SOME LINEAR AND NON-LINEAR PROBLEMS ON 'DEFLECTIONS' AND 'VIBRATIONS' OF THIN ELASTIC PLATES

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PREFACE

This thesis which contains five chapters is concerned with some linear and non-linear problems of thin elastic plates.

The first chapter contains two linear problems on uniformly compressed thin plates of variable thickness. Basic simplifying assumptions for the development of the theory remain the same as in the case of a plate of uniform thickness. In the first paper of this chapter the symmetrical bending of an annular plate whose thickness varies inversly as the distance from the centre has been investigated. The same problem was solved by Basuli (1961) with linearly varying thickness. At the outer boundary both the deflection and slope are zero while at the inner boundary only the slope vanishes. It is assumed that the normal load is either distributed uniformly round the hole with the remaining part of the surface free or that the load is distributed uniformly over the plate surface. In the absence of any force in the middle plane of the plate, the symmetrical bending of circular plates of variable thickness was first discussed by Holzer (1918). Since then, many authors have investigated the problem, the outstanding investigation amongst them being those of Pitchler (1928) and Olsson (1937). The last named author (1939) has also solved the problem of unsymmetrical bending of circular plates. The second paper of this chapter is deveted to the bending of involving Bessel's function which is convergent. In the second paper of this chapter the large deflection of a circular plate subjected to a concentrated load at a distance from the centre has been investigated. The corresponding result (Basuli, 1961) for the problem with the load at the centre has also been deduced. The third paper of this chapter deals with the large deflection of an orthotropic circular plate under a concentrated load at the centre. The corresponding problem on the application of this technique of Berger to the case of orthotropic plates under uniform load was given by Iwinski and Nowinski (1957). The fourth paper of this chapter is concerned with the large deflection of a semi-circular plate, simply-supported along the boundary. The deflection is obtained in terms of Bessel functions and Lommel functions. The fifth paper of this chapter is devoted to the large deflection of an elliptic plate with clamped edges. The deflection is obtained in terms of Mathieu functions. The sixth paper of this chapter deals with the large deflection of an isoceles right-angled triangular plate under uniform load. The deflection of an orthotropic plate has been investigated first. Then the deflection in the case of isotropic plate has also been deduced.

The third chapter contains two non-linear plate problems of variable thickness based on the approximate method of Berger (1955). The first paper of this chapter consists of the large deflection of a square plate whose thickness varies uni-directionally as the cube root of the distance of the parallel strip from the origin and an infinite strip plate whose thickness varies along the breadth. The second paper of this chapter deals with the large deflection of a circular plate with thickness varying as the cube root of the distance from the centre. The corresponding problem with linearly varying thickness is due to Basuli (1961).

The fourth chapter consists of only one problem on time-hardening and time-softening elastic plates. The free vibrations of different elastic plates have been considered, Young's modulus being assumed to be a function of time of the asymptotic type. The deflections of rectangular and triangular plates of such materials are obtained in terms of Bessel's functions of purely imaginary orders.

The fifth and the last chapter consists of only one problem on large amplitude free vibrations of different elastic plates. Applying the technique offered by Berger (1955), Nash and Modeer (1960) investigated the large amplitude free vibrations of rectangular and circular plates. In this paper the large amplitude free vibrations of triangular, elliptic and semi-circular plates have been considered on applying the same technique offered by Berger.

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CHAPTER - I.

Bending of uniformly compressed circular plates of variable thickness.*

PAPER - I.

Nomenclature :

The following nomenclature are used in this paper.

a = radius of the plate,

b = radius of the inner boundary,

h = thickness of the plate at a distance r from the centre,

D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,

E = Young's modulus,

σ = Poisson's ratio.

Introduction :

Holzer (1918) discussed first the problem of symmetrical bending of circular plates of variable thickness. Since then, many authors have investigated the problem, outstanding of which are the investigations of Pichler (1928) and Olsson (1937). The last named author (1939) has also solved the problem of unsymmetrical bending of circular plates. Conway (1948) investigated the problem of symmetrically loaded circular plates of variable thickness with various types of thickness variations.

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Basuli (1961) solved the problem of bending of uniformly compressed annular plates, thickness of which varies linearly from the centre.

In this paper an attempt has been made to solve the same problem with the thickness varying inversely as the distance from the centre.

Theory :

Let $T =$ uniform pressure per unit length and thickness of the section of the deflection surface bounded by two concentric cylindrical surfaces of radii r and $r + dr$ and two radial planes including a small angle $d\theta$ at the centre of the plate, at a distance r; $M_r, M_\theta =$ bending moments per unit length of the section perpendicular to radius and tangent, $Q_r =$ shearing forces per unit length, acting normally to the middle plane, $\phi =$ slope at a distance $r = -\frac{dw}{dr}$, w being the corresponding displacement.

Considering the equilibrium of the element and taking moments we have the following differential equation,

$$D\frac{d}{dr}\left(\frac{d\phi}{dr}+\frac{\phi}{r}\right)+\frac{dD}{dr}\left(\frac{d\phi}{dr}+\sigma\frac{\phi}{r}\right)+Th\phi = -Q_r \qquad \ldots (1)$$

where the flexural rigidity D is a variable quantity.

[Basuli. S (1961)]

Problem :

(a) Outer boundary clamped and supported, inner boundary clamped. Line load along the inner boundary.

Let us consider an annular plate whose thickness at a distance r is given by

$$h = h_0 r^{-1} \qquad \ldots(2)$$

subjected to a total normal load P distributed uniformly round the radius of the hole.

Then the differential equation (1) will take the form

$$r^2\frac{d^2\phi}{dr^2} - 2r\frac{d\phi}{dr} + \left(\alpha^2 r^4 - 3\sigma - 1\right)\phi = -\frac{\alpha^2 r^4 P}{2\pi h_0 T} \qquad \ldots(3)$$

where $\alpha^2 = \frac{12\left(1-\sigma^2\right)T}{E h_0^2}$, $Q_r = \frac{P}{2\pi r}$.

The complementary function for the equation (3) can be put in the form $r^{3/2}\left[A\,J_\mu\left(\frac{\alpha r^2}{2}\right) + B\,Y_\mu\left(\frac{\alpha r^2}{2}\right)\right]$

where $\mu^2 = \frac{13+12\sigma}{16}$; A, B being constants and $J_\mu\left(\frac{\alpha r^2}{2}\right)$, $Y_\mu\left(\frac{\alpha r^2}{2}\right)$ being the Bessel functions of 1st and 2nd kind of order μ.

[Forsyth, A.R.(1929)]

The particular integral is $-\frac{P\alpha^{3/4}\cdot r^{3/2}\cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{\pi h_0 T\cdot 2^{7/4}}$

where $$S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right) = \sum_{m=0}^{\infty}\frac{(-1)^m\cdot\left(\frac{\alpha r^2}{2}\right)^{1/4+1+2m}}{\left\{\left(1/4+1\right)^2-\mu^2\right\}\cdots\left\{\left(1/4+1+2m\right)^2-\mu^2\right\}}$$

is the Lommel's function.

[Erdelyi, A. (1953)]

Hence the general solution is

$$\phi = r^{3/2}\left[A J_\mu\left(\frac{\alpha r^2}{2}\right) + B Y_\mu\left(\frac{\alpha r^2}{2}\right) - \frac{P\alpha^{3/4} S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{7/4}\pi T h_0}\right] \qquad \ldots (4)$$

If the outer boundary be clamped and supported and the inner boundary be clamped

boundary conditions are $\phi = 0$ **when** $r = a,\ r = b,\ W = 0$ **at**

$$r = a \qquad \ldots (5)$$

Considering equations (4) and (5) and solving for the constants, we get

$$A = \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\left[\frac{S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha a^2}{2}\right)}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - J_\mu\left(\frac{\alpha b^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha a^2}{2}\right)}\right] \qquad \ldots (6)$$

$$B = \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\left[\frac{S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right) J_\mu\left(\frac{\alpha b^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)\cdot J_\mu\left(\frac{\alpha a^2}{2}\right)}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - J_\mu\left(\frac{\alpha b^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha a^2}{2}\right)}\right] \qquad \ldots (7)$$

Using equations (4), (6) and (7) we get ϕ **in the form**

$$\begin{aligned}\phi = \frac{P r^{3/2}\alpha^{3/4}}{2^{7/4}\pi T h_0}\Bigg[& J_\mu\left(\frac{\alpha r^2}{2}\right)\frac{S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right) Y_\mu\left(\frac{\alpha b^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha a^2}{2}\right)}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)} \\ & + Y_\mu\left(\frac{\alpha r^2}{2}\right)\frac{S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)\cdot J_\mu\left(\frac{\alpha a^2}{2}\right)}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)} \\ & - S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)\Bigg] \qquad \ldots (8)\end{aligned}$$

Considering equation (4) we get

$$\begin{aligned}M_r = D\left[\frac{d\phi}{dr} + \sigma\frac{\phi}{r}\right] = \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\cdot D\Bigg[& A\left\{\frac{3}{2} r^{1/2} J_\mu\left(\frac{\alpha r^2}{2}\right) + r^{5/2}\alpha\cdot J'_\mu\left(\frac{\alpha r^2}{2}\right)\right\} + B\left\{\frac{3}{2} r^{1/2} Y_\mu\left(\frac{\alpha r^2}{2}\right) + r^{5/2}\alpha\cdot Y'_\mu\left(\frac{\alpha r^2}{2}\right)\right\} \\ & + \sigma r^{1/2}\left\{A J_\mu\left(\frac{\alpha r^2}{2}\right) + B Y_\mu\left(\frac{\alpha r^2}{2}\right) - S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)\right\} \\ & - \alpha r S'_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)\Bigg] \qquad \ldots (9)\end{aligned}$$

The same equation also determines

$$M_\theta = D\left[\frac{\phi}{r} + \sigma\frac{d\phi}{dr}\right] \qquad \ldots (10)$$

To get the deflection we know that $\phi = -\frac{dw}{dr}$ ... (11)

On integrating equation (11) we get,

$$\begin{aligned} W = & -(2/\alpha)^{1/4}\Bigg[A\Big\{(\mu-3/4)\,J_\mu\left(\tfrac{\alpha r^2}{2}\right) S_{-3/4,\mu-1}\left(\tfrac{\alpha r^2}{2}\right) \\ & - J_{\mu-1}\left(\tfrac{\alpha r^2}{2}\right) S_{1/4,\mu}\left(\tfrac{\alpha r^2}{2}\right)\Big\}\tfrac{1}{2}r^2 \\ & + B\Big\{(\mu-3/4)\,Y_\mu\left(\tfrac{\alpha r^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha r^2}{2}\right) \\ & - Y_{\mu-1}\left(\tfrac{\alpha r^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha r^2}{2}\right)\Big\}\tfrac{1}{2}r^2 \\ & - \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\left\{\sum_{m=0}^{\infty}\frac{(-1)^m\cdot r^{5+4m}\cdot\left(\frac{\alpha}{2}\right)^{5/4+2m}}{(5+4m)\{(5/4)^2-\mu^2\}\cdots\{(5/4+2m)^2-\mu^2\}}\right\}\Bigg] \\ & + K_1 \text{ (constant)} \end{aligned} \qquad \ldots (12)$$

The boundary condition is $W=0$ at $r=a$ ... (13)

Hence

$$\begin{aligned} K_1 = & (2/\alpha)^{1/4}\Bigg[\frac{Aa^2}{2}\Big\{(\mu-3/4)\,J_\mu\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha a^2}{2}\right) \\ & - J_{\mu-1}\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha a^2}{2}\right)\Big\} + \frac{Ba^2}{2}\Big\{(\mu-3/4)\,Y_\mu\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\tfrac{\alpha a^2}{2}\right) \\ & - Y_{\mu-1}\left(\tfrac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\left(\tfrac{\alpha a^2}{2}\right)\Big\} \\ & - \frac{P\alpha^{3/4}}{2^{7/4}\pi T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\cdot a^{5+4m}\cdot\left(\frac{\alpha}{2}\right)^{5/4+2m}}{(5+4m)\{(5/4)^2-\mu^2\}\cdots\{(5/4+2m)^2-\mu^2\}}\Bigg] \end{aligned} \qquad \ldots (14)$$

Therefore W is determined.

(b) Outer boundary clamped and supported, inner boundary clamped. Load uniform.

For the same plate if the load be uniformly distributed with intensity q then

$$Q_r = \frac{1}{2\pi r}\int_b^r q\cdot 2\pi r\,dr = \frac{q}{2r}\left(r^2 - b^2\right) \qquad \dots (15)$$

With this value of Q_r the differential equation (1) takes the form

$$r^2\frac{d^2\phi}{dr^2} - 2r\frac{d\phi}{dr} + \left(\alpha^2 r^4 - 3\sigma - 1\right)\phi = \frac{q\alpha^2}{2Th_0}\left(b^2 r^4 - r^6\right) \qquad \dots (16)$$

Solution of equation (16) can be put in the form

$$\phi = r^{3/2}\left[C\,J_\mu\left(\frac{\alpha r^2}{2}\right) + D\,Y_\mu\left(\frac{\alpha r^2}{2}\right) + \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right) q}{2^{7/4}\,Th_0} - \frac{q\cdot S_{5/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{3/4}\alpha^{1/4}\,Th_0}\right] \qquad \dots (17)$$

Boundary conditions are $\phi = 0$ when $r = a$, $r = b$ $\qquad \dots (18)$

Considering equations (17) and (18) and solving for the constants, we get

$$C = \frac{Y_\mu\left(\frac{\alpha b^2}{2}\right)\cdot\frac{q}{Th_0}\left[\frac{S_{5/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right] - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot\frac{q}{Th_0}\left[\frac{S_{5/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right]}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)}$$

$$D = \frac{J_\mu\left(\frac{\alpha a^2}{2}\right)\frac{q}{Th_0}\left[\frac{S_{5/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right] - J_\mu\left(\frac{\alpha b^2}{2}\right)\frac{q}{Th_0}\left[\frac{S_{5/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right]}{J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)} \qquad \dots (19)$$

... (20)

Combining equations (17), (19) and (20), ϕ is determined in the form

$$\begin{aligned}
\phi = r^{3/2}\Bigg[& J_\mu\left(\frac{\alpha r^2}{2}\right)\Bigg\{ Y_\mu\left(\frac{\alpha b^2}{2}\right)\frac{q}{Th_0}\left(\frac{S_{5/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right) \\
& - Y_\mu\left(\frac{\alpha a^2}{2}\right)\frac{q}{Th_0}\left(\frac{S_{5/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right)\Bigg\} \\
& \Big/\left\{ J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)\right\} \\
& + Y_\mu\left(\frac{\alpha r^2}{2}\right)\Bigg\{ J_\mu\left(\frac{\alpha a^2}{2}\right)\frac{q}{Th_0}\left(\frac{S_{5/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right) \\
& - J_\mu\left(\frac{\alpha b^2}{2}\right)\frac{q}{Th_0}\left(\frac{S_{5/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right)\Bigg\} \\
& \Big/\left\{ J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot Y_\mu\left(\frac{\alpha b^2}{2}\right) - Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot J_\mu\left(\frac{\alpha b^2}{2}\right)\right\} \\
& + \frac{\alpha^{3/4}\cdot q\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{7/4}\cdot Th_0} \\
& - \frac{q\cdot S_{5/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}\cdot Th_0}\Bigg] \qquad \ldots(21)
\end{aligned}$$

To get the deflection w we know that $\phi = -\frac{dw}{dr}$

Hence

$$\begin{aligned}
\frac{dw}{dr} = -r^{3/2}\Bigg[& C J_\mu\left(\frac{\alpha r^2}{2}\right) + D Y_\mu\left(\frac{\alpha r^2}{2}\right) + \frac{\alpha^{3/4}\cdot q\cdot b^2\cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{7/4}\cdot Th_0} \\
& - \frac{q\cdot S_{5/4,\mu}\left(\frac{\alpha r^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}\cdot Th_0}\Bigg] \qquad \ldots(22)
\end{aligned}$$

Integrating equation (22) we get

$$
\begin{aligned}
W = {} & -\left(2/\alpha\right)^{1/4}\Bigg[C\left\{\left(\mu-3/4\right)J_\mu\left(\frac{\alpha r^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\frac{\alpha r^2}{2}\right)\right. \\
& \left.-J_{\mu-1}\left(\frac{\alpha r^2}{2}\right)\cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)\right\}\frac{1}{2}r^2 \\
& +D\left\{\left(\mu-3/4\right)Y_\mu\left(\frac{\alpha r^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\frac{\alpha r^2}{2}\right)\right. \\
& \left.-Y_{\mu-1}\left(\frac{\alpha r^2}{2}\right)\cdot S_{1/4,\mu}\left(\frac{\alpha r^2}{2}\right)\right\}\frac{1}{2}r^2 \\
& -\frac{\alpha^{3/4}\cdot q\cdot b^2}{2^{7/4}\cdot T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\cdot\left(\frac{\alpha}{2}\right)^{5/4+2m}\cdot r^{5+4m}}{(5+4m)\left\{\left(5/4\right)^2-\mu^2\right\}\cdots\left\{\left(5/4+2m\right)^2-\mu^2\right\}} \\
& +\frac{q}{2^{3/4}\cdot\alpha^{1/4}\cdot T h_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\cdot\left(\frac{\alpha}{2}\right)^{9/4+2m}\cdot r^{7+4m}}{(7+4m)\left\{\left(9/4\right)^2-\mu^2\right\}\cdots\left\{\left(9/4+2m\right)^2-\mu^2\right\}}\Bigg] \\
& +K_2 \qquad \cdots (23)
\end{aligned}
$$

Boundary condition is $W=0$ at $r=a$

Therefore the constant K_2 **is given by**

$$
\begin{aligned}
K_2 = {} & \left(2/\alpha\right)^{1/4}\Bigg[C\left\{\left(\mu-3/4\right)J_\mu\left(\frac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\frac{\alpha a^2}{2}\right)\right. \\
& \left.-J_{\mu-1}\left(\frac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)\right\}\frac{1}{2}a^2 \\
& +D\left\{\left(\mu-3/4\right)Y_\mu\left(\frac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\left(\frac{\alpha a^2}{2}\right)\right. \\
& \left.-Y_{\mu-1}\left(\frac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\left(\frac{\alpha a^2}{2}\right)\right\}\frac{1}{2}a^2
\end{aligned}
$$

$$- \frac{\alpha^{3/4} \cdot q \cdot b^2}{2^{7/4} \cdot T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot a^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$+ \frac{q}{2^{3/4} \alpha^{1/4} \cdot T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot a^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}} \Bigg] \qquad \ldots (24)$$

Substituting the value of K_2 from equation (24) in (23) W is determined.

(c) Outer boundary clamped, inner boundary clamped and supported. Load uniform.

For the same plate if the inner boundary be clamped and supported and the outer boundary be clamped, boundary conditions are $\phi = 0$ at $r = a$, $r = b$ and $W = 0$ at $r = b$.

If the load be uniformly distributed with intensity q, equation (23) reduces to

$$W = -(2/\alpha)^{1/4} \Bigg[C\left\{ (\mu - 3/4) J_\mu\left(\frac{\alpha r^2}{2}\right) \cdot S_{-3/4, \mu-1}\left(\frac{\alpha r^2}{2}\right) - J_{\mu-1}\left(\frac{\alpha r^2}{2}\right) \cdot S_{1/4, \mu}\left(\frac{\alpha r^2}{2}\right) \right\} \frac{1}{2} r^2 + D\left\{ (\mu - 3/4) Y_\mu\left(\frac{\alpha r^2}{2}\right) \cdot S_{-3/4, \mu-1}\left(\frac{\alpha r^2}{2}\right) - Y_{\mu-1}\left(\frac{\alpha r^2}{2}\right) \cdot S_{1/4, \mu}\left(\frac{\alpha r^2}{2}\right) \right\} \frac{1}{2} r^2$$

$$- \frac{\alpha^{3/4} \cdot q \cdot b^2}{2^{7/4} \cdot T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot r^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$+ \frac{q}{2^{3/4} \alpha^{1/4} T h_0} \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot r^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}} \Bigg]$$

$$+\left(2/\alpha\right)^{1/4}\left[\frac{Cb^2}{2}\left\{(\mu-3/4)\,J_\mu\!\left(\frac{\alpha b^2}{2}\right)\cdot S_{-3/4,\mu-1}\!\left(\frac{\alpha b^2}{2}\right)-J_{\mu-1}\!\left(\frac{\alpha b^2}{2}\right)\cdot S_{1/4,\mu}\!\left(\frac{\alpha b^2}{2}\right)\right\}\right.$$

$$+\frac{Db^2}{2}\left\{(\mu-3/4)\,Y_\mu\!\left(\frac{\alpha b^2}{2}\right)\cdot S_{-3/4,\mu-1}\!\left(\frac{\alpha b^2}{2}\right)-Y_{\mu-1}\!\left(\frac{\alpha b^2}{2}\right)\cdot S_{1/4,\mu}\!\left(\frac{\alpha b^2}{2}\right)\right\}$$

$$-\frac{\alpha^{3/4}\,q\cdot b^2}{2^{7/4}\,Th_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(\alpha/2)^{5/4+2m}\cdot b^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$\left.+\frac{q}{2^{3/4}\cdot\alpha^{1/4}\cdot Th_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(\alpha/2)^{9/4+2m}\cdot b^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}}\right]\qquad\ldots\ (25)$$

which is the deflection under uniform load at a distance η.

Deflection will be maximum at the outer boundary $\eta = a$.

Thus we get,

$$(W)_{max}=\left(2/\alpha\right)^{1/4}\left[\frac{Cb^2}{2}\left\{(\mu-3/4)\,J_\mu\!\left(\frac{\alpha b^2}{2}\right)\cdot S_{-3/4,\mu-1}\!\left(\frac{\alpha b^2}{2}\right)\right.\right.$$

$$\left.-J_{\mu-1}\!\left(\frac{\alpha b^2}{2}\right)\cdot S_{1/4,\mu}\!\left(\frac{\alpha b^2}{2}\right)\right\}+\frac{Db^2}{2}\left\{(\mu-3/4)\,Y_\mu\!\left(\frac{\alpha b^2}{2}\right)\cdot S_{-3/4,\mu-1}\!\left(\frac{\alpha b^2}{2}\right)\right.$$

$$\left.-Y_{\mu-1}\!\left(\frac{\alpha b^2}{2}\right)\cdot S_{1/4,\mu}\!\left(\frac{\alpha b^2}{2}\right)\right\}$$

$$-\frac{\alpha^{3/4}\cdot q\cdot b^2}{2^{7/4}\cdot Th_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(\alpha/2)^{5/4+2m}\cdot b^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$\left.+\frac{q}{2^{3/4}\alpha^{1/4}\cdot Th_0}\cdot\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(\alpha/2)^{9/4+2m}\cdot b^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}}\right]$$

$$-\left(2/\alpha\right)^{1/4}\left[\frac{Ca^2}{2}\left\{(\mu-3/4)\,J_\mu\!\left(\frac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\!\left(\frac{\alpha a^2}{2}\right)\right.\right.$$

$$\left.-J_{\mu-1}\!\left(\frac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\!\left(\frac{\alpha a^2}{2}\right)\right\}+\frac{Da^2}{2}\left\{(\mu-3/4)\,Y_\mu\!\left(\frac{\alpha a^2}{2}\right)\cdot S_{-3/4,\mu-1}\!\left(\frac{\alpha a^2}{2}\right)\right.$$

$$\left.-Y_{\mu-1}\!\left(\frac{\alpha a^2}{2}\right)\cdot S_{1/4,\mu}\!\left(\frac{\alpha a^2}{2}\right)\right\}$$

$$- \frac{\alpha^{3/4} \cdot q \cdot b^2}{2^{7/4} \cdot T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot a^{5+4m}}{(5+4m)\left\{(5/4)^2-\mu^2\right\}\cdots\left\{(5/4+2m)^2-\mu^2\right\}}$$

$$+ \frac{q}{2^{3/4} \alpha^{1/4} T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot a^{7+4m}}{(7+4m)\left\{(9/4)^2-\mu^2\right\}\cdots\left\{(9/4+2m)^2-\mu^2\right\}} \Bigg] \qquad \ldots (26)$$

In particular, if $\sigma = 0{\cdot}25$, the deflection and the maximum deflection can be obtained from equations (25) and (26) as given below.

$$W = -(2/\alpha)^{1/4} \Bigg[C_1 r^2 \left\{ \tfrac{1}{8} J_1\left(\tfrac{\alpha r^2}{2}\right) \cdot S_{-3/4,0}\left(\tfrac{\alpha r^2}{2}\right) - \tfrac{1}{2} J_0\left(\tfrac{\alpha r^2}{2}\right) \cdot S_{1/4,1}\left(\tfrac{\alpha r^2}{2}\right) \right\}$$

$$+ D_1 r^2 \left\{ \tfrac{1}{8} Y_1\left(\tfrac{\alpha r^2}{2}\right) \cdot S_{-3/4,0}\left(\tfrac{\alpha r^2}{2}\right) - \tfrac{1}{2} Y_0\left(\tfrac{\alpha r^2}{2}\right) \cdot S_{1/4,1}\left(\tfrac{\alpha r^2}{2}\right) \right\}$$

$$- \frac{\alpha^{3/4} q \cdot b^2}{2^{7/4} T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot r^{5+4m}}{(5+4m)\left\{(5/4)^2-1\right\}\cdots\left\{(5/4+2m)^2-1\right\}}$$

$$+ \frac{q}{2^{3/4} \alpha^{1/4} T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot r^{7+4m}}{(7+4m)\left\{(9/4)^2-1\right\}\cdots\left\{(9/4+2m)^2-1\right\}} \Bigg]$$

$$+ (2/\alpha)^{1/4} \Bigg[C_1 b^2 \left\{ \tfrac{1}{8} J_1\left(\tfrac{\alpha b^2}{2}\right) \cdot S_{-3/4,0}\left(\tfrac{\alpha b^2}{2}\right) - \tfrac{1}{2} J_0\left(\tfrac{\alpha b^2}{2}\right) \cdot S_{1/4,1}\left(\tfrac{\alpha b^2}{2}\right) \right\}$$

$$+ D_1 b^2 \left\{ \tfrac{1}{8} Y_1\left(\tfrac{\alpha b^2}{2}\right) \cdot S_{-3/4,0}\left(\tfrac{\alpha b^2}{2}\right) - \tfrac{1}{2} Y_0\left(\tfrac{\alpha b^2}{2}\right) \cdot S_{1/4,1}\left(\tfrac{\alpha b^2}{2}\right) \right\}$$

$$- \frac{\alpha^{3/4} q \cdot b^2}{2^{7/4} T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{5/4+2m} \cdot b^{5+4m}}{(5+4m)\left\{(5/4)^2-1\right\}\cdots\left\{(5/4+2m)^2-1\right\}}$$

$$+ \frac{q}{2^{3/4} \alpha^{1/4} T h_0} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(\frac{\alpha}{2}\right)^{9/4+2m} \cdot b^{7+4m}}{(7+4m)\left\{(9/4)^2-1\right\}\cdots\left\{(9/4+2m)^2-1\right\}} \Bigg] \qquad \ldots (27)$$

where

$$C_1 = \frac{\frac{q}{T h_0}\cdot Y_1\left(\frac{\alpha b^2}{2}\right)\left[\frac{S_{5/4,1}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4}\cdot b^2\cdot S_{1/4,1}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right] - \frac{q}{T h_0}\cdot Y_1\left(\frac{\alpha a^2}{2}\right)\left[\frac{S_{5/4,1}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2\cdot S_{1/4,1}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right]}{J_1\left(\frac{\alpha a^2}{2}\right)\cdot Y_1\left(\frac{\alpha b^2}{2}\right) - Y_1\left(\frac{\alpha a^2}{2}\right)\cdot J_1\left(\frac{\alpha b^2}{2}\right)}$$

$$D_1 = \frac{\frac{q}{T h_0}\cdot J_1\left(\frac{\alpha a^2}{2}\right)\left[\frac{S_{5/4,1}\left(\frac{\alpha b^2}{2}\right)}{2^{3/4}\cdot\alpha^{1/4}} - \frac{\alpha^{3/4} b^2\cdot S_{1/4,1}\left(\frac{\alpha b^2}{2}\right)}{2^{7/4}}\right] - \frac{q}{T h_0}\cdot J_1\left(\frac{\alpha b^2}{2}\right)\left[\frac{S_{5/4,1}\left(\frac{\alpha a^2}{2}\right)}{2^{3/4}\alpha^{1/4}} - \frac{\alpha^{3/4} b^2\cdot S_{1/4,1}\left(\frac{\alpha a^2}{2}\right)}{2^{7/4}}\right]}{J_1\left(\frac{\alpha a^2}{2}\right)\cdot Y_1\left(\frac{\alpha b^2}{2}\right) - Y_1\left(\frac{\alpha a^2}{2}\right)\cdot J_1\left(\frac{\alpha b^2}{2}\right)}$$

$$\begin{aligned}
(W)_{max} = {} & (2/\alpha)^{1/4}\Bigg[C_1 b^2\left\{\frac{1}{8}J_1\left(\frac{\alpha b^2}{2}\right)\cdot S_{-3/4,0}\left(\frac{\alpha b^2}{2}\right) - \frac{1}{2}J_0\left(\frac{\alpha b^2}{2}\right)\cdot S_{1/4,1}\left(\frac{\alpha b^2}{2}\right)\right\} \\
& + D_1 b^2\left\{\frac{1}{8}Y_1\left(\frac{\alpha b^2}{2}\right)\cdot S_{-3/4,0}\left(\frac{\alpha b^2}{2}\right) - \frac{1}{2}Y_0\left(\frac{\alpha b^2}{2}\right)\cdot S_{1/4,1}\left(\frac{\alpha b^2}{2}\right)\right\} \\
& - \frac{\alpha^{3/4} q\cdot b^2}{2^{7/4}\, T h_0}\sum_{m=0}^{\infty}\frac{(-1)^m\cdot\left(\frac{\alpha}{2}\right)^{5/4+2m}\cdot b^{5+4m}}{(5+4m)\left\{(5/4)^2-1\right\}\cdots\left\{(5/4+2m)^2-1\right\}} \\
& + \frac{q}{2^{3/4}\alpha^{1/4}\, T h_0}\sum_{m=0}^{\infty}\frac{(-1)^m\cdot\left(\frac{\alpha}{2}\right)^{9/4+2m}\cdot b^{7+4m}}{(7+4m)\left\{(9/4)^2-1\right\}\cdots\left\{(9/4+2m)^2-1\right\}}\Bigg] \\
& - (2/\alpha)^{1/4}\Bigg[C_1 a^2\left\{\frac{1}{8}J_1\left(\frac{\alpha a^2}{2}\right)\cdot S_{-3/4,0}\left(\frac{\alpha a^2}{2}\right) - \frac{1}{2}J_0\left(\frac{\alpha a^2}{2}\right)\cdot S_{1/4,1}\left(\frac{\alpha a^2}{2}\right)\right\} \\
& + D_1 a^2\left\{\frac{1}{8}Y_1\left(\frac{\alpha a^2}{2}\right)\cdot S_{-3/4,0}\left(\frac{\alpha a^2}{2}\right) - \frac{1}{2}Y_0\left(\frac{\alpha a^2}{2}\right)\cdot S_{1/4,1}\left(\frac{\alpha a^2}{2}\right)\right\} \\
& - \frac{\alpha^{3/4} q\cdot b^2}{2^{7/4}\cdot T h_0}\sum_{m=0}^{\infty}\frac{(-1)^m\cdot\left(\frac{\alpha}{2}\right)^{5/4+2m}\cdot a^{5+4m}}{(5+4m)\left\{(5/4)^2-1\right\}\cdots\left\{(5/4+2m)^2-1\right\}} \\
& + \frac{q}{2^{3/4}\alpha^{1/4}\, T h_0}\sum_{m=0}^{\infty}\frac{(-1)^m\cdot\left(\frac{\alpha}{2}\right)^{9/4+2m}\cdot a^{7+4m}}{(7+4m)\left\{(9/4)^2-1\right\}\cdots\left\{(9/4+2m)^2-1\right\}}\Bigg] \quad \ldots (28)
\end{aligned}$$

Numerical calculation :

Let us assume $\alpha = 1,\ b = 5,\ a = 10$

Putting the above values of α, b and a in (28) we have, $(W)_{max} = \frac{q}{T h_0}\times 0{\cdot}4135$

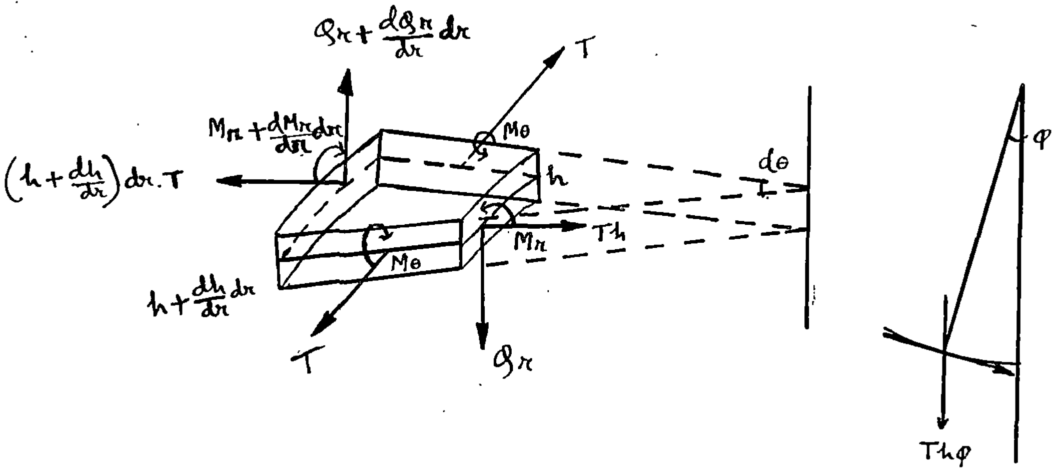

FIG. 1

Stresses and moments on an element bounded by two adjacent cylindrical surfaces and radial planes.

