CHAPTER - III

Large deflection of a square plate of variable thickness under uniform load.*

PAPER - I

Nomenclature :

The following nomenclature are used in this paper.

q = uniform load,

W = lateral displacement,

u, v = components in the middle plane of the plate,

h = thickness of the plate,

D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,

σ = Poisson's ratio,

E = Young's modulus,

Introduction :

For the large deflection of a plate we usually get non-linear equations which cannot be exactly solved. For a uniformly thin plate, Berger (1955) has shown that if in deriving the differential equation from strain energy, the strain energy due to the second strain invariant in the middle plane of the plate is neglected, a simple fourth order differential equation together with a non-linear second order

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*Accepted for publication in the Journal of Revue Roumanie Des Sciences Techniques Mecanique Appliquee, Romania.

equation is obtained. Following Berger, the corresponding equation for a square plate of variable thickness has been deduced. Basuli (1961) solved the problem of an annular plate of variable thickness with linearly varying thickness.

In this paper an attempt has been made to investigate the large deflection of a square plate of uni-directionally variable thickness and an infinite strip whose thickness varies along the breadth.

Analysis :

Let us consider a square plate of side $2a$. Let the centre of the plate be taken as origin.

Total strain energy of the plate with the present approximation is given by

$$V_2 = \frac{1}{2}\iint D\left\{\left[(\nabla^2 w)^2 + \frac{12}{h^2}e^2\right] - 2(1-\sigma)\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dx\,dy - \iint q w\,dx\,dy \qquad \ldots(1)$$

where

$$e = \epsilon_x + \epsilon_y\,,\quad \epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\quad \epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \ldots(2)$$

Using Euler's differential equation of variation we have (If $F(x,y)$ correspond to minimum value of V_2)

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\cdot\frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y}\cdot\frac{\partial F}{\partial u_y} = 0 \qquad \ldots(3)$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\cdot\frac{\partial F}{\partial v_x} - \frac{\partial}{\partial y}\cdot\frac{\partial F}{\partial v_y} = 0 \qquad \ldots(4)$$

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\cdot\frac{\partial F}{\partial w_x} - \frac{\partial}{\partial y}\cdot\frac{\partial F}{\partial w_y} + \frac{\partial^2}{\partial x^2}\cdot\frac{\partial F}{\partial w_{xx}} + \frac{\partial^2}{\partial x\partial y}\cdot\frac{\partial F}{\partial w_{xy}} + \frac{\partial^2}{\partial y^2}\cdot\frac{\partial F}{\partial w_{yy}} = 0 \qquad \ldots(5)$$

Let us assume that

$$h = h_0\cdot x^{1/3} \quad , \qquad \text{where} \quad D = D_0\cdot x \qquad \ldots(6)$$

Combining all the equations (1), (2), (3), (4), (5) and (6)

We have,

$$\frac{\partial}{\partial x}(he) = 0\ , \ \frac{\partial}{\partial y}(he) = 0 \qquad \text{Hence} \quad he = \text{constant} = \frac{\alpha^2 h_0^2}{12}$$

Thus we get the following differential equations.

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h_0^2}{12h} \qquad \ldots(7)$$

$$\nabla^2\left(x\nabla^2 - \alpha^2/h_0\right)w = \frac{q}{D_0} \qquad \ldots(8)$$

For the complementary function of equation (8) let us put $w = w_1 + w_2$

So that for the equation

$$\nabla^2\left(x\nabla^2 - \alpha^2/h_0\right)w = 0$$

We have,

$$\nabla^2 W_1 = 0 \qquad \ldots(9) \quad \text{and} \quad \chi \nabla^2 W_2 - \frac{\alpha^2}{h_0} W_2 = 0 \qquad \ldots(10)$$

To solve equation (9)

let us put

$$W_1 = \sum_{n=1,3,\ldots}^{\infty} W_n(x) \sin\frac{n\pi y}{a} \qquad \ldots(11)$$

boundary conditions being

$$\left.\begin{aligned} u = W = \frac{\partial^2 W}{\partial x^2} = 0 \quad & \text{at } x = b,\ x = c,\ b > c > 0. \\ v = W = \frac{\partial^2 W}{\partial y^2} = 0 \quad & \text{at } y = \pm a,\ b - c = 2a \end{aligned}\right\} \qquad \ldots(12)$$

Putting (11) in equation (9) and solving we get

$$W_n = A_1 e^{\frac{n\pi x}{a}} + A_2 e^{-\frac{n\pi x}{a}}$$

Again to solve equation (10)

let us put

$$W_2 = \sum_{n=1,3,\ldots}^{\infty} W_n'(x) \sin\frac{n\pi y}{a} \qquad \ldots(13)$$

On substitution we get,

$$x\frac{d^2 W_n'}{dx^2} - W_n'\left(x\frac{n^2\pi^2}{a^2} + \frac{\alpha^2}{h_0}\right) = 0 \qquad \ldots(14)$$

To solve equation (14) let us put $W_n' = V'\cdot x$

The transformed equation is

$$x\frac{d^2 V'}{dx^2} + 2\frac{dV'}{dx} - \left(\frac{n^2\pi^2}{a^2}x + \frac{\alpha^2}{h_0}\right)V' = 0 \qquad \ldots(15)$$

The solution of the above equation can be put in the form

$$V' = e^{-\frac{n\pi x}{a}}\left[A_3\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi x}{a}\right)^m + A_4\left\{\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi x}{a}\right)^m \cdot \operatorname{Log}\left(\frac{2n\pi x}{a}\right)\right.\right.$$

$$\left.\left. + \sum_{m=1}^{\infty}\mu_m\left(\frac{2n\pi x}{a}\right)^m + \frac{\Gamma(\lambda-1)}{\Gamma(\lambda)}\cdot\frac{a}{2n\pi x}\right\}\right] \qquad \ldots(16)$$

[Murphy, George M.: Ordinary differential equations and their solutions PP. 331.]

Where,

$$\lambda_m = \frac{\lambda(\lambda+1)\cdots(\lambda+m-1)}{2.3.\cdots(2+m-1)\,m!}, \quad \lambda = 1 + \frac{a\alpha^2}{2n\pi h_0}$$

$$\mu_m = \frac{\Gamma(\lambda+m)\cdot H_m}{m!\,(m+1)!\,\Gamma(\lambda)}, \quad H_m = \sum_{r=0}^{m-1}\left[\frac{1}{\lambda+r} - \frac{1}{2+r} - \frac{1}{1+r}\right]$$

Hence

$$W_n' = x e^{-\frac{n\pi x}{a}} \left[A_3 \sum_{m=0}^{\infty} \lambda_m \left(\frac{2n\pi x}{a}\right)^m + A_4 \left\{ \sum_{m=0}^{\infty} \lambda_m \left(\frac{2n\pi x}{a}\right) . \log\left(\frac{2n\pi x}{a}\right) \right.\right.$$

$$\left.\left. + \sum_{m=1}^{\infty} \mu_m \left(\frac{2n\pi x}{a}\right)^m + \frac{\Gamma(\lambda-1)}{\Gamma(\lambda)} \cdot \frac{a}{2n\pi x} \right\} \right]$$

$$= A_3 \phi_n(x) + A_4 \psi_n(x) \qquad \ldots(17)$$

Particular integral of equation (8) can be taken as

$$- \frac{q x^2 h_0}{2\alpha^2 D_0}$$

Now q can be expanded by Fourier series in the form

$$q(y) = \frac{4q}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi y}{a} \qquad \ldots(18)$$

Hence.

$$W = \sum_{n=1,3,\ldots}^{\infty} \left[A_1 e^{\frac{n\pi x}{a}} + A_2 e^{-\frac{n\pi x}{a}} + A_3 \phi_n(x) + A_4 \psi_n(x) \right.$$

$$\left. - \frac{2 q x^2 h_0}{\alpha^2 D_0 \pi n} \right] \sin \frac{n\pi y}{a} \qquad \ldots(19)$$

is determined.

Considering boundary conditions on W and solving for the constants we get,

$$A_1 = \frac{\lambda_3'\lambda_5' - \lambda_6'\lambda_2'}{\lambda_1'\lambda_5' - \lambda_2'\lambda_4'} \quad , \quad A_2 = \frac{\lambda_1'\lambda_6' - \lambda_3'\lambda_4'}{\lambda_1'\lambda_5' - \lambda_2'\lambda_4'}$$

$$A_3 = \frac{\psi_n(b)\mu_2' - \mu_1'\psi_n(c)}{\phi_n(b)\psi_n(c) - \psi_n(b)\phi_n(c)} \quad , \quad A_4 = \frac{\mu_1'\phi_n(c) - \mu_2'\phi_n(b)}{\phi_n(b)\psi_n(c) - \psi_n(b)\phi_n(c)}$$

where,

$$\mu_1' = A_1 e^{\frac{n\pi b}{a}} + A_2 e^{-\frac{n\pi b}{a}} - \frac{2qb^2h_o}{\alpha^2 D_o n\pi}$$

$$\mu_2' = A_1 e^{\frac{n\pi c}{a}} + A_2 e^{-\frac{n\pi c}{a}} - \frac{2qc^2h_o}{\alpha^2 D_o n\pi}$$

