CHAPTER - III

MIXED BOUNDARY VALUE PROBLEMS IN VISCOELASTIC MEDIA

Paper - 7 : Moving punch on a viscoelastic semi-infinite medium.

Paper - 8 : Antiplane dynamic crack propagation in an inhomogeneous viscoelastic solid.

MOVING PUNCH ON A VISCOELASTIC SEMI-INFINITE MEDIUM

1. INTRODUCTION

Problems involving the motion of a punch on the surface of an elastic half-space or on the free boundaries of long strips are extremely important in view of their application in road construction technology and also in geophysical research. Punch problems within the classical theory of elasticity have been studied extensively by Galin (1961) and by Gladwell (1980) in their books. The motiom of a rough punch on an elastic half-space has been treated in detail by Suhubi (1972). Recently problems involving antiplane motion due to punches moving along the surfaces of an elastic strip have been solved by complex variable methods by Tait and Moodie (1981). An analytical solution to the problem of a long rigid punch moving rapidly on a strip of a highly orthotropic elastic layer has been solved by Georgiadis (1987) using integral transforms and the Wiener-Hopf techniques (1958).

However, natural or artificial materials have generally dissipative behaviour which often can be taken into account by viscoelastic models. Accordingly, problems involving the motion of a punch on a viscoelastic medium have drawn the attention of many

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scientists. The problem of a rigid cylinder rolling on the surface of a viscoelastic half space has been solved by Hunter (1961). The contact problem of rigid cylinder rolling slowly on a thin viscoelastic layer has been treated by Alblas and Kuipers (1970) assuming that the layer thickness is small compared to the width of the contact region of the cylinder. The problem of a plane punch sliding without friction on a viscoelastic half space has been considered by Golden (1977).

In the present paper, we have examined the stress and displacement field produced by a long punch moving on the boundary of a semi-infinite viscoelastic medium and producing Horizontal Shear waves. Two types of viscoelastic models viz. Maxwell Solid and Standard Linear Solid have been considered and loading is assumed to be such that Mode III conditions prevail. The mathematical technique which is employed here consists of the application of integral transforms and the solution of the resulting Wiener-Hopf equations for the transformed unknown variables. Both the steady and nonsteady solutions of the problem have been derived. Displacement and stress on the free surface and at points below the punch have been derived analytically and the nature of their variations with the velocity of the moving punch has been shown by means of graphs.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION FOR STEADY STATE MOTION .

Let us consider a semi-infinite viscoelastic medium which was set

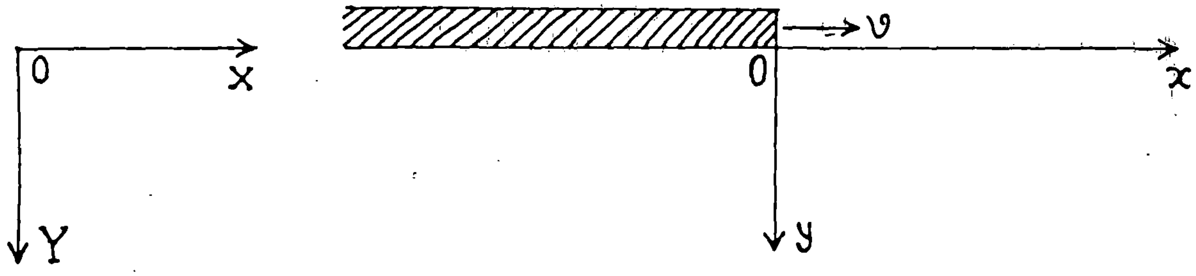

Fig. 1. The Geometry of the problem.

into motion by semi-infinite rigid punch moving with a constant velocity v in the direction of the x-axis. The y-axis is taken vertically downwards into the medium (Fig.1).

For horizontal shear waves, the displacements along X and Y directions are zero and only the displacement W = W(X,Y,t) along Z-direction exists. The stresses under the punch are

$$\sigma_{13} = \sigma_{13}(X,Y,t) \quad \text{and} \quad \sigma_{23} = \sigma_{23}(X,Y,t) \tag{1}$$

The non-vanishing strains are

$$e_{13} = \frac{1}{2}\frac{\partial W}{\partial X} \quad \text{and} \quad e_{23} = \frac{1}{2}\frac{\partial W}{\partial Y} \tag{2}$$

Considering a 'Standard Linear Solid' as the viscoelastic model, the stress strain relations are

$$\frac{\partial \sigma_{i3}}{\partial t} + \beta\sigma_{i3} = 2\mu\left[\frac{\partial e_{i3}}{\partial t} + \alpha e_{i3}\right], \quad i=1,2 \tag{3}$$

where α,β are positive constants and μ is the instantaneous elastic modulus of rigidity of the material.

The equation of motion is

$$\frac{\partial \sigma_{13}}{\partial X} + \frac{\partial \sigma_{23}}{\partial Y} = \rho\frac{\partial^2 W}{\partial t^2} \tag{4}$$

where ρ is the density of the material.

The boundary conditions of the problem are

$$\begin{aligned} W(X,0,t) &= w_0 , & X-vt<0 \\ W(X,\infty,t) &= 0 , & -\infty<X<\infty \\ \sigma_{23}(X,0,t) &= 0 , & X-vt>0 \end{aligned} \tag{5}$$

Since we are going to investigate the steady state propagation of a punch, it is convenient to define a moving co-ordinate system (x,y) whose origin coincides with the tip of the punch and whose axes are parallel to the fixed (X,Y) axes respectively (Fig.1). Hence putting $x = X-vt$, $y = Y$ equations (1) to (4) become respectively

$$\sigma_{13} = \sigma_{13}(x,y) \quad \text{and} \quad \sigma_{23} = \sigma_{23}(x,y) \tag{6}$$

$$e_{13} = \frac{1}{2}\frac{\partial}{\partial x}W(x,y) \quad \text{and} \quad e_{23} = \frac{1}{2}\frac{\partial}{\partial y}W(x,y) \tag{7}$$

$$\begin{aligned} -v\frac{\partial\sigma_{13}}{\partial x} + \beta\sigma_{13} &= \mu\left[-v\frac{\partial^2 W}{\partial x^2} + \alpha\frac{\partial W}{\partial x}\right] \\ -v\frac{\partial\sigma_{23}}{\partial x} + \beta\sigma_{23} &= \mu\left[-v\frac{\partial^2 W}{\partial x\partial y} + \alpha\frac{\partial W}{\partial y}\right] \end{aligned} \tag{8}$$

and

$$\frac{\partial\sigma_{13}}{\partial x} + \frac{\partial\sigma_{23}}{\partial y} = \rho v^2\frac{\partial^2 W}{\partial x^2} \tag{9}$$

The boundary conditions (5), now become

$$W(x,0) = w_0 , \qquad x<0$$

$$W(x,\infty) = 0 , \qquad -\infty<x<\infty \tag{10}$$

$$\sigma_{23}(x,0) = 0 , \qquad x>0$$

Now introduce Fourier transform

$$\bar{f}(\xi,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,y)\exp(i\xi x)dx$$

(11)

so that $$f(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(\xi,y)\exp(-i\xi x)d\xi$$

Taking Fourier transform of (8) and (9) we get

$$(i\xi v + \beta)\bar{\sigma}_{13} = \mu(\xi^2 v - i\xi\alpha)\bar{W} \tag{12}$$

$$(i\xi v + \beta)\bar{\sigma}_{23} = \mu(i\xi v + \alpha)\frac{d\bar{W}}{dy} \tag{13}$$

and

$$i\xi\bar{\sigma}_{13} + \frac{d\bar{\sigma}_{23}}{dy} = -\rho v^2\xi^2\bar{W} \tag{14}$$

Eliminating $\bar{\sigma}_{13}$, $\bar{\sigma}_{23}$ from (12), (13) and (14)) we obtain,

$$\frac{d^2\bar{W}}{dy^2} - \gamma^2\bar{W} = 0 \tag{15}$$

where

$$\gamma^2 = \frac{\xi^2}{\left(\xi - \frac{i\alpha}{v}\right)}\left[\left(1 - \frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right] , \quad c^2 = \frac{\mu}{\rho} \tag{16}$$

The branches of γ are so chosen that

$$\mathrm{Re}(\gamma)>0 \quad \text{for} \quad -a<\mathrm{Im}(\xi)<0$$

where

$$a = \left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right) \tag{17}$$

Now the solution of equation (15) bounded as $y \to \infty$ is

$$\bar{W}(\xi,y) = B(\xi)e^{-\gamma y} \tag{18}$$

Let us consider

$$W(x,0) = w_0 = W_0\, e^{\varepsilon x}, \quad x<0,\ \varepsilon>0 \text{ and } \varepsilon \text{ will be made to tend to zero finally}$$

$$= W_0 p(x) \quad (\text{say}), \quad x>0 \tag{19}$$

$$\sigma_{23}(x,0) = 0, \quad x>0$$

$$= W_0 t(x) \quad (\text{say}), \quad x<0 \tag{20}$$

where $p(x)$ and $t(x)$ are unknown functions such that

$$p(x) \sim O\left(e^{-k_1 x}\right) \quad \text{as} \quad x \to \infty, \quad k_1>0$$

$$t(x) \sim O\left(e^{+k_2 x}\right) \quad \text{as} \quad x \to -\infty, \quad k_2>0.$$

Taking Fourier transform of (19)

$$\bar{W}(\xi,0) = \frac{W_0}{\sqrt{(2\pi)}(\varepsilon+i\xi)} + \frac{W_0}{\sqrt{(2\pi)}} P_+(\xi) \tag{21}$$

where

$$P_+(\xi) = \int_0^{\infty} p(x)\exp(i\xi x)dx\ , \quad (\xi=\sigma+i\tau) \tag{22}$$

In (21) the first term on the right hand side is analytic in the lower half plane $\mathrm{Im}(\xi) = \tau < \varepsilon$ and $P_+(\xi)$ is analytic in the upper half plane $\tau > -k_1$ ($k_1 < a$, say).

Again taking Fourier transforms of (20)

$$\bar{\sigma}_{23}(\xi,0) = \frac{W_0}{\sqrt{(2\pi)}}T_-(\xi) \tag{23}$$

where
$$T_-(\xi) = \int_{-\infty}^{0} t(x)\exp(i\xi x)dx \tag{24}$$

$T_-(\xi)$ is analytic in the lower half plane $\tau < k_2$. Therefore, $\bar{W}(\xi,0)$ is analytic for $-k_1 < \tau < \varepsilon$ and $\bar{\sigma}_{23}(\xi,0)$ is analytic in the lower half plane $\tau < k_2$.

From (13),

$$\Big[(i\xi v + \beta)\bar{\sigma}_{23} \Big]_{y=0} = \Big[\mu(i\xi v + \alpha)\frac{d\bar{W}}{dy} \Big]_{y=0}$$

Using (18), (21) and (23) this becomes

$$T_-(\xi) = -H(\xi)\Big[P_+(\xi) - \frac{i}{\xi - i\varepsilon} \Big] \tag{25}$$

where

$$H(\xi) = \frac{\mu\xi\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \left[\left(1 - \frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \tag{26}$$

It may be noted that the problem has been reduced to a form suitable for the application of the Wiener-Hopf technique.

Now $H(\xi)$ can be written as

$$H(\xi) = H_+(\xi)H_-(\xi) \tag{27}$$

where

$$H_+(\xi) = \mu\left[\left(1 - \frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \tag{28}$$

and

$$H_-(\xi) = \frac{\xi\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \tag{29}$$

$H_+(\xi)$ is analytic in the upper half plane $\tau > -a$ and $H_-(\xi)$ is analytic in the lower half plane $\tau < 0$.

Introducing (27) in equation (25), we obtain after a little algebraic simplification,

$$R_-(\xi) - \frac{T_-(\xi)}{H_-(\xi)} = H_+(\xi)P_+(\xi) - R_+(\xi) \tag{30}$$

where

$$R_+(\xi) = \frac{i\left[H_+(\xi) - H_+(i\varepsilon)\right]}{\xi - i\varepsilon} \tag{31}$$

and
$$R_-(\xi) = \frac{i\,H_+(i\varepsilon)}{\xi - i\varepsilon} \tag{32}$$

The functions $R_+(\xi)$ and $R_-(\xi)$ are such that each is analytic and non-zero in some upper and lower half planes respectively.

The functions on the R.H.S. of (30) are analytic and non-zero in the upper half plane $\tau > -a$ and the functions on the L.H.S. are analytic and non-zero in the lower half plane $\tau < 0$.

Since, both the functions are analytic in the strip $-a<\tau<0$, the principle of analytic continuation states that each represents an entire function $M(\xi)$ in the whole ξ-plane.

Now near the tip of the punch,

$\sigma_{23}(x,0) \sim o\left[-\frac{1}{x}\right]^{1/2}$ as $x \to 0^-$, so $T_-(\xi) \sim o\left[\xi^{-1/2}\right]$, as $|\xi| \to \infty$

and $R_-(\xi) \sim o\left[\xi^{-1}\right]$ as $|\xi| \to \infty$.

Thus the L.H.S. of (30) approaches zero as $|\xi| \to \infty$. It may be concluded by Liouville's theorem that $M(\xi) = 0$ and therefore

$$T_-(\xi) = R_-(\xi)H_-(\xi) \tag{33}$$

and

$$P_+(\xi) = \frac{R_+(\xi)}{H_+(\xi)} \tag{34}$$

Now,

$$T_-(\xi) = i\mu\left[i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \quad \text{as } \varepsilon \to 0.$$

So from (23)

$$\bar{\sigma}_{23}(\xi,0) = \frac{W_0}{\sqrt{(2\pi)}}\, i\mu\left[i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)}$$

Therefore for $x<0$

$$\sigma_{23}(x,0) = \frac{i\mu W_0}{2\pi}\left[i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \exp(-i\xi x)\,d\xi \tag{35}$$

Considering a branch cut along the positive imaginary axis starting from $\xi = \frac{i\alpha}{v}$ and changing the path of integration from real ξ-axis to the path around the branch cut as shown in Fig.2, it can easily be shown that the integral

$$I = \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \exp(-i\xi x)\,d\xi \qquad (\text{assuming } \beta>\alpha)$$

can be converted to the following integral

$$I = 2\, e^{\frac{\pi i}{4}}\, e^{-\frac{\alpha}{v}x_1} \fint_0^{\infty} \frac{\sqrt{u}\; e^{-ux_1}}{u - \left(\frac{\beta}{v} - \frac{\alpha}{v}\right)}\, du \tag{36}$$

where x has been replaced by $-x_1$, $\fint_0^{\infty}$ denotes the principal value of the integral.

For large values of $\frac{\beta-\alpha}{v}\, x_1 = mx_1$, where $m = \frac{\beta-\alpha}{v}$, the integral (36) can be evaluated in the form

$$I = -2\, e^{\frac{\pi i}{4}}\, e^{-\frac{\alpha}{v}x_1} (x_1)^{-1/2} \left[\frac{\Gamma(3/2)}{mx_1} + \frac{\Gamma(5/2)}{m^2 x_1^2} + \frac{\Gamma(7/2)}{m^3 x_1^3} + \ldots\ldots \right] \tag{37}$$

and for small values of mx_1 it can be shown that

$$I = 2\, e^{\frac{\pi i}{4}}\, e^{-\frac{\alpha}{v}x_1} \sqrt{\frac{\pi}{x_1}} \tag{38}$$

The details of the evaluation of the integral I has been shown in the Appendix-1.

