CHAPTER-IV

LARGE DEFLECTIONS OF ELLIPTIC PLATES EXHIBITING RECTILINEAR ORTHOTROPY *

INTRODUCTION

Treatments of the large deflections of orthotropic elliptic plates placed on elastic foundations are not found in literature. Only a few treatments of the large deflections of isotropic elliptic plates are found in literature and they are due to Perry [1950], Weil and Newmark [1956] and Nash and Cooley [1959]. These investigators employed various numerical methods for solution of the problem without ascertaining the appropriate stress function.

This paper deals with the large amplitude deflections and induced stresses in a clamped edge thin orthotropic elliptic plate under uniform load and placed on an elastic foundation of the Winkler type. The investigation is based on Von Kármans equations generalised to the rectilinearly orthotropic case. After postulating the shape of the deflection surface, the stress function is found from the compatibility equation. The final solution for deflection is obtained by applying

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Galerkin's method. Results for isotropic elliptic and circular plates have been deduced from those obtained for orthotropic elliptic plates. These results are presented in the form of graphs and compared with other known results. The results obtained for the isotropic case are more accurate than the other known results and are in excellent agreement with the practical values given by Nash and Cooley [1959]. The results obtained for the orthotropic case could not be compared in absence of any known result.

ANALYSIS

Consider a flat elliptic rectilinearly orthotropic elastic plate with major axis 2a and minor axis 2b. Let the origin 0 of a cartesian rectangular co-ordinate system x, y, z be located at the centre of the plate. Consider that the plate is clamped along the boundary and subjected to a uniform load of intensity q. The plate is placed on an elastic foundation of the Winkler type having the foundation reaction k_1 per unit area per unit deflection. Transverse deflections of the plate are considered to be large, that is of the order of magnitude of the plate thickness, h. Let the flexural and torsional rigidities of the plate be denoted by D_1, D_2, and D_3 respectively,

$$D_1 = \frac{E_1 I}{1-\nu_1\nu_2}, \quad D_2 = \frac{E_2 I}{1-\nu_1\nu_2}, \quad D_3 = \frac{1}{2}\left(D_1\nu_2 + D_2\nu_1\right) + 2D_k \qquad \ldots (4.1)$$

where $I = \frac{h^3}{12}$, $D_k = G_{12} I$, E_1 and E_2 denote the Young's moduli in the x and y directions respectively, G_{12} denotes the shear modulus,

ν_1 represents the contraction in the y direction influenced by the tension in the x direction and ν_2 is given by the familiar relation $E_1\nu_2 = E_2\nu_1 = E''$. In what follows the following symbolism will be utilised :

$$k^2 = \frac{D_2}{D_1} = \frac{E_2}{E_1}, \quad l^2 = \frac{D_3}{D_1} = 2\frac{G_{12}}{E_1}\left(1 - k^2\nu_1^2\right), \quad p^2 = \frac{E_2}{G_{12}} - 2\nu_2 \qquad \ldots (4.2)$$

With the above notations, the familiar von Kármán equations generalised to rectilinearly orthotropic case take the following form :

$$\frac{\partial^4 w}{\partial x^4} + 2l^2\frac{\partial^4 w}{\partial x^2 \partial y^2} + k^2\frac{\partial^4 w}{\partial y^4} + \frac{k_1 w}{D_1} = \frac{h}{D_1}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2}\right) \qquad \ldots (4.3)$$

$$\frac{\partial^4 \phi}{\partial x^4} + p^2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + k^2\frac{\partial^4 \phi}{\partial y^4} = E_2\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \qquad \ldots (4.4)$$

where w is the transverse deflection of the middle plane of the plate, ϕ is the stress function by means of which the membrane forces are represented in the customary form :

$$N_x = h\frac{\partial^2 \phi}{\partial y^2}, \quad N_y = h\frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = -h\frac{\partial^2 \phi}{\partial x \partial y}$$

Let the deflection function w (x,y) be taken in the following form satisfying the clamped edge boundary conditions

$$w(x,y) = w_o \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2 \qquad \ldots (4.5)$$

Substitution of Eq.(4.5) into Eq.(4.4) yields

$$\frac{\partial^4 \phi}{\partial x^4} + p^2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + k^2 \frac{\partial^4 \phi}{\partial y^4} = \frac{16 E_2 w_o^2}{a^2 \; b^2} \left[\frac{4x^2}{a^2} + \frac{4y^2}{b^2} - \frac{3x^4}{a^4} - \right.$$

$$\left. - \frac{3y^4}{b^4} - \frac{6x^2y^2}{a^2 \, b^2} - 1 \right] \qquad \ldots (4.6)$$

As a solution of Eq.(4.6), the stress function, ϕ (x,y) is taken in the following form

$$\phi(x,y) = \frac{1}{30.56}(A_1 x^8 + A_2 y^8) + \frac{1}{12.30}(A_3 a^2 x^6 + A_4 b^2 y^6) +$$

$$+ \frac{1}{24}(A_5 \; a^4 x^4 + A_6 b^4 \; y^4) + \frac{1}{2}(F_1 \; a^6 x^2 + F_2 \; b^6 y^2) + \frac{1}{360} \times$$

$$\times (B_1 \; x^6 y^2 + B_2 \; x^2 y^6) + \frac{1}{24}(H_1 \; a^2 x^4 y^2 + H_2 \; b^2 x^2 y^4) + \frac{1}{24} \times$$

$$\times B_3 \; x^4 y^4 + \frac{1}{4} B_4 \; a^2 \; b^2 \; x^2 \; y^2 \qquad \ldots (4.7)$$

The 14 co-efficients, A_1, A_2, A_3 etc. in Eq.(4.7) are to be determined from Eq.(4.6) and the prescribed boundary conditions. By virtue of Eq.(4.6) the following six relations exist :

$$-\frac{3\lambda_1}{a^4} = A_1 + k^2 B_3 + \frac{p^2 B_1}{6}$$

$$-\frac{3\lambda_1}{b^4} = B_3 + A_2 k^2 + \frac{B_2 p^2}{6}$$

$$\frac{4\lambda_1}{a^2} = A_3 a^2 + H_2 b^2 k^2 + H_1 a^2 p^2$$

$$\frac{4\lambda_1}{b^2} = H_1 a^2 + k^2 A_4 b^2 + H_2 p^2 b^2 \qquad \ldots (4.8)$$

$$-\frac{6\lambda_1}{a^2 b^2} = B_1 + B_2 k^2 + 6B_3 p^2$$

$$-\lambda_1 = A_5 a^4 + k^2 A_6 b^4 + B_4 p^2 a^2 b^2$$

where

$$\lambda_1 = \frac{16E_2 w_o^2}{a^2 b^2}$$

For the immovable edges, the inplane displacements u and v vanish on the boundary i.e.