$$\lambda_1' = \Big[\Big\{e^{\frac{n\pi b}{a}}\psi_n(c) - e^{\frac{n\pi c}{a}}\psi_n(b)\Big\}\Big\{\phi_n''(b)\psi_n''(c) - \psi_n''(b)\phi_n''(c)\Big\}$$
$$-\Big\{\frac{n^2\pi^2}{a^2}e^{\frac{n\pi b}{a}}\psi_n''(c) - \frac{n^2\pi^2}{a^2}e^{\frac{n\pi c}{a}}\psi_n''(b)\Big\}\Big\{\phi_n(b)\psi_n(c) - \psi_n(b)\phi_n(c)\Big\}\Big]$$

$$\lambda_2' = \Big[\Big\{e^{-\frac{n\pi b}{a}}\psi_n(c) - e^{-\frac{n\pi c}{a}}\psi_n(b)\Big\}\Big\{\phi_n''(b)\psi_n''(c) - \psi_n''(b)\phi_n''(c)\Big\}$$
$$-\Big\{\frac{n^2\pi^2}{a^2}e^{-\frac{n\pi b}{a}}\psi_n''(c) - \frac{n^2\pi^2}{a^2}e^{-\frac{n\pi c}{a}}\psi_n''(b)\Big\}\Big\{\phi_n(b)\psi_n(c) - \psi_n(b)\phi_n(c)\Big\}\Big]$$

$$\lambda_3' = \frac{2 q h_0}{\alpha^2 D_0 n\pi}\Big[\Big\{b^2\psi_n(c) - c^2\psi_n(b)\Big\}\Big\{\phi_n''(b)\psi_n''(c) - \psi_n''(b)\phi_n''(c)\Big\}$$

$$- 2\Big\{\psi_n''(c) - \psi_n''(b)\Big\}\Big\{\psi_n(c)\phi_n(b) - \phi_n(c)\psi_n(b)\Big\}\Big]$$

$$\lambda_4' = \Big[\Big\{e^{\frac{n\pi b}{a}}\cdot\psi_n''(b) - \frac{n^2\pi^2}{a^2}e^{\frac{n\pi b}{a}}\psi_n(b)\Big\}\Big\{\phi_n(c)\psi_n''(c) - \phi_n''(c)\psi_n(c)\Big\}$$

$$-\Big\{e^{\frac{n\pi c}{a}}\psi_n''(c) - \frac{n^2\pi^2}{a^2}e^{\frac{n\pi c}{a}}\psi_n(c)\Big\}\Big\{\phi_n(b)\psi_n''(b) - \phi_n''(b)\psi_n(b)\Big\}\Big]$$

$$\lambda_5' = \Big[\Big\{e^{-\frac{n\pi b}{a}}\cdot\psi_n''(b) - \frac{n^2\pi^2}{a^2}e^{-\frac{n\pi b}{a}}\cdot\psi_n(b)\Big\}\Big\{\phi_n(c)\psi_n''(c) - \phi_n''(c)\psi_n(c)\Big\}$$

$$-\Big\{e^{-\frac{n\pi c}{a}}\cdot\psi_n''(c) - \frac{n^2\pi^2}{a^2}e^{-\frac{n\pi c}{a}}\cdot\psi_n(c)\Big\}\Big\{\phi_n(b)\psi_n''(b) - \phi_n''(b)\psi_n(b)\Big\}\Big]$$

$$\lambda_6' = \frac{2 q h_0}{\alpha^2 D_0 n\pi}\Big[\Big\{b^2\psi_n''(b) - 2\psi_n(b)\Big\}\Big\{\phi_n(c)\psi_n''(c) - \phi_n''(c)\psi_n(c)\Big\}$$

$$-\Big\{c^2\psi_n''(c) - 2\psi_n(c)\Big\}\Big\{\phi_n(b)\psi_n''(b) - \phi_n''(b)\psi_n(b)\Big\}\Big]$$

To determine α , we know from (7)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h_0^2}{12h}$$

Let us assume

$$u = \sum_{k=0}^{\infty} g_k(x) \cos \frac{k\pi y}{a} \qquad \ldots(20)$$

$$v = \sum_{k=1}^{\infty} L_k(x) \sin \frac{k\pi y}{a} \qquad \ldots(21)$$

Combining equations(7), (19), (20) and (21), we get,

$$\sum_{k=0}^{\infty} g_k'(x) \cos \frac{k\pi y}{a} + \sum_{k=1}^{\infty} L_k(x) \frac{k\pi}{a} \cos \frac{k\pi y}{a}$$

$$+\frac{1}{2}\left[\sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi y}{a}\left(\frac{n\pi}{a} A_1 e^{\frac{n\pi x}{a}} - \frac{n\pi}{a} A_2 e^{-\frac{n\pi x}{a}} + A_3 \Phi_n'(x) + A_4 \Psi_n'(x) - \frac{4 q v x h_0}{\alpha^2 D_0 n\pi}\right)\right]^2$$

$$+\frac{1}{2}\left[\sum_{n=1,3,\ldots}^{\infty} \left(A_1 e^{\frac{n\pi x}{a}} + A_2 e^{-\frac{n\pi x}{a}} + A_3 \Phi_n(x) + A_4 \Psi_n(x) - \frac{2 q v x^2 h_0}{\alpha^2 D_0 n\pi}\right)\frac{n\pi}{a} \cos \frac{n\pi y}{a}\right]^2$$

$$= \frac{\alpha^2 h_0}{12} \cdot x^{-1/3} \qquad \ldots(22)$$

Equating the terms independent of y , and integrating with respect to x between the limits b and c , we get the following biquadratic equation for the determination of α .

$$A_1^2\,\frac{n\pi}{a}\left(e^{\frac{2n\pi b}{a}} - e^{\frac{2n\pi c}{a}}\right) - A_2^2\,\frac{n\pi}{a}\left(e^{-\frac{2n\pi b}{a}} - e^{-\frac{2n\pi c}{a}}\right)$$

$$+ A_3^2\,\frac{n^2\pi^2}{a^2}\left[\sum_{m=0}^{\infty}\lambda_m^2\left(\frac{2n\pi}{a}\right)^{2m+1}\cdot e^{-\frac{2n\pi c}{a}}\left\{c^{2m+2} - \frac{(2m+2)\,c^{2m+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots\right.\right.$$

$$\left.+ \frac{(-1)^{2m+2}\cdot\lfloor\underline{2m+2}}{\left(-\frac{a}{2n\pi}\right)^{2m+2}}\right\} - \sum_{m=0}^{\infty}\lambda_m^2\left(\frac{2n\pi}{a}\right)^{2m+1}\cdot e^{\frac{-2n\pi b}{a}}\left\{b^{2m+2} - \frac{(2m+2)\,b^{2m+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots\right.$$

$$\left.+ \frac{(-1)^{2m+2}\cdot\lfloor\underline{2m+2}}{\left(-\frac{a}{2n\pi}\right)^{2m+2}}\right\}$$

$$+ \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty}\lambda_m\cdot\lambda_s\cdot\left(\frac{2n\pi}{a}\right)^{m+s+1}\cdot e^{-\frac{2n\pi c}{a}}\left\{c^{m+s+2} - \frac{(m+s+2)\,c^{m+s+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots\right.$$

$$\left.+ \frac{(-1)^{m+s+2}\cdot\lfloor\underline{m+s+2}}{\left(-\frac{a}{2n\pi}\right)^{m+s+2}}\right\}$$

$$- \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty}\lambda_m\cdot\lambda_s\cdot\left(\frac{2n\pi}{a}\right)^{m+s+1}\cdot e^{-\frac{2n\pi b}{a}}\left\{b^{m+s+2} - \frac{(m+s+2)\,b^{m+s+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots\right.$$

$$\left.\left.+ \frac{(-1)^{m+s+2}\cdot\lfloor\underline{m+s+2}}{\left(-\frac{a}{2n\pi}\right)^{m+s+2}}\right\}\right]$$

$$+ A_3^2\left[e^{-\frac{2n\pi c}{a}}\cdot\left\{\sum_{m=0}^{\infty}\left(\frac{2n\pi}{a}\right)^{2m+1}(m+1)^2\lambda_m^2\left(c^{2m} - \frac{2m\cdot c^{2m-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots\right.\right.\right.$$

$$\left.\left.+ \frac{(-1)^{2m}\cdot\lfloor\underline{2m}}{\left(-\frac{a}{2n\pi}\right)^{2m}}\right)\right\} - e^{-\frac{2n\pi b}{a}}\left\{\sum_{m=0}^{\infty}\left(\frac{2n\pi}{a}\right)^{2m+1}\cdot(m+1)^2\cdot\lambda_m^2\left(b^{2m} - \frac{2m\cdot b^{2m-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\right.\right.$$

$$\left.\left.+ \frac{(-1)^{2m}\cdot\lfloor\underline{2m}}{\left(-\frac{a}{2n\pi}\right)^{2m}}\right)\right\}$$

$$+ e^{-\frac{2n\pi c}{a}} \left\{ \sum_{\substack{m=0 \\ s=0 \\ m\neq s}}^{\infty} \lambda_m \lambda_s \left(\frac{2n\pi}{a}\right)^{m+s+1} \cdot (m+1)(s+1) \left(c^{m+s} - \frac{(m+s)\, c^{m+s-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+s} \cdot \lfloor m+s}{\left(-\frac{a}{2n\pi}\right)^{m+s}} \right) \right\}$$