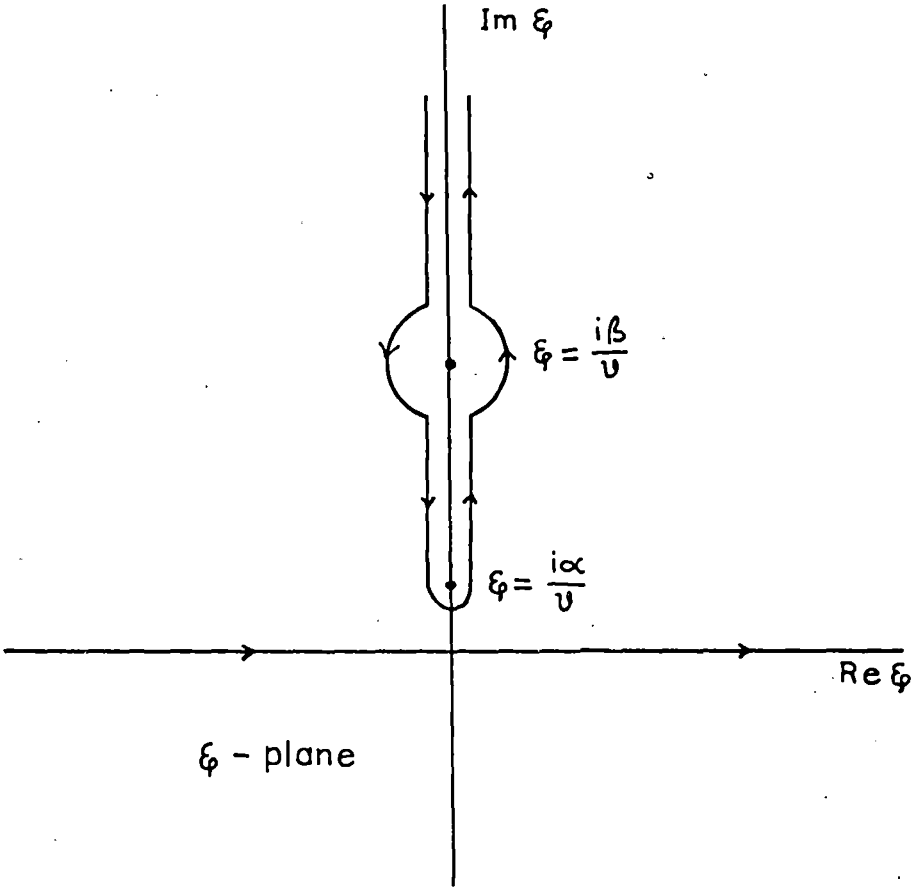

Fig.2. Path of Integration to evaluate I .

Therefore using (35) and (36) we obtain for x<0,

$$\sigma_{23}(x,0) = -\frac{\mu W_o}{\pi}\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)^{1/2} e^{-\frac{\alpha}{v}x_1}\int_0^{\infty}\frac{\sqrt{u}\;e^{-ux_1}}{u-\left(\frac{\beta}{v}-\frac{\alpha}{v}\right)}\,du \quad , \quad x<0 \tag{39}$$

Using the value of the integral arising in (39) by (38), we get for small values of mx_1,

$$\sigma_{23}(x,0) = -\frac{\mu W_o}{\sqrt{\pi m x_1}}\left[m\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)\right]^{1/2} e^{-\frac{\alpha}{v}x_1} \quad , \quad x_1 \to 0^+ \tag{40}$$

Also with the help of (39) and (37), for large values of mx_1 (x<0) we have,

$$\sigma_{23}(x,0) = \frac{\mu W_o}{\pi\sqrt{m x_1}}\left[m\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)\right]^{1/2} e^{-\frac{\alpha}{v}x_1} \times$$

$$\times\left[\frac{\Gamma(3/2)}{mx_1}+\frac{\Gamma(5/2)}{m^2x_1^2}+\frac{\Gamma(7/2)}{m^3x_1^3}+\ldots\ldots\right] \tag{41}$$

Now, from (28), (31) and (39)

$$P_+(\xi) = \frac{i}{\xi} - \frac{i\left[i\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)\right]^{1/2}}{\xi\left[\left(1-\frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)\right]^{1/2}} \quad , \quad \varepsilon \to 0.$$

Using this result in (21) we get

$$\overline{W}(\xi,0) = -\frac{iW_o}{2\pi}\sqrt{ia}\,\frac{1}{\xi\sqrt{(\xi+ia)}} \quad , \quad a = \left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right) \Big/ \left(1-\frac{v^2}{c^2}\right)$$

Taking inverse Fourier transform

$$W(x,0) = -\frac{iW_o}{2\pi}\sqrt{ia}\int_{-\infty-id}^{\infty-id}\frac{1}{\xi\sqrt{(\xi+ia)}}\exp(-i\xi x)d\xi \ , \quad x>0 \quad (0<d<a) \tag{42}$$

Transforming the integral in (42) to an integral along the contour around the branch cut from -ia to -i∞, it can be shown that

$$W(x,0) = -\frac{iW_o}{\pi}\sqrt{iax}\; e^{-ax}\; e^{\frac{\pi i}{4}}\int_0^{\infty}\frac{e^{-U}\,U^{-1/2}}{U+ax}\,dU \qquad (x>0)$$

which can be written as

$$W(x,0) = \frac{W_o}{\sqrt{\pi}}\, e^{-ax/2}\,(ax)^{-1/4}\, W_{-1/4,-1/4}(ax) \ , \ (x>0) \tag{43}$$

where $W_{k,m}$ is the Whittaker function (1969).

Using the results that

$$W_{k,m}(z) \sim \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}\, z^{\frac{1}{2}+m}\, e^{-z/2} + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\, z^{\frac{1}{2}-m}\, e^{-z/2}$$

for small z,

and $\quad W_{k,m}(z) \sim e^{-z/2}\,(z)^k \quad$ for large z,

in (43) we obtain for small values of ax (x>0)

$$W(x,0) = W_o\, e^{-ax} - \frac{2W_o}{\sqrt{\pi}}\, e^{-ax}\sqrt{(ax)} \ , \quad x \to 0^+ \tag{44}$$

and for large values of ax (x>0)

$$W(x,0) = \frac{W_o}{\sqrt{\pi}} \frac{e^{-ax}}{\sqrt{(ax)}} \quad , \quad x \to \infty \ (x>0) \tag{45}$$

3. STEADY STATE SOLUTION FOR MAXWELL SOLID

For 'Maxwell Solid' the stress strain relations obtained from (3) putting $\alpha = 0$ are

$$\frac{\partial \sigma_{i3}}{\partial t} + \beta \sigma_{i3} = 2\mu \frac{\partial e_{i3}}{\partial t} \quad , \quad i=1,2 \tag{46}$$

The stress can be found by putting $\alpha = 0$ in (39) as (for $x<0$, $y=0$)

$$\sigma_{23}(x,0) = -\frac{\mu W_o}{\pi}\left[\frac{v\beta}{c^2}\right]^{1/2} \int_0^{\infty} \frac{\sqrt{u}\, e^{-ux_1}}{u - \frac{\beta}{v}}\, du \tag{47}$$

For small values of $\frac{\beta}{v} x$, $x<0$, putting $\alpha = 0$ in (40) we get

$$\sigma_{23}(x,0) = -\frac{\mu W_o}{\sqrt{\pi x_1}}\left[\frac{v\beta}{c^2}\right]^{1/2} \quad , \quad x_1 \to 0^+ \quad , \quad (x_1=-x) \tag{48}$$

Again for large values of $\beta x/v$, $(x<0)$, from (41)

$$\sigma_{23}(x,0) = \frac{\mu W_o}{\pi\sqrt{x_1}}\left[\frac{v\beta}{c^2}\right]^{1/2}\left[\frac{v}{\beta x_1}\Gamma(\tfrac{3}{2}) + \frac{v^2}{\beta^2 x_1^2}\Gamma(\tfrac{5}{2}) + \frac{v^3}{\beta^3 x_1^3}\Gamma(\tfrac{7}{2}) + \ldots\right] \tag{49}$$

Putting $\alpha = 0$ the displacement on the free surface ($y=0$, $x>0$) is obtained from (43) as

$$W(x,0) = \frac{W_o}{\sqrt{\pi}} e^{-kx/2} (kx)^{-1/4} W_{-1/4,-1/4}(kx) , \quad (x>0) \qquad (50)$$

where

$$k = \left(\frac{v\beta}{c^2} \right) / \left(1 - \frac{v^2}{c^2} \right)$$

which for small values of kx>0 becomes by help of (44)

$$W(x,0) = W_o e^{-kx} - \frac{2W_o}{\sqrt{\pi}} e^{-kx} \sqrt{(kx)} , \quad x \to 0^+ \qquad (51)$$

and for large values of kx>0, using (45), we obtain

$$W(x,0) = \frac{W_o}{\sqrt{\pi}} \frac{e^{-kx}}{\sqrt{(kx)}} , \quad x \to \infty \ (x>0) \qquad (52)$$

4. SOLUTION OF THE PROBLEM FOR NON STEADY STATE MOTION

In this case it is assumed that at time t=0 a semi-infinite punch starts to move with a constant velocity v at X=Y=0 on the surface of the semi-infinite viscoelastic medium.

The 'Standard Linear Solid' is taken as the viscoelastic model. Shifting the origin at X=vt and putting X-vt=x and Y=y so that $\frac{\partial}{\partial X} = \frac{\partial}{\partial x}$, $\frac{\partial}{\partial Y} = \frac{\partial}{\partial y}$ and time derivative equal to $-v \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ the stress displacement relations given by (8) become in this case

$$-v\frac{\partial\sigma_{13}}{\partial x}+\frac{\partial\sigma_{13}}{\partial t}+\beta\sigma_{13}=\mu\left[-v\frac{\partial^{2}W}{\partial x^{2}}+\frac{\partial^{2}W}{\partial t\partial x}+\alpha\frac{\partial W}{\partial x}\right]$$

$$\tag{53}$$

$$-v\frac{\partial\sigma_{23}}{\partial x}+\frac{\partial\sigma_{23}}{\partial t}+\beta\sigma_{23}=\mu\left[-v\frac{\partial^{2}W}{\partial x\partial y}+\frac{\partial^{2}W}{\partial t\partial y}+\alpha\frac{\partial W}{\partial y}\right]$$

Both these equations can be reduced to ordinary differential equations by the application of the Laplace transform over t and the Fourier transform over x.

Let us denote the Laplace transform by a single bar

$$\bar{f}\equiv\bar{f}(x,y,p)=\int_{0}^{\infty}f(x,y,t)\exp(-pt)dt \tag{54}$$

and Fourier transform by two bars

$$\bar{\bar{f}}\equiv\bar{\bar{f}}(\xi,y,p)=\int_{-\infty}^{\infty}\bar{f}(x,y,p)\exp(i\xi x)dx \tag{55}$$

Applying these transforms to (53) we get

$$(i\xi v+p+\beta)\bar{\bar{\sigma}}_{13}=\mu(\xi^{2}v-i\xi p-i\xi\alpha)\bar{\bar{W}} \tag{56}$$

$$(i\xi v+p+\beta)\bar{\bar{\sigma}}_{23}=\mu(i\xi v+p+\alpha)\frac{d\bar{\bar{W}}}{dy} \tag{57}$$

Now the equation of motion given by (4) becomes

$$\frac{\partial\sigma_{13}}{\partial x}+\frac{\partial\sigma_{23}}{\partial y}=\rho\left[v^{2}\frac{\partial^{2}W}{\partial x^{2}}-2v\frac{\partial^{2}W}{\partial x\partial t}+\frac{\partial^{2}W}{\partial t^{2}}\right]$$

which after taking Laplace and Fourier transforms takes the form

$$- i\xi\bar{\sigma}_{13} + \frac{d\bar{\sigma}_{23}}{dy} = -\rho(-v^2\xi^2 + 2vi\xi p + p^2)\bar{W} \tag{58}$$

Substituting for $\bar{\sigma}_{13}$ and $\bar{\sigma}_{23}$ from (56) and (57) in (58) we have

$$\frac{d^2\bar{W}}{dy^2} - \gamma^2 \bar{W} = 0 \tag{59}$$

where

$$\gamma^2 = \frac{1}{(vi\xi+p+\alpha)} \left[\xi^2(vi\xi+p+\alpha) + \frac{\rho}{\mu}(vi\xi+p)^2(vi\xi+p+\beta)\right] \tag{60}$$

The branches of γ are defined by $\mathrm{Re}(\gamma)>0$.

Since the stresses are bounded as $y \to \infty$, $W(x,y,t)$ and hence also $\bar{W}(\xi,y,p)$ must remain bounded as $y \to \infty$.

Hence, the solution of the equation (59) is given by

$$\bar{W}(\xi,y,p) = A(\xi,p)\, e^{-\gamma y}.$$

Now the boundary conditions are

$$\begin{aligned} W(x,0,t) &= W_0H(t), & x<0 \\ W(x,\infty,t) &= 0, & -\infty<x<\infty \\ \sigma_{23}(x,0,t) &= 0. & \end{aligned} \tag{61}$$

Taking Laplace transform with respect to t, these boundary conditions become

$$\overline{W}(x,0,p) = \frac{W_o}{p} \quad , \quad x<0$$

$$\bar{\sigma}_{23}(x,0,p) = 0 \quad , \quad x>0 \tag{62}$$

Let us consider

$$\overline{W}(x,0,p) = W_o p(x) \quad \text{(say)} \quad , \quad x>0$$

and $$\bar{\sigma}_{23}(x,0,p) = \mu W_o t(x) \quad \text{(say)} \quad , \quad x<0 \tag{63}$$

The functions p(x) and t(x) are such that

$$p(x) \sim O\left[e^{-k_1 x} \right] \quad \text{as} \quad x \to \infty \,, \quad k_1>0$$

$$t(x) \sim O\left[e^{+k_2 x} \right] \quad \text{as} \quad x \to -\infty \,, \quad k_2>0.$$

Taking Fourier transform of (63) and (64) we obtain

$$\overline{W}(\xi,0,p) = \frac{W_o}{i\xi p} + W_o P_+(\xi) \tag{64}$$

where $$P_+(\xi) = \int_0^\infty p(x)\exp(i\xi x)dx \quad , \quad (\xi=\sigma+i\tau)$$

and $$\bar{\sigma}_{23}(\xi,0,p) = \mu W_o T_-(\xi) \tag{65}$$

where $$T_-(\xi) = \int_{-\infty}^0 t(x)\exp(i\xi x)dx \,.$$

The integral of $\overline{W}(\xi,0,p)$ over $(-\infty,0)$ converges if and only if $\mathrm{Im}(\xi) = \tau < 0$ and integral over $(0,\infty)$ converges if $\tau>-k_1$. $\bar{\sigma}_{23}$ is analytic over $(-\infty,0)$ if $\tau<k_2$.

Now (57) becomes with the help of (61), (64) and (65)

$$-\frac{(vi\xi+p+\beta)\,T_-(\xi)}{(vi\xi+p+\alpha)\gamma}=\frac{1}{i\xi p}+P_+(\xi) \tag{66}$$

In this form of equation Wiener-Hopf technique can easily be applied.

5. NON STEADY STATE SOLUTION FOR MAXWELL SOLID

For general α and β, γ does not readily factorize. Expressions for the roots of $\gamma = 0$ can be obtained but these are difficult to handle. We discuss here the case of the Maxwell Solid, where $\alpha=0$. In this case γ^2 reduces to

$$\gamma^2=\left(1-\frac{v^2}{c^2}\right)\left[\xi^2+\frac{vi(2p+\beta)}{c^2\left(1-\frac{v^2}{c^2}\right)}\xi+\frac{p(p+\beta)}{c^2\left(1-\frac{v^2}{c^2}\right)}\right],\quad c^2=\frac{\mu}{\rho} \tag{67}$$

The quadratic within the second bracket equated to zero has the complex roots

$$\frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[-\frac{vi(2p+\beta)}{c^2}\pm\frac{2i}{c}\sqrt{p(p+\beta)+\frac{v^2\beta^2}{4c^2}}\right] \tag{68}$$

one positive and one negative for v<c.

Hence

$$\gamma=\left(1-\frac{v^2}{c^2}\right)^{1/2}(\xi+iX_1)^{1/2}(\xi-iX_2)^{1/2},\quad \mathrm{Re}\ X_1,\ X_2>0 \tag{69}$$

where

$$X_1 = \frac{1}{2\left(1 - \frac{v^2}{c^2}\right)}\left[\frac{v(2p+\beta)}{c^2} + \frac{2}{c}\sqrt{p(p+\beta) + \frac{v^2\beta^2}{4c^2}}\right]$$

$$X_2 = \frac{1}{2\left(1 - \frac{v^2}{c^2}\right)}\left[-\frac{v(2p+\beta)}{c^2} + \frac{2}{c}\sqrt{p(p+\beta) + \frac{v^2\beta^2}{4c^2}}\right] \tag{70}$$

Branches are chosen so that $\gamma \to +\infty$ as $\xi \to \pm\infty$.