Note on the deflection of a square plate of variable thickness under a variable load and uniform tension in the middle plane of the plate.*

PAPER - II

Nomenclature :

The following nomenclature are used in this paper.

a = side of the plate,

D = flexural rigidity of the plate $= \frac{Eh^3}{12(1-\sigma^2)}$,

h = thickness at a distance x $= h_0 e^{\lambda \frac{x}{a}}$ where λ is a parameter, small in magnitude,

E = Young's modulus,

q = load,

T = Uniform tension in the middle plane of the plate,

σ = Poisson's ratio,

w = deflection, normal to the plate.

Introduction :

Several problems of bending of rectangular plates of uniform thickness under the combined action of lateral loads and forces in the middle plane of the plate have been discussed by Timoshenko and Woinowsky - Krieger (1959), Conway H.D.(1949), Chang C.C. and Conway H.D.(1952). The

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object of this paper is to solve the problem of a square plate of variable thickness under the combined action of a variable load and uniform tension in the middle plane of the plate, thickness varying exponentially. It may be mentioned that the same problem without any compressive force was discussed by Favre H. and Glig.B (1952). The corresponding problem with linearly varying thickness was due to Basuli (1961).

Theory :

The figure (1) represents an element of the plate. Prejecting normal and shearing forces on the z-axis and considering the equation of equilibrium we obtain the following differential equation (Timoshenko and Woinowsky-Krieger; Theory of plates and shells. Page - 379):-

$$D\nabla\nabla W+2\frac{dD}{dx}\cdot\frac{\partial}{\partial x}\cdot\nabla W+\frac{d^2D}{dx^2}\cdot\nabla W-(1-\sigma)\cdot\frac{d^2D}{dx^2}\cdot\frac{\partial^2 w}{\partial y^2}=q+Th\frac{\partial^2 w}{\partial x^2} \qquad \ldots(1)$$

Problem :

Let us consider a plate stretched in the direction of x- axis.

Hence $N_x=T,\ N_y=N_{xy}=N_{yx}=0$. Let us assume load q to be hydrostatic, represented by $q=q_0\cdot\frac{x}{a}$.

Let $W=\sum_{m=0}^{\infty}W_m\lambda^m$ where λ is a parameter defined earlier.

Substituting this in (1) and equating the coefficients of successive powers of λ we have the following sequence of differential equations

$$D_0 \nabla\nabla W_0 - Th_0 \frac{\partial^2 W_0}{\partial x^2} - q = 0 \qquad \ldots(2{\cdot}0)$$

$$D_0 3\frac{x}{a} \nabla\nabla W_0 + D_0 \nabla\nabla W_1 + \frac{6D_0}{a} \cdot \frac{\partial}{\partial x} \nabla W_0 - Th_0 \frac{x}{a} \frac{\partial^2 W_0}{\partial x^2} - Th_0 \frac{\partial^2 W_1}{\partial x^2} = 0 \qquad \ldots(2{\cdot}1)$$

$$\frac{9}{2} D_0 \frac{x^2}{a^2} \nabla\nabla W_0 + D_0 \nabla\nabla W_2 + \frac{6}{a} D_0 \frac{\partial}{\partial x} \nabla W_1 + 18 D_0 \frac{x}{a^2} \frac{\partial}{\partial x} \nabla W_0 +$$

$$+ \frac{9D_0}{a^2} \left\{ \nabla W_0 - (1-\sigma) \frac{\partial^2 W_0}{\partial y^2} \right\} - Th_0 \frac{x^2}{2a^2} \frac{\partial^2 W_0}{\partial x^2} - Th_0 \frac{\partial^2 W_2}{\partial x^2} - q = 0 \qquad \ldots(2{\cdot}2)$$

and so on.

For a simply supported plate using the method of M. Levy we take the solution in the form

$$W_0 = \sum_{n=1,3,\cdots}^{\infty} X_{0n} \sin \frac{n\pi y}{a} \qquad \ldots(3{\cdot}0)$$

$$W_1 = \sum_{n=1,3,\cdots}^{\infty} X_{1n} \sin \frac{n\pi y}{a} \qquad \ldots(3{\cdot}1)$$

. . . .

$$W_m = \sum_{n=1,3,\cdots}^{\infty} X_{mn} \sin \frac{n\pi y}{a} \qquad \ldots(3{\cdot}m)$$

($m = 0, 1, 2, \cdots$) X_{mn} being some functions of x only.

We can finally represent the load q by

$$q = \frac{4q_0 x}{\pi a} \sum_{n=1,3,\cdots}^{\infty} \frac{1}{n} \sin \frac{n\pi y}{a} \qquad \ldots(4)$$

Considering equations (2·0), (3·0) and (4) we have the following differential equations of 4th order for X_{on}.

$$\frac{d^4X_{on}}{dx^4} - \frac{d^2X_{on}}{dx^2}\left(\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2}\right) + X_{on}\frac{n^4\pi^4}{a^4} = \frac{4q_0 x}{\pi . a . n . D_0} \quad \ldots(5)$$

Solving we get

$$X_{on} = A_{01}e^{m_1x} + A_{02}e^{m_2x} + A_{03}e^{m_3x} + A_{04}e^{m_4x} + \frac{4q_0 x a^3}{n^5\pi^5 D_0} \quad \ldots(6)$$

where A_{01}, A_{02}, A_{03}, A_{04} are arbitrary constants.

Using boundary conditions

$X_{on} = 0$ at $x = 0$ and $x = a$

$X''_{on} = 0$ at $x = 0$ and $x = a$

one gets

$$A_{01} = \frac{a^4 . 4q_0}{n^5\pi^5 D_0}\left[\frac{P + Q + R}{P(e^{m_2a} - e^{m_1a}) + Q(e^{m_3a} - e^{m_1a}) + R(e^{m_4a} - e^{m_1a})}\right] \quad \ldots(7{\cdot}1)$$

$$A_{02} = -\frac{4q_0 P a^4}{n^5\pi^5 D_0}\left[\frac{1}{P(e^{m_2a} - e^{m_1a}) + Q(e^{m_3a} - e^{m_1a}) + R(e^{m_4a} - e^{m_1a})}\right] \quad \ldots(7{\cdot}2)$$

$$A_{03} = -\frac{4q_0 Q a^4}{n^5\pi^5 D_0}\left[\frac{1}{P(e^{m_2a} - e^{m_1a}) + Q(e^{m_3a} - e^{m_1a}) + R(e^{m_4a} - e^{m_1a})}\right] \quad \ldots(7{\cdot}3)$$

$$A_{04} = -\frac{4q_0 R a^4}{n^5\pi^5 D_0}\left[\frac{1}{P(e^{m_2a} - e^{m_1a}) + Q(e^{m_3a} - e^{m_1a}) + R(e^{m_4a} - e^{m_1a})}\right] \quad \ldots(7{\cdot}4)$$

where

$$m_1 = +\sqrt{\frac{\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2} + \sqrt{\frac{T^2h_0^2}{D_0^2} + \frac{4n^2\pi^2}{a^2}\cdot\frac{Th_0}{D_0}}}{2}}$$

$$m_2 = +\sqrt{\frac{\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2} - \sqrt{\frac{T^2h_0^2}{D_0^2} + \frac{4n^2\pi^2}{a^2}\cdot\frac{Th_0}{D_0}}}{2}}$$

$$m_3 = -\sqrt{\frac{\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2} + \sqrt{\frac{T^2h_0^2}{D_0^2} + \frac{4n^2\pi^2}{a^2}\cdot\frac{Th_0}{D_0}}}{2}}$$

$$m_4 = -\sqrt{\frac{\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2} - \sqrt{\frac{T^2h_0^2}{D_0^2} + \frac{4n^2\pi^2}{a^2}\cdot\frac{Th_0}{D_0}}}{2}}$$

and

$$P = m_4^2(m_3^2 - m_1^2)(e^{m_4a} - e^{m_1a}) - (m_4^2 - m_1^2)\cdot m_3^2(e^{m_3a} - e^{m_1a})$$

$$Q = m_2^2(m_4^2 - m_1^2)(e^{m_2a} - e^{m_1a}) - m_4^2(e^{m_4a} - e^{m_1a})(m_2^2 - m_1^2)$$

$$R = m_3^2(m_2^2 - m_1^2)(e^{m_3a} - e^{m_1a}) - m_2^2(m_3^2 - m_1^2)(e^{m_2a} - e^{m_1a})$$

With these values of $A_{01}, A_{02}, A_{03}, A_{04}$ **and combining equations (**6**) and (**3·0**)** W_0 **is obtained.**

Considering equations (2·1**),(**3·1**) and (**4**) we have**

$$\frac{d^4X_{1n}}{dx^4} - \frac{d^2X_{1n}}{dx^2}\left(\frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2}\right) + X_{1n}\frac{n^4\pi^4}{a^4}$$

$$= -\Bigg[2\frac{x}{a}\cdot\frac{Th_0}{D_0}\left\{A_{01}m_1^2e^{m_1x} + A_{02}m_2^2e^{m_2x} + A_{03}\cdot m_3^2e^{m_3x} + A_{04}\cdot m_4^2e^{m_4x}\right\} +$$

$$+ 12\frac{x^2}{a^2}\cdot\frac{q_0}{D_0}\cdot\frac{1}{\pi n} + \frac{6}{a}\Big\{A_{01}e^{m_1x}\left(m_1^3 - m_1\frac{n^2\pi^2}{a^2}\right) + A_{02}e^{m_2x}\left(m_2^3 - m_2\frac{n^2\pi^2}{a^2}\right) +$$

$$+ A_{03}e^{m_3x}\left(m_3^3 - m_3\frac{n^2\pi^2}{a^2}\right) +$$

$$+ A_{04}e^{m_4x}\left(m_4^3 - m_4\frac{n^2\pi^2}{a^2}\right)\Big\}\Bigg] \qquad \text{...(8)}$$

Solution of equation (8) can be put in the form

$$
\begin{aligned}
X_{1n} = {} & A_{11}e^{m_1x} + A_{12}e^{m_2x} + A_{13}e^{m_3x} + A_{14}e^{m_4x} \\
& - \left[2\frac{Th_0}{D_0 a}\left\{ \frac{A_{01}m_1e^{m_1x}}{4m_1^2-2\lambda_1^2}\left(\frac{x^2}{2} - \frac{6m_1^2-\lambda_1^2}{4m_1^3-2\lambda_1^2m_1}\cdot x\right) \right.\right. \\
& + \frac{A_{02}m_2e^{m_2x}}{4m_2^2-2\lambda_1^2}\left(\frac{x^2}{2} - \frac{6m_2^2-\lambda_1^2}{4m_2^3-2\lambda_1^2m_2}\cdot x\right) \\
& + \frac{A_{03}m_3e^{m_3x}}{4m_3^2-2\lambda_1^2}\left(\frac{x^2}{2} - \frac{6m_3^2-\lambda_1^2}{4m_3^3-2\lambda_1^2m_3}\cdot x\right) \\
& \left. + \frac{A_{04}m_4e^{m_4x}}{4m_4^2-2\lambda_1^2}\left(\frac{x^2}{2} - \frac{6m_4^2-\lambda_1^2}{4m_4^3-2\lambda_1^2m_4}\cdot x\right)\right\} \\
& + \frac{12q_0a^4}{D_0n^5\pi^5}\left\{\frac{x^2}{a^2} + \frac{\left(\frac{2Th_0}{D_0}+\frac{4n^2\pi^2}{a^2}\right)a^2}{n^4\pi^4}\right\} + \frac{6}{a}\left\{\frac{A_{01}e^{m_1x}}{4m_1^2-2\lambda_1^2}\left(m_1^2-\frac{n^2\pi^2}{a^2}\right)x\right. \\
& + \frac{A_{02}e^{m_2x}}{4m_2^2-2\lambda_1^2}\left(m_2^2-\frac{n^2\pi^2}{a^2}\right)x + \frac{A_{03}e^{m_3x}}{4m_3^2-2\lambda_1^2}\left(m_3^2-\frac{n^2\pi^2}{a^2}\right)x \\
& \left.\left. + \frac{A_{04}e^{m_4x}}{4m_4^2-2\lambda_1^2}\left(m_4^2-\frac{n^2\pi^2}{a^2}\right)x\right\}\right] \\
= {} & A_{11}e^{m_1x} + A_{12}e^{m_2x} + A_{13}e^{m_3x} + A_{14}e^{m_4x} + F(x) \qquad \ldots(9)
\end{aligned}
$$

where

$$\lambda_1^2 = \frac{Th_0}{D_0} + \frac{2n^2\pi^2}{a^2}$$

Using boundary conditions and solving for the constants, we have

$$A_{11} = F(0) - \frac{F''(0) - m_1^2 F(0)}{m_2^2 - m_1^2} - \left(\frac{m_3^2 - m_1^2}{m_2^2 - m_1^2} + 1\right)\left(\frac{LM_2 - L_2 M}{L_1 M_2 - L_2 M_1}\right)$$

$$- \left(\frac{m_4^2 - m_1^2}{m_2^2 - m_1^2} + 1\right)\left(\frac{LM_1 - L_1 M}{L_2 M_1 - L_1 M_2}\right)$$

$$A_{12} = \frac{F''(0) - m_1^2 F(0) - (m_3^2 - m_1^2)\left\{\frac{LM_2 - L_2 M}{L_1 M_2 - L_2 M_1}\right\} - (m_4^2 - m_1^2)\left\{\frac{LM_1 - L_1 M}{L_2 M_1 - L_1 M_2}\right\}}{m_2^2 - m_1^2}$$

$$A_{13} = \frac{LM_2 - L_2 M}{L_1 M_2 - L_2 M_1}, \qquad A_{14} = \frac{LM_1 - L_1 M}{L_2 M_1 - L_1 M_2}$$

where

$$L = F''(a) - m_1^2 F(a) - F''(0) e^{m_2 a} + m_1^2 e^{m_2 a} F(0)$$

$$M = F''(a)(m_2^2 - m_1^2) - e^{m_1 a}(m_2^2 - m_1^2) F''(0) - m_2^2 (e^{m_2 a} - e^{m_1 a}) F''(0)$$

$$+ m_1^2 m_2^2 (e^{m_2 a} - e^{m_1 a}) F(0)$$

$$L_1 = (m_3^2 - m_1^2)(e^{m_3 a} - e^{m_2 a})$$

$$M_1 = m_3^2 (m_2^2 - m_1^2)(e^{m_3 a} - e^{m_1 a}) - m_2^2 (m_3^2 - m_1^2)(e^{m_2 a} - e^{m_1 a})$$

$$L_2 = (m_4^2 - m_1^2)(e^{m_4 a} - e^{m_2 a})$$

$$M_2 = m_4^2 (m_2^2 - m_1^2)(e^{m_4 a} - e^{m_1 a}) - m_2^2 (m_4^2 - m_1^2)(e^{m_2 a} - e^{m_1 a})$$

Thus W_1 is determined completely. By similar procedure we can have $W_2, W_3, \ldots W_m$. Hence W is obtained.

A graph is plotted showing $\frac{wD_0}{q_0 a^4}$ against $\frac{x}{a}$ of the section $\frac{y}{a} =$ with $\lambda = 0{\cdot}1$; $\sigma = 0{\cdot}25$, and $\frac{a^2 T h_0}{D_0} = 1$.

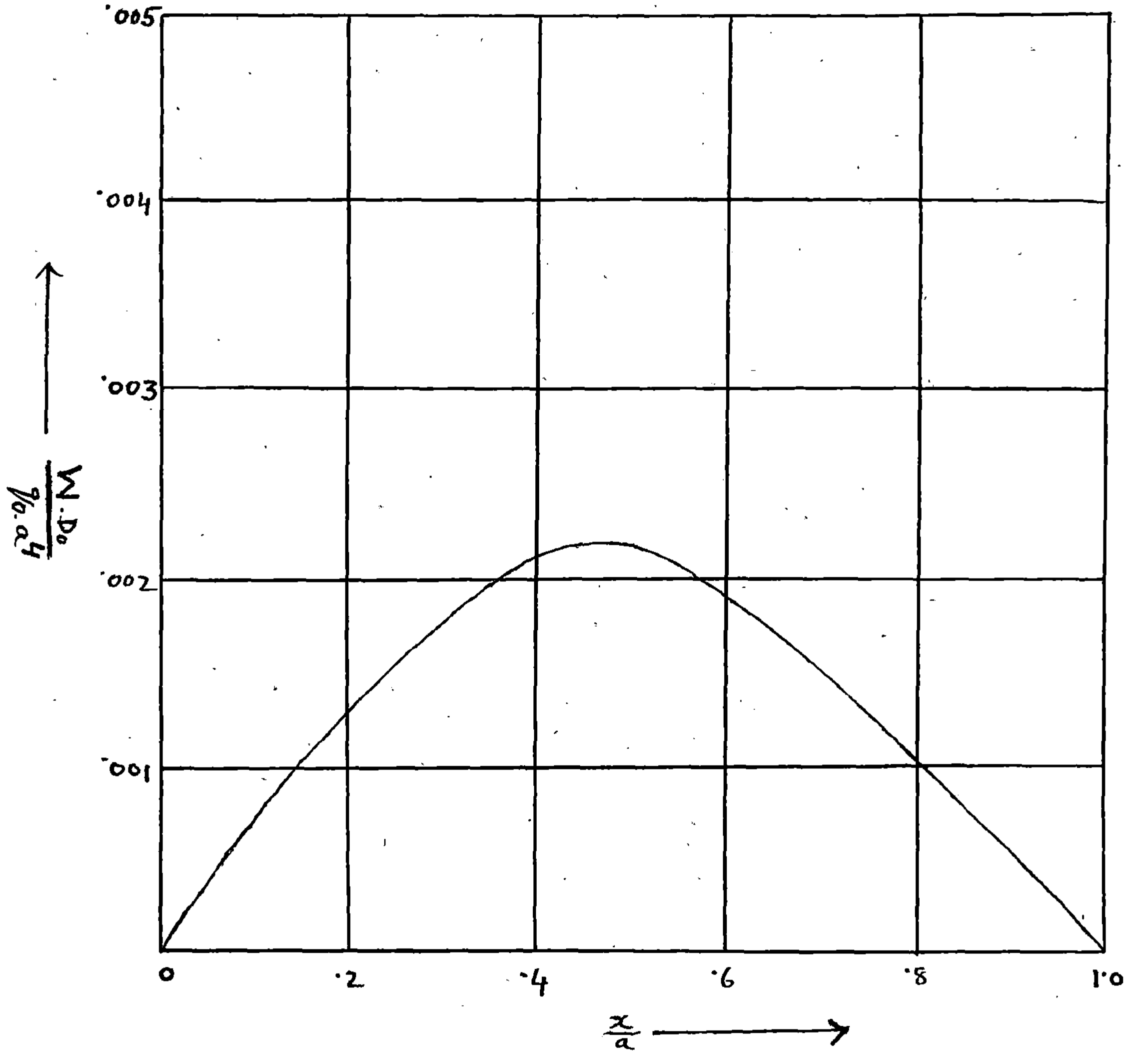

FIG. 2

Deflection of the section $y = \frac{a}{2}$

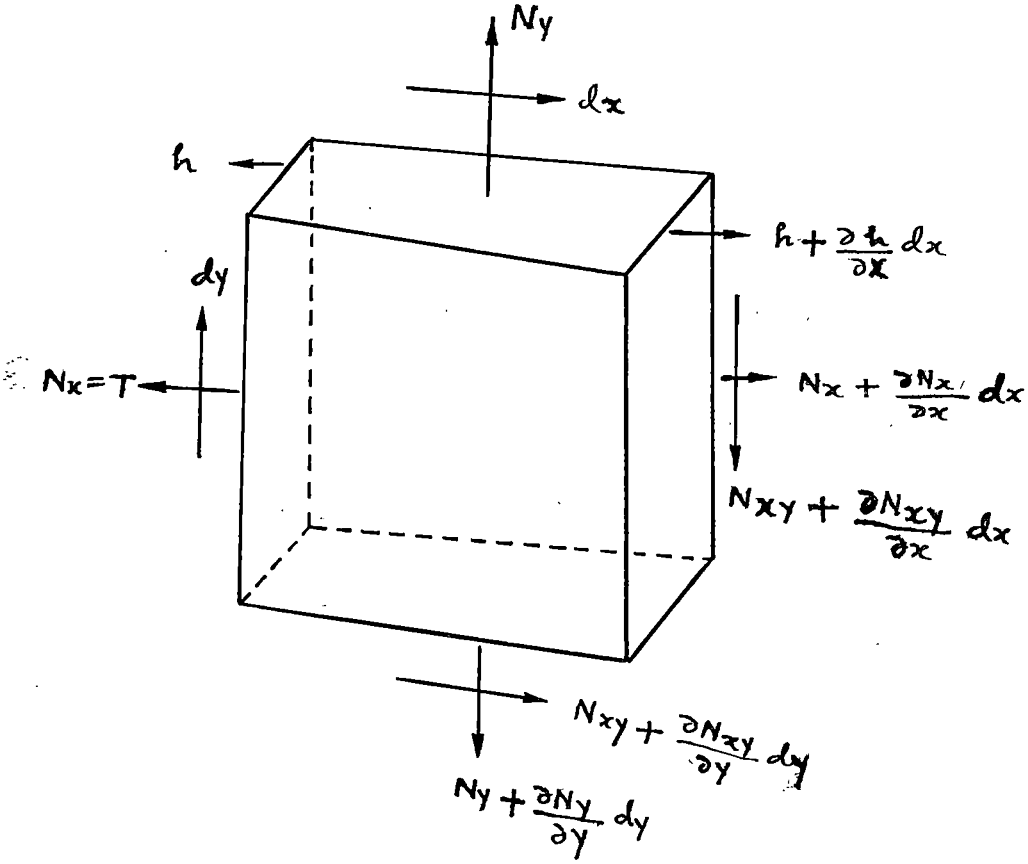

FIG. 3

Stresses on an element of the plate.

CHAPTER - II

Note on the large deflection of a circular plate with clamped edge under symmetrical load.*

PAPER - I

Nomenclature :

The following nomenclature are used in this paper.

w = deflection, normal to the middle plane,

u = radial displacement,

a = radius of the plate,

D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,

h = thickness of the plate,

σ = Poisson's ratio,

E = Young's modulus.

Introduction :

An approximate method for investigating the large deflection of initially flat isotropic plates has been proposed by Berger (1955). Essentially, the method is based upon neglecting the second invariant of the middle surface strains in the expression of the total potential energy of the system. The application of variational technique on the simplified energy expression yields approximate equations of equilibrium of the plate. For

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the several cases of static loading of initially flat plates investigated by Berger, the approximate equilibrium equations are in an uncoupled form. Although no complete explanation of this method is set forth, the stresses and deflections obtained for both rectangular and circular plates agree well with those found from more precise analysis. An application of this technique to the case of orthotropic plates has been considered by Iwinski and Nowinski (1957) and further boundary value problems associated with circular and rectangular plates have been investigated by Nowinski (1958). Basuli(1962) has shown that the large deflection of a cylindrical shell panel can also be obtained quite elegantly following Berger.

Nash and Modeer (1959) have shown that the method will be accurate for the problems for which there is a symmetry about the axis and for which radial membrane stress is approximately uniform. However problems without symmetry have also been treated by Berger (1955), Nash and Modeer (1957) and Sinha (1963). They are found to be in good agreement with the results known earlier.

The present author's endeavour is to find the large deflection of a clamped circular plate under symmetrical load. The corresponding linear problem was due to Sen (1935).

Analysis :

Following Berger's (1955) method, the differential equation for deflection takes the form

$$\nabla^2(\nabla^2-\alpha^2)W = \frac{\phi(r)}{D} = f(r) \quad \text{(say)} \ldots (1)$$

Where $f(r)$ is the load function,

α is a constant given by the equation

$$\frac{du}{dr} + \frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = \frac{\alpha^2 h^2}{12} \qquad \ldots(2)$$

Let us assume $W = \sum_{s=1}^{\infty} A_s \left[J_0(p_s r) - J_0(p_s a)\right]$ ...(3)

$p_s a$ being the sth root of $J_1(pa) = 0$

It is evident that the following boundary conditions are satisfied by the above equations.

$$W = \frac{dw}{dr} = 0 \quad at \ r = a .$$

Substituting equation (3) in equation (1) we have $\sum A_s p_s^2 (\alpha^2 + p_s^2) J_0(p_s r) = f(r)$

If it is possible to expand $f(r)$ in a series of Bessel functions, we get $\int_0^a \sum_{s=1}^{\infty} A_s p_s^2 (p_s^2+\alpha^2) J_0^2(p_s r) r\,dr = \int_0^a f(r) J_0(p_s r) r\,dr$ on integretion which leads to

$$A_s a^2 p_s^2 (p_s^2+\alpha^2) \cdot \frac{J_0^2(p_s a)}{2} = \int_0^a f(r) J_0(p_s r) r\,dr$$

Hence $A_s = \dfrac{2}{a^2 p_s^2 (p_s^2+\alpha^2) \cdot J_0^2(p_s a)} \cdot \displaystyle\int_0^a f(r) J_0(p_s r) r\,dr$ ...(4)

As an example let us suppose that the load varies as $(b^2 - r^2)^{1/2}$ over a concentric circular area of radius $b < a$.

In this case $f(r) = C(b^2 - r^2)^{1/2}$ when $r < b < a$

$= 0$ when $b < r < a$

where C is a constant.

Now equation (4) becomes $A_s = \dfrac{2C}{a^2 p_s^2 (p_s^2+\alpha^2) \cdot J_0^2(p_s a)} \cdot \displaystyle\int_0^a J_0(p_s r)(b^2 - r^2)^{1/2} r\,dr$

Putting $r = b \sin\theta$ we have

$$\int_0^a J_0(p_s r)(b^2 - r^2)^{1/2} r\,dr = b^3 \int_0^{\pi/2} J_0(p_s \cdot b \cdot \sin\theta) \cos^2\theta \cdot \sin\theta \, d\theta .$$

Using the expansion for $J_0(p_s \cdot b \sin\theta)$ and integrating term by term we have the right hand side as $b^3 P(p_s b)$

where $P(p_s b) = \frac{1}{3}\left[1 - \frac{p_s^2 b^2}{2.5} + \frac{p_s^4 b^4}{2.4.5.7} - \cdots\right]$

Hence $A_s = \dfrac{2 b^3 c\, P(p_s b)}{\alpha^2 p_s^2 (p_s^2+\alpha^2) J_0^2(p_s a)}$ ...(5)

Combining equations (3) and (5) we have

$$W = \sum_{s=1}^{\infty} A_s \left[J_0(p_s r) - J_0(p_s a)\right]$$

$$= \frac{2b^3 c}{a^2} \sum_{s=1}^{\infty} \frac{P(p_s b)}{p_s^2 (p_s^2+\alpha^2) J_0^2(p_s a)} \cdot \left[J_0(p_s r) - J_0(p_s a)\right] \quad \text{...(6)}$$

which is convergent.

To determine the displacement u we have from (2) and (3)

$$\frac{du}{dr} + \frac{u}{r} = \frac{\alpha^2 h^2}{12} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

$$= \frac{\alpha^2 h^2}{12} - \frac{1}{2}\sum_{s=1}^{\infty} A_s^2 \cdot p_s^2 \cdot J_1^2(p_s r) - \frac{1}{2}\sum_{s=1}^{\infty} \sum_{m=1,\ s\neq m}^{\infty} A_s \cdot A_m \cdot p_s \cdot p_m \cdot J_1(p_s r) \cdot J_1(p_m r)$$

Integrating with respect to r, one gets

$$u r = \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{2}\sum_{s=1}^{\infty} A_s^2\, p_s^2 \left[\frac{r^2}{2}\left\{\left(1 - \frac{1}{p_s^2 r^2}\right) J_1^2(p_s r) + J_1'^2(p_s r)\right\}\right]$$

$$- \frac{1}{2}\sum_{s=1}^{\infty} \sum_{m=1,\ s\neq m}^{\infty} A_s \cdot A_m \cdot p_s \cdot p_m \left[r \left\{\frac{p_s \cdot J_2(p_s r) \cdot J_1(p_m r) - p_m J_1(p_s r) \cdot J_2(p_m r)}{p_s^2 - p_m^2}\right\}\right]$$

$$+ K \quad \text{...(7)}$$

K being the constant of integration.

Using $u = 0$, at $r = a$

$$K = \frac{1}{4}\sum_{s=1}^{\infty} A_s^2 p_s^2 a^2 J_0^2(p_s a) - \frac{\alpha^2 h^2 a^2}{24} \qquad \text{since} \quad J_1(p_s a) = 0$$

To determine α we know that as $r \to 0, u \to 0$ from symmetry, and then equation (7) leads to

$$\frac{\alpha^2 h^2 a^2}{24} = \frac{1}{4}\sum_{s=1}^{\infty} A_s^2 p_s^2 a^2 J_0^2(p_s a) \qquad \ldots(8)$$

Putting $\alpha = 0$ the differential equation (1) corresponds to that of small deflection equation. Now as α tends to zero, equation (6) leads to

$$W = \frac{2b^3 c}{a^2}\sum_{s=1}^{\infty} \frac{P(p_s b)\left[J_0(p_s r) - J_0(p_s a)\right]}{p_s^4 \cdot J_0^2(p_s a)} \qquad \ldots(9)$$

as obtained by Sen (1935),

where $p_s a$ is the sth root of $J_1(pa) = 0$

The deflection will be maximum at the centre of the plate. From equation (6) maximum deflection can be obtained by putting $r = 0$ as

$$W_0 = \frac{2b^3 c}{a^2}\sum_{s=1}^{\infty} \frac{P(p_s b)}{p_s^2(p_s^2+\alpha^2) J_0^2(p_s a)}\left[1 - J_0(p_s a)\right] \qquad \ldots(10)$$

whereas for small deflection W_0 will be given by

$$W_0 = \frac{2b^3 c}{a^2}\sum_{s=1}^{\infty} \frac{P(p_s b)}{p_s^4 J_0^2(p_s a)}\left[1 - J_0(p_s a)\right] \qquad \ldots(11)$$

from either equation (9) or equation (10).

In figure (4) W_0/h has been plotted against $b^5 c/h$ for both large and small deflection assuming $a = 2b$.

Discussion :

Small deflection theory of a plate which assumes the deflections small as compared with the thickness of the plate is based on the neglect of middle surface strains. In cases in which the deflections are no longer small in comparison with the thickness of the plate but are still small as compared with the other dimensions, the analysis of the problem must be extended to include the strain of the middle plane of the plate. For such problems, strain displacement relations are non-linear. In the present problem the latter theory is investigated.

The graph is plotted against cb^5/h for central deflection w_0/h. In calculating the deflection one has to start from equation (8) with an assumed value of αa leading to a particular value for the load function cb^5/h .

These values of αa and cb^5/h determine w_0/h from equation (10), Here αa has been assumed $1, 2, 3,$ etc.

It is clear that as αa increases, the load increases, also the central deflection.

For small deflection, w_0/h has been calculated from equation (11) for different values of cb^5/h and has been plotted side by side for comparison.

