the solution of a set of four integral equations. These integral equations have been solved by using finite Hilbert transform technique and Cook's result. The analytical forms of crack opening displacements and stress intensity factors have been derived for low frequency vibration. Making the length of the central crack tend to zero, the corresponding results for two Griffith cracks have been obtained. Numerical results for stress intensity factors and crack opening displacements have been plotted against dimensionless frequency and distance respectively.

for different orthotropic materials which have been shown graphically.

The last problem of this chapter deals with the problem of diffraction of normally incident elastic waves due to four coplanar Griffith cracks in an infinite orthotropic elastic medium. The faces of each of the cracks do not come into contact during small deformation of the solid because a small distance are assumed to be separated. By the use of the Fourier integral, the mixed boundary value problem has been reduced to solving a set of five integral equations which have been solved by finite Hilbert transform technique. This enables us to obtain approximate value of the crack opening displacements and stress intensity factors at the crack tips. The effect of stress intensity factors and crack opening displacements at the edges of the cracks for several orthotropic materials have been calculated and plotted by means of graphs. Also letting the distance of the inner cracks tend to zero, the corresponding results for three cracks have been obtained.

With this much of introduction, we now present the thesis chapterwise. References given in the thesis do not include all the previous workers in this line. But attempt has been made to include most of them.