$$
\left.
\begin{aligned}
u &= \int_0^x \left\{ \left[\frac{E_2 \frac{\partial^2\phi}{\partial y^2} - E'' \frac{\partial^2\phi}{\partial x^2}}{E_1E_2 - (E'')^2} \right] - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right\} dx = 0 \\
&\qquad \text{on } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\
v &= \int_0^y \left\{ \left[\frac{E_1 \frac{\partial^2\phi}{\partial x^2} - E'' \frac{\partial^2\phi}{\partial y^2}}{E_1E_2 - (E'')^2} \right] - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right\} dy = 0 \\
&\qquad \text{on } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\end{aligned}
\right\} \qquad \ldots (4.9)
$$

Eq.(4.9) will yield the following eight additional conditions :

$$
\frac{8\,E_2A_2}{525} + \frac{2\,E_2A_4}{45} + \frac{A_6E_2}{3} + F_2E_2 + B_1\left[\frac{a^6}{b^6}\,\frac{E_2}{1260} - \frac{a^4}{b^4}\,\frac{E''}{210}\right] +
$$

$$
+ B_2\left[\frac{2\,a^2}{b^2}\,\frac{E_2}{315} - \frac{4\,E''}{1575}\right] + H_1\left[\frac{a^6}{b^6}\,\frac{E_2}{60} - \frac{a^4}{b^4}\,\frac{E''}{15}\right] + H_2\left[\frac{E_2a^2/b^2}{15} - \frac{2E''}{45}\right]
$$

$$
+ B_3\left[\frac{a^4}{b^4}\,\frac{E_2}{35} - \frac{a^2}{b^2}\,\frac{4E''}{105}\right] + B_4\left[\frac{a^4}{b^4}\,\frac{E_2}{6} - \frac{a^2}{b^2}\,\frac{E''}{3}\right] - \frac{A_1E''}{210}\,\frac{a^6}{b^6} -
$$

$$
- \frac{A_3\,E''}{60}\,\frac{a^6}{b^6} - \frac{A_5\,E''}{6}\,\frac{a^6}{b^6} - F_1\,E''\,\frac{a^6}{b^6} = 0 \qquad \ldots (4.10a)
$$

$$
- \frac{A_2\,E_2}{210} + B_1\left[\frac{a^6}{b^6}\,\frac{E_2}{1260} + \frac{E''}{84}\,\frac{a^4}{b^4}\right] + B_2\left[\frac{E_2}{84}\,\frac{a^2}{b^2} + \frac{E''}{1260}\right] -
$$

$$
- \frac{A_1\,E''}{210}\,\frac{a^6}{b^6} - B_3\left[\frac{E_2}{14}\,\frac{a^4}{b^4} + \frac{E''}{14}\,\frac{a^2}{b^2}\right] = 0 \qquad \ldots (4.10b)
$$

$$\frac{A_2E_2}{175} + \frac{A_4E_2}{60} + B_2\left[\frac{a^2}{b^2}\frac{E_2}{420} - \frac{E''}{1050}\right] + B_1\left[\frac{E_2}{420}\frac{a^6}{b^6} + \frac{2E''}{105}\frac{a^4}{b^4}\right] -$$

$$- B_3\left[\frac{4\,E_2}{35}\frac{a^4}{b^4} + \frac{E''}{70}\frac{a^2}{b^2}\right] - \frac{A_1\,E''}{70}\frac{a^6}{b^6} - \frac{A_3\,E''}{60}\frac{a^6}{b^6} + H_1\,x$$

$$x\left[\frac{a^6}{b^6}\frac{E_2}{60} + \frac{E''}{10}\frac{a^4}{b^4}\right] - H_2\left[\frac{E_2}{10}\frac{a^2}{b^2} + \frac{E''}{60}\right] = 0 \qquad \ldots \text{(4.10c)}$$

$$- \frac{4\,A_2\,E_2}{525} - A_4\frac{E_2}{45} - A_6\frac{E_2}{6} + B_2\left[\frac{2\,E''}{1575} - \frac{E_2}{315}\frac{a^2}{b^2}\right] +$$

$$+ H_2\left[\frac{E''}{45} - \frac{E_2}{30}\frac{a^2}{b^2}\right] +$$

$$+ B_4\left[\frac{E_2}{6}\frac{a^4}{b^4} + \frac{E''}{6}\frac{a^2}{b^2}\right] - A_6\frac{E''}{6}\frac{a^6}{b^6} + H_1\left[\frac{E_2}{30}\frac{a^6}{b^6} + \frac{E''}{30}\frac{a^4}{b^4}\right] +$$

$$+ B_3\left[\frac{2\,E''}{105}\frac{a^2}{b^2} - \frac{1}{70}E_2\frac{a^4}{b^4}\right] -$$

$$- \frac{A_1\,E''}{70}\frac{a^6}{b^6} - A_3\frac{E''}{30}\frac{a^6}{b^6} + B_1\left[\frac{E_2}{420}\frac{a^6}{b^6} + \frac{E''}{420}\frac{a^4}{b^4}\right] = 0$$

$$\ldots \text{(4.10d)}$$

$$\frac{8\,A_1E_1}{525} + B_1\left[\frac{2\,E_1}{315}\,\frac{b^2}{a^2} - \frac{4\,E''}{1575}\right] + \frac{2\,A_3E_1}{45} + H_1\left[\frac{E_1}{15}\,\frac{b^2}{a^2} - \frac{2\,E''}{45}\right] +$$

$$+ A_5\,\frac{E_1}{3} + F_1\,E_1 -$$

$$- \frac{A_2\,E''}{210}\,\frac{b^6}{a^6} - \frac{A_6\,E''}{6}\,\frac{b^6}{a^6} - F_2E''\,\frac{b^6}{a^6} + B_3\left[\frac{E_1}{35}\,\frac{b^4}{a^4} - \frac{4\,E''}{105}\,\frac{b^2}{a^2}\right] +$$

$$+ B_4\left[\frac{E_1}{6}\,\frac{b^4}{a^4} - \frac{E''}{3}\,\frac{b^2}{a^2}\right]$$

$$+ B_2\left[\frac{E_1}{1260}\,\frac{b^6}{a^6} - \frac{E''}{210}\,\frac{b^4}{a^4}\right] + H_2\left[\frac{E_1}{60}\,\frac{b^6}{a^6} - \frac{E''}{15}\,\frac{b^4}{a^4}\right] -$$

$$- \frac{A_4\,E''}{60}\,\frac{b^6}{a^6} = 0 \qquad \ldots (4.10e)$$

$$- \frac{A_1\,E_1}{210} + B_1\left[-\frac{E''}{1260} + \frac{E_1}{84}\,\frac{b^2}{a^2}\right] - B_3\left[-\frac{E''}{14}\,\frac{b^2}{a^2} + \frac{E_1}{14}\,\frac{b^4}{a^4}\right] +$$