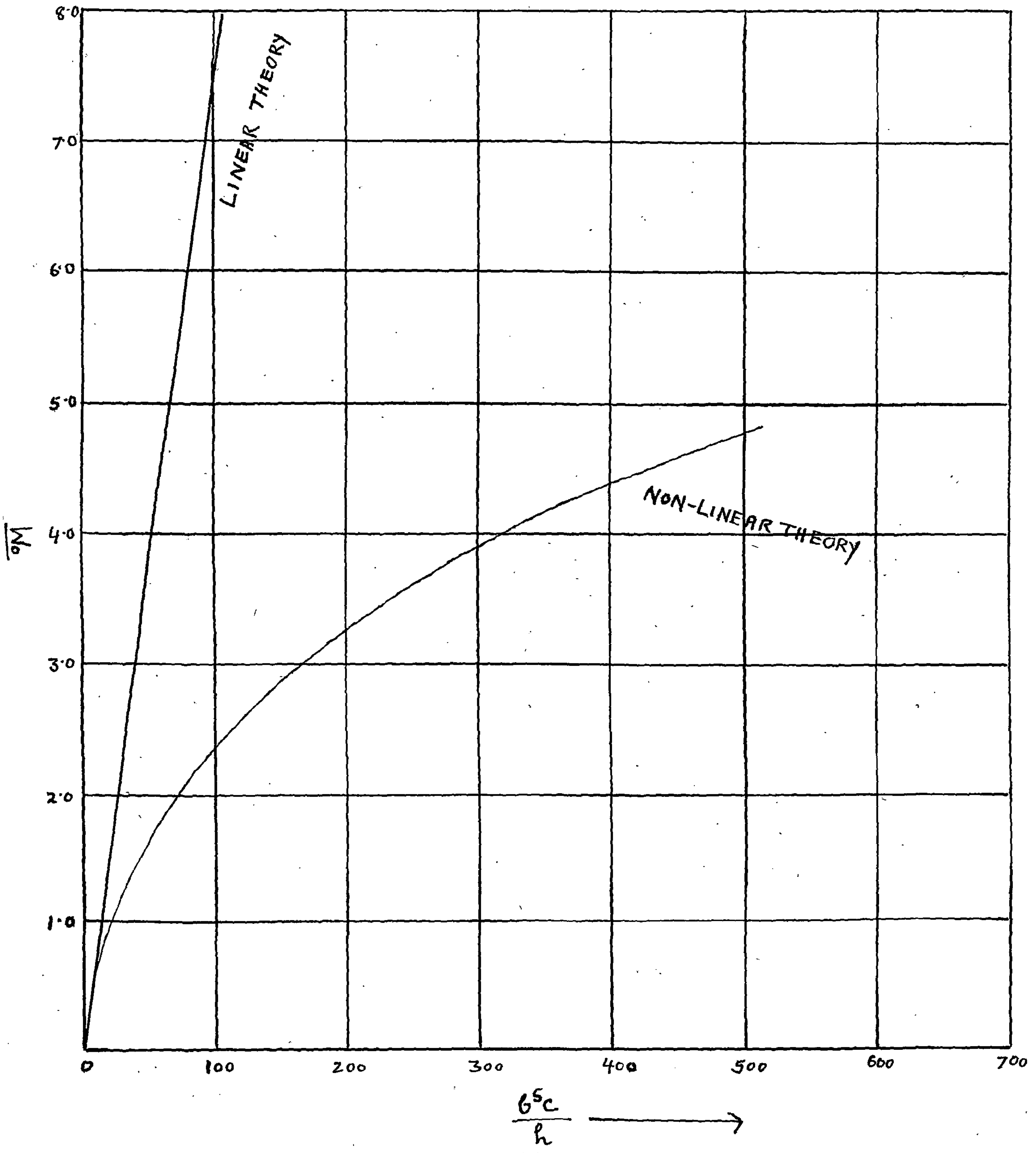

FIG. 4

Graph showing central deflections for various values of the load function $\frac{b^5c}{h}$

Note on the large deflection of a circular plate under concentrated load *

PAPER - II

Nomenclature :

The following nomenclature are used in this paper.

a = radius of the plate,

w = lateral displacement,

U, V = radial and cross-radial displacements,

h = thickness of the plate,

D = flexural rigidity of the plate $= Eh^3/12(1-\sigma^2)$,

E = Young's modulus,

σ = Poisson's ratio,

P = concentrated load at a distance b from the centre.

Introduction :

Following Berger (1955), many problems on the large deflection of initially flat isotropic plates have been investigated. In this paper the large deflection of a clamped circular plate under a concentrated load at a distance from the centre has been investigated.

The corresponding problem with the load at the centre was obtained by Basuli (1961).

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Fundamental Equations and Solution of the Problem :

Following Berger [1955], the deflection w of the plate (except at the load) satisfies the equation

$$\nabla_1^2(\nabla_1^2-\alpha^2)w=0 \qquad \ldots (1)$$

where

$$\frac{\alpha^2 h^2}{12}=\frac{\partial u}{\partial r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad \ldots (2)$$

u, v being radial and crossradial displacements and α being supposed to be constant.

Let the concentrated load P be placed at a distance b from the centre of the plate and let the radius of the plate be a. To solve the problem let us divide the plate by a concentric cylindrical surface of radius b passing through the load. Taking the line joining the centre of the plate and the load as the initial line and centre of the plate as pole, the equation (1) can be written as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}-\alpha^2 w\right)=0 \ldots (3)$$

As solutions of (3) we assume

$$w=w_1=R_0+\sum_{m=1}^{\infty}R_m\cos m\theta, \qquad \text{for } r>b \quad \ldots (4)$$

$$w=w_2=R_0'+\sum_{m=1}^{\infty}R_m'\cos m\theta, \qquad \text{for } r<b \quad \ldots (5)$$

where R_0, R_0', R_m, R_m' are functions of r only.

Now substituting (4) in (3) and considering the contributions of R_0 and R_0' only, we see that they satisfy the equation of the form

$$\left(\frac{d^2}{d\eta^2}+\frac{1}{\eta}\frac{d}{d\eta}\right)\left(\frac{d^2R}{d\eta^2}+\frac{1}{\eta}\frac{dR}{d\eta}-\alpha^2R\right)=0 \quad \text{.... (6)}$$

The solutions of (6) may be put in the form

$$R=R_0=A_0I_0(\alpha\eta)+B_0K_0(\alpha\eta)+C_0+D_0\operatorname{Log}\eta\,, \quad \text{for} \quad \eta>b \quad \text{... (7)}$$

$$R=R_0'=A_0'I_0(\alpha\eta)+C_0'\,, \quad \text{for} \quad \eta<b \quad \text{... (8)}$$

where $I_0(\alpha\eta)$, $K_0(\alpha\eta)$ are the Modified Bessel functions of the 1st and 2nd kind of order zero.

If the boundary be clamped,

$$W_1=\frac{\partial W_1}{\partial\eta}=0\,, \qquad \text{on} \quad \eta=a \quad \text{... (9)}$$

As the deflections, slope and bending moment will be continuous on the dividing circle, we get,

$$W_1=W_2\,;\ \frac{\partial W_1}{\partial\eta}=\frac{\partial W_2}{\partial\eta}\,;\ \frac{\partial^2W_1}{\partial\eta^2}=\frac{\partial^2W_2}{\partial\eta^2} \qquad \text{on} \quad \eta=b \quad \text{... (10)}$$

These continuity conditions ensure the same α for W_1 and W_2. The equations (7) and (8) contain altogether six constants and relations in (9) and

(10) are five in number. To get the sixth relation we shall have to consider the shearing force on the dividing circle, and this is continuous at every point of that circle except at the concentrated load.

Representing the load in the form of an infinite series

$$\frac{P}{\pi b}\left\{\frac{1}{2}+\sum_{m=1}^{\infty}\cos m\theta\right\}$$

the discontinuity in the shearing force is given by

$$\left[D\frac{\partial}{\partial r}\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\alpha^2\right\}w_1\right]_{r=b}$$

$$-\left[D\frac{\partial}{\partial r}\left\{\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\alpha^2\right\}w_2\right]_{r=b}$$

$$=\frac{P}{\pi b}\left\{\frac{1}{2}+\sum_{m=1}^{\infty}\cos m\theta\right\} \qquad \ldots (11)$$

where D is flexural rigidity of the plate and $= Eh^3/12(1-\sigma^2)$, σ is Poisson's ratio, E is Young's modulus, h is thickness of the plate. Substituting equations (7) and (8) in (9), (10) and (11) and solving for A_0, B_0, C_0, D_0, C_0', A_0' we get,

$$R_0=\frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\Big[I_0(\alpha r)+I_0(\alpha b)+\alpha a I_1(\alpha a)\operatorname{Log} a/r$$

$$-I_0(\alpha r)I_0(\alpha b)K_1(\alpha a)\alpha a-I_0(\alpha a)-I_1(\alpha a)K_0(\alpha r)I_0(\alpha b)\alpha a\Big]$$

$$R_0' = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\Big[I_0(\alpha r) + I_0(\alpha b) - I_0(\alpha a) - \alpha a I_1(\alpha a)\operatorname{Log} b/a$$

$$- \alpha a I_0(\alpha r) I_1(\alpha a) K_0(\alpha b) - I_0(\alpha r)\alpha a I_0(\alpha b) K_1(\alpha a)\Big]$$

Now taking the equations (3), (4) and (5) and contributions of R_m, R_m' only we see that they satisfy the equation of the form

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{m^2}{r^2}R - \alpha^2 R\right) = 0$$

This equation is satisfied if

$$R = R_m = A_m I_m(\alpha r) + B_m K_m(\alpha r) + C_m r^m + D_m r^{-m}, \quad r > b \qquad \dots (12)$$

$$R = R_m' = A_m' I_m(\alpha r) + C_m' r^m, \qquad r < b \qquad \dots (13)$$

where $I_m(\alpha r)$; $K_m(\alpha r)$ are Modified Bessel functions of order m. Considering the equations (9),(10),(11) (12) and (13) and solving for the constants A_m, B_m, C_m, D_m, A_m', C_m', we get

$$R_m = \frac{P}{\pi D\alpha^3 a I_{m+1}(\alpha a)}\Big[(b/a)^m I_m(\alpha r) - \alpha a I_m(\alpha b) K_{m+1}(\alpha a) I_m(\alpha r)$$

$$- I_m(\alpha b) K_m(\alpha r)\alpha a I_{m+1}(\alpha a) + \left(\frac{r}{a}\right)^m\left\{I_m(\alpha b) - \left(\frac{b}{a}\right)^m I_{m-1}(\alpha a)\frac{\alpha a}{2m}\right\}$$

$$+ \left(\frac{b}{r}\right)^m \frac{\alpha a}{2m} I_{m+1}(\alpha a)\Big]$$

$$R_m' = \frac{P}{\pi D\alpha^3 a I_{m+1}(\alpha a)}\Big[I_m(\alpha r)\left(\frac{b}{a}\right)^m - I_m(\alpha r) K_m(\alpha b)\alpha a I_{m+1}(\alpha a)$$

$$- I_m(\alpha r) I_m(\alpha b) K_{m+1}(\alpha a)\alpha a + \left(\frac{r}{a}\right)^m\Big\{I_m(\alpha b) + \left(\frac{a}{b}\right)^m \frac{\alpha a}{2m} I_{m+1}(\alpha a)$$

$$- \left(\frac{b}{a}\right)^m \frac{\alpha a}{2m} I_{m-1}(\alpha a)\Big\}\Big]$$

Therefore

$$w_1 = R_0 + \sum_{m=1}^{\infty} R_m \cos m\theta \,, \qquad r > b$$
$$w_2 = R_0' + \sum R_m' \cos m\theta \,, \qquad r < b$$

are determined.

To find the constant α , we have the following equation for α

$$\frac{\alpha^2 h^2}{12} = \frac{\partial U}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 .$$

Let U_1 , V_1 be the radial, cross-radial displacements when $r > b$ and U_2, V_2 be those displacements for $r < b$.

Let

$$\left.\begin{aligned} U_1 &= U_0(r) + \sum_{m=1}^{\infty} U_m(r) \cos m\theta \\ V_1 &= V_0(r) + \sum_{m=1}^{\infty} V_m(r) \sin m\theta \end{aligned}\right\} \quad r > b$$

$$\left.\begin{aligned} U_2 &= U_0'(r) + \sum_{m=1}^{\infty} U_m'(r) \cos m\theta \\ V_2 &= V_0'(r) + \sum_{m=1}^{\infty} V_m'(r) \sin m\theta \end{aligned}\right\} \quad r < b$$

As we have no interest in the radial and cross-radial displacements U, V we eliminate them by multiplying the last equation by $r\,d\theta\,dr$ and integrating between the limits b to a and 0 to 2π. For the outer portion we have,

$$\frac{\alpha^2 h^2}{12}\int_0^{2\pi}\int_b^a r\,dr\,d\theta = \int_0^{2\pi}\int_b^a \frac{\partial U_0(r)}{\partial r}\, r\,dr\,d\theta + \sum_{m=1}^{\infty}\int_0^{2\pi}\int_b^a \frac{\partial U_m(r)}{\partial r}\, r\cos m\theta\,dr\,d\theta$$

$$+\int_0^{2\pi}\int_b^a U_0(r)\,dr\,d\theta + \sum_{m=1}^{\infty}\int_0^{2\pi}\int_b^a U_m(r)\cos m\theta\,dr\,d\theta$$

$$+\frac{1}{2}\int_0^{2\pi}\int_b^a \left(\frac{\partial w_1}{\partial r}\right)^2 r\,dr\,d\theta + \frac{1}{2}\int_0^{2\pi}\int_b^a \frac{1}{r}\left(\frac{\partial w_1}{\partial \theta}\right)^2 dr\,d\theta$$

$$+\sum_{m=1}^{\infty} m\int_0^{2\pi}\int_b^a V_m(r)\cos m\theta\,dr\,d\theta$$

which leads to

$$\frac{\alpha^2 h^2 \pi (a^2-b^2)}{12} = 2\pi\left[a U_0(a) - b U_0(b)\right]$$

$$+\frac{1}{2}\int_0^{2\pi}\int_b^a \left(\frac{\partial w_1}{\partial r}\right)^2 r\,dr\,d\theta + \frac{1}{2}\int_0^{2\pi}\int_b^a \frac{1}{r}\left(\frac{\partial w_1}{\partial \theta}\right)^2 dr\,d\theta \qquad \ldots (14)$$

Similarly for the inner portion we have,

$$\frac{\alpha^2 h^2}{12}\int_0^{2\pi}\int_0^b r\,d\theta\,dr = \frac{\pi\alpha^2 h^2 b^2}{12} = 2\pi b U_0'(b) + \frac{1}{2}\int_0^{2\pi}\int_0^b \left(\frac{\partial w_2}{\partial r}\right)^2 r\,dr\,d\theta$$

$$+\frac{1}{2}\int_0^{2\pi}\int_0^b \frac{1}{r}\left(\frac{\partial w_2}{\partial \theta}\right)^2 dr\,d\theta \qquad \ldots (15)$$

Now, on $r=b$, $U_0(a)=0$, $U_0(b)=U_0'(b)$. Using these in equations (14) and (15) and adding together we get on

substitution the expressions for W_1 **and** W_2 **;**

$$
\begin{aligned}
\frac{\alpha^2\hbar^2a^2}{6} = {} & 2A_0^2\Big[\frac{\alpha^2a^2}{2}\Big\{I_1^2(\alpha a) - I_0^2(\alpha a)\Big\} + \alpha a I_1(\alpha a) I_0(\alpha a) \\
& - \frac{\alpha^2b^2}{2}\Big\{I_1^2(\alpha b) - I_0^2(\alpha b)\Big\} - \alpha b I_1(\alpha b) I_0(\alpha b)\Big] \\
& + 2B_0^2\Big[\frac{\alpha^2a^2}{2}\Big\{K_1^2(\alpha a) - K_0^2(\alpha a)\Big\} - \alpha a K_1(\alpha a) K_0(\alpha a) \\
& - \frac{\alpha^2b^2}{2}\Big\{K_1^2(\alpha b) - K_0^2(\alpha b)\Big\} + \alpha b K_1(\alpha b) K_0(\alpha b)\Big] \\
& + 2D_0^2 \log\frac{a}{b} - 4A_0B_0\Big[\frac{1}{2}\alpha^2a^2\Big\{I_1(\alpha a)K_1(\alpha a) + I_0(\alpha a)K_0(\alpha a)\Big\} \\
& - \frac{1}{2}(\alpha a)\Big\{I_1(\alpha a)K_0(\alpha a) - I_0(\alpha a)K_1(\alpha a)\Big\} \\
& - \frac{1}{2}\alpha^2b^2\Big\{I_1(\alpha b)K_1(\alpha b) + I_0(\alpha b)K_0(\alpha b)\Big\} + \frac{1}{2}\alpha b\Big\{I_1(\alpha b)K_0(\alpha b) \\
& - I_0(\alpha b)K_1(\alpha b)\Big\}\Big] + 4A_0D_0\Big[I_0(\alpha a) - I_0(\alpha b)\Big] \\
& + 4B_0D_0\Big[K_0(\alpha a) - K_0(\alpha b)\Big] \\
& + 2A_0'^2\Big[\frac{\alpha^2b^2}{2}\Big\{I_1^2(\alpha b) - I_0^2(\alpha b)\Big\} + \alpha b I_1(\alpha b) I_0(\alpha b)\Big] \\
& + \sum_{m=1}^{\alpha} 2m A_m C_m \Big[a^m I_m(\alpha a) - b^m I_m(\alpha b)\Big] \\
& + \sum_{m=1}^{\alpha} 2m A_m' C_m' b^m I_m(\alpha b) \\
& + \sum_{m=1}^{\alpha} 2m B_m C_m \Big[a^m K_m(\alpha a) - b^m K_m(\alpha b)\Big] \\
& + \sum_{m=1}^{\alpha} 2m B_m D_m \Big[K_m(\alpha b) b^{-m} - K_m(\alpha a) a^{-m}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{m=1}^{\infty} 2m A_m D_m \left[I_m(\alpha a) a^{-m} - I_m(\alpha b) b^{-m} \right] \\
&+\sum_{m=1}^{\infty} A_m^2 \Big[\tfrac{1}{2}\alpha^2 a^2 I_{m+1}^2(\alpha a) + (m+1)\alpha a I_m(\alpha a) I_{m+1}(\alpha a) \\
&\qquad - \tfrac{1}{2}\alpha^2 a^2 I_m^2(\alpha a) - \tfrac{1}{2}\alpha^2 b^2 I_{m+1}^2(\alpha b) - (m+1)\alpha b I_m(\alpha b) I_{m+1}(\alpha b) + \tfrac{1}{2}\alpha^2 b^2 I_m^2(\alpha b) \Big] \\
&+\sum_{m=1}^{\infty} B_m^2 \Big[\tfrac{1}{2}\alpha^2 a^2 K_{m+1}^2(\alpha a) - (m+1)\alpha a K_m(\alpha a) K_{m+1}(\alpha a) - \tfrac{1}{2}\alpha^2 a^2 K_m^2(\alpha a) \\
&\qquad - \tfrac{1}{2}\alpha^2 b^2 K_{m+1}^2(\alpha b) + (m+1)\alpha b K_m(\alpha b) K_{m+1}(\alpha b) + \tfrac{1}{2}\alpha^2 b^2 K_m^2(\alpha b) \Big] \\
&+\sum_{m=1}^{\infty} A_m'^2 \left[\tfrac{1}{2}\alpha^2 b^2 I_{m+1}^2(\alpha b) + (m+1)\alpha b I_m(\alpha b) I_{m+1}(\alpha b) - \tfrac{1}{2}\alpha^2 b^2 I_m^2(\alpha b) \right] \\
&+\sum_{m=1}^{\infty} m A_m^2 \left[I_m^2(\alpha a) - I_m^2(\alpha b) \right] + \sum_{m=1}^{\infty} m A_m'^2 I_m^2(\alpha b) \\
&\qquad\qquad + \sum_{m=1}^{\infty} m B_m^2 \left[K_m^2(\alpha a) - K_m^2(\alpha b) \right] \\
&+\sum_{m=1}^{\infty} 2m A_m B_m \left[I_m(\alpha a) K_m(\alpha a) - I_m(\alpha b) K_m(\alpha b) \right] \\
&-\sum_{m=1}^{\infty} 2 A_m B_m \Big[\tfrac{1}{2}\alpha^2 a^2 \left\{ I_{m+1}(\alpha a) K_{m+1}(\alpha a) + I_m(\alpha a) K_m(\alpha a) \right\} \\
&\qquad - (m+1)\frac{\alpha a}{2} \left\{ I_{m+1}(\alpha a) K_m(\alpha a) - I_m(\alpha a) K_{m+1}(\alpha a) \right\} \\
&\qquad - \tfrac{1}{2}\alpha^2 b^2 \left\{ I_{m+1}(\alpha b) K_{m+1}(\alpha b) + I_m(\alpha b) K_m(\alpha b) \right\} \\
&\qquad + (m+1)\frac{\alpha b}{2} \left\{ I_{m+1}(\alpha b) K_m(\alpha b) - I_m(\alpha b) K_{m+1}(\alpha b) \right\} \Big] \\
&+\sum_{m=1}^{\infty} m D_m^2 \left[b^{-2m} - a^{-2m} \right] + \sum_{m=1}^{\infty} m C_m^2 \left[a^{2m} - b^{2m} \right] \\
&+\sum_{m=1}^{\infty} \left[m C_m'^2 b^{2m} + 2m^2 C_m D_m \log \frac{b}{a} \right]
\end{aligned}
$$

$$\cdots (16)$$

where

$$A_0 = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\left[1 - I_0(\alpha b) K_1(\alpha a)\alpha a\right]$$

$$B_0 = -\frac{P I_0(\alpha b)}{2\pi D\alpha^2}$$

$$C_0 = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\left[I_0(\alpha b) - I_0(\alpha a) + \alpha a I_1(\alpha a)\operatorname{Log} a\right]$$

$$D_0 = -\frac{P}{2\pi D\alpha^2}$$

$$A_0' = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\left[1 - \alpha a I_1(\alpha a) K_0(\alpha b) - \alpha a I_0(\alpha a) K_1(\alpha a)\right]$$

$$C_0' = \frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\left[I_0(\alpha b) - I_0(\alpha a) - \alpha a I_0(\alpha a)\operatorname{Log}\frac{b}{a}\right]$$

$$A_m = \frac{P}{\pi D\alpha^3 a I_{m+1}(\alpha a)}\left[\left(\frac{b}{a}\right)^m - \alpha a I_m(\alpha b) K_{m+1}(\alpha a)\right]$$

$$B_m = -\frac{P I_m(\alpha b)}{\pi D\alpha^2}$$

$$C_m = \frac{P}{\pi D\alpha^3 a^{m+1} I_{m+1}(\alpha a)}\left[I_m(\alpha b) - \left(\frac{b}{a}\right)^m I_{m-1}(\alpha a)\frac{\alpha a}{2m}\right]$$

$$D_m = \frac{P b^m}{\pi D\alpha^2 2m}$$

$$A_m' = \frac{P}{\pi D\alpha^3 a I_{m+1}(\alpha a)}\left[\left(\frac{b}{a}\right)^m - K_m(\alpha b)\alpha a I_{m+1}(\alpha a) - I_m(\alpha b) K_{m+1}(\alpha a)\alpha a\right]$$

$$C_m' = \frac{P}{\pi D\alpha^3 a^{m+1} I_{m+1}(\alpha a)}\left[I_m(\alpha b) + \left(\frac{a}{b}\right)^m \frac{\alpha a}{2m} I_{m+1}(\alpha a) - \left(\frac{b}{a}\right)^m \frac{\alpha a}{2m} I_{m-1}(\alpha a)\right]$$

Now the large deflection of the plate under a concentrated load at the centre can be obtained from W_1 by making b tend to zero everywhere in the form [Basuli (1961)]

$$W = -\frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\Big[\alpha a\big\{K_1(\alpha a)\,I_0(\alpha r) + K_0(\alpha r)\,I_1(\alpha a)\big\}$$

$$+\,\alpha a\, I_1(\alpha a)\,\mathrm{Log}\,\frac{r}{a} - I_0(\alpha r) + I_0(\alpha a) - 1\Big].$$

Also the value of the constant α under a concentrated load at the centre can also be obtained from (16) taking limit as $b \to 0$ in the form

$$\left(\frac{Pa^2}{\pi D h}\right)^2 = \frac{\frac{1}{3}(\alpha a)^6}{\gamma + \mathrm{Log}\,\frac{\alpha a}{2} - \frac{I_0(\alpha a) + \alpha a K_1(\alpha a) - 2}{\alpha a I_1(\alpha a)} - \frac{1}{2}\left\{\frac{I_0(\alpha a) - 1}{I_1(\alpha a)}\right\}^2}.$$

where γ = Euler's constant.

The deflection is obtained for a plate with $\frac{a}{b} = 2$ for various loads. The figure shows the deflection $\left(\frac{W_2}{h}\right)_{r=b}$ against $\frac{Pa^2}{\pi D h}$. In calculating the deflection one has to start from equation (16) with assumed values of αa and αb leading to the particular value for the load function $\frac{Pa^2}{\pi D h}$. These values of αa and αb together with $\frac{Pa^2}{\pi D h}$ determine corresponding $\left(\frac{W_2}{h}\right)_{r=b}$.

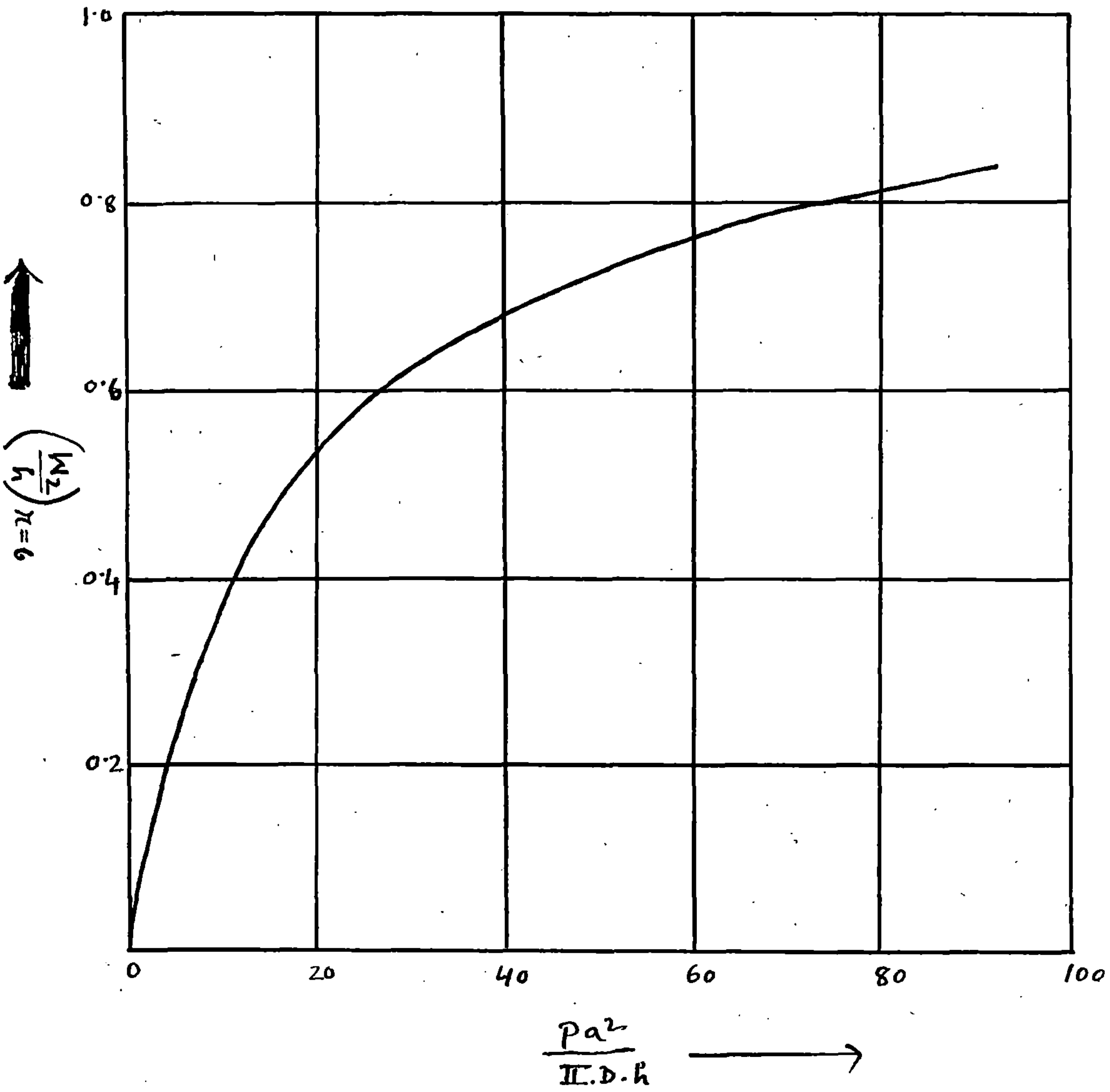

FIG. 5

Graph showing deflections for various values of $\frac{Pa^2}{\Pi D h}$

Note on the large deflection of an orthotropic circular plate under a concentrated load.*

PAPER - III

Nomenclature :

The following nomenclature are used in this paper.

P = concentrated load at the centre,
u = radial displacement,
w = deflection, normal to the plane,
a = radius of the plate,
h = thickness of the plate,
D_r = average flexural rigidity of the plate,
σ_r, σ_t = Poisson's ratios corresponding to radial and cross - radial directions.
$k^2 = \frac{\sigma_t}{\sigma_r}$

Introduction :

Following Berger's (1955) approximate method, numerous problems have been solved with remarkable ease and satisfactory results.

Iwinski and Nowinski (1957) generalized the procedure of Berger to orthotropic plates and found out the deflections of circular and rectangular plates under uniform load with different boundary conditions. In this

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paper the above method has been applied in the case of an orthotropic circular plate under a concentrated load at the centre.

Analysis :

In the case of circular symmetry if h is the thickness of the plate, w the displacement perpendicular to the middle plane, u the radial displacement in the middle plane under a concentrated load at the centre, then the differential equation for w and u will be (Iwinski and Nowinski, 1957)

$$\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{K^2}{r^2}\left(\frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr}\right) - \frac{12}{h^2 r}\frac{d}{dr}\left(e_1^* r \frac{dw}{dr}\right) = 0 \qquad \ldots(1)$$

except at the load.

and

$$\frac{de_1^*}{dr} + \frac{1-K}{r}e_1^* = 0 \qquad \ldots(2)$$

where $$e_1^* = \frac{du}{dr} + K\frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \qquad \ldots(3)$$

Again considering the radial stress and shearing stress on a concentric circular ring of radius r, the concentrated load P at the centre, and since u and $\frac{dw}{dr}$ are both zero at the centre, we have,

$$D_r \cdot \lim_{r \to 0} r\left[\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{K^2}{r^2}\frac{dw}{dr} - \frac{12}{h^2}e_1^* \cdot \frac{dw}{dr}\right] = \frac{P}{2\pi} \qquad \ldots(4)$$

Solving (2) we have

$$e_1^* = c r^{K-1} \qquad \ldots(5)$$

Hence we have the following differential equation for w

$$r^3\frac{d^4w}{dr^4} + 2r^2\frac{d^3w}{dr^3} - r\left(K^2 + \chi^2 r^{K+1}\right)\frac{d^2w}{dr^2} + \left(K^2 - \chi^2 K r^{K+1}\right)\frac{dw}{dr} = 0 \qquad \ldots(6)$$

where $\chi^2 = \frac{12c}{h^2}$

After changing the variables the equation takes the form

$$\frac{d^3z}{dr^3} + 2r^{-1}\frac{d^2z}{dr^2} - r^{-2}\left(K^2 + \chi^2 r^{K+1}\right)\frac{dz}{dr} + r^{-3}\left(K^2 - \chi^2 K r^{K+1}\right)z = 0 \qquad \ldots(7)$$

The above equation can be put in the form

$$\left(\frac{d}{dr} + \frac{1}{r}\right)\left[\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} - \left(K^2 r^{-2} + \chi^2 r^{K-1}\right)z\right] = 0 \qquad \ldots(8)$$

This equation can be represented by a system of two differential equations,

$$\frac{d^2z}{d\eta^2} + \frac{1}{\eta}\frac{dz}{d\eta} - \left(K^2\eta^{-2} + \lambda^2\eta^{K-1}\right)z = P'(\eta) \qquad \ldots(9)$$

$$\frac{dP'}{d\eta} + P'\eta^{-1} = 0 \qquad \ldots(10)$$

Solving (10) we get

$$P' = \frac{C_3'}{\eta} \qquad \ldots(11)$$

Hence equation (9) is equivalent to

$$\eta^2\frac{d^2z}{d\eta^2} + \eta\frac{dz}{d\eta} - \left(K^2 + \lambda^2\eta^{K+1}\right)z = C_3'\eta \qquad \ldots(12)$$

Solving (12) the deflection w can be put in the form

$$w = C_1\eta^{\frac{1-K}{2}} I_{\frac{K-1}{K+1}}\left(\frac{2\lambda\eta^{\frac{1+K}{2}}}{1+K}\right) + C_2\left[\eta^{\frac{1-K}{2}} K_{\frac{K-1}{K+1}}\left(\frac{2\lambda\eta^{\frac{1+K}{2}}}{1+K}\right) - \right.$$

$$\left. - \mu\left\{\frac{a^{1-K}}{1-K} - \frac{\eta^{1-K}}{1-K}\right\}\right] + C_3 \qquad \ldots(13)$$

where $\mu = \frac{1}{2}\Gamma\left(\frac{2K}{1+K}\right)(\lambda)^{1-\frac{2K}{1+K}}(1+K)^{\frac{2K}{1+K}}$

and I and K represent Modified

Bessel functions of first and second kind.

