CHAPTER 1
INTRODUCTION

1.1 Basic Definitions

We review here the basic terminology about graphs, directed graphs and relations used throughout this dissertation.

Given a graph $G(V, E)$, $V= V(G)$ will denote its vertex set and $E=E(G)$ its edge set. A graph G' is a subgraph of G if $V(G') \subset V(G)$ and $E(G') \subset E(G)$. A subgraph G' is a *generated subgraph* or an *induced subgraph* of G if $V(G') \subset V(G)$ and two vertices are adjacent in G' if and only if they are adjacent in G. The complement $\bar{G}(V, E)$ of a graph G has the same vertex set as G and two vertices are adjacent in $\bar{G}$ iff they are not adjacent in G.

A set of vertices of a graph G is a *stable* set (or *independent* set) if no two vertices in the set are adjacent. A *bipartite graph* is a graph $H(V, E)$ whose vertex set V can be partitioned into two stable sets, say X and Y. in this case we write $H=H(X, Y, E)$. A *complete bipartite graph* is a bipartite graph $H(X, Y, E)$ where every vertex in X is adjacent to every vertex in Y.

Let $V=\{v_1, v_2, \ldots, v_n\}$ be the vertex set of an undirected graph G. Then *characteristic vector* of any subset $V' \subseteq V$ is the vector $x = (x_1, x_2, \ldots, x_n)$ where

$$x_i = \begin{cases} 1 & \text{if } v_i \in V' \\ 0 & \text{if } v_i \notin V' \end{cases} \quad \text{for all } i.$$

It is clear that the family of all subsets of V has one-to-one correspondence with the corners of the unit hypercube in R^n.

A directed graph (digraph) $D(V, E)$ consists of finite set V of vertices and a set E ordered pairs of vertices called edges (arcs). In this dissertation, sometimes we will write

$u \rightarrow v$ to mean $uv \in E$ and $u \not\rightarrow v$ to mean $uv \notin E$. All throughout we will assume that a digraph may have loops (i.e. edges of the form (u, u))but no multiple edges.

The *successor* set of a vertex v is the set of vertices u such that vu is an edge of the digraph $D(V, E)$. The *predecessor* set of a vertex v is the set of vertices u such that uv is an edge of the digraph $D(V, E)$.

The digraph $D(V, E)$ is an *acyclic digraph* when it contains no directed cycle.

The adjacency matrix of a di(graph) on the vertex set $V=\{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ 0, 1-matrix with 1 in the (i, j) position if only if (v_i, v_j) is an edge of the (di)graph.

The union of two digraphs $D_1=(V_1, E_1)$ and $D_2=(V_2, E_2)$ is the is the digraph $D(V, E)$ whose vertex set $V = V_1 \cup V_2$ and edge $E=E_1 \cup E_2$; similarly the intersection of two digraphs D_1 and D_2 is a digraph D consisting only of those vertices and edges that are in both D_1 and D_2.

Other definitions on graph theory will be given as and when required in the thesis. For any undefined terms one is referred to Roberts [1976], Golumbic [1980] or West [1996].

Graphs, digraphs and partial orders are all binary relations. A binary relation R on a set A is a subset of $A \times A$. A binary relation (A, R) is *reflexive* if aRa for all $a \in A$. A binary relation (A, R) is *symmetric* if $aRb \Rightarrow bRa$ for all $a, b \in A$. An undirected graph is a symmetric related set. A binary relation (A, R) is *transitive* if for all $a, b, c, \in A$, aRb and bRc implies aRc. A binary relation R on A is a partial order if R *irreflexive* and transitive. A partial order (A, R) is a linear order if for elements $a, b, \in A$, either aRb or bRa holds.. More background materials on relations can be found in Fishburn [1985].

1.2 Notations

For convenience, the most frequently used notations are listed here.

$G(V, E)$	A graph whose vertex set is V and edge set is E.
$D(V, E)$	A digraph whose vertex set is V and edge set is E.
$H(X, Y, E)$	A bipartite graph whose vertex set is partitioned into two stable sets X and Y and whose edge set is E.
xy	edge whose vertices are x and y .
$\bar{D}$	Complement of D.
$H(D)$	Associated graph of D.
$G\backslash G_1$	Subgraph of G induced by the vertex set $V(G) - V(G_1)$, for any subgraph G_1 of G.
C_n	Cycle of length n.
K_n	Complete graph with n vertices .
$K_{m, n}$	Complete bipartite graph, the size of whose stable sets are m and n.
$f(D)$	Ferrers dimension of the digraph D.
I	The set of all isolated vertices of $H(D)$.
$H_b(D)$	The graph obtained from $H(D)$ by deleting its isolated vertices.
$B(xy)$	The subgraph induced by $adj(x) + adj(y)$.
$\lfloor n \rfloor$	greatest integer $\leq n$
$\lceil n \rceil$	smallest integer $\geq n$.
$\lvert S \rvert$	Cardinality of the set S.
$A(D)$	Adjacency matrix of the digraph D.
$Adj(H)$	Biadjacency matrix of the bigraph H.

1.3 Preliminaries

The subject of intersection graph is an important area of study in graph theory. The intersection graph of a family $\mathfrak{F}$ of sets is an undirected graph $G(V, E)$ such that there is a one-to-one correspondence between $\mathfrak{F}$ and V and two vertices of V have an edge between them when the corresponding sets have a non-empty intersection. It was shown by Marczewski [1945] that all graphs are intersection graphs; because to each vertex v of G, if we associate the set of all edges incident to v, then G is the intersection graph of this family of sets. Interesting problems arise only when the families of sets have some specific topological or other properties. The families of sets most often considered in connection with intersection graphs are the families of intervals on the real line and then the corresponding intersection graph is known as *interval graph*. Detailed discussion on this topic can be found in the Golumbic [1980] and Fishburn [1985]. A recent survey by Trotter [1997] summaries a variety of recent results and open problems.

The study of directed graphs (digraphs) from the viewpoint of intersection representation began only during the last decade. Since the undirected graphs arc special types of directed graphs, the problem of finding a natural translation of the theory of intersection graphs along with its abundant literature to directed graphs is very well worth investigating. Intersection digraphs of a family of ordered pair of intervals on the real line and of arcs of a circle, respectively called *interval digraphs* and *circular arc digraphs* and related topics have been extensively studied by Sen, West and others. These results provide the background materials for this dissertation.

For a very nice treatment of the several aspects of intersection (di) graph and their applications to different fields such as, biology, computing, psychology, statistics etc. with many latest references, see the book by Mckee and McMorris [1999].

In the second chapter, first we characterize interval digraphs in terms of the interior edges with reference to a bicolouration of its associated bipartite *graph H(D)*. This gives us a complete list of forbidden configurations of an interval digraph. We then apply this characterization to obtain a recognition algorithm of an interval digraph in time which is much less than that given by Müller [1997].

Lekkerkerker and Boland [1962] used the linear order of points on the real line to characterize an interval graph which states, interalia, that any three vertices of an interval graph can be ordered in such a way that every path from the first vertex to the third vertex passes through a neighbour of the second. To obtain an analogous characterization of an interval digraph H. Müller [1997] introduced the notion of asteroidal triple of edges

(ATE) and conjectured that a digraph is an interval digraph if and only if the corresponding bipartite graph is free from ATE and other special class of bigraphs (insects).

In this context we note that an interval digraph is necessarily of Ferrers dimension ≤ 2 and its corresponding bigraph is ATE free so the question arises as to the relationship between these two classes of graphs. In chapter 3 we obtain the result that a digraph of $f(D)$ at most 2 is necessarily bichordal and ATE-free. But the converse problem is much involved. In the same chapter, we have considered this problem in the particular case, when the graph contains a strong bismplicial edge.

A base interval is an ordered pair (S, p) where S is a closed interval and p is a point of S. Previously assigning an ordered pair of base intervals to every vertex, the model of overlap base interval digraphs (Robin digraphs) was introduced and its adjacency matrix characterized.

In chapter 4 we have extended the notion of intersection to the base intervals, *base interval digraphs* and characterized the adjacency matrices of such digraphs. We have also characterized a base interval digraph in terms of intersection of two interval digraphs and then we have obtained an analogous result for undirected graph.

