

## **CHAPTER - 2**

### **RADIATIVE TRANSFER PROBLEMS**

**IN**

### **SEMI - INFINITE ATMOSPHERES**

## 2.1 Solution of a radiative transfer problem in semi - infinite atmosphere with Isotropic scattering using a modified form of spherical harmonic method.

### 2.1.1. The problem and development of equations

The equation of transfer appropriate to the problem is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 p(\mu, \mu') I(\tau, \mu') d\mu' \quad (2.1.1.1)$$

where  $p(\mu, \mu')$  is the phase function which gives the measure through which a pencil of radiation is scattered from  $(\mu, \phi)$  direction to  $(\mu', \phi')$  direction, and this is defined by

$$p(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} p(\mu, \mu'; \phi, \phi') d\phi' \quad (2.1.1.2)$$

$\phi$  being the azimuthal angle,  $I$  is the specific intensity,  $\tau$  is the optical thickness defined by

$$\tau = \int_z^{\infty} \kappa \rho dz,$$

$\kappa$  is the absorption coefficient,  $\rho$  is the density of the medium,  $\mu = \cos(\theta)$ ,  $\theta$  is the angle made by the incident radiation with the outward drawn normal.

The equation (2.1.1.1) is to be solved subject to the boundary conditions

$$I(0, -\mu) = 0, \quad \text{for } 0 \leq \mu \leq 1 \quad (2.1.1.3a)$$

$$I(\tau, \mu) e^{-\tau} \rightarrow 0 \quad \text{as } \tau \rightarrow \infty \quad (2.1.1.3b)$$

For isotropic scattering, we know that

$$p(\mu, \mu') = 1 \quad (2.1.1.4)$$

We consider the following two forms of intensity

$$I^+(\tau, \mu) = I(0,0) \left[ \phi(\tau) + \psi(\mu) + \sum_{l=0}^{l=L} (2l+1) I_l^+(\tau) \mu P_l(2\mu-1) \right], \quad 0 \leq \mu \leq 1, \quad (2.1.1.5a)$$

$$I^-(\tau, \mu) = I(0,0) \left[ \phi(\tau) + \psi(\mu) + \sum_{l=0}^{l=L} (2l+1) I_l^-(\tau) \mu P_l(2\mu+1) \right], \quad -1 \leq \mu \leq 0, \quad (2.1.1.5b)$$

where  $I(0,0)$  is some constant,  $\phi(\tau)$  is a function of  $\tau$  only and

$$\psi(\mu) = \begin{cases} 1 & \text{if } 0 \leq \mu \leq 1 \\ 0 & \text{if } -1 \leq \mu \leq 0 \end{cases} \quad (2.1.1.6)$$

Therefore the form of the equation of transfer now becomes,

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2} \left[ \int_{-1}^0 I^-(\tau, \mu') d\mu' + \int_0^1 I^+(\tau, \mu') d\mu' \right] \quad (2.1.1.7a)$$

$$\mu \frac{dI^-(\tau, \mu)}{d\tau} = I^-(\tau, \mu) - \frac{1}{2} \left[ \int_{-1}^0 I^-(\tau, \mu') d\mu' + \int_0^1 I^+(\tau, \mu') d\mu' \right] \quad (2.1.1.7b)$$

Using (2.1.1.4) and (2.1.1.5) in equations (2.1.1.7) we obtain the following ordinary differential equations

$$\begin{aligned} \mu \left[ \phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) I_l^{+'}(\tau) P_l(2l-1) \right] &= \\ &= \psi(\mu) - \frac{1}{2} + \sum_{l=0}^{l=L} (2l+1) I_l^{-}(\tau) \mu P_l(2\mu-1), \quad l=0,1,2,\dots,L \end{aligned} \quad (2.1.1.8a)$$

and

$$\begin{aligned} \mu \left[ \phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) I_l^{+'}(\tau) P_l(2l+1) \right] &= \\ &= \psi(\mu) - \frac{1}{2} + \sum_{l=0}^{l=L} (2l+1) I_l^{-}(\tau) \mu P_l(2\mu+1), \quad l=0,1,2,\dots,L \end{aligned} \quad (2.1.1.8b)$$

We make use of the recurrence formula for Legendre polynomials

$$\mu P_l(2\mu \pm 1) = \frac{1}{2l+1} \left[ \frac{l+1}{2} P_{l+1}(2\mu \pm 1) \mp \frac{2l+1}{2} P_l(2\mu \pm 1) + \frac{l}{2} P_{l-1}(2\mu \pm 1) \right]$$

Multiplying both sides of (2.1.1.8a) by  $P_l(2\mu-1)$  and integrating over  $[0,1]$  we obtain

$$\begin{aligned} \phi'(\tau) \int_0^1 \mu P_l(2\mu-1) d\mu + \frac{1}{4(2l+1)} \left[ \frac{l^2-l}{2l-1} I_{l-2}^{+'} + 2l I_{l-1}^{+'} + \right. \\ \left. + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1) I_{l+1}^{+'} + \frac{l^2+3l+2}{2l+3} I_{l+2}^{+'} \right] &= \\ = \int_0^1 \psi(\mu) P_l(2\mu-1) d\mu + \frac{1}{2(2l+1)} \left[ l I_{l-1}^{-} + (2l+1) I_l^{-} + (l+1) I_{l+1}^{-} \right] &- \end{aligned}$$

$$-\frac{1}{2}\delta_{0l} - \frac{1}{4}(I_0^+ - I_0^- + I_1^+ + I_1^-), \quad l=0,1,2,\dots,L \quad (2.1.1.9a)$$

Similarly multiplying both sides of (2.1.1.8b) by  $P_l(2\mu + 1)$  and integrating over  $[-1,0]$  we obtain the following equation

$$\begin{aligned} & \Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu + 1) d\mu + \frac{1}{4(2l+1)} \left[ \frac{l^2 - l}{2l-1} I_{l-2}^- - 2l I_{l-1}^- + \right. \\ & \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^- - 2(l+1) I_{l-1}^- + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^- \right] = \\ & = \int_{-1}^0 \Psi(\mu) P_l(2\mu + 1) d\mu + \frac{1}{2(2l+1)} \left[ l I_{l-1}^- - (2l+1) I_l^- + (l+1) I_{l-1}^- \right] - \\ & - \frac{1}{2}\delta_{0l} - \frac{1}{4}(I_0^+ - I_0^- + I_1^+ + I_1^-), \quad l=0,1,2,\dots,L \quad (2.1.1.9b) \end{aligned}$$

where  $\delta_{0l}$  is the Kronecker delta. The equations (2.1.1.9 a,b) are to be solved subject to the boundary conditions which are given below

$$I_l^-(0) \equiv 0 \quad (2.1.1.10)$$

$$I_l^+(\tau) e^{-\tau} \rightarrow 0 \quad \text{as } \tau \rightarrow 0 \quad (2.1.1.11a)$$

$$I_l^+(\tau) e^{-\tau} \rightarrow 0 \quad \text{as } \tau \rightarrow 0 \quad (2.1.1.11b)$$

In case of N-th approximations we assume

$$I_{N-1}^+ = I_{N-1}^- = 0 \quad (2.1.1.12)$$

Also we assume a trial solution of the form

$$I_l^+(\tau) = g_{l,\alpha} e^{-k\tau} + g_{l,\beta} \quad (2.1.1.13a)$$

$$I_l^-(\tau) = h_{l,\alpha} e^{-k\tau} + h_{l,\beta} \quad (2.1.1.13b)$$

where  $g_{l,\alpha}, g_{l,\beta}, h_{l,\alpha}, h_{l,\beta}$  are constants which are to be determined.

Using the boundary condition (2.1.1.10) and (2.1.1.11) we can write the solution as

$$I_l^+(\tau) = \sum_{r=1}^{n-1} g_{l,\alpha}^{(r)} e^{-k_r \tau} + g_{l,\beta} \quad (2.1.1.14a)$$

$$I_l^-(\tau) = \sum_{r=1}^{n-1} h_{l,\alpha}^{(r)} e^{-k_r \tau} + h_{l,\beta} \quad (2.1.1.14b)$$

Further, from (2.1.1.10) we have

$$\sum_{r=1}^{n-1} h_{l,\alpha}^{(r)} + h_{l,\beta} = 0 \quad (2.1.1.15b)$$

where  $k_r$  are the  $n-2$  roots of the determinant of order  $N$ . It will be found that there will be two zero roots for all order of approximation  $N$  (Chandrasekhar [1960]). Here will consider two approximations, viz  $L = 1$  and  $L = 2$ . Also we know net flux is given by

$$F = 2 \int_{-1}^1 I^+(\tau, \mu) \mu d\mu = 2 \left[ \int_{-1}^0 I^-(\tau, \mu) \mu d\mu + \int_0^1 I^+(\tau, \mu) \mu d\mu \right] \quad (2.1.1.16)$$

Also from the definition of source function and mean intensity  $J$  we have,

$$J(\tau) = \frac{1}{2} \int_{-1}^1 I(\tau, \mu) d\mu = \mathfrak{S}(\tau) \quad (2.1.1.17a)$$

and

$$K(\tau) = \frac{1}{2} \int_{-1}^1 I(\tau, \mu) \mu^2 d\mu \quad (2.1.1.17b)$$

The law of darkening ( or the emergent intensity) is given by

$$I(0, \mu) = \int_0^{\infty} \mathfrak{S}(\tau, \mu) e^{-\frac{\tau}{\mu}} \frac{d\tau}{\mu} \quad (2.1.1.18)$$

$$\text{We assume } \phi(\tau) = A\tau \quad (2.1.1.19)$$

where  $A$  is some constant which will be determined.