Boundary conditions on w are

$$w = \frac{dw}{dr} = 0 \quad \text{at } r = a \qquad \ldots(14)$$

Considering equations (4) and (13) we have

$$C_2 = -\frac{P}{\mu 2\pi \lambda^2 D_r} \qquad \ldots(15)$$

Combining equations(13), (14) and (15) we have

$$C_1 = C_2 \left[\frac{\lambda a^k K_{\frac{2k}{1+k}}\left(\frac{2\lambda a^{\frac{1+k}{2}}}{1+k}\right) - \mu}{\lambda a^k I_{\frac{2k}{1+k}}\left(\frac{2\lambda a^{\frac{1+k}{2}}}{1+k}\right)} \right]$$

$$C_3 = C_2 \cdot \frac{1}{\lambda a^k I_{\frac{2k}{1+k}}\left(\frac{2\lambda a^{\frac{1+k}{2}}}{1+k}\right)} \left[\mu a^{\frac{1-k}{2}} I_{\frac{k-1}{k+1}}\left(\frac{2\lambda a^{\frac{1+k}{2}}}{1+k}\right) - \frac{1+k}{2} \right]$$

To determine the displacement u we have from equation (2)

$$\frac{du}{dr} + \frac{k}{r} u = c r^{k-1} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

Substituting the expression for w from (13) and solving for u one gets,

$$r^K u = \frac{\lambda^2 h^2}{24K} r^{2K} - \frac{1}{2}\int r^K \Big[c_1^2 \lambda^2 I^2_{\frac{2K}{1+K}}(z_1)$$

$$+ c_2^2 \Big\{ \lambda^2 K^2_{\frac{2K}{1+K}}(z_1) + r^{-2K}\mu^2 - 2\lambda\mu K_{\frac{2K}{1+K}}(z_1) r^{-K} \Big\}$$

$$+ 2c_1 c_2 \Big\{ \mu\lambda r^{-K} I_{\frac{2K}{1+K}}(z_1) - \lambda^2 I_{\frac{2K}{1+K}}(z_1) K_{\frac{2K}{1+K}}(z_1) \Big\} \Big] dr$$

$$+ K_1$$

where $z_1 = \dfrac{2\lambda r^{\frac{1+K}{2}}}{1+K}$

After evaluating the integrals we have,

$$r^K u = \frac{\lambda^2 h^2}{24K} r^{2K} - \frac{1}{2} c_1^2 \Big[\frac{\lambda^2 r^{1+K}}{1+K} \Big\{ \Big[I_{\frac{2K}{1+K}}(z_1) \Big]^2 \Big[1 + \frac{K^2}{\lambda^2} r^{-1-K} \Big]$$

$$- \Big[I'_{\frac{2K}{1+K}}(z_1) \Big]^2 \Big\} \Big] -$$

$$- \frac{C_2^2}{2} \Bigg[\frac{\lambda^2 r^{1+K}}{1+K} \bigg\{ \Big[K_{\frac{2K}{1+K}}(Z_1) \Big]^2 \Big[1 + \frac{K^2}{\lambda^2} r^{-1-K} \Big] - \Big[K'_{\frac{2K}{1+K}}(Z_1) \Big]^2 \bigg\} \Bigg]$$

$$- \frac{C_2^2}{2} \frac{r^{1-K}}{1-K} \mu^2 - \mu C_2^2 r^{\frac{1-K}{2}} K_{\frac{K-1}{K+1}}(Z_1) - C_1 C_2 \mu r^{\frac{1-K}{2}} I_{\frac{K-1}{K+1}}(Z_1)$$

$$+ C_1 C_2 \frac{\lambda^2 r^{1+K}}{1+K} \Big[I_{\frac{2K}{1+K}}(Z_1) K_{\frac{2K}{1+K}}(Z_1) + I_{\frac{2K}{1+K}-1}(Z_1) K_{\frac{2K}{1+K}-1}(Z_1)$$

$$+ \frac{K}{\lambda} r^{\frac{-1-K}{2}} \Big\{ K_{\frac{2K}{1+K}}(Z_1) I_{\frac{2K}{1+K}-1}(Z_1) - I_{\frac{2K}{1+K}}(Z_1) K_{\frac{2K}{1+K}-1}(Z_1) \Big\} \Big]$$

$$+ K_1 \qquad \ldots(16)$$

Using the boundary condition $r \rightarrow a$, $u \rightarrow 0$ the integration constant K_1 can be evaluated as

$$K_1 = \frac{C_1^2}{2} \Bigg[\frac{\lambda^2 a^{1+K}}{1+K} \bigg\{ \Big[I_{\frac{2K}{1+K}}(Z_2) \Big]^2 \Big[1 + \frac{K^2}{\lambda^2} a^{-1-K} \Big] - \Big[I'_{\frac{2K}{1+K}}(Z_2) \Big]^2 \bigg\} \Bigg] +$$

$$+\frac{c_2^2}{2}\left[\frac{\lambda^2 a^{1+K}}{1+K}\left\{\left[K_{\frac{2K}{1+K}}(z_2)\right]^2\left[1+\frac{K^2}{\lambda^2}a^{-1-K}\right]-\left[K'_{\frac{2K}{1+K}}(z_2)\right]^2\right\}\right]$$

$$+\frac{c_2^2}{2}\frac{a^{1-K}}{1-K}\mu^2+\mu c_2^2 a^{\frac{1-K}{2}}K_{\frac{K-1}{K+1}}(z_2)+c_1c_2\mu a^{\frac{1-K}{2}}I_{\frac{K-1}{K+1}}(z_2)$$

$$-c_1c_2\frac{\lambda^2 a^{1+K}}{1+K}\left[I_{\frac{2K}{1+K}}(z_2)K_{\frac{2K}{1+K}}(z_2)+I_{\frac{2K}{1+K}-1}(z_2)K_{\frac{2K}{1+K}-1}(z_2)\right.$$

$$\left.+\frac{K}{\lambda}a^{\frac{-1-K}{2}}\left\{K_{\frac{2K}{1+K}}(z_2)I_{\frac{2K}{1+K}-1}(z_2)-I_{\frac{2K}{1+K}}(z_2)K_{\frac{2K}{1+K}-1}(z_2)\right\}\right]$$

$$-\frac{\lambda^2 h^2}{24K}a^{2K} \qquad \ldots(17)$$

where $z_2 = \frac{2\lambda a^{\frac{1+K}{2}}}{1+K}$

To determine the constant λ we shall use the condition that $u \to 0$ as $r \to 0$. Thus we have

$$\frac{c_1c_2K}{2}-\frac{\mu c_1c_2\left(\frac{\lambda}{1+K}\right)^{\frac{K-1}{K+1}}}{\Gamma\left(\frac{2K}{1+K}\right)}+\frac{\mu c_2^2\pi\left(\frac{\lambda}{1+K}\right)^{\frac{K-1}{K+1}}}{2\sin\left(\frac{K-1}{K+1}\pi\right)\Gamma\left(1+\frac{K-1}{K+1}\right)}$$

$$-\frac{c_2^2\pi K\Gamma\left(\frac{2K}{1+K}\right)}{4\sin\left(\frac{K-1}{K+1}\pi\right)}\left[\frac{\lambda^{\frac{2-2K}{1+K}}(1+K)^{\frac{2K-2}{1+K}}}{\Gamma\left(1+\frac{K-1}{K+1}\right)}\right]+K_1=0 \qquad \ldots(18)$$

As $K \rightarrow 1$, equation (13) reduces to the corresponding deflection for isotropic plate under a concentrated load at the centre as obtained by Basuli (1961) in the form

$$W = -\frac{P}{2\pi D\alpha^3 a I_1(\alpha a)}\Big\{\alpha a\left[K_1(\alpha a) I_0(\alpha r) + K_0(\alpha r) I_1(\alpha a)\right] + \alpha a I_1(\alpha a)\operatorname{Log}\frac{r}{a} - I_0(\alpha r) + I_0(\alpha a) - 1\Big\}$$

replacing λ by α

Also in the above case, equation (18) to determine α reduces to (Basuli 1961)

$$\left(\frac{Pa^2}{\pi D h}\right)^2 = \frac{\frac{1}{3}(\alpha a)^6}{\gamma + \operatorname{Log}\frac{\alpha a}{2} - \frac{I_0(\alpha a) + \alpha a K_1(\alpha a) - 2}{\alpha a I_1(\alpha a)} - \frac{1}{2}\left(\frac{I_0(\alpha a) - 1}{I_1(\alpha a)}\right)^2}$$

γ = Euler's constant.

Numerical calculation :

Let us take $\lambda = 1{\cdot}5$, $a = 10$, $K = 1/3$

Putting all these valus in (18) we get the load function in the form

$$\frac{P.10^4}{2\pi D_r h} = 76{\cdot}4$$

For this value of the load function the maximum deflection (deflection at the centre) is given from (13) in the form,

$$\frac{w_0}{h} = 2{\cdot}14$$

Large deflection of a semi-circular plate under a uniform load.*

PAPER IV

Nomenclature :

The following nomenclature are used throught this paper :

q = uniform lateral load,
u, v = radial and cross-radial displacements,
h = thickness of the plate,
D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,
E = Young's modulus,
σ = Poisson's ratio,
a = radius of the plate,
W = lateral displacement.

Introduction :

Approximate equations governing the non-linear behaviour of the plates (flat) have been given first by Berger (1955). Following Berger, a large number of non-linear problems have been solved by different authors. The present author's attempt is to apply this method to a semi-circular plate, simply-supported along the bounding diameter.

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Analysis :

Let us consider a plate in the form of a semicircle, simply-supported along the boundary.

Let us take the centre as pole and the bounding diameter as initial line. Following Berger, the differential equation satisfying the lateral displacement W is

$$\nabla^4 W - \alpha^2 \nabla^2 W = \frac{q}{D} \qquad \ldots(1)$$

where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad \ldots(2)$$

Expanding the load into the appropriate Fourier series we have

$$q = \frac{4q}{\pi} \sum_{m=1,3,\ldots} \frac{\sin m\theta}{m} \qquad \ldots(3)$$

Now, assuming

$$W = \sum R_m \sin m\theta \qquad \ldots(4)$$

where R_m is a function of r only and substituting the expressions for q and W (Equations (3) and (4)) into Equation (1) we get

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\left(\frac{d^2 R_m}{dr^2} + \frac{1}{r}\frac{dR_m}{dr} - \frac{m^2}{r^2}R_m - \alpha^2 R_m\right) = \frac{4q}{m\pi D} \qquad \ldots(5)$$

The solution of the above equation can be written in the form

$$R_m = A_m r^m + B_m r^{-m} + C_m J_m(i\alpha r) + D Y_m(i\alpha r) + \frac{4q S_{3,m}(i\alpha r)}{m\pi D\alpha^4(2^2-m^2)} \quad \ldots(6)$$

where J_m and Y_m are the Bessel functions of 1st and 2nd kind of order m and

$$S_{3,m}(i\alpha r) = \sum_{n=0}^{\infty} \frac{(-1)^n (i\alpha r)^{3+1+2n}}{\{(3+1)^2-m^2\}\cdots\{(3+1+2n)^2-m^2\}}$$

is the Lommel function.

The solution satisfying the boundary condition along the diameter is

$$R_m = A_m r^m + C_m J_m(i\alpha r) + \frac{4q S_{3,m}(i\alpha r)}{m\pi D\alpha^4(2^2-m^2)} \quad \ldots(7)$$

Hence,

$$W = \sum_{m=1,3,\ldots}^{\infty} \left[A_m r^m + C_m J_m(i\alpha r) + \frac{4q S_{3,m}(i\alpha r)}{m\pi D\alpha^4(2^2-m^2)}\right] \sin m\theta \quad \ldots(8)$$

In the case of a simply-supported plate, boundary conditions are as follows :

$$(u)_{r=a} = (W)_{r=a} = 0 \quad \ldots(9)$$

$$\left[\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right]_{r=a} = 0 \quad \ldots(10)$$

Combining equations (8), (9) and (10) and solving for the constants, we get

$$A_m = \frac{\dfrac{4q\,J_m(i\alpha a)}{m\pi D(2^2-m^2)}\left[\dfrac{\sigma i\,S'_{3,m}(i\alpha a)}{a\alpha^3} - \dfrac{S''_{3,m}(i\alpha a)}{\alpha^2}\right] - \dfrac{4q\,S_{3,m}(i\alpha a)}{m\pi D\alpha^4(2^2-m^2)}\left[\dfrac{\sigma i\alpha}{a} J'_m(i\alpha a) - \alpha^2 J''_m(i\alpha a)\right]}{a^m\left[\dfrac{\sigma i\alpha}{a} J'_m(i\alpha a) - \alpha^2 J''_m(i\alpha a)\right] - J_m(i\alpha a)\,a^{m-2}(m^2-m+m\sigma)} \qquad \text{...(11)}$$

$$C_m = \frac{\dfrac{4q\,S_{3,m}(i\alpha a)}{m\pi D\alpha^4(2^2-m^2)}\left[a^{m-2}(m^2-m+m\sigma)\right] - \dfrac{4q\,a^m}{m\pi D(2^2-m^2)}\left[\dfrac{\sigma i\,S'_{3,m}(i\alpha a)}{a\alpha^3} - \dfrac{S''_{3,m}(i\alpha a)}{\alpha^2}\right]}{a^m\left[\dfrac{\sigma i\alpha}{a} J'_m(i\alpha a) - \alpha^2 J''_m(i\alpha a)\right] - J_m(i\alpha a)\,a^{m-2}(m^2-m+m\sigma)} \qquad \text{...(12)}$$

To determine α, let us assume :

$$u = \sum U(r)\cos m\theta \qquad \text{...(13)}$$

$$v = \sum V(r)\sin m\theta \qquad \text{...(14)}$$

Multiplying equation (2) by $r\,d\theta\,dr$ and integrating within the limits 0 to a and 0 to π we have,

$$\int_0^a\int_0^\pi r\sum U'(r)\cos m\theta\,d\theta\,dr + \int_0^a\int_0^\pi \sum U(r)\cos m\theta\,d\theta\,dr$$

$$+ \int_0^a\int_0^\pi \sum m\,V(r)\cos m\theta\,d\theta\,dr + \frac{1}{2}\int_0^a\int_0^\pi \left(\frac{\partial w}{\partial r}\right)^2 r\,d\theta\,dr$$

$$+ \frac{1}{2}\int_0^a\int_0^\pi \frac{1}{r}\left(\frac{\partial w}{\partial \theta}\right)^2 d\theta\,dr = \frac{\alpha^2 h^2}{12}\int_0^a\int_0^\pi r\,d\theta\,dr$$

After evaluating the integrals, we obtain the following equation determining α

$$A_m^2 m a^{2m} + C_m^2\left[\frac{\alpha^2 a^2}{2} J_m^2(i\alpha a) + m J_m^2(i\alpha a) - \frac{\alpha^2 a^2}{2} J_{m+1}^2(i\alpha a) + \right.$$

$$\left. + i\alpha(m+1) J_{m+1}(i\alpha a) J_m(i\alpha a)\right] + 2m A_m C_m a^m J_m(i\alpha a) +$$

$$+ \frac{8A_m q a^m S_{3,m}(i\alpha a)}{\pi D\alpha^4(2^2-m^2)} - \frac{16q^2}{m^2\pi^2 D^2\alpha^6(2^2-m^2)^2} \times$$

$$\times\left[\sum_{\substack{n=0\\s=0\\n\neq s}}^{\infty}\left\{\frac{(i\alpha)^{6+4n}\cdot(4+2n)^2\cdot a^{8+4n}}{(8+4n)\left[(4^2-m^2)\cdots\left\{(4+2n)^2-m^2\right\}\right]^2} + \right.\right.$$

$$+ \frac{(-1)^{\eta}(-1)^{s}\cdot(i\alpha)^{6+2\eta+2s}\cdot(4+2\eta)(4+2s)\cdot a^{8+2\eta+2s}}{(8+2\eta+2s)\left[(4^2-m^2)\cdots\{(4+2\eta)^2-m^2\}\right]\left[(4^2-m^2)\cdots\{(4+2s)^2-m^2\}\right]}\Bigg\}\Bigg]$$

$$+ \frac{16q^2}{\pi^2 D^2\alpha^8(2^2-m^2)^2}\Bigg[\sum_{\substack{\eta=0\\ s=0\\ \eta\neq s}}^{\infty}\Bigg\{\frac{(i\alpha)^{8+4\eta}\cdot a^{8+4\eta}}{(8+4\eta)\left[(4^2-m^2)\cdots\{(4+2\eta)^2-m^2\}\right]} +$$

$$+ \frac{(-1)^{\eta}\cdot(-1)^{s}(i\alpha)^{8+2\eta+2s}\cdot a^{8+2\eta+2s}}{(8+2\eta+2s)\left[(4^2-m^2)\cdots\{(4+2\eta)^2-m^2\}\right]\left[(4^2-m^2)\cdots\{(4+2s)^2-m^2\}\right]}\Bigg\}\Bigg]$$

$$- \frac{8C_m q}{m\pi D\alpha^2(2^2-m^2)}\Bigg[\sum_{\eta=0}^{\infty}\Bigg\{\frac{(-1)^{\eta}(i\alpha)^{2+2\eta}\cdot(4+2\eta)\cdot a^{4+2\eta}}{(4^2-m^2)\cdots\{(4+2\eta)^2-m^2\}}\times J_m(i\alpha a) -$$

$$- \frac{(-1)^{\eta}\cdot(4+2\eta)^2\cdot\frac{a}{i\alpha}\left[(2+m+2\eta)\cdot J_m(i\alpha a)\cdot S_{2+2\eta,m-1}(i\alpha a) - J_{m-1}(i\alpha a)\cdot S_{3+2\eta,m}(i\alpha a)\right]}{(4^2-m^2)\cdots\{(4+2\eta)^2-m^2\}}\Bigg\}\Bigg]$$

$$+ \frac{8C_m q}{m\pi D\alpha^4(2^2-m^2)}\Bigg[\sum_{\eta=0}^{\infty}\frac{(-1)^{\eta}(i\alpha a)\left[(2+2\eta+m)J_m(i\alpha a)\cdot S_{2+2\eta,m-1}(i\alpha a) - J_{m-1}(i\alpha a)\cdot S_{3+2\eta,m}(i\alpha a)\right]}{(4^2-m^2)\cdots\{(4+2\eta)^2-m^2\}}\Bigg]$$

$$= \frac{\alpha^2 a^2 h^2}{6} \qquad \cdots (15)$$

As $\alpha \longrightarrow 0$ equation (8) reduces to

$$W = \frac{qa^4}{D}\cdot\sum_{m=1,3,\ldots}^{\infty}\Bigg[\frac{4r^4}{a^4}\cdot\frac{1}{m\pi(4-m^2)(16-m^2)} + \frac{r^m}{a^m}\cdot\frac{m+5+\sigma}{m\pi(2+m)(16-m^2)\left(m+\frac{1}{2}+\frac{\sigma}{2}\right)}$$

$$- \frac{r^{m+2}}{a^{m+2}}\cdot\frac{m+3+\sigma}{m\pi(4+m)(4-m^2)\left(m+\frac{1}{2}+\frac{\sigma}{2}\right)}\Bigg]\sin m\theta$$

as obtained by Timoshenko and Woinowsky-Krieger (1959) for the corresponding problem of small deflections.

The deflection is obtained for a plate with $a = 10$, $\sigma = 0{\cdot}25$ at $\theta = \frac{\pi}{4}$, $r = \frac{a}{2}$ for various loads.

The figure shows the deflection w/h against $q \cdot 10^4/\pi D h$. In calculating the deflection one has to start from equation (15) with an assumed value of $i\alpha$ leading to a particular value for the load function $q \cdot 10^4/\pi D h$. These values of $i\alpha$ and $q \cdot 10^4/\pi D h$ determine the corresponding w/h from equation (8). Here the values of $i\alpha$ have been assumed to be equal to $0{\cdot}1$, $0{\cdot}2$, etc.

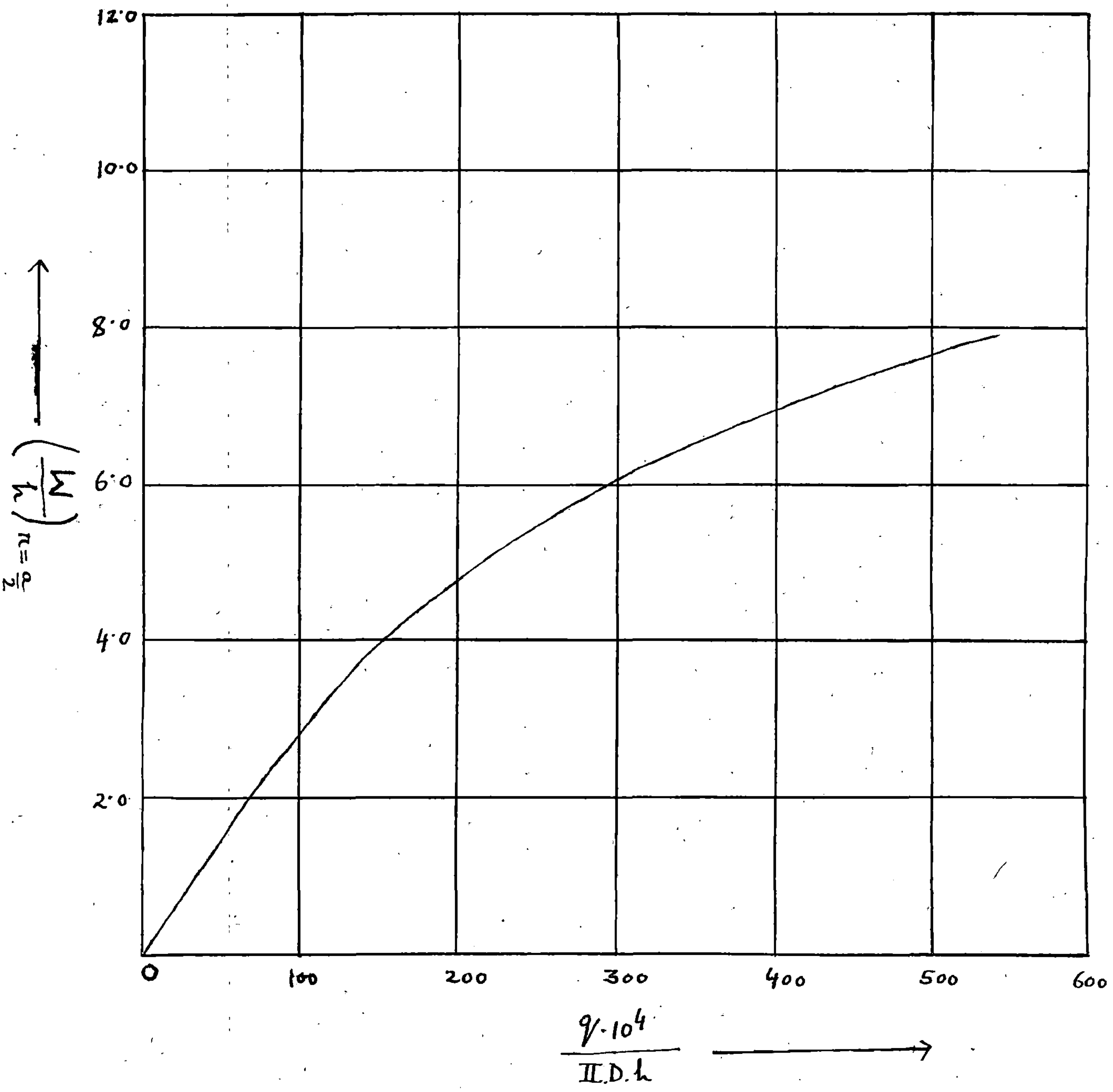

FIG. 6

Graph showing deflections for various values of the load function $\frac{q\cdot 10^4}{\Pi . D . h}$

Note on the large deflection of elliptic plates.*

PAPER - V

Nomenclature :

The following nomenclature are used in this paper.

$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$,

D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,

h = thickness of the plate,

E = Young's modulus,

σ = Poisson's ratio,

q = uniform load, normal to the plane,

w = deflection, normal to the plane,

u, v = displacements corresponding to x and y axes.

Introduction :

Following Berger's (1955) approximate method for large deflection, an attempt has been made to investigate the large deflection of elliptic plates with clamped edges. The general solution is obtained in terms of Mathieu functions of zero and even orders. Retaining only zero order, the deflection is obtained and with usual limiting process the known results for corresponding circular plates have also been deduced.

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Analysis :

Following Berger (1955) the deflection W of an elastic plate satisfies the differential equation

$$\nabla_1^2(\nabla_1^2-\alpha^2)W=\frac{q}{D} \qquad \ldots(1)$$

where α is a constant given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2=\frac{\alpha^2 h^2}{12} \qquad \ldots(2)$$

A particular integral W_0 of (1) is given by

$$W_0=-\frac{q}{4D\alpha^2}(x^2+y^2) \qquad \ldots(3)$$

Transferring to elliptic co-ordinates (ξ,η) defined by $x+iy=d\cosh(\xi+i\eta)$, where $2d$ is the interfocal distance of the ellipse,

the particular integral becomes

$$W_0=-\frac{qd^2}{8D\alpha^2}(\cosh 2\xi+\cos 2\eta) \qquad \ldots(4)$$

For the complementary function let us assume $W=W_1+W_2$

such that $\nabla_1^2 W_1=0$ and $\nabla_1^2 W_2-\alpha^2 W_2=0$ $\qquad \ldots(5)$

Changing to elliptic co-ordinates we have

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)W_1 = 0 \qquad \text{...(6)}$$

and $$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)W_2 - \frac{\alpha^2 d^2}{2}\left(\cosh 2\xi - \cos 2\eta\right)W_2 = 0 \qquad \text{...(7)}$$

Periodic solutions of (6) and (7) which are symmetric about the centre can be represented by

$$W_1 = \sum_{m=0}^{\infty} C_{2m} \cosh 2m\xi . \cos 2m\eta \qquad \text{...(8)}$$

$$W_2 = \sum_{m=0}^{\infty} \bar{C}_{2m} \mathbf{Ce}_{2m}(\xi, -q') \, ce_{2m}(\eta, -q') \qquad \text{...(9)}$$

where $ce_{2m}(\eta,-q')$ and $\mathbf{Ce}_{2m}(\xi,-q')$ are Mathieu function and Modified Mathieu function of the first kind of order $2m$ and

$$q' = \frac{\alpha^2 d^2}{4}$$

Combining equations(4), (8) and (9), the general solution can be written as

$$W = \sum_{m=0}^{\infty} C_{2m} \cosh 2m\xi . \cos 2m\eta$$

$$+ \sum_{m=0}^{\infty} \bar{C}_{2m} \mathbf{Ce}_{2m}(\xi,-q') \, ce_{2m}(\eta,-q') - \frac{q d^2}{8D\alpha^2}\left(\cosh 2\xi + \cos 2\eta\right) \qquad \text{...(10)}$$

While solving a problem of bending of a plate with elliptic hole, instead of taking Mathieu function of all orders, taking a single Mathieu function of second order, Naghdi (1955) has shown that the results obtained are satisfactory for larger elliptic hole. In our present problem we also make similar approximation by taking a single Mathieu function of order zero.

Hence on this approximation equation (10) reduces to

$$W = c_1 \mathbf{Ce}_0(\xi, -q')\, ce_0(\eta, -q') - \frac{qd^2}{8D\alpha^2}\left(\cosh 2\xi + \cos 2\eta\right) + c_2 \qquad \ldots(11)$$

If the outer boundary of the plate $\xi = \xi_0$ be clamped, we have $W = \frac{\partial W}{\partial \xi} = 0$, when $\xi = \xi_0$ . ...(12)

Using the above boundary conditions, the equations to determine the contstants will be

$$c_1 \mathbf{Ce}_0(\xi_0, -q')\, ce_0(\eta, -q') - \frac{qd^2}{8D\alpha^2}\left(\cosh 2\xi_0 + \cos 2\eta\right) + c_2 = 0 \qquad \ldots(13)$$

$$c_1 \mathbf{Ce}'_0(\xi_0, -q')\, ce_0(\eta, -q') - \frac{qd^2}{4D\alpha^2}\sinh 2\xi_0 = 0 \qquad \ldots(14)$$

Multiplying these equations by $ce_0(\eta, -q')$ and integrating w.r.t. η from 0 to 2π and using the orthogonality relations and normalization (Mclachlan, 1947, P-24), we get

$$\left.\begin{aligned} c_1 &= \frac{qd^2}{D\alpha^2}\,\frac{A_0(0)\sinh 2\xi_0}{\mathbf{Ce}'_0(\xi_0, -q')} \\ c_2 &= -\frac{qd^2}{4D\alpha^2}\left[\frac{\sinh 2\xi_0\, \mathbf{Ce}_0(\xi_0, -q')}{\mathbf{Ce}'_0(\xi_0, -q')} - \frac{1}{2}\cosh 2\xi_0 + \frac{1}{4}\frac{A_2(0)}{A_0(0)}\right] \end{aligned}\right\} \qquad \ldots(15)$$

$A_0(0)$ and $A_2(0)$ being the first two Fourier co-efficients in the expression of $ce_0(\eta,-q')$. Hence deflection is given by

$$W = \frac{qd^2}{4D\alpha^2}\left[\frac{2A_0(0)\sinh 2\xi_0}{Ce_0'(\xi_0,-q')}\, Ce_0(\xi,-q')\, ce_0(\eta,-q') - \frac{\sinh 2\xi_0\, Ce_0(\xi_0,-q')}{Ce_0'(\xi_0,-q')} + \frac{1}{2}\cosh 2\xi_0 - \frac{1}{4}\frac{A_2(0)}{A_0(0)} - \frac{1}{2}\left(\cosh 2\xi + \cos 2\eta\right)\right] \qquad \text{...(16)}$$

To determine α we know that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h^2}{12}$$

In elliptic co-ordinates, the above equation reduces to

$$h_1h_2\left[\frac{\partial}{\partial\xi}\left(\frac{u_\xi}{h_2}\right) + \frac{\partial}{\partial\eta}\left(\frac{u_\eta}{h_1}\right)\right] + \frac{1}{2}h_1h_2\left[\left(\frac{\partial w}{\partial\xi}\right)^2 + \left(\frac{\partial w}{\partial\eta}\right)^2\right] = \frac{\alpha^2 h^2}{12} \qquad \text{...(17)}$$

where $h_1 = h_2 = \dfrac{1}{d\sqrt{\sinh^2\xi + \sin^2\eta}}$

Boundary conditions for u_ξ and u_η are $u_\xi = 0 = u_\eta$ at $\xi = \xi_0$.

Let us assume that

$$\left.\begin{aligned} u_\xi &= \sum_{\eta=0}^{\infty} P(\xi)\cos 2n\eta \\ u_\eta &= \sum_{\eta=1}^{\infty} G(\xi)\sin 2n\eta \end{aligned}\right\} \qquad \ldots(18)$$

subject to the conditions $P(\xi_0) = G(\xi_0) = 0$

Integrating equation (17) over the surface of the plate we have

$$\int_0^{2\pi}\int_0^{\xi_0}\left[\left(\frac{\partial w}{\partial \xi}\right)^2+\left(\frac{\partial w}{\partial \eta}\right)^2\right]d\xi\, d\eta = \frac{\alpha^2 h^2 d^2}{6}\int_0^{2\pi}\int_0^{\xi_0}\left(\sinh^2\xi+\sin^2\eta\right)d\xi\, d\eta$$

OR,

$$\frac{c_1^2}{2}\int_0^{\xi_0}\int_0^{2\pi}\left[\mathbf{C}^2e_0'(\xi,-q')\,ce_0(\eta,-q')+\mathbf{C}^2e_0(\xi,-q')\,ce_0'(\eta,-q')\right]d\xi\, d\eta$$

$$+\frac{1}{2}\int_0^{\xi_0}\int_0^{2\pi}\frac{q^2d^4\sinh^2 2\xi}{16D^2\alpha^4}\,d\xi\, d\eta+\frac{1}{2}\int_0^{\xi_0}\int_0^{2\pi}\frac{q^2d^4\sin^2 2\eta}{16D^2\alpha^4}\,d\xi\, d\eta$$

$$-c_1\int_0^{\xi_0}\int_0^{2\pi}\mathbf{C}e_0'(\xi,-q')\,ce_0(\eta,-q')\,\frac{qd^2}{4D\alpha^2}\sinh 2\xi\,d\xi\, d\eta +$$

$$+C_1\int_0^{\xi_0}\int_0^{2\pi}\mathbf{Ce}_0(\xi,-q')\,ce_0'(\eta,-q')\,\frac{qd^2}{4D\alpha^2}\sin 2\eta\,d\xi\,d\eta$$

$$=\frac{\alpha^2h^2d^2}{12}\int_0^{\xi_0}\int_0^{2\pi}\left(\sin^2\eta+\sin^2 h\xi\right)d\xi\,d\eta \qquad \ldots(19)$$

After evaluating the integrals, we get the equations to determine α in the form

$$\frac{C_1qd^2}{D\alpha^2}\Bigg[\frac{1}{2}\{A_0^{(0)}\}^2\sinh 2\xi_0-A_0^{(0)}\sinh 2\xi_0\,\mathbf{Ce}_0(\xi_0,-q')$$

$$+\frac{1}{2}A_0^{(0)}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\left(A_{2r+2}^{(0)}+A_{2r-2}^{(0)}\right)\sinh 2r\xi_0-A_0^{(0)}A_2^{(0)}(\xi_0)\Bigg]$$

$$+C_1^2\Bigg[\sum_{r=1}^{\infty}\{A_{2r}^{(0)}\}^2 r\sinh 4r\xi_0+\sum_{r=1}^{\infty}\sum_{\substack{s=1\\ r\neq s}}^{\infty}\frac{(-1)^r(-1)^sA_{2r}^{(0)}A_{2s}^{(0)}\,4rs}{r^2-s^2}$$

$$\times\left\{r\sinh 2r\xi_0\cosh 2s\xi_0-s\sinh 2s\xi_0\cosh 2r\xi_0\right\}$$

$$+\sum_{r=1}^{\infty}4r^2\left(A_{2r}^{(0)}\right)^2\Bigg\{\sum_{r=1}^{\infty}\left(A_{2r}^{(0)}\right)^2\left(\xi_0+\frac{\sinh 4r\xi_0}{4r}\right)$$

$$+2A_0^{(0)}\sum_{r=1}^{\infty}(-1)^rA_{2r}^{(0)}\frac{\sinh 2r\xi_0}{r}+\sum_{r=1}^{\infty}\sum_{\substack{s=1\\ r\neq s}}^{\infty}\frac{(-1)^r(-1)^sA_{2r}^{(0)}A_{2s}^{(0)}}{r^2-s^2}\times$$

$$\times\left(r \sinh 2s\xi_0 \cosh 2r\xi_0 - s \sinh 2r\xi_0 \cosh 2s\xi_0\right)\Big\}\Big]$$

$$+ \frac{q^2 d^4 \sinh 4\xi_0}{64 D^2 \alpha^4}$$

$$= \frac{\alpha^2 h^2 d^2 \sinh 2\xi_0}{12} \qquad \ldots(20)$$

In the limiting case, when an elliptic plate of semi-major axis a tends to a circulate plate of radius a, $\xi \to \infty$, $d \to 0$.