$$- e^{-\frac{2n\pi b}{a}} \left\{ \sum_{\substack{m=0 \\ s=0 \\ m\neq s}}^{\infty} \lambda_m \lambda_s \left(\frac{2n\pi}{a}\right)^{m+s+1} \cdot (m+1)(s+1) \left(b^{m+s} - \frac{(m+s)\, b^{m+s-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+s} \cdot \lfloor m+s}{\left(-\frac{a}{2n\pi}\right)^{m+s}} \right) \right\}$$

$$+ \frac{n^2\pi^2}{a^2} \cdot e^{-\frac{2n\pi c}{a}} \left\{ \sum_{m=0}^{\infty} \lambda_m^2 \left(\frac{2n\pi}{a}\right)^{2m+1} \left(c^{2m+2} - \frac{(2m+2)\, c^{2m+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{2m+2} \cdot \lfloor 2m+2}{\left(-\frac{a}{2n\pi}\right)^{2m+2}} \right) \right\}$$

$$- \frac{n^2\pi^2}{a^2} e^{-\frac{2n\pi b}{a}} \left\{ \sum_{m=0}^{\infty} \lambda_m^2 \left(\frac{2n\pi}{a}\right)^{2m+1} \left(b^{2m+2} - \frac{(2m+2)\, b^{2m+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{2m+2} \cdot \lfloor 2m+2}{\left(-\frac{a}{2n\pi}\right)^{2m+2}} \right) \right\}$$

$$+ \frac{n\pi}{a} e^{-\frac{2n\pi c}{a}} \left\{ \sum_{\substack{m=0 \\ s=0 \\ m\neq s}}^{\infty} \lambda_m \lambda_s \left(\frac{2n\pi}{a}\right)^{m+s+1} \left(c^{m+s+1} - \frac{(m+s+1)\, c^{m+s}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+s+1} \cdot \lfloor m+s+1}{\left(-\frac{a}{2n\pi}\right)^{m+s+1}} \right) \right\}$$

$$- \frac{n\pi}{a} e^{-\frac{2n\pi b}{a}} \left\{ \sum_{\substack{m=0 \\ s=0 \\ m\neq s}}^{\infty} \lambda_m \lambda_s \left(\frac{2n\pi}{a}\right)^{m+s+1} \left(b^{m+s+1} - \frac{(m+s+1)\, b^{m+s}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+s+1} \cdot \lfloor m+s+1}{\left(-\frac{a}{2n\pi}\right)^{m+s+1}} \right) \right\}$$

$$+ \frac{2n\pi}{a} e^{-\frac{2n\pi b}{a}} \left\{ \sum_{\substack{m=0 \\ p=0 \\ m\neq p}}^{\infty} \lambda_m \lambda_p \left(\frac{2n\pi}{a}\right)^{m+p+1} (p+1)(m+1) \left(b^{m+p+1} - \frac{(m+p+1)\, b^{m+p}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+p+1} \cdot \lfloor m+p+1}{\left(-\frac{a}{2n\pi}\right)^{m+p+1}} \right) \right\}$$

$$- \frac{2n\pi}{a} e^{-\frac{2n\pi c}{a}} \left\{ \sum_{\substack{m=0 \\ p=0 \\ m\neq p}}^{\infty} \lambda_m \lambda_p \left(\frac{2n\pi}{a}\right)^{m+p+1} (p+1)(m+1) \left(c^{m+p+1} - \frac{(m+p+1)\, c^{m+p}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+p+1} \cdot \lfloor m+p+1}{\left(-\frac{a}{2n\pi}\right)^{m+p+1}} \right) \right\} \Bigg]$$

$$+ A_4^2 \left[\left\{ \frac{\Gamma(\lambda-1)}{\Gamma(\lambda)} \right\}^2 \cdot \frac{a}{2n\pi} \left\{ e^{-\frac{2n\pi c}{a}} - e^{-\frac{2n\pi b}{a}} \right\} + \cdots\cdots \right] + \frac{16\, q^2 h_0^2}{3\alpha^4 D_0^2 n^2 \pi^2} (b^3 - c^3)$$

$$+ \frac{4\, q^2 h_0^2}{5\alpha^4 D_0^2 a^2} (b^5 - c^5) + 2 A_3 A_1 \left[e^{\frac{n\pi b}{a}} \phi_n(b) - e^{\frac{n\pi c}{a}} \phi_n(c) \right] \frac{n\pi}{a}$$

$$+ 2A_4A_1\frac{n\pi}{a}\left[e^{\frac{n\pi b}{a}}\psi_n(b) - e^{\frac{n\pi c}{a}}\psi_n(c)\right] + \frac{4q h_0 A_1}{\alpha^2 D_0 a}\left[b^2 e^{\frac{n\pi b}{a}} - c^2 e^{\frac{n\pi c}{a}}\right]$$

$$+ 2A_2A_3\frac{n\pi}{a}\left[e^{-\frac{n\pi c}{a}}\phi_n(c) - e^{-\frac{nb\pi}{a}}\phi_n(b)\right] + \frac{4q h_0 A_2}{\alpha^2 D_0 a}\left[b^2 e^{-\frac{n\pi b}{a}} - c^2 e^{-\frac{n\pi c}{a}}\right]$$

$$+ 2A_2A_4\frac{n\pi}{a}\left[e^{-\frac{n\pi c}{a}}\psi_n(c) - e^{-\frac{n\pi b}{a}}\psi_n(b)\right]$$

$$- \frac{4A_3 n\pi q h_0}{\alpha^2 D_0 a^2}\left[\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot\frac{e^{-\frac{n\pi c}{a}}}{\frac{a}{n\pi}}\left\{c^{m+3} - \frac{(m+3)c^{m+2}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+3}\cdot\underline{\lfloor m+3}}{\left(-\frac{a}{n\pi}\right)^{m+3}}\right\}\right.$$

$$\left.- \sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot\frac{e^{-\frac{n\pi b}{a}}}{\frac{a}{n\pi}}\left\{b^{m+3} - \frac{(m+3)b^{m+2}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+3}\cdot\underline{\lfloor m+3}}{\left(-\frac{a}{n\pi}\right)^{m+3}}\right\}\right]$$

$$- \frac{8q h_0 A_3}{\alpha^2 D_0 \pi n}\left[\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot(m+1)\frac{e^{-\frac{n\pi c}{a}}}{\frac{a}{n\pi}}\left\{c^{m+1} - \frac{(m+1)c^{m}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+1}\cdot\underline{\lfloor m+1}}{\left(-\frac{a}{n\pi}\right)^{m+1}}\right\}\right.$$

$$- \sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot(m+1)\frac{e^{-\frac{n\pi b}{a}}}{\frac{a}{n\pi}}\left\{b^{m+1} - \frac{(m+1)b^{m}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+1}\cdot\underline{\lfloor m+1}}{\left(-\frac{a}{n\pi}\right)^{m+1}}\right\}$$

$$+ \sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot\frac{e^{-\frac{n\pi b}{a}}}{\frac{a}{n\pi}}\left\{b^{m+2} - \frac{(m+2)b^{m+1}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+2}\cdot\underline{\lfloor m+2}}{\left(-\frac{a}{n\pi}\right)^{m+2}}\right\}$$

$$\left.- \sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot\frac{e^{-\frac{n\pi c}{a}}}{\frac{a}{n\pi}}\left\{c^{m+2} - \frac{(m+2)c^{m+1}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+2}\cdot\underline{\lfloor m+2}}{\left(-\frac{a}{n\pi}\right)^{m+2}}\right\}\right]$$

$$- \frac{4A_4 q h_0 n\pi}{\alpha^2 D_0 a^2}\left[\frac{e^{-\frac{n\pi c}{a}}}{\frac{a}{n\pi}}\sum_{m=1}^{\infty}\mu_m\left(\frac{2n\pi}{a}\right)^m\left\{c^{m+3} - \frac{(m+3)c^{m+2}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+3}\cdot\underline{\lfloor m+3}}{\left(-\frac{a}{n\pi}\right)^{m+3}}\right\}\right.$$

$$\left.- \frac{e^{-\frac{n\pi b}{a}}}{\frac{a}{n\pi}}\sum_{m=1}^{\infty}\mu_m\left(\frac{2n\pi}{a}\right)^m\left\{b^{m+3} - \frac{(m+3)b^{m+2}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+3}\cdot\underline{\lfloor m+3}}{\left(-\frac{a}{n\pi}\right)^{m+3}}\right\} + \cdots\cdots\right]$$

$$- \frac{8q h_0}{\alpha^2 D_0 n\pi}\left[\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\left\{b^{m+2}\cdot e^{-\frac{n\pi b}{a}}\cdot \mathrm{Log}\frac{2n\pi b}{a} - c^{m+2}\cdot e^{-\frac{n\pi c}{a}}\cdot \log\frac{2n\pi c}{a}\right\}\right.$$

$$+\sum_{m=1}^{\infty} u_m \left(\frac{2n\pi}{a}\right)^m \left\{ b^{m+2} \cdot e^{-\frac{n\pi b}{a}} - c^{m+2} \cdot e^{-\frac{n\pi c}{a}} \right\}$$

$$+ \frac{\Gamma(\lambda-1)}{\Gamma(\lambda)} \cdot \frac{a}{2n\pi} \left\{ b e^{-\frac{n\pi b}{a}} - c e^{-\frac{n\pi c}{a}} \right\} + \cdots\cdots \Bigg]$$