Thus for a Maxwell Solid, (66) can be written after simplication as

$$-\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \frac{\left(\xi - \frac{i(p+\beta)}{v}\right)}{\left(\xi - \frac{ip}{v}\right)} \frac{T_-(\xi)}{(\xi - iX_2)^{1/2}} - \frac{(iX_1)^{1/2}}{i\xi p}$$

$$= \frac{(\xi+iX_1)^{1/2} - (iX_1)^{1/2}}{i\xi p} + (\xi+iX_1)^{1/2} P_+(\xi) \tag{71}$$

The function on the left hand side of (71) is analytic for $\tau<0$ and the function on the right hand side of (71) is analytic for $\tau>-k_1$. As they are both equal for $-k_1<\tau<0$ the principle of analytic continuation gives that either side represents the same entire function $M(\xi)$, say. Further we had previously noted that

$$T_-(\xi) \to 0 \quad , \quad |\xi| \to \infty \quad , \quad \tau<0$$

$$P_+(\xi) \to 0 \quad , \quad |\xi| \to \infty \quad , \quad \tau>-k_1.$$

Liouville's theorem gives that $M(\xi) = 0$ and thus

$$P_+(\xi) = \frac{(iX_1)^{1/2}}{i\xi p(\xi + iX_1)^{1/2}} - \frac{1}{i\xi p} \tag{72}$$

and

$$T_-(\xi) = -\left(1 - \frac{v^2}{c^2}\right)^{1/2} \frac{(iX_1)^{1/2}}{i\xi p} \frac{\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{1/2}}{\left(\xi - \frac{i(p+\beta)}{v}\right)} \tag{73}$$

Therefore, $\overline{W}(\xi,0,p)$ given in (64), with the help of (72), takes the form

$$\overline{W}(\xi,0,p) = \frac{W_o(iX_1)^{1/2}}{i\xi p(\xi + iX_1)^{1/2}} \tag{74}$$

Taking inverse transforms one get,

$$W(x,0,t) = \frac{W_o}{i} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(iX_1)^{1/2}}{p} e^{pt}\, dp \, \frac{1}{2\pi} \int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}}{\xi(\xi + iX_1)^{1/2}}\, d\xi$$

$$0<d<k_1, \quad x>0 \tag{75}$$

Taking the path of integration around the branch cut along negative imaginary axis from $-iX_1$ to $-i\infty$ the integral

$$I = \int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}}{\xi(\xi + iX_1)^{1/2}}\, d\xi$$

can be converted to the integral

$$I = 2e^{\pi i/4}\, e^{-X_1 x}\, \sqrt{x} \int_0^{\infty} \frac{e^{-U}\, U^{-1/2}}{U + xX_1}\, dU$$

which is finally evaluated as

$$I = 2\sqrt{\pi}\, e^{\pi i/4}\, e^{-X_1 x/2}\, \sqrt{x}\, (xX_1)^{-3/4}\, W_{-1/4,-1/4}(xX_1).$$

Putting this value of the integral in (75) we obtain

$$W(x,0,t) = \frac{W_0}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}\, e^{-xX_1/2}}{p} (xX_1)^{-1/4}\, W_{-1/4,-1/4}(xX_1)\, dp$$

$$, \quad x>0 \qquad (76)$$

Now for small p,

$$X_1 = \left[\frac{v\beta}{c^2} \right] / \left[1 - \frac{v^2}{c^2} \right] = k \text{ (say)} \qquad (77)$$

and for large p,

$$X_1 = \frac{p}{c - v},$$

therefore for large $\frac{px}{c - v}$,

$$W_{-1/4,-1/4}\left[\frac{px}{c - v}\right] \sim \exp\left[- \frac{1}{2} \frac{px}{c - v}\right] \left[\frac{px}{c - v}\right]^{-1/4} \qquad (78)$$

So in equation (76) putting the value of X_1 for small p given by (77), we obtain for large time t,

$$W(x,0,t) = \frac{W_0}{\sqrt{\pi}}\, e^{-kx/2}\, (kx)^{-1/4}\, W_{-1/4,-1/4}(kx) \qquad (79)$$

which is same as the result for the steady state case for all $x>0$ given by (50).

For large p, i.e. for small time t and for all finite x such that $px/(c-v)$ is large, using (78) we obtain from (76)

$$W(x,0,t) = \frac{2W_o}{\pi}\sqrt{\frac{c-v}{x}}\left[t - \frac{x}{c-v}\right]^{1/2} H\left[t - \frac{x}{c-v}\right] \tag{80}$$

Now, using (73), (65) becomes

$$\bar{\sigma}_{23}(\xi,0,p) = -\mu W_o\left[1 - \frac{v^2}{c^2}\right]^{1/2}\frac{(iX_1)^{1/2}}{i\xi p}\,\frac{\left[\xi - \frac{ip}{v}\right](\xi - iX_2)^{1/2}}{\left[\xi - \frac{i(p+\beta)}{v}\right]}$$

After taking inverse transforms it converts to

$$\sigma_{23}(x,0,t) = -\frac{\mu W_o}{i}\left[1 - \frac{v^2}{c^2}\right]^{1/2}\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{pt}}{p}(iX_1)^{1/2}dp \;\times$$

$$\times\;\frac{1}{2\pi}\int_{-\infty-id}^{\infty-id}\frac{\left[\xi - \frac{ip}{v}\right](\xi - iX_2)^{1/2}}{\xi\left[\xi - \frac{i(p+\beta)}{v}\right]}\exp(-i\xi x)d\xi \tag{81}$$

Reversing the order of integration

$$\sigma_{23}(x,0,t) = -\frac{\mu W_o}{i}\left[1 - \frac{v^2}{c^2}\right]^{1/2}\frac{1}{2\pi}\int_{-\infty-id}^{\infty-id}F(\xi)\exp(-i\xi x)d\xi \tag{82}$$

where

$$F(\xi) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{\left[\xi - \frac{ip}{v}\right](\xi - iX_2)^{1/2}}{\xi\left[\xi - \frac{i(p+\beta)}{v}\right]} \frac{e^{pt}}{p} (iX_1)^{1/2} dp \tag{83}$$

For large ξ, (83) becomes

$$F(\xi) = \sqrt{\frac{i}{\xi}}\, B \tag{84}$$

where

$$B = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p} (iX_1)^{1/2} dp \tag{85}$$

Putting the value of $F(\xi)$ from (84) in (82) we get

$$\sigma_{23}(x,0,t) = -\frac{\mu W_0}{\sqrt{\pi x_1}}\left[1 - \frac{v^2}{c^2}\right]^{1/2} B \;, \quad \text{as } -x = x_1 \to 0^+ \text{ (for all } t>0) \tag{86}$$

The evaluation of B for all t (t>0) has been done in the Appendix-2.

For small p i.e., for large t, using from (77) that $X_1 \sim k$, we have

$$B = \sqrt{k} = \sqrt{\left[\frac{v\beta}{c^2}\right] / \left[1 - \frac{v^2}{c^2}\right]} \tag{87}$$

Substituting the value of B given by (87) in (86) we obtain

$$\sigma_{23}(x,0,t) = -\frac{\mu W_0}{\sqrt{\pi x_1}} \left[\frac{v\beta}{c^2}\right]^{1/2} \;, \quad x_1 \to 0^+$$

which is same as the result for the steady state case given by (48).

The variation of the nondimensional values of B given by

$$B^* = \sqrt{\frac{c}{\beta}}\left(1 - \frac{v^2}{c^2}\right)^{1/2} B$$

has been plotted against nondimensional time $t_1=\beta t$ for various values of $v/c = 0.5$, 0.7 and 0.9 and has been shown by means of graphs in Fig.3.

Now for all values of x i.e., for general value of ξ the integral

$$\frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id} \frac{\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{1/2}}{\xi\left(\xi - \frac{i(p+\beta)}{v}\right)} \exp(-i\xi x)\,d\xi$$

appearing in equation (81) can be converted to the integral

$$I_1 = e^{3\pi i/4}\frac{p}{p+\beta}(X_1)^{1/2} + \frac{e^{\pi i/4}\, e^{X_2 x}}{\pi i}\int_0^{\infty}\frac{e^{ux}\sqrt{u}\left(u+X_2 - \frac{p}{v}\right)}{(u+X_2)\left(u+X_2 - \frac{p+\beta}{v}\right)}\,du\ ,$$

$$\text{since}\quad \frac{p+\beta}{v} > X_2 \tag{88}$$

considering the path of integration around the branch cut along positive imaginary axis from iX_2 to $i\infty$ as shown in Fig.4.

So using (88), (81) becomes

$$\sigma_{23}(x,0,t) = -\frac{\mu W_0}{\pi}\left(1 - \frac{v^2}{c^2}\right)^{1/2}\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{pt}}{p}(X_1)^{1/2}e^{X_2 x}\,dp\ \times$$

$$\times\int_0^{\infty}\frac{e^{ux}\sqrt{u}\left(u+X_2 - \frac{p}{v}\right)}{(u+X_2)\left(u+X_2 - \frac{p+\beta}{v}\right)}\,du + \mu W_0\left(1 - \frac{v^2}{c^2}\right)^{1/2}\ \times$$

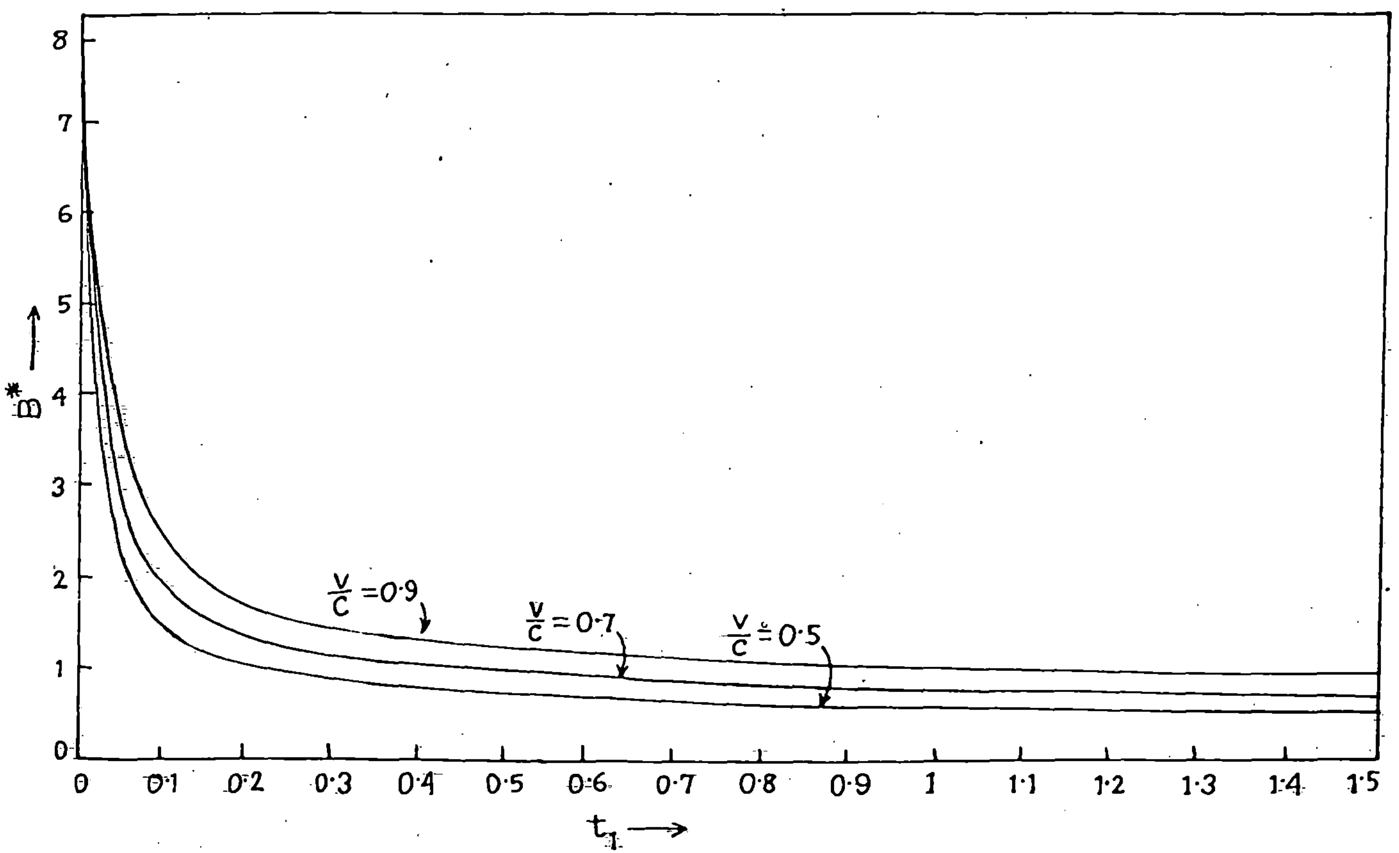

Fig. 3. Variation of B^* vs t_1 in the non steady state case of the Maxwell Solid

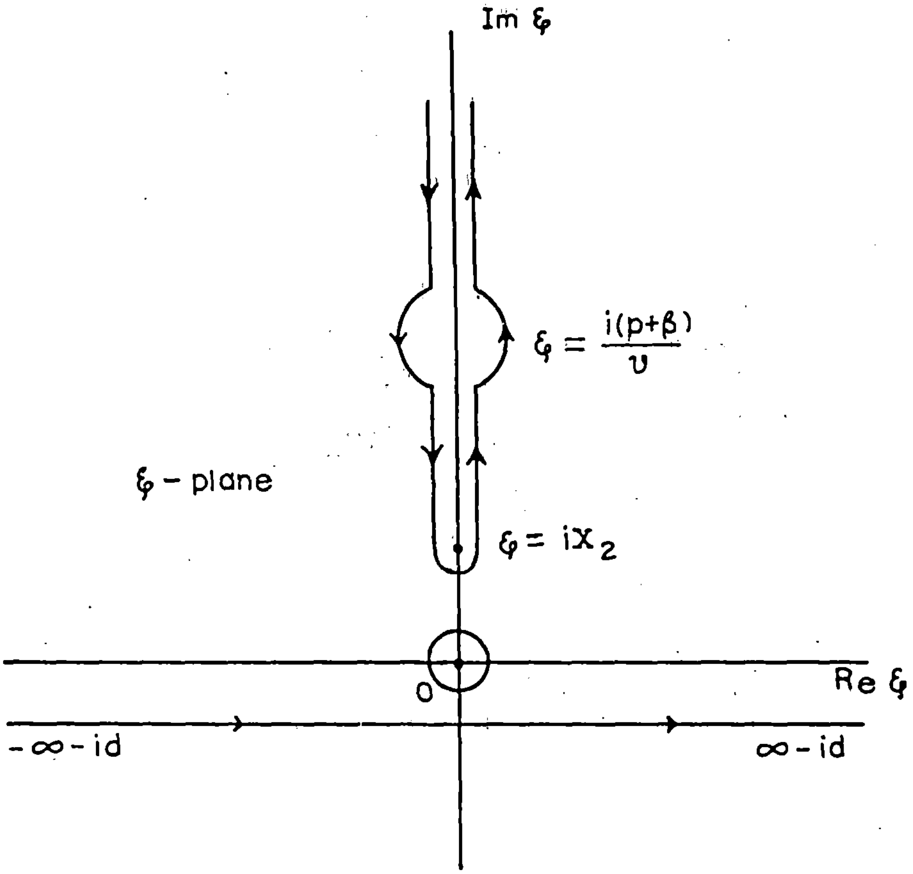

Fig 4. Path of Integration to evaluate I_1

$$\times \quad \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p+\beta} (X_1 X_2)^{1/2}\, dp \tag{89}$$

For large t (i.e.,for small p using $X_1=k$, $X_2=0$) and for all $x(x<0)$ we obtain from (89)

$$\sigma_{23}(x,0,t) = -\frac{\mu W_0}{\pi}\left(\frac{v\beta}{c^2}\right)^{1/2} \int_0^\infty \frac{e^{ux}\sqrt{u}}{u-\frac{\beta}{v}}\, du$$

where

$$k = \left(\frac{v\beta}{c^2}\right) \Big/ \left(1-\frac{v^2}{c^2}\right)$$

which is same as the solution for the steady state case for all values of $x>0$ given by (47).