$$+ B_2\left[-\frac{E''}{84}\,\frac{b^4}{a^4} + \frac{E_1}{1260}\,\frac{b^6}{a^6}\right] - A_2\,\frac{E''}{210}\,\frac{b^6}{a^6} = 0 \qquad \ldots (4.10f)$$

$$\frac{A_1 E_1}{175} + B_1 \left[\frac{E_1}{420} \frac{b^2}{a^2} - \frac{E''}{1050} \right] + B_2 \left[\frac{E_1}{420} \frac{b^6}{a^6} + \frac{2E''}{105} \frac{b^4}{a^4} \right] -$$

$$- A_2 \frac{E''}{70} \frac{b^6}{a^6} + A_3 \frac{E_1}{60} -$$

$$- B_3 \left[\frac{E''}{70} \frac{b^2}{a^2} + \frac{4E_1}{35} \frac{b^4}{a^4} \right] - A_4 \frac{E''}{60} \frac{b^6}{a^6} - H_1 \left[\frac{E_1}{10} \frac{b^2}{a^2} + \frac{E''}{60} \right] +$$

$$+ H_2 \left[\frac{E''}{10} \frac{b^4}{a^4} + \frac{E_1}{60} \frac{b^6}{a^6} \right] = 0 \qquad \ldots (4.10g)$$

$$- \frac{4A_1 E_1}{525} - \frac{A_2 E''}{70} \frac{b^6}{a^6} - \frac{A_3 E_1}{45} - \frac{A_5 E_1}{6} - A_6 \frac{E''}{6} \frac{b^6}{a^6} - A_4 \frac{E''}{30} \frac{b^6}{a^6}$$

$$+ B_1 \left[\frac{2E''}{1575} - \frac{E_1}{315} \frac{b^2}{a^2} \right] + B_2 \left[\frac{E''}{420} \frac{b^4}{a^4} + \frac{E_1}{420} \frac{b^6}{a^6} \right] + B_3 \left[\frac{2E''}{105} \frac{b^2}{a^2} - \frac{E_1}{70} \frac{b^4}{a^4} \right]$$

$$+ B_4 \left[\frac{E_1}{6} \frac{b^4}{a^4} + \frac{E''}{6} \frac{b^2}{a^2} \right] + H_1 \left[\frac{E''}{45} - \frac{E_1}{30} \frac{b^2}{a^2} \right] + H_2 \left[\frac{E''}{30} \frac{b^4}{a^4} + \frac{E_1}{30} \frac{b^6}{a^6} \right] = 0$$

$$\ldots (4.10h)$$

The equations in (4.8) and (4.10) are 14 in number. After solving these fourteen simultaneous equations one gets the following values of the fourteen unknown constants :

$$B_1 = \frac{\lambda\left[\left(\frac{2\beta_6}{a^2b^2} + 6\alpha_2 p^2\right)\left(\beta_3\beta_6 + \beta_4\beta_5\right) - \left(\alpha_1\beta_6 + \alpha_2\beta_5\right)\left(\beta_6 K^2 + 6\beta_4 p^2\right)\right]}{\left[\left(\beta_6 + 6\beta_2 p^2\right)\left(\beta_3\beta_6 + \beta_4\beta_5\right) - \left(\beta_1\beta_6 + \beta_2\beta_5\right)\left(\beta_6 K^2 + 6\beta_4 p^2\right)\right]} \qquad \ldots (4.11)$$

$$B_2 = \frac{\lambda\left[\left(\alpha_1\beta_6 + \alpha_2\beta_5\right)\left(\beta_6 + 6\beta_2 p^2\right) - \left(\frac{2\beta_6}{a^2b^2} + 6\alpha_2 p^2\right)\left(\beta_1\beta_6 + \beta_2\beta_5\right)\right]}{\left[\left(\beta_6 + 6\beta_2 p^2\right)\left(\beta_3\beta_6 + \beta_4\beta_5\right) - \left(\beta_1\beta_6 + \beta_2\beta_5\right)\left(\beta_6 K^2 + 6\beta_4 p^2\right)\right]} \qquad \ldots (4.12)$$

where $\lambda = -49 \frac{E_2}{a^2} \frac{W_0^2}{b^2}$

$$\beta_1 = \frac{E^*}{1260} + \frac{E_1}{84}\frac{b^2}{a^2} + \frac{E_1 p^2}{1260}$$

$$\beta_2 = \frac{E_2}{1260}\frac{a^6}{b^6} + \frac{E^*}{84}\frac{a^4}{b^4} + \frac{E''}{1260}\frac{p^2 a^6}{b^6}$$

$$\beta_3 = \frac{E^*}{84}\frac{b^4}{a^4} + \frac{E_1}{1260}\frac{b^6}{a^6} + \frac{E''}{1260}\frac{p^2}{k^2}\frac{b^6}{a^6}$$

$$\beta_4 = \frac{E_2}{84}\frac{a^2}{b^2} + \frac{E''}{1260} + \frac{E_2}{1260}\frac{p^2}{k^2}$$

$$\beta_5 = \frac{E_1 k^2}{210} - \frac{E''}{14}\,\frac{b^2}{a^2} - \frac{E_1}{14}\,\frac{b^4}{a^4} + \frac{E''}{210}\,\frac{b^6}{k^2 a^6}$$

$$\beta_6 = \frac{E_2}{14}\,\frac{a^4}{b^4} + \frac{E''}{14}\,\frac{a^2}{b^2} - \frac{E''}{210}\,\frac{k^2 a^6}{b^6} - \frac{E_2}{210}\,k^2$$

$$\alpha_1 = \frac{E_1}{210\,a^4} + \frac{E''}{210}\,\frac{b^2}{k^2 a^6}$$

$$\alpha_2 = \frac{E_2}{210\,k^2 b^4} + \frac{E''}{210}\,\frac{a^2}{b^6}$$

$$B_3 = \frac{\lambda}{3\,p^2 a^2 b^2} - B_1\,\frac{1}{6p^2} - B_2\,\frac{k^2}{6p^2} \qquad \ldots \quad (4.13)$$

$$A_1 = \frac{\lambda}{a^4} - B_1\,\frac{p^2}{6} - B_3\,k^2 \qquad \ldots \quad (4.14)$$

$$A_2 = \frac{\lambda}{k^2\,b^4} - B_2\,\frac{p^2}{6\,k^2} - \frac{B_3}{k^2} \qquad \ldots \quad (4.15)$$

$$H_1 = \frac{\frac{\lambda}{45}\left(\alpha_3\phi_4 - \alpha_4\phi_3\right) - \left(M\phi_4 - N\phi_3\right)}{\phi_1\phi_4 - \phi_2\phi_3} \qquad \ldots \quad (4.16)$$

$$H_2 = \frac{\frac{\lambda}{45}\left(\alpha_4\phi_1 - \alpha_3\phi_2\right) - \left(N\phi_1 - M\phi_2\right)}{\phi_1\phi_4 - \phi_2\phi_3} \qquad \ldots \quad (4.17)$$