$$+ 2A_3A_4 \frac{n^2\pi^2}{a^2} \left[\frac{\Gamma(\lambda-1)}{\Gamma(\lambda)} \cdot \frac{a}{2n\pi} \cdot \sum_{m=0}^{\infty} \lambda_m \left(\frac{2n\pi}{a}\right)^{m+1} \left\{ e^{-\frac{2n\pi c}{a}} \left(c^{m+1} - \frac{(m+1)c^m}{\left(-\frac{a}{2n\pi}\right)} + \cdots \right.\right.\right.$$

$$\left.\left.\left. + \frac{(-1)^{m+1} \cdot \lfloor m+1}{\left(-\frac{a}{2n\pi}\right)^{m+1}} \right) - e^{-\frac{2n\pi b}{a}} \left(b^{m+1} - \frac{(m+1)b^m}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+1} \cdot \lfloor m+1}{\left(-\frac{a}{2n\pi}\right)^{m+1}} \right) \right\} + \cdots \right]$$

$$+ 2A_3A_4 \left[\frac{\Gamma(\lambda-1)}{2\Gamma(\lambda)} \cdot \sum_{m=0}^{\infty} \lambda_m \left(\frac{2n\pi}{a}\right)^{m+1} \cdot (m+1) \left\{ e^{-\frac{2n\pi b}{a}} \left(b^m - \frac{m \cdot b^{m-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots \right.\right.\right.$$

$$\left.\left.\left. + \frac{(-1)^m \cdot \lfloor m}{\left(-\frac{a}{2n\pi}\right)^m} \right) - e^{-\frac{2n\pi c}{a}} \left(c^m - \frac{m \cdot c^{m-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^m \cdot \lfloor m}{\left(-\frac{a}{2n\pi}\right)^m} \right) \right\} + \cdots \right]$$

$$= \frac{\alpha^2 h_0}{2} \left(b^{2/3} - c^{2/3} \right) \qquad \cdots\cdots (27)$$

If the plate is infinite in y direction only, then the differential equation (8) will take the form

$$x\frac{d^4w}{dx^4} + 2\frac{d^3w}{dx^3} - \frac{\alpha^2}{h_o}\frac{d^2w}{dx^2} = \frac{q}{D_o} \qquad \ldots(23)$$

Solution of the above equation can be put as

$$W = \frac{x^{1/2}h_o}{\alpha^2}\left[A_1 J_1(2id_o x^{1/2}) + A_2 Y_1(2id_o x^{1/2})\right]$$

$$+ c_1 x + c_2 - \frac{qx^2}{2d_o^2 D_o} \qquad \ldots(24)$$

Where $-\frac{qx^2}{2d_o^2 D_o}$ is taken as the particular solution of (23) and $d_o^2 = \alpha^2/h_o$, J_1 and Y_1 being the Bessel functions of 1st and 2nd kind.

Boundary conditions are

$$\left.\begin{aligned} u = W &= 0 \quad , \text{at } x = b,\ x = c \\ \frac{d^2w}{dx^2} &= 0 \quad \text{at } x = b,\ x = c \end{aligned}\right\} \qquad \ldots(25)$$

Considering boundary conditions on w and solving for the constants we get,

$$A_1 = \frac{q}{d_o^2 D_o}\left[\frac{\lambda_8 - \lambda_6}{\lambda_5\lambda_8 - \lambda_7\lambda_6}\right], \quad A_2 = \frac{q}{d_o^2 D_o}\left[\frac{\lambda_7 - \lambda_5}{\lambda_7\lambda_6 - \lambda_5\lambda_8}\right]$$

$$c_1 = \frac{q(b+c)}{2d_o^2 D_o} - \frac{A_1(\lambda_1 - \lambda_3) + A_2(\lambda_2 - \lambda_4)}{b - c}$$

$$c_2 = \frac{qb^2}{2d_o^2 D_o} - A_1\lambda_1 - A_2\lambda_2 - c_1 b$$

Where,

$$\lambda_1 = \frac{b^{1/2} J_1(2 i d_0 b^{1/2})}{d_0^2}, \quad \lambda_2 = \frac{b^{1/2} Y_1(2 i d_0 b^{1/2})}{d_0^2}$$

$$\lambda_3 = \frac{c^{1/2} J_1(2 i d_0 c^{1/2})}{d_0^2}, \quad \lambda_4 = \frac{c^{1/2} Y_1(2 i d_0 c^{1/2})}{d_0^2}$$

$$\lambda_5 = b^{-1/2} J_1(2 i d_0 b^{1/2}), \quad \lambda_6 = b^{-1/2} Y_1(2 i d_0 b^{1/2})$$

$$\lambda_7 = c^{-1/2} J_1(2 i d_0 c^{1/2}), \quad \lambda_8 = c^{-1/2} Y_1(2 i d_0 c^{1/2})$$

To determine α, we know that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h_0^2}{12 h} \quad \ldots(26)$$

In this case equation (26) reduces to

$$\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{\alpha^2 h_0 x^{-1/3}}{12} \quad \ldots(27)$$

If $u = U(x)$ we have,

$$U'(x) + \frac{1}{2}\left[c_1 + \frac{qx}{d_0^2 D_0} + \frac{i}{d_0}\left\{A_1 J_0(2 i d_0 x^{1/2}) + A_2 Y_0(2 i d_0 x^{1/2})\right\}\right]^2$$

$$= \frac{\alpha^2 h_0 x^{-1/3}}{12} \quad \ldots(28)$$

Integrating the above equation between the limits b and c we get the following equation to determine α

$$\frac{A_1^2}{8d_0^4}\left[z_1^2\left\{J_0^2(z_1)+J_1^2(z_1)\right\}-z_2^2\left\{J_0^2(z_2)+J_1^2(z_2)\right\}\right]$$

$$+\frac{A_2^2}{8d_0^4}\left[z_1^2\left\{Y_0^2(z_1)+Y_1^2(z_1)\right\}-z_2^2\left\{Y_0^2(z_2)+Y_1^2(z_2)\right\}\right]$$

$$+\frac{iA_1A_2}{4d_0^3}\left[z_1^2\left\{J_0(z_1)Y_0(z_1)+J_1(z_1)\cdot Y_1(z_1)\right\}-z_2^2\left\{J_0(z_2)Y_0(z_2)+J_1(z_2)Y_1(z_2)\right\}\right]$$

$$+\frac{c_1^2}{2}(b-c)+\frac{q^2}{6d_0^4D_0^2}\left(b^3-c^3\right)+\frac{c_1q}{2d_0^2D_0}\left(c^2-b^2\right)$$

$$+\frac{A_1c_1}{2id_0^3}\left[z_1J_1(z_1)-z_2J_1(z_2)\right]+\frac{A_2c_1}{2id_0^3}\left[z_1Y_1(z_1)-z_2Y_1(z_2)\right]$$

$$+\frac{A_1q}{8id_0^7D_0}\left[z_1^3J_1(z_1)-2z_1^2J_2(z_1)-z_2^3J_1(z_2)+2z_2^2J_2(z_2)\right]$$

$$+\frac{A_2q}{8id_0^7D_0}\left[z_1^3Y_1(z_1)-2z_1^2Y_2(z_1)-z_2^3Y_1(z_2)+2z_2^2Y_2(z_2)\right]$$

$$=\frac{\alpha^2h_0}{8}\left(b^{2/3}-c^{2/3}\right) \qquad \ldots(29)$$

Where

$$z_1=2id_0b^{1/2}\,,\quad z_2=2id_0c^{1/2}$$

<u>Numerical calculations :</u>

(a) Square plate.

Let us take $a = 10,\ b = 30,\ c = 10,\ \alpha^2 = 10,\ h_0 = 1$

Putting these values in the biquadratic equation for the determination of α, we have the load function in the form

$$\frac{q}{D_0} = 108 \cdot 27 \times 10^{-4}$$

If $x/a = 2,\ y/a = 0.50$, then from equation (19)

We get, $W = 2.71$

(b) Plate of infinite strip.

Let us take $b = 30,\ c = 10,\ i\alpha = 1.5,\ h_0 = 1$

Putting these values in equation (29), we get,

$$\frac{q}{D_0} = 87 \cdot 15 \times 10^{-3}$$

If $x = 20$, then from equation (24)

$$W = 1.703$$

Large deflection of a circular plate of variable thickness under uniform load.*

PAPER - II

Nomenclature :

The following nomenclature are used in this paper.

q = uniform lateral load,

u, w = radial and lateral displacements,

h = thickness of the plate at a distance r from the centre,

D = flexural rigidity of the plate = $\dfrac{Eh^3}{12(1-\sigma^2)}$,

σ = Poisson's ratio,

E = Young's modulus,

$e_{rr} = \dfrac{du}{dr} + \dfrac{1}{2}\left(\dfrac{dw}{dr}\right)^2$,

$e_{\theta\theta} = \dfrac{u}{r}$,

$e = e_{rr} + e_{\theta\theta}$

$S_{x,y}(z)$ = Lommel's function = $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z^{x+1+2m}}{\{(x+1)^2-y^2\}\cdots\{(x+1+2m)^2-y^2\}}$,

Introduction :

Following Berger (1955), the corresponding equation for a plate of variable thickness has been deduced. Basuli (1961) solved the above problem with linearly varying thickness. In this paper an attempt has been made to solve

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*Accepted for publication in the Bulletin of the Calcutta Mathematical Society.

this problem with thickness varying as the cube root of the distance from the origin.