Finally we have considered overlap base interval digraphs where the intervals are of unit length. It has been proved in this chapter that this class of digraphs is actually the same as the class of overlap digraphs. The notion of overlap digraphs was introduced earlier [Sen, Sanyal and West, 1995].

1.4 Interval graphs

1.4.1 Interval graphs and their applications

An *interval graph* is the intersection graph of some family of intervals on the real line. That is, $G=G(V, E)$ is an interval graph if and only if there is an assignment $\boldsymbol{I}$ of a real intervals I_v to each $v \in V$ such that for all $u, v \in V$,

$$uv \in E \Leftrightarrow I_u \cap I_v \neq \varphi.$$

Interval graphs have a long and rich history. The topic is so important that a special issue of Discrete Mathematics [1985] is devoted to discuss interval graphs and related topics. They arose independently from purely mathematical consideration by Hajos [1957] in combinatorics and by the renowned molecular biologist Benzer [1959] from a problem of genetics. Benzer's problem was whether the fine structure (subelements) inside the gene were linked together in a linear arrangement. He answered this question in affirmative by studying the overlap data of mutations. In terms of graph theory a graph G can be defined by considering the substructures (mutations) to be the vertices of G with an edge between two vertices when two corresponding substructures overlap. Then Benzer's problem reduced to finding out if G was an interval graph or not.

Interval graphs are among the most useful mathematical structures for modeling real world problem. It had been used in seriation problem by Kendall [1971] and Hubert [1974], in archaeology by Skrien [1980, 1984] and in developmental psychology by Coombs and Smith [1973] to name a few. In arriving at a solution to general traffic phasing problem, stoffers [1968] and Roberts [1978, 1979] used interval graphs. Nicholson [1992] applied interval graphs to computing a protein model. More in-depth treatment and recent views in this field can be found in [Goldberg, Golumbic, Kaplan and Shamir, 1995] and [Waterman, 1995].

Recently in a relevant work [Zang, to appear] introduces the following generalization of interval graphs called a *probe interval graph* to deal with the physical mapping of DNA. A graph G is a *probe interval graph* if $V(G)$ can be partitioned into subsets P and N and each $v \in V(G)$ can be assigned an interval I_v such that $uv \in E(G)$ if and only if both $I_u \cap I_v \neq \varphi$ and at least one of u and v is in P. Interval graphs are simply probe interval graphs with $N = \varphi$.

1.4.2 Some characterizations of interval graphs

Not all graphs are interval graphs (for example, it is easy to check that C_4 is not an interval graph). First characterization of interval graphs was due to Lekkerkerker and

Boland [1962]. They first defined an asteroidal triple of vertices and characterized an interval graph in terms of this notion. A collection of three pairwise non-adjacent vertices v_1, v_2, v_3 of a graph G is an *asteroidal triple* of G if for every pair of vertices of the triple, there exists a path joining them which does not pass through a neighbour of the third.

Theorem 1.1 [Lekkerkerker and Boland, 1962]. *A graph G is an interval graph iff G is triangulated and does not contain any asteroidal triple.*

In that paper, they also provided a complete set of forbidden subgraphs for interval graphs depicted in Fig. 1.1. Thus a graph is an interval graph iff it does not contain any graph of Fig. 1.1. as induced subgraph. [Corneil, Olariu and Stewart, 1997] gives an excellent survey and synthesis of all aspects of asteroidal triple-free graphs. In that paper they showed that interval, permutation, trapezoidal, cocomparability graphs are all special classes of asteroidal triple-free graph.

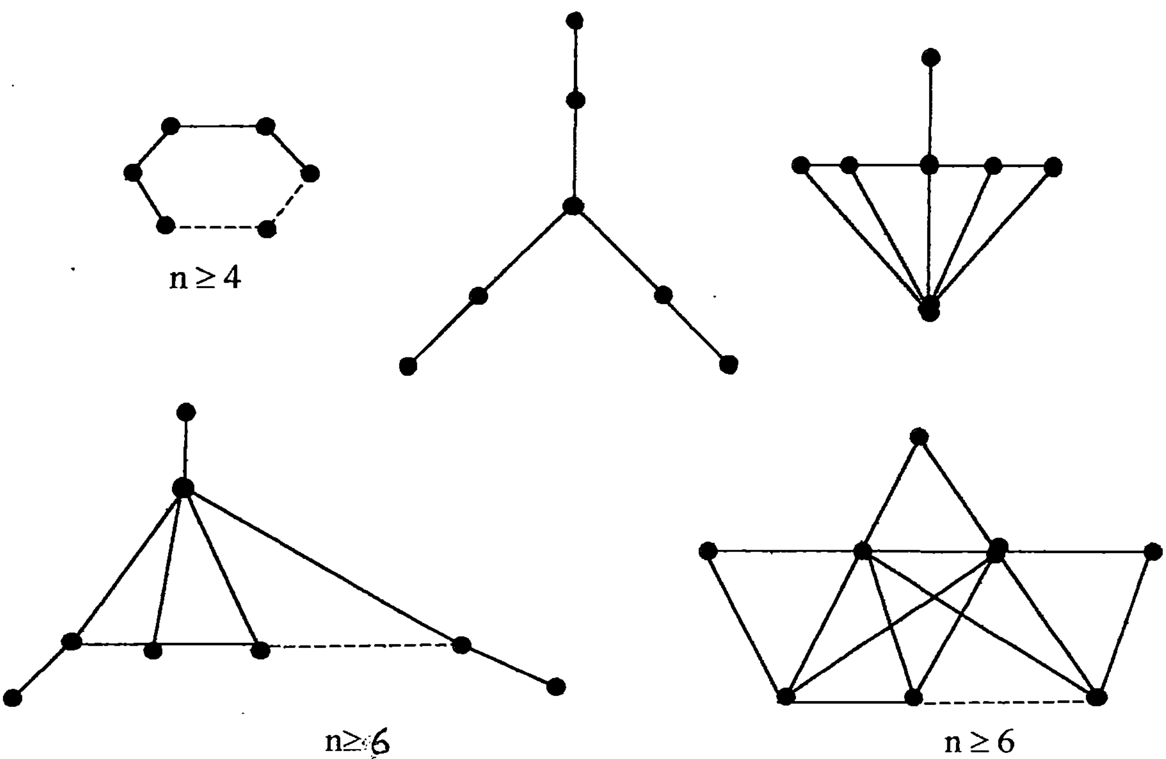


Fig 1.1 Minimal non-interval graphs

Another characterization of interval graph was given by Gilmore and Hoffman [1964] in which they had related interval digraph to what is called a *comparability* graph (or *transitive orientable* graph). A digraph is *transitive* if whenever there is an arc from *u* to v and an arc from v to w, $u \neq w$, then there is an arc from u to w. An orientation of a graph is an assignment of a direction to each of the edges of the graph to get a digraph. A graph has a transitive orientation of its edges if there is an orientation of its edges so that the resulting digraph is transitive. A graph which has a transitive orientation is called a *comparability* graph.

Theorem 1.2 [Gilmore and Hoffman, 1964]. *A graph G is an interval graph if and only if G is chordal and its complement* $\bar{G}$ *is a comparability graph.*

A maximal clique of a graph is a complete induced subgraph which is not contained in any larger such subgraph. For a graph G, its vertex-clique incident matrix $M = (m_{ij})$ is the matrix whose rows and columns correspond to the vertices and the maximal cliques respectively of the graph and

$$m_{ij} = \begin{cases} 1 \ , & \text{if the ith vertex belongs to jth clique} \\ 0 \ , & \text{otherwise} \end{cases}$$

A matrix is said to have consecutive one's property for rows if there is a permutation of the columns so that the 1's in each row appear consecutively.

Theorem 1.3 [Fulkerson and Gross; 1965]. *A graph G is an interval graph if and only if its vertex-clique incident matrix has a consecutive one's property for rows*

Booth and Leuker [1976] used the vertex-clique characterization to obtain a linear time algorithm for interval graph recognition. See Golumbic [1980] for an excellent exposition of the algorithm.[Simon, 1991], [Hsu, 1993] and [Corneil, Olariu, and Stewart, Proc. Ninth ACM-SIAM (SODA), 1998] contain more recent recognition algorithms.