Hence

$$\frac{\mathbf{C}e_0(\xi_0,-q')}{\mathbf{C}e_0'(\xi_0,-q')} \to \frac{I_0(\alpha a)}{\alpha a\, I_0'(\alpha a)}, \quad \frac{\mathbf{C}e_0(\xi,-q')}{\mathbf{C}e_0'(\xi,-q')} \longrightarrow \frac{I_0(\alpha r)}{\alpha r\, I_0'(\alpha r)}$$

and $ce_0(\eta,-q') \Rightarrow \frac{1}{\sqrt{2}}$, $A_0^{(0)} \Rightarrow \frac{1}{\sqrt{2}}$, $A_2(0) \longrightarrow 0$, $d^2 \sinh 2\xi_0 \to 2a^2$,
$A_{2r}^{(0)} \longrightarrow 0$ and $\cosh 2\xi\, d\xi \longrightarrow \frac{2r\,dr}{d^2}$

Then the equation (16) reduces to

$$W = \frac{q a^2}{2D\alpha^3 a I_1(\alpha a)}\left[I_0(\alpha r) - I_0(\alpha a)\right] + \frac{q}{4D\alpha^2}\left(a^2 - r^2\right) \qquad \ldots(21)$$

which gives the large deflection of a uniformly loaded circular plate of radius a.

Also the equation to determine α reduces to in the limiting case

$$\frac{q^2 a I_1(\alpha a) I_0(\alpha a)}{2D^2\alpha I_1^2(\alpha a)} + \frac{q^2\int_0^a r I_0^2(\alpha r)\,dr}{2D^2 I_1^2(\alpha a)} + \frac{q^2 a^2}{8D^2}$$

$$- \frac{q^2}{\alpha D^2 I_1(\alpha a)}\int_0^a r^2 I_1(\alpha r)\,dr = \frac{\alpha^6 h^2}{6}$$

OR,
$$\frac{\alpha^2 h^2 a^2}{24} = \frac{q^2 a^2}{8D^2\alpha^6 I_1^2(\alpha a)}\left[\frac{\alpha^2 a^2}{2}\left\{I_1^2(\alpha a) - I_0^2(\alpha a)\right\} + \alpha a I_0(\alpha a) I_1(\alpha a)\right]$$

$$+ \frac{q^2 a^4}{32 D^2\alpha^4} - \frac{q^2 a^3 I_2(\alpha a)}{4D^2\alpha^5 I_1(\alpha a)}$$

which is the equation for α in the case of uniformly loaded circular plate.

Numerical Calculation :

Putting $d = 2,\ \alpha = 2\sqrt{2},\ \xi_0 = 3,\ \xi = 2{\cdot}2$ and $\eta = \pi/4$

in equation (20), the value of the load function is found in the form

$$\frac{10^4 q}{Dh} = 110{\cdot}12$$

Putting this value of the load function in (16) we get the deflection in the form

$$\frac{W}{h} = 2{\cdot}13$$

Large deflection of an isoceles right-angled triangular plate. *

PAPER - VI

Nomenclature :

The following nomenclature are used in this paper.

q = uniform load,

u, v = displacements along x and y axes,

w = deflection, normal to the middle plane,

a = equal sides of the plate,

E_1, E_2 = Young's modulus corresponding to the directions of x and y,

σ_1, σ_2 = Poisson's ratios corresponding to the directions of x and y,

h = thickness of the plate,

D_1 = average flexural rigidity $= \frac{(EI)_1}{1-\sigma_1\sigma_2}$,

D_2 = average flexural rigidity $= \frac{(EI)_2}{1-\sigma_1\sigma_2}$,

$D_3 = \frac{1}{2}(\sigma_1 D_2 + \sigma_2 D_1) + 2D_K$,

D_K = average torsional rigidity,

$L^2 = D_3/D_1$,

$K^2 = D_2/D_1$.

Introduction :

Following Berger's approximate method for large deflection, a good number of problems have been solved.

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Iwinski and Nowinski (1957) extended this method to the case of orthotropic plates and arrived at the satisfactory results. The present author's attempt is to apply this method to the case of orthotropic isoceles right - angled triangular plate. The corresponding deflection of isotropic triangular plate has also been deduced.

Analysis :

Let us consider an isoceles right - angled triangular plate of equal sides a . The equation governing the deflection of an orthotropic plate in cartesian co-ordinates can be written as (Iwinski and Nowinski, 1957)

$$\frac{\partial^4 w}{\partial x^4} + 2L^2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + K^2 \frac{\partial^4 w}{\partial y^4} - \frac{12c}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + K\frac{\partial^2 w}{\partial y^2}\right) = \frac{q}{D_1} \quad \ldots (1)$$

where $e_1^* = \bar{\epsilon}_x + K\bar{\epsilon}_y = \frac{\partial u}{\partial x} + K\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{K}{2}\left(\frac{\partial w}{\partial y}\right)^2 = c \quad \ldots (2)$

and $e_1 = \epsilon_x + \epsilon_y$

Let the boundary be simply - supported with the following edge conditions.

$$\left.\begin{aligned} u = w = \frac{\partial^2 w}{\partial x^2} = 0 \quad &\text{at} \quad x = 0 \\ v = w = \frac{\partial^2 w}{\partial y^2} = 0 \quad &\text{at} \quad y = 0 \\ u + v = w = \frac{\partial^2 w}{\partial \nu^2} = 0 \quad &\text{at} \quad x + y = a \end{aligned}\right\} \quad \ldots (3)$$

where $\frac{\partial}{\partial \nu} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$

Assuming the deflection in the form

$$w = \sum A_m \left(\sin\frac{2m\pi x}{a} \sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) \quad \ldots \quad (4)$$

where m is an odd integer, A_m being constant, the boundary conditions on w can be satisfied identically.

Substituting (4) in (1) we have,

$$\sum A_m \left(C_m \sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + D_m \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) = \frac{q}{D_1} \quad \ldots \quad (5)$$

where

$$\left.\begin{aligned} C_m &= \frac{m^4\pi^4}{a^4}\left(K^2+8l^2+16\right) + \frac{12C}{h^2}\,\frac{m^2\pi^2}{a^2}\left(4+K\right) \\ D_m &= \frac{m^4\pi^4}{a^4}\left(16K^2+8l^2+1\right) + \frac{12C}{h^2}\,\frac{m^2\pi^2}{a^2}\left(4K+1\right) \end{aligned}\right\} \quad \ldots \quad (6)$$

Now the load can be expanded in the form,

$$q = \sum q_m \left(C_m \sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + D_m \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) \quad \ldots \quad (7)$$

Multiplying both sides of equation (7) by $\left[C_m \sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + D_m \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right]$ and integrating over the surface of the plate we have,

$$q_m = \frac{32q\left(C_m + D_m\right)}{3m^2\pi^2\left(C_m^2 + D_m^2\right)} \quad \ldots \quad (8)$$

Substituting (8) and (7) in (5) we have

$$A_m = \frac{32q\left(C_m + D_m\right)}{3m^2\pi^2\left(C_m^2 + D_m^2\right)D_1} \qquad \ldots (9)$$

Hence $$W = \sum A_m\left(\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right) \qquad \ldots (10)$$

is known.

To determine the constant c, we know from (2)

$$\frac{\partial u}{\partial x} + K\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{K}{2}\left(\frac{\partial w}{\partial y}\right)^2 = c$$

The boundary conditions $u=0$ at $x=0$, $v=0$ at $y=0$ and $u+v=0$ at $x+y=a$ are satisfied by the functions

$$u = \sum_{n=1,3,5,\ldots}^{\infty} B_n \sin\frac{n\pi x}{a}\left(\cos\frac{n\pi y}{a} + \sin\frac{n\pi x}{a} - \frac{n\pi}{4}\right) \qquad \ldots (11)$$

$$v = \sum_{n=1,3,5,\ldots}^{\infty} B_n \sin\frac{n\pi y}{a}\left(\cos\frac{n\pi x}{a} - \sin\frac{n\pi y}{a} + \frac{n\pi}{4}\right) \qquad \ldots (12)$$

where B_n is a constant.

Integrating (2) with respect to x and y over the surface of the plate we have

$$\int_0^a\int_0^{a-y} \sum B_n \frac{n\pi}{a}\left[\cos\frac{n\pi x}{a}\cdot\cos\frac{n\pi y}{a} + \sin\frac{2n\pi x}{a} - \frac{n\pi}{4}\cos\frac{n\pi x}{a}\right]dx\,dy$$

$$+ K\int_0^a\int_0^{a-y} B_n \frac{n\pi}{a}\left[\cos\frac{n\pi x}{a}\cdot\cos\frac{n\pi y}{a} - \sin\frac{2n\pi y}{a} + \frac{n\pi}{4}\cos\frac{n\pi y}{a}\right]dx\,dy$$

$$+\frac{1}{2}\int_0^a\int_0^{a-y}\left[\sum A_m\left(\cos\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a}\cdot\frac{2m\pi}{a} + \frac{m\pi}{a}\sin\frac{2m\pi y}{a}\cos\frac{m\pi x}{a}\right)\right]^2 dx\,dy$$

$$+\frac{K}{2}\int_0^a\int_0^{a-y}\left[\sum A_m\left(\sin\frac{2m\pi x}{a}\cos\frac{m\pi y}{a}\cdot\frac{m\pi}{a} + \frac{2m\pi}{a}\cos\frac{2m\pi y}{a}\sin\frac{m\pi x}{a}\right)\right]^2 dx\,dy$$

$$= c\int_0^a\int_0^{a-y} dx\,dy \qquad \ldots (13)$$

After evaluating the integrals we have

$$\sum A_m^2(1+K)m^2\pi^2 = \frac{8Ca^2}{5} \qquad \ldots (14)$$

If the plate be isotropic, we have

$$K = L = 1,\quad D_1 = D_2 = D_3 = D,$$
$$\sigma_1 = \sigma_2 = \sigma,\quad E_1 = E_2 = E,\quad C = \frac{\alpha^2 h^2}{12}$$

In that case the differential equation (1) reduces to

$$\nabla^2(\nabla^2-\alpha^2)W=\frac{q}{D} \quad \cdots (15), \quad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Corresponding deflection is given by

$$W=\sum B_m\left(\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a}+\sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right)$$

$$=\frac{32qa^2}{15D\pi^4}\sum\frac{\left(\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a}+\sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right)}{m^4\left(\frac{5m^2\pi^2}{a^2}+\alpha^2\right)} \quad \ldots (16)$$

where

$$B_m=\frac{32qa^2}{15D\pi^4}\cdot\frac{1}{m^4\left(\frac{5m^2\pi^2}{a^2}+\alpha^2\right)}$$

The corresponding equation to determine α reduces to

$$\sum B_m^2\, m^2\pi^2=\frac{\alpha^2 h^2 a^2}{15} \quad \ldots (17)$$

If $\alpha \to 0$, we get the corresponding small deflection for isotropic right - angled isoceles triangular plate with simply - supported edges in the form,

$$W=\frac{32qa^4}{75D\pi^6}\sum\frac{1}{m^6}\left(\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a}+\sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right) \quad \ldots (18)$$

Which is numerically equal to that obtained by Timoshenko. S and S. Woinowsky-Krieger (1959) in the form

$$W=\frac{16qa^4}{\pi^6 D}\left[\sum_{m=1,3,\ldots}^{\infty}\ \sum_{n=2,4,\ldots}^{\infty}\frac{n\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{a}}{m(n^2-m^2)(m^2+n^2)^2}+\right.$$

$$+ \sum_{m=2,4,\ldots}^{\alpha} \; \sum_{n=1,3,\ldots}^{\alpha} \frac{m \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a}}{n\left(m^2-n^2\right)\left(m^2+n^2\right)^2} \Bigg] \quad \ldots \quad (19)$$

The deflection is obtained for a plate at the point

$$\frac{x}{a} = \frac{y}{a} = \cdot 25$$

The graph is ploted showing the deflection $\frac{w}{h}$ of the isotropic plate against $\frac{qa^4}{\pi^4 Dh}$. In calculating the deflection one has to start from equation (17) with an assumed value of αa leading to a particular value for the load function $\frac{qa^4}{\pi^4 Dh}$. These values of αa and $\frac{qa^4}{\pi^4 Dh}$ determine corresponding $\frac{w}{h}$ from equation (16). Here αa has been assumed 1, 3, 5, 7, 9 etc.

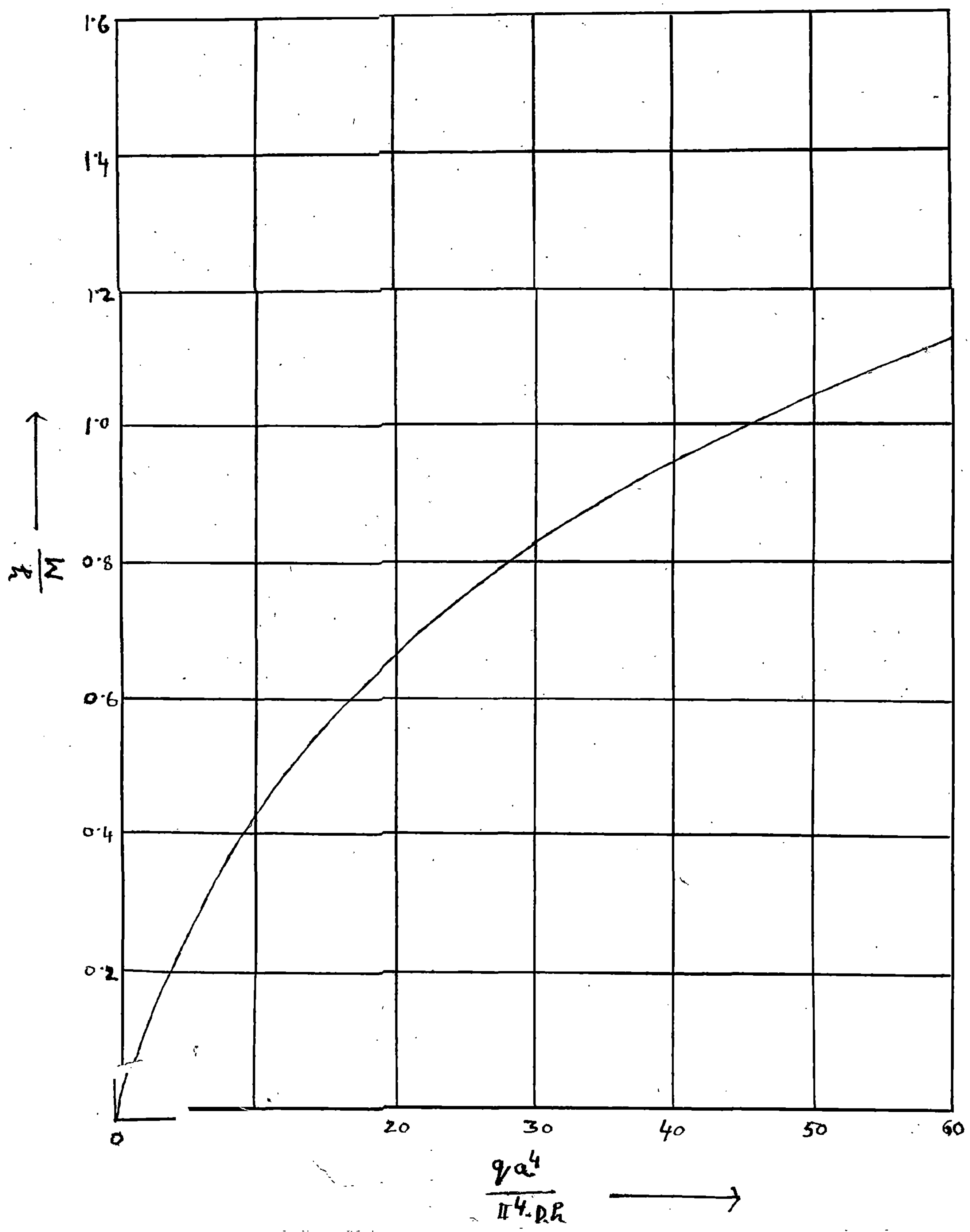

FIG.7

Graph showing deflections for various values of the load function $\frac{qa^4}{\pi^4 D h}$

CHAPTER - III

Large deflection of a square plate of variable thickness under uniform load.*

PAPER - I

Nomenclature :

The following nomenclature are used in this paper.

q = uniform load,
w = lateral displacement,
u, v = components in the middle plane of the plate,
h = thickness of the plate,
D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,
σ = Poisson's ratio,
E = Young's modulus,

Introduction :

For the large deflection of a plate we usually get non-linear equations which cannot be exactly solved. For a uniformly thin plate, Berger (1955) has shown that if in deriving the differential equation from strain energy, the strain energy due to the second strain invariant in the middle plane of the plate is neglected, a simple fourth order differential equation together with a non-linear second order

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equation is obtained. Following Berger, the corresponding equation for a square plate of variable thickness has been deduced. Basuli (1961) solved the problem of an annular plate of variable thickness with linearly varying thickness.

In this paper an attempt has been made to investigate the large deflection of a square plate of uni-directionally variable thickness and an infinite strip whose thickness varies along the breadth.

Analysis :

Let us consider a square plate of side $2a$. Let the centre of the plate be taken as origin.

Total strain energy of the plate with the present approximation is given by

$$V_2 = \frac{1}{2}\iint D\left\{\left[\left(\nabla^2 w\right)^2 + \frac{12}{h^2}e^2\right] - 2(1-\sigma)\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dx\,dy - \iint q\,w\,dx\,dy \qquad \ldots(1)$$

where

$$e = \epsilon_x + \epsilon_y\,,\quad \epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\quad \epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \ldots(2)$$

Using Euler's differential equation of variation we have (If $F(x,y)$ correspond to minimum value of V_2)

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\cdot\frac{\partial F}{\partial u_x} - \frac{\partial}{\partial y}\cdot\frac{\partial F}{\partial u_y} = 0 \qquad \ldots(3)$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\cdot\frac{\partial F}{\partial v_x} - \frac{\partial}{\partial y}\cdot\frac{\partial F}{\partial v_y} = 0 \qquad \ldots(4)$$

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\cdot\frac{\partial F}{\partial w_x} - \frac{\partial}{\partial y}\cdot\frac{\partial F}{\partial w_y} + \frac{\partial^2}{\partial x^2}\cdot\frac{\partial F}{\partial w_{xx}} + \frac{\partial^2}{\partial x\,\partial y}\cdot\frac{\partial F}{\partial w_{xy}} + \frac{\partial^2}{\partial y^2}\cdot\frac{\partial F}{\partial w_{yy}} = 0 \qquad \ldots(5)$$

Let us assume that

$$h = h_0\cdot x^{1/3}, \qquad \text{where} \quad D = D_0\cdot x \qquad \ldots(6)$$

Combining all the equations (1), (2), (3), (4), (5) and (6)

We have,

$$\frac{\partial}{\partial x}(he) = 0, \ \frac{\partial}{\partial y}(he) = 0 \quad \text{Hence} \quad he = \text{constant} = \frac{\alpha^2 h_0^2}{12}$$

Thus we get the following differential equations.

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h_0^2}{12h} \qquad \ldots(7)$$

$$\nabla^2\left(x\nabla^2 - \alpha^2/h_0\right)w = \frac{q}{D_0} \qquad \ldots(8)$$

For the complementary function of equation (8) let us put $w = w_1 + w_2$

So that for the equation

$$\nabla^2\left(x\nabla^2 - \alpha^2/h_0\right)w = 0$$

We have,

$$\nabla^2 W_1 = 0 \quad \text{....(9)} \quad \text{and} \quad x\nabla^2 W_2 - \frac{\alpha^2}{\hbar_0} W_2 = 0 \quad \text{...(10)}$$

To solve equation (9)

let us put

$$W_1 = \sum_{\eta=1,3,\cdots}^{\infty} W_\eta(x) \sin\frac{\eta\pi y}{a} \quad \text{...(11)}$$

boundary conditions being

$$\left.\begin{aligned} u = w = \frac{\partial^2 w}{\partial x^2} = 0 \quad &\text{at } x=b,\ x=c,\ b>c>0. \\ v = w = \frac{\partial^2 w}{\partial y^2} = 0 \quad &\text{at } y=\pm a,\ b-c=2a \end{aligned}\right\} \quad \text{...(12)}$$

Putting (11) in equation (9) and solving we get

$$W_\eta = A_1 e^{\frac{\eta\pi x}{a}} + A_2 e^{-\frac{\eta\pi x}{a}}$$

Again to solve equation (10)

let us put

$$W_2 = \sum_{\eta=1,3,\cdots}^{\infty} W_\eta'(x) \sin\frac{\eta\pi y}{a} \quad \text{...(13)}$$

On substitution we get,

$$x\frac{d^2W_n'}{dx^2} - W_n'\left(x\frac{n^2\pi^2}{a^2}+\frac{\alpha^2}{h_0}\right) = 0 \qquad \ldots(14)$$

To solve equation (14) let us put $W_n' = V'\cdot x$

The transformed equation is

$$x\frac{d^2V'}{dx^2} + 2\frac{dV'}{dx} - \left(\frac{n^2\pi^2}{a^2}x+\frac{\alpha^2}{h_0}\right)V' = 0 \qquad \ldots(15)$$

The solution of the above equation can be put in the form

$$V' = e^{-\frac{n\pi x}{a}}\left[A_3\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi x}{a}\right)^m + A_4\left\{\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi x}{a}\right)^m\cdot\operatorname{Log}\left(\frac{2n\pi x}{a}\right)\right.\right.$$

$$\left.\left.+\sum_{m=1}^{\infty}\mu_m\left(\frac{2n\pi x}{a}\right)^m + \frac{\Gamma(\lambda-1)}{\Gamma(\lambda)}\cdot\frac{a}{2n\pi x}\right\}\right] \qquad \ldots(16)$$

[Murphy, George M.: Ordinary differential equations and their solutions PP. 331.]

Where,

$$\lambda_m = \frac{\lambda(\lambda+1)\cdots(\lambda+m-1)}{2.3.\cdots(2+m-1)\,m!}, \quad \lambda = 1+\frac{a\alpha^2}{2n\pi h_0}$$

$$\mu_m = \frac{\Gamma(\lambda+m)\cdot H_m}{m!\,(m+1)!\,\Gamma(\lambda)}, \quad H_m = \sum_{r=0}^{m-1}\left[\frac{1}{\lambda+r}-\frac{1}{2+r}-\frac{1}{1+r}\right]$$

Hence

$$W_n' = x e^{-\frac{n\pi x}{a}} \left[A_3 \sum_{m=0}^{\infty} \lambda_m \left(\frac{2n\pi x}{a}\right)^m + A_4 \left\{ \sum_{m=0}^{\infty} \lambda_m \left(\frac{2n\pi x}{a}\right) . \operatorname{Log}\left(\frac{2n\pi x}{a}\right) \right.\right.$$

$$\left.\left. + \sum_{m=1}^{\infty} \mu_m \left(\frac{2n\pi x}{a}\right)^m + \frac{\Gamma(\lambda-1)}{\Gamma(\lambda)} \cdot \frac{a}{2n\pi x} \right\} \right]$$

$$= A_3 \phi_n(x) + A_4 \psi_n(x) \qquad \text{...(17)}$$

Particular integral of equation (8) can be taken as

$$- \frac{q x^2 h_0}{2\alpha^2 D_0}$$

Now q can be expanded by Fourier series in the form

$$q(y) = \frac{4q}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \operatorname{Sin} \frac{n\pi y}{a} \qquad \text{...(18)}$$

Hence

$$W = \sum_{n=1,3,\ldots}^{\infty} \left[A_1 e^{\frac{n\pi x}{a}} + A_2 e^{-\frac{n\pi x}{a}} + A_3 \phi_n(x) + A_4 \psi_n(x) \right.$$

$$\left. - \frac{2 q x^2 h_0}{\alpha^2 D_0 \pi n} \right] \operatorname{Sin} \frac{n\pi y}{a} \qquad \text{...(19)}$$

is determined.

Considering boundary conditions on W and solving for the constants we get,

$$A_1 = \frac{\lambda_3'\lambda_5' - \lambda_6'\lambda_2'}{\lambda_1'\lambda_5' - \lambda_2'\lambda_4'} \quad , \quad A_2 = \frac{\lambda_1'\lambda_6' - \lambda_3'\lambda_4'}{\lambda_1'\lambda_5' - \lambda_2'\lambda_4'}$$

$$A_3 = \frac{\psi_n(b)\mu_2' - \mu_1'\psi_n(c)}{\phi_n(b)\psi_n(c) - \psi_n(b)\phi_n(c)} \quad , \quad A_4 = \frac{\mu_1'\phi_n(c) - \mu_2'\phi_n(b)}{\phi_n(b)\psi_n(c) - \psi_n(b)\phi_n(c)}$$

where,

$$\mu_1' = A_1 e^{\frac{n\pi b}{a}} + A_2 e^{\frac{-n\pi b}{a}} - \frac{2qb^2h_0}{\alpha^2 D_0 n\pi}$$

$$\mu_2' = A_1 e^{\frac{n\pi c}{a}} + A_2 e^{-\frac{n\pi c}{a}} - \frac{2qc^2h_0}{\alpha^2 D_0 n\pi}$$

$$\lambda_1' = \Big[\Big\{e^{\frac{n\pi b}{a}}\cdot\psi_n(c) - e^{\frac{n\pi c}{a}}\cdot\psi_n(b)\Big\}\Big\{\phi_n''(b)\psi_n''(c) - \psi_n''(b)\phi_n''(c)\Big\}$$

$$-\Big\{\frac{n^2\pi^2}{a^2}e^{\frac{n\pi b}{a}}\cdot\psi_n''(c) - \frac{n^2\pi^2}{a^2}e^{\frac{n\pi c}{a}}\cdot\psi_n''(b)\Big\}\Big\{\phi_n(b)\psi_n(c) - \psi_n(b)\phi_n(c)\Big\}\Big]$$

$$\lambda_2' = \Big[\Big\{e^{-\frac{n\pi b}{a}}\psi_n(c) - e^{-\frac{n\pi c}{a}}\cdot\psi_n(b)\Big\}\Big\{\phi_n''(b)\psi_n''(c) - \psi_n''(b)\phi_n''(c)\Big\}$$

$$-\Big\{\frac{n^2\pi^2}{a^2}e^{-\frac{n\pi b}{a}}\cdot\psi_n''(c) - \frac{n^2\pi^2}{a^2}e^{-\frac{n\pi c}{a}}\psi_n''(b)\Big\}\Big\{\phi_n(b)\psi_n(c) - \psi_n(b)\phi_n(c)\Big\}\Big]$$

$$\lambda_3' = \frac{2 q h_0}{\alpha^2 D_0 n\pi}\Big[\big\{b^2\psi_n(c) - c^2\psi_n(b)\big\}\big\{\phi_n''(b)\psi_n''(c) - \psi_n''(b)\phi_n''(c)\big\}$$

$$- 2\big\{\psi_n''(c) - \psi_n''(b)\big\}\big\{\psi_n(c)\phi_n(b) - \phi_n(c)\psi_n(b)\big\}\Big]$$

$$\lambda_4' = \Big[\Big\{e^{\frac{n\pi b}{a}}\cdot\psi_n''(b) - \frac{n^2\pi^2}{a^2}e^{\frac{n\pi b}{a}}\psi_n(b)\Big\}\big\{\phi_n(c)\psi_n''(c) - \phi_n''(c)\psi_n(c)\big\}$$

$$-\Big\{e^{\frac{n\pi c}{a}}\psi_n''(c) - \frac{n^2\pi^2}{a^2}e^{\frac{n\pi c}{a}}\psi_n(c)\Big\}\big\{\phi_n(b)\psi_n''(b) - \phi_n''(b)\psi_n(b)\big\}\Big]$$

$$\lambda_5' = \Big[\Big\{e^{-\frac{n\pi b}{a}}\cdot\psi_n''(b) - \frac{n^2\pi^2}{a^2}e^{-\frac{n\pi b}{a}}\cdot\psi_n(b)\Big\}\big\{\phi_n(c)\psi_n''(c) - \phi_n''(c)\psi_n(c)\big\}$$

$$-\Big\{e^{-\frac{n\pi c}{a}}\cdot\psi_n''(c) - \frac{n^2\pi^2}{a^2}e^{-\frac{n\pi c}{a}}\cdot\psi_n(c)\Big\}\big\{\phi_n(b)\psi_n''(b) - \phi_n''(b)\psi_n(b)\big\}\Big]$$

$$\lambda_6' = \frac{2 q h_0}{\alpha^2 D_0 n\pi}\Big[\big\{b^2\psi_n''(b) - 2\psi_n(b)\big\}\big\{\phi_n(c)\psi_n''(c) - \phi_n''(c)\psi_n(c)\big\}$$

$$-\big\{c^2\psi_n''(c) - 2\psi_n(c)\big\}\big\{\phi_n(b)\psi_n''(b) - \phi_n''(b)\psi_n(b)\big\}\Big]$$

To determine α, we know from (7)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2=\frac{\alpha^2 h_0^2}{12h}$$

Let us assume

$$u=\sum_{k=0}^{\infty} g_k(x)\cos\frac{k\pi y}{a} \qquad \ldots(20)$$

$$v=\sum_{k=1}^{\infty} L_k(x)\sin\frac{k\pi y}{a} \qquad \ldots(21)$$

Combining equations(7), (19), (20) and (21), we get,

$$\sum_{k=0}^{\infty} g_k'(x)\cos\frac{k\pi y}{a}+\sum_{k=1}^{\infty} L_k(x)\frac{k\pi}{a}\cos\frac{k\pi y}{a}$$

$$+\frac{1}{2}\left[\sum_{n=1,3,\ldots}^{\infty}\sin\frac{n\pi y}{a}\left(\frac{n\pi}{a}A_1e^{\frac{n\pi x}{a}}-\frac{n\pi}{a}A_2e^{-\frac{n\pi x}{a}}+A_3\Phi_n'(x)+A_4\Psi_n'(x)-\frac{4qxh_0}{\alpha^2 D_0 n\pi}\right)\right]^2$$

$$+\frac{1}{2}\left[\sum_{n=1,3,\ldots}^{\infty}\left(A_1e^{\frac{n\pi x}{a}}+A_2e^{-\frac{n\pi x}{a}}+A_3\Phi_n(x)+A_4\Psi_n(x)-\frac{2qx^2h_0}{\alpha^2 D_0 n\pi}\right)\frac{n\pi}{a}\cos\frac{n\pi y}{a}\right]^2$$

$$=\frac{\alpha^2 h_0}{12}\cdot x^{-1/3} \qquad \ldots(22)$$

Equating the terms independent of y, and integrating with respect to x between the limits b and c, we get the following biquadratic equation for the determination of α.