### 2.1.2. First approximation

Here  $L = 1$ . Taking successively  $l = 0, 1$  in equations (2.1.1.9a) and (2.1.1.9b) we obtain following differential equations.

$$\left( \frac{4}{3} I_0^{+'} + 2I_1^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\phi'(\tau) \quad (2.1.2.1)$$

$$\left( 2I_0^{+'} + \frac{24k}{5} I_1^{+'} \right) - 2(I_0^+ + 3I_1^+) = -2\phi'(\tau) \quad (2.1.2.2)$$

$$\left( \frac{4}{3} I_0^{-'} - 2I_1^{-'} \right) + (I_0^- + I_1^- + I_0^+ - I_1^+) = -2 + 2\phi'(\tau) \quad (2.1.2.3)$$

$$\left( -2I_0^{-'} + \frac{24k}{5} I_1^{-'} \right) - 2(I_0^- - 3I_1^-) = -2\phi'(\tau) \quad (2.1.2.4)$$

Using the forms of  $\phi(\tau)$  and  $I_l^+(\tau), I_l^-(\tau)$  in equations (2.1.2.1) to (2.1.2.4) and comparing the coefficients of  $e^{-k\tau}$  and the constant's terms we obtain the following set of equations

$$\left(\frac{4k}{3} + 1\right)g_{0,\alpha} + (2k + 1)g_{1,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \quad (2.1.2.5i)$$

$$(2k + 2)g_{0,\alpha} + \left(\frac{24k}{5} + 6\right)g_{1,\alpha} = 0 \quad (2.1.2.5ii)$$

$$g_{0,\alpha} + g_{1,\alpha} + \left(-\frac{4k}{3} + 1\right)h_{0,\alpha} + (2k - 1)h_{1,\alpha} = 0 \quad (2.1.2.5iii)$$

$$(2k - 2)h_{0,\alpha} + \left(6 - \frac{24k}{5}\right)h_{1,\alpha} = 0 \quad (2.1.2.5iv)$$

And equating the coefficients of the constant terms, we obtain following linear algebraic equations.

$$g_{0,\beta} + g_{1,\beta} + h_{0,\beta} - h_{1,\beta} = -2 + 2A \quad (2.1.2.6i)$$

$$2g_{0,\beta} + 6g_{1,\beta} = 2A \quad (2.1.2.6ii)$$

$$2h_{0,\beta} - 6h_{1,\beta} = 2A \quad (2.1.2.6iii)$$

Also from the boundary condition we find that

$$h_{0,\alpha} + h_{0,\beta} = 0 \quad (2.1.2.7a)$$

$$h_{1,\alpha} + h_{1,\beta} = 0 \quad (2.1.2.7b)$$

The equation (2.1.2.4) will have a nontrivial solution if

$$D_1(k) = 0 \quad (2.1.2.7)$$

where

$$D_1(k) = \begin{vmatrix} \frac{4k}{3} + 1 & 2k + 1 & 1 & -1 \\ 2k + 2 & \frac{24k}{5} + 6 & 0 & 0 \\ 1 & 1 & -\frac{4k}{3} + 1 & 2k - 1 \\ 0 & 0 & 2k - 2 & -\frac{24k}{5} + 6 \end{vmatrix}$$

So we have  $k = 0, 0, \pm 1.8257$ , and we will consider the positive root only.

We see that there are 9 unknown constants to be determined from 8 equations. For an extra equation we consider the first moment of H-function. We have [Chandrasekhar (1960)]

$$\alpha_0 = \int_0^1 H(\mu) d\mu = 2 \quad (2.1.2.8)$$

or

$$1 + \frac{1}{2}I_0^+(0) + \frac{1}{2}I_1^+(0) = 2$$

which gives

$$g_{0,\alpha} + g_{1,\alpha} + g_{0,\beta} + g_{1,\beta} = 2 \quad (2.1.2.9)$$

Solving (2.1.2.5), (2.1.2.6), (2.1.2.7) and (2.1.2.9) we obtain [(Bishnu 1969)]

$$g_{0,\alpha} = -0.5293, \quad h_{0,\alpha} = 2.1765$$

$$g_{1,\alpha} = -0.2026, \quad h_{1,\alpha} = 1.3006$$

$$g_{0,\beta} = -2.6273, \quad h_{0,\beta} = -2.1765$$

$$g_{1,\beta} = -0.3006, \quad h_{1,\beta} = -1.3006$$

$$A = 1.7254$$

Therefore, we can calculate the values of  $I_0^+, I_1^+, I_0^-, I_1^-$ . From (2.1.17a) we have

$$J(\tau) = I(0,0) \left[ A\tau + \frac{1}{2} + \frac{1}{4} (I_0^+ + I_1^+ - I_0^- + I_1^-) \right]$$

$$\text{Or, } J(\tau) = I(0,0) [1.7253\tau + 1.3006 - 0.300065e^{-k\tau}] \quad (2.1.2.10)$$

The net flux is given by

$$F = 2I(0,0) \left[ \frac{1}{2} + \frac{1}{3} (I_0^+ + I_0^-) + \frac{1}{2} (I_1^+ - I_1^-) \right] \quad (2.1.2.11)$$

From which we get

$$I(0,0) = 0.6927 F$$

The source function now becomes

$$\mathfrak{S}(\tau, \mu) = J(\tau, \mu)$$

Hence we obtain the emergent intensity as

$$I(0, \mu) = I(0, 0) [1.7253\tau + 1.3006 - 0.300065e^{-k\tau}] \quad (2.1.2.11)$$

Now expressing  $I(0, 0)$  in terms of  $F$  we can find the ratio  $I(0, \mu)/F$  and  $I(0, \mu)/I(0, 1)$ . The results are shown in the table (2.1.1).

### 2.1.3. Second approximation.

Here  $L = 2$ . Putting successively  $l = 0, 1, 2$  in the equations (2.1.1.9a) and (2.1.1.9b) we obtain in the usual manner, the following differential equations

$$\left( \frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\phi'(\tau) \quad (2.1.3.1)$$

$$\left( 2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} \right) - 2(I_0^+ + 3I_1^+ + 2I_2^+) = -2\phi'(\tau) \quad (2.1.3.2)$$

$$\left( \frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} \right) - (2I_1^+ + 5I_2^+) = 0 \quad (2.1.3.3)$$

$$\left( \frac{4}{3}I_0^{-'} - 2I_1^{-'} + \frac{2}{3}I_2^{-'} \right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2 + 2\phi'(\tau) \quad (2.1.3.4)$$

$$\left( -2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'} \right) - 2(I_0^- - 3I_1^- + 2I_2^-) = -2\phi'(\tau) \quad (2.1.3.5)$$

$$\left( \frac{1}{3}I_0^- - 2I_1^- + \frac{80}{21}I_2^- \right) - (2I_1^- - 5I_2^-) = 0 \quad (2.1.3.6)$$

Using the forms of  $\phi(\tau)$  and  $I_l^+(\tau), I_l^-(\tau)$  in equations (2.1.3.1) to (2.1.3.6) and comparing the coefficients of  $e^{-k\tau}$  and the constant's terms we obtain the following set of equations

$$\left( \frac{4k}{3} + 1 \right) g_{0,\alpha} + (2k+1)g_{1,\alpha} + \frac{2k}{3}g_{2,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \quad (2.1.3.7a)$$

$$(2k+2)g_{0,\alpha} + \left( \frac{24k}{5} + 6 \right) g_{1,\alpha} + (4k+4)g_{2,\alpha} = 0 \quad (2.1.3.7b)$$

$$\frac{k}{3}g_{0,\alpha} + (2k+2)g_{1,\alpha} + \left( \frac{80k}{21} + 5 \right) g_{2,\alpha} = 0 \quad (2.1.3.7c)$$

$$g_{0,\alpha} + g_{1,\alpha} + \left( -\frac{4k}{3} + 1 \right) h_{0,\alpha} + (2k-1)h_{1,\alpha} - \frac{2k}{3}h_{2,\alpha} = 0 \quad (2.1.3.7d)$$

$$(2k-2)h_{0,\alpha} + \left( -\frac{24k}{5} + 6 \right) h_{1,\alpha} + (4k-4)h_{2,\alpha} = 0 \quad (2.1.3.7e)$$

$$-\frac{k}{3}h_{0,\alpha} + (2k-2)h_{1,\alpha} + \left( -\frac{80k}{21} + 5 \right) h_{2,\alpha} = 0 \quad (2.1.3.7f)$$

and comparing the coefficients of the constants terms we get,

$$g_{0,\beta} + g_{1,\beta} + h_{0,\beta} - h_{1,\beta} = -2 + 2A \quad (2.1.3.8a)$$

$$g_{0,\beta} + 3g_{1,\beta} + 2g_{2,\beta} = A \quad (2.1.3.8b)$$

$$2g_{1,\beta} + 5g_{2,\beta} = 0 \quad (2.1.3.8c)$$

$$h_{0,\beta} - 3h_{1,\beta} + 2h_{2,\beta} = A \quad (2.1.3.8d)$$

$$2h_{1,\beta} - 5h_{2,\beta} = 0 \quad (2.1.3.8e)$$

The set of equations (2.1.3.7) will have nontrivial solution if  $D_2(k) = 0$  where

$$D_2(k) = \begin{vmatrix} \frac{4k}{3} + 1 & 2k + 1 & \frac{2k}{3} & 1 & -1 & 0 \\ 2k + 2 & \frac{24k}{5} + 6 & 4k + 4 & 0 & 0 & 0 \\ \frac{k}{3} & 2k + 2 & \frac{80k}{21} + 5 & 0 & 0 & 0 \\ 1 & 1 & 0 & -\frac{4k}{3} + 1 & 2k - 1 & -\frac{2k}{3} \\ 0 & 0 & 0 & 2k - 2 & -\frac{24k}{5} + 6 & 4k - 4 \\ 0 & 0 & 0 & -\frac{k}{3} & 2k - 2 & -\frac{80k}{21} + 5 \end{vmatrix}$$