6. RESULTS AND DISCUSSION

The stress $\sigma_{23}(x,0)$ just below the punch $(x<0)$ and the displacement $W(x,0)$ on the free surface $(y=0,\ x>0)$ have been computed numerically from equations (39) and (43) for different values of parameters v/c and α/β. The case $\alpha/\beta=0$ corresponds to Maxwell Solid. In Fig.5 non dimensional stress $\tau^* = \dfrac{\sigma_{23}(x,0)}{\mu W_0\beta/c}$ has been plotted against nondimensional distance $x_1^* = \beta x_1/c$ for values of the parameter $v/c = 0.5,\ 0.9$ and for values of the parameter $\alpha/\beta = 0$ and 0.2.

For the same sets of the parameter values nondimensional displacement $W^* = W/W_o$ has been plotted versus nondimensional distance $x^* = \beta x/c$ in Fig.6. W^* varies from 1 to zero as x^* changes gradually from zero to ∞.

It may be noted from the graphs that variation of the values of W^* with x^* is rapid with the increase in the values of the parameter v/c. Further it is found that the graphs become steeper with the decrease in the values of the parameter α/β. From Fig.5 it is found that nondimensional stress τ^* changes rapidly with the decrease in the values of v/c where as for a fixed value of v/c graphs become flat with the increase in the values of α/β.

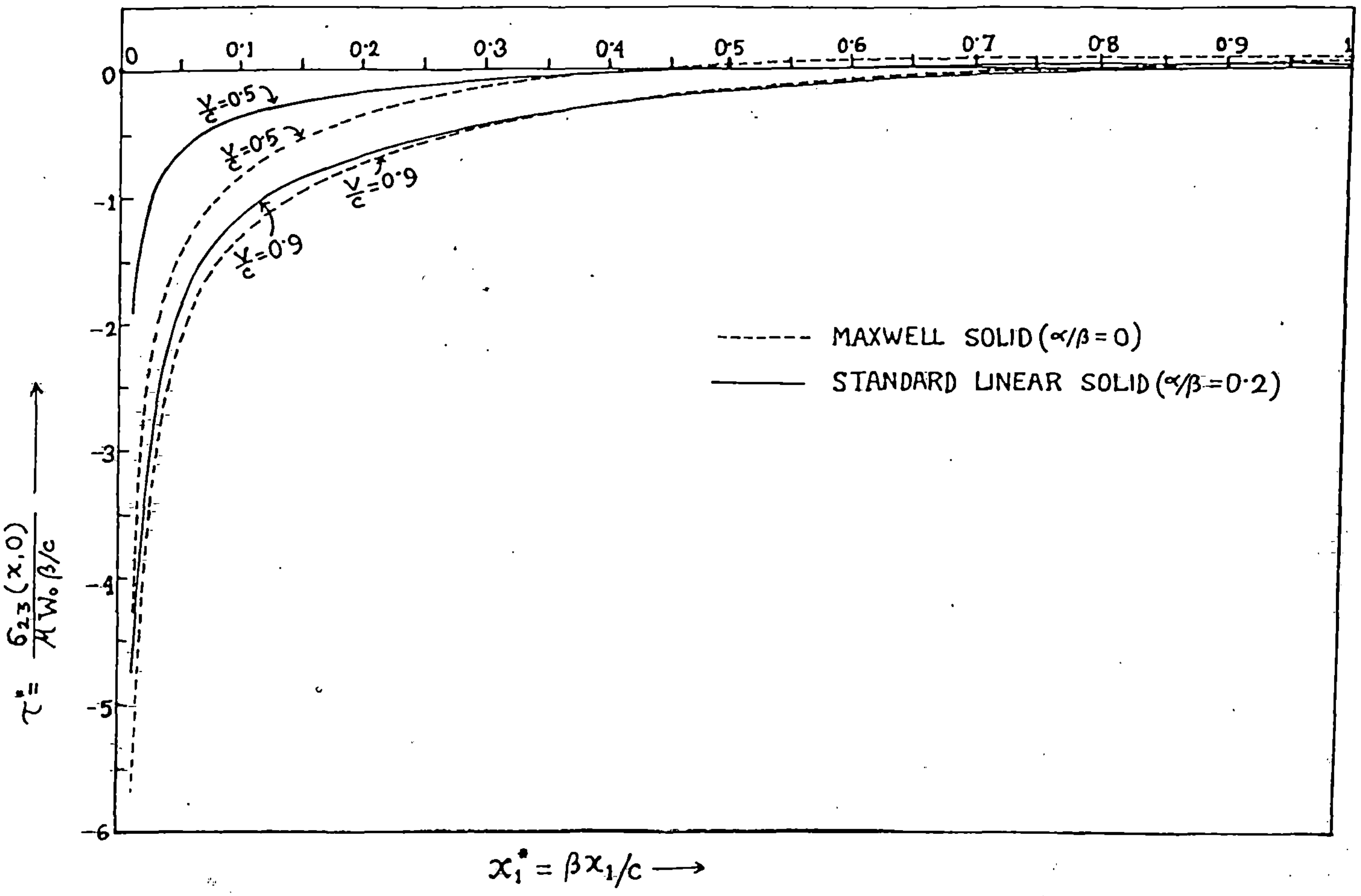

Fig. 5 Nondimentional stress τ^* vs. nondimentional distance x_1^* ($x<0$) just below the punch in the steady state case

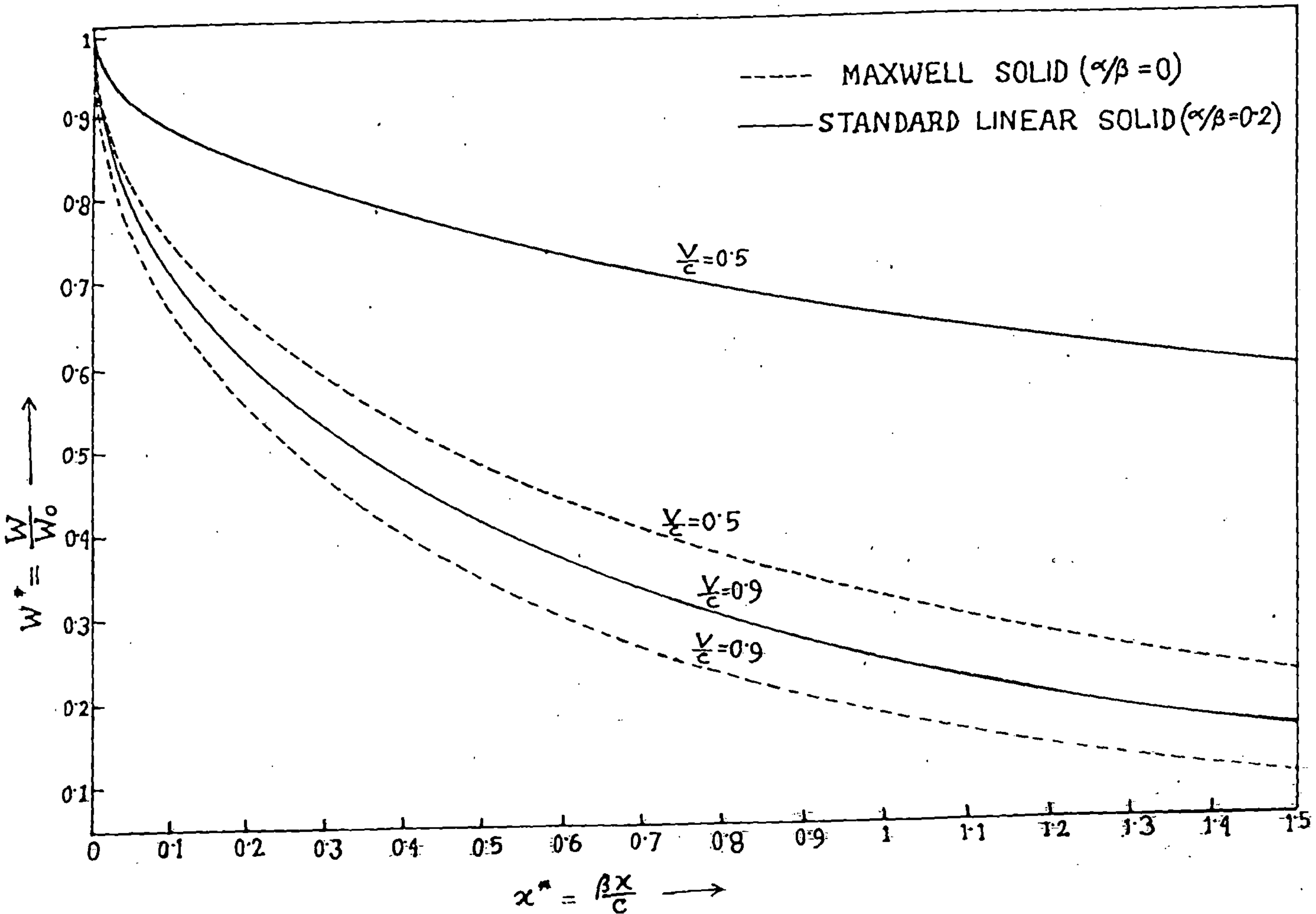

Fig. 6 Nondimensional displacement W^* vs nondimensional distance x^* ($x > 0, y = 0$) in the steady state case

APPENDIX - 1

Evaluation of the integral

$$I = \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \exp(-i\xi x)\, d\xi$$

The contour of integration is shown in Fig.2.

Putting $\xi = \frac{i\alpha}{v} + iu$, $0<u<\infty$, the integral can be written as

$$I = 2\, e^{\frac{\pi i}{4}}\, e^{\frac{\alpha}{v}x} \fint_0^{\infty} \frac{\sqrt{u}\; e^{-ux_1}}{u - \left(\frac{\beta}{v} - \frac{\alpha}{v}\right)}\, du \quad , \quad x=-x_1 \;,\; x<0$$

where $\fint$ means the principal value of the integral.

Now, for large $\left(\frac{\beta}{v} - \frac{\alpha}{v}\right)x_1$ $(x<0)$, putting $ux_1 = z$ the integral I becomes

$$I = \frac{2e^{\pi i/4}\; e^{\alpha x/v}}{\sqrt{x_1}} \int_0^{\infty} \frac{\sqrt{z}\; e^{-z}}{z - \left(\frac{\beta}{v} - \frac{\alpha}{v}\right)x_1}\, dz$$

$$= \frac{2e^{\pi i/4}\; e^{\alpha x/v}}{\sqrt{x_1}} \left[\int_0^{\infty} \frac{e^{-z}}{\sqrt{z}}\, dz - \int_0^{\infty} \frac{e^{-z}}{\sqrt{z}}\, dz \left\{ 1 + \frac{z}{\left(\frac{\beta-\alpha}{v}x_1\right)} + \frac{z^2}{\left(\frac{\beta-\alpha}{v}x_1\right)^2} + \ldots\ldots \right\} \right]$$

$$= - \frac{2e^{\pi i/4} \, e^{\alpha x/v}}{\sqrt{x_1}} \left[\frac{v}{(\beta-\alpha)\, x_1} \Gamma(\tfrac{3}{2}) + \frac{v^2}{(\beta-\alpha)^2 x_1^2} \Gamma(\tfrac{5}{2}) + \right.$$

$$\left. + \frac{v^3}{(\beta-\alpha)^3 x_1^3} \Gamma(\tfrac{7}{2}) + \ldots\ldots\ldots \right]$$

Again for small $\left(\frac{\beta}{v} - \frac{\alpha}{v}\right) x_1$, putting $u = \left(\frac{\beta}{v} - \frac{\alpha}{v}\right) z$, I can be converted to the integral

$$I = 2\, e^{\frac{\pi i}{4}} \, e^{-\frac{\alpha}{v} x_1} \left(\frac{\beta-\alpha}{v}\right)^{1/2} \int_0^\infty \frac{\sqrt{z}}{z - 1} \exp\left[-z\left(\frac{\beta-\alpha}{v}\right) x_1\right] dz$$

$$= 2\, e^{\frac{\pi i}{4}} \, e^{-\frac{\alpha}{v} x_1} \sqrt{\frac{\pi}{x_1}} + 2\, e^{\frac{\pi i}{4}} \, e^{-\frac{\alpha}{v} x_1} \left(\frac{\beta-\alpha}{v}\right)^{1/2} \times$$

$$\times \int_0^\infty \frac{1}{t^2 - 1} \exp\left[-t^2\left(\frac{\beta-\alpha}{v}\right) x_1\right] dt$$

$$= 2\, e^{\frac{\pi i}{4}} \, e^{-\frac{\alpha}{v} x_1} \sqrt{\frac{\pi}{x_1}} + O\left[\left(\frac{\beta-\alpha}{v}\right) x_1\right] \quad \text{where} \quad z = t^2$$

So,

$$I = 2\, e^{\frac{\pi i}{4}} \, e^{-\frac{\alpha}{v} x_1} \sqrt{\frac{\pi}{x_1}} \qquad \text{as} \qquad \left[\left(\frac{\beta-\alpha}{v}\right) x_1\right] \to 0.$$

APPENDIX - 2

EVALUATION OF THE INTEGRAL B.

The integral in (85)

$$B = \frac{1}{\sqrt{c}}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p} \sqrt{\frac{v}{c}\left(p + \frac{\beta}{2}\right) + \sqrt{p(p+\beta) + \frac{v^2\beta^2}{4c^2}}}\, dp.$$

has a simple pole at $p = 0$ and branch points at $p = -\beta$,

$$p = \alpha_1 = \frac{\beta}{2}\left(-1 + \sqrt{1 - \frac{v^2}{c^2}}\right)$$

and

$$p = \alpha_2 = \frac{\beta}{2}\left(-1 - \sqrt{1 - \frac{v^2}{c^2}}\right).$$

Taking the branch cut along the negative real axis from α_1 to $-\infty$ the integral can be considered as a contour integral around the path as shown in Fig.7.

Let,

$$I = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p} \sqrt{\frac{v}{c}\left(p + \frac{\beta}{2}\right) + \sqrt{(p-\alpha_1)(p-\alpha_2)}}\, dp$$

which can be written as

$I = \sqrt{\frac{v\beta}{c}} + I_1$ where $\sqrt{\frac{v\beta}{c}}$ is the contribution to the integral from pole at $p = 0$.