Analysis :

Let us consider an annular plate of outer radius b. Let the centre of the plate be taken as origin. Strain energy due to pure bending and stretching of the middle plane of the plate on combination gives

$$V = \frac{1}{2}\int D\left[\left(\nabla^2 w\right)^2 + \frac{12e^2}{h^2} - \frac{2(1-\sigma)}{r}\cdot\frac{dw}{dr}\cdot\frac{d^2w}{dr^2}\right] r\,dr\,d\theta - \int q w r\,dr\,d\theta \qquad \ldots(1)$$

If F be an extremum of V, Euler's variational equations are

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial r}\cdot\frac{\partial F}{\partial u_r} = 0 \qquad \ldots(2)$$

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial r}\cdot\frac{\partial F}{\partial w_r} + \frac{\partial^2}{\partial r^2}\cdot\frac{\partial F}{\partial w_{rr}} = 0 \qquad \ldots(3)$$

Considering (1) and (2) we can write the differential equation in the form

$$\frac{d}{dr}(he) = 0, \quad he = \text{constant} = \frac{\alpha^2 h_0^2}{12}$$

so that $$e = \frac{u}{r} + \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = \frac{\alpha^2 h_0^2}{12h} \qquad \ldots(4)$$

Let $h = h_0 r^{1/3}$, where $D = D_0 r$ ...(5)

From (1) and (3) we have

$$r^2 \frac{d^4 w}{dr^4} + 4r \frac{d^3 w}{dr^3} + \frac{d^2 w}{dr^2}(A - Br) - B \frac{dw}{dr}$$
$$= \frac{qr}{D_0}$$

Where $A = 1 + \sigma$, $B = \frac{\alpha^2}{h_0}$

Setting $\frac{dw}{dr} = z$, we have

$$r^2 \frac{d^3 z}{dr^3} + 4r \frac{d^2 z}{dr^2} + \frac{dz}{dr}(A - Br) - Bz = \frac{qr}{D_0} \qquad \text{...(6)}$$

Clearly $z = -\frac{q}{2B^2 D_0}(A + Br)$ is a particular solution of (6)
Now the equation

$$r^2 \frac{d^3 z}{dr^3} + 4r \frac{d^2 z}{dr^2} + \frac{dz}{dr}(A - Br) - Bz = 0 \qquad \text{...(7)}$$

is an exact equation which on integration gives

$$r^2 \frac{d^2 z}{dr^2} + 2r \frac{dz}{dr} + z(\sigma - 1 - Br) = K \qquad \text{...(8)}$$

Where K is a constant.

The solution of the above equation (8) can be put in the form

$$Z = r^{-1/2}\left[c_1 J_\mu\left(2iB^{1/2}r^{1/2}\right) + c_2 Y_\mu\left(2iB^{1/2}r^{1/2}\right)\right] + c_3 r^{-1/2} \cdot S_{0,\mu}\left(2iB^{1/2} \cdot r^{1/2}\right)$$

(Forsyth, A.R; A treatise on differential equation pp. 202).

Where $\frac{\mu^2}{4} = \frac{1}{4} - \sigma + 1$ and $J_\mu\left(2iB^{1/2} \cdot r^{1/2}\right)$, $Y_\mu\left(2iB^{1/2} \cdot r^{1/2}\right)$ are the Bessel functions of the first and second kind. Hence the complete solution of (6) is

$$Z = \frac{dw}{dr} = r^{-1/2}\left[c_1 J_\mu(p) + c_2 Y_\mu(p) + c_3 S_{0,\mu}(p)\right] - \frac{q}{2B^2 D_0}(A + Br) \quad \ldots(9)$$

where $p = 2iB^{1/2} \cdot r^{1/2}$

Integrating equation (9) with respect to r we get the deflection in the form

$$W = 2c_1 r^{1/2}\left[(\mu-1) J_\mu(p) \cdot S_{-1,\mu-1}(p) - J_{\mu-1}(p) \cdot S_{0,\mu}(p)\right]$$
$$+ 2c_2 r^{1/2}\left[(\mu-1) Y_\mu(p) \cdot S_{-1,\mu-1}(p) - Y_{\mu-1}(p) \cdot S_{0,\mu}(p)\right]$$
$$+ c_3\left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(2iB^{1/2}\right)^{2m+1} \cdot r^{m+1}}{(m+1)\left\{(1^2-\mu^2)\right\}\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]$$
$$- \frac{qr}{2B^2 D_0}(A + Br) + c_4 \quad \ldots(10)$$

For an annular plate clamped at the inner boundary $r = c$ and at the outer boundary $r = b$,

boundary conditions are

$$\left.\begin{aligned} u = w = 0, &\quad at\ r = b,\ r = c \\ \frac{dw}{dr} = 0, &\quad at\ r = b,\ r = c \end{aligned}\right\} \qquad \ldots(11)$$

Solving for the constants we get,

$$C_1 = \frac{\begin{aligned}&\left[\lambda_4''\lambda_3''' - \lambda_3''\lambda_4'''\right]\left[(\lambda_2-\lambda_2')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_2''-\lambda_2''')\right] \\ &\quad -\left[\lambda_2''\lambda_3''' - \lambda_3''\lambda_2'''\right]\left[(\lambda_4-\lambda_4')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_4''-\lambda_4''')\right]\end{aligned}}{\begin{aligned}&\left[\lambda_2''\lambda_3''' - \lambda_3''\lambda_2'''\right]\left[(\lambda_1-\lambda_1')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_1''-\lambda_1''')\right] \\ &\quad -\left[\lambda_1''\lambda_3''' - \lambda_3''\lambda_1'''\right]\left[(\lambda_2-\lambda_2')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_2''-\lambda_2''')\right]\end{aligned}}$$

$$C_2 = \frac{\begin{aligned}&\left[\lambda_1''\lambda_3''' - \lambda_3''\lambda_1'''\right]\left[(\lambda_4-\lambda_4')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_4''-\lambda_4''')\right] \\ &\quad -\left[\lambda_4''\lambda_3''' - \lambda_3''\lambda_4'''\right]\left[(\lambda_1-\lambda_1')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_1''-\lambda_1''')\right]\end{aligned}}{\begin{aligned}&\left[\lambda_2''\lambda_3''' - \lambda_3''\lambda_2'''\right]\left[(\lambda_1-\lambda_1')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_1''-\lambda_1''')\right] \\ &\quad -\left[\lambda_1''\lambda_3''' - \lambda_3''\lambda_1'''\right]\left[(\lambda_2-\lambda_2')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_2''-\lambda_2''')\right]\end{aligned}}$$

$$C_3 = \frac{-\left[C_1 \lambda_1'' + C_2 \lambda_2'' + \lambda_4''\right]}{\lambda_3''}$$

$$C_4 = -\left[\lambda_4 + C_1 \lambda_1 + C_2 \lambda_2 + C_3 \lambda_3\right]$$

Where

$$\lambda_1 = 2b^{1/2}\left[(\mu-1)\cdot J_\mu(z_1)\cdot S_{-1,\mu-1}(z_1) - J_{\mu-1}(z_1)\cdot S_{0,\mu}(z_1)\right]$$

$$\lambda_2 = 2b^{1/2}\left[(\mu-1)\cdot Y_\mu(z_1)\cdot S_{-1,\mu-1}(z_1) - Y_{\mu-1}(z_1)\cdot S_{0,\mu}(z_1)\right]$$

$$\lambda_3 = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(2iB^{1/2}\right)^{2m+1} \cdot b^{m+1}}{(m+1)\left\{1^2-\mu^2\right\}\cdots\left\{(1+2m)^2-\mu^2\right\}}$$

$$\lambda_3 = -\frac{q}{2B^2 D_0}\left[Ab + \frac{Bb^2}{2}\right]$$

$$\lambda_1' = 2c^{1/2}\left[(\mu-1)\cdot J_\mu(z_2)\cdot S_{-1,\mu-1}(z_2) - J_{\mu-1}(z_2)\cdot S_{0,\mu}(z_2)\right]$$

$$\lambda_2' = 2c^{1/2}\left[(\mu-1)\,Y_\mu(z_2)\cdot S_{-1,\mu-1}(z_2) - Y_{\mu-1}(z_2)\cdot S_{0,\mu}(z_2)\right]$$

$$\lambda_3' = \sum_{m=0}^{\infty} \frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot c^{m+1}}{(m+1)\{1^2-\mu^2\}\cdots\{(1+2m)^2-\mu^2\}}$$

$$\lambda_4' = -\frac{q}{2B^2D_0}\left[cA + \frac{c^2B}{2}\right]$$

$$\lambda_1'' = b^{-1/2}\cdot J_\mu(z_1)\ ,\quad \lambda_2'' = b^{-1/2}\cdot Y_\mu(z_1)\ ,\quad \lambda_3'' = b^{-1/2}\cdot S_{0,\mu}(z_1)$$

$$\lambda_4'' = -\frac{q(A+Bb)}{2B^2D_0}.$$

$$\lambda_1''' = c^{-1/2}\cdot J_\mu(z_2)\ ,\quad \lambda_2''' = c^{-1/2}\,Y_\mu(z_2)\ ,\quad \lambda_3''' = c^{-1/2}\,S_{0,\mu}(z_2)$$

$$\lambda_4''' = -\frac{q(A+Bc)}{2B^2D_0}$$

$$z_1 = 2iB^{1/2}b^{1/2}\ ,\qquad z_2 = 2iB^{1/2}c^{1/2}$$

To determine u, we know that

$$\frac{du}{dr} + \frac{u}{r} = \frac{\alpha^2 h_0^2}{12h} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