So we have

$$k = 0, 0, \pm 1.3988, \pm 3.6116$$

As usual we will take only the positive roots

$$k_1 = 3.6116, \quad k_2 = 1.3988 \quad (2.1.3.9)$$

Next, from the boundary condition we have,

$$h_{0,\alpha}^{(1)} + h_{0,\alpha}^{(2)} + h_{0,\beta} = 0 \quad (2.1.3.10a)$$

$$h_{1,\alpha}^{(1)} + h_{1,\alpha}^{(2)} + h_{1,\beta} = 0 \quad (2.1.3.10b)$$

$$h_{2,\alpha}^{(1)} + h_{2,\alpha}^{(2)} + h_{2,\beta} = 0 \quad (2.1.3.10c)$$

As in case of first approximation, following the same procedure we can evaluate the constants and then obtain the source function as

$$\mathfrak{S}(\tau) = J(\tau) = I(0,0) \left[ 1.730759\tau + 1.231444 - 0.22227e^{-k_1\tau} - .009135e^{-k_2\tau} \right]$$

The law of darkening is given by

$$I(0,\mu) = I(0,0) \left[ 1.730759\mu + 1.231444 - \frac{.022227}{1+k_1\mu} - \frac{.009135}{1+k_2\mu} \right]$$

In the following table we give our results.

Table 2.1.1

$P_1$ - Approximation			$P_2$ - Approximation	
$\mu$	$I(0,\mu) / F$	$I(0,\mu) / I(0,1)$	$I(0,\mu) / F$	$I(0,\mu) / I(0,1)$
.05	.48312	.38067	.48585	.38526
.10	.52987	.41750	.53440	.42376
.15	.57529	.45329	.58039	.46022
.20	.61965	.48824	.62461	.49529
.25	.66315	.52252	.66758	.52937
.30	.70594	.55624	.70961	.56270
.35	.74815	.58949	.75093	.59546
.40	.78985	.62235	.79169	.62778
.45	.83114	.65488	.83201	.65975
.50	.87206	.68713	.87179	.69144
.55	.91268	.71913	.91164	.72290
.60	.95302	.75092	.95107	.75416
.65	.99312	.78251	.99029	.78526
.70	1.03302	.81395	1.02933	.81622
.75	1.07273	.84524	1.06823	.84706
.80	1.11228	.87640	1.10700	.87781
.85	1.15168	.90745	1.14566	.90846
.90	1.19095	.93839	1.18422	.93904
.95	1.23010	.96924	1.22270	.96955
1.00	1.26914	1.0000	1.26110	1.0000

Here the data in the parentheses indicates that due to Chandrasekhar. We find that second approximation, i.e.,  $P_2$  method is superior to that of first approximation, i.e.,  $P_1$  for higher values of  $\mu$ .

## 2.2 Solution of a radiative transfer problem in semi - infinite atmosphere with Rayleigh phase function using a modified form of spherical harmonic method.

### 2.2.1 The problem and development of equations.

The equation of transfer in case of plane parallel atmosphere with spherical symmetry is given by

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 p(\mu, \mu') I(\tau, \mu') d\mu' \quad (2.2.1.1)$$

where the symbols have their usual meanings, which are described in section 2.2.1.

The equation (2.2.1.1) is to be solved subject to the boundary conditions

$$I(0, -\mu) \equiv 0, \quad \text{for } 0 \leq \mu \leq 1 \quad (2.2.1.2a)$$

$$I(\tau, \mu) e^{-\tau} \rightarrow 0 \quad \text{as } \tau \rightarrow \infty \quad (2.2.1.2b)$$

The Rayleigh phase function is given by

$$p(\mu, \mu') = 1 + P_2(\mu)P_2(\mu') = \frac{3}{8} \left[ (3 - \mu^2) + (3\mu^2 - 1)\mu'^2 \right] \quad (2.2.1.3)$$

As in section 2.1 we consider the following two forms of intensity

$$I^+(\tau, \mu) = I(0, 0) \left[ \phi(\tau) + \psi(\mu) + \sum_{l=0}^{l=L} (2l+1) I_l^+ \mu P_l(2\mu-1) \right], \quad 0 \leq \mu \leq 1, \quad (2.2.1.4a)$$

$$I^-(\tau, \mu) = I(0,0) \left[ \phi(\tau) + \psi(\mu) + \sum_{l=0}^{l=L} (2l+1) I_l^- \mu P_l(2\mu+1) \right], \quad -1 \leq \mu \leq 0, \quad (2.2.1.4b)$$

where  $I(0,0)$  is some constant,  $\phi(\tau)$  is a function of  $\tau$  only and

$$\psi(\mu) = \begin{cases} 1 & \text{if } 0 \leq \mu \leq 1 \\ 0 & \text{if } -1 \leq \mu \leq 0 \end{cases} \quad (2.2.1.5)$$

Therefore the form of the equation of transfer now becomes,

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2} \left[ \int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' + \int_0^1 p(\mu, \mu') I^+(\tau, \mu') d\mu' \right] \quad (2.2.1.6a)$$

$$\mu \frac{dI^-(\tau, \mu)}{d\tau} = I^-(\tau, \mu) - \frac{1}{2} \left[ \int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' + \int_0^1 p(\mu, \mu') I^+(\tau, \mu') d\mu' \right] \quad (2.2.1.6b)$$

Using (2.2.1.4a) and (2.2.2.4b) in equations (2.2.1.6) we obtain respectively the following ordinary differential equations

$$\mu \left[ \phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) I_l^+(\tau) P_l(2\mu-1) \right] = \psi(\mu) - \frac{1}{2} + \sum_{l=0}^{l=L} (2l+1) I_l^+(\tau) \mu P_l(2\mu-1)$$

$$- \frac{3}{64} (\mu^2 + 5) (I_0^+ - I_0^-) - \frac{3}{320} (17\mu^2 + 21) (I_1^+ + I_1^-)$$

$$+ (3\mu^2 - 1) \left[ - \frac{3}{64} (I_2^+ - I_2^-) - \frac{3}{320} (I_3^+ + I_3^-) \right] \quad (2.2.1.7a)$$

and

$$\begin{aligned} \mu \left[ \phi'(\tau) + \sum_{l=0}^{l=L} (2l+1) I_l^{\prime}(\tau) P_l(2\mu+1) \right] &= \Psi(\mu) - \frac{1}{2} + \sum_{l=0}^{l=L} (2l+1) I_l^{\prime}(\tau) \mu P_l(2\mu+1) - \\ &- \frac{3}{64}(\mu^2+5)(I_0^+ - I_0^-) - \frac{3}{320}(17\mu^2+21)(I_1^+ + I_1^-) + \\ &+ (3\mu^2-1) \left[ -\frac{3}{64}(I_2^+ - I_2^-) - \frac{3}{320}(I_3^+ + I_3^-) \right] \end{aligned} \quad (2.2.1.7b)$$

We make use of the recurrence formula for Legendre polynomials

$$\mu P_l(2\mu \pm 1) = \frac{1}{2l+1} \left[ \frac{l+1}{2} P_{l+1}(2\mu \pm 1) \mp \frac{2l+1}{2} P_l(2\mu \pm 1) + \frac{l}{2} P_{l-1}(2\mu \pm 1) \right]$$

Multiplying both sides of (2.2.1.7a) by  $P_l(2\mu-1)$  and integrating over  $[0,1]$  and making use of orthogonal property of Legendre polynomials in  $[0,1]$  we obtain

$$\begin{aligned} \phi'(\tau) \int_0^1 \mu P_l(2\mu-1) d\mu + \frac{1}{4(2l+1)} \left[ \frac{l^2-l}{2l-1} I_{l-2}^{\prime} + 2l I_{l-1}^{\prime} + \right. \\ \left. + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)} I_l^{\prime} + 2(l+1) I_{l+1}^{\prime} + \frac{l^2+3l+2}{2l+3} I_{l-2}^{\prime} \right] = \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \psi(\mu) P_l(2\mu - 1) d\mu + \frac{1}{2(2l + 1)} \left[ H_{l-1}^+ + (2l + 1)I_l^+ + (l + 1)I_{l+1}^+ \right] - \\
&\quad - \frac{1}{2} \delta_{0l} - S^+, \quad l=0, 1, 2, \dots, L
\end{aligned} \tag{2.2.1.8a}$$