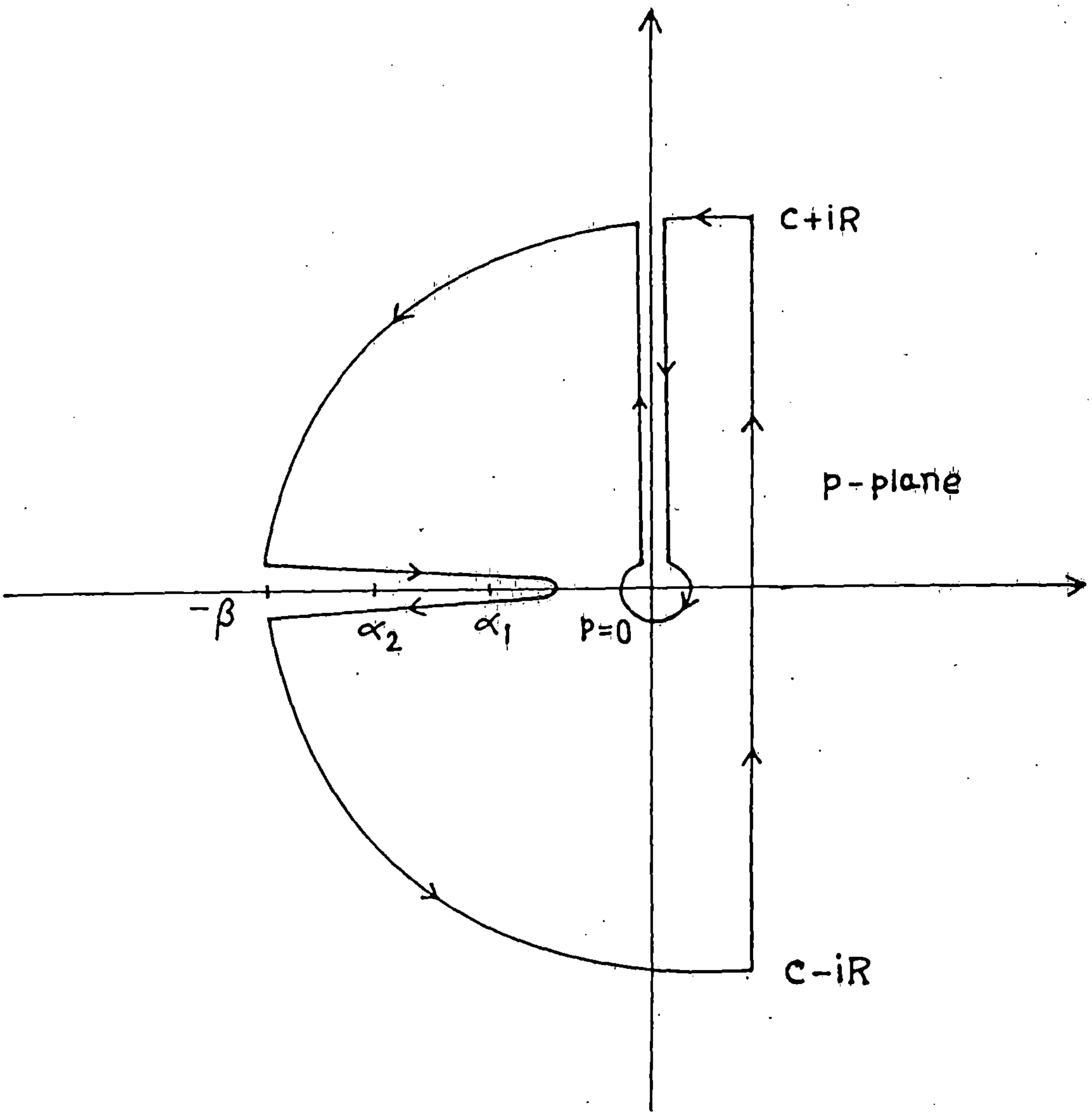

Fig. 7. The integration contour to evaluate B for the Maxwell Solid.

ANTIPLANE DYNAMIC CRACK PROPAGATION IN AN INHOMOGENEOUS VISCOELASTIC SOLID

1. INTRODUCTION

Until now many authors, Baker (1962), Cherepanov and Afanasev (1974) and others have investigated the dynamic crack propagation in a homogeneous elastic medium. This problem presents an interest for a better understanding of the brittle behaviour of the material. However, natural or artificial materials are usually inhomogeneous. There exist very few solutions to the problem of dynamic crack propagation in inhomogeneous elastic media. Atkinson and List (1978) and Atkinson (1977) considered steady-state crack propagation in different types of inhomogeneous elastic media. In addition, if the materials are dissipative, that effect can be taken into account by considering the material to be viscoelastic. Crack propagation in viscoelastic medium has been studied by Willis (1972), Atkinson and List (1972), Coussy (1987) and others. Willis (1972) considered steady-state Mode III crack propagation for a standard linear solid under general type of loading on the crack surfaces. Atkinson and List (1972) studied nonsteady SH-wave type crack propagation starting at t=0 and moving with a constant velocity in the "Maxwell Solid" or using the viscoelastic model suggested by Achenbach and Chao. Finally, Sills and Benveniste

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PP. 373 - 392 , 1990.

Let $I_1 = I_2 + I_3$ where I_2 is the value of the integral I_1 around the branch cut from α_1 to α_2 and I_3 is its value round the branch cut from α_2 to $-\infty$.

Now it can be shown that

$$I_2 = \frac{1}{\pi}\sqrt{\frac{\beta}{2}}\int_0^b \frac{\sqrt{(R^* - x^*)}\; e^{t_1(\alpha_1^* - r)}}{(\alpha_1^* - r)}\, dr$$

In the interval $(\alpha_2, -\infty)$

$$I_3 = \frac{\sqrt{\beta}}{\pi}\int_b^\infty \frac{\sqrt{(-x^{**})}\; e^{t_1(\alpha_1^* - r)}}{(\alpha_1^* - r)}\, dr$$

where $\alpha_1 = \beta\alpha_1^*$, $\alpha_2 = \beta\alpha_2^*$, $t_1 = \beta t$, $b = \sqrt{1 - \frac{v^2}{c^2}}$,

$$x^* = \frac{v}{c}\left[\frac{b}{2} - r\right] \quad , \quad y^* = \sqrt{r(b-r)}$$

$$R^* = \sqrt{(x^*)^2 + (y^*)^2}$$

$$x^{**} = -\frac{v}{c}\left[r - \frac{b}{2}\right] - \sqrt{r(r-b)}$$

Finally, we obtain

$$B = \frac{1}{\sqrt{c}}\left[1 - \frac{v^2}{c^2}\right]^{-1/2}\left[\sqrt{\frac{v\beta}{c}} + \frac{1}{\pi}\sqrt{\frac{\beta}{2}}\int_0^b \frac{\sqrt{(R^* - x^*)}\; e^{t_1(\alpha_1^* - r)}}{(\alpha_1^* - r)}\, dr + \right.$$

$$\left. + \frac{\sqrt{\beta}}{\pi}\int_b^\infty \frac{\sqrt{(-x^{**})}\; e^{t_1(\alpha_1^* - r)}}{(\alpha_1^* - r)}\, dr\right].$$

(1969) and Coussy (1987) studied steady state crack propagation of SH-type at the interface between two viscoelastic media.

In our case we have considered steady and nonsteady cases of Mode III crack propagation in an inhomogeneous viscoelastic medium. Two types of viscoelastic models, namely Maxwell Solid and Standard Linear Solid have been considered. Material properties have been assumed to vary exponentially in the direction perpendicular to the direction of crack propagation. We have studied how the material inhomogeneity affects the stress intensity factor and also the crack opening displacement when a Mode III type crack propagates through the inhomogeneous viscoelastic medium.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION FOR NONSTEADY CASE IN MAXWELL SOLID

Let us consider an inhomogeneous viscoelastic medium which was set in motion by a semi-infinite crack suddenly appearing at t=0 and moving with a constant velocity V in the direction of the X-axis. The Y-axis is taken perpendicular to the X-axis(fig.1). For SH-waves, the displacements along X and Y directions are zero and only the displacement $W = W(X,Y,t)$ along the Z-direction exists. The shear modulus is

$$\mu(Y) = \mu_0 \exp(2\beta Y) \quad \text{and density} \quad \rho(Y) = \rho_0 \exp(2\beta Y),$$

where β, μ_0 and ρ_0 are constants.

The non-zero stresses are

$$\sigma_{YZ} = \sigma_{YZ}(X,Y,t) \quad \text{and} \quad \sigma_{XZ} = \sigma_{XZ}(X,Y,t) \tag{1}$$

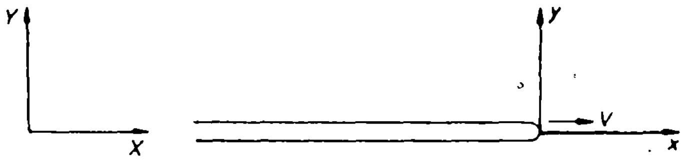

FIG. 1. The crack geometry.

and nonvanishing strains are

$$e_{XZ} = \frac{1}{2}\frac{\partial W}{\partial X} , \quad e_{YZ} = \frac{1}{2}\frac{\partial W}{\partial Y} \tag{2}$$

Considering a Maxwell Solid as the viscoelastic model, the stress-strain relations are

$$\begin{aligned} \frac{\partial\sigma_{YZ}}{\partial t} + \beta_1\sigma_{YZ} &= 2\mu(Y)\frac{\partial e_{YZ}}{\partial t} \\ \frac{\partial\sigma_{XZ}}{\partial t} + \beta_1\sigma_{XZ} &= 2\mu(Y)\frac{\partial e_{XZ}}{\partial t} \end{aligned} \tag{3}$$

where β_1 is a positive constant.

The equation of motion has the form

$$\frac{\partial\sigma_{XZ}}{\partial X} + \frac{\partial\sigma_{YZ}}{\partial Y} = \rho(Y)\frac{\partial^2 W}{\partial t^2} \tag{4}$$

and the boundary conditions of the problem are

$$\begin{aligned} &W(X,0,t) = 0 , \quad X-Vt>0, \quad t>0 \\ &\sigma_{YZ}(X,0,t) = -\sigma H(t) , \quad X-Vt<0 , \quad t>0 \\ &\sigma_{YZ}(X,Y,t) \to 0 \quad \text{as} \quad X^2+Y^2 \to \infty \end{aligned} \tag{5}$$

It is convenient to shift the origin of co-ordinates to the tip of the crack at X=Vt. New co-ordinate axes (x,y) are parallel to the respective fixed ones (X,Y).

Hence, putting x=X-Vt, y=Y, we obtain $\frac{\partial}{\partial X} = \frac{\partial}{\partial x}$, $\frac{\partial}{\partial Y} = \frac{\partial}{\partial y}$ and the time derivative transforms to $-V\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$. Equations (1), (2), (3) and (4) become

$$\sigma_{yz} = \sigma_{yz}(x,y,t) \quad \text{and} \quad \sigma_{xz} = \sigma_{xz}(x,y,t) \tag{6}$$

$$e_{xz} = \frac{1}{2}\frac{\partial}{\partial x} W(x,y,t) \quad , \quad e_{yz} = \frac{1}{2}\frac{\partial}{\partial y} W(x,y,t) \tag{7}$$

$$-V\frac{\partial\sigma_{yz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial t} + \beta_1\sigma_{yz} = \mu(y)\left[-V\frac{\partial^2 W}{\partial x\partial y} + \frac{\partial^2 W}{\partial t\partial y}\right]$$

(8)

$$-V\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{xz}}{\partial t} + \beta_1\sigma_{xz} = \mu(y)\left[-V\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial t\partial x}\right]$$

and

$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} = \rho(y)\left[V^2\frac{\partial^2 W}{\partial x^2} - 2V\frac{\partial^2 W}{\partial x\partial t} + \frac{\partial^2 W}{\partial t^2}\right] \tag{9}$$

The boundary conditions (5) now assume the form

$$\begin{aligned} &W(x,0,t) = 0 \ , \quad x>0 \\ &\sigma_{yz}(x,0,t) = -\sigma H(t) \ , \quad x<0 \\ &\sigma_{yz}(x,y,t) \to 0 \quad \text{as} \quad x^2+y^2 \to \infty \end{aligned} \tag{10}$$

Let us denote the Laplace transform by a single bar

$$\bar{f} \equiv \bar{f}(x,y,p) = \int_0^\infty f(x,y,t)\exp(-pt)dt$$

and the Fourier transform by two bars

$$\bar{\bar{f}} \equiv \bar{\bar{f}}(\xi,y,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(x,y,p)\exp(i\xi x)dx$$

Applying these transforms to equations (8) and (9), we get

$$(i\xi V+p+\beta_1)\bar{\bar{\sigma}}_{yz} = \mu(y)(Vi\xi+p)\frac{d\bar{\bar{W}}}{dy} \tag{11}$$

$$(i\xi V+p+\beta_1)\bar{\bar{\sigma}}_{xz} = \mu(y)(V\xi^2-i\xi p)\bar{\bar{W}} \tag{12}$$

and $$-i\xi\bar{\bar{\sigma}}_{xz} + \frac{d}{dy}\bar{\bar{\sigma}}_{yz} = \rho(y)(-V^2\xi^2+2Vi\xi p+p^2)\bar{\bar{W}} \tag{13}$$

Eliminating $\bar{\sigma}_{xz}$, $\bar{\sigma}_{yz}$ from equations (11), (12) and (13) we obtain

$$\frac{d^2\overline{W}}{dy^2} + 2\beta \frac{d\overline{W}}{dy} - \gamma^2\overline{W} = 0 \tag{14}$$

where

$$\gamma^2 = \xi^2 + \frac{(Vi\xi+p)(Vi\xi+p+\beta_1)}{c^2}$$
$$c^2 = \frac{\mu_0}{\rho_0} \tag{15}$$

The branches of γ are chosen so that $\mathrm{Re}(\gamma)>0$.

Since $\overline{W}$ must remain bounded as $y \to \pm\infty$, so solutions of (14) are

$$\overline{W}^{(1)} = A_1 \exp\left[-\left(\beta+\sqrt{\beta^2+\gamma^2}\right)y\right], \quad y>0 \tag{16}$$

and

$$\overline{W}^{(2)} = A_2 \exp\left[\left(-\beta+\sqrt{\beta^2+\gamma^2}\right)y\right], \quad y<0 \tag{17}$$

where $W^{(1)}$ and $W^{(2)}$ denote the displacement in the upper and lower half-plane respectively.

Let us consider on $y=0$

$$\begin{aligned}\overline{W}^{(1)} - \overline{W}^{(2)} &= h(x,p), \quad x<0\\ &= 0, \quad x>0\end{aligned} \tag{18}$$

where $h(x,p)$ is an unknown function such that

$$h(x,p) \sim o\left[\exp(k_1x)\right] \quad \text{as} \quad x \to -\infty, \quad k_1>0.$$

Applying the Fourier transform to equation (18), we get

$$\overline{W}^{(1)} - \overline{W}^{(2)} = A_1-A_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x,p)\exp(i\xi x)dx$$

$$= H_-(\xi,p) \tag{19}$$

where $H_-(\xi,p)$ is an analytic function in the lower half-plane $\tau<k_1$ and $\xi=\sigma+i\tau$.

Now from equations (11), (16) and (17) we obtain

$$\bar{\bar{\sigma}}^{(1)}_{yz} = \mu(y)\,\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\,\frac{\partial}{\partial y}\,\bar{\bar{W}}^{(1)}$$

$$= -\mu(y)A_1\left[\beta+\sqrt{\beta^2+\gamma^2}\,\right]\exp\left[-\left(\beta+\sqrt{\beta^2+\gamma^2}\,\right)y\right]\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\ ,\quad y>0$$

$$\bar{\bar{\sigma}}^{(2)}_{yz} = \mu(y)A_2\left[-\beta+\sqrt{\beta^2+\gamma^2}\,\right]\exp\left[\left(-\beta+\sqrt{\beta^2+\gamma^2}\,\right)y\right]\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\ ,\quad y<0 \qquad (20)$$

where $\sigma^{(1)}_{yz}$ and $\sigma^{(2)}_{yz}$ are the stresses on the upper and lower surfaces of the crack.

Since the stresses are continuous on y=0,

$$\bar{\bar{\sigma}}^{(1)}_{yz} = \bar{\bar{\sigma}}^{(2)}_{yz}$$

Using equations (20) we obtain

$$A_1 = -\,\frac{-\beta+\sqrt{(\beta^2+\gamma^2)}}{\beta+\sqrt{(\beta^2+\gamma^2)}}\,A_2 \qquad (21)$$

Using equation (21), (19) becomes

$$H_-(\xi,p) = -\,\frac{2\sqrt{(\beta^2+\gamma^2)}}{\beta+\sqrt{(\beta^2+\gamma^2)}}\,A_2 \qquad (22)$$

Again let us assume that on y=0

$$\bar{\sigma}_{yz} = \bar{\sigma}^{(1)}_{yz} = \bar{\sigma}^{(2)}_{yz} = -\,\frac{\sigma_0\exp(\lambda x)}{p}\ ,\qquad x<0$$

$$= e(x)\ ,\qquad x>0 \qquad (23)$$

Here $e(x)$ is an unknown function such that

$$e(x) \sim o\left[\exp(-k_2 x)\right] \quad \text{as} \quad x \to \infty, \quad k_2>0.$$

Taking Fourier transforms of equation (23) we get

$$\bar{\sigma}^{(2)}_{yz} = \mu_0 A_2\left(-\beta+\sqrt{\beta^2+\gamma^2}\right) \frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \bar{\sigma}^{(2)}_{yz} \exp(i\xi x)dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \bar{\sigma}^{(2)}_{yz} \exp(i\xi x)dx$$

$$= E_+(\xi,p) - \frac{\sigma_0}{\sqrt{2\pi}\,(\lambda+i\xi)p} \tag{24}$$

where

$$E_+(\xi,p) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \bar{\sigma}^{(2)}_{yz} \exp(i\xi x)dx \tag{25}$$

and is an analytic function in the upper half-plane $\tau>-k_2$ and $\dfrac{\sigma_0}{i\sqrt{2\pi}\,(\xi-i\lambda)p}$ is analytic in the lower half-plane $\tau<\lambda$.