where $\alpha_3 = \dfrac{E_2}{k^2 b^4} - \dfrac{E'' a^2}{b^6}$

$$\alpha_4 = \frac{E_1}{a^4} - \frac{E'' b^2}{k^2 a^6}$$

$$\phi_1 = \frac{E_2}{60}\frac{a^6}{b^6} + \frac{E''}{10}\frac{a^4}{b^4} + \frac{E''}{60}\frac{\beta^2 a^6}{b^6} - \frac{E_2}{60}\frac{a^2}{k^2 b^2}$$

$$\phi_2 = \frac{E''}{60}\frac{b^4}{k^2 a^4} - \frac{E_1}{10}\frac{b^2}{a^2} - \frac{E''}{60} - \frac{E_1}{60}\beta^2$$

$$\phi_3 = \frac{E''}{60}\frac{k^2 a^4 b^2}{b^6} - \frac{E_2}{10}\frac{a^2}{b^2} - \frac{E''}{60} - \frac{E_2}{60}\frac{\beta^2}{k^2}$$

$$\phi_4 = \frac{E''}{60}\frac{\beta^2 b^6}{k^2 a^6} + \frac{E''}{10}\frac{b^4}{a^4} + \frac{E_1}{60}\frac{b^6}{a^6} - \frac{E_1}{60}\frac{k^2 b^2}{a^2}$$

$$M = -A_1 \frac{E''}{70}\frac{a^6}{b^6} + A_2 \frac{E_2}{175} + B_1\left[-\frac{E_2}{420}\frac{a^6}{b^6} + \frac{2E''}{105}\frac{a^4}{b^4}\right] +$$

$$+ B_2\left[-\frac{E_2}{420}\frac{a^2}{b^2} - \frac{E''}{1050}\right] - B_3\left[-\frac{4E_2}{35}\frac{a^4}{b^4} + \frac{E''}{70}\frac{a^2}{b^2}\right]$$

$$N = A_1 \frac{E_1}{175} - A_2 \frac{E''}{70}\frac{b^6}{a^6} + B_1\left[-\frac{E_1}{420}\frac{b^2}{a^2} - \frac{E''}{1050}\right] +$$

$$+ B_2 \left[- \frac{B_1}{420} \frac{b^6}{a^6} + \frac{2E''}{105} \frac{b^4}{a^4} \right] - B_3 \left[- \frac{E''}{70} \frac{b^2}{a^2} + \frac{4E_2}{35} \frac{b^4}{a^4} \right]$$

$$A_3 = - \left(\frac{4}{3} \frac{\lambda}{a^4} + H_1 \beta^2 + H_2 k^2 \frac{b^2}{a^2} \right) \qquad \ldots (4.18)$$

$$A_4 = - \left(\frac{4}{3} \frac{\lambda}{k^2 b^4} + H_1 \frac{a^2}{k^2 b^2} + H_2 \frac{\beta^2}{k^2} \right) \qquad \ldots (4.19)$$

$$A_5 = \frac{(P\phi_8 - Q\phi_7) - \frac{\lambda}{3}(\alpha_6 \phi_7 - \alpha_5 \phi_8)}{\phi_5 \phi_8 - \phi_6 \phi_7} \qquad \ldots (4.20)$$

$$A_6 = \frac{(Q\phi_5 - P\phi_6) - \frac{\lambda}{3}(\alpha_5 \phi_6 - \alpha_6 \phi_5)}{\phi_5 \phi_8 - \phi_6 \phi_7} \qquad \ldots (4.21)$$

where

$$P = - A_1 \frac{E''}{70} \frac{a^6}{b^6} - A_2 \frac{4E_2}{525} - A_3 \frac{E''}{30} \frac{a^6}{b^6} - A_4 \frac{E_2}{45} + B_1 \left[\frac{E_2}{420} \frac{a^6}{b^6} + \frac{E''}{420} \frac{a^4}{b^4} \right]$$

$$+ B_2 \left[\frac{2E''}{1575} - \frac{E_2}{315} \frac{a^2}{b^2} \right] + B_3 \left[\frac{2E''}{105} \frac{a^2}{b^2} - \frac{E_2}{70} \frac{a^4}{b^4} \right] +$$

$$+ H_1 \left[- \frac{E_2}{30} \frac{a^6}{b^6} + \frac{E''}{30} \frac{a^4}{b^4} \right] + H_2 \left[- \frac{E''}{45} - \frac{E_2}{30} \frac{a^2}{b^2} \right]$$

$$Q = - A_1 \frac{4E_1}{525} - A_2 \frac{E''}{70} \frac{b^6}{a^6} - A_3 \frac{E_1}{45} - A_4 \frac{E''}{30} \frac{b^6}{a^6} + B_1 \left[-\frac{2E''}{1575} - \frac{E_1}{315} \frac{b^2}{a^2} \right] +$$

$$+ B_2 \left[-\frac{E''}{420} \frac{b^4}{a^4} + \frac{E_1}{420} \frac{b^6}{a^6} \right] + B_3 \left[-\frac{2E''}{105} \frac{b^2}{a^2} - \frac{E_1}{70} \frac{b^4}{a^4} \right] + H_1 \left[-\frac{E''}{45} - \frac{E_1}{30} \frac{b^2}{a^2} \right] +$$

$$+ H_2 \left[-\frac{E''}{30} \frac{b^4}{a^4} + \frac{E_1}{30} \frac{b^6}{a^6} \right]$$

$$\phi_5 = \frac{E''}{6} \frac{a^6}{b^6} + \frac{E_2}{6p^2} \frac{a^6}{b^6} + \frac{E''}{6p^2} \frac{a^4}{b^4}$$

$$\phi_6 = \frac{E_1}{6} + \frac{E_1}{6p^2} \frac{b^2}{a^2} + \frac{E''}{6p^2}$$

$$\phi_7 = \frac{E_2}{6} + \frac{E_2}{6p^2} \frac{k^2 a^2}{b^2} + \frac{E'' k^2}{6p^2}$$

$$\phi_8 = \frac{E''}{6} \frac{b^6}{a^6} + \frac{E_1}{6p^2} \frac{k^2 b^6}{a^6} + \frac{E''}{6} \frac{k^2 b^4}{p^2 a^4}$$

$$\alpha_5 = \frac{E_2}{6p^2} \frac{a^2}{b^6} + \frac{E''}{6 p^2 b^4}$$

$$\alpha_6 = \frac{E_1}{6p^2} \frac{b^2}{a^6} + \frac{E''}{6 p^2 a^4}$$

$$B_4 = \frac{\lambda}{3\beta^2 a^2 b^2} - A_5 \frac{a^2}{\beta^2 b^2} - A_5 \frac{k^2}{\beta^2} \frac{b^2}{a^2} \qquad \ldots (4.22)$$

$$F_1 = \frac{R\ E'' \frac{b^6}{a^6} + T\ E_2}{(E'')^2 - E_1 E_2} \qquad \ldots (4.23)$$