Similarly multiplying both sides of (2.2.1.7b) by  $P_l(2\mu + 1)$  and integrating over  $[-1, 0]$  we obtain the following equation

$$\begin{aligned}
&\Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu - 1) d\mu + \frac{1}{4(2l + 1)} \left[ \frac{l^2 - l}{2l - 1} I_{l-2}^- - 2H_{l-1}^- + \right. \\
&\quad \left. - \frac{12l^3 + 18l^2 - 2l - 4}{(2l + 3)(2l - 1)} I_l^- - 2(l + 1)I_{l+1}^- + \frac{l^2 + 3l + 2}{2l + 3} I_{l+2}^- \right] = \\
&= \int_{-1}^0 \psi(\mu) P_l(2\mu + 1) d\mu + \frac{1}{2(2l + 1)} \left[ H_{l-1}^- - (2l + 1)I_l^- + (l + 1)I_{l+1}^- \right] - \\
&\quad - \frac{1}{2} \delta_{0l} - S^-, \quad l=0, 1, 2, \dots, L
\end{aligned} \tag{2.2.1.8b}$$

where

$$S^+ = \frac{3}{64} (I_0^+ - I_0^-) \int_0^1 (\mu^2 + 5) P_l(2\mu - 1) d\mu -$$

$$\begin{aligned}
& - \frac{3}{320} (I_1^+ + I_1^-) \int_0^1 (17\mu^2 + 21) P_1(2\mu - 1) d\mu - \\
& - \frac{3}{64} \left[ (I_2^+ - I_2^-) + \frac{I_3^+ + I_3^-}{5} \right] \int_0^1 (3\mu^2 - 1) P_1(2\mu - 1) d\mu
\end{aligned} \tag{2.2.1.9a}$$

and

$$\begin{aligned}
S^- &= \frac{3}{64} (I_0^+ - I_0^-) \int_{-1}^0 (\mu^2 + 5) P_1(2\mu + 1) d\mu - \\
& - \frac{3}{320} (I_1^+ + I_1^-) \int_{-1}^0 (17\mu^2 + 21) P_1(2\mu + 1) d\mu - \\
& - \frac{3}{64} \left[ (I_2^+ - I_2^-) + \frac{I_3^+ + I_3^-}{5} \right] \int_{-1}^0 (3\mu^2 - 1) P_1(2\mu + 1) d\mu
\end{aligned} \tag{2.2.1.9b}$$

and  $\delta_{0l}$  is the Kronecker delta and dash denotes derivatives' w.r.t optical thickness  $\tau$ .

The equations (2.2.1.8a) and (2.2.1.8b) are to be solved subject to the boundary conditions which are restated below

$$I_l^-(0) = 0 \tag{2.2.1.10}$$

$$I_l^+(\tau) e^{-\tau} \rightarrow 0 \text{ as } \tau \rightarrow \infty \tag{2.2.1.11a}$$

$$I_l^-(\tau)e^{-k\tau} \rightarrow 0 \quad \text{as } \tau \rightarrow \infty \quad (2.2.1.11b)$$

In case of N-th approximations we assume

$$I_{N+1}^+ = I_{N+1}^- = 0 \quad (2.2.1.12)$$

Also we assume a trial solution of the form

$$I_l^+(\tau) = g_{l,\alpha}e^{-k\tau} + g_{l,\beta} \quad (2.2.1.13a)$$

$$I_l^-(\tau) = h_{l,\alpha}e^{-k\tau} + h_{l,\beta} \quad (2.2.1.13b)$$

where  $g_{l,\alpha}, g_{l,\beta}, h_{l,\alpha}, h_{l,\beta}$  are constants which are to be determined.

Using the boundary condition (2.2.1.11a) and (2.2.1.11b) we can write the solution as

$$I_l^+(\tau) = \sum_{r=1}^{n-1} g_{l,\alpha}^{(r)} e^{-k_r \tau} + g_{l,\beta} \quad (2.2.1.14a)$$

$$I_l^-(\tau) = \sum_{r=1}^{n-1} h_{l,\alpha}^{(r)} e^{-k_r \tau} + h_{l,\beta} \quad (2.2.1.14b)$$

And from (2.2.1.10) we have

$$\sum_{r=1}^{n-1} h_{l,\alpha}^{(r)} + h_{l,\beta} = 0 \quad (2.2.1.14c)$$

where  $k_r$  are the  $n-2$  roots of the determinant of order  $N$ . It will be found that there will be two zero roots for all order of approximation  $N$ . Here will consider two approximations, viz.  $N=1$  and  $N=2$ .

Now the quantities  $F$  (net flux),  $J(\tau)$  (mean intensity) are defined in section (2.1.1) and the source function in this case is

$$\mathfrak{S}(\tau, \mu) = \frac{3}{8} \left[ (3 - \mu^2)J(\tau) + (3\mu^2 - 1)K(\tau) \right] \quad (2.2.1.15)$$

where

$$K(\tau) = \frac{1}{2} \int_{-1}^1 I(\tau, \mu) \mu^2 d\mu \quad (2.2.1.16)$$

The law of darkening ( or the emergent intensity) is given by

$$I(0, \mu) = \int_0^{\infty} \mathfrak{S}(\tau, \mu) e^{-\frac{\tau}{\mu}} \frac{d\tau}{\mu} \quad (2.2.1.17)$$

In this case also we take the form of  $\phi(\tau)$  as

$$\phi(\tau) = A\tau, \quad \text{where } A \text{ being some constant which is to be determined.}$$

### 2.2.2. First Approximation.

In this case  $L = 1$  and we neglect  $I_2^+$  and  $I_2^-$ .

For  $l = 0, 1$  we have from (2.2.1.8a)

$$\left( \frac{4}{3}I_0^+ + 2I_1^+ \right) - \left( I_0^+ + I_1^+ + I_0^- - I_1^- \right) = 2 - 2\phi'(\tau) \quad (2.2.2.1)$$

$$\left( 2I_0^{+'} + \frac{24}{5}I_1^{+'} \right) - \frac{61}{32}I_0^+ - \frac{909}{160}I_1^+ - \frac{3}{32}I_0^- + \frac{51}{160}I_1^- = -2\phi'(\tau) \quad (2.2.2.2)$$

For  $l = 0, 1$  from (2.2.1.8b) we get

$$\left( \frac{4}{3}I_0^{-'} - 2I_1^{-'} \right) + \left( I_0^+ + I_1^+ + I_0^- - I_1^- \right) = -2 + 2\phi'(\tau) \quad (2.2.2.3)$$

$$\left( -2I_0^{-'} + \frac{24}{5}I_1^{-'} \right) - \frac{3}{32}I_0^+ - \frac{51}{160}I_1^+ - \frac{61}{32}I_0^- + \frac{909}{160}I_1^- = -2\phi'(\tau) \quad (2.2.2.4)$$

Inserting equations (2.2.1.13a) and (2.2.1.13b) in (2.2.2.1) to (2.2.2.4) and equating the coefficients of  $e^{-k\tau}$  we obtain the following set of equations

$$\left( \frac{4k}{3} + 1 \right) g_{0,\alpha} + (2k + 1)g_{1,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \quad (2.2.2.5i)$$

$$\left( 2k + \frac{61}{32} \right) g_{0,\alpha} + \left( \frac{24k}{5} + \frac{909}{160} \right) g_{1,\alpha} + \frac{3}{32}h_{0,\alpha} - \frac{51}{160}h_{1,\alpha} = 0 \quad (2.2.2.5ii)$$

$$g_{0,\alpha} + g_{1,\alpha} + \left( -\frac{4k}{3} + 1 \right) h_{0,\alpha} + (-2k - 1)h_{1,\alpha} = 0 \quad (2.2.2.5iii)$$

$$\frac{3}{32}g_{0,\alpha} + \frac{51}{160}g_{1,\alpha} + \left(-2k + \frac{61}{32}\right)h_{0,\alpha} + \left(\frac{24k}{5} - \frac{909}{160}\right)h_{1,\alpha} = 0 \quad (2.2.2.5iv)$$

The equation (2.2.2.5) will have a nontrivial solution if

$$D_1(k) = 0 \quad (2.2.2.6)$$

where

$$D_1(k) = \begin{vmatrix} \frac{4k}{3} + 1 & 2k + 1 & 1 & -1 \\ 2k + \frac{61}{32} & \frac{24k}{5} + \frac{909}{160} & \frac{3}{32} & -\frac{51}{160} \\ 1 & 1 & -\frac{4k}{3} + 1 & 2k - 1 \\ \frac{3}{32} & \frac{51}{160} & -2k + \frac{61}{32} & \frac{24k}{5} - \frac{909}{160} \end{vmatrix}$$

Therefore from (2.2.2.6) we have

$$k = 0, 0, \pm 2.9167 \quad (2.2.2.7)$$

To satisfy the boundary condition we will take only the positive root.

Again equating the coefficients of constants terms we get

$$g_{0,\beta} + g_{1,\beta} + h_{0,\beta} - h_{1,\beta} = 2A - 2 \quad (2.2.2.8i)$$

$$\frac{61}{32}g_{0,\beta} + \frac{909}{160}g_{1,\beta} + \frac{3}{32}h_{0,\beta} - \frac{51}{160}h_{1,\beta} = 2A \quad (2.2.2.8ii)$$

$$\frac{3}{32}g_{0,\beta} + \frac{51}{160}g_{1,\beta} + \frac{61}{32}h_{0,\beta} - \frac{909}{160}h_{1,\beta} = 2A \quad (2.2.2.8iii)$$

From the boundary condition (2.2.1.14c) we have,

$$\left. \begin{aligned} h_{0,\alpha} + h_{0,\beta} &= 0 \\ h_{1,\alpha} + h_{1,\beta} &= 0 \end{aligned} \right\} \quad (2.2.2.9)$$

We see that there are 9 unknown constants to be determined from 8 equations, viz., (2.2.2.5), (2.2.2.8) and (2.2.2.9). For an extra equation we consider the first moment of H-function. We have

$$\alpha_0 = \int_0^1 H(\mu) d\mu = 2.06088 \quad (2.2.2.10)$$

which gives

$$g_{0,\alpha} + g_{1,\alpha} + g_{0,\beta} + g_{1,\beta} = 2.12176 \quad (2.2.2.11)$$