From equations (22) and (24) we get

$$-\frac{\mu_0(Vi\xi+p)\gamma^2 H_-(\xi,p)}{2(Vi\xi+p+\beta_1)\sqrt{(\beta^2+\gamma^2)}} = E_+(\xi,p) - \frac{\sigma_0}{\sqrt{2\pi}\,(\lambda+i\xi)p} \tag{26}$$

It may be noted that the problem has been reduced to a form suitable for application of the Wiener-Hopf technique.

Now

$$\gamma^2 = \left(1-\frac{V^2}{c^2}\right)(\xi+iX_1)(\xi-iX_2) \tag{27}$$

where

$$X_1 = \frac{1}{2(1-V^2/c^2)}\left[\frac{(2p+\beta_1)V}{c^2} + \sqrt{\frac{(2p+\beta_1)^2V^2}{c^4} + \frac{4p(p+\beta_1)(1-V^2/c^2)}{c^2}}\right] \tag{28}$$

$$X_2 = \frac{1}{2(1-V^2/c^2)}\left[-\frac{(2p+\beta_1)V}{c^2} + \sqrt{\frac{(2p+\beta_1)^2V^2}{c^4} + \frac{4p(p+\beta_1)(1-V^2/c^2)}{c^2}}\right] \tag{29}$$

and $$\sqrt{(\beta^2+\gamma^2)} = (\xi+iY_1)^{1/2}(\xi-iY_2)^{1/2}(1-V^2/c^2)^{1/2} \tag{30}$$

where

$$Y_1 = \frac{1}{2(1-V^2/c^2)}\left[\frac{(2p+\beta_1)V}{c^2} + \sqrt{\frac{(2p+\beta_1)^2V^2}{c^4} + 4\left\{\frac{p(p+\beta_1)}{c^2} + \beta^2\right\}(1-V^2/c^2)}\right] \tag{31}$$

$$Y_2 = \frac{1}{2(1-V^2/c^2)}\left[-\frac{(2p+\beta_1)V}{c^2} + \sqrt{\frac{(2p+\beta_1)^2V^2}{c^4} + 4\left\{\frac{p(p+\beta_1)}{c^2} + \beta^2\right\}(1-V^2/c^2)}\right] \tag{32}$$

Using equations (27) and (30), (26) becomes

$$-\frac{\mu_0(1-V^2/c^2)^{1/2}(\xi-ip/V)(\xi-iX_2)H_-(\xi,p)}{2\left[\xi-i(p+\beta_1)/V\right](\xi-iY_2)^{1/2}} + \frac{\sigma_0(i\lambda+iY_1)^{1/2}}{\sqrt{2\pi}\; i(\xi-i\lambda)(i\lambda+iX_1)p}$$

$$= \frac{(\xi+iY_1)^{1/2}\,E_+(\xi;p)}{(\xi+iX_1)} - \frac{\sigma_0}{i\sqrt{2\pi}\;(\xi-i\lambda)p}\left[\frac{(\xi+iY_1)^{1/2}}{(\xi+iX_1)} - \frac{(i\lambda+iY_1)^{1/2}}{(i\lambda+iX_1)}\right] \tag{33}$$

The functions on the R.H.S. of (33) are analytic and non-zero in the upper half-plane $\tau>-k_2$, and functions on the L.H.S. are analytic and non-zero in the lower half-plane $\tau<\lambda$ ($\lambda<k_1$). Since both the functions are analytic in the strip $-k_2<\tau<\lambda$, the principle of an analytic continuation states that each of them represents an entire function $M(\xi)$ in the whole ξ-plane.

Now, the L.H.S. of (33) approaches zero as $|\xi| \to \infty$. It may then be concluded by Liouville's theorem that $M(\xi) = 0$ and therefore

$$H_-(\xi,p) = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(\xi-iY_2)^{1/2}[\xi-i(p+\beta_1)/V]}{\mu_0\sqrt{2\pi}\ ip(\xi-i\lambda)(i\lambda+iX_1)(\xi-iX_2)(\xi-ip/V)(1-V^2/c^2)^{1/2}} \qquad (34)$$

and

$$E_+(\xi,p) = \frac{\sigma_0}{i\sqrt{2\pi}\ (\xi-i\lambda)p} - \frac{\sigma_0(i\lambda+iY_1)^{1/2}(\xi+iX_1)}{i\sqrt{2\pi}\ (\xi-i\lambda)p(i\lambda+iX_1)(\xi+iY_1)^{1/2}} \qquad (35)$$

From equation (34) it follows that

$$H_-(\xi,p) = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}}{\mu_0\sqrt{2\pi}\ ip(i\lambda+iX_1)(1-V^2/c^2)^{1/2}}\,\xi^{-3/2}, \quad \xi \to \infty.$$

Application of the Inverse Fourier transform yields

$$h(x,p) = \frac{4\sigma_0}{\mu_0}\sqrt{-\frac{x}{\pi}}\ \frac{1}{(1-V^2/c^2)^{1/2}}\ \frac{(\lambda+Y_1)^{1/2}}{(\lambda+X_1)p}, \qquad x \to 0^- .$$

Again, taking the Inverse Laplace transform, the displacement jump across the surface of the crack near the crack tip is

$$W^{(1)} - W^{(2)} = \frac{4\sigma_0}{\mu_0}\sqrt{-\frac{x}{\pi}}\ \frac{1}{(1-V^2/c^2)^{1/2}}\ \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{(\lambda+Y_1)^{1/2}}{(\lambda+X_1)p}\,e^{pt}\,dp$$

(36)

From (35)

$$E_+(\xi,p) = -\frac{\sigma_0(i\lambda+iY_1)^{1/2}}{i\sqrt{2\pi}\ p(i\lambda+iX_1)}\,\xi^{-1/2}, \quad \xi \to \infty .$$

Taking the Inverse Fourier transform we obtain

$$e(x,p) = \frac{\sigma_o}{\sqrt{\pi x}} \frac{(\lambda+Y_1)^{1/2}}{p(\lambda+X_1)} , \quad x \to 0^+ .$$

Again, taking the inverse Laplace transform

$$\sigma_{yz} = \frac{\sigma_o}{\sqrt{\pi x}} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda+Y_1)^{1/2}}{p(\lambda+X_1)} e^{pt}\, dp \tag{37}$$

If ΔW is the displacement jump, then the crack opening displacement near the crack tip ia given by

$$\mu_o \Delta W = 4\sigma_o \sqrt{-\frac{x}{\pi}} \frac{1}{(1-V^2/c^2)^{1/2}} A , \quad (1 \ll x < 0) \tag{38}$$

and the stress near the crack tip is

$$\sigma_{yz} = \frac{\sigma_o}{\sqrt{\pi x}} A , \quad (0 < x \ll 1) \tag{39}$$

where

$$A = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda+Y_1)^{1/2}}{p(\lambda+X_1)} e^{pt}\, dp = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(Y_1)^{1/2}}{pX_1} e^{pt}\, dp , \quad \lambda \to 0 \tag{40}$$

Evaluation of the integral A given by (40) corresponding to constant stress $-\sigma_o$ on the crack surfaces is presented in the appendix.

In the fracture mechanics, it is customary to write $\sigma_{yz} = (0^+,0,t)$ in the form $K/\sqrt{(2\pi x)}$, where K is stress intensity factor.

In our case

$$K = \sqrt{2}\, \sigma_o A \tag{41}$$

Putting $\beta=0$ in the expression for A, we obtain the stress intensity factor in a homogeneous viscoelastic medium as

$$K = \sqrt{2}\, \sigma_o A_1$$

where

$$A_1 = \frac{1}{2\pi i}\sqrt{2(1-V^2/c^2)}\ \times$$

$$\times \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}\,dp}{p\left[(2p+\beta_1)V/c^2+\sqrt{(2p+\beta_1)^2V^2/c^4+4p(p+\beta_1)(1-V^2/c^2)/c^2}\,\right]}$$

which agrees with the results of Atkinson and List (1972).

3. STEADY STATE CASE FOR MAXWELL SOLID

Steady state solutions are the results of Sec.2 corresponding to the case of t approaching infinity. So for the steady state case, passing to the limit $p \to 0$ and using the Tauberian theorem we obtain from equation (34)

$$H_-(\xi,p) = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(\xi-iY_2)^{1/2}\ (\xi-i\beta_1/V)}{\mu_0\sqrt{2\pi}\ i(\xi-i\lambda)(i\lambda+iX_1)\xi^2(1-V^2/c^2)^{1/2}}$$

Applying the Inverse Fourier transform we obtain

$$W^{(1)} - W^{(2)} = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}}{\mu_0(i\lambda+iX_1)(1-V^2/c^2)^{1/2}}\ \times$$

$$\times\ \frac{1}{2\pi i}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{(\xi-iY_2)^{1/2}(\xi-i\beta_1/V)}{(\xi-i\lambda)\xi^2}\ \exp(-i\xi x)\ d\xi$$

$$= \frac{2\sigma_0(i\lambda+iY_1)^{1/2}}{2\pi i\mu_0(i\lambda+iX_1)(1-V^2/c^2)^{1/2}}\ I \qquad (42)$$

where

$$I = \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{(\xi - iY_2)^{1/2}(\xi - i\beta_1/V)}{(\xi - i\lambda)\xi^2} \exp(-i\xi x)\, d\xi$$

For $x<0$, the above integral can be replaced with the integral taken along the positive imaginary ξ-axis round the branch point at $\xi = iY_2$, together with the contribution from the poles at $\xi = 0$ and $\xi = i\lambda$ as shown in Fig.2.

Thus it can be shown that

$$I = \exp\Big[-\Big(\frac{\pi i}{4} + x_1 Y_2\Big)\Big]\Bigg[\frac{2}{\lambda}\Big(\frac{\beta_1}{\lambda V} - 1\Big)\int_0^{\infty} \frac{u^{1/2}\exp(-ux_1)}{u+Y_2}\, du +$$

$$+ \frac{2\beta_1}{\lambda V}\int_0^{\infty} \frac{u^{1/2}\exp(-ux_1)}{(u+Y_2)^2}\, du - \frac{2}{\lambda}\Big(\frac{\beta_1}{\lambda V} - 1\Big)\int_0^{\infty} \frac{u^{1/2}\exp(-ux_1)}{u+(Y_2-\lambda)}\, du\Bigg] +$$

$$+ \frac{2\pi}{\lambda}\Bigg[\Bigg[\Big(\frac{\beta_1}{\lambda V} - 1\Big)\sqrt{-iY_2} + \frac{\beta_1}{V}\Big(x_1\sqrt{-iY_2} + \frac{1}{2\sqrt{-iY_2}}\Big) -$$

$$- \Big(\frac{\beta_1}{\lambda V} - 1\Big)\sqrt{(i\lambda - iY_2)}\, \exp(-\lambda x_1)\Bigg]$$

$$= \frac{1}{\lambda}\exp(-\pi i/4)\Bigg[\Bigg[\Big(\frac{\beta_1}{\lambda V} - 1\Big)\sqrt{\frac{\pi}{x_1}}\,(x_1Y_2)^{-1/4}\exp(-x_1Y_2/2)\, \times$$

$$\times W_{-3/4,1/4}(x_1Y_2) + \frac{\beta_1}{V}\sqrt{(\pi x_1)}\,(x_1Y_2)^{-3/4}\exp(-x_1Y_2/2)\, \times$$

$$\times W_{-5/4,-1/4}(x_1Y_2) + \Big(1 - \frac{\beta_1}{\lambda V}\Big)\sqrt{\frac{\pi}{x_1}}\Big[x_1(Y_2-\lambda)\Big]^{-1/4}\exp\Big[-\frac{x_1(Y_2+\lambda)}{2}\Big]\times$$

$$\times W_{-3/4,1/4}\Big[x_1(Y_2-\lambda)\Big]\Bigg] +$$

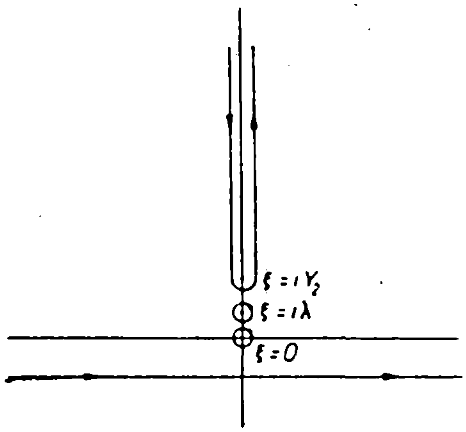

FIG. 2. The path of integration of the integral I.

$$+ \frac{2\pi}{\lambda} \Big[\Big[\Big(\frac{\beta_1}{\lambda V} - 1 \Big] \sqrt{-iY_2} + \frac{i\beta_1}{V} \Big[i x_1 \sqrt{-iY_2} + \frac{1}{2\sqrt{-iY_2}} \Big] -$$

$$- \Big(\frac{\beta_1}{\lambda V} - 1 \Big] \sqrt{(i\lambda - iY_2)} \exp(-\lambda x_1) \Big] \qquad (43)$$

where $W_{k,m}$ is the Whittaker function (1969).

Therefore the displacement jump ΔW across the surface of the crack $(x<0)$ is given by

$$\mu_0 \Delta W = - \frac{\sigma_0 (\lambda + Y_1)^{1/2}}{\pi(\lambda + X_1)(1 - V^2/c^2)^{1/2}} \exp(\pi i/4) \; I \qquad (44)$$

where I is given by equation (43).

Using the result that

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m - k)} (z)^{1/2+m} \exp(-z/2) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2} + m + k)} (z)^{1/2-m} \exp(-z/2)$$

for small z,

we find that for small $(x_1 Y_2)$, equation (43) yields

$$I = -4\sqrt{(\pi x_1)} \exp(-\pi i/4) \qquad (45)$$

Substituting the value of I from equation (45) in to equation (44) we get

$$\mu_0 \Delta W = - \frac{4\sigma_0}{(1 - V^2/c^2)^{1/2}} \sqrt{- \frac{x}{\pi}} \; \frac{(\lambda + \alpha_1)^{1/2}}{(\lambda + \alpha_2)} , \quad -1 \ll x < 0 \qquad (46)$$

where

$$\alpha_1 = \frac{1}{2(1 - V^2/c^2)} \left[\frac{\beta_1 V}{c^2} + \sqrt{\frac{\beta_1^2 V^2}{c^4} + 4\beta^2 (1 - V^2/c^2)} \right] \qquad (47)$$

and $$\alpha_2 = \frac{\beta_1 V}{(1 - V^2/c^2) c^2} \qquad (48)$$

Again, letting $p \to 0$ and using the Tauberian theorem we find from equations (35) and (24) that in the steady state case

$$\bar{\sigma}_{yz} = - \frac{\sigma_0 (i\lambda + iY_1)^{1/2} (\xi + iX_1)}{i\sqrt{2\pi}\ (\xi - i\lambda)(i\lambda + iX_1)(\xi + iY_1)^{1/2}} \tag{49}$$

Taking the inverse Fourier transform we obtain

$$\sigma_{yz} = - \frac{\sigma_0 (i\lambda + iY_1)^{1/2}}{2\pi i (i\lambda + iX_1)} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{(\xi + iX_1)\exp(-i\xi x)}{(\xi - i\lambda)(\xi + iY_1)^{1/2}} d\xi \cdot$$

$$= - \frac{\sigma_0 (i\lambda + iY_1)^{1/2}}{2\pi i (i\lambda + iX_1)} I_1 \tag{50}$$

where
$$I_1 = \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{(\xi + iX_1)\exp(-i\xi x)}{(\xi - i\lambda)(\xi + iY_1)^{1/2}} d\xi$$

$$= 2\sqrt{\pi}\ \exp(-\pi i/4) \left[\frac{\exp(-Y_1 x)}{\sqrt{x}} - \frac{(\lambda + X_1)\sqrt{x}}{\left[x(\lambda + Y_1)\right]^{3/4}} \times \right.$$

$$\left. \times \exp\left(\frac{x(\lambda - Y_1)}{2} \right) W_{-1/4,-1/4}\left[x(\lambda + Y_1)\right] \right] \tag{51}$$

Thus the stress at $y=0$ for all x ($x>0$) is given by (50).