Multiplying the above equation by η and integrating with respect to η, we get

$$u\eta = \frac{\alpha^2 h_0 \eta^{5/3}}{20} + \frac{c_1^2}{8B}\left[\left\{p J_\mu'(p)\right\}^2 + (p^2-\mu^2) J_\mu^2(p)\right]$$

$$+ \frac{c_2^2}{8B}\left[\left\{p Y_\mu'(p)\right\}^2 - (p^2-\mu^2) Y_\mu^2(p)\right]$$

$$- \frac{c_3^2}{2}\left[\sum_{m=0}^{\infty} \frac{(2iB^{1/2})^{2+4m} \cdot \eta^{2+2m}}{(2+2m)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]^2}\right.$$

$$\left. + \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m \cdot (-1)^s \cdot (2iB^{1/2})^{2m+2s+2} \cdot \eta^{2+m+s}}{(m+s+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]\left[(1^2-\mu^2)\cdots\left\{(1+2s)^2-\mu^2\right\}\right]}\right]$$

$$- \frac{q^2}{8B^4 D_0^2}\left[\frac{A^2\eta^2}{2} + \frac{B^2\eta^4}{4} + \frac{2AB\eta^3}{3}\right]$$

$$+ \frac{c_1 c_2}{2B}\left[\frac{p^2}{4}\left\{2J_\mu(p)Y_\mu(p) - J_{\mu-1}(p)Y_{\mu+1}(p) - J_{\mu+1}(p)Y_{\mu-1}(p)\right\}\right]$$

$$+ \frac{c_1 c_3}{2B}\left[\sum_{m=0}^{\infty} \frac{(-1)^m\left\{(2m+1+\mu)\, p J_\mu(p)\cdot S_{2m+1,\mu-1}(p) - p J_{\mu-1}(p)\cdot S_{2m+2,\mu}(p)\right\}}{(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$+\frac{c_2 c_3}{2B}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\, p\left\{(2m+1+\mu)\, Y_\mu(p)\cdot S_{2m+1,\mu-1}(p) - Y_{\mu-1}(p)\cdot S_{2m+2,\mu}(p)\right\}}{(1^2-\mu^2)\ \cdots\ \left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$+\frac{\gamma C_1}{2B^2 D_0}\left[\frac{A r^{1/2}}{2B}\left\{J_{\mu-1}(p)\cdot S_{2,\mu}(p) - (\mu+1)\, J_\mu(p)\cdot S_{1,\mu-1}(p)\right\}\right.$$

$$\left.+\frac{r^{1/2}}{8B}\left\{(3+\mu)\, J_\mu(p)\cdot S_{3,\mu-1}(p) - J_{\mu-1}(p)\cdot S_{4,\mu}(p)\right\}\right]$$

$$+\frac{\gamma C_2}{2B^2 D_0}\left[\frac{A r^{1/2}}{2B}\left\{Y_{\mu-1}(p)\cdot S_{2,\mu}(p) - (\mu+1)\cdot Y_\mu(p)\cdot S_{1,\mu-1}(p)\right\}\right.$$

$$\left.+\frac{r^{1/2}}{8B}\left\{(3+\mu)\, Y_\mu(p)\cdot S_{3,\mu-1}(p) - Y_{\mu-1}(p)\cdot S_{4,\mu}(p)\right\}\right]$$

$$+\frac{\gamma C_3}{2B^2 D_0}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot (2iB^{1/2})^{2m+1}\cdot r^{m+2}}{(m+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right.$$

$$\left.+B\sum_{m=0}^{\infty}\frac{(-1)^m\cdot (2iB^{1/2})^{2m+1}\cdot r^{m+3}}{(m+3)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right]$$

$$+C_5 \qquad \ldots(12)$$

Where C_5 **is the constant of integration and** $p = 2iB^{1/2}r^{1/2}$.

But as $r \to b,\ u \to 0$. **Hence**

$$C_5 = -\frac{\alpha^2 h_0 b^{5/3}}{20} - \frac{c_1^2}{8B}\left[\left\{z_1 J_\mu'(z_1)\right\}^2 + (z_1^2-\mu^2)\, J_\mu^2(z_1)\right]$$

$$- \frac{C_2^2}{8B}\left[\left\{z_1 Y_\mu'(z_1)\right\}^2 + (z_1^2-\mu^2)\, Y_\mu^2(z_1)\right]$$

$$+ \frac{C_3^2}{2}\left[\sum_{m=0}^{\infty} \frac{(2iB^{1/2})^{2+4m} \cdot b^{2+2m}}{(2+2m)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]^2}\right.$$

$$\left. + \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m\cdot(-1)^s\cdot(2iB^{1/2})^{2m+2s+2} \cdot b^{2+m+s}}{(m+s+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]\left[(1^2-\mu^2)\cdots\left\{(1+2s)^2-\mu^2\right\}\right]}\right]$$

$$+ \frac{q^2}{8B^4D_0^2}\left[\frac{A^2b^2}{2} + \frac{B^2b^4}{4} + \frac{2ABb^3}{3}\right]$$

$$- \frac{C_1C_2}{2B}\left[\frac{z_1^2}{4}\left\{2J_\mu(z_1)Y_\mu(z_1) - J_{\mu-1}(z_1)\,Y_{\mu+1}(z_1) - J_{\mu+1}(z_1)Y_{\mu-1}(z_1)\right\}\right]$$

$$- \frac{C_1C_3}{2B}\left[\sum_{m=0}^{\infty} \frac{z_1(-1)^m\left\{(2m+1+\mu)J_\mu(z_1)\cdot S_{2m+1,\mu-1}(z_1) - J_{\mu-1}(z_1)\cdot S_{2m+2,\mu}(z_1)\right\}}{(1^2-\mu^2)\cdots\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$- \frac{C_2C_3}{2B}\left[\sum_{m=0}^{\infty} \frac{z_1(-1)^m\left\{(2m+1+\mu)Y_\mu(z_1)\cdot S_{2m+1,\mu-1}(z_1) - Y_{\mu-1}(z_1)\cdot S_{2m+2,\mu}(z_1)\right\}}{(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$- \frac{qC_1}{2B^2D_0}\left[\frac{Ab^{1/2}}{2B}\left\{J_{\mu-1}(z_1)\cdot S_{2,\mu}(z_1) - (\mu+1)J_\mu(z_1)\cdot S_{1,\mu-1}(z_1)\right\}\right.$$

$$\left. + \frac{b^{1/2}}{8B}\left\{(3+\mu)\cdot J_\mu(z_1)\cdot S_{3,\mu-1}(z_1) - J_{\mu-1}(z_1)\cdot S_{4,\mu}(z_1)\right\}\right]$$

$$-\frac{\gamma C_2}{2B^2 D_0}\left[\frac{Ab^{1/2}}{2B}\left\{Y_{\mu-1}(z_1)\cdot S_{2,\mu}(z_1)-(\mu+1)Y_\mu(z_1)\cdot S_{1,\mu-1}(z_1)\right\}\right.$$

$$\left.+\frac{b^{1/2}}{8B}\left\{(3+\mu)Y_\mu(z_1)\cdot S_{3,\mu-1}(z_1)-Y_{\mu-1}(z_1)\cdot S_{4,\mu}(z_1)\right\}\right]$$

$$-\frac{\gamma C_3}{2B^2 D_0}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot b^{m+2}}{(m+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right.$$

$$\left.+B\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot b^{m+3}}{(m+3)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right] \qquad \ldots (13)$$

Also as $r \to c$, $u \to 0$ **. Hence the equation to determine** α **leads to**

$$\frac{\alpha^2 h_0 c^{5/3}}{20}+\frac{C_1^2}{8B}\left[\left\{z_2 J_\mu'(z_2)\right\}^2+(z_2^2-\mu^2)J_\mu^2(z_2)\right]$$

$$+\frac{C_2^2}{8B}\left[\left\{z_2 Y_\mu'(z_2)\right\}^2+(z_2^2-\mu^2)Y_\mu^2(z_2)\right]$$

$$-\frac{C_3^2}{2}\left[\sum_{m=0}^{\infty}\frac{(2iB^{1/2})^{2+4m}\cdot c^{2+2m}}{(2+2m)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]^2}\right.$$

$$\left.+\sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty}\frac{(-1)^m\cdot(-1)^s\cdot(2iB^{1/2})^{2m+2s+2}\cdot c^{m+s+2}}{(m+s+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]\left[(1^2-\mu^2)\cdots\left\{(1+2s)^2-\mu^2\right\}\right]}\right]$$

$$- \frac{q^2}{8B^4D_0^2}\left[\frac{A^2c^2}{2} + \frac{B^2c^4}{4} + \frac{2AB.c^3}{3}\right]$$

$$+ \frac{C_1C_2}{2B}\left[\frac{z_2^2}{4}\left\{2J_\mu(z_2)\cdot Y_\mu(z_2) - J_{\mu-1}(z_2)\cdot Y_{\mu+1}(z_2) - J_{\mu+1}(z_2)\cdot Y_{\mu-1}(z_2)\right\}\right]$$