Therefore taking  $k = 2.9167$  and solving the equations (2.2.2.5), (2.2.2.8) and (2.2.2.9) we obtain

$$g_{0,\alpha} = 0.2144, \quad h_{0,\alpha} = 2.1162$$

$$g_{1,\alpha} = -.5273, \quad h_{1,\alpha} = .9968$$

$$g_{0,\beta} = 0.8231, \quad h_{0,\beta} = -2.1162$$

$$g_{1,\beta} = 0.0331, \quad h_{1,\beta} = -0.09968$$

$$A = 0.8535$$

We can calculate the values of  $I_0^+, I_1^+, I_0^-, I_1^-$ . Therefore, we find that

$$J(\tau) = I(0,0) \left[ A\tau + \frac{1}{2} + \frac{1}{4} (I_0^+ + I_1^+ - I_0^- + I_1^-) \right]$$

$$\text{Or, } J(\tau) = I(0,0) [0.8535\tau + 0.9939 - 0.3583e^{-k\tau}] \quad (2.2.2.12)$$

$$K(\tau) = I(0,0) \left[ \frac{A\tau}{3} + \frac{1}{6} + \frac{5(I_0^+ - I_0^-) + 9(I_1^+ + I_1^-)}{40} \right]$$

$$\text{Or, } K(\tau) = I(0,0) \left[ \frac{0.2845\tau}{3} + 0.3173 - 0.0581e^{-k\tau} \right] \quad (2.2.2.13)$$

The net flux is given by

$$F = 2I(0,0) \left[ \frac{1}{2} + \frac{1}{3} (I_0^+ + I_0^-) + \frac{1}{2} (I_1^+ - I_1^-) \right] \quad (2.2.2.14)$$

From which we get  $I(0,0) = 0.6927 F$

The source function now becomes

$$\mathfrak{S}(\tau, \mu) = \frac{3}{8} I(0,0) [2.276\tau + 2.6644 - 0.042\mu^2 - 1.0168e^{-k\tau} + 0.184\mu^2 e^{-k\tau}]$$

Hence we obtain the emergent intensity as

$$I(0, \mu) = \frac{3}{8} I(0,0) \left[ 2.276\mu + 2.6644 - 0.042\mu^2 - \frac{1.0168}{1+k\mu} + \frac{0.184\mu^2}{1+k\mu} \right] \quad (2.2.2.15)$$

Now expressing  $I(0,0)$  in terms of  $F$  we can find the ratio  $I(0,\mu)/F$  and  $I(0,\mu)/I(0,1)$ . The results are shown in the Table 2.2.1

### 2.2.3 Second approximation.

Here  $L = 2$ . As in case of first approximation, putting successively  $l = 0, 1, 2$  in equations (2.2.1.8a) and (2.2.1.8b) we obtain,

$$\left( \frac{4}{3} I_0^{+'} + 2I_1^{+'} + \frac{2}{3} I_2^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\Phi'(\tau) \quad (2.2.3.1)$$

$$\begin{aligned} \left( 2I_0^{+'} + \frac{24}{5} I_1^{+'} + 4I_2^{+'} \right) - \frac{61}{32} I_0^+ - \frac{909}{160} I_1^+ - \frac{119}{32} I_2^+ - \\ - \frac{3}{32} I_0^- + \frac{51}{160} I_1^- - \frac{9}{32} I_2^- = -2\Phi'(\tau) \end{aligned} \quad (2.2.3.2)$$

$$\left( \frac{1}{3} I_0^{+'} + 2I_1^{+'} + \frac{80}{21} I_2^{+'} \right) + \frac{1}{64} I_0^+ - \frac{623}{320} I_1^+ - \frac{317}{64} I_2^+ -$$

$$-\frac{1}{64}I_0^- + \frac{17}{320}I_1^- - \frac{3}{64}I_2^- = 0 \quad (2.2.3.3)$$

$$\left(\frac{4}{3}I_0^{-'} - 2I_1^{-'} + \frac{2}{3}I_2^{-'}\right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2 + 2\phi'(\tau) \quad (2.2.3.4)$$

$$\left(-2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'}\right) - \frac{3}{32}I_0^+ - \frac{51}{160}I_1^+ - \frac{9}{32}I_2^+ -$$

$$-\frac{61}{32}I_0^- + \frac{909}{160}I_1^- - \frac{119}{32}I_2^- = -2\phi'(\tau) \quad (2.2.3.5)$$

$$\left(\frac{1}{3}I_0^{-'} - 2I_1^{-'} + \frac{80}{21}I_2^{-'}\right) + \frac{1}{64}I_0^+ - \frac{17}{320}I_1^+ - \frac{3}{64}I_2^+ -$$

$$-\frac{1}{64}I_0^- - \frac{623}{320}I_1^- + \frac{317}{64}I_2^- = 0 \quad (2.2.3.6)$$

Inserting equations (2.2.1.13a) and (2.2.1.13b) in equations (2.2.3.1) to (2.2.3.6) and equating the coefficients of  $e^{-k\tau}$  we get,

$$\left(\frac{4k}{3} + 1\right)g_{0,\alpha} + (2k + 1)g_{1,\alpha} + \frac{2k}{3}g_{2,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \quad (2.2.3.7a)$$

$$\begin{aligned} & \left(2k + \frac{61}{32}\right)g_{0,\alpha} + \left(\frac{24k}{5} + \frac{909}{160}\right)g_{1,\alpha} + \left(4k + \frac{119}{32}\right)g_{2,\alpha} + \\ & + \frac{3}{32}h_{0,\alpha} - \frac{51}{160}h_{1,\alpha} + \frac{9}{32}h_{2,\alpha} = 0 \end{aligned} \quad (2.2.3.7b)$$

$$\begin{aligned} & \left(\frac{k}{3} - \frac{1}{64}\right)g_{0,\alpha} + \left(2k + \frac{623}{320}\right)g_{1,\alpha} + \left(\frac{80k}{21} + \frac{317}{64}\right)g_{2,\alpha} + \\ & + \frac{1}{64}h_{0,\alpha} - \frac{17}{320}h_{1,\alpha} + \frac{3}{64}h_{2,\alpha} = 0 \end{aligned} \quad (2.2.3.7c)$$

$$g_{0,\alpha} + g_{1,\alpha} + \left(-\frac{4k}{3} + 1\right)h_{0,\alpha} + (2k-1)h_{1,\alpha} - \frac{2k}{3}h_{2,\alpha} = 0 \quad (2.2.3.7d)$$

$$\begin{aligned} & \frac{3}{32}g_{0,\alpha} + \frac{51}{160}g_{1,\alpha} + \frac{9}{32}g_{2,\alpha} + \\ & + \left(-2k + \frac{61}{32}\right)h_{0,\alpha} + \left(\frac{24k}{5} - \frac{909}{160}\right)h_{1,\alpha} + \left(-4k + \frac{119}{32}\right)h_{2,\alpha} = 0 \end{aligned} \quad (2.2.3.7e)$$

$$\frac{1}{64}g_{0,\alpha} + \frac{17}{320}g_{1,\alpha} + \frac{3}{64}g_{2,\alpha} + \left(-\frac{k}{3} - \frac{1}{64}\right)h_{0,\alpha} + \left(2k - \frac{623}{320}\right)h_{1,\alpha} + \left(-\frac{80k}{21} + \frac{317}{64}\right)h_{2,\alpha} = 0 \quad (2.2.3.7f)$$

The above set of equations will have nontrivial solution  $D_2(k) = 0$  if, where

$$D_2(k) = \begin{vmatrix} \frac{4k}{3} + 1 & 2k + 1 & \frac{2k}{3} & 1 & -1 & 0 \\ 2k + \frac{61}{32} & \frac{24k}{5} + \frac{909}{160} & 4k + \frac{119}{32} & \frac{3}{32} & -\frac{51}{160} & \frac{9}{32} \\ \frac{k}{3} - \frac{1}{64} & 2k + \frac{623}{320} & \frac{80k}{21} + \frac{317}{64} & \frac{1}{64} & -\frac{17}{320} & \frac{3}{64} \\ 1 & 1 & 0 & -\frac{4k}{3} + 1 & 2k - 1 & -\frac{2k}{3} \\ \frac{3}{32} & \frac{51}{160} & \frac{9}{32} & -2k + \frac{61}{32} & \frac{24k}{5} - \frac{909}{160} & -4k + \frac{119}{32} \\ \frac{1}{64} & \frac{17}{320} & \frac{3}{64} & -\frac{k}{3} - \frac{1}{64} & 2k - \frac{623}{320} & -\frac{80k}{21} + \frac{317}{64} \end{vmatrix}$$

$$\text{This gives } D_2(k) = 7.5193K^6 - 104.9134k^4 + 155.4331k^2$$

Therefore,  $k = 0, 0, \pm 1.298, \pm 3.503$ . As usual we will take only the positive roots to satisfy the boundary conditions

$$k_1 = 1.298, \quad k_2 = 3.503 \quad (2.2.3.8)$$

Next, equating the coefficients of constant terms we obtain

$$g_{0,\beta} + g_{1,\beta} + h_{0,\beta} - h_{1,\beta} = 2A - 2 \quad (2.2.3.9a)$$

$$\frac{61}{32}g_{0,\beta} + \frac{909}{160}g_{1,\beta} + \frac{119}{32}g_{2,\beta} + \frac{3}{32}h_{0,\beta} - \frac{51}{160}h_{1,\beta} + \frac{9}{32}h_{2,\beta} = 2A \quad (2.2.3.9b)$$

$$\frac{1}{64}g_{0,\beta} - \frac{623}{320}g_{1,\beta} - \frac{317}{64}g_{2,\beta} - \frac{1}{64}h_{0,\beta} + \frac{17}{320}h_{1,\beta} - \frac{3}{64}h_{2,\beta} = 0 \quad (2.2.3.9c)$$

$$\frac{3}{32}g_{0,\beta} + \frac{51}{160}g_{1,\beta} + \frac{9}{32}g_{2,\beta} + \frac{61}{32}h_{0,\beta} - \frac{909}{160}h_{1,\beta} + \frac{119}{32}h_{2,\beta} = 2A \quad (2.2.3.9d)$$

$$\frac{1}{64}g_{0,\beta} + \frac{17}{320}g_{1,\beta} + \frac{3}{64}g_{2,\beta} - \frac{1}{64}h_{0,\beta} - \frac{623}{320}h_{1,\beta} + \frac{317}{64}h_{2,\beta} = 0 \quad (2.2.3.9e)$$