Now for small $(\lambda + Y_1)x$

$$I_1 = 2\sqrt{\frac{\pi}{x}}\ \exp(-\pi i/4) \tag{52}$$

so from (50) it follows that

$$\sigma_{yz} = \frac{\sigma_0}{\sqrt{\pi x}} \frac{(\lambda + \alpha_1)^{1/2}}{(\lambda + \alpha_2)}, \quad 0 < x \ll 1\ ,\ y=0 \tag{53}$$

Stress intensity factor K is given by

$$K = \sqrt{2}\,\sigma_0 B \tag{54}$$

where $$B = \frac{(\lambda+\alpha_1)^{1/2}}{(\lambda+\alpha_2)}$$

Now putting $\beta_1=0$ in the expression for α_1 and α_2 we get from (44) and (53) the displacement jump and stress intensity factor in an inhomogeneous elastic medium as

$$\mu_0 \Delta W = -\frac{4\sigma_0}{\lambda(1-V^2/c^2)^{1/2}} \sqrt{-\frac{x}{\pi}} \left[\lambda + \frac{\beta}{(1-V^2/c^2)^{1/2}} \right]^{1/2} \tag{55}$$

and $$\sigma_{yz} = \frac{\sigma_0}{\lambda\sqrt{\pi x}} \left[\lambda + \frac{\beta}{(1-V^2/c^2)^{1/2}} \right]^{1/2} \tag{56}$$

which agree with the results derived by Atkinson (1977).

4. STEADY STATE SOLUTION FOR STANDARD LINEAR SOLID

In this case the stress strain relations are

$$\begin{aligned} \frac{\partial\sigma_{YZ}}{\partial t} + \beta_1\sigma_{YZ} &= 2\mu(Y)\left[\frac{\partial e_{YZ}}{\partial t} + \alpha e_{YZ}\right] \\ \frac{\partial\sigma_{XZ}}{\partial t} + \beta_1\sigma_{XZ} &= 2\mu(Y)\left[\frac{\partial e_{XZ}}{\partial t} + \alpha e_{XZ}\right] \end{aligned} \tag{57}$$

where β_1 and α are constants.

Equation of motion has the form

$$\frac{\partial\sigma_{XZ}}{\partial X} + \frac{\partial\sigma_{YZ}}{\partial Y} = \rho(Y)\frac{\partial^2 W}{\partial t^2} \tag{58}$$

Now, putting x=X-Vt and y=Y so that

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial t} = -V\frac{\partial}{\partial x},$$

equations (57) and (58) become

$$-V\frac{\partial\sigma_{yz}}{\partial x}+\beta_1\sigma_{yz}=\mu(y)\left[-V\frac{\partial^2 W}{\partial x\partial y}+\alpha\frac{\partial W}{\partial y}\right]$$

$$-V\frac{\partial\sigma_{xz}}{\partial x}+\beta_1\sigma_{xz}=\mu(y)\left[-V\frac{\partial^2 W}{\partial x^2}+\alpha\frac{\partial W}{\partial x}\right] \tag{59}$$

and

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}=\rho(y)V^2\frac{\partial^2 W}{\partial x^2} \tag{60}$$

Introducing the Fourier transform denoted by

$$\overline{f}\equiv\overline{f}(\xi,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x,y)\exp(i\xi x)dx \tag{61}$$

equations (59) and (60) can be transformed to

$$(i\xi V+\beta_1)\overline{\sigma}_{yz}=\mu(y)(Vi\xi+\alpha)\frac{d\overline{W}}{dy} \tag{62}$$

$$(i\xi V+\beta_1)\overline{\sigma}_{xz}=\mu(y)(V\xi^2-i\xi\alpha)\overline{W} \tag{63}$$

and $$i\xi\overline{\sigma}_{xz}+\frac{d}{dy}\overline{\sigma}_{yz}=-\rho(y)V^2\xi^2\overline{W} \tag{64}$$

Eliminating $\overline{\sigma}_{xz}$, $\overline{\sigma}_{yz}$ from equations (11), (12) and (13) we obtain

$$\frac{d^2\overline{W}}{dy^2}+2\beta\frac{d\overline{W}}{dy}-\gamma^2\overline{W}=0 \tag{65}$$

where

$$\gamma^2=\frac{\xi^2\left[(1-V^2/c^2)\xi+i(V\beta_1/c^2-\alpha/V)\right]}{(\xi-i\alpha/V)} \tag{66}$$

Branches of γ are chosen so that $\mathrm{Re}(\gamma)>0$.

Since $\overline{W}$ must remain bounded as $y\to\pm\infty$, so solutions of (65) are

$$\overline{W}^{(1)} = A_1 \exp\left[-\left(\beta+\sqrt{\beta^2+\gamma^2}\right)y\right], \quad y>0$$

and (67)

$$\overline{W}^{(2)} = A_2 \exp\left[\left(-\beta+\sqrt{\beta^2+\gamma^2}\right)y\right], \quad y<0$$

where $W^{(1)}$ and $W^{(2)}$ denote the displacement in the upper and lower half-plane respectively.

Let us consider on $y=0$

$$\begin{aligned} W^{(1)} - W^{(2)} &= h(x), \quad x<0 \\ &= 0, \quad x>0 \end{aligned} \tag{68}$$

where $h(x)$ is an unknown function such that

$$h(x) \sim o\left[\exp(k_1 x)\right] \quad \text{as} \quad x \to -\infty, \quad k_1>0$$

and

$$\begin{aligned} \sigma_{yz} &= -\sigma_0 \exp(\lambda x), \quad x<0, \\ &= e(x), \quad x>0 \end{aligned} \tag{69}$$

where $e(x)$ is an unknown function satisfying the condition

$$e(x) \sim o\left[\exp(-k_2 x)\right] \quad \text{as} \quad x \to \infty, \quad k_2>0.$$

In this case equation (26) becomes

$$-\frac{\mu_0 (Vi\xi+\alpha)\gamma^2 H_-(\xi)}{2(Vi\xi+\beta_1)\sqrt{(\beta^2+\gamma^2)}} = E_+(\xi) - \frac{\sigma_0}{\sqrt{2\pi}\; i(\xi-i\lambda)} \tag{70}$$

This equation holds in the region of regularity of the functions appearing in equation (71).

Owing to our former assumptions regarding the behaviour of $e(x)$ and $h(x)$ at infinity, this region is represented by the inequality $-k_2<\tau<\lambda<k_1$ where $\xi=\sigma+i\tau$.

Now equation (71) is suitable for the application of the Wiener-Hopf technique. Again,

$$\gamma^2 = \frac{\xi^2(1-V^2/c^2)(\xi+ia)}{(\xi - i\alpha/V)} \tag{71}$$

where

$$a = \frac{(V\beta_1/c^2 - \alpha/V)}{(1-V^2/c^2)}$$

and

$$\gamma^2 + \beta^2 = \frac{[\xi^3(1-V^2/c^2) + i(V\beta_1/c^2 - \alpha/V)\xi^2 + \beta^2(\xi - i\alpha/V)]}{(\xi - i\alpha/V)} \tag{72}$$

Since it is difficult to factorise $\sqrt{(\gamma^2+\beta^2)}$, i.e. to represent it as a product of two functions, one analytic in the upper half plane and other analytic in the lower half plane, we follow the approximate method of Koiter (1954) of solving Wiener-Hopf type equations. Accordingly, we write

$$P(\xi) = \sqrt{(\gamma^2+\beta^2)} \quad \text{in the form} \quad P(\xi) = \overline{P}(\xi)P_1(\xi) ,$$

where the function $\overline{P}(\xi)$ is required to behave at $|\xi| \to \infty$ and at $|\xi| \to 0$ in the same manner as $P(\xi)$. The auxiliary function $P_1(\xi)$ should be non-zero and should have no singularity within the strip $-k_2<-\tau_1<\tau<\tau_2<\lambda$; it has to be suitably chosen such that $P(\xi)$ is non-zero and possesses no singularity within the strip $-\tau_1<\tau<\tau_2$. Now we note that

$$P(\xi) = \sqrt{(\gamma^2+\beta^2)} \approx [(1-V^2/c^2)\xi^2 + i(V\beta_1/c^2 - \alpha/V)\xi]^{1/2} \quad \text{as} \quad |\xi| \to \infty$$

and $$\sqrt{(\tau^2+\beta^2)} \approx \beta \quad \text{as} \quad |\xi| \to 0.$$

Therfore we choose $\overline{P}(\xi)$ in the form

$$\overline{P}(\xi) = \left[(1-V^2/c^2)\xi^2 + i(V\beta_1/c^2 - \alpha/V)\xi + \beta^2\right]^{1/2} \tag{73}$$

which behaves in the same manner as $P(\xi)$ for $|\xi| \to \infty$ and $|\xi| \to 0$. Now $\overline{P}(\xi)$ can be written as

$$\overline{P}(\xi) = (1-V^2/c^2)^{1/2}(\xi-ia_2)^{1/2}(\xi+ia_1)^{1/2} \tag{74}$$

where

$$a_1 = \frac{1}{2(1-V^2/c^2)}\left[\left(\frac{V\beta_1}{c^2} - \frac{\alpha}{V}\right) + \sqrt{\left(\frac{V\beta_1}{c^2} - \frac{\alpha}{V}\right)^2 + 4\beta^2(1-V^2/c^2)}\right] \tag{75}$$

and

$$a_2 = \frac{1}{2(1-V^2/c^2)}\left[-\left(\frac{V\beta_1}{c^2} - \frac{\alpha}{V}\right) + \sqrt{\left(\frac{V\beta_1}{c^2} - \frac{\alpha}{V}\right)^2 + 4\beta^2(1-V^2/c^2)}\right] \tag{76}$$

It consequently follows that the assumptions concerning $P_1(\xi)$ are satisfied and in view of the fact that $P_1(\xi) \to 1$ in the strip $-\tau_1<\tau<\tau_2$ for $\xi \to \infty$, the function may be represented in the form

$$P_1(\xi) = P_1^+(\xi)P_1^-(\xi) \tag{77}$$

where

$$P_1^+(\xi) = \exp\left[\frac{1}{2\pi i}\int_{-\infty+id_2}^{\infty+id_2} \frac{\ln P_1(\eta)}{\eta-\xi}\,d\eta\right]$$

$$\tag{78}$$

$$P_1^-(\xi) = \exp\left[-\frac{1}{2\pi i}\int_{-\infty+id_1}^{\infty+id_1} \frac{\ln P_1(\eta)}{\eta-\xi}\,d\eta\right]$$

where $-\tau_1<d_1<d_2<\tau_2$ and the functions $P_1^{\pm}(\xi)$ are regular in the respective half planes $\tau>-\tau_1$ and $\tau<\tau_2$.

It follows from (79) and from the fact $P_1(0) = P_1(\infty) = 1$ that these functions satisfy the additional condition $P_1^{\pm}(0) = P_1^{\pm}(\infty) = 1$ with the help of (71), (74), (77) and the relation

$$P(\xi) = \overline{P}(\xi)P_1(\xi),$$

equation (69) becomes

$$-\frac{\mu_o(1-V^2/c^2)^{1/2}H_-(\xi)\xi^2}{2(\xi-i\beta_1/V)(\xi-ia_2)^{1/2}P_1^-(\xi)} + \frac{\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\ i(\xi-i\lambda)(i\lambda+ia)} =$$

$$= \frac{P_1^+(\xi)(\xi+ia_1)^{1/2}E_+(\xi)}{(\xi+ia)} - \frac{\sigma_o}{\sqrt{(2\pi)}\ i(\xi-i\lambda)} \times$$

$$\times \left[\frac{P_1^+(\xi)(\xi+ia_1)^{1/2}}{(\xi+ia)} - \frac{P_1^+(i\lambda)(i\lambda+a_1)^{1/2}}{(i\lambda+ia)}\right] \qquad (79)$$

Using the same arguments as in equation (33) we get

$$H_-(\xi) = -\frac{2\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}(\xi-ia_2)^{1/2}P_1^-(\xi)(\xi-i\beta_1/V)}{\mu_o\sqrt{(2\pi)}\ (\xi-i\lambda)(\lambda+a)(1-V^2/c^2)^{1/2}\xi^2}$$

$$= -\frac{2\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\mu_o\sqrt{(2\pi)}\ (\lambda+a)(1-V^2/c^2)^{1/2}}\,\xi^{-3/2} \quad \text{as} \quad \xi \to \infty \qquad (80)$$

and

$$E_+(\xi) = \frac{\sigma_o}{\sqrt{(2\pi)}\ i(\xi-i\lambda)} - \frac{\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}(\xi+ia)}{\sqrt{(2\pi)}\ i(\xi-i\lambda)(i\lambda+ia)P_1^+(\xi)(\xi+ia_1)^{1/2}}$$

$$= -\frac{\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\ i(i\lambda+ia)}\,\xi^{-1/2} \quad \text{as} \quad \xi \to \infty \qquad (81)$$

Now taking the Inverse Fourier transform we get from (80) and (81)

$$h(x) = \frac{4\sigma_o P_1^+(i\lambda)(\lambda+a_1)^{1/2}}{\sqrt{\pi}\;\mu_o(\lambda+a)(1-V^2/c^2)^{1/2}}(-x)^{1/2}, \quad -1\ll x<0 \tag{82}$$

and

$$e(x) = \frac{\sigma_o P_1^+(i\lambda)(\lambda+a_1)^{1/2}}{\sqrt{\pi}\;(\lambda+a)}(x)^{-1/2}, \quad 0<x\ll 1 \tag{83}$$

The corresponding results for the case of constant loading $\sigma_{yz} = -\sigma_o$ $(x<0)$ on the crack surface are obtained by putting $\lambda=0$ in the above equation. If ΔW is the displacement jump then the crack opening displacement in this case is given by

$$\mu_o \Delta W = \frac{4\sigma_o\sqrt{(-xa_1)}}{\sqrt{\pi}\;a(1-V^2/c^2)^{1/2}}, \quad -1\ll x<0 \tag{84}$$

and also the stress near the crack tip is

$$\sigma_{yz} = \frac{\sigma_o\sqrt{a_1}}{a\sqrt{\pi x}}, \quad 0<x\ll 1 \quad (\text{since } P_1^+(0)=1) \tag{85}$$

Therefore the stress intensity factor is equal to

$$K = \sqrt{2}\,\sigma_o B_1 \quad \text{where} \quad B_1 = \frac{\sqrt{a_1}}{a} \tag{86}$$

Now, putting $\alpha=0$ in equations (84) and (86) we get the crack opening displacement and stress intensity factor for the Maxwell Solid

$$\mu_o \Delta W = \frac{4\sigma_o\sqrt{(-x\alpha_1)}}{\sqrt{\pi}\;\alpha_2(1-V^2/c^2)^{1/2}}, \quad -1\ll x<0 \tag{87}$$

and $$K = \sqrt{2}\,\sigma_o\,B \qquad \text{where} \quad B = \frac{\sqrt{\alpha_1}}{\alpha_2} \tag{88}$$

which agree with the results given by (53) and (54) in the Maxwell Solid corresponding to $\lambda=0$.

5. RESULTS AND DISCUSSION

5.1. The Maxwell Solid.

In this case time variation of the stress intensity factor is given by $K = \sqrt{2}\,\sigma_o A$ where A is given by equation (40) and has been evaluated in the Appendix.

The dimensionless stress intensity factor $K^* = (K/\sigma_o)(\beta_1/c)^{1/2}$ has been plotted against $t_1 = \beta_1 t$ for the range of values of V/c = 0.1, 0.3, 0.5, 0.7 and 0.9 for different values of the inhomogeneity factor $\beta^* = 4\beta^2 c^2/\beta_1^2$.

It is interesting to note by inspecting the graphs given in Fig.3, Fig.4 and Fig.5 that the effect of inhomogeneity of the medium introduced through the factor β^* in the stress intensity factor K^* becomes more significant for small values of V/c, whereas for values of V/c differing slightly from unity, the effect of inhomogeneity of the medium on the stress intensity factor is negligible.