$$A_1^2\,\frac{n\pi}{a}\left(e^{\frac{2n\pi b}{a}} - e^{\frac{2n\pi c}{a}}\right) - A_2^2\,\frac{n\pi}{a}\left(e^{-\frac{2n\pi b}{a}} - e^{-\frac{2n\pi c}{a}}\right)$$

$$+ A_3^2\,\frac{n^2\pi^2}{a^2}\Bigg[\sum_{m=0}^{\infty}\lambda_m^2\left(\frac{2n\pi}{a}\right)^{2m+1}\cdot e^{-\frac{2n\pi c}{a}}\Bigg\{c^{2m+2} - \frac{(2m+2)c^{2m+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots$$

$$+ \frac{(-1)^{2m+2}\cdot\lfloor\underline{2m+2}}{\left(-\frac{a}{2n\pi}\right)^{2m+2}}\Bigg\} - \sum_{m=0}^{\infty}\lambda_m^2\left(\frac{2n\pi}{a}\right)^{2m+1}\cdot e^{\frac{-2n\pi b}{a}}\Bigg\{b^{2m+2} - \frac{(2m+2)b^{2m+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots$$

$$+ \frac{(-1)^{2m+2}\cdot\lfloor\underline{2m+2}}{\left(-\frac{a}{2n\pi}\right)^{2m+2}}\Bigg\}$$

$$+ \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty}\lambda_m\cdot\lambda_s\cdot\left(\frac{2n\pi}{a}\right)^{m+s+1}\cdot e^{-\frac{2n\pi c}{a}}\Bigg\{c^{m+s+2} - \frac{(m+s+2)c^{m+s+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots$$

$$+ \frac{(-1)^{m+s+2}\cdot\lfloor\underline{m+s+2}}{\left(-\frac{a}{2n\pi}\right)^{m+s+2}}\Bigg\}$$

$$- \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty}\lambda_m\cdot\lambda_s\cdot\left(\frac{2n\pi}{a}\right)^{m+s+1}\cdot e^{-\frac{2n\pi b}{a}}\Bigg\{b^{m+s+2} - \frac{(m+s+2)b^{m+s+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots$$

$$+ \frac{(-1)^{m+s+2}\cdot\lfloor\underline{m+s+2}}{\left(-\frac{a}{2n\pi}\right)^{m+s+2}}\Bigg\}\Bigg]$$

$$+ A_3^2\Bigg[e^{-\frac{2n\pi c}{a}}\cdot\Bigg\{\sum_{m=0}^{\infty}\left(\frac{2n\pi}{a}\right)^{2m+1}(m+1)^2\lambda_m^2\Bigg(c^{2m} - \frac{2m\cdot c^{2m-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots$$

$$+ \frac{(-1)^{2m}\cdot\lfloor\underline{2m}}{\left(-\frac{a}{2n\pi}\right)^{2m}}\Bigg)\Bigg\} - e^{-\frac{2n\pi b}{a}}\Bigg\{\sum_{m=0}^{\infty}\left(\frac{2n\pi}{a}\right)^{2m+1}\cdot(m+1)^2\cdot\lambda_m^2\Bigg(b^{2m} - \frac{2m\cdot b^{2m-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots$$

$$+ \frac{(-1)^{2m}\cdot\lfloor\underline{2m}}{\left(-\frac{a}{2n\pi}\right)^{2m}}\Bigg)\Bigg\}$$

$$+ e^{-\frac{2n\pi c}{a}} \left\{ \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\propto} \lambda_m \lambda_s \left(\frac{2n\pi}{a}\right)^{m+s+1} \cdot (m+1)(s+1) \left(c^{m+s} - \frac{(m+s)\, c^{m+s-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+s} \cdot \lfloor\!\underline{m+s}}{\left(-\frac{a}{2n\pi}\right)^{m+s}} \right) \right\}$$

$$- e^{-\frac{2n\pi b}{a}} \left\{ \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\propto} \lambda_m \lambda_s \left(\frac{2n\pi}{a}\right)^{m+s+1} \cdot (m+1)(s+1) \left(b^{m+s} - \frac{(m+s)\, b^{m+s-1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+s} \cdot \lfloor\!\underline{m+s}}{\left(-\frac{a}{2n\pi}\right)^{m+s}} \right) \right\}$$

$$+ \frac{n^2\pi^2}{a^2} \cdot e^{-\frac{2n\pi c}{a}} \left\{ \sum_{m=0}^{\propto} \lambda_m^2 \left(\frac{2n\pi}{a}\right)^{2m+1} \left(c^{2m+2} - \frac{(2m+2)\, c^{2m+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{2m+2} \cdot \lfloor\!\underline{2m+2}}{\left(-\frac{a}{2n\pi}\right)^{2m+2}} \right) \right\}$$

$$- \frac{n^2\pi^2}{a^2} e^{-\frac{2n\pi b}{a}} \left\{ \sum_{m=0}^{\propto} \lambda_m^2 \left(\frac{2n\pi}{a}\right)^{2m+1} \left(b^{2m+2} - \frac{(2m+2)\, b^{2m+1}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{2m+2} \cdot \lfloor\!\underline{2m+2}}{\left(-\frac{a}{2n\pi}\right)^{2m+2}} \right) \right\}$$

$$+ \frac{n\pi}{a} e^{-\frac{2n\pi c}{a}} \left\{ \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\propto} \lambda_m \lambda_s \left(\frac{2n\pi}{a}\right)^{m+s+1} \left(c^{m+s+1} - \frac{(m+s+1)\, c^{m+s}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+s+1} \cdot \lfloor\!\underline{m+s+1}}{\left(-\frac{a}{2n\pi}\right)^{m+s+1}} \right) \right\}$$

$$- \frac{n\pi}{a} e^{-\frac{2n\pi b}{a}} \left\{ \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\propto} \lambda_m \lambda_s \left(\frac{2n\pi}{a}\right)^{m+s+1} \left(b^{m+s+1} - \frac{(m+s+1)\, b^{m+s}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+s+1} \cdot \lfloor\!\underline{m+s+1}}{\left(-\frac{a}{2n\pi}\right)^{m+s+1}} \right) \right\}$$

$$+ \frac{2n\pi}{a} e^{-\frac{2n\pi b}{a}} \left\{ \sum_{\substack{m=0\\ p=0\\ m\neq p}}^{\propto} \lambda_m \lambda_p \left(\frac{2n\pi}{a}\right)^{m+p+1} (p+1)(m+1) \left(b^{m+p+1} - \frac{(m+p+1)\, b^{m+p}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+p+1} \cdot \lfloor\!\underline{m+p+1}}{\left(-\frac{a}{2n\pi}\right)^{m+p+1}} \right) \right\}$$

$$- \frac{2n\pi}{a} e^{-\frac{2n\pi c}{a}} \left\{ \sum_{\substack{m=0\\ p=0\\ m\neq p}}^{\propto} \lambda_m \lambda_p \left(\frac{2n\pi}{a}\right)^{m+p+1} (p+1)(m+1) \left(c^{m+p+1} - \frac{(m+p+1)\, c^{m+p}}{\left(-\frac{a}{2n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+p+1} \cdot \lfloor\!\underline{m+p+1}}{\left(-\frac{a}{2n\pi}\right)^{m+p+1}} \right) \right\} \Bigg]$$

$$+ A_4^2 \left[\left\{ \frac{\Gamma(\lambda-1)}{\Gamma(\lambda)} \right\}^2 \cdot \frac{a}{2n\pi} \left\{ e^{-\frac{2n\pi c}{a}} - e^{-\frac{2n\pi b}{a}} \right\} + \cdots\cdots \right] + \frac{16\, q^2 h_0^2}{3\alpha^4 D_0^2 n^2 \pi^2} \left(b^3 - c^3\right)$$

$$+ \frac{4\, q^2 h_0^2}{5\alpha^4 D_0^2 a^2} \left(b^5 - c^5\right) + 2 A_3 A_1 \left[e^{\frac{n\pi b}{a}} \phi_n(b) - e^{\frac{n\pi c}{a}} \phi_n(c) \right] \frac{n\pi}{a}$$

$$+ 2A_4A_1\frac{n\pi}{a}\left[e^{\frac{n\pi b}{a}}\psi_n(b) - e^{\frac{n\pi c}{a}}\psi_n(c)\right] + \frac{4q\nu\hbar_0A_1}{\alpha^2D_0a}\left[b^2e^{\frac{n\pi b}{a}} - c^2e^{\frac{n\pi c}{a}}\right]$$

$$+ 2A_2A_3\frac{n\pi}{a}\left[e^{-\frac{n\pi c}{a}}\phi_n(c) - e^{-\frac{nb\pi}{a}}\phi_n(b)\right] + \frac{4q\nu\hbar_0A_2}{\alpha^2D_0a}\left[b^2e^{-\frac{n\pi b}{a}} - c^2e^{-\frac{n\pi c}{a}}\right]$$

$$+2A_2A_4\frac{n\pi}{a}\left[e^{-\frac{n\pi c}{a}}\psi_n(c) - e^{-\frac{n\pi b}{a}}\psi_n(b)\right]$$

$$- \frac{4A_3n\pi q\nu\hbar_0}{\alpha^2D_0a^2}\left[\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot\frac{e^{-\frac{n\pi c}{a}}}{\frac{a}{n\pi}}\left\{c^{m+3} - \frac{(m+3)c^{m+2}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+3}\cdot\lfloor m+3}{\left(-\frac{a}{n\pi}\right)^{m+3}}\right\}\right.$$

$$\left.- \sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot\frac{e^{-\frac{n\pi b}{a}}}{\frac{a}{n\pi}}\left\{b^{m+3} - \frac{(m+3)b^{m+2}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+3}\cdot\lfloor m+3}{\left(-\frac{a}{n\pi}\right)^{m+3}}\right\}\right]$$

$$- \frac{8q\nu\hbar_0A_3}{\alpha^2D_0\pi n}\left[\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot(m+1)\frac{e^{-\frac{n\pi c}{a}}}{\frac{a}{n\pi}}\left\{c^{m+1} - \frac{(m+1)c^{m}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+1}\cdot\lfloor m+1}{\left(-\frac{a}{n\pi}\right)^{m+1}}\right\}\right.$$

$$- \sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot(m+1)\frac{e^{-\frac{n\pi b}{a}}}{\frac{a}{n\pi}}\left\{b^{m+1} - \frac{(m+1)b^{m}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+1}\cdot\lfloor m+1}{\left(-\frac{a}{n\pi}\right)^{m+1}}\right\}$$

$$+ \sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot\frac{e^{-\frac{n\pi b}{a}}}{\frac{a}{n\pi}}\left\{b^{m+2} - \frac{(m+2)b^{m+1}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+2}\cdot\lfloor m+2}{\left(-\frac{a}{n\pi}\right)^{m+2}}\right\}$$

$$\left.- \sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\cdot\frac{e^{-\frac{n\pi c}{a}}}{\frac{a}{n\pi}}\left\{c^{m+2} - \frac{(m+2)c^{m+1}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+2}\cdot\lfloor m+2}{\left(-\frac{a}{n\pi}\right)^{m+2}}\right\}\right]$$

$$- \frac{4A_4q\nu\hbar_0n\pi}{\alpha^2D_0a^2}\left[\frac{e^{-\frac{n\pi c}{a}}}{\frac{a}{n\pi}}\sum_{m=1}^{\infty}\mu_m\left(\frac{2n\pi}{a}\right)^m\left\{c^{m+3} - \frac{(m+3)c^{m+2}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+3}\cdot\lfloor m+3}{\left(-\frac{a}{n\pi}\right)^{m+3}}\right\}\right.$$

$$\left.- \frac{e^{-\frac{n\pi b}{a}}}{\frac{a}{n\pi}}\sum_{m=1}^{\infty}\mu_m\left(\frac{2n\pi}{a}\right)^m\left\{b^{m+3} - \frac{(m+3)b^{m+2}}{\left(-\frac{a}{n\pi}\right)} + \cdots\cdots + \frac{(-1)^{m+3}\cdot\lfloor m+3}{\left(-\frac{a}{n\pi}\right)^{m+3}}\right\} + \cdots\cdots\right]$$

$$- \frac{8q\nu\hbar_0}{\alpha^2D_0n\pi}\left[\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^m\left\{b^{m+2}\cdot e^{-\frac{n\pi b}{a}}\cdot \mathrm{Log}\frac{2n\pi b}{a} - c^{m+2}\cdot e^{-\frac{n\pi c}{a}}\cdot \log\frac{2n\pi c}{a}\right\}\right.$$

$$+\sum_{m=1}^{\infty}\mu_m\left(\frac{2n\pi}{a}\right)^m\left\{b^{m+2}\cdot e^{-\frac{n\pi b}{a}}-c^{m+2}\cdot e^{-\frac{n\pi c}{a}}\right\}$$

$$+\frac{\Gamma(\lambda-1)}{\Gamma(\lambda)}\cdot\frac{a}{2n\pi}\left\{be^{-\frac{n\pi b}{a}}-ce^{-\frac{n\pi c}{a}}\right\}+\cdots\cdots\Bigg]$$

$$+2A_3A_4\frac{n^2\pi^2}{a^2}\left[\frac{\Gamma(\lambda-1)}{\Gamma(\lambda)}\cdot\frac{a}{2n\pi}\cdot\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^{m+1}\left\{e^{-\frac{2n\pi c}{a}}\left(c^{m+1}-\frac{(m+1)c^m}{\left(-\frac{a}{2n\pi}\right)}+\cdots\cdots\right.\right.\right.$$

$$\left.\left.\left.+\frac{(-1)^{m+1}\cdot\lfloor\underline{m+1}}{\left(-\frac{a}{2n\pi}\right)^{m+1}}\right)-e^{-\frac{2n\pi b}{a}}\left(b^{m+1}-\frac{(m+1)b^m}{\left(-\frac{a}{2n\pi}\right)}+\cdots\cdots+\frac{(-1)^{m+1}\cdot\lfloor\underline{m+1}}{\left(-\frac{a}{2n\pi}\right)^{m+1}}\right)\right\}+\cdots\right]$$

$$+2A_3A_4\left[\frac{\Gamma(\lambda-1)}{2\Gamma(\lambda)}\cdot\sum_{m=0}^{\infty}\lambda_m\left(\frac{2n\pi}{a}\right)^{m+1}\cdot(m+1)\left\{e^{-\frac{2n\pi b}{a}}\left(b^m-\frac{m\cdot b^{m-1}}{\left(-\frac{a}{2n\pi}\right)}+\cdots\cdots\right.\right.\right.$$

$$\left.\left.\left.+\frac{(-1)^m\cdot\lfloor\underline{m}}{\left(-\frac{a}{2n\pi}\right)^m}\right)-e^{-\frac{2n\pi c}{a}}\left(c^m-\frac{m\cdot c^{m-1}}{\left(-\frac{a}{2n\pi}\right)}+\cdots\cdots+\frac{(-1)^m\cdot\lfloor\underline{m}}{\left(-\frac{a}{2n\pi}\right)^m}\right)\right\}+\cdots\right]$$

$$=\frac{\alpha^2h_0}{2}\left(b^{2/3}-c^{2/3}\right)\qquad\cdots\cdots(27)$$

If the plate is infinite in y direction only, then the differential equation (8) will take the form

$$x\frac{d^4w}{dx^4} + 2\frac{d^3w}{dx^3} - \frac{\alpha^2}{h_o}\frac{d^2w}{dx^2} = \frac{q}{D_o} \qquad \ldots(23)$$

Solution of the above equation can be put as

$$W = \frac{x^{1/2}h_o}{\alpha^2}\left[A_1 J_1\left(2i d_o x^{1/2}\right) + A_2 Y_1\left(2i d_o x^{1/2}\right)\right]$$

$$+ C_1 x + C_2 - \frac{qx^2}{2d_o^2 D_o} \qquad \ldots(24)$$

Where $-\frac{qx^2}{2d_o^2 D_o}$ is taken as the particular solution of(23) and $d_o^2 = \alpha^2/h_o$, J_1 and Y_1 being the Bessel functions of 1st and 2nd kind.

Boundary conditions are

$$\left.\begin{aligned} u = W &= 0 \ , \quad at \ x = b, \ x = c \\ \frac{d^2W}{dx^2} &= 0 \qquad at \ x = b, \ x = c \end{aligned}\right\} \qquad \ldots(25)$$

Considering boundary conditions on W and solving for the constants we get,

$$A_1 = \frac{q}{d_o^2 D_o}\left[\frac{\lambda_8 - \lambda_6}{\lambda_5\lambda_8 - \lambda_7\lambda_6}\right] , \quad A_2 = \frac{q}{d_o^2 D_o}\left[\frac{\lambda_7 - \lambda_5}{\lambda_7\lambda_6 - \lambda_5\lambda_8}\right]$$

$$C_1 = \frac{q(b+c)}{2d_o^2 D_o} - \frac{A_1(\lambda_1 - \lambda_3) + A_2(\lambda_2 - \lambda_4)}{b-c}$$

$$C_2 = \frac{qb^2}{2d_o^2 D_o} - A_1\lambda_1 - A_2\lambda_2 - C_1 b$$

Where,

$$\lambda_1 = \frac{b^{1/2} J_1(2 i d_0 b^{1/2})}{d_0^2}, \qquad \lambda_2 = \frac{b^{1/2} Y_1(2 i d_0 b^{1/2})}{d_0^2}$$

$$\lambda_3 = \frac{c^{1/2} J_1(2 i d_0 c^{1/2})}{d_0^2}, \qquad \lambda_4 = \frac{c^{1/2} Y_1(2 i d_0 c^{1/2})}{d_0^2}$$

$$\lambda_5 = b^{-1/2} J_1(2 i d_0 b^{1/2}), \qquad \lambda_6 = b^{-1/2} Y_1(2 i d_0 b^{1/2})$$

$$\lambda_7 = c^{-1/2} J_1(2 i d_0 c^{1/2}), \qquad \lambda_8 = c^{-1/2} Y_1(2 i d_0 c^{1/2})$$

To determine α , we know that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h_0^2}{12 h} \qquad \ldots(26)$$

In this case equation (26) reduces to

$$\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = \frac{\alpha^2 h_0 x^{-1/3}}{12} \qquad \ldots(27)$$

If $u = U(x)$ we have,

$$U'(x) + \frac{1}{2}\left[C_1 + \frac{q x}{d_0^2 D_0} + \frac{i}{d_0}\left\{A_1 J_0(2 i d_0 x^{1/2}) + A_2 Y_0(2 i d_0 x^{1/2})\right\}\right]^2$$

$$= \frac{\alpha^2 h_0 x^{-1/3}}{12} \qquad \ldots(28)$$

Integrating the above equation between the limits b and c we get the following equation to determine α

$$\frac{A_1^2}{8d_0^4}\left[z_1^2\left\{J_0^2(z_1)+J_1^2(z_1)\right\}-z_2^2\left\{J_0^2(z_2)+J_1^2(z_2)\right\}\right]$$

$$+\frac{A_2^2}{8d_0^4}\left[z_1^2\left\{Y_0^2(z_1)+Y_1^2(z_1)\right\}-z_2^2\left\{Y_0^2(z_2)+Y_1^2(z_2)\right\}\right]$$

$$+\frac{iA_1A_2}{4d_0^3}\left[z_1^2\left\{J_0(z_1)Y_0(z_1)+J_1(z_1)\cdot Y_1(z_1)\right\}-z_2^2\left\{J_0(z_2)Y_0(z_2)+J_1(z_2)Y_1(z_2)\right\}\right]$$

$$+\frac{c_1^2}{2}(b-c)+\frac{q^2}{6d_0^4D_0^2}(b^3-c^3)+\frac{c_1q}{2d_0^2D_0}(c^2-b^2)$$

$$+\frac{A_1c_1}{2id_0^3}\left[z_1J_1(z_1)-z_2J_1(z_2)\right]+\frac{A_2c_1}{2id_0^3}\left[z_1Y_1(z_1)-z_2Y_1(z_2)\right]$$

$$+\frac{A_1q}{8id_0^7D_0}\left[z_1^3J_1(z_1)-2z_1^2J_2(z_1)-z_2^3J_1(z_2)+2z_2^2J_2(z_2)\right]$$

$$+\frac{A_2q}{8id_0^7D_0}\left[z_1^3Y_1(z_1)-2z_1^2Y_2(z_1)-z_2^3Y_1(z_2)+2z_2^2Y_2(z_2)\right]$$

$$=\frac{\alpha^2h_0}{8}\left(b^{2/3}-c^{2/3}\right) \qquad \ldots(29)$$

Where

$$z_1=2id_0b^{1/2}\quad,\quad z_2=2id_0c^{1/2}$$

Numerical calculations :

(a) Square plate.

Let us take $a = 10,\ b = 30,\ c = 10,\ \alpha^2 = 10,\ h_0 = 1$

Putting these values in the biquadratic equation for the determination of α , we have the load function in the form

$$\frac{q}{D_0} = 108{\cdot}27 \times 10^{-4}$$

If $x/a = 2,\ y/a = 0{\cdot}50$, then from equation (19)

We get, $W = 2{\cdot}71$

(b) Plate of infinite strip.

Let us take $b = 30,\ c = 10,\ i\alpha = 1{\cdot}5,\ h_0 = 1$

Putting these values in equation (29), we get,

$$\frac{q}{D_0} = 87{\cdot}15 \times 10^{-3}$$

If $x = 20$, then from equation (24)

$$W = 1{\cdot}703$$

Large deflection of a circular plate of variable thickness under uniform load.*

PAPER - II

Nomenclature :

The following nomenclature are used in this paper.

q = uniform lateral load,

u, w = radial and lateral displacements,

h = thickness of the plate at a distance r from the centre,

D = flexural rigidity of the plate = $\frac{Eh^3}{12(1-\sigma^2)}$,

σ = Poisson's ratio,

E = Young's modulus,

$e_{rr} = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2$,

$e_{\theta\theta} = \frac{u}{r}$,

$e = e_{rr} + e_{\theta\theta}$

$s_{x,y}(z)$ = Lommel's function = $\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z^{x+1+2m}}{\{(x+1)^2 - y^2\} \cdots \{(x+1+2m)^2 - y^2\}}$,

Introduction :

Following Berger (1955), the corresponding equation for a plate of variable thickness has been deduced. Basuli (1961) solved the above problem with linearly varying thickness. In this paper an attempt has been made to solve

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this problem with thickness varying as the cube root of the distance from the origin.

Analysis :

Let us consider an annular plate of outer radius b. Let the centre of the plate be taken as origin. Strain energy due to pure bending and stretching of the middle plane of the plate on combination gives

$$V = \frac{1}{2}\int D\left[\left(\nabla^2 w\right)^2 + \frac{12e^2}{h^2} - \frac{2(1-\sigma)}{r}\cdot\frac{dw}{dr}\cdot\frac{d^2w}{dr^2}\right] r\,dr\,d\theta - \int q w r\,dr\,d\theta \qquad \ldots(1)$$

If F be an extremum of V, Euler's variational equations are

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial r}\cdot\frac{\partial F}{\partial u_r} = 0 \qquad \ldots(2)$$

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial r}\cdot\frac{\partial F}{\partial w_r} + \frac{\partial^2}{\partial r^2}\cdot\frac{\partial F}{\partial w_{rr}} = 0 \qquad \ldots(3)$$

Considering (1) and (2) we can write the differential equation in the form

$$\frac{d}{dr}(he) = 0, \quad he = \text{constant} = \frac{\alpha^2 h_0^2}{12}$$

so that

$$e = \frac{u}{r} + \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = \frac{\alpha^2 h_0^2}{12h} \qquad \ldots(4)$$

Let $h = h_0 r^{1/3}$, where $D = D_0 r$ ...(5)

From (1) and (3) we have

$$r^2 \frac{d^4 w}{dr^4} + 4r \frac{d^3 w}{dr^3} + \frac{d^2 w}{dr^2}\left(A - Br\right) - B\frac{dw}{dr}$$
$$= \frac{qr}{D_0}$$

Where $A = 1 + \sigma$, $B = \frac{\alpha^2}{h_0}$

Setting $\frac{dw}{dr} = z$, **we have**

$$r^2 \frac{d^3 z}{dr^3} + 4r \frac{d^2 z}{dr^2} + \frac{dz}{dr}\left(A - Br\right) - Bz = \frac{qr}{D_0} \qquad \text{...(6)}$$

Clearly $z = -\frac{q}{2B^2 D_0}\left(A + Br\right)$ **is a particular solution of (6)**
Now the equation

$$r^2 \frac{d^3 z}{dr^3} + 4r \frac{d^2 z}{dr^2} + \frac{dz}{dr}\left(A - Br\right) - Bz = 0 \qquad \text{...(7)}$$

is an exact equation which on integration gives

$$r^2 \frac{d^2 z}{dr^2} + 2r \frac{dz}{dr} + z\left(\sigma - 1 - Br\right) = K \qquad \text{...(8)}$$

Where K **is a constant.**

The solution of the above equation (8) can be put in the form

$$Z = r^{-1/2}\left[C_1 J_\mu\left(2iB^{1/2}r^{1/2}\right) + C_2 Y_\mu\left(2iB^{1/2}r^{1/2}\right)\right]$$
$$+ C_3 r^{-1/2} \cdot S_{0,\mu}\left(2iB^{1/2} \cdot r^{1/2}\right)$$

(Forsyth, A.R; A treatise on differential equation pp. 202).

Where $\frac{\mu^2}{4} = \frac{1}{4} - \sigma + 1$ and $J_\mu\left(2iB^{1/2} \cdot r^{1/2}\right)$, $Y_\mu\left(2iB^{1/2} \cdot r^{1/2}\right)$ are the Bessel functions of the first and second kind. Hence the complete solution of (6) is

$$Z = \frac{dw}{dr} = r^{-1/2}\left[C_1 J_\mu(p) + C_2 Y_\mu(p) + C_3 S_{0,\mu}(p)\right] - \frac{q}{2B^2 D_0}\left(A + Br\right) \quad \ldots (9)$$

where $p = 2iB^{1/2} \cdot r^{1/2}$

Integrating equation (9) with respect to r we get the deflection in the form

$$W = 2C_1 r^{1/2}\left[(\mu-1) J_\mu(p) \cdot S_{-1,\mu-1}(p) - J_{\mu-1}(p) \cdot S_{0,\mu}(p)\right]$$
$$+ 2C_2 r^{1/2}\left[(\mu-1) Y_\mu(p) \cdot S_{-1,\mu-1}(p) - Y_{\mu-1}(p) \cdot S_{0,\mu}(p)\right]$$
$$+ C_3\left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot \left(2iB^{1/2}\right)^{2m+1} \cdot r^{m+1}}{(m+1)\left\{(1^2-\mu^2)\right\} \cdots \left\{(1+2m)^2-\mu^2\right\}}\right]$$
$$- \frac{q r}{2B^2 D_0}\left(A + Br\right) + C_4 \quad \ldots (10)$$

For an annular plate clamped at the inner boundary $r = c$ and at the outer boundary $r = b$,

boundary conditions are

$$\left.\begin{aligned} u = w &= 0, \quad \text{at } r = b,\ r = c \\ \frac{dw}{dr} &= 0, \quad \text{at } r = b,\ r = c \end{aligned}\right\} \qquad \ldots(11)$$

Solving for the constants we get,

$$C_1 = \frac{\begin{aligned}&\left[\lambda_4''\lambda_3''' - \lambda_3''\lambda_4'''\right]\left[(\lambda_2-\lambda_2')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_2''-\lambda_2''')\right] \\ &\quad - \left[\lambda_2''\lambda_3''' - \lambda_3''\lambda_2'''\right]\left[(\lambda_4-\lambda_4')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_4''-\lambda_4''')\right]\end{aligned}}{\begin{aligned}&\left[\lambda_2''\lambda_3''' - \lambda_3''\lambda_2'''\right]\left[(\lambda_1-\lambda_1')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_1''-\lambda_1''')\right] \\ &\quad - \left[\lambda_1''\lambda_3''' - \lambda_3''\lambda_1'''\right]\left[(\lambda_2-\lambda_2')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_2''-\lambda_2''')\right]\end{aligned}}$$

$$C_2 = \frac{\begin{aligned}&\left[\lambda_1''\lambda_3''' - \lambda_3''\lambda_1'''\right]\left[(\lambda_4-\lambda_4')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_4''-\lambda_4''')\right] \\ &\quad - \left[\lambda_4''\lambda_3''' - \lambda_3''\lambda_4'''\right]\left[(\lambda_1-\lambda_1')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_1''-\lambda_1''')\right]\end{aligned}}{\begin{aligned}&\left[\lambda_2''\lambda_3''' - \lambda_3''\lambda_2'''\right]\left[(\lambda_1-\lambda_1')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_1''-\lambda_1''')\right] \\ &\quad - \left[\lambda_1''\lambda_3''' - \lambda_3''\lambda_1'''\right]\left[(\lambda_2-\lambda_2')(\lambda_3''-\lambda_3''') - (\lambda_3-\lambda_3')(\lambda_2''-\lambda_2''')\right]\end{aligned}}$$

$$C_3 = \frac{-\left[C_1\lambda_1'' + C_2\lambda_2'' + \lambda_4''\right]}{\lambda_3''}$$

$$C_4 = -\left[\lambda_4 + C_1\lambda_1 + C_2\lambda_2 + C_3\lambda_3\right]$$

Where

$$\lambda_1 = 2b^{1/2}\left[(\mu-1)\cdot J_\mu(z_1)\cdot S_{-1,\mu-1}(z_1) - J_{\mu-1}(z_1)\cdot S_{0,\mu}(z_1)\right]$$

$$\lambda_2 = 2b^{1/2}\left[(\mu-1)\cdot Y_\mu(z_1)\cdot S_{-1,\mu-1}(z_1) - Y_{\mu-1}(z_1)\cdot S_{0,\mu}(z_1)\right]$$

$$\lambda_3 = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot b^{m+1}}{(m+1)\{1^2-\mu^2\}\cdots\{(1+2m)^2-\mu^2\}}$$

$$\lambda_3 = -\frac{q}{2B^2D_0}\left[Ab + \frac{Bb^2}{2}\right]$$

$$\lambda_1' = 2c^{1/2}\left[(\mu-1)\cdot J_\mu(z_2)\cdot S_{-1,\mu-1}(z_2) - J_{\mu-1}(z_2)\cdot S_{0,\mu}(z_2)\right]$$

$$\lambda_2' = 2c^{1/2}\left[(\mu-1)\,Y_\mu(z_2)\cdot S_{-1,\mu-1}(z_2) - Y_{\mu-1}(z_2)\cdot S_{0,\mu}(z_2)\right]$$

$$\lambda_3' = \sum_{m=0}^{\infty} \frac{(-1)^m\cdot\left(2iB^{1/2}\right)^{2m+1}\cdot c^{m+1}}{(m+1)\left\{1^2-\mu^2\right\}\cdots\left\{(1+2m)^2-\mu^2\right\}}$$

$$\lambda_4' = -\frac{q}{2B^2D_0}\left[cA+\frac{c^2B}{2}\right]$$

$$\lambda_1'' = b^{-1/2}\cdot J_\mu(z_1)\ ,\quad \lambda_2'' = b^{-1/2}\cdot Y_\mu(z_1)\ ,\quad \lambda_3'' = b^{-1/2}\cdot S_{0,\mu}(z_1)$$

$$\lambda_4'' = -\frac{q(A+Bb)}{2B^2D_0}.$$

$$\lambda_1''' = c^{-1/2}\cdot J_\mu(z_2)\ ,\quad \lambda_2''' = c^{-1/2}Y_\mu(z_2)\ ,\quad \lambda_3''' = c^{-1/2}S_{0,\mu}(z_2)$$

$$\lambda_4''' = -\frac{q(A+Bc)}{2B^2D_0}$$

$$z_1 = 2iB^{1/2}b^{1/2}\ ,\qquad z_2 = 2iB^{1/2}c^{1/2}$$

To determine u, we know that

$$\frac{du}{dr} + \frac{u}{r} = \frac{\alpha^2h_0^2}{12h} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

Multiplying the above equation by η and integrating with respect to η, we get

$$u\eta = \frac{\alpha^2 h_0 \eta^{5/3}}{20} + \frac{c_1^2}{8B}\left[\left\{p J_\mu'(p)\right\}^2 + \left(p^2-\mu^2\right) J_\mu^2(p)\right)\right]$$

$$+ \frac{c_2^2}{8B}\left[\left\{p Y_\mu'(p)\right\}^2 - \left(p^2-\mu^2\right) Y_\mu^2(p)\right]$$

$$- \frac{c_3^2}{2}\left[\sum_{m=0}^{\infty} \frac{\left(2iB^{1/2}\right)^{2+4m} \cdot \eta^{2+2m}}{(2+2m)\left[\left(1^2-\mu^2\right)\cdots\left\{\left(1+2m\right)^2-\mu^2\right\}\right]^2}\right.$$

$$\left. + \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m \cdot (-1)^s \cdot \left(2iB^{1/2}\right)^{2m+2s+2} \cdot \eta^{2+m+s}}{(m+s+2)\left[\left(1^2-\mu^2\right)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]\left[\left(1^2-\mu^2\right)\cdots\left\{(1+2s)^2-\mu^2\right\}\right]}\right]$$

$$- \frac{q^2}{8B^4D_0^2}\left[\frac{A^2\eta^2}{2} + \frac{B^2\eta^4}{4} + \frac{2AB\eta^3}{3}\right]$$

$$+ \frac{c_1c_2}{2B}\left[\frac{p^2}{4}\left\{2J_\mu(p)Y_\mu(p) - J_{\mu-1}(p)Y_{\mu+1}(p) - J_{\mu+1}(p)Y_{\mu-1}(p)\right\}\right]$$

$$+ \frac{c_1c_3}{2B}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\left\{(2m+1+\mu)\,p\,J_\mu(p)\cdot S_{2m+1,\mu-1}(p) - p\,J_{\mu-1}(p)\cdot S_{2m+2,\mu}(p)\right\}}{\left(1^2-\mu^2\right)\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$+\frac{c_2c_3}{2B}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\, p\left\{(2m+1+\mu)\,Y_\mu(p)\cdot S_{2m+1,\mu-1}(p)-Y_{\mu-1}(p)\cdot S_{2m+2,\mu}(p)\right\}}{(1^2-\mu^2)\cdots\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$+\frac{\gamma c_1}{2B^2D_0}\left[\frac{A r^{1/2}}{2B}\left\{J_{\mu-1}(p)\cdot S_{2,\mu}(p)-(\mu+1)\,J_\mu(p)\cdot S_{1,\mu-1}(p)\right\}\right.$$

$$\left.+\frac{r^{1/2}}{8B}\left\{(3+\mu)\,J_\mu(p)\cdot S_{3,\mu-1}(p)-J_{\mu-1}(p)\cdot S_{4,\mu}(p)\right\}\right]$$

$$+\frac{\gamma c_2}{2B^2D_0}\left[\frac{A r^{1/2}}{2B}\left\{Y_{\mu-1}(p)\cdot S_{2,\mu}(p)-(\mu+1)\cdot Y_\mu(p)\cdot S_{1,\mu-1}(p)\right\}\right.$$

$$\left.+\frac{r^{1/2}}{8B}\left\{(3+\mu)\,Y_\mu(p)\cdot S_{3,\mu-1}(p)-Y_{\mu-1}(p)\cdot S_{4,\mu}(p)\right\}\right]$$

$$+\frac{\gamma c_3}{2B^2D_0}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot r^{m+2}}{(m+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right.$$

$$\left.+B\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot r^{m+3}}{(m+3)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right]$$

$$+C_5 \qquad \ldots(12)$$

Where C_5 is the constant of integration and $p = 2iB^{1/2}r^{1/2}$.