Next, from the boundary condition we have,

$$\left. \begin{aligned} h_{0,\alpha}^{(1)} + h_{0,\alpha}^{(2)} + h_{0,\beta} &= 0 \\ h_{1,\alpha}^{(1)} + h_{1,\alpha}^{(2)} + h_{1,\beta} &= 0 \\ h_{2,\alpha}^{(1)} + h_{2,\alpha}^{(2)} + h_{2,\beta} &= 0 \end{aligned} \right\} \quad (2.2.3.10)$$

As in case of first approximation we obtain the required constants

$$g_{0,\alpha}^{(1)} = -0.0171, \quad h_{0,\alpha}^{(1)} = 0.0624$$

$$g_{0,\alpha}^{(2)} = -0.2459, \quad h_{0,\alpha}^{(2)} = 2.7750$$

$$g_{1,\alpha}^{(1)} = 0.0077, \quad h_{1,\alpha}^{(1)} = 0.0431$$

$$g_{1,\alpha}^{(2)} = -0.0729, \quad h_{1,\alpha}^{(2)} = 2.0588$$

$$g_{2,\alpha}^{(1)} = -0.0004, \quad h_{2,\alpha}^{(1)} = -0.0122$$

$$g_{2,\alpha}^{(2)} = -0.0399, \quad h_{2,\alpha}^{(2)} = 0.8522$$

$$g_{0,\beta} = 2.7369, \quad h_{0,\beta} = -2.8365$$

$$g_{1,\beta} = -0.4333, \quad h_{1,\beta} = -2.1000$$

$$g_{2,\beta} = 0.1733, \quad h_{2,\beta} = -0.8400$$

$$A = 1.7835$$

Therefore the  $J$  and  $K$  integrals are given by

$$J(\tau) = I(0,0) \left[ 1.7835\tau + 1.26 - .0072e^{-k_1\tau} - .2223e^{-k_2\tau} \right],$$

and

$$K(\tau) = I(0,0) \left[ .5945\tau + .42 + .0030e^{-k_1\tau} + .0005e^{-k_2\tau} \right]$$

The law of darkening in this case is

$$I(0,\tau) = \frac{3}{8} I(0,0) \left[ 4.756\mu + 3.36 + \frac{.0162\mu^2 - .0246}{1 + k_1\mu} + \frac{.2238\mu^2 - .6674}{1 + k_2\mu} \right]$$

In the following table we give our results.

Table 2.2.1

$P_1$ - Approximation			$P_2$ - Approximation	
$\mu$	$I(0,\mu) / F$	$I(0,\mu) / I(0,1)$	$I(0,\mu) / F$	$I(0,\mu) / I(0,1)$
.05	.49124	.40359	.4767 (.48028)	.3753 (.37981)
.10	.54701	.44940	.5265 (.52961)	.4145 (.41881)
.15	.59756	.49093	.5736 (.57562)	.4516 (.45520)
.20	.64431	.52934	.6190 (.61981)	.4873 (.49014)
.25	.68821	.56541	.6630 (.66282)	.5220 (.52416)
.30	.72992	.59968	.7061 (.70501)	.5559 (.55752)
.35	.76990	.63252	.7484 (.74660)	.5892 (.59041)
.40	.80848	.66422	.7902 (.78771)	.6220 (.62292)
.45	.84592	.69498	.8314 (.82846)	.6545 (.65514)
.50	.88241	.72496	.8723 (.86891)	.6867 (.68971)
.55	.91811	.75429	.9129 (.90912)	.7187 (.71893)
.60	.95313	.78306	.9533 (.94912)	.7504 (.75056)
.65	.98756	.81135	.9934 (.98896)	.7820 (.78026)
.70	1.02148	.83922	1.0333 (1.02864)	.8135 (.81345)
.75	1.05496	.86672	1.0731 (1.06820)	.8448 (.84473)
.80	1.08804	.89390	1.1128 (1.10764)	.8760 (.87592)
.85	1.12077	.92079	1.1523 (1.14698)	.9071 (.90703)
.90	1.15319	.94742	1.1917 (1.18624)	.9381 (.93808)
.95	1.18532	.97382	1.2310 (1.22543)	.9691 (.96906)
1.00	1.21719	1.0000	1.2703 (1.26455)	1.0000 (1.00000)

Here the data in the parentheses indicates that due to Chandrasekhar. We find that second approximation, i.e.  $P_2$  method is superior to that of first approximation, i.e.  $P_1$  for almost all values of  $\mu$ .

## 2.3 Solution of a radiative transfer problem in semi-infinite atmosphere with General phase function using a modified form of spherical harmonic method.

### 2.3.1 The problem and the equations

The equation of transfer in for plane parallel scattering atmosphere with axial symmetry is given by

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^1 p(\mu, \mu') I(\tau, \mu') d\mu' \quad (2.3.1.1)$$

where the phase function is, as usual defined by

$$p(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} p(\mu, \mu'; \phi, \phi') d\phi' \quad (2.3.1.2)$$

The other symbols have already been explained in section 2.1.1. The equation (2.3.1.1) is solved subject to the appropriate boundary conditions. These boundary conditions are

(a) Absence of incident radiation from outside at the free surface  $\tau = 0$ , i.e.,

$$I(0, -\mu) = 0, \quad \text{for } 0 \leq \mu \leq 1 \quad (2.3.1.3a)$$

(b) the convergence of intensity. i.e.

$$I(\tau, \mu) e^{-\tau} \rightarrow 0 \quad \text{as } \tau \rightarrow \infty \quad (2.3.1.3b)$$

We represent  $I(\tau, \mu)$  by two different expansions given by

$$I^*(\tau, \mu) = I(0, 0) \left[ \phi(\tau) + \psi(\mu) \sum_{l=0}^{l=L} (2l+1) I_l^*(\tau) \mu P_l(2\mu-1) \right] \quad 0 \leq \mu \leq 1 \quad (2.3.1.4a)$$

$$I^-(\tau, \mu) = I(0,0) \left[ \phi(\tau) + \psi(\mu) \sum_{l=0}^{l=L} (2l+1) I_l^-(\tau) \mu P_l(2\mu+1) \right] \quad -1 \leq \mu \leq 0 \quad (2.3.1.4b)$$

where  $I(0,0)$  is some constant,  $\phi(\tau)$  is a function of  $\tau$  only and we define  $\psi(\mu)$  by

$$\psi(\mu) = \begin{cases} 1 & \text{if } 0 \leq \mu \leq 1 \\ 0 & \text{if } -1 \leq \mu \leq 0 \end{cases} \quad (2.3.1.6)$$

The function  $\phi(\tau)$  will be specified later. Therefore, with these two forms of intensity, the equation of transfer now becomes,

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2} \left[ \int_0^1 p(\mu, \mu') I^+(\tau, \mu') d\mu' + \int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' \right] \quad (2.3.1.7a)$$

and

$$\mu \frac{dI^-(\tau, \mu)}{d\tau} = I^-(\tau, \mu) - \frac{1}{2} \left[ \int_0^1 p(\mu, \mu') I^+(\tau, \mu') d\mu' + \int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu') d\mu' \right] \quad (2.3.1.7b)$$

We take the phase function represented in the following manner

$$p(\mu, \mu') = \sum_{k=0}^{\infty} w_k P_k(\mu) P_k(\mu') \quad (2.3.1.8)$$

where  $w_k$  are assumed to be simply constants and  $P_k$  are Legendre polynomials and they satisfy the well-known recurrence relation

$$\mu P_l(2\mu \pm 1) = \frac{1}{2l+1} \left[ \frac{l+1}{2} P_{l+1}(2\mu \pm 1) \mp \frac{2l+1}{2} P_l(2\mu \pm 1) + \frac{l}{2} P_{l-1}(2\mu \pm 1) \right]$$

Let us now use the following notations

$$S_{l,k}^+ = \int_0^1 P_k(\mu) P_l(2\mu - 1) d\mu \quad (2.3.1.9a)$$

$$S_{l,k}^- = \int_{-1}^0 P_k(\mu) P_l(2\mu + 1) d\mu \quad (2.3.1.9b)$$

Using these two notations (2.3.1.9) and the recurrence formula we obtain,

$$\begin{aligned} \int_0^1 p(\mu, \mu') I^+(\tau, \mu) d\mu' &= I(0,0) \sum_{k=0}^{\infty} w_k P_k(\mu) \left[ \{\Phi(\tau) + 1\} \int_0^1 P_k(\mu') d\mu' + \right. \\ &\quad \left. + \sum_{l=0}^{l=L} \frac{I_l^+(\tau)}{2} \{(I+1)S_{l+1,k}^+ + (2l+1)S_{l,k}^+ + lS_{l-1,k}^+\} \right] \end{aligned} \quad (2.3.1.10a)$$

and

$$\begin{aligned} \int_{-1}^0 p(\mu, \mu') I^-(\tau, \mu) d\mu' &= I(0,0) \sum_{k=0}^{\infty} w_k P_k(\mu) \left[ \Phi(\tau) \int_{-1}^0 P_k(\mu') d\mu' + \right. \\ &\quad \left. + \sum_{l=0}^{l=L} \frac{I_l^-(\tau)}{2} \{(I+1)S_{l+1,k}^- - (2l+1)S_{l,k}^- + lS_{l-1,k}^-\} \right] \end{aligned} \quad (2.3.1.10b)$$