$$F_2 = \frac{T\ E'' \frac{a^6}{b^6} + R\ E_1}{(E'')^2 - E_1 E_2} \qquad \ldots (4.24)$$

where

$$R = -A_1 \frac{E''}{210} \frac{a^6}{b^6} + A_2 \frac{8\ E_2}{525} - A_3 \frac{E''}{60} \frac{a^6}{b^6} + A_4 \frac{2E_2}{45} - A_5 \frac{E''}{6} \frac{a^6}{b^6} + A_6 \frac{E_2}{3}$$

$$+ B_1 \left[\frac{E_2}{1260} \frac{a^6}{b^6} - \frac{E''}{210} \frac{a^4}{b^4} \right] + B_2 \left[\frac{2E_2}{315} \frac{a^2}{b^2} - \frac{4E''}{1575} \right] + B_3 \left[\frac{E_2}{35} \frac{a^4}{b^4} - \frac{4E''}{105} \frac{a^2}{b^2} \right]$$

$$+ B_4 \left[\frac{E_2}{6} \frac{a^4}{b^4} - \frac{E''}{3} \frac{a^2}{b^2} \right] + H_1 \left[\frac{E_2}{60} \frac{a^6}{b^6} - \frac{E''}{15} \frac{a^4}{b^4} \right] + H_2 \left[\frac{E_2}{15} \frac{a^2}{b^2} - \frac{2E''}{45} \right]$$

$$T = A_1 \frac{8E_1}{525} - A_2 \frac{E''}{210} \frac{b^6}{a^6} + A_3 \frac{2E_1}{45} - A_4 \frac{E''}{60} \frac{b^6}{a^6} + A_5 \frac{E_1}{3} - A_6 \frac{E''}{6} \frac{b^6}{a^6} +$$

$$+ B_1 \left[\frac{2E_1}{315} \frac{b^2}{a^2} - \frac{4E''}{1575} \right] + B_2 \left[\frac{E_1}{1260} \frac{b^6}{a^6} - \frac{E''}{210} \frac{b^4}{a^4} \right] +$$

$$+ B_3 \left[\frac{E_1}{35} \frac{b^4}{a^4} - \frac{4E''}{105} \frac{b^2}{a^2} \right] + B_4 \left[\frac{E_1}{6} \frac{b^4}{a^4} - \frac{E''}{3} \frac{b^2}{a^2} \right] +$$

$$+ H_1 \left[\frac{E_1}{15} \frac{b^2}{a^2} - \frac{2E''}{45} \right] + H_2 \left[\frac{E_1}{60} \frac{b^6}{a^6} - \frac{E''}{15} \frac{b^4}{a^4} \right]$$

This completes the determination of the stress function $\phi\ (x, y)$. We now apply the procedure of Galerkin to obtain the deflection function $w\ (x, y)$. As a final result of a lengthy calculation we obtain the following cubic equation determining the central deflection w_0

$$\left(\frac{24}{a^4} + \frac{16\,\lambda^2}{a^2 b^2} + \frac{24\,k^2}{b^4} + 0.634\,\frac{k_1}{D_1} \right) w_0 D_1 + 24 \times 10^{-4}\,\frac{w_0 h}{ab}$$

$$\Big[0.3 \left(A_1 \frac{a^7}{b} + A_2 \frac{b^7}{a} \right) + 15 \left(A_3 \frac{a^7}{b} + A_4 \frac{b^7}{a} \right) - 70 \left(A_5 \frac{a^7}{b} + A_6 \frac{b^7}{a} \right) -$$

$$- 9 \left(B_1 a^5 b + B_2 a b^5 \right) + 70 \left(H_1 a^5 b + H_2 a b^5 \right) - 5.7\, B_3 a^3 b^3 +$$

$$+ 280\, B_4 a^3 b^3 + 415 \left(F_1 \frac{a^7}{b} + F_2 \frac{b^7}{a} \right) \Big] = q \qquad \ldots (4.25)$$

Thus the deflection function is completely determined.

RESULTS

Numerical results are presented for two real wooden materials (plywood and delta product) having roughly the following elastic properties [Nowinski (1963)] and taking a = 2 b for the ellipse.

Case	E_1	E_2	G_{12}	ν_1	ν_2	k^2	l^2	p^2
1	1×10^5	0.5×10^5	0.1×10^5	0.05	0.025	0.5	0.223	5
2	1×10^5	0.05×10^5	0.05×10^5	0.2	0.01	.05	0.103	1
3	E	E	G	0.3	0.3	1	1	2

For a strong anisotropy ($E_2 : E_1 = \frac{1}{20}$) Eq.(4.25) reduces to

$$(0.26 + 0.057\ K_F)\left(\frac{w_o}{h}\right) + 0.027\left(\frac{w_o}{h}\right)^3 = \left(\frac{q\ b^4}{E_1\ h^4}\right) \quad \ldots (4.26)$$

where the nondimensional foundation modulus $K_F = \frac{k_1\ b^4}{D_1}$.

For a weaker anisotropy ($E_2 : E_1 = \frac{1}{2}$) Eq.(4.25) reduces to

$$(1.2 + 0.057\ K_F)\left(\frac{w_o}{h}\right) + 0.08\left(\frac{w_o}{h}\right)^3 = \left(\frac{q\ b^4}{E_1\ h^4}\right) \quad \ldots (4.27)$$

and for isotropy,

$$(2.71 + 0.059\ K_F)\left(\frac{w_o}{h}\right) + 1.59\left(\frac{w_o}{h}\right)^3 = \left(\frac{q\ b^4}{E\ h^4}\right) \quad \ldots (4.28)$$

Equations (4.26) and (4.27) are presented in Fig.4.1. Equation (4.28) is also presented in the same figure for comparison. The stress function and the deflection function being known, membrane and bending stresses at any point in the plate can be easily computed. The peak stress occurs in the outer fibers at the ends of the minor axis of the ellipse in the direction of that axis, and the sum of the dimensionless membrane and bending stresses at that point by the theory of the present writer is shown in Fig.4.2.

As a check on the accuracy of the data for deflections and stresses for the elliptic plate, the deflections and the total nondimensional membrane and bending stresses, $\left(\frac{\sigma_r b^2}{E h^2}\right)$ obtained for isotropic case are presented in Fig.4.3 and 4.4 respectively for comparison with the corresponding results obtained by [Weil and Newmark (1956) and Nash and Cooley (1959)]. It is observed that the deflections obtained by the present writer are in excellent agreement with the experimental values. But the stresses are at variance with those obtained by [Nash and Cooley].