Scheinerman [1988] in a fundamental paper introduced two equivalent models for random interval graphs. Several results about number of edges, degrees, chromatic number and other indices of almost all interval graphs were also established in that paper.

1.5 Indifference graphs

Motivated by the theory of preference and indifference in economics and psychology, indifference graphs were introduced and studied by Roberts [1969]. An undirected graph G is an *indifference graph* if given $\delta > 0$ there exists a real-valued function f on the vertices of G such that vertices u, v are adjacent if and only if $|f(u) - f(v)| \leq \delta$. Then f is an *indifference representation* of G. Roberts characterized indifference graphs and proved that they are equivalent to proper interval graphs and to unit interval graphs. A *proper interval graph* is the intersection graph of intervals in which no interval properly contains other. A *unit interval graph* is the intersection graph of intervals of unit length.

Theorem 1.4 [Roberts, 1969]. *For a graph G, the following conditions are equivalent* :

i) *G is an indifference graph ;*

ii) *G is a unit interval graph ;*

iii) *G is a proper interval graph ;*

iv) *G is an interval graph and does not contain* $K_{1,3}$ *as an induced subgraph.*

1.6 Containment Graphs and overlap Graphs

These two classes of graphs are generated by the models related to intersection. Given a collection of intervals on a real line, each pair of intervals will satisfy exactly one of the following properties, concerning the question of their intersection :

Overlap : the two intervals intersect but neither properly contains the other.

Containment : one of the two intervals properly contains the other.

Disjointness : the two intervals have empty intersection.

A graph G is an *overlap graph* if its vertices can be put into one-to-one correspondence with a collection of real intervals such that two vertices are adjacent in G if and only if their corresponding intervals overlap. The concept of overlap graph were introduced by Even and Itai [971] and were studied by Gavril [1973], Fournier [1978] and Buckingram [1980]. It turns out, nevertheless, that this class of graphs is exactly the same as the class of *circle graphs*, the intersection graph of a finite collection of chords on a circle.

Similarly, in a *containment graph* edges correspond to containment of intervals $I_v \subset I_u$ *or* $I_u \subset I_v$. Containment graphs were studied by Golumbic [1984] and also by Golumbic and Scheinerman [1985]. In [1985] they defined containment graphs in terms of partially ordered sets and showed that the class of containment graph is equivalent to the class of all comparability graphs.

1.7 Ferrers digraphs and Ferrers dimension

Ferrers digraphs play an important role in our study. This special class of digraphs was introduced independently by Guttman [1944] and Riguet [1951]. Riguet defined a *Ferrers digraph* to be a digraph $D(V, E)$ in which for all x, y, z and $t \in V$; $xy, zt \in E \Rightarrow xt \in E$ *or* $zy \in E$ (inclusive) (vertices need not be disjoint). Riguet characterized Ferrers digraphs as those whose successor sets or predecessor sets are linearly ordered by inclusion. This condition is equivalent to the transformability of the adjacency matrix $A(D)$ by independent row and column permutations to a 0, 1-matrix in which the 1's (or 0's) are clustered in the lower left in the shape of a Ferrers diagram (see Fig. 1.2).

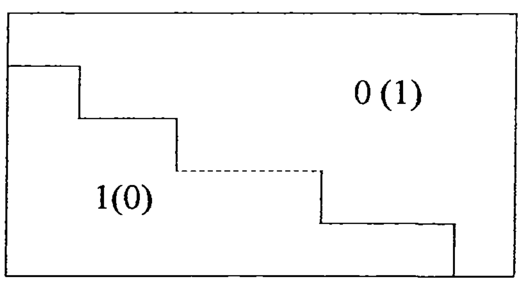


Fig. 1.2 Ferrers digraph

Riguet also showed that the above conditions are equivalent to the condition that adjacency matrix of the digraph has no 2 by 2 submatrix that is a permutation matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Such a forbidden matrix is called an *obstruction* and the two zeros are said to form a *couple* [Sen, Das, Roy and West 1989a]

Intersection of Ferrers digraphs was studied by Bouchet [1971] and he showed that any digraph *D* is the intersection of a family of Ferrers digraphs containing *D*. This induces one to introduce the important notion of Ferrers dimension. The *Ferrers dimension f(D)* of a digraph *D* is the minimum number of Ferrers digraphs whose intersection is *D*. Bouchet [1971, 1984] also obtained several interesting result on Ferrers dimension.

The digraphs with Ferrers dimension 2 have been characterized independently by Cogis [1979] and Doignon, Ducamp and Falmagne [1984] in different contexts. In order to characterize a digraph *D* with $f(D)=2$, Cogis associated an undirected graph *H(D)* with *D*, whose vertices correspond to the 0's of the adjacency matrix of *D*, and two such vertices joined by an edge if the corresponding 0's belong to an obstruction. He proved that *D* has Ferrers dimension at most 2 if and only if *H(D)* is bipartite. This characterization yields a polynomial time algorithm for testing if a digraph has Ferrers dimension at most 2. Sen *et al.*, [1989a] translated the Cogis condition to an adjacency matrix condition and proved the following theorem.

Theorem 1.5 [Sen *et al*, 1989a] and [Cogis, 1979]. *For a digraph the following conditions are equivalent*

i) *D has Ferrers dimension at most 2*

ii) *The rows and columns of A(D) can be (independently) permuted so that no 0 has a 1 both below it and to its right.*

iii) *The graph H(D) of couples in D is bipartite.*

In a later work [Sen, Sanyal and West, 1995] introduced containment digraphs and showed that the class of containment digraphs is equivalent to the class of digraphs with Ferrers dimension at most 2.

Threshold graphs, a concept analogous to Ferrers digraphs were introduced by Chvatal and Hammer [1977]. The *threshold dimension* $\theta(G)$ of the graph $G = G(V, E)$ is defined to be the minimum number k of linear inequalities

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_3 + \dots\dots\dots + a_{1n}x_n \leq t_1 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (1) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&a_{k1}x_1 + a_{k2}x_2 + \dots\dots\dots + a_{kn}x_n \leq t_k
\end{aligned}
$$

such the $X \subseteq V(G)$ is a stable set if and only if its characteristic vector $X = \{x_1, x_2, \dots, x_n\}$ satisfies (1)

An undirected graph G whose threshold dimension $\theta(G) \leq 1$ is a *threshold graph*. Equivalently $G(V, E)$ is a threshold graph if there exists a threshold assignment $[a;t]$ consisting of a labeling of the vertices by the non-negative integers and an integer threshold t such that

$$X \text{ is stable } \Leftrightarrow \sum_{x \in X} a(x) \leq t \qquad (X \subseteq V(G))$$

An excellent survey of these two topics of Ferrers digraphs and threshold graphs can be found in [Cogis, 1982a]. Though these two concepts look completely different, still they are related to each other in a peculiar manner. Cogis deduced a set of known characterization of Ferrers digraphs by investigating the connection between symmetric Ferrers digraphs and threshold graph. Then in the same paper he brought about precisely the connection between symmetric Ferrers digraphs and threshold graphs. A broad overview of this topic is to be found in [Golumbic, 1980]. For a very nice and comprehensive study see [Mahadev and Peled, 1995].

The notion of dimension of a partial order was introduced by Dushnik and Miller [1941]; it is the minimum number of linear orders whose intersection is the given partial order. Bouchet showed (by proving that linear orders can equivalently be changed into Ferrers digraphs) that Ferrers dimension of a digraph is an extension of order dimension of a partial order. [Cogis, 1982 b] gave an alternative proof of this result for finite case and showed that the problem of finding the Ferrers dimension of a digraph and the problem of finding the order dimension of a partial order are polynomially equivalent. [Yannakakis, 1982] showed that the 2-dimensional posets are polynomially recognizable but the problem of designing efficient algorithms for order dimension exceeding 2 is NP-complete.

A very good summary of the notion of order dimension and analogous parameters for graphs and digraphs such as boxicity, threshold dimension and Ferrers dimension to name only a few is given in a review paper by West [1985].