Further we assume,

$$\Lambda_l = (I+1)S_{l+1,k}^+ + (2l+1)S_{l,k}^+ + lS_{l-1,k}^+ \quad (2.3.1.11a)$$

$$\Lambda_2 = (l+1)S_{l+1,k}^- - (2l+1)S_{l,k}^- + lS_{l-1,k}^- \quad (2.3.1.11a)$$

Hence the equation of transfer takes the following forms

$$\mu \frac{dI^+(\tau, \mu)}{d\tau} = I^+(\tau, \mu) - \frac{1}{2}I(0,0) \sum_{k=0}^{\infty} w_k P_k(\mu) \Pi \quad (2.3.1.12)$$

$$\mu \frac{dI^-(\tau, \mu)}{d\tau} = I^-(\tau, \mu) - \frac{1}{2}I(0,0) \sum_{k=0}^{\infty} w_k P_k(\mu) \Pi \quad (2.3.1.13)$$

where

$$\Pi = \phi(\tau) \int_{-1}^1 P_k(\mu') d\mu' + \int_{-1}^1 \psi(\mu') P_k(\mu') d\mu' + \sum_{l=0}^{l=L} \frac{I_l^+(\tau)}{2} \Lambda_1 + \sum_{l=0}^{l=L} \frac{I_l^-(\tau)}{2} \Lambda_2 \quad (2.3.1.14)$$

Now, using the form of intensity (vide.eq. 2.3.1.4a) in equation (2.3.1.12) and then multiplying by  $P_l(2\mu - 1)$  and integrating over  $[0,1]$  we obtain

$$\begin{aligned} \phi'(\tau) \int_0^1 \mu P_l(2\mu - 1) d\mu + \frac{1}{4(2l+1)} \left[ \frac{l^2 - l}{2l-1} I_{l-2}^{+'} + 2l I_{l-1}^{+'} + \right. \\ \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1) I_{l+1}^{+'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{+'} \right] = [\phi(\tau) + 1] \delta_{0,l} + \\ + \frac{1}{2(2l+1)} \left[ l I_{l-1}^{+'} + (2l+1) I_l^{+'} + (l+1) I_{l+1}^{+'} \right] - \frac{1}{2} \sum_{k=0}^{\infty} w_k S_{l,k}^{+'} \Pi, \quad l = 0, 1, 2, \dots, L \quad (2.3.1.15a) \end{aligned}$$

Similarly, using (2.3.1.4b) in (2.3.1.13) and then multiplying it by  $P_l(2\mu + 1)$  and integrating over  $[-1,0]$  we obtain

$$\begin{aligned}
& \Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu + 1) d\mu + \frac{1}{4(2l+1)} \left[ \frac{l^2 - l}{2l-1} I_{l-2}^- - 2l I_{l-1}^- + \right. \\
& \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^- - 2(l+1) I_{l+1}^- + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^- \right] = \Phi(\tau) \delta_{0,l} + \\
& + \frac{1}{2(2l+1)} \left[ l I_{l-1}^- - (2l+1) I_l^- + (l+1) I_{l+1}^- \right] - \frac{1}{2} \sum_{k=0}^{\infty} w_k S_{l,k}^- \Pi, \quad l=0,1,2,\dots,L \quad (2.3.15b)
\end{aligned}$$

We note that in equations (2.3.1.15a) and (2.3.1.15b) we have used the recurrence formulae for the Legendre polynomials and their orthogonality relations in respective intervals. The equations (2.3.1.15a) and (2.3.1.15b) are two systems of ordinary differential equations which are to be solved subject to the boundary conditions given in (2.3.1.3a) and (2.3.1.3b)

Let us set  $w_0 = 1$ . Carrying out simple calculations we find that

$$\sum_{k=0}^{\infty} w_k S_{l,k}^+ \Pi = \delta_{0l} \left[ 2\Phi(\tau) + 1 + \frac{1}{2} (I_0^+ + I_1^+ - I_0^- + I_1^-) \right] + \sum_{k=1}^{\infty} w_k S_{l,k}^+ \Pi \quad (2.3.1.16)$$

and

$$\sum_{k=0}^{\infty} w_k S_{l,k}^- \Pi = \delta_{0l} \left[ 2\Phi(\tau) + 1 + \frac{1}{2} (I_0^+ + I_1^+ - I_0^- + I_1^-) \right] + \sum_{k=1}^{\infty} w_k S_{l,k}^- \Pi \quad (2.3.1.17)$$

For  $l = 0$  we get from (2.3.1.15a) and (2.3.1.15b) the following equations

$$2\Phi'(\tau) + \left( \frac{4}{3} I_0^+ + 2I_1^+ + \frac{2}{3} I_2^+ \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) =$$

$$= 2 - 2 \sum_{k=1}^{\infty} w_k S_{0,k}^+ \Pi \quad (2.3.1.18a)$$

and

$$\begin{aligned} -2\Phi'(\tau) + \left( \frac{4}{3} I_0^- - 2I_1^- + \frac{2}{3} I_2^- \right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = \\ = -2 - 2 \sum_{k=1}^{\infty} w_k S_{0,k}^- \Pi \quad (2.3.1.18b) \end{aligned}$$

and for  $l \neq 0$  we obtain respectively from the same equations [(2.3.1.15a) and (2.3.1.15b)] the following ones

$$\begin{aligned} \Phi'(\tau) \int_0^1 \mu P_l(2\mu - 1) d\mu + \frac{1}{4(2l+1)} \left[ \frac{l^2 - l}{2l-1} I_{l-2}^{+'} + 2l I_{l-1}^{+'} + \right. \\ \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{+'} + 2(l+1) I_{l+1}^{+'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{+'} \right] = \\ = \frac{1}{2(2l+1)} \left[ l I_{l-1}^+ + (2l+1) I_l^+ + (l+1) I_{l+1}^+ \right] - \frac{1}{2} \sum_{k=1}^{\infty} w_k S_{l,k}^+ \Pi, \quad l = 1, 2, \dots, L \quad (2.3.1.19a) \end{aligned}$$

and

$$\begin{aligned} \Phi'(\tau) \int_{-1}^0 \mu P_l(2\mu + 1) d\mu + \frac{1}{4(2l+1)} \left[ \frac{l^2 - l}{2l-1} I_{l-2}^{-'} - 2l I_{l-1}^{-'} + \right. \\ \left. + \frac{12l^3 + 18l^2 - 2l - 4}{(2l+3)(2l-1)} I_l^{-'} - 2(l+1) I_{l+1}^{-'} + \frac{l^2 + 3l + 2}{2l+3} I_{l+2}^{-'} \right] = \end{aligned}$$

$$= \frac{1}{2(2l+1)} \left[ II_{l-1}^- - (2l+1)I_l^- + (l+1)I_{l+1}^- \right] - \frac{1}{2} \sum_{k=1}^{\infty} w_k S_{l,k}^- \Pi, \quad l=1,2,\dots,L \quad (2.3.1.19b)$$

The equations (2.3.1.19a) and (2.3.1.19b) will be solved subject to the boundary conditions (2.3.1.3a) and (2.3.1.3b). As before, while working in the  $L$  th approximation, we will neglect  $I_{L+1}^+, I_{L+1}^-$ . We will discuss two phase functions, namely Isotropic and Rayleigh in both first and second approximations and deduce the spherical harmonics equations from (2.3.1.15a) and (2.3.1.15b)

## 2.3.2. Different cases.

### 2.3.2.1. Case 1. Isotropic scattering.

The phase function in this case is

$$p(\mu, \mu') = P_0(\mu)P_0(\mu') = 1 \quad (2.3.2.1)$$

From (2.3.1.8) and (2.3.2.1), on comparing we find that

$$w_0 = 1, \quad \text{and } w_k = 0, \quad \text{for } k \geq 1 \quad (2.3.2.2)$$

### A. First approximations.

Here  $L = 1$ ,

For  $l = 0$  from (2.3.1.18a) we obtain the equation

$$\left( \frac{4}{3}I_0^{+'} + 2I_1^{+'} \right) - \left( I_0^+ + I_1^+ + I_0^- - I_1^- \right) = 2 - 2\phi'(\tau) \quad (2.3.2.2)$$

For  $l = 1$  from (2.3.1.19a) we get

$$\left( 2I_0^{+'} + \frac{24k}{5}I_1^{+'} \right) - 2\left( I_0^+ + 3I_1^+ \right) = -2\phi'(\tau) \quad (2.3.2.3)$$

For  $l = 0$  from (2.3.1.18b) we have

$$\left( \frac{4}{3}I_0^{-'} - 2I_1^{-'} \right) + \left( I_0^{-} + I_1^{+} + I_0^{-} - I_1^{-} \right) = -2 + 2\phi'(\tau) \quad (2.3.2.4)$$

And for  $l = 1$  from (2.3.1.19b) we get

$$\left( -2I_0^{-'} + \frac{24k}{5}I_1^{-'} \right) - 2\left( I_0^{-} - 3I_1^{-} \right) = -2\phi'(\tau) \quad (2.3.2.5)$$

The equations (2.3.2.2) to (2.3.2.5) are identical with the equations (2.1.3.1) to (2.3.1.4) which have already been obtained in section 2.1.3 and their solutions are also found out.