As a further check on the accuracy of the data for deflection with foundation Eq.(4.25) is reduced for circular boundary and isotropy to the following equation

$$(5.86 + 0.055\ K_F)\left(\frac{w_o}{h}\right) + 2.78\left(\frac{w_o}{h}\right)^3 = \left(\frac{q\, a^4}{E h^4}\right) \qquad \ldots (4.29)$$

Equation (4.29) is presented in Fig.4.5 for comparison with the corresponding results obtained by Way [1934] and Bolton [1972]. It is observed from Fig.4.5 that the deflections obtained for $K_F = 0$ for circular boundary are in excellent agreement with an exact solution obtained by Way [1934] and agree very well with those results obtained by Bolton [1972] even for higher values of load function. For $K_F \to 0$ the results of Fig.4.5 agree very well with those obtained by Bolton and Sinha [1963]. Nondimensional membrane stresses, $(\frac{\sigma_m a^2}{E h^2})$ for circular boundary are also calculated and they are presented in Fig.4.6. It is observed that the stresses obtained are in excellent agreement with the exact solution obtained by Way [1934].

The results obtained for isotropic elliptic and circular plates are more accurate than the results obtained by other investigators. This is due to the fact that the stress functions for the elliptic plate have been determined very accurately. But the results obtained are still approximate because the deflection functions have been determined with the help Galerkin's procedure.

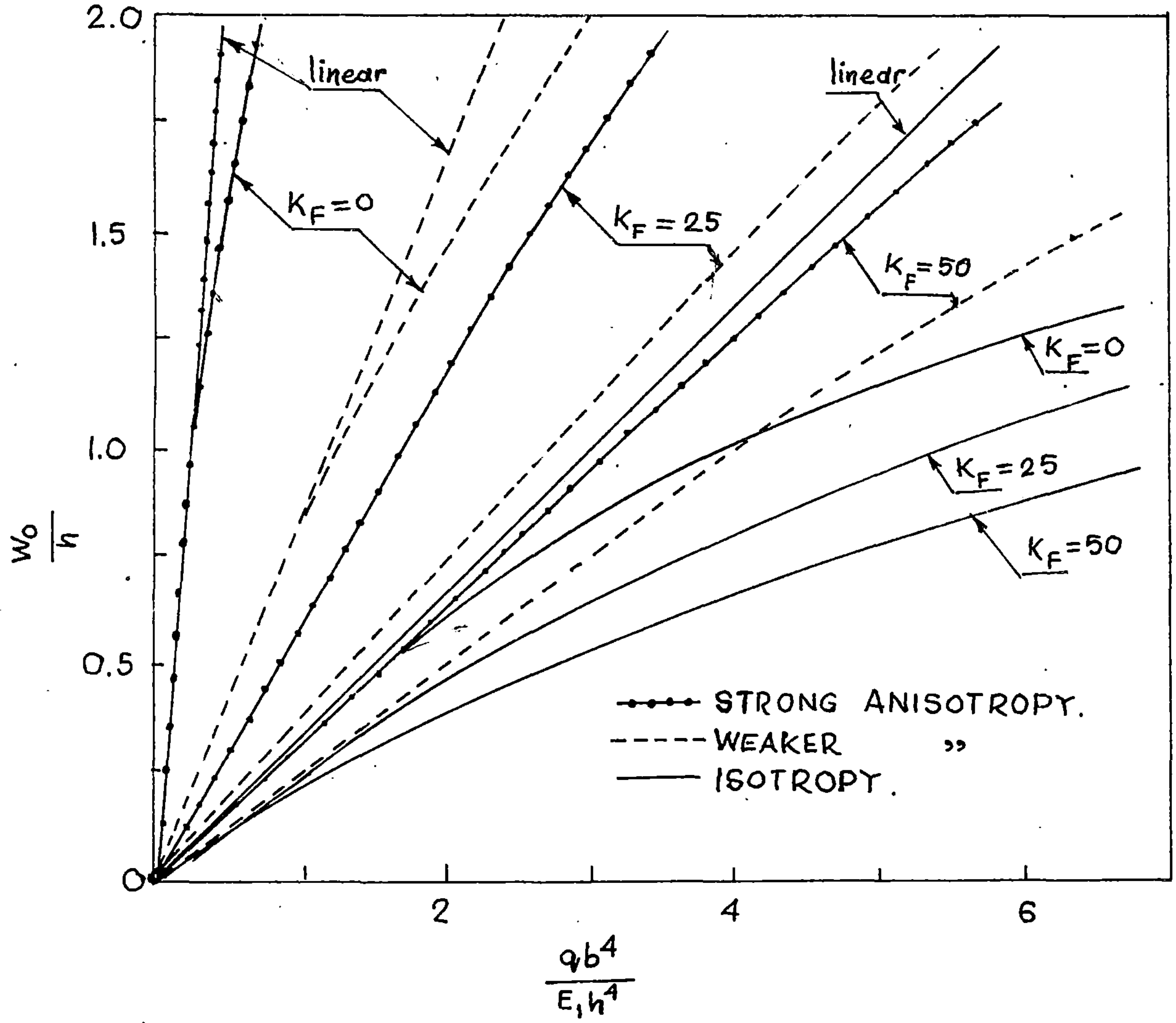

FIG. 4·1 CENTRAL DEFLECTIONS IN ORTHOTROPIC ELLIPTIC PLATES, $a=2b$.