1.8 Chordal bipartite graphs

The notion of chordal bipartite graph was introduced and studied by Golumbic and Goss [1978]. For an excellent survey of this topic see Golumbic [1980]. A bipartite graph $H = H(X, Y, E)$ in which every cycle of length strictly greater than 4 has a chord is a *chordal bipartite graph or bichordal graph.*

A pair of edges x_1y_1 and x_2y_2 of H is separable if there exists a set S of vertices whose removal from H causes x_1y_1 and x_2y_2 to lie in distinct connected component of H\S. The set S is an *edge separator* for x_1y_1 and x_2y_2; S is *minimal* if no proper subset of S is an edge separator for x_1y_1 and x_2y_2, with this concept the following theorem characterizes a bichordal graph.

Theorem 1.6 [Golumbic and Goss, 1978]. *A bipartite graph $H = H(X, Y, E)$ is chordal bipartite if and only if every minimal edge separator induces a complete bipartite graph.*

It can be observed that every vertex in the cycle C_{2n} $(n \geq 3)$ is the centre of a chordless path with five vertices (P_5). With this results in mind Hammer *et al.* [1989] gave a more elegant characterization of chordal bipartite graph.

Theorem 1.7 [Hammer, Maffray and Preissmann, 1989]. *Let H be a bipartite graph. Then H is chordal bipartite if and only if every induced subgraph has a vertex which is not the centre of a P_5.*

The notion of bisimplicial edge was also introduced by Golumbic and Goss [1978], and was motivated by the study of a Gaussian elimination in (0, 1)-matrices. An edge xy of a bipartite graph $H = H(X, Y, E)$ is called bisimplicial if every neighbour of x is adjacent to every neighbour of y. They consequently introduced *bipartite edge elimination scheme*, which consists in successively deleting pairs of vertices which form a bisimplicial edge until the remaining graph has no more edge. In relation to the chordal bipartite graphs they proved that every chordal bipartite graph has a bipartite edge elimination scheme but the converse is not true. Golumbic [1980] gave an example of bipartite graph which has bipartite edge elimination scheme but is not bichordal. This means that existence of bipartite edge elimination scheme is not a characteristic property of chordal bipartite graph.

Hammer *et al.* [1989] proved that every chordal bipartite graph has a vertex which is not the centre of a P_5. From this result they derived a bipartite vertex elimination scheme, which consists, in successively deleting vertices which are not the centre of a P_5 until there is no vertex. They proved that the existence of a bipartite vertex elimination scheme is a characteristic property of a chordal bipartite graph.

Perfect edge elimination ordering (a related but different notion of "bipartite edge elimination scheme") were introduced (with different names) in [Brandstadt, 1993] and [Bakonyi and Bono, 1997]. An ordering $\{e_1, \ldots, e_m\}$ of all the edges of G is *a perfect edge elimination ordering* of G if, for each $i \in \{1, \ldots, m\}$, e_i is a bisimplicial edge of the spanning subgraph of G having edge set $e_i, \ldots, e_m$. Müller [1997] and Kloks and Kratsch [1995] discuss algorithmic aspects of perfect edge elimination ordering and recognition of chordal bipartite graph.

Observe that the complement of a chordal graph can not contain an induced cycle of length greater than four. This motivates the following definition. A graph is *weakly chordal* (very often called *weakly triangulated*) if neither it nor its complement contains an induced cycle of length greater than four. Thus every chordal graph is weakly chordal. For characterization and further result on weakly chordal graph see [Sprindal and Sritharan, 1995] and [Hayward, 1996].

A chordal graph is *strongly chordal* if it has the additional property that every cycle of even length at least six has a chord that divides C into two odd length paths. Strongly chordal graphs form an intermediate family between the families of interval graphs and chordal graphs. They have been particularly important because certain graph theoretical problems have efficient computational solutions for sub-families of the family of strongly chordal graph. For structural properties of these graphs, see the fundamental paper by Faber [1983]. Roychauduri [1988] gave an algorithm for the intersection number of strongly chordal graph.

1.9 Intersection digraphs

The concept of intersection digraph was introduced independently in different contents by Beineke and Zamfirescu [1982] and Sen *et al.* [1989a]. For digraphs, the distinction between heads and tails of edges are crucial. To capture this, they assign to each vertex v of the digraph $D(V, E)$ a source set S_v and a sink set T_v. By analogy with undirected graph, a collection of ordered pairs $\{(S_v, T_v) : v \in V\}$ is an intersection representation of D when $uv \in E$ if and only if $S_u \cap T_v \neq \varphi$. Similar to the case of graphs, any digraph is an intersection digraph of some family of ordered pair of sets. This can be seen by taking the source set S_v of the vertex v to be the set of edges with v as source and the terminal set T_v to be the set of the edges with v as terminus. Naturally the problem is posed to minimize the number of elements in a universal set which will determine the intersection representation of the digraph. The intersection number $i(G)$ of an undirected graph G is the minimum size of a set U such that G is the intersection graph of the subsets of U. Erdos, Goodman and Posa [1966] showed that the intersection number of G equals the minimum number of complete subgraph needed to cover its edges. They also proved that $i(G) \leq \lfloor n^2/4 \rfloor$ for an n-vertex graph, the equality being achieved by the bipartite graph $K_{p,q}$, where $p = \lfloor n/2 \rfloor$ and $q = \lceil n/2 \rceil$. More results on this topic can be found in West [1996].

To develop analogous result for digraphs, Sen *et al.* [1989a] defined a generalized complete bipartite subdigraph (abbreviated GBS) to be a sub-digraph generated by the vertex sets X, Y, whose edges are all xy such that $x \in X$ and $y \in Y$. They said "generalized" because X, Y need not be disjoint, which means that loops may arise. The intersection number $i(D)$ of a digraph is the minimum size of U such that D is the intersection digraph of ordered pair of subsets of U. The analogue of the Erdos-Goodman-Posa result is follows :

Theorem 1.8 [Sen *et al.*,1989a]. *The intersection number of a digraph equals the minimum number of GBS's required to cover its edges and the best possible upper bound on this is n for an n-vertex digraph.*

Harary, Kabell and McMorris [1982] introduced the idea of bipartite intersection graph or intersection bigraph. Let $H = H(X, Y, E)$ be a bipartite graph with the bipartite vertex sets X and Y. If two families of sets $\{S_x; x \in X\}$ and $\{T_y : y \in Y\}$ be assigned to the vertices of X and Y respectively such that $xy \in E$ iff $S_x \cap T_y \neq \varphi$, then $H = H(X, Y, E)$ is the *intersection bigraph* of the pair of families $\{S_x\}$ and $\{T_x\}$. The two concepts of intersection bigraphs and intersection digraphs introduced independently are essentially the same; this is explained in the following manner :

Let $D(V, E)$ be an intersection digraph with the representation $\{(S_v, T_v) : v \in V\}$ Let $V=\{v_1, v_2, \ldots, v_n\}$ and let $X=\{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two copies of V. Now consider a bipartite graph $H = H(X, Y, E)$ such that two vertices $x_i \in X$ and $y_j \in Y$ have an edge in H iff $v_i v_j \in E$ in D. The bipartite graph so constructed is the bipartite representation of the digraph D and is denoted by $B(D)$. It is now obvious that a digraph $D(V, E)$ is an intersection digraph of a family $\{(S_v, T_v) : v \in V\}$ iff the bipartite graph $B(D)$ is an intersection bigraph of the pair of families $\{S_x : x \in X\}$ and $\{T_y : y \in Y\}$. On the other hand $H(X, Y, E)$ is an intersection digraph iff the directed graph $D(X \cup Y, E)$, obtained from H by directing all the edges from X to Y is an intersection digraph.

Thus to study intersection bigraphs is essentially the same as to study intersection digraphs. Note that the adjacency matrix of D is the biadjacency matrix of $B(D)$ with rows and columns arranged in the obvious way and determines the bigraph completely.

1.9.1 Interval digraphs and their characterizations

By analogy with interval graphs, a digraph is an interval digraph if it has an intersection representation in which every source set and sink set is an interval on the real line. Sen, Sanyal and West [1995] proved that interval digraph is a generalized concept of interval graph. In particular, they proved the following result.