### B. Second approximations.

Here  $L = 2$

For  $l = 0$  from (2.3.1.18a) we get

$$\left( \frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} \right) - \left( I_0^{+} + I_1^{+} + I_0^{-} - I_1^{-} \right) = 2 - 2\phi'(\tau) \quad (2.3.2.6)$$

For  $l = 1, 2$  we get successively from (2.3.1.19a)

$$\left( 2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} \right) - 2\left( I_0^{+} + 3I_1^{+} + 2I_2^{+} \right) = -2\phi'(\tau) \quad (2.3.2.7)$$

and

$$\left( \frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} \right) - (2I_1^+ + 5I_2^+) = 0 \quad (2.3.2.8)$$

For  $l = 0$  from (2.3.1.18b)

$$\left( \frac{4}{3}I_0^{-'} - 2I_1^{-'} + \frac{2}{3}I_2^{-'} \right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2 + 2\phi'(\tau) \quad (2.3.2.9)$$

For  $l = 1, 2$  from (2.3.1.19b) we obtain successively

$$\left( -2I_0^{-'} + \frac{24}{5}I_1^{-'} - 4I_2^{-'} \right) - 2(I_0^- - 3I_1^- + 2I_2^-) = -2\phi'(\tau) \quad (2.3.2.10)$$

and

$$\left( \frac{1}{3}I_0^{-'} - 2I_1^{-'} + \frac{80}{21}I_2^{-'} \right) - (2I_1^- - 5I_2^-) = 0 \quad (2.3.2.11)$$

Again these equations (i.e., (2.3.2.6) to (2.3.2.11)) are identical with the equations (2.1.3.1) to (2.1.3.6) which we have already obtained and their results were calculated.

### 2.3.1.2. Case II. Rayleigh phase function.

The Rayleigh phase function is given by

$$p(\mu, \phi; \mu', \phi') = \frac{3}{4}(1 + \cos^2(\Theta)),$$

where  $\Theta$  is the angle of scattering and is given by

$$\cos\Theta = \left[ \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\Phi - \Phi') \right]$$

$$P(\mu, \mu') = 1 + \frac{1}{8}(3\mu^2 - 1)(3\mu'^2 - 1) = 1 + P_2(\mu)P_2(\mu') \quad (2.3.3.1)$$

From (2.3.3.1) and (2.3.1.8) we find that

$$w_0 = 1, w_1 = 0, w_2 = \frac{1}{2} \text{ and } w_k = 0, \text{ for } k \geq 3$$

The spherical harmonics equations are obtained in the following manner.

#### A. First approximations.

Here  $L=1$

For  $l=0$ ; the equation (2.3.1.18a) directly gives

$$\left( \frac{4}{3}I_0^{+'} + 2I_1^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\Phi'(\tau) \quad (2.3.3.2)$$

For  $l=1$ , from (2.3.1.19a) we obtain

$$\left( 2I_0^{+'} + \frac{24k}{5}I_1^{+'} \right) - 2(I_0^+ + 3I_1^+) = -2\Phi'(\tau) - 6 \sum_{k=1}^2 w_k S_{1,k}^+ \Pi$$

We have obtained, 
$$\Pi = \frac{1}{2} \left[ \frac{I_0^+ - I_0^-}{4} + \frac{17}{20}(I_1^+ + I_1^-) \right]$$

and this gives after little calculation

$$\left( 2I_0^{+'} + \frac{24}{5}I_1^{+'} \right) - \frac{61}{32}I_0^+ - \frac{909}{160}I_1^+ - \frac{3}{32}I_0^- + \frac{51}{160}I_1^- = -2\Phi'(\tau) \quad (2.3.3.3)$$

Next for  $l=0$  directly from (2.3.1.18b) we get

$$\left( \frac{4}{3}I_0^{-'} - 2I_1^{-'} \right) + \left( I_0^+ + I_1^+ + I_0^- - I_1^- \right) = -2 + 2\Phi'(\tau) \quad (2.3.3.4)$$

For  $l=1$ , from (2.3.1.19b) we obtain

$$\left( -2I_0^{-'} + \frac{24k}{5}I_1^{-'} \right) - 2\left( I_0^- - 3I_1^- \right) = -2\Phi'(\tau) - 6 \sum_{k=1}^2 w_k S_{1,k}^- \Pi,$$

and using the value of  $\Pi$  we obtain as before

$$\left( -2I_0^{-'} + \frac{24}{5}I_1^{-'} \right) - \frac{3}{32}I_0^+ - \frac{51}{160}I_1^+ - \frac{61}{32}I_0^- + \frac{909}{160}I_1^- = -2\Phi'(\tau) \quad (2.3.3.5)$$

So we see that the equations (2.3.3.2) to (2.3.3.5) are identical with the equations (2.2.3.1) to (2.2.3.4) and we have already solved the later ones.

## B. Second approximations.

Here  $L=2$

For  $l=0$ , from (2.3.1.8a) we have

$$\left( \frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} \right) - \left( I_0^+ + I_1^+ + I_0^- - I_1^- \right) = 2 - 2\Phi'(\tau) - 2 \sum_{k=1}^2 w_k S_{0,k}^+ \Pi$$

We have found 
$$\Pi = \frac{1}{2} \left[ \frac{I_0^+ - I_0^-}{4} + \frac{17}{20}(I_1^+ + I_1^-) + \frac{3}{4}(I_2^+ - I_2^-) \right]$$

The quantities  $S_{l,k}^+$  are easily calculated from (2.3.1.9a) and then we get

$$\left( \frac{4}{3}I_0^{+'} + 2I_1^{+'} + \frac{2}{3}I_2^{+'} \right) - (I_0^+ + I_1^+ + I_0^- - I_1^-) = 2 - 2\phi'(\tau) \quad (2.3.3.6)$$

For  $l=1$  from (2.3.1.19a)

$$\left( 2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} \right) - 2(I_0^+ + 3I_1^+ + 2I_2^+) = -2\phi'(\tau) - 6 \sum_{k=1}^2 w_k S_{1,k}^+ \Pi$$

and using  $S_{1,k}^+, \Pi$ , we get on simplification

$$\begin{aligned} \left( 2I_0^{+'} + \frac{24}{5}I_1^{+'} + 4I_2^{+'} \right) - \frac{61}{32}I_0^+ - \frac{909}{160}I_1^+ - \frac{119}{32}I_2^+ - \\ - \frac{3}{32}I_0^- + \frac{51}{160}I_1^- - \frac{9}{32}I_2^- = -2\phi'(\tau) \end{aligned} \quad (2.3.3.7)$$

For  $l=2$  from (2.3.1.19a) again

$$\frac{1}{20} \left( \frac{2}{3}I_0^{+'} + 4I_1^{+'} + \frac{160}{21}I_2^{+'} \right) = \frac{1}{10} (2I_1^+ + 5I_2^+) - \frac{1}{2} \sum_{k=1}^2 w_k S_{2,k}^+ \Pi$$

and on simplifying we get

$$\begin{aligned} \left( \frac{1}{3}I_0^{+'} + 2I_1^{+'} + \frac{80}{21}I_2^{+'} \right) + \frac{1}{64}I_0^+ - \frac{623}{320}I_1^+ - \frac{317}{64}I_2^+ - \\ - \frac{1}{64}I_0^- + \frac{17}{320}I_1^- - \frac{3}{64}I_2^- = 0 \end{aligned} \quad (2.3.3.8)$$

Again for  $l = 0$  from (2.3.1.18b) we have straightway

$$\left( \frac{4}{3}I_0^- - 2I_1^- + \frac{2}{3}I_2^- \right) + (I_0^+ + I_1^+ + I_0^- - I_1^-) = -2 + 2\phi'(\tau) \quad (2.3.3.9)$$

Further for  $l = 1$  from (2.3.1.19b)

$$\left( -2I_0^- + \frac{24}{5}I_1^- - 4I_2^- \right) = 2(I_0^- - 3I_1^- + 2I_2^-) - 2\phi'(\tau) - 6 \sum_{k=1}^2 w_k S_{1,k}^- \Pi$$

and using  $S_{1,k}^-$ ,  $\Pi$ , we get

$$\begin{aligned} \left( -2I_0^- + \frac{24}{5}I_1^- - 4I_2^- \right) - \frac{3}{32}I_0^+ - \frac{51}{160}I_1^+ - \frac{9}{32}I_2^+ - \\ - \frac{61}{32}I_0^- + \frac{909}{160}I_1^- - \frac{119}{32}I_2^- = -2\phi'(\tau) \end{aligned} \quad (2.3.3.10)$$

and for  $l = 2$  from (2.3.1.19b)

$$\frac{1}{20} \left( \frac{1}{3}I_0^- - 2I_1^- + \frac{80}{21}I_2^- \right) = \frac{1}{10} (2I_1^- - 5I_2^-) - \frac{1}{2} \sum_{k=1}^2 w_k S_{2,k}^- \Pi$$

on simplification the last equation gives

$$\left( \frac{1}{3}I_0^- - 2I_1^- + \frac{80}{21}I_2^- \right) + \frac{1}{64}I_0^+ - \frac{17}{320}I_1^+ - \frac{3}{64}I_2^+ -$$

$$- \frac{1}{64}I_0^- - \frac{623}{320}I_1^- + \frac{317}{64}I_2^- = 0 \quad (2.3.3.11)$$

These equations (i.e., from (2.3.3.6) to (2.3.3.11)) are identical with those in section (2.2.3) and those have already been solved.