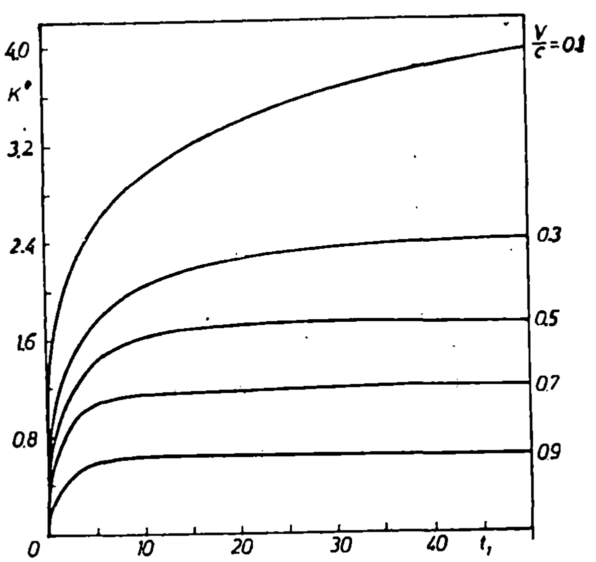

FIG. 3. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0$ (homogeneous medium).

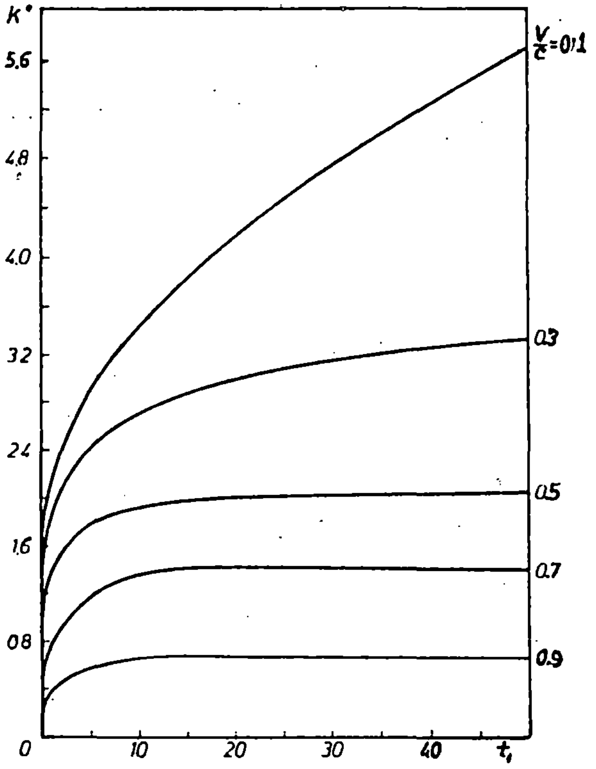

FIG. 4. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0.1$.

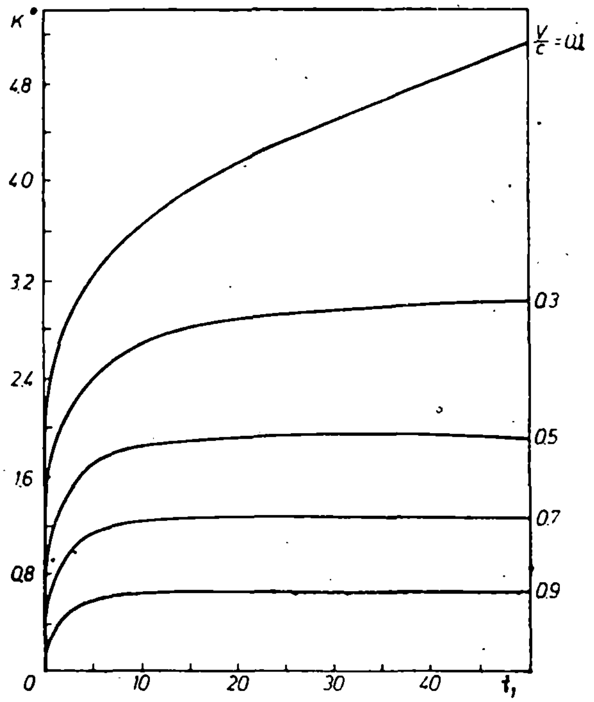

FIG. 5. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0.2$.

5.2. Standard Linear Solid.

In this case the stress intensity factor for the steadily propagating crack is given by $K = \sqrt{2}\,\sigma_0 B_1$, where B_1 is given by equation (86).

We have plotted also the stress intensity factor $K^* = (K/\sigma_0)(\beta_1/c)^{1/2}$ against β^* for various values of V/c, V/c=0.5, 0.6, 0.7, 0.8 and 0.9, and for different values of $\alpha/\beta_1 = 0, 0.1, 0.2$. The case $\alpha/\beta_1=0$ corresponds to the steady state values of K^* for the Maxwell solid. It is evident from the graphs given in Fig.6, Fig.7 and Fig.8 that at large values of α/β_1, values of K^* increase rapidly with the increase in values of β^* if V/c is very small. But for values of V/c close to unity the variation of K^* with the change in the value of β^* is small showing that the inhomogeneity effect is negligible in this case. This is also evident from the expressions (87) and (75).

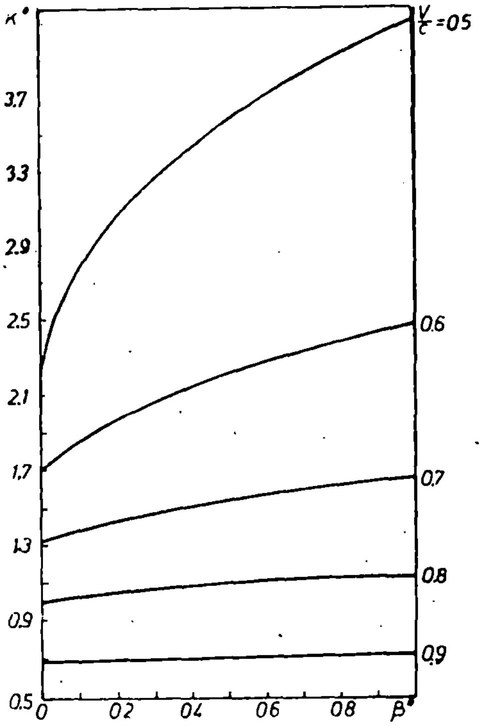

FIG. 6. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0$ (Maxwell solid).

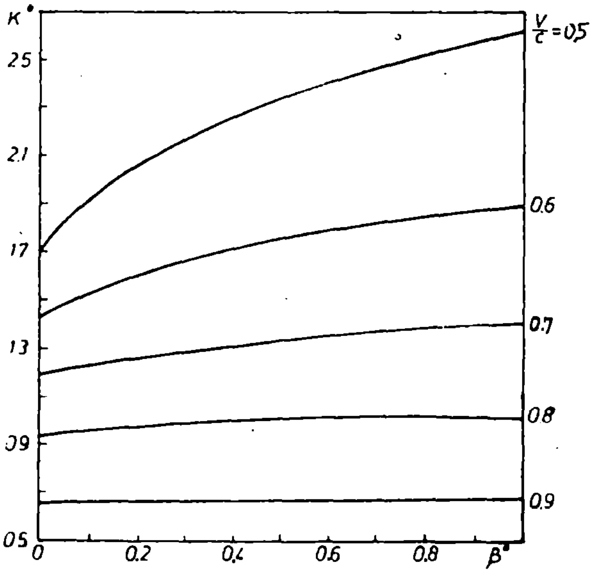

FIG. 7. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0.1$.

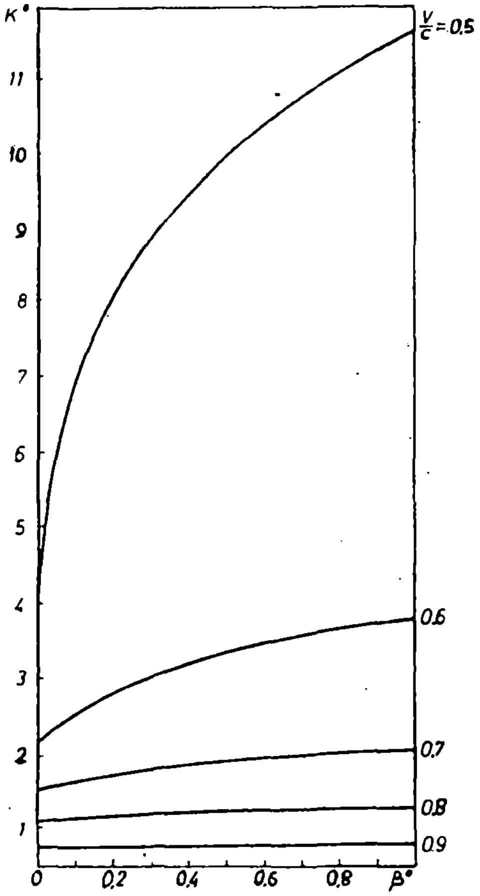

FIG. 8. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0.2$.

APPENDIX

EVALUATION OF THE INTEGRAL A IN EQUATION (40).

The integral

$$A = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(Y_1)^{1/2}}{pX_1} e^{pt}\, dp$$

The integrand has poles at p=0 and also at $p= -\beta_1$ which correspond to the zero of X_1.

Further the integrand has branch points at

$$\delta_1 = \frac{\beta_1}{2}\left[-1 + \sqrt{(1-V^2/c^2)(1-4z)} \right]$$

$$\delta_2 = \frac{\beta_1}{2}\left[-1 - \sqrt{(1-V^2/c^2)(1-4z)} \right]$$

$$\delta_3 = \frac{\beta_1}{2}\left[-1 - \sqrt{(1-4z)} \right]$$

$$\delta_4 = \frac{\beta_1}{2}\left[-1 + \sqrt{(1-V^2/c^2)} \right]$$

$$\delta_5 = \frac{\beta_1}{2}\left[-1 - \sqrt{(1-V^2/c^2)} \right]$$

where $z = \beta^2 c^2/\beta_1^2$ which is assumed to be less than 1/4.

Evidently, $\delta_4 > \delta_1 > \delta_2 > \delta_5 > \delta_3$.

Now taking the branch cut along the negative real axis from δ_4 to

$-\infty$, the integral can be considered as a contour integral around the path shown in Fig.9.

Now,

$$A = \sqrt{2(1-V^2/c^2)} \times$$

$$\times \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{\left[(2p+\beta_1)V/c^2 + (2/c)\sqrt{(p-\delta_1)(p-\delta_2)}\,\right]^{1/2}}{p\left[(2p+\beta_1)V/c^2 + (2/c)\sqrt{(p-\delta_4)(p-\delta_5)}\,\right]}\, e^{pt}\, dp$$

It can be shown that

$$A = \sqrt{2(1-V^2/c^2)}\left[\frac{1}{2}\,\frac{c}{V}\sqrt{\frac{c}{\beta_1}}\sqrt{\frac{V}{c}+\sqrt{\frac{V^2}{c^2}+4z(1-V^2/c^2)}}\,\right] +$$

$$+ \sqrt{2(1-V^2/c^2)}\,\frac{1}{\pi}\sqrt{\frac{c}{\beta_1}}\left(I_1+I_2-I_3-I_4\right)$$

where

$$I_1 = \int_0^{b_1} \frac{\sqrt{(x_1^{**})}\; y_1^*}{(\delta_4^*-r)R_1^*}\exp\left[(\delta_4^*-r)t_1\right]dr$$

$$I_2 = \int_0^{b_2} \frac{\sqrt{\left[\,R_2^{**}-(x_2^*)^3+(y_2^{**})^2x_2^* - 2x_2^*y_2^*y_2^{**}\,\right]/2}}{(\delta_1^*-r)R_2^*}\exp\left[(\delta_1^*-r)t_1\right]dr$$

$$I_3 = \int_{b_2}^{b_3} \frac{\sqrt{(x_3^{**})}\; x_3^*}{(\delta_1^*-r)R_3^*}\exp\left[(\delta_1^*-r)t_1\right]dr$$

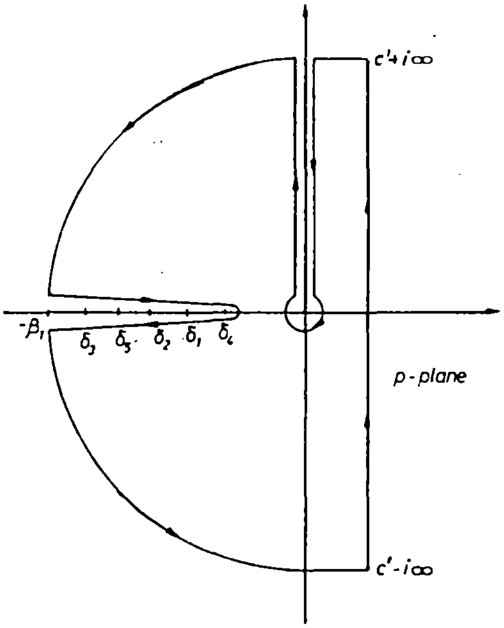

FIG. 9. The integration contour to evaluate A for the Maxwell solid.

$$I_4 = \int_{b_3}^{\infty} \frac{\sqrt{(x_3^{**})}}{(\delta_1^*-r)x_4^*} \exp\left[(\delta_1^*-r)t_1\right]dr$$

where $\delta_1=\beta_1\delta_1^*$, $\delta_2=\beta_1\delta_2^*$, $\delta_3=\beta_1\delta_3^*$, $\delta_4=\beta_1\delta_4^*$, $\delta_5=\beta_1\delta_5^*$, $t_1=\beta_1 t$,

$$b_1 = \frac{1}{2}\sqrt{(1-V^2/c^2)}\left[1 - \sqrt{(1-4z)}\right]$$

$$b_2 = \sqrt{(1-V^2/c^2)(1-4z)}$$

$$b_3 = \frac{1}{2}\sqrt{(1-V^2/c^2)}\left[1 + \sqrt{(1-4z)}\right]$$

$$x_1^{**} = \left[\sqrt{(1-V^2/c^2)} - 2r\right]\frac{V}{c} + 2\sqrt{\left[r^2 - r\sqrt{(1-V^2/c^2)} + (1-V^2/c^2)z\right]}$$

$$x_1^* = \left[\sqrt{(1-V^2/c^2)} - 2r\right]\frac{V}{c}$$

$$y_1^* = 2\sqrt{r\sqrt{(1-V^2/c^2)} - r^2}$$

$$R_1^* = (x_1^*)^2 + (y_1^*)^2$$

$$x_2^* = \left[\sqrt{(1-V^2/c^2)(1-4z)} - 2r\right]\frac{V}{c}$$

$$y_2^* = 2\sqrt{r\sqrt{(1-V^2/c^2)(1-4z)} - r^2}$$

$$y_2^{**} = 2\sqrt{-r^2 + r\sqrt{(1-V^2/c^2)(1-4z)} + z(1-V^2/c^2)}$$

$$R_2^{**} = \left[(x_2^*)^2 + (y_2^{**})^2 \right]\left[(x_2^*)^2 + (y_2^*)^2 \right]^{1/2}$$

$$R_2^* = (x_2^*)^2 + (y_2^{**})^2$$

$$x_3^{**} = -\left[\sqrt{(1-V^2/c^2)(1-4z)} - 2r \right] \frac{V}{c} + 2\sqrt{r^2 - r\sqrt{(1-V^2/c^2)(1-4z)}}$$

$$x_3^* = \left[\sqrt{(1-V^2/c^2)(1-4z)} - 2r \right] \frac{V}{c}$$

$$y_3^* = 2\sqrt{z(1-V^2/c^2) + r\sqrt{(1-V^2/c^2)(1-4z)} - r^2}$$

$$R_3^* = (x_3^*)^2 + (y_3^*)^2$$

$$x_4^* = \left[\sqrt{(1-V^2/c^2)(1-4z)} - 2r \right] \frac{V}{c} -$$

$$- 2\sqrt{r^2 - z(1-V^2/c^2) - r\sqrt{(1-V^2/c^2)(1-4z)}} \; .$$