Theorem 1.9 [Sen, Sanyal and West, 1995]. *An undirected graph G is an interval graph if and only if the corresponding symmetric digraph D(G) with loops at every vertex is an interval digraph.*

Necessity of the above result is trivial. If G is an interval graph with interval I_v assigned to the vertex v, then by setting $S_v=T_v=I_v$ yields an intersection representation of $D(G)$. Conversly if $\{(S_v, T_v) : v \in V(G)\}$ is an interval representation of $D(G)$, where $S_v = [a_v, b_v]$ and $T_v = [c_v, d_v]$ then they showed that $[a_v + c_v, b_v + d_v]$ yields an interval representation of G. Sen, West *et al.* [1989a, 1989b, 1996] gave several characterizations of interval digraphs. In [Sen *et al.*, 1989a] a characterization of interval digraph is obtained which is analogous to the Fulkerson and Gross characterization of interval graph. To state the result let $B\{(X_k, Y_k)\}$be a collection of GBS's whose union is the digraph D. Then define the vertex-source incident matrix for B (abbreviated V, X-matrix) to be the incident matrix between the vertices and the source sets $\{X_k\}$. Similarly, the vertex-terminal incident matrix for B (abbreviated V, Y-matrix) is the incident matrix between the vertices and the terminal sets $\{Y_k\}$. Then the following theorem gives the first characterization of interval digraphs.

Theorem 1.10 [Sen *et al.*, 1989a]. *D is an interval digraph if and only if there is a numbering of the GBS's in some covering B of D such that 1's in a row appear consecutively for both the V, X-matrix and V, Y-matrix of D.*

But more interesting characterization of interval digraph is its adjacency matrix characterization. For a given digraph D, we use $\bar{D}$ to denote the digraph whose adjacency matrix is the difference between $A(D)$ and the matrix of all ones (i.e., the complement of D). Now we state the first adjacency matrix characterization of interval digraphs.

Theorem 1.11 [Sen *et al.*, 1989a]. *The following conditions are equivalent for a digraph D.*

1) *D is an interval digraph;*

2) $\bar{D}$ *is the union of two disjoint Ferrers digraphs ;*

3) *The rows and columns of the adjacency matrix of D can be permuted independently such that each 0 can be labeled with R or C in such a way that every position to the right of an R is an R and every position below a C is a C.*

A matrix satisfying condition (3) above has the *partitionable zero property*, and such a matrix is a *zero partitionable matrix*.

From the definition of Ferrers digraphs, it is obvious that the complement of any Ferrers digraph is also a Ferrers digraph. Hence Ferrers dimension of *D* also equals the minimum number of Ferrers digraphs whose union is $\bar{D}$. From the above theorem, it is clear that an interval digraph is of Ferrers dimension at most 2. Naturally, the question arises about the converse of the statement. Sen *et al.* [1989a] gave an example of a seven vertex digraph which is of Ferrers dimension 2 but not an interval digraph. In the next chapter we have shown that the smallest such digraph has 6-vertex. Thus the class of interval digraphs form a proper subclass of the class of digraphs of Ferrers dimension at most 2.

Sen, Sanyal and West [1995] gave a less obvious characterization of Ferrers digraph using the existence of biorder representation : *D* is a Ferrers digraph if and only if there exist two real valued function *f, g* on *V(D)* such that $uv \in E$ if and only if $f(u) \geq g(v)$. Recently, West [1998] has used this biorder representation of Ferrers digraphs to give a short proof of the above characterization of interval digraphs.

An alternative way of describing the *(R, C)* partition of a binary matrix is in terms of the generalized linear ones (glo) property [Sen *et al.*, 1989b]. To describe this property we need an additional concept. A *stair partition* of a matrix is a partition of its position into two subsets *(L, U)* by a polygonal path from the upper left to the lower right such the set *L* is closed under leftward or downward movement and the set *U* is closed under rightward or upward movement. Equivalently, U corresponds to the positions in some upper triangular matrix and L to the positions in the lower triangular matrix (see Fig 1.3).

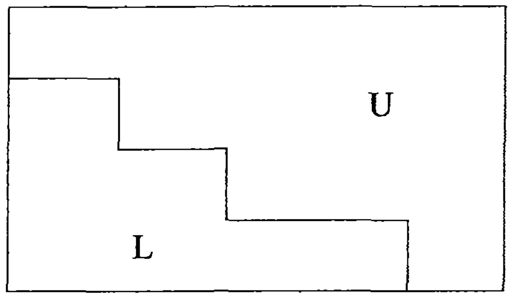


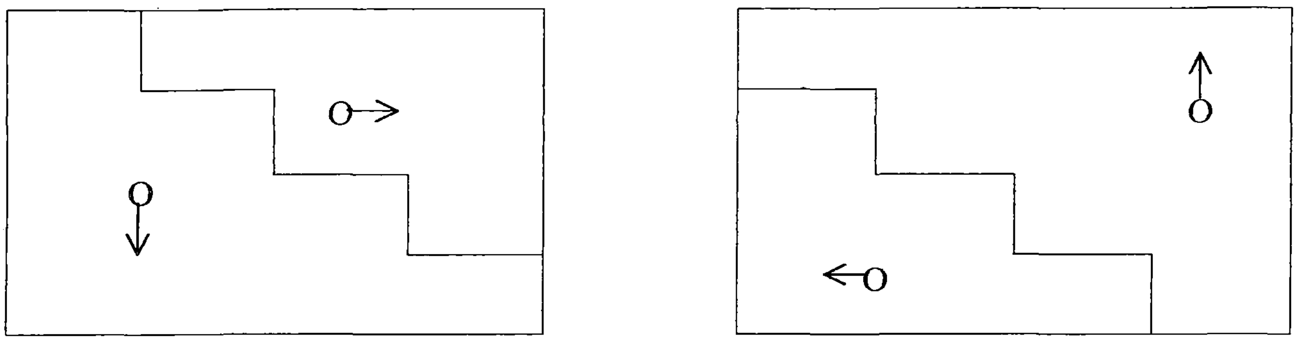
Fig. 1.3 Stair Partition

A 0, 1-matrix has the first generalized linear ones property (glop I) if it has a stair partition (L, U) such that the 1's in U are consecutive and appear leftmost in each row and the 1's in L are consecutive and appear topmost in each column. Similarly a 0, 1-matrix has the second generalized linear ones property (glop II) if it has a stair partition (L, U) such that 1's in U are consecutive and appear down most in each column and 1's in L are consecutive and appear rightmost in each row. This is illustrated in Figure 1.4. It is not difficult to see that two notions of glop I and glop II are equivalent and also that they are equivalent to the idea of (R, C) partition of a (0, 1) –matrix. So as a consequence of the previous theorem, one obtains :

Corollary 1.1 [M. Sen, P. Talukder, S. Das; Preprint]. *For a digraph D the following are equivalent :*

i) *D is an interval digraph.*

ii) *The adjacency matrix A(D) of D has the generalized linear one's property of either first kind or second kind.*



glop I glop II

Fig. 1.4 Generalized linear ones property.

Most characterization of interval graphs and interval digraphs so far involve an order of their vertices. Sanyal and Sen [1996] posed the question, "Is there any ordering among the edges of a (di)graph that characterizes an interval (di)graph?" And they answered this question in the affirmative. For this, they introduced the notion of a consistent ordering of the edges of a (di)graph.

The set of all edges of a digraph $D(V, E)$ is said to have a *consistent ordering* if E has a linear ordering (<) such that for $pq, pu, tq \in E$.

i) $pq < rs < pu \Rightarrow ps \in E$ $(q \neq u)$

ii) $pq < rs < tq \Rightarrow rq \in E$ $(p \neq t)$

Theorem 1.12 [Sanyal and Sen, 1996]. *A digraph D(V, E) is an interval digraph if and only if its edge set has a consistent ordering*

Then by appropriate changes in the definition of consistent ordering, they obtained an analogous result for interval graphs. The set E of all edges of a graph $G(V, E)$ has a consistent ordering if E has a linear ordering (<) such that for $pq, rs, pu, tq \in E$

$$pq < rs < pu \Rightarrow ps \text{ and } pr \in E.$$

Theorem 1.13 [Sanyal and Sen, 1996]. *A graph G(V, E) is an interval graph if and only if its edge set E has a consistent ordering.*

1.9.2 Subclasses of Interval digraphs

We recall that Roberts [1969] introduced indifference graphs and proved that they are equivalent to the unit interval graphs and proper interval graphs.