But as $r \to b,\ u \to 0$. Hence

$$C_5 = -\frac{\alpha^2 h_0 b^{5/3}}{20}-\frac{c_1^2}{8B}\left[\left\{z_1 J_\mu'(z_1)\right\}^2+(z_1^2-\mu^2)\,J_\mu^2(z_1)\right]$$

$$- \frac{C_2^2}{8B}\left[\left\{z_1 Y'_\mu(z_1)\right\}^2 + (z_1^2-\mu^2)\, Y_\mu^2(z_1)\right]$$

$$+ \frac{C_3^2}{2}\left[\sum_{m=0}^{\infty} \frac{(2iB^{1/2})^{2+4m} \cdot b^{2+2m}}{(2+2m)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]^2}\right.$$

$$\left.+ \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m \cdot (-1)^s \cdot (2iB^{1/2})^{2m+2s+2} \cdot b^{2+m+s}}{(m+s+2)\left[(1^2-\mu^2)\cdots\cdots\left\{(1+2m)^2-\mu^2\right\}\right]\left[(1^2-\mu^2)\cdots\left\{(1+2s)^2-\mu^2\right\}\right]}\right]$$

$$+ \frac{q^2}{8B^4D_0^2}\left[\frac{A^2b^2}{2} + \frac{B^2b^4}{4} + \frac{2ABb^3}{3}\right]$$

$$- \frac{C_1C_2}{2B}\left[\frac{z_1^2}{4}\left\{2J_\mu(z_1)Y_\mu(z_1) - J_{\mu-1}(z_1)\,Y_{\mu+1}(z_1) - J_{\mu+1}(z_1)Y_{\mu-1}(z_1)\right\}\right]$$

$$- \frac{C_1C_3}{2B}\left[\sum_{m=0}^{\infty} \frac{z_1(-1)^m\left\{(2m+1+\mu)J_\mu(z_1)\cdot S_{2m+1,\mu-1}(z_1) - J_{\mu-1}(z_1)\cdot S_{2m+2,\mu}(z_1)\right\}}{(1^2-\mu^2)\cdots\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$- \frac{C_2C_3}{2B}\left[\sum_{m=0}^{\infty} \frac{z_1(-1)^m\left\{(2m+1+\mu)Y_\mu(z_1)\cdot S_{2m+1,\mu-1}(z_1) - Y_{\mu-1}(z_1)\cdot S_{2m+2,\mu}(z_1)\right\}}{(1^2-\mu^2)\cdots\cdots\left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$- \frac{qC_1}{2B^2D_0}\left[\frac{Ab^{1/2}}{2B}\left\{J_{\mu-1}(z_1)\cdot S_{2,\mu}(z_1) - (\mu+1)J_\mu(z_1)\cdot S_{1,\mu-1}(z_1)\right\}\right.$$

$$\left.+ \frac{b^{1/2}}{8B}\left\{(3+\mu)\cdot J_\mu(z_1)\cdot S_{3,\mu-1}(z_1) - J_{\mu-1}(z_1)\cdot S_{4,\mu}(z_1)\right\}\right]$$

$$-\frac{qC_2}{2B^2D_0}\left[\frac{Ab^{1/2}}{2B}\left\{Y_{\mu-1}(z_1)\cdot S_{2,\mu}(z_1)-(\mu+1)Y_\mu(z_1)\cdot S_{1,\mu-1}(z_1)\right\}\right.$$

$$\left.+\frac{b^{1/2}}{8B}\left\{(3+\mu)Y_\mu(z_1)\cdot S_{3,\mu-1}(z_1)-Y_{\mu-1}(z_1)\cdot S_{4,\mu}(z_1)\right\}\right]$$

$$-\frac{qC_3}{2B^2D_0}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot b^{m+2}}{(m+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right.$$

$$\left.+B\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot b^{m+3}}{(m+3)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]}\right] \qquad \ldots (13)$$

Also as $r\to c$, $u\to 0$ **. Hence the equation to determine** α **leads to**

$$\frac{\alpha^2 h_0 c^{5/3}}{20}+\frac{C_1^2}{8B}\left[\left\{z_2J'_\mu(z_2)\right\}^2+(z_2^2-\mu^2)J_\mu^2(z_2)\right]$$

$$+\frac{C_2^2}{8B}\left[\left\{z_2Y'_\mu(z_2)\right\}^2+(z_2^2-\mu^2)Y_\mu^2(z_2)\right]$$

$$-\frac{C_3^2}{2}\left[\sum_{m=0}^{\infty}\frac{(2iB^{1/2})^{2+4m}\cdot c^{2+2m}}{(2+2m)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]^2}\right.$$

$$\left.+\sum_{\substack{m=0\\s=0\\m\neq s}}^{\infty}\frac{(-1)^m(-1)^s\cdot(2iB^{1/2})^{2m+2s+2}\cdot c^{m+s+2}}{(m+s+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]\left[(1^2-\mu^2)\cdots\left\{(1+2s)^2-\mu^2\right\}\right]}\right]$$

$$- \frac{q^2}{8B^4 D_0^2}\left[\frac{A^2 C^2}{2} + \frac{B^2 C^4}{4} + \frac{2AB.C^3}{3}\right]$$

$$+ \frac{C_1 C_2}{2B}\left[\frac{z_2^2}{4}\left\{2 J_\mu(z_2) \cdot Y_\mu(z_2) - J_{\mu-1}(z_2) \cdot Y_{\mu+1}(z_2) - J_{\mu+1}(z_2) \cdot Y_{\mu-1}(z_2)\right\}\right]$$

$$+ \frac{C_1 C_3}{2B}\left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z_2 \left\{(2m+1+\mu) \cdot J_\mu(z_2) \cdot S_{2m+1,\mu-1}(z_2) - J_{\mu-1}(z_2) \cdot S_{2m+2,\mu}(z_2)\right\}}{(1^2-\mu^2) \cdots \left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$+ \frac{C_2 C_3}{2B}\left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z_2 \left\{(2m+1+\mu) Y_\mu(z_2) \cdot S_{2m+1,\mu-1}(z_2) - Y_{\mu-1}(z_2) \cdot S_{2m+2,\mu}(z_2)\right\}}{(1^2-\mu^2) \cdots \left\{(1+2m)^2-\mu^2\right\}}\right]$$

$$+ \frac{q C_1}{2B^2 D_0}\left[\frac{A C^{1/2}}{2B}\left\{J_{\mu-1}(z_2) \cdot S_{2,\mu}(z_2) - (\mu+1) J_\mu(z_2) \cdot S_{1,\mu-1}(z_2)\right\}\right.$$

$$\left. + \frac{C^{1/2}}{8B}\left\{(3+\mu) J_\mu(z_2) \cdot S_{3,\mu-1}(z_2) - J_{\mu-1}(z_2) \cdot S_{4,\mu}(z_2)\right\}\right]$$

$$+ \frac{q C_2}{2B^2 D_0}\left[\frac{A C^{1/2}}{2B}\left\{Y_{\mu-1}(z_2) \cdot S_{2,\mu}(z_2) - (\mu+1) Y_\mu(z_2) \cdot S_{1,\mu-1}(z_2)\right\}\right.$$

$$\left. + \frac{C^{1/2}}{8B}\left\{(3+\mu) Y_\mu(z_2) \cdot S_{3,\mu-1}(z_2) - Y_{\mu-1}(z_2) \cdot S_{4,\mu}(z_2)\right\}\right]$$

$$+ \frac{q C_3}{2B^2 D_0}\left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot C^{m+2}}{(m+2)\left[(1^2-\mu^2) \cdots \left\{(1+2m)^2-\mu^2\right\}\right.}\right.$$

$$\left. + B \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot C^{m+3}}{(m+3)\left[(1^2-\mu^2) \cdots \left\{(1+2m)^2-\mu^2\right\}\right]}\right]$$

$$
\begin{aligned}
& -\frac{\alpha^2 h_0 b^{5/3}}{20} - \frac{c_1^2}{8B}\left[\left\{z_1 J_\mu'(z_1)\right\}^2 + (z_1^2-\mu^2)\, J_\mu^2(z_1)\right] \\
& - \frac{c_2^2}{8B}\left[\left\{z_1 Y_\mu'(z_1)\right\}^2 + (z_1^2-\mu^2)\, Y_\mu^2(z_1)\right] \\
& + \frac{c_3^2}{2}\left[\sum_{m=0}^{\infty} \frac{(2iB^{1/2})^{2+4m} \cdot b^{2+2m}}{(2+2m)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]^2}\right. \\
& \left. + \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m\cdot(-1)^s\cdot(2iB^{1/2})^{2m+2s+2}\cdot b^{m+s+2}}{(m+s+2)\left[(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}\right]\left[(1^2-\mu^2)\cdots\left\{(1+2s)^2-\mu^2\right\}\right]}\right] \\
& + \frac{q^2}{8B^4 D_0^2}\left[\frac{A^2b^2}{2} + \frac{B^2b^4}{4} + \frac{2AB.b^3}{3}\right] \\
& - \frac{c_1c_2}{2B}\left[\frac{z_1^2}{4}\left\{2J_\mu(z_1)\cdot Y_\mu(z_1) - J_{\mu-1}(z_1)\cdot Y_{\mu+1}(z_1) - J_{\mu+1}(z_1)\, Y_{\mu-1}(z_1)\right\}\right] \\
& - \frac{c_1c_3}{2B}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot z_1\left\{(2m+1+\mu)J_\mu(z_1)\cdot S_{2m+1,\mu-1}(z_1) - J_{\mu-1}(z_1)\, S_{2m+2,\mu}(z_1)\right\}}{(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}}\right] \\
& - \frac{c_2c_3}{2B}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot z_1\left\{(2m+1+\mu)Y_\mu(z_1)\cdot S_{2m+1,\mu-1}(z_1) - Y_{\mu-1}(z_1)\cdot S_{2m+2,\mu}(z_1)\right\}}{(1^2-\mu^2)\cdots\left\{(1+2m)^2-\mu^2\right\}}\right] \\
& - \frac{qc_1}{2B^2D_0}\left[\frac{Ab^{1/2}}{2B}\left\{J_{\mu-1}(z_1)\cdot S_{2,\mu}(z_1) - (\mu+1)\,J_\mu(z_1)\cdot S_{1,\mu-1}(z_1)\right\}\right. \\
& \left. + \frac{b^{1/2}}{8B}\left\{(3+\mu)\,J_\mu(z_1)\cdot S_{3,\mu-1}(z_1) - J_{\mu-1}(z_1)\cdot S_{4,\mu}(z_1)\right\}\right]
\end{aligned}
$$

$$- \frac{q c_2}{2B^2 D_0} \left[\frac{A b^{1/2}}{2B} \left\{ Y_{\mu-1}(z_1) \cdot S_{2,\mu}(z_1) - (\mu+1) Y_\mu(z_1) \cdot S_{1,\mu-1}(z_1) \right\} \right.$$

$$\left. + \frac{b^{1/2}}{2B} \left\{ (3+\mu) Y_\mu(z_1) \cdot S_{3,\mu-1}(z_1) - Y_{\mu-1}(z_1) \cdot S_{4,\mu}(z_1) \right\} \right]$$

$$- \frac{q c_3}{2B^2 D_0} \left[\sum_{m=0}^{\infty} \frac{(-1)^m \, (2iB^{1/2})^{2m+1} \cdot b^{m+2}}{(m+2)\left[(1^2-\mu^2) \cdots \left\{(1+2m)^2-\mu^2\right\}\right]} \right.$$

$$\left. + B \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot b^{m+3}}{(m+3)\left[(1^2-\mu^2) \cdots \left\{(1+2m)^2-\mu^2\right\}\right]} \right] = 0 \qquad \ldots(14)$$

In particular, if $\sigma = 0{\cdot}25$ **, the corresponding deflection of the plate is given by**

$$W = 2c_1' r^{1/2} \left[J_2(p) \cdot S_{-1,1}(p) - J_1(p) \cdot S_{0,2}(p) \right]$$

$$+ 2c_2' r^{1/2} \left[Y_2(p) \cdot S_{-1,1}(p) - Y_1(p) \cdot S_{0,2}(p) \right]$$

$$+ c_3' \left[\sum_{m=0}^{\infty} \frac{(-1)^m \, (2iB^{1/2})^{2m+1} \cdot r^{m+1}}{(m+1)\left[(1^2-2^2) \cdots \left\{(1+2m)^2-2^2\right\}\right]} \right]$$

$$- \frac{q}{2B^2 D_0} \left[\frac{5}{4} r + \frac{B r^2}{2} \right] + c_4' \qquad \ldots(15)$$

Where

$$C_1' = \frac{\left[\mu_4''\mu_3''' - \mu_3''\mu_4'''\right]\left[(\mu_2-\mu_2')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_2''-\mu_2''')\right] - \left[\mu_2''\mu_3''' - \mu_3''\mu_2'''\right]\left[(\mu_4-\mu_4')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_4''-\mu_4''')\right]}{\left[\mu_2''\mu_3''' - \mu_3''\mu_2'''\right]\left[(\mu_1-\mu_1')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_1''-\mu_1''')\right] - \left[\mu_1''\mu_3''' - \mu_3''\mu_1'''\right]\left[(\mu_2-\mu_2')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_2''-\mu_2''')\right]}$$

$$C_2' = \frac{\left[\mu_1''\mu_3''' - \mu_3''\mu_1'''\right]\left[(\mu_4-\mu_4')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_4''-\mu_4''')\right] - \left[\mu_4''\mu_3''' - \mu_3''\mu_4'''\right]\left[(\mu_1-\mu_1')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_1''-\mu_1''')\right]}{\left[\mu_2''\mu_3''' - \mu_3''\mu_2'''\right]\left[(\mu_1-\mu_1')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_1''-\mu_1''')\right] - \left[\mu_1''\mu_3''' - \mu_3''\mu_1'''\right]\left[(\mu_2-\mu_2')(\mu_3''-\mu_3''') - (\mu_3-\mu_3')(\mu_2''-\mu_2''')\right]}$$

$$C_3' = \frac{-\left[C_1'\mu_1'' + C_2'\mu_2'' + \mu_4''\right]}{\mu_3''} \quad , \quad C_4' = -\left[\mu_4 + C_1'\mu_1 + C_2'\mu_2 + C_3'\mu_3\right]$$

Where

$$\mu_1 = 2b^{1/2}\left[J_2(z_1)\, S_{-1,1}(z_1) - J_1(z_1)\cdot S_{0,2}(z_1)\right]$$

$$\mu_2 = 2b^{1/2}\left[Y_2(z_1)\cdot S_{-1,1}(z_1) - Y_1(z_1)\cdot S_{0,2}(z_1)\right]$$

$$\mu_3 = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot b^{m+1}}{(m+1)\{(1^2-2^2)\}\cdots\{(1+2m)^2-2^2\}}$$

$$\mu_4 = -\frac{q}{2B^2D_0}\left[\frac{s}{4}b + \frac{Bb^2}{2}\right]$$

$$\mu_1' = 2c^{1/2}\left[J_2(z_2)\cdot S_{-1,1}(z_2) - J_1(z_2)\cdot S_{0,2}(z_2)\right]$$

$$\mu_2' = 2c^{1/2}\left[Y_2(z_2)\cdot S_{-1,1}(z_2) - Y_1(z_2)\cdot S_{0,2}(z_2)\right]$$

$$\mu_3' = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot c^{m+1}}{(m+1)\{(1^2-2^2)\}\cdots\{(1+2m)^2-2^2\}}$$

$$\mu_4' = -\frac{q}{2B^2D_0}\left[\frac{s}{4}c + \frac{Bc^2}{2}\right]$$

$$\mu_1'' = b^{-1/2}\cdot J_2(z_1)\ ,\ \mu_2'' = b^{-1/2}\cdot Y_2(z_1),\ \mu_3'' = b^{-1/2}\cdot S_{0,2}(z_1)$$

$$\mu_4'' = -\frac{q}{2B^2D_0}\left[\frac{s}{4} + Bb\right]$$

$$\mu_1''' = c^{-1/2}\cdot J_2(z_2),\ \mu_2''' = c^{-1/2}Y_2(z_2),\ \mu_3''' = c^{-1/2}\cdot S_{0,2}(z_2)$$

$$\mu_4''' = -\frac{q}{2B^2D_0}\left[\frac{s}{4} + Bc\right]$$

The equation to determine α **leads to**

$$
\begin{aligned}
&\frac{\alpha^2 h_0 c^{5/3}}{20} + \frac{c_1'^2}{8B}\Big[\{z_2 J_2'(z_2)\}^2 + (z_2^2 - 2^2) J_2^2(z_2)\Big] \\
&+ \frac{c_2'^2}{8B}\Big[\{z_2 Y_2'(z_2)\}^2 + (z_2^2 - 2^2) Y_2^2(z_2)\Big] \\
&- \frac{c_3'^2}{2}\Bigg[\sum_{m=0}^{\infty} \frac{(2iB^{1/2})^{2+4m} \cdot c^{2+2m}}{(2+2m)\big[(1^2-2^2)\cdots\{(1+2m)^2-2^2\}\big]^2} \\
&+ \sum_{\substack{m=0\\ s=0\\ m\neq s}}^{\infty} \frac{(-1)^m \cdot (-1)^s \cdot (2iB^{1/2})^{2m+2s+2} \cdot c^{m+s+2}}{(m+s+2)\big[(1^2-2^2)\cdots\{(1+2m)^2-2^2\}\big]\big[(1^2-2^2)\cdots\{(1+2s)^2-2^2\}\big]}\Bigg] \\
&- \frac{q^2}{8B^4 D_0^2}\Big[\frac{25}{32}c^2 + \frac{B^2}{4}c^4 + \frac{5B}{6}c^3\Big] \\
&+ \frac{c_1' c_2'}{2B}\Big[\frac{z_2^2}{4}\Big\{2 J_2(z_2) \cdot Y_2(z_2) - J_1(z_2) Y_3(z_2) - J_3(z_2) Y_1(z_2)\Big\}\Big] \\
&+ \frac{c_1' c_3'}{2B}\Bigg[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z_2 \{(2m+3) J_2(z_2) \cdot S_{2m+1,1}(z_2) - J_1(z_2) S_{2m+2,2}(z_2)\}}{(1^2-2^2)\cdots\{(1+2m)^2-2^2\}}\Bigg] \\
&+ \frac{c_2' c_3'}{2B}\Bigg[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z_2 \{(2m+3) Y_2(z_2) S_{2m+1,1}(z_2) - Y_1(z_2) S_{2m+2,2}(z_2)\}}{(1^2-2^2)\cdots\cdots\{(1+2m)^2-2^2\}}\Bigg] \\
&+ \frac{q c_1'}{2B^2 D_0}\Bigg[\frac{5c^{1/2}}{8B}\Big\{J_1(z_2) S_{2,2}(z_2) - 3 J_2(z_2) S_{1,1}(z_2)\Big\} \\
&+ \frac{c^{1/2}}{8B}\Big\{5 J_2(z_2) \cdot S_{3,1}(z_2) - J_1(z_2) S_{4,2}(z_2)\Big\}\Bigg]
\end{aligned}
$$

$$+\frac{qC_2'}{2B^2D_0}\left[\frac{5c^{1/2}}{8B}\left\{Y_1(z_2)\,S_{2,2}(z_2)-3Y_2(z_2)\,S_{1,1}(z_2)\right\}\right.$$

$$\left.+\frac{c^{1/2}}{8B}\left\{5Y_2(z_2)\,S_{3,1}(z_2)-Y_1(z_2)\,S_{4,2}(z_2)\right\}\right]$$

$$+\frac{qC_3'}{2B^2D_0}\left[\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\cdot c^{m+2}}{(m+2)\left[(1^2-2^2)\cdots\left\{(1+2m)^2-2^2\right\}\right]}\right.$$

$$\left.+B\sum_{m=0}^{\infty}\frac{(-1)^m\cdot(2iB^{1/2})^{2m+1}\,c^{m+3}}{(m+3)\left[(1^2-2^2)\cdots\cdots\left\{(1+2m)^2-2^2\right\}\right]}\right]$$

$$-\frac{\alpha^2h_0b^{5/3}}{20}-\frac{C_1'^2}{8B}\left[\left\{z_1J_2'(z_1)\right\}^2+\left(z_1^2-2^2\right)J_2^2(z_2)\right]$$

$$-\frac{C_2'^2}{8B}\left[\left\{z_1Y_2'(z_1)\right\}^2+\left(z_1^2-2^2\right)Y_2^2(z_1)\right]$$

$$+\frac{C_3'^2}{2}\left[\sum_{m=0}^{\infty}\frac{(2iB^{1/2})^{2+4m}\cdot b^{2+2m}}{(2+2m)\left[(1^2-2^2)\cdots\cdots\left\{(1+2m)^2-2^2\right\}\right]^2}\right.$$

$$\left.+\sum_{\substack{m=0\\s=0\\m\neq s}}^{\infty}\frac{(-1)^m\cdot(-1)^s\cdot(2iB^{1/2})^{2m+2s+2}\cdot b^{m+s+2}}{(m+s+2)\left[(1^2-2^2)\cdots\left\{(1+2m)^2-2^2\right\}\right]\left[(1^2-2^2)\cdots\left\{(1+2s)^2-2^2\right\}\right]}\right]$$

$$+\frac{q^2}{8B^4D_0^2}\left[\frac{25}{32}b^2+\frac{B^2}{4}b^4+\frac{5B}{6}b^3\right]$$

$$-\frac{C_1'C_2'}{2B}\left[\frac{z_1^2}{4}\left\{2J_2(z_1)Y_2(z_1)-J_1(z_1)Y_3(z_1)-J_3(z_1)Y_1(z_1)\right\}\right]$$

$$- \frac{c_1' c_3'}{2B} \left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z_1 \left\{ (2m+3) J_2(z_1)\, S_{2m+1,1}(z_1) - J_1(z_1)\, S_{2m+2,2}(z_1) \right\}}{(1^2-2^2) \cdots \cdots \left\{ (1+2m)^2 - 2^2 \right\}} \right]$$

$$- \frac{c_2' c_3'}{2B} \left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot z_1 \left\{ (2m+3) Y_2(z_1)\, S_{2m+1,1}(z_1) - Y_1(z_1)\, S_{2m+2,2}(z_1) \right\}}{(1^2-2^2) \cdots \cdots \left\{ (1+2m)^2 - 2^2 \right\}} \right]$$

$$- \frac{\gamma c_1'}{2B^2 D_0} \left[\frac{5 b^{1/2}}{8B} \left\{ J_1(z_1)\, S_{2,2}(z_1) - 3 J_2(z_1)\, S_{1,1}(z_1) \right\} \right.$$

$$\left. + \frac{b^{1/2}}{8B} \left\{ 5 J_2(z_1)\, S_{3,1}(z_1) - J_1(z_1)\, S_{4,2}(z_1) \right\} \right]$$

$$- \frac{\gamma c_2'}{2B^2 D_0} \left[\frac{5 b^{1/2}}{8B} \left\{ Y_1(z_1)\, S_{2,2}(z_1) - 3 Y_2(z_1)\, S_{1,1}(z_1) \right\} \right.$$

$$\left. + \frac{b^{1/2}}{8B} \left\{ 5 Y_2(z_1)\, S_{3,1}(z_1) - Y_1(z_1)\, S_{4,2}(z_1) \right\} \right]$$

$$- \frac{\gamma c_3'}{2B^2 D_0} \left[\sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot b^{m+2}}{(m+2) \left[(1^2-2^2) \cdots \cdots \left\{ (1+2m)^2 - 2^2 \right\} \right]} \right.$$

$$\left. + B \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (2iB^{1/2})^{2m+1} \cdot b^{m+3}}{(m+3) \left[(1^2-2^2) \cdots \cdots \left\{ (1+2m)^2 - 2^2 \right\} \right]} \right]$$

$$= \quad 0 \qquad \cdots (16)$$

Numerical calculations :

Let us take $i\alpha = 1{\cdot}5,\ b = 20,\ c = 10,\ h_0 = 1,\ n = 15$

Putting all these values in (16) we get, $\frac{q}{D_0} = 118{\cdot}0 \times 10^{-3}$

Substituting this value of $\frac{q}{D_0}$ in (15) we get, $W = 2{\cdot}32$

CHAPTER - IV

Time - hardening and time - softening elastic plates.*

Nomenclature :

The following nomenclature are used in this paper.

w = deflection, normal to the middle plane,

D = flexural rigidity of the plate $= \frac{Eh^3}{12(1-\sigma^2)}$,

ρ = density, supposed constant,

h = thickness of the plate,

E = Young's modulus,

σ = Poisson's ratio.

Introduction :

Earlier periods in the development of the theory of Elasticity were mainly concerned with materials of homogeneous elastic properties. After then, the generalization was done from isotropy to anisotropy especially in problems on crystal Elasticity.

A recent generalization has been made from homogenety to non-homogenety of the material in which elastic properties vary from point to point. This has helped much in the improvement of the design of structures.

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A still another generalization of the elastic properties has been made by Paria (1966). The Young's modulus has been assumed to be function of time but independent of co-ordinates. If the Young's modulus increases as time increases, the material will be said to be time-hardening. In the time-softening material, Young's modulus is assumed to decrease with increasing time. The physical justification for such assumption may be found in concrete, for example, in which the elastic properties vary during its aging periods. The elastic properties of a material may vary from season to season during the year due to variations in temperature, moisture contents and similar other varying factors. To illustrate his assumption Paria (1966) has solved the problem of free vibration of elastic rod with Young's modulus as the function of time.

In this paper the auther has attempted to introduce this new idea to the case of different elastic plates.

Theory.

Let us consider the transverse vibration of elastic plates. The differential equation for such free vibration is

[Love, A.E.H. (1927), PP.464]

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \qquad \ldots (1)$$

If the Young's modulus depends on time, the above equation reduces to

$$\nabla^4 W + \frac{\lambda}{E(t)} \cdot \frac{\partial^2 W}{\partial t^2} = 0 \qquad \ldots (2)$$

where

$$\lambda = \frac{12 \rho (1-\sigma^2)}{h^2} = \qquad \text{constant}$$

Let

$$E(t) = E_0 \left(1 + \alpha e^{-t/t_0}\right) \qquad \ldots (3)$$

Where E_0 and t_0 are constants, α being a parameter. $E(t)$ varies from $E_0(1+\alpha)$ at $t=0$ to E_0 as $t \to \infty$. If α is negative, the material is time-hardening and if α is positive, it is time - softening. Putting (3) in (2) we have,

$$E_0\left(1+\alpha e^{-t/t_0}\right) \nabla^4 W + \lambda \frac{\partial^2 W}{\partial t^2} = 0 \qquad \ldots (4)$$

Let

$$W = \omega(x, y) \cdot \phi(t) \qquad \ldots (5)$$

Substituting (5) in (4) we have,

$$\frac{d^2\phi}{dt^2} + \frac{\nabla^4 \omega}{\omega} \cdot \frac{E_0\left(1+\alpha e^{-t/t_0}\right)}{\lambda} \phi = 0 \qquad \ldots (6)$$

1. Free vibration of a rectangular plate.

We consider the transverse vibration of a rectangular plate having its boundary simply-supported.

Let the centre of the plate be taken as the origin. Let us assume $w(x,y)$ in the form

$$w(x,y) = A \cos\frac{(2m+1)\pi x}{2a} \cdot \cos\frac{(2n+1)\pi y}{2b} \qquad \ldots (7)$$

It is evident that this form of $w(x,y)$ satisfies the following boundary conditions for the simply-supported edges.

$$\left.\begin{aligned} w = \frac{\partial^2 w}{\partial x^2} &= 0 \quad \text{at} \quad x = \pm a \\ w = \frac{\partial^2 w}{\partial y^2} &= 0 \quad \text{at} \quad y = \pm b \end{aligned}\right\} \qquad \ldots (8)$$

Substituting (7) in (6) we have,

$$\frac{d^2\phi}{dt^2} + \frac{K_{mn}^2}{\lambda} \cdot E_0\left(1+\alpha e^{-t/t_0}\right)\phi = 0 \qquad \ldots (9)$$

where

$$K_{mn}^2 = \frac{\pi^4}{16}\left[\frac{(2m+1)^2}{a^2} + \frac{(2n+1)^2}{b^2}\right]^2$$

To solve (9), let us put

$$\tau = e^{-t/2t_0} \qquad \ldots (10)$$

Substituting (10) in (9), we have

$$\frac{d^2\phi}{d\tau^2} + \frac{1}{\tau}\frac{d\phi}{d\tau} + \frac{4K_{mn}^2 E_0 t_0^2}{\lambda}\left(\alpha + \frac{1}{\tau^2}\right)\phi = 0 \qquad \ldots (11)$$

If α is positive, we write $\alpha = \alpha_1^2$. The equation (11) reduces to

$$\frac{d^2\phi}{d\tau^2} + \frac{1}{\tau}\frac{d\phi}{d\tau} + q_{mn}^2\left(\alpha_1^2 + \frac{1}{\tau^2}\right)\phi = 0 \qquad \ldots (12)$$

where

$$q_{mn}^2 = \frac{4K_{mn}^2 E_0 t_0^2}{\lambda}$$

If α is negative, we write $\alpha = -\alpha_2^2$. Then the equation (11) reduces to

$$\frac{d^2\phi}{d\tau^2} + \frac{1}{\tau}\frac{d\phi}{d\tau} - q_{mn}^2\left(\alpha_2^2 - \frac{1}{\tau^2}\right)\phi = 0 \qquad \ldots (13)$$

We shall consider the case only when α is positive. The case when α is negative can be treated in the similar manner by replacing Bessel functions by Modified Bessel functions.

The solution of (12) can be put in the form

$$\phi = A_1 J_{iq_{mn}}(z_1) + A_2 J_{-iq_{mn}}(z_1) \qquad \dots (14)$$

where

$$z_1 = \alpha_1 q_{mn} \tau = \alpha_1 q_{mn} e^{-t/2t_0} \qquad \dots (15)$$

Hence

$$W = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos\frac{(2m+1)\pi x}{2a} \cdot \cos\frac{(2n+1)\pi y}{2b} \left[A_{mn} J_{iq_{mn}}(z_1) + B_{mn} J_{-iq_{mn}}(z_1) \right] \qquad \dots (16)$$

Let the initial conditions be

$$\left. \begin{aligned} W(x,y,0) &= \sum \sum C_{mn} \cos\frac{(2m+1)\pi x}{2a} \cdot \cos\frac{(2n+1)\pi y}{2b} \\ \frac{\partial}{\partial t} W(x,y,0) &= \sum \sum D_{mn} \cos\frac{(2m+1)\pi x}{2a} \cdot \cos\frac{(2n+1)\pi y}{2b} \end{aligned} \right\} \qquad \dots (17)$$

where C_{mn} and D_{mn} are known constants.