$$+ \frac{C_1C_3}{2B}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot z_2\left\{(2m+1+\mu)\cdot J_\mu(z_2)\cdot S_{2m+1,\mu-1}(z_2) - J_{\mu-1}(z_2)\cdot S_{2m+2,\mu}(z_2)\right\}}{(1^2-\mu^2)\cdot\cdots\cdot\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$+ \frac{C_2C_3}{2B}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot z_2\left\{(2m+1+\mu)Y_\mu(z_2)\cdot S_{2m+1,\mu-1}(z_2) - Y_{\mu-1}(z_2)\cdot S_{2m+2,\mu}(z_2)\right\}}{(1^2-\mu^2)\cdot\cdots\cdot\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$+ \frac{qC_1}{2B^2D_0}\left[\frac{Ac^{1/2}}{2B}\left\{J_{\mu-1}(z_2)\cdot S_{2,\mu}(z_2) - (\mu+1)J_\mu(z_2)\cdot S_{1,\mu-1}(z_2)\right\}\right.$$

$$\left. + \frac{c^{1/2}}{8B}\left\{(3+\mu)J_\mu(z_2)\cdot S_{3,\mu-1}(z_2) - J_{\mu-1}(z_2)\cdot S_{4,\mu}(z_2)\right\}\right]$$

$$+ \frac{qC_2}{2B^2D_0}\left[\frac{Ac^{1/2}}{2B}\left\{Y_{\mu-1}(z_2)\cdot S_{2,\mu}(z_2) - (\mu+1)Y_\mu(z_2)\cdot S_{1,\mu-1}(z_2)\right\}\right.$$

$$\left. + \frac{c^{1/2}}{8B}\left\{(3+\mu)Y_\mu(z_2)\cdot S_{3,\mu-1}(z_2) - Y_{\mu-1}(z_2)\cdot S_{4,\mu}(z_2)\right\}\right]$$

$$+ \frac{qC_3}{2B^2D_0}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot c^{m+2}}{(m+2)\left[(1^2-\mu^2)\cdot\cdots\cdot\left\{(1+2m)^2-\mu^2\right\}\right]}\right.$$

$$\left. + B\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot c^{m+3}}{(m+3)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right]$$

$$
-\frac{\alpha^2 h_0 b^{5/3}}{20} - \frac{c_1^2}{8B}\left[\left\{z_1 J_\mu'(z_1)\right\}^2 + \left(z_1^2-\mu^2\right) J_\mu^2(z_1)\right]
$$

$$
-\frac{c_2^2}{8B}\left[\left\{z_1 Y_\mu'(z_1)\right\}^2 + \left(z_1^2-\mu^2\right) Y_\mu^2(z_1)\right]
$$

$$
+\frac{c_3^2}{2}\left[\sum_{m=0}^{\infty} \frac{\left(2iB^{1/2}\right)^{2+4m} \cdot b^{2+2m}}{(2+2m)\left[\left(1^2-\mu^2\right)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]^2}\right.
$$

$$
\left.+\sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m \cdot (-1)^s \cdot \left(2iB^{1/2}\right)^{2m+2s+2} \cdot b^{m+s+2}}{(m+s+2)\left[\left(1^2-\mu^2\right)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]\left[\left(1^2-\mu^2\right)\cdots\left\{(1+2s)^2-\mu^2\right\}\right]}\right]
$$

$$
+\frac{q^2}{8B^4 D_0^2}\left[\frac{A^2 b^2}{2} + \frac{B^2 b^4}{4} + \frac{2AB.b^3}{3}\right]
$$

$$
-\frac{c_1 c_2}{2B}\left[\frac{z_1^2}{4}\left\{2 J_\mu(z_1).Y_\mu(z_1) - J_{\mu-1}(z_1).Y_{\mu+1}(z_1) - J_{\mu+1}(z_1) Y_{\mu-1}(z_1)\right\}\right]
$$

$$
-\frac{c_1 c_3}{2B}\left[\sum_{m=0}^{\infty} \frac{(-1)^m . z_1\left\{(2m+1+\mu) J_\mu(z_1) \cdot S_{2m+1,\mu-1}(z_1) - J_{\mu-1}(z_1) S_{2m+2,\mu}(z_1)\right\}}{\left(1^2-\mu^2\right)\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]
$$

$$
-\frac{c_2 c_3}{2B}\left[\sum_{m=0}^{\infty} \frac{(-1)^m . z_1\left\{(2m+1+\mu) Y_\mu(z_1) \cdot S_{2m+1,\mu-1}(z_1) - Y_{\mu-1}(z_1) \cdot S_{2m+2,\mu}(z_1)\right\}}{\left(1^2-\mu^2\right)\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]
$$

$$
-\frac{q c_1}{2B^2 D_0}\left[\frac{A b^{1/2}}{2B}\left\{J_{\mu-1}(z_1) \cdot S_{2,\mu}(z_1) - (\mu+1) J_\mu(z_1) \cdot S_{1,\mu-1}(z_1)\right\}\right.
$$

$$
\left.+\frac{b^{1/2}}{8B}\left\{(3+\mu) J_\mu(z_1) \cdot S_{3,\mu-1}(z_1) - J_{\mu-1}(z_1) \cdot S_{4,\mu}(z_1)\right\}\right]
$$

$$- \frac{q c_2}{2B^2 D_0} \left[\frac{A b^{1/2}}{2B} \left\{ Y_{\mu-1}(z_1) \cdot S_{2,\mu}(z_1) - (\mu+1) Y_\mu(z_1) \cdot S_{1,\mu-1}(z_1) \right\} \right.$$

$$\left. + \frac{b^{1/2}}{2B} \left\{ (3+\mu) Y_\mu(z_1) \cdot S_{3,\mu-1}(z_1) - Y_{\mu-1}(z_1) \cdot S_{4,\mu}(z_1) \right\} \right]$$

$$- \frac{q c_3}{2B^2 D_0} \left[\sum_{m=0}^{\infty} \frac{(-1)^m (2iB^{1/2})^{2m+1} \cdot b^{m+2}}{(m+2)\left[(1^2-\mu^2) \cdots \{(1+2m)^2-\mu^2\}\right]} \right.$$

$$\left. + B \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot b^{m+3}}{(m+3)\left[(1^2-\mu^2) \cdots \{(1+2m)^2-\mu^2\}\right]} \right] = 0 \qquad \ldots(14)$$

In particular, if $\sigma = 0 \cdot 25$, the corresponding deflection of the plate is given by

$$W = 2c_1' r^{1/2} \left[J_2(p) \cdot S_{-1,1}(p) - J_1(p) \cdot S_{0,2}(p) \right]$$

$$+ 2c_2' r^{1/2} \left[Y_2(p) \cdot S_{-1,1}(p) - Y_1(p) \cdot S_{0,2}(p) \right]$$

$$+ c_3' \left[\sum_{m=0}^{\infty} \frac{(-1)^m (2iB^{1/2})^{2m+1} \cdot r^{m+1}}{(m+1)\left[(1^2-2^2) \cdots \{(1+2m)^2-2^2\}\right]} \right]$$

$$- \frac{q}{2B^2 D_0} \left[\frac{5}{4} r + \frac{B r^2}{2} \right] + c_4' \qquad \ldots(15)$$

Where

$$C_1' = \frac{\left[\mu_4''\mu_3''' - \mu_3''\mu_4'''\right]\left[(\mu_2-\mu_2')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_2''-\mu_2''')\right] - \left[\mu_2''\mu_3''' - \mu_3''\mu_2'''\right]\left[(\mu_4-\mu_4')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_4''-\mu_4''')\right]}{\left[\mu_2''\mu_3''' - \mu_3''\mu_2'''\right]\left[(\mu_1-\mu_1')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_1''-\mu_1''')\right] - \left[\mu_1''\mu_3''' - \mu_3''\mu_1'''\right]\left[(\mu_2-\mu_2')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_2''-\mu_2''')\right]}$$

$$C_2' = \frac{\left[\mu_1''\mu_3''' - \mu_3''\mu_1'''\right]\left[(\mu_4-\mu_4')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_4''-\mu_4''')\right] - \left[\mu_4''\mu_3''' - \mu_3''\mu_4'''\right]\left[(\mu_1-\mu_1')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_1''-\mu_1'')\right]}{\left[\mu_2''\mu_3''' - \mu_3''\mu_2'''\right]\left[(\mu_1-\mu_1')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_1''-\mu_1'')\right] - \left[\mu_1''\mu_3''' - \mu_3''\mu_1'''\right]\left[(\mu_2-\mu_2')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_2''-\mu_2''')\right]}$$

$$C_3' = \frac{-\left[C_1'\mu_1'' + C_2'\mu_2'' + \mu_4''\right]}{\mu_3''} , \quad C_4' = -\left[\mu_4 + C_1'\mu_1 + C_2'\mu_2 + C_3'\mu_3\right]$$