Sen and Sanyal [1994] generalized the above mentioned undirected graph families by placing analogous constraintson source and sink intervals for interval digraphs. *Unit interval digraphs* are interval digraphs with interval representations such that all the source and sink intervals have unit length. *Proper interval digraphs* are interval digraphs with representation such that no source interval properly contains other sink interval. *Indifference digraphs* are those for which there exists an ordered pair of real valued functions f, g on the vertices such that uv is an edge if and only if $|f(u) - g(v)| \leq 1$.

Sen and Sanyal (1994) characterized the graphs in these families and proved that the families are equivalent, generalizing the results of Roberts (1969, 1976). These characterizations are generalizations of those for undirected graphs when an undirected graph is viewed as a symmetric digraph with loops. The adjacency matrix of the corresponding digraph is obtained by adding 1's on the diagonal; this is called the *augmented adjacency matrix* $A^*(G)$ for an undirected graph G. A symmetric digraph with loops has an indifference representation with $f=g$, because the symmetry implies that averaging f and g will not change the resulting edges. Conversely, every indifference representation with $f=g$ yields a symmetric digraph with loops. This establishes a bijection between indifference graphs and indifference digraphs representable using $f=g$. It therefore also establishes a bijection between unit interval graph and unit interval digraphs representable by giving every vertex the same source and sink interval.

Sen *et al.* [1989a] characterized the adjacency matrix of an interval digraph in terms of *partitionable zeros property* (described earlier). Since unit interval digraphs is a subclass of interval digraphs, it is hoped that the adjacency matrix characterization for unit interval digraphs is a more restrictive version of the partitionable zeros property. *A 0,* 1-matrix has a *monotone consecutive arrangement* if there exist independent row and column permutations exhibiting the following structure : the 0's of the resulting matrix can be labeled *R* or *C* such that every position above or to the right of an *R* is an *R*, and every position below or to the left of a *C* is a *C*. The resulting matrix and the labeling is a monotone consecutive arrangement, abbreviated *MCA* (see Fig. 1.5).

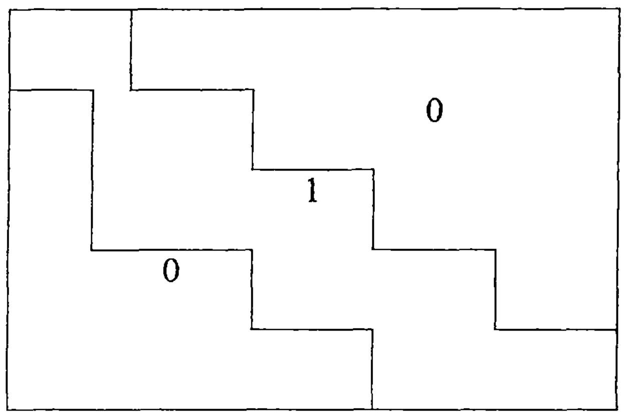


Fig. 1.5 Monotone consecutive arrangement (MCA)

Below we state the characterization of the different digraph classes by Sen and Sanyal [1994].

Theorem 1.14 [Sen and Sanyal, 1994]. *For a digraph D, the following are equivalent.*

1) D is a unit interval digraph;

2) D is a proper interval digraph;

3) D is an indifference digraph;

4) The adjacency matrix of D has monotone consecutive arrangement

Recently, West [1998] has given a short inductive proof of the equivalence of (3) and (4)

Sen and Sanyal [1994, 1996] gave several other characterizations, but they did not give a forbidden submatrix characterization for adjacency matrix of unit interval digraphs. Motivated by Roberts [1969] that an interval graph is an unit interval graph if and only if it does not contain the bipartite graph $K_{1,3}$ as an induced subgraph, Lin, Sen and West [1995, 1997] characterized the interval digraph that are unit interval digraphs. Particularly, they prove the following :

Theorem 1.15 [Lin & West; 1995 and Lin, Sen & West; 1997]. *A zero partitionable matrix has a MCA if and only if it does not contain any of the three 3 by 4 matrices listed below and their transposes.*

$$F_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Steiner [1996] gave a nice algorithm to recognize an indifference digraph (bigraph) in linear time. His recognition algrorithim uses permutation graph. A graph $G(V, E)$ on $V = \{v_1, v_2, \ldots, v_n\}$ is a permutation graph if there is a labeling $\ell: V \to \{1,2, \ldots, n\}$ and a permutation π of $\{1, 2, \ldots, n\}$ such that $v_iv_j \in E$ if an only if $\ell(v_i) < \ell(v_j)$ and $\ell(v_j)$ precedes $\ell(v_i)$ in π. The function ℓ is called the permutation labeling of G and π is the defining permutation.

Theorem 1.16 [Steiner, 1996]. *A digraph $D(V, E)$ is an indifference digraph if and only if its bipartite graph $B(D)$ is a permutation graph*

Since a bipartite permutation graph can be recognized in linear time [Spinrad, Brandstadt; 1987], he concluded that :

Theorem 1.17 [Steiner, 1996]. *An indifference digraph can be recognized in linear time.*

As a subclass of interval digraphs, *interval nest digraphs* were introduced by Prisner (1994). It has a representation $\{(S_v, T_v): v \in V\}$ where each T_v is contained in S_v. He gave some application of interval nest digraph model to the real world situation. In the same paper, he had shown that some parameters for interval nest digraphs can be computed in polynomial time.

Prisner had also studied reflexive interval digraphs. Using the adjacency matrix characterization of interval digraph (Theorem 1.11) it easily follows that a digraph is a reflexive interval digraph iff its complement is (edge) disjoint union of two loopless Ferrers digraphs. He also showed that loopless Ferrers digraphs are precisely interval order. A poset $P(V,<)$ is an interval order if there is some family of intervals $\{I_v : v \in V\}$ such that $x<y$ if and only if I_x lies completely to the left of I_y.

Theorem 1.18 [Prinser, Preprint]. *A digraph D is a reflexive interval digraph if and only if D is the disjoint union of two interval orders.*

The underlying graph $U(D)$ of a digraph $D(V,E)$ has V as vertex set and two vertices x and y are adjacent in $U(D)$ whenever $xy \in E$ or $yx \in E$. The characterization of an interval digraph that it is the intersection of two Ferrers digraphs whose union is complete has no analogue for reflexive interval digraphs, because the complement of an interval order is no longer an interval order. Nevertheless, the underlying graphs of reflexive interval digraphs can be expressed by the intersection of two interval orders.

A *trapezoid graph* is the intersection graph of a family of trapezoids whose parallel sides all lie on a single pair of parallel lines. The complements of trapezoid graphs are exactly the comparability graphs of posets of interval dimension at most 2, that is intersection of two interval orders. So we have the following theorem due to Prisner.

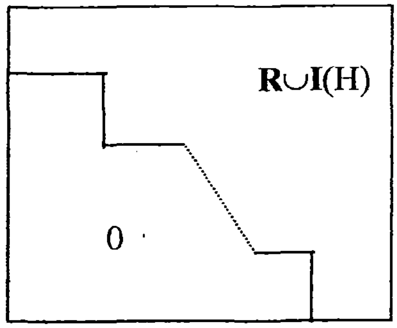
Theorem 1.19 [Prisner, Preprint]. *The underlying graph of a reflexive interval graph is a trapezoid graph.*

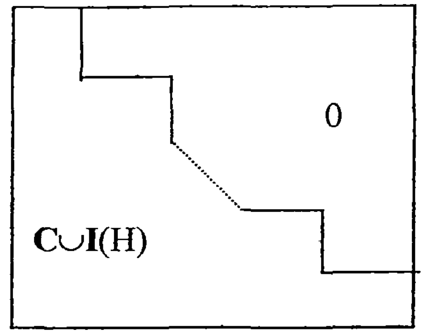
Since trapezoid graphs are weakly triangulated [Dagman, Golumbic, and Pinter; 1988] it follows that the underlying graph of a reflexive interval graph is weakly triangulated. Prisner then uses the result to compute several parameters of this digraph more efficiently.