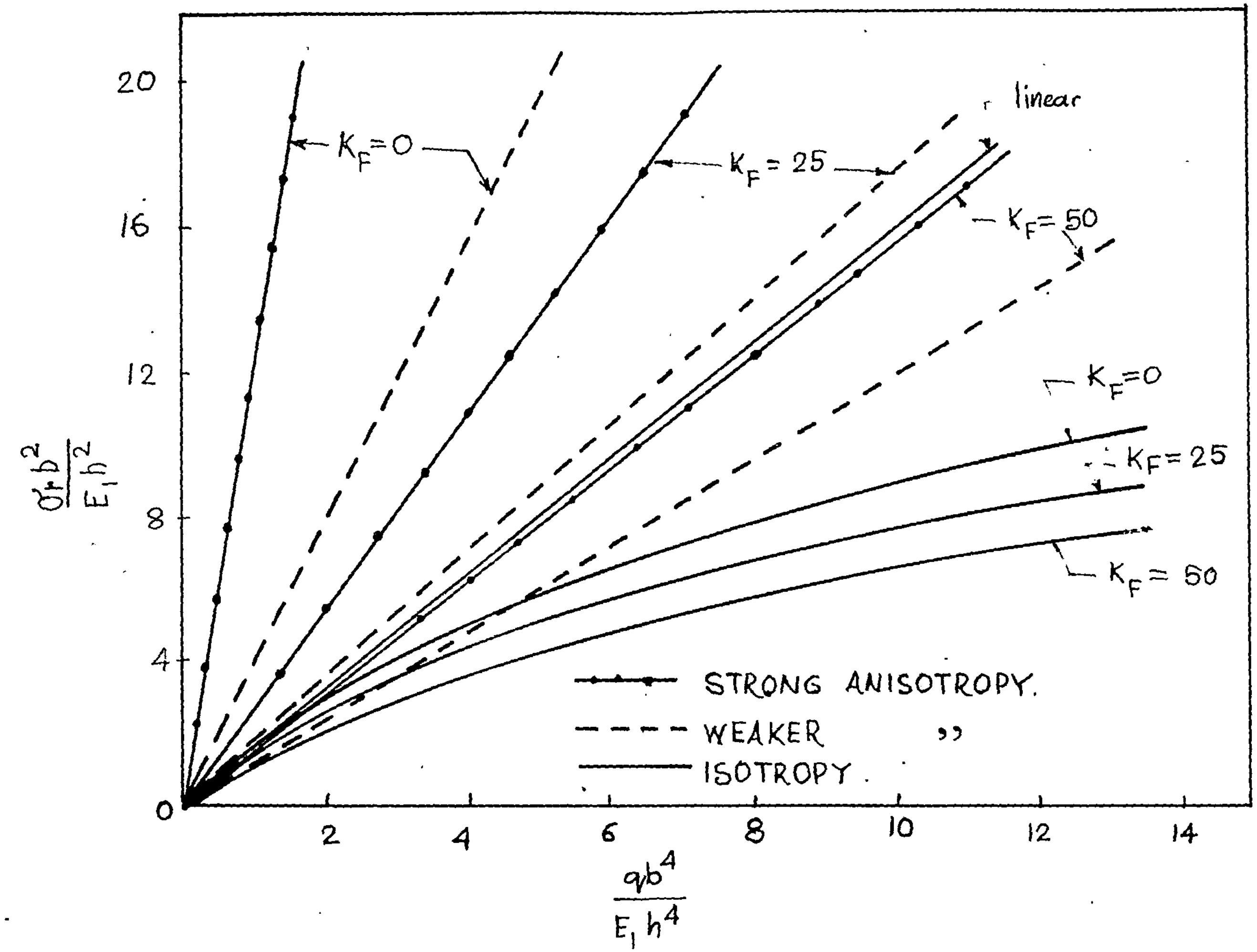

FIG. 4.2 STRESSES IN ORTHOTROPIC ELLIPTIC PLATES, $a=2b$

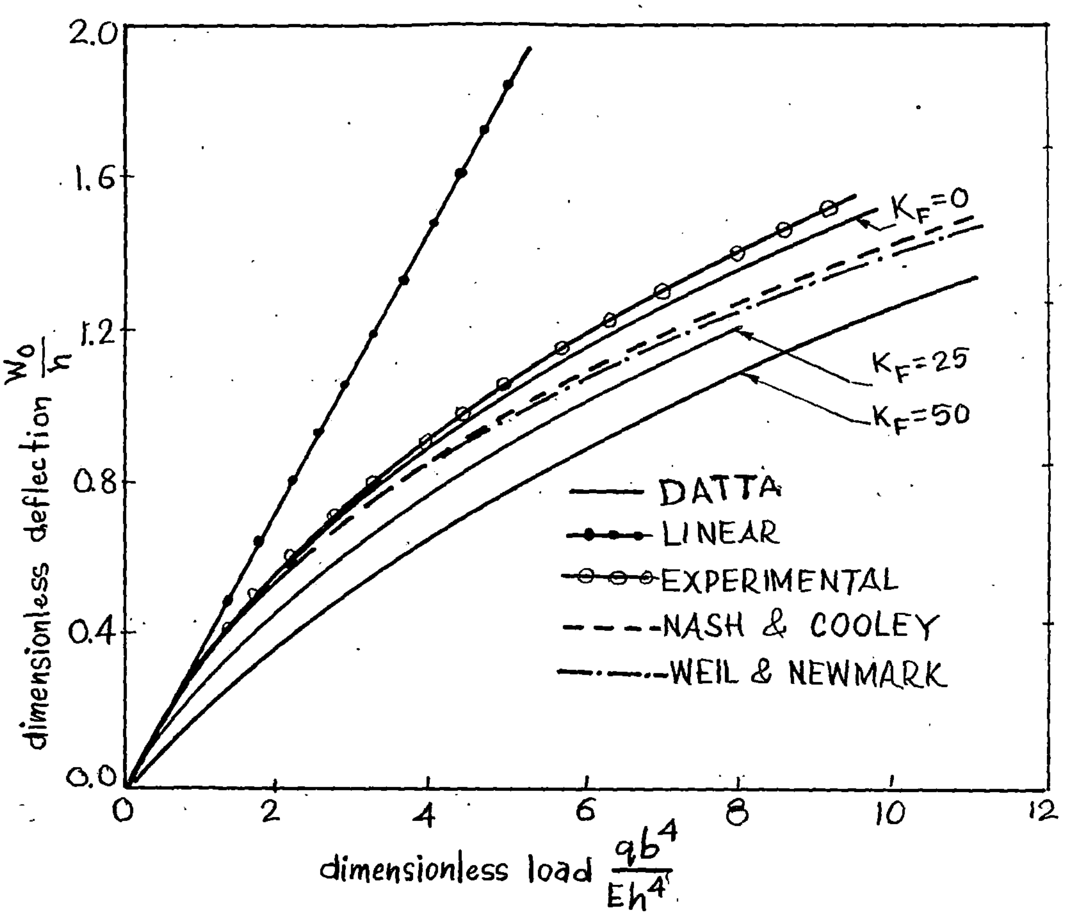

FIG. 43 CENTRAL DEFLECTION CURVE FOR ISOTROPIC ELLIPTIC PLATE, $a=2b$.