Where

$$\mu_1 = 2b^{1/2}\left[J_2(z_1)\, S_{-1,1}(z_1) - J_1(z_1)\cdot S_{0,2}(z_1)\right]$$

$$\mu_2 = 2b^{1/2}\left[Y_2(z_1)\cdot S_{-1,1}(z_1) - Y_1(z_1)\cdot S_{0,2}(z_1)\right]$$

$$\mu_3 = \sum_{m=0}^{\infty} \frac{(-1)^m \, (2iB^{1/2})^{2m+1} \cdot b^{m+1}}{(m+1)\{(1^2-2^2)\}\cdots\{(1+2m)^2-2^2\}}$$

$$\mu_4 = -\frac{q}{2B^2 D_0}\left[\frac{s}{4}b + \frac{Bb^2}{2}\right]$$

$$\mu_1' = 2c^{1/2}\left[J_2(z_2)\cdot S_{-1,1}(z_2) - J_1(z_2)\cdot S_{0,2}(z_2)\right]$$

$$\mu_2' = 2c^{1/2}\left[Y_2(z_2)\cdot S_{-1,1}(z_2) - Y_1(z_2)\cdot S_{0,2}(z_2)\right]$$

$$\mu_3' = \sum_{m=0}^{\infty} \frac{(-1)^m \, (2iB^{1/2})^{2m+1} \cdot c^{m+1}}{(m+1)\{(1^2-2^2)\}\cdots\{(1+2m)^2-2^2\}}$$

$$\mu_4' = -\frac{q}{2B^2 D_0}\left[\frac{s}{4}c + \frac{Bc^2}{2}\right]$$

$$\mu_1'' = b^{-1/2}\cdot J_2(z_1), \quad \mu_2'' = b^{-1/2}\cdot Y_2(z_1), \quad \mu_3'' = b^{-1/2}\cdot S_{0,2}(z_1)$$

$$\mu_4'' = -\frac{q}{2B^2 D_0}\left[\frac{s}{4} + Bb\right]$$

$$\mu_1''' = c^{-1/2}\cdot J_2(z_2), \quad \mu_2''' = c^{-1/2}\cdot Y_2(z_2), \quad \mu_3''' = c^{-1/2}\cdot S_{0,2}(z_2)$$

$$\mu_4''' = -\frac{q}{2B^2 D_0}\left[\frac{s}{4} + Bc\right]$$

The equation to determine α leads to

$$\frac{\alpha^2 h_0 C^{5/3}}{20} + \frac{C_1'^2}{8B}\left[\left\{z_2 J_2'(z_2)\right\}^2 + \left(z_2^2 - 2^2\right) J_2^2(z_2)\right]$$

$$+ \frac{C_2'^2}{8B}\left[\left\{z_2 Y_2'(z_2)\right\}^2 + \left(z_2^2 - 2^2\right) Y_2^2(z_2)\right]$$

$$- \frac{C_3'^2}{2}\left[\sum_{m=0}^{\infty} \frac{(2iB^{1/2})^{2+4m} \cdot C^{2+2m}}{(2+2m)\left[(1^2-2^2)\cdots\{(1+2m)^2-2^2\}\right]^2}\right.$$

$$\left. + \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m \cdot (-1)^s \cdot (2iB^{1/2})^{2m+2s+2} \cdot C^{m+s+2}}{(m+s+2)\left[(1^2-2^2)\cdots\{(1+2m)^2-2^2\}\right]\left[(1^2-2^2)\cdots\{(1+2s)^2-2^2\}\right]}\right]$$

$$- \frac{q^2}{8B^4 D_0^2}\left[\frac{25}{32}C^2 + \frac{B^2}{4}C^4 + \frac{5B}{6}C^3\right]$$

$$+ \frac{C_1' C_2'}{2B}\left[\frac{z_2^2}{4}\left\{2J_2(z_2)\cdot Y_2(z_2) - J_1(z_2) Y_3(z_2) - J_3(z_2) Y_1(z_2)\right\}\right]$$

$$+ \frac{C_1' C_3'}{2B}\left[\sum_{m=0}^{\infty} \frac{(-1)^m z_2\left\{(2m+3) J_2(z_2)\cdot S_{2m+1,1}(z_2) - J_1(z_2) S_{2m+2,2}(z_2)\right\}}{(1^2-2^2)\cdots\{(1+2m)^2-2^2\}}\right]$$

$$+ \frac{C_2' C_3'}{2B}\left[\sum_{m=0}^{\infty} \frac{(-1)^m z_2\left\{(2m+3) Y_2(z_2) S_{2m+1,1}(z_2) - Y_1(z_2) S_{2m+2,2}(z_2)\right\}}{(1^2-2^2)\cdots\{(1+2m)^2-2^2\}}\right]$$

$$+ \frac{q C_1'}{2B^2 D_0}\left[\frac{5C^{1/2}}{8B}\left\{J_1(z_2) S_{2,2}(z_2) - 3J_2(z_2) S_{1,1}(z_2)\right\}\right.$$

$$\left. + \frac{C^{1/2}}{8B}\left\{5J_2(z_2)\cdot S_{3,1}(z_2) - J_1(z_2) S_{4,2}(z_2)\right\}\right]$$

$$+ \frac{q C_2'}{2B^2 D_0}\left[\frac{5c^{1/2}}{8B}\left\{Y_1(z_2)\, S_{2,2}(z_2) - 3Y_2(z_2)\, S_{1,1}(z_2)\right\}\right.$$

$$\left. + \frac{c^{1/2}}{8B}\left\{5Y_2(z_2)\, S_{3,1}(z_2) - Y_1(z_2)\, S_{4,2}(z_2)\right\}\right]$$

$$+ \frac{q C_3'}{2B^2 D_0}\left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot c^{m+2}}{(m+2)\left[(1^2-2^2)\cdots\{(1+2m)^2-2^2\}\right]}\right.$$

$$\left. + B\sum_{m=0}^{\infty} \frac{(-1)^m (2iB^{1/2})^{2m+1}\, c^{m+3}}{(m+3)\left[(1^2-2^2)\cdots\{(1+2m)^2-2^2\}\right]}\right]$$

$$- \frac{\alpha^2 h_0 b^{5/3}}{20} - \frac{C_1'^2}{8B}\left[\left\{z_1 J_2'(z_1)\right\}^2 + (z_1^2 - 2^2)\, J_2^2(z_2)\right]$$

$$- \frac{C_2'^2}{8B}\left[\left\{z_1 Y_2'(z_1)\right\}^2 + (z_1^2 - 2^2)\, Y_2^2(z_1)\right]$$

$$+ \frac{C_3'^2}{2}\left[\sum_{m=0}^{\infty} \frac{(2iB^{1/2})^{2+4m} \cdot b^{2+2m}}{(2+2m)\left[(1^2-2^2)\cdots\{(1+2m)^2-2^2\}\right]^2}\right.$$

$$\left. + \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m (-1)^s \cdot (2iB^{1/2})^{2m+2s+2} \cdot b^{m+s+2}}{(m+s+2)\left[(1^2-2^2)\cdots\{(1+2m)^2-2^2\}\right]\left[(1^2-2^2)\cdots\{(1+2s)^2-2^2\}\right]}\right]$$

$$+ \frac{q^2}{8B^4 D_0^2}\left[\frac{25}{32} b^2 + \frac{B^2}{4} b^4 + \frac{5B}{6} b^3\right]$$

$$- \frac{C_1' C_2'}{2B}\left[\frac{z_1^2}{4}\left\{2J_2(z_1) Y_2(z_1) - J_1(z_1) Y_3(z_1) - J_3(z_1) Y_1(z_1)\right\}\right]$$

$$- \frac{C_1' C_3'}{2B} \left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z_1 \left\{ (2m+3) J_2(z_1) S_{2m+1,1}(z_1) - J_1(z_1) S_{2m+2,2}(z_1) \right\}}{(1^2-2^2) \cdots \left\{ (1+2m)^2 - 2^2 \right\}} \right]$$

$$- \frac{C_2' C_3'}{2B} \left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z_1 \left\{ (2m+3) Y_2(z_1) S_{2m+1,1}(z_1) - Y_1(z_1) S_{2m+2,2}(z_1) \right\}}{(1^2-2^2) \cdots \left\{ (1+2m)^2 - 2^2 \right\}} \right]$$

$$- \frac{q C_1'}{2B^2 D_0} \left[\frac{5 b^{1/2}}{8B} \left\{ J_1(z_1) S_{2,2}(z_1) - 3 J_2(z_1) S_{1,1}(z_1) \right\} \right.$$

$$\left. + \frac{b^{1/2}}{8B} \left\{ 5 J_2(z_1) S_{3,1}(z_1) - J_1(z_1) S_{4,2}(z_1) \right\} \right]$$

$$- \frac{q C_2'}{2B^2 D_0} \left[\frac{5 b^{1/2}}{8B} \left\{ Y_1(z_1) S_{2,2}(z_1) - 3 Y_2(z_1) S_{1,1}(z_1) \right\} \right.$$

$$\left. + \frac{b^{1/2}}{8B} \left\{ 5 Y_2(z_1) S_{3,1}(z_1) - Y_1(z_1) S_{4,2}(z_1) \right\} \right]$$

$$- \frac{q C_3'}{2B^2 D_0} \left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot b^{m+2}}{(m+2) \left[(1^2-2^2) \cdots \left\{ (1+2m)^2 - 2^2 \right\} \right]} \right.$$

$$\left. + B \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot b^{m+3}}{(m+3) \left[(1^2-2^2) \cdots \left\{ (1+2m)^2 - 2^2 \right\} \right]} \right]$$

$$= 0 \qquad \cdots (16)$$

Numerical calculations :

Let us take $i\alpha = 1{\cdot}5$, $b = 20$, $c = 10$, $h_0 = 1$, $n = 15$

Putting all these values in (16) we get, $\frac{q}{D_0} = 118{\cdot}0 \times 10^{-3}$

Substituting this value of $\frac{q}{D_0}$ in (15) we get, $W = 2{\cdot}32$