1.9.3 Interior edges

The notion of interior edges was introduced by Das and Sen [1993]. In may be recalled from section 1.3 that to characterize a digraph D of $f(D) \leq 2$, Cogis [1982] associated an undirected graph $H(D)$ with D and proved that D is of $f(D) \leq 2$ if and only if $H(D)$ is bipartite. Also it is known that for a given digraph D with $f(D) = 2$, the complement $\bar{D}$ is the union of two Ferrers digraphs (not necessarily disjoint). These two Ferrers digraphs are realizations of $\bar{D}$. And obviously realizations of $\bar{D}$ is not unique. Now we note that $H(D)$ may have more than one connected components; besides it may have one or more isolated vertices (corresponding to the 0's of $A(D)$ which do not belong to any obstruction). The graph obtained by deleting the isolated vertices from H(D) denoted by $H_b(D)$, is the *bare* graph associated with D [Doignon *et al.*, 1984]; and the set of all isolated vertices is denoted by $\boldsymbol{I(H)}$ or $\boldsymbol{I}$.

For a digraph D with $f(D) = 2$ any bicolouration of $H_b(D)$ does not lead to a covering of $\bar{D}$ by two Ferrers digraphs. But to prove the characterization of a digraph of $f(D)$ at most 2, Cogis adopted a constructive method to show that there always exists a suitable bicoluration of $H_b(D)$ that yields a realization of $\bar{D}$ as the union of two Ferrers digraphs. As a matter of fact, he obtained the particular bicoluration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ in such a way that adjoining all the edges of $\boldsymbol{I(H)}$ to each $\boldsymbol{R}$ and $\boldsymbol{C}$ yielded the required Ferrers digraphs realization G_1 and G_2 so that $\bar{D} = G_1 \cup G_2$, $G_1 = \boldsymbol{R} \cup \boldsymbol{I(H)}$ and $G_2 = \boldsymbol{C} \cup \boldsymbol{I(H)}$. Such a bicoluration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ was termed a satisfactory bicoluration by Das and Sen [1993]. Let $(\boldsymbol{R}, \boldsymbol{C})$ be a satisfactory bicolouration of $H_b(D)$ leading to a realization of $\bar{D} = G_1(V, E_1) \cup G_2(V, E_2)$ where $E_1 = \boldsymbol{R} \cup \boldsymbol{I(H)}$ and $E_2 = \boldsymbol{C} \cup \boldsymbol{I(H)}$. Let the rows and columns of $A(G_1)$ be so arranged that all the ones are clustered in the upper right. Similarly, the rows and columns of $A(G_2)$ are so arranged that all the ones are clustered in the lower left.

$A(G_1)$ = 

$A(G_2)$ = 

Then an edge corresponding to an $I \in G_1$ is said to be an *interior edge* of G_1, denoted by I_r, if there exists a configuration of the form

$$\begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$$

in $A(G_1)$; Similarly, an $I \in G_2$ is said to be an *interrior edge* of G_2, denoted by I_c, if there exists a configuration of the form

$$\begin{pmatrix} C & 0 \\ I & C \end{pmatrix}$$

in $A(G_2)$.The set of all interior edges of G_1 is denoted by $\boldsymbol{I_r}$ (G_1) or $\boldsymbol{I_r}$, and the set of all interior edges of G_2 is denoted by $\boldsymbol{I_c}$. Note that the sets $\boldsymbol{I_r}$ and $\boldsymbol{I_c}$ are identified with reference to a particular realization of $\bar{D}$ and will change if the realization changes.

With this notion of interior edges, Das and Sen [1993] proved that if a digraph D of $f(D) = 2$ is an interval digraph then for any satisfactory bicolouration of $H_b(D)$, $\boldsymbol{I_r} \cap \boldsymbol{I_c} = \varphi$.

But the converse is not true. They gave an example of an eight vertex digraph with $f(D) = 2$ which is not an interval digraph and for which $\boldsymbol{I_r} \cap \boldsymbol{I_c} = \varphi$. In chapter 2 of the present thesis, we continue from their work, and have been able to solve the problem of finding all possible configuration of a digraph having $f(D)=2$ but which is not an interval digraph. Incidentally, this leads us to a recognition algorithm of an interval digraph in a more efficient way.

1.9.4 Recognition Algorithm

Recognition of interval digraphs is a major open problem is this area. Recently Müller [1997] has found a polynomial algorithm to recognize an interval digraph. He has used a dynamic programming method to recognize an interval bigraph and accordingly interval digraph. A similar approach was used earlier in [Bodlaender, Kloks and Kratsch; 1995] and [Deogun, Kloks and Kratsch and Müller; 1994] to compute the tree-width and the vertex ranking number respectively, of permutation graphs. To describe the method we require a number of new ideas and definition.

Analogous to the result on interval graph that an interval graph is chordal, it can be easily seen that an interval bigraph is bichordal. For pair a, b of non-adjacent vertices of $G(V,E)$, a set $S \subset V$ is a *minimal a-b separator* if

i) a and b belong to different connected components of the subgraph $G \setminus S$, and

ii) no proper subset of S has the above property.

A minimal a-b separator is also called a minimal separator. Muller has then used the notion of a complete separator and introduced an *anchored segment*. A set of vertices S is a *complete separator* if

i) either $S = \varphi$ or S is a separator such that the subgraph $G(S)$ induced by S is a complete bipartite graph and

ii) S is either a minimal separator or at most the union of two minimal separators of G.

Let now L and R be two complete separators of G. Then clearly $G\setminus(L \cup R)$ is a disconnected graph. If C is a connected component of $G\setminus L \cup R$, then the subgraph H' induced by the vertices of C and the vertices in $L \cup R$ is called a segment of G and the triples (L, H', R) *anchored segment* of G. The algorithm to recognize an interval bigraph is briefly described below.

Assume that a chordal bipartite graph is given. Find out all the complete separators and the edges of G.

If a segment forms a complete bipartite subgraph of G, then the unique, short interval representation of this segment is just a point only. Müller defined length of a bipartite interval representation of an interval bigraph as the sum of the lengths of its intervals.

Then he called an interval representation *short* if all the intervals are closed, have integer end points and have minimum length. Müller showed how to combine two or more segments with short interval representation to obtain a large one and this is the most crucial step in the algorithm. It turns out that the important parts of the bipartite interval representation are the complete bipartite subgraphs bounding it on both sides. If for a subgraph H' the anchored segment (L, H', R) is not unique, the different bipartite interval representation of H exist. An anchored segment obtained by suitably enlarging two GBS's of a segment is a *realization*. In the same paper he has shown that a chordal bipartite graph is an interval bigraph iff the anchored segment (φ, G, φ) is realizable. He had also shown that the algorithm takes time of order $O(nm^6(m+n)\log n)$, where n is the number of vertices and m is the number of edges of the graph.

In the present thesis we have obtained a recognition algorithm from a different viewpoint. An interval digraph is first characterized in terms of interior edges with reference to a bicolouration of its associated bipartite graph $H(D)$. From this, a complete list of forbidden configurations of an interval digraph is obtained. Then this characterization is applied to obtain a recognition algorithm of an interval digraph in time $O(n^3)$.

1.10 Containment digraphs and overlap digraphs

Sen, Sanyal and West [1995] extended the containment and overlap model to representations of digraphs. The containment digraph of a family $\mathcal{F} = \{(S_v, T_v) : v \in V\}$ is the digraph with vertex set V in which there is an edge from u to v if and only if S_u property contains T_v. It is easy to represent any digraph as a containment digraph; given vertex set $V = \{v_i\}$, let $T_{v_i} = \{i\}$, and let S_{v_i} be the element - i together with $\{j : v_j \in N^+(v_i)\}$, where $N^+(v_i)$ is the set of successor (out-neighbour) of v_i. Naturally, one has to restrict the pair of sets in $\mathfrak{I}$ to obtain interesting class of digraphs. If $\mathfrak{I}$ is a family of order pairs of intervals, then the resulting containment digraph is an *interval containment digraph*. The characterization of interval containment digraph uses Ferrers digraphs.