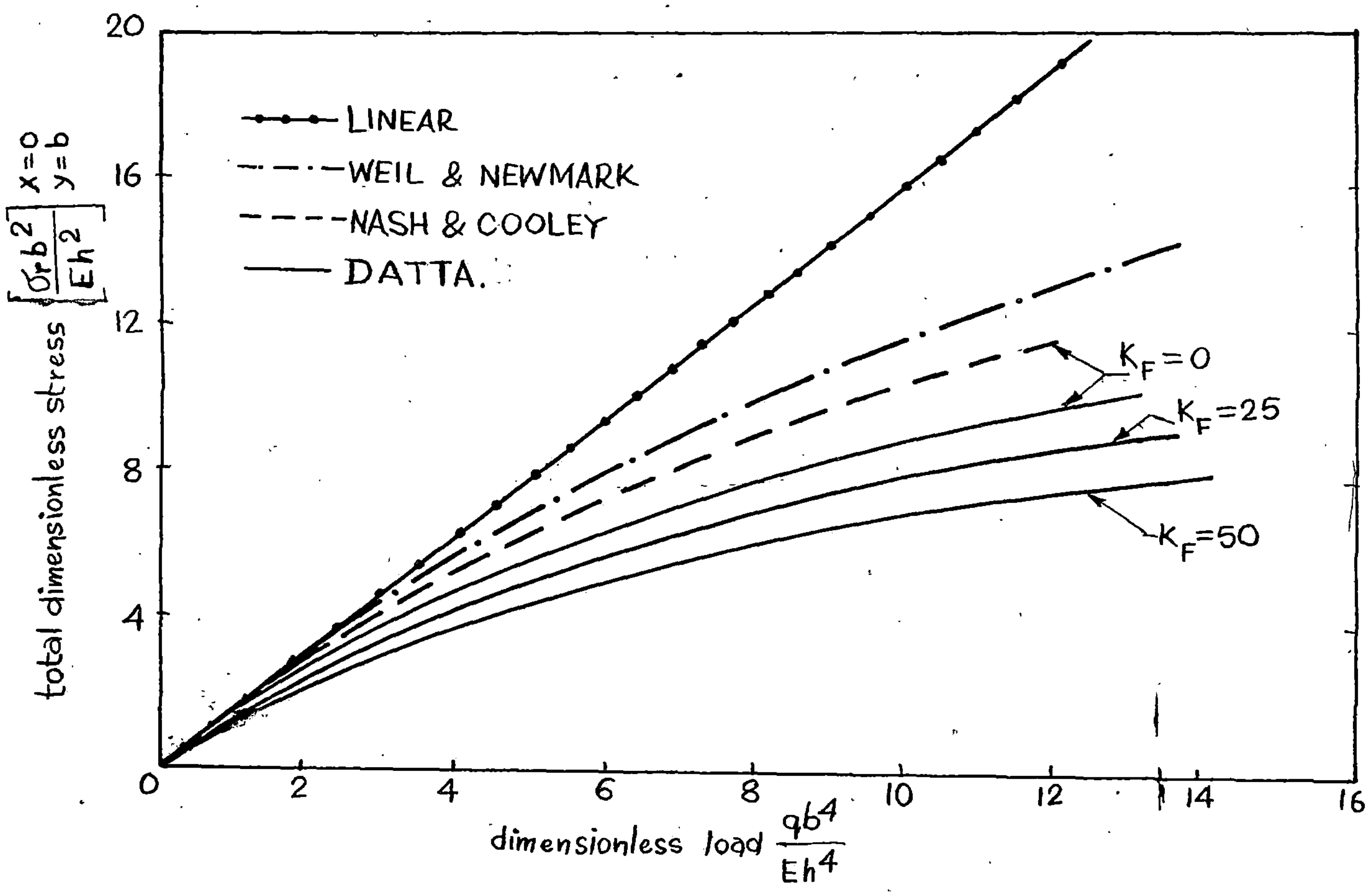

FIG. 4·4 STRESSES IN ISOTROPIC ELLIPTIC PLATES, $a=2b$.

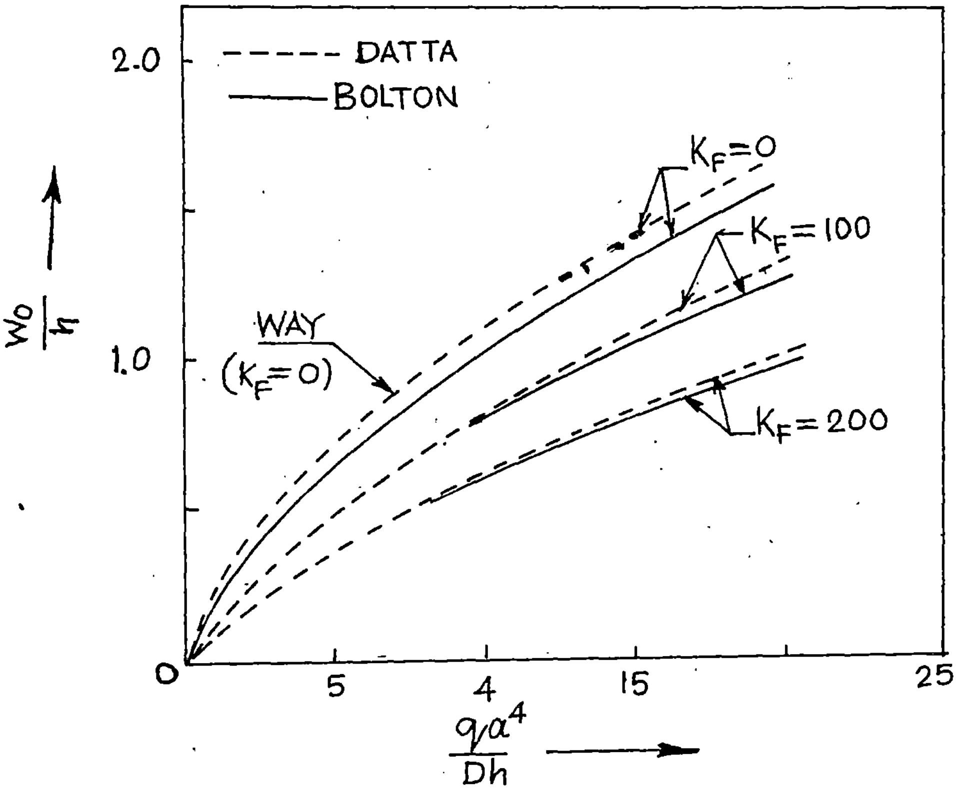

FIG. 4.5 CENTRAL DEFLECTIONS IN CIRCULAR ISOTROPIC PLATES.

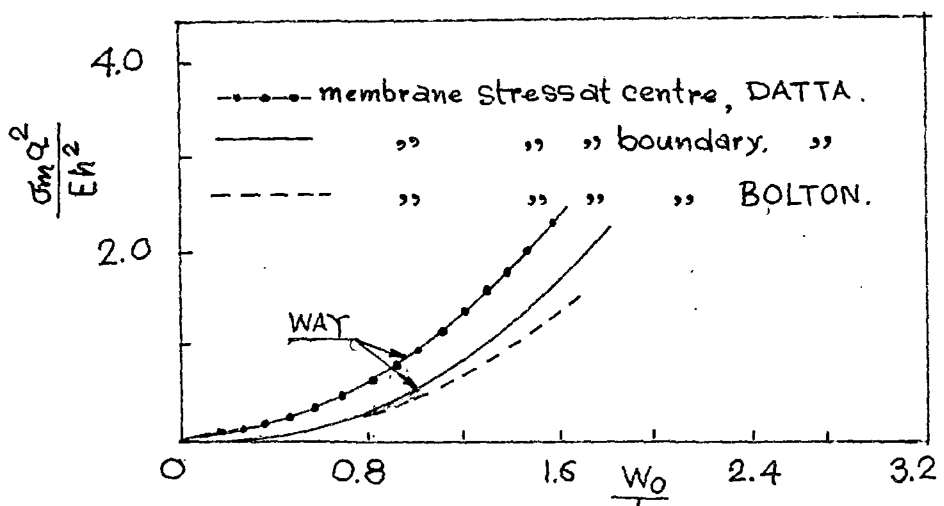

FIG. 4.6 MEMBRANE STRESES IN CIRCULAR PLATE.