Just as interval digraphs are closely related to interval graphs, so interval containment digraphs are closely related to interval containment posets. A containment representation of a poset assigns to each element $x \in p$ a set S_x such that $x < y$ it and only if $S_x \subset S_y$. It is well known that interval containment posets are precisely the posets of dimension 2 [Dushnik and Miller, 1941] and [Madej and West, 1991]. Furthermore, the Ferrers dimension of the comparability digraph of a poset (a digraph that is irreflexive, and transitive) equals the order dimension the poset [Bouchet, 1971 and Doignon *et al.*, 1984]. Hence it is not surprising that the interval containment digraphs are precisely the digraphs of Ferrers dimension 2, which was proved in [Sen, Sanyal & West, 1995].

To study overlap digraphs represented by intervals they [1995] have used a more restrictive model than the direct analogue with undirected graphs. *A right overlap digraph* (ROI-digraph) is a digraph represented by a family $\mathfrak{I}$ of ordered pairs of intervals such that there is an edge from u to v if and only if (i) S_u and T_v overlap (no containment) and (ii) inf S_u <inf T_v. To characterize the adjacency matrix of these digraphs they gave the following definition : A 0, 1-matrix has a *P, R-partition* if its rows and columns can be permuted independently so that its 0's can be labeled P or R such that (1) the position to the right *and* the positions above any R are also 0's labeled R, and (2) positions to the left *or* the positions below any P are also 0's labeled P (Fig. 1.6).

As illustrated in the Fig. 1.6, it is easy to see that the R's constitute a Ferrers digraph and the P's constitute the union of two Ferrers digraphs. Hence any digraph whose adjacency matrix has a P-R partition is a digraph of Ferrers dimension at most 3, and they precisely proved that a digraph is a ROI-digraph if and only if its adjacency matrix has a P-R partition. They also observed that the digraphs having right-overlap interval representations in which all the intervals have unit length, are same as the unit interval digraphs or indifference digraphs.

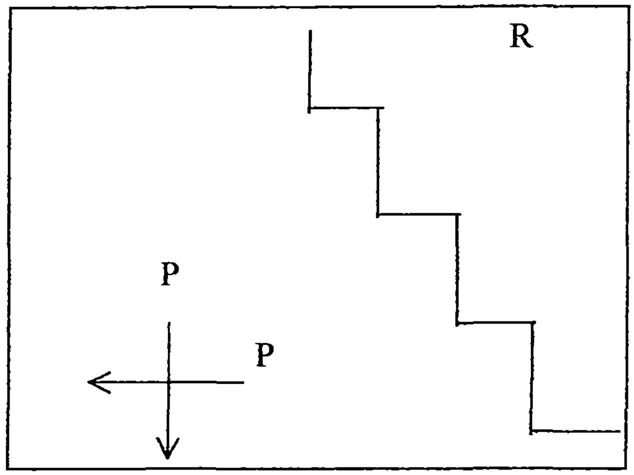


Fig 1.6

In this context we can similarly introduce left overlap interval digraph (LOI-digraph) replacing the condition (ii) of the definition of a ROI by the condition (ii)′ inf T_v < inf S_u. Then we observe that if we take the mirror image of intervals with respect to any point on the real line, an ROI-digraph becomes an LOI-digraph and vice versa. Thus the class of ROI-digraphs coincides with the class of LOI –digraph.

1.11 Robin digraphs

Sanyal [1994] posed the question of characterizing a binary matrix having independent row and column permutations where 0's (in the matrix) are such that a 0 has all positions 0 throughout any one of the four directions, *viz*, left, right, above and below (see fig 1.7). such a matrix is a *4-directable binary matrix*.

Also a square matrix having consecutive 1's property for rows was characterized by Sen *et al.* (1989a) in terms of an interval-point digraph; it is represented by a family $\mathcal{J} = \{(S_v, p_v) : v \in V\}$ of ordered pairs where S_v is a closed interval and p_v is any arbitrary point and $uv \in E$ if and only if $p_v \in S_u$. Such an ordered pair (S_v, p_v) was called a *pointed interval*. Now it can be easily seen that a four directable matrix is the adjacency matrix of the intersection of two digraphs D_1 and D_2, where $A(D_1)$ has consecutive 1's property for rows and $A(D_2)$ has consecutive 1's property for columns. So it is easy to infer that adjacency matrix of a digraph D is four directable iff $D = D_1 \cap D'_2$ where D_1 and D_2 are two interval-point digraphs and D'_2 is the converse of D_2.

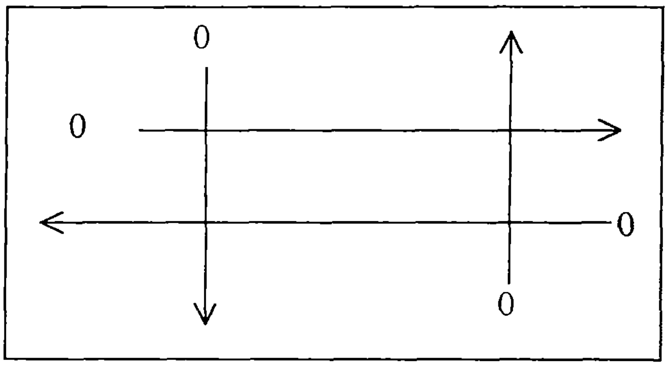


Fig 1.7   4 – directable matrix

To give another characterization of the 4-directable matrix, he considered p_u as a member of the closed interval S_u and called it a base point of S_u. Such an ordered pair (S_v, p_v) is called a *base interval*. Then a generalized version of an overlap digraph was introduced in the following way : Let $\mathfrak{I} = \{((S_v, p_v), (T_v, q_v)) : v \in V\}$ be a family of ordered pairs of base *intervals* (S_v, p_v) and (T_v, q_v) then a digraph $D(V, E)$ is a *right overlap base interval digraph (Robin digraph)* when $uv \in E$ if and only if

i) S_u and T_v overlap (no containment)

ii) Inf $S_u <$ Inf T_v and

iii) $p_u, q_v \in S_u \cap T_v$.

Then he proved that a digraph has a overlap base interval representation (right or left) iff its adjacency matrix has 4-directable property.

The idea of representation of a digraph by the intersection model of the ordered pair of base interval was given by Sanyal [1994]. In the Chapter 4 of the present thesis, we characterize the adjacency matrix of these digraphs (called the *base interval digraphs*) by showing that it is a particular form of 4-directable binary matrix. This is a modified version of the characterization given by Sanyal in his thesis [1994]. Then we characterize a base interval digraph in terms of intersection of two interval digraphs and overlap base interval digraph in terms of intersection of two containment digraphs. Lastly, we consider the particular case when all the intervals of Robin digraph are of unit length and prove that Robin digraphs with unit length interval are precisely the ROI – digraph.

1.12 Miscellaneous

Lin *et al.* [1999] has shown that every digraph is the intersection digraph of a family of pairs of subtrees of a star. For any digraph, we can let each sink set T_v consist of a distinct leaf of the star and let each source set S_v consist of the centre and the leaves assigned to the out-neighbors (i.e. successors) of v. Hence for an arbitrary digraph D, it makes sense to define the parameter $L(D)$, the leafage of D, to be the minimum number of leaves in the host tree in any subtree representation of D. They also proved that the interval digraphs are precisely the digraphs with leafage 2. They also defined catch representation of a digraph D in the following manner : In a subtree representation of a digraph if every sink set is restricted to be a singleton vertex, then this is a *catch representation* of D, and D is a *catch-tree digraph*. The *catch leafage* of a digraph D is the minimum number of leaves in the host tree in any catch representation of D. then they studied upper and lower bounds for the leafage and catch leafage and showed that the bounds are best possible, but can be arbitrarily weak.

It is well known that a representation of an interval digraph is not unique and an interval (di)graph may have many representations differing not only in length, but more importantly, in the relative positions of the intervals. To study the relative positions of the intervals in interval representation of an interval (di)graph is the problem of chronological orderings of an interval (di)graph. Skrien [1980, 1984] studied the problem for undirected graph. In a recent paper Sen *et al.* (Preprint) studied the corresponding problem for directed graph. Actually their results generalize the corresponding results on interval graph by Skrien and describes how, given an interval digraph, the order of intervals of one representation differs from another. In other words it describes the various interrelations between the end points of the intervals for a digraph to be an interval digraph.