Combining equations (16) and (17) and solving for the constants we have,

$$A_{mn} = \frac{1}{\alpha_1 \Delta_1} \left[\frac{2 D_{mn} t_0}{q_{mn}} J_{-iq_{mn}}(\alpha_1 q_{mn}) + \alpha_1 C_{mn} J'_{-iq_{mn}}(\alpha_1 q_{mn}) \right]$$

$$B_{mn} = -\frac{1}{\alpha_1 \Delta_1} \left[\frac{2 D_{mn} t_0}{q_{mn}} J_{iq_{mn}}(\alpha_1 q_{mn}) + \alpha_1 C_{mn} J'_{iq_{mn}}(\alpha_1 q_{mn}) \right]$$

where $\Delta_1 = -\dfrac{2i \sinh \pi q_{mn}}{\pi q_{mn} \alpha_1}$

Hence

$$W(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \cos\frac{(2m+1)\pi x}{2a} \cdot \cos\frac{(2n+1)\pi y}{2b}$$

$$\times \frac{\pi q_{mn}}{2i\sinh \pi q_{mn}} \Bigg[\frac{2D_{mn}t_0}{q_{mn}}\Big\{J_{iq_{mn}}(z_0)\cdot J_{-iq_{mn}}(z_1)$$

$$- J_{iq_{mn}}(z_1)\, J_{-iq_{mn}}(z_0)\Big\} + \alpha_1 C_{mn}\Big\{J_{-iq_{mn}}(z_1)\, J'_{iq_{mn}}(z_0)$$

$$- J_{iq_{mn}}(z_1)\, J'_{-iq_{mn}}(z_0)\Big\}\Bigg] \qquad \ldots (18)$$

where $\quad z_1 = \alpha_1 q_{mn} e^{-t/2t_0} \quad , \quad z_0 = \alpha_1 q_{mn}$

Now for numerical computation it will be convenient to replace the products of Bessel functions of imaginary orders by the equivalent series involving Hypergeometric functions as follows.

$$J_{-iq_{mn}}(\alpha_1 q_{mn})\, J_{iq_{mn}}(\alpha_1 q_{mn} e^{-t/2t_0})$$

$$= \frac{e^{\frac{-itq_{mn}}{2t_0}}}{\Gamma(1+iq_{mn})} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2r}}{\underline{|r}\,\Gamma(1-iq_{mn}+r)} \times {}_2F_1\left(-r, iq_{mn}-r; iq_{mn}+1; e^{-t/t_0}\right) \ldots (19)$$

Similarly

$$J_{iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{-iq_{mn}}\left(\alpha_1 q_{mn} e^{-t/2t_0}\right)$$

$$= \frac{e^{\frac{itq_{mn}}{2t_0}}}{\Gamma(1-iq_{mn})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2n}}{\underline{|n} \cdot \Gamma(1+iq_{mn}+n)}$$

$$\times {}_2F_1\left(-n, -iq_{mn}-n; -iq_{mn}+1; e^{-t/t_0}\right) \qquad \ldots (20)$$

Also

$$J'_{-iq_{mn}}(z_0)\, J_{iq_{mn}}(z_1) - J'_{iq_{mn}}(z_0) \cdot J_{-iq_{mn}}(z_1)$$

$$= \frac{1}{2}\Big[\left\{J_{-1-iq_{mn}}(z_0) \cdot J_{iq_{mn}}(z_1) - J_{-1+iq_{mn}}(z_0) \cdot J_{-iq_{mn}}(z_1)\right\}$$

$$-\left\{J_{1-iq_{mn}}(z_0) \cdot J_{iq_{mn}}(z_1) - J_{1+iq_{mn}}(z_0) \cdot J_{-iq_{mn}}(z_1)\right\}\Big] \qquad \ldots (21)$$

But

$$J_{-1-iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{iq_{mn}}\left(\alpha_1 q_{mn} e^{-t/2t_0}\right)$$

$$= \frac{2e^{\frac{-itq_{mn}}{2t_0}}}{\alpha_1 q_{mn}\Gamma(1+iq_{mn})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2n}}{\underline{|n} \cdot \Gamma(n-iq_{mn})} \times {}_2F_1\left(-n, 1-n+iq_{mn}; 1+iq_{mn}; e^{-t/t_0}\right) \qquad \ldots (22)$$

$$J_{-1+iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{-iq_{mn}}\left(\alpha_1 q_{mn} e^{-t/2t_0}\right)$$

$$= \frac{2e^{\frac{itq_{mn}}{2t_0}}}{\alpha_1 q_{mn}\Gamma(1-iq_{mn})} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2n}}{\underline{|n} \cdot \Gamma(n+iq_{mn})} \times {}_2F_1\left(-n, 1-n-iq_{mn}; 1-iq_{mn}; e^{-t/t_0}\right) \qquad \ldots (23)$$

$$J_{1-iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{iq_{mn}}\left(\alpha_1 q_{mn} e^{-t/2t_0}\right)$$

$$= \frac{\alpha_1 q_{mn} e^{\frac{-itq_{mn}}{2t_0}}}{2\Gamma(1+iq_{mn})} \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2r}}{\lfloor\underline{r}\, \Gamma(r+2-iq_{mn})} \times {}_2F_2\left(-r, -1-r+iq_{mn}; 1+iq_{mn}; e^{-t/t_0}\right) \quad \ldots (24)$$

$$J_{1+iq_{mn}}(\alpha_1 q_{mn}) \cdot J_{-iq_{mn}}\left(\alpha_1 q_{mn} e^{-t/2t_0}\right)$$

$$= \frac{\alpha_1 q_{mn} e^{\frac{itq_{mn}}{2t_0}}}{2\Gamma(1-iq_{mn})} \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{\alpha_1 q_{mn}}{2}\right)^{2r}}{\lfloor\underline{r}\, \Gamma(r+2+iq_{mn})} \times {}_2F_2\left(-r, -1-r-iq_{mn}; 1-iq_{mn}; e^{-t/t_0}\right) \quad \ldots (25)$$

If $\alpha \to 0$, equation (18) reduces to (setting $r = 0$ in the equivalent Hypergeometric series)

$$W = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos(2m+1)\frac{\pi x}{2a} \cdot \cos(2n+1)\frac{\pi y}{2b} \times$$

$$\times \left[\frac{2D_{mn}t_0}{q_{mn}} \sin\frac{q_{mn}t}{2t_0} + C_{mn} \cdot \cos\frac{q_{mn} \cdot t}{2t_0} \right] \quad \ldots (26)$$

which is the classical result.

If the parameter α in the assumed law is assumed to be small, the perturbed deflection of various orders may be obtained by collecting the different powers of α_1 in the given Hypergeometric series as follows.

The first perturbation is obtained by putting $r = 1$ in (19), (20), (22) and (23) and $r = 0$ in (24) and (25).

Thus the first perturbed deflection is found to be

$$W(x,y,t) = -\frac{\alpha_1^2}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_{mn}^2 \cos(2m+1)\frac{\pi x}{2a}\cdot\cos(2n+1)\frac{\pi y}{2b}$$

$$\times\left[\frac{D_{mn}2t_0}{q_{mn}(1+q_{mn}^2)}\left\{\left(1+e^{-t/t_0}\right)\sin\frac{q_{mn}t}{2t_0} - \left(1-e^{-t/t_0}\right)\cos\frac{q_{mn}t}{2t_0}\right\}\right.$$

$$+\frac{q_{mn}C_{mn}}{1+q_{mn}^2}\left\{q_{mn}\left(\sin\frac{q_{mn}t}{2t_0} + \cos\frac{q_{mn}t}{2t_0}\right)\right.$$

$$\left.\left.+e^{-t/t_0}\left(\cos\frac{q_{mn}t}{2t_0} - q_{mn}\sin\frac{q_{mn}t}{2t_0}\right)\right\}\right] \quad \ldots (27)$$

Collecting terms of higher order of α in like manner, the higher perturbations may be obtained.

2. Free-vibration of isoceles right-angled triangular Plate.

Let us consider the transverse vibration of an isoceles right-angled triangular plate having its boundary simply-supported. The equal sides of the plate are considered to be of lengths a in the directions of X and Y. Let us assume $w(x,y)$ in the form

$$w = c\left[\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right] \quad \ldots (28)$$

where m is an odd integer.

It is evident that this form of ω satisfies the following boundary conditions for the simply-supported edges.

$$\left.\begin{aligned}
&\omega = \frac{\partial^2 \omega}{\partial x^2} = 0 \qquad \text{at } x = 0 \\
&\omega = \frac{\partial^2 \omega}{\partial y^2} = 0 \qquad \text{at } y = 0 \\
&\omega = \frac{\partial^2 \omega}{\partial v^2} = 0 \qquad \text{at } x + y = a \\
&\text{where } \frac{\partial}{\partial v} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)
\end{aligned}\right\} \qquad \ldots (29)$$

Substituting (28) in equation (6) and proceeding as before ϕ is determined in the form,

$$\phi = C_1 J_{iq_m}(z_2) + C_2 J_{-iq_m}(z_2) \qquad \ldots (30)$$

where

$$q_m^2 = \frac{4 p_m^2 E_0 t_0^2}{\lambda}, \quad p_m^2 = \frac{49 \pi^4 m^4}{a^4}$$

$$z_2 = \alpha_1 q_m e^{-t/2t_0}$$

Hence
$$W = \sum_{m=1,3,5,\ldots}^{\infty} \left(\sin\frac{2m\pi x}{a} \cdot \sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a} \cdot \sin\frac{m\pi x}{a} \right) \times$$

$$\times \left[A_m J_{iq_m}(z_2) + B_m J_{-iq_m}(z_2) \right] \qquad \ldots (31)$$

Let the initial conditions be

$$\left.\begin{aligned} W(x,y,0) &= \sum C_m\left(\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a}+\sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right) \\ \frac{\partial}{\partial t}W(x,y,0) &= \sum D_m\left(\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a}+\sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right) \end{aligned}\right\} \quad \ldots (32)$$

where C_m and D_m are known constants.

Putting (31) in (32) and solving for the constants, we have

$$A_m = \frac{1}{\alpha_1\Delta_2}\left[\frac{D_m t_0}{q_m}J_{-iq_m}(\alpha_1 q_m)+\alpha_1 C_m J'_{-iq_m}(\alpha_1 q_m)\right]$$

$$B_m = -\frac{1}{\alpha_1\Delta_2}\left[\frac{D_m t_0}{q_m}J_{iq_m}(\alpha_1 q_m)+\alpha_1 C_m J'_{iq_m}(\alpha_1 q_m)\right]$$

where

$$\Delta_2 = -\frac{2i\sinh\pi q_m}{\pi q_m\alpha_1}$$

Hence $W(x,y,t)$ is known.

To obtain the classical result as well as the numerical computation together with different perturbed deflections, it will be convenient to replace the products of Bessel functions of imaginary orders by the equivalent series involving Hypergeometric functions as shown in the case of rectangular plate.

CHAPTER - V

Large amplitude free vibrations of elastic plates.

Nomenclature :

The following nomenclature are used throughout this paper.

w = deflection, normal to the middle plane,

u, v = displacements corresponding to the directions of co-ordinate axes,

h = thickness of the plate,

D = flexural rigidity of the plate $= \dfrac{Eh^3}{12(1-\sigma^2)}$

E = Young's modulus,

σ = Poisson's ratio,

ρ = density of the plate material.

Introduction :

Berger's (1955) approximate plate theory for the large deflection of isotropic plates has been extended to orthotropic plate problems by Iwinski and Nowinski (1957). Nowinski (1958) has also solved some boundary value problems associated with circular and rectangular plates undergoing large deflections. Nash and Modeer (1960) found the large amplitude free vibrations of rectangular and circular plates applying the technique exhibited by Berger.

In this paper an attempt has been made to investigate the large amplitude free vibrations of triangular, elliptic and semi-circular plates.

Theory :

Let us consider the free vibrations of flat elastic plates with hinged, immovable edges. The deflections are considered to have the order of magnitude of the plate thickness.

The sum of the membrane and bending energies in a thin plate undergoing large deflections can be written in the form,

$$V = \frac{D}{2}\iint\left\{\left[(\nabla^2 w)^2 + \frac{12}{h^2}e^2\right] - 2(1-\sigma)\left[\frac{12}{h^2}e_2 + \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}dx\,dy \qquad \ldots (1)$$

The kinetic energy of the plate is

$$T = \frac{\rho h}{2}\iint\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right)dx\,dy \qquad \ldots (2)$$

It is now possible to form the Lagrangian function

$$L = T - V \qquad \ldots (3)$$

According to Hamilton's principle $\delta\int_{t_1}^{t_2} L\,dt = 0$ ... (4)

If we set $A = \int_{t_1}^{t_2} L\,dt$, then $\delta A = 0$ ... (5)

Neglecting e_2 and applying Euler's variational equations we get the following equations

$$\nabla^4 w - \alpha^2 f(t)\cdot\nabla^2 w + \frac{12}{h^2 e_\xi^2}\cdot\frac{\partial^2 w}{\partial t^2} = 0 \qquad \ldots (6)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h^2}{12} f(t) \qquad \ldots (7)$$

[Nash and Modeer (1960)]

where $\alpha =$ constant,

$e_\xi^{-2} = \dfrac{\rho h^3}{12 D}$

It is to be noted that the terms corresponding to inertia effects in the plane of the plate have been neglected for the equations (6) and (7).

Problem :

1. **Large amplitude free-vibration of an isoceles right-angled triangular plate.***

Let us consider the free vibration of a flat isoceles right-angled triangular plate with hinged, immovable edges. The equal sides of the plate are considered to be of lengths a in the directions of x and y .

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For the simply-supported edges, the boundary conditions are

$$\left.\begin{aligned} u = w = \frac{\partial^2 w}{\partial x^2} &= 0 \quad \text{at} \quad x = 0 \\ v = w = \frac{\partial^2 w}{\partial y^2} &= 0 \quad \text{at} \quad y = 0 \\ u + v = w = \frac{\partial^2 w}{\partial \nu^2} &= 0 \quad \text{at} \quad x + y = a \end{aligned}\right\} \qquad \ldots(8)$$

$$\text{where} \quad \frac{\partial}{\partial \nu} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$$

The boundary conditions are satisfied by the configurations of the form

$$u(x,y,t) = \sum_{K=1,3,5,\ldots}^{\infty} B_K \sin\frac{K\pi x}{a}\left(\cos\frac{K\pi y}{a} + \sin\frac{K\pi x}{a} - \frac{K\pi}{4}\right) H(t) \qquad \ldots(9)$$

$$v(x,y,t) = \sum_{K=1,3,5,\ldots}^{\infty} B_K \sin\frac{K\pi y}{a}\left(\cos\frac{K\pi x}{a} - \sin\frac{K\pi y}{a} + \frac{K\pi}{4}\right) G(t) \qquad \ldots(10)$$

$$w(x,y,t) = \sum_{m=1,3,\ldots}^{\infty} A_m\left(\sin\frac{2m\pi x}{a}\cdot\sin\frac{m\pi y}{a} + \sin\frac{2m\pi y}{a}\cdot\sin\frac{m\pi x}{a}\right) F(t) \qquad \ldots(11)$$

The equations (9), (10) and (11) may now be substituted in equation (7) to yield

$$F^2(t) = G(t) = H(t) = f(t) \qquad \ldots(12)$$

Let us investigate the fundamental mode of vibration by putting $m = 1$ in equation (11).

Substituting equations(9), (10) and (11) in (7), considering equation (12) and integrating over the surface of the plate we have,

$$\alpha^2 = \frac{15 A_1^2 \pi^2}{a^2 h^2} \qquad \ldots(13)$$

If we now substitute equations (11), (12) and (13) in (6) with $m = 1$, we obtain,

$$\frac{12}{h^2 c_p^2} \cdot \frac{d^2 F}{dt^2} + \frac{49\pi^4}{a^4} \cdot F(t) + \frac{75 A_1^2 \pi^4}{a^4 h^2} F^3(t) = 0 \qquad \ldots(14)$$

This equation is of the form

$$\ddot{F} + \lambda F + \mu F^3 = 0 \qquad \ldots(15)$$

which is to be solved subject to the initial conditions

$$F(0) = 1, \quad \dot{F}(0) = 0 \qquad \ldots(16)$$

Solution of (15) can be put in the form

$$F(t) = C_n(\omega_1 t, K) \qquad \ldots(17)$$

[Nash and Modeer (1960)]

where $$\omega_1^2 = \lambda + \mu = \frac{c_p^2 \pi^4 h^2}{12 a^4}\left(49 + 75\frac{A_1^2}{h^2}\right) \qquad \ldots(18)$$

$$K^2 = \frac{\mu}{2(\lambda + \mu)} = \frac{75 A_1^2}{2h^2\left(49 + 75\frac{A_1^2}{h^2}\right)} \qquad \ldots(19)$$

Here ω_1 and K are positive constants and c_n is Jacobi's elliptic function.

The period T' is given by $T' = \frac{4K}{\omega_1}$ ... (20)

where K is the complete elliptic integral of the first kind.

Hence $$T' = \frac{4Ka^2\sqrt{12}}{c_p h \pi^2 \left(49 + 75\frac{A_1^2}{h^2}\right)^{1/2}} \qquad \text{... (21)}$$

The usual linear period $T = \frac{2\pi}{\omega_2}$ ... (22)

is found from the equation

$$\left.\begin{aligned} &\nabla^4 w + \frac{12\ddot{w}}{h^2 c_p^2} = 0 \\ &\text{with} \quad w = A_1\left[\sin\frac{2\pi x}{a}\cdot\sin\frac{\pi y}{a} + \sin\frac{2\pi y}{a}\cdot\sin\frac{\pi x}{a}\right]\cos\omega_2 t \end{aligned}\right\} \qquad \text{... (23)}$$

in the form $$T = \frac{2\pi a^2\sqrt{12}}{\pi^2 h c_p \cdot 7} \qquad \text{... (24)}$$

Hence $$\frac{T'}{T} = \frac{\frac{2K}{\pi}}{\left[1 + \frac{75}{49}\cdot\frac{A_1^2}{h^2}\right]^{1/2}} = \frac{\frac{2K}{\pi}}{\left[1 + \frac{75}{49}\cdot\beta_1^2\right]^{1/2}} \qquad \text{... (25)}$$

where $$\beta_1 = \frac{A_1}{h} \qquad \text{... (26)}$$

The ratio T'/T given by (25) is plotted against various values of β_1 in (26). The graph is shown in the figure No.8

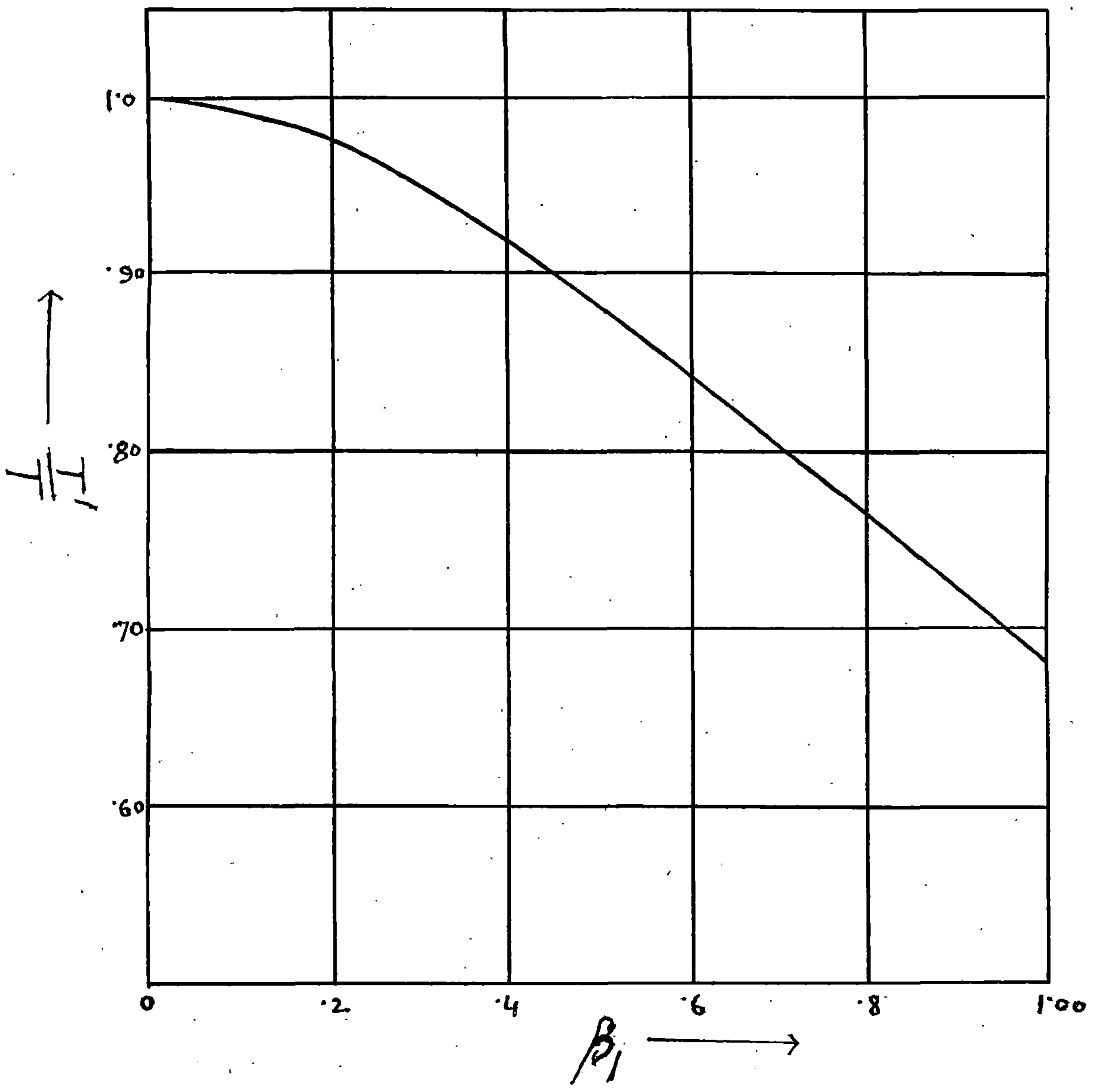

FIG. 8

Graph showing $\frac{T'}{T}$ against β_1

2. Large amplitude free-vibrations of elliptic plates.*

Let us consider the free vibration of an elliptic plate having its boundary elastically restrained against rotation.

For this case, let us assume w in the form

$$w = W(x, y) \cdot F(t) \qquad \ldots(27)$$

Substituting equation (27) in (6) we have

$$\nabla^4 W \cdot F(t) - \alpha^2 F^3(t)\, \nabla^2 W + \frac{12}{h^2 c_\rho^2} \cdot \frac{d^2 F}{dt^2} \cdot W = 0 \qquad \ldots(28)$$

where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y}\right)$ and $F^2(t) = f(t)$

A solution of equation (28) is possible if

$$\frac{\nabla^4 W}{W} = K^4 \,, \quad \frac{\nabla^2 W}{W} = -K^2 \qquad \ldots(29)$$

From (29)

$$\nabla^2 W + K^2 W = 0 \qquad \ldots(30)$$

Changing into elliptic co-ordinates (ξ, η), we have

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) W + 2p^2 \left(\cosh 2\xi - \cos 2\eta\right) W = 0 \qquad \ldots(31)$$

where $p = \frac{kd}{2}$, $2d$ being the interfocal distance of the ellipse.

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Solution of (31) can be written as

$$W = \sum_{m=0}^{\infty} \bar{C}_{2m} \mathrm{Ce}_{2m}(\xi, q)\, ce_{2m}(\eta, q) \qquad \ldots (32)$$

where $\mathrm{Ce}_{2m}(\xi,q)$ and $ce_{2m}(\eta,q)$ are the Modified Mathieu function and Mathieu function of the first kind of order $2m$ and $q = p^2 = \frac{k^2 d^2}{4}$.

While solving a problem of bending of a plate with elliptic hole, instead of taking Mathieu functions of all orders, taking a single Mathieu function of 2nd order, Naghdi (1955) has shown that the results obtained are satisfactory for larger elliptic hole. In our present problem also we can make similar approximation by taking a single Mathieu function of Zero order.

Hence (32) reduces to

$$W = c_1 \mathrm{Ce}_0(\xi, q)\, ce_0(\eta, q) \qquad \ldots (33)$$

Combining equations (28) and (29) we have the following differential equation for determining $F(t)$.

$$\frac{d^2F}{dt^2} + \frac{h^2 c_p^2 k^4}{12} F(t) + \frac{\alpha^2 h^2 c_p^2 \cdot k^2}{12} F^3(t) = 0 \qquad \ldots (34)$$

The equation is of the form

$$\ddot{F} + \lambda_1 F + \mu F^3 = 0 \qquad \ldots (35)$$

which is to be solved subject to initial conditions

$$F(0) = 1, \quad \dot{F}(0) = 0 \qquad \ldots (36)$$

Solution of (35) can be put in the form

$$F(t) = C_n(\omega_3 t, \lambda_2) \qquad \text{... (37)}$$

where
$$\left.\begin{aligned}(\omega_3)^2 &= \frac{h^2 c_p^2 \cdot K^4}{12}\left(1+\frac{\alpha^2}{K^2}\right)\\ \lambda_2^2 &= \frac{1}{2\left(1+\frac{K^2}{\alpha^2}\right)}\end{aligned}\right\} \qquad \text{... (38)}$$

Here ω_3 and λ_2 are positive contants and C_n is Jacobi's elliptic function.

The period T_1 is given by $T_1 = \frac{4K}{\omega_3}$ ... (39)

K being the complete elliptic integral of the first kind.

Hence $w = c_1 Ce_0(\xi, q)\, ce_0(\eta, q)\, C_n(\omega_3 t, \lambda_2)$ ... (40) is known.

For w to vanish on the boundary $\xi = \xi_0$

q must be root of $Ce_0(\xi_0, q) = 0$ ... (41)

To determine α, we know that

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{\alpha^2 h^2}{12} f(t)$$

Changing into elliptic co-ordinates, the above equation reduces to

$$h_1 h_2\left[\frac{\partial}{\partial \xi}\left(\frac{u_\xi}{h_2}\right) + \frac{\partial}{\partial \eta}\left(\frac{u_\eta}{h_1}\right)\right] + \frac{1}{2} h_1 h_2\left[\left(\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{\partial w}{\partial \eta}\right)^2\right]$$

$$= \frac{\alpha^2 h^2}{12} f(t) \qquad \text{... (42)}$$

where

$$h_1 = h_2 = \frac{1}{d\sqrt{\sin^2 h\xi + \sin^2\eta}}$$

Boundary conditions for u_ξ and u_η are

$$u_\xi = u_\eta = 0 \quad \text{at} \quad \xi = \xi_0 \qquad \dots (43)$$

Let us assume that

$$\left.\begin{aligned} u_\xi &= \sum_{n=0}^{\infty} P(\xi)\cos 2n\eta\, F^2(t) \\ u_\eta &= \sum_{n=1}^{\infty} G(\xi)\sin 2n\eta\, F^2(t) \end{aligned}\right\} \qquad \dots (44)$$

Subject to the conditions $P(\xi_0) = G(\xi_0) = 0$

Combining equations (37), (40), (42) and (44) and integrating equation (42) over the surface of the plate we have

$$\int_0^{2\pi}\int_0^{\xi_0}\left[\left(\frac{\partial\omega}{\partial\xi}\right)^2 + \left(\frac{\partial\omega}{\partial\eta}\right)^2\right] d\xi\, d\eta$$

$$= \frac{\alpha^2 h^2 d^2}{6}\int_0^{2\pi}\int_0^{\xi_0}\left(\sin^2 h\xi + \sin^2\eta\right) d\xi\, d\eta \qquad \dots (45)$$

After evaluating the integrals we get the following equation to determine α ;

$$C_1^2\Bigg[\sum_{r=1}^{\infty}\left\{A_{2r}^{(0)}\right\}^2 r\,\text{Sinh}\,4r\xi_0+\sum_{r=1}^{\infty}\sum_{\substack{s=1\\ r\neq s}}^{\infty}\frac{A_{2r}^{(0)}\cdot A_{2s}^{(0)}\cdot 4rs}{r^2-s^2}$$

$$\times\left\{r\,\text{Sinh}\,2r\xi_0\,\text{Cosh}\,2s\xi_0-s\,\text{Sinh}\,2s\xi_0\,\text{Cosh}\,2r\xi_0\right\}$$

$$+\sum_{r=1}^{\infty}4r^2\left(A_{2r}^{(0)}\right)^2\Bigg\{\sum_{r=1}^{\infty}\left(A_{2r}^{(0)}\right)^2\left(\xi_0+\frac{\text{Sinh}\,4r\xi_0}{4r}\right)$$

$$+\,2A_0^{(0)}\sum_{r=1}^{\infty}\frac{A_{2r}^{(0)}\,\text{Sinh}\,2r\xi_0}{r}+\sum_{r=1}^{\infty}\sum_{\substack{s=1\\ r\neq s}}^{\infty}\frac{A_{2r}^{(0)}\cdot A_{2s}^{(0)}}{r^2-s^2}$$

$$\times\left(r\,\text{Sinh}\,2s\xi_0\,\text{Cosh}\,2r\xi_0-s\,\text{Sinh}\,2r\xi_0\,\text{Cosh}\,2s\xi_0\right)\Bigg\}\Bigg]$$

$$=\frac{\alpha^2h^2d^2\,\text{Sinh}\,2\xi_0}{12}$$

OR,

$$C_1^2F_1(\xi_0)=\frac{\alpha^2h^2d^2\,\text{Sinh}\,2\xi_0}{12}\qquad\ldots(46)$$

where $A_{2r}^{(0)}$ are the Fourier coefficients in the expansion of $ce_0(\xi,q)$

Considering equations (38), (39) and (46) we get

$$T_1 = \frac{4K\sqrt{12}}{h c_p k^2\left(1+\frac{12C_1^2 F_1(\xi_0)}{k^2h^2d^2\sinh 2\xi_0}\right)^{1/2}} \qquad \ldots(47)$$

The usual linear period is given by

$$T_2 = \frac{2\pi}{\omega_4} \qquad \ldots(48)$$

ω_4 being found out from the following equation

$$\nabla^4 w + \frac{12}{h^2c_p^2}\ddot{w} = 0 \qquad \ldots(49)$$

in the form

$$\omega_4^2 = \frac{k^4h^2c_p^2}{12} \qquad \ldots(50)$$

Hence

$$\frac{T_1}{T_2} = \frac{\frac{2K}{\pi}}{\left(1+\frac{12C_1^2F_1(\xi_0)}{k^2h^2d^2\sinh 2\xi_0}\right)^{1/2}} \qquad \ldots(51)$$

If $d \to 0$, $\xi \to \infty$ the ellipse degenerates to a circle of radius R (say). In that case

$$Ce_0(\xi, q) \longrightarrow P_0' J_0(kr)$$

where $P_0' = \dfrac{Ce_0(0,q)\, ce_0(\pi/2, q)}{A_0(0)}$

and $ce_0(\eta, q) \longrightarrow A_0(0) \longrightarrow \frac{1}{\sqrt{2}}$

Hence in the limiting case, equation (40) reduces to

$$W = C_1 P_0' J_0(kr) \frac{1}{\sqrt{2}} cn(\omega^* t, \lambda)$$

$$= A J_0(kr)\, cn(\omega^* t, \lambda) \qquad \text{...(52)}$$

Also equation (45) can be written as

$$C_1^2 \left[\left\{ ce_0'(\xi, q)\, ce_0(\xi, q) \right\}_0^{\xi_0} + 2q \int_0^{\xi_0} c^2e_0(\xi, q) \left\{ \cosh 2\xi - \Theta_0 \right\} d\xi \right]$$

$$= \frac{\alpha^2 \hbar^2 d^2 \sinh 2\xi_0}{12} \qquad \text{...(53)}$$

where $\Theta_0 = A_0^{(0)} . A_2^{(0)} + \sum_{r=0}^{\infty} A_{2r}^{(0)} . A_{2r+2}^{(0)}$

Since $ce_0(\xi_0, q) = 0; \; d^2 \sinh 2\xi_0 \to 2R^2, \; \Theta_0 \to 0, \; A_{2r}^{(0)} \to 0$

and $\cosh 2\xi \, d\xi \to 2r\,dr/d^2$ as $\xi \to \infty, \; d \to 0,$

equation (53) reduces to, in the limiting case,

$$C_1^2 \left[K^2 P_0'^2 \int_0^R r J_0^2(kr)\, dr \right] = \frac{\alpha^2 \hbar^2 R^2}{6}$$

OR, $\dfrac{\alpha^2 \hbar^2}{6} = A^2 K^2 J_1^2(KR)$ ...(54)

Since $J_0(KR) = 0$

Hence

$$T_1/T_2 = \frac{2K/\pi}{\left(1+6A^2J_1^2(kR)/h^2\right)^{1/2}} \quad \ldots(55)$$

Equations(52), (54) and (55) are the corresponding results for a circular plate as obtained by Willam A. Nash and James R. Modeer (1960).

Numerical results :

Since $ce_0(\xi_0, q) = 0\ , \quad q = 6{\cdot}4$

when $\xi_0 = 3{\cdot}12$ and $d = 2{\cdot}11$

Let $\alpha = 2\sqrt{2}$

Putting all these values in (46)

$$\frac{c_1}{h} = {\cdot}74$$

Corresponding T_1/T_2 is found from equation (55) in the form

$$\frac{T_1}{T_2} = {\cdot}97$$

3. Large amplitude free-vibrations of semi-circular plates.

Let us consider a plate in the form of a semi - circle having its boundary elastically restrained against rotation. Let us take the centre as pole and the bounding diameter as initial line.

For the above case, let us assume w in the form

$$w = W(r, \theta) \cdot F(t) \qquad \ldots (56)$$

Substituting (56) in (6) we have

$$\nabla^4 W \cdot F(t) - \alpha^2 F^3(t) \cdot \nabla^2 W + \frac{12}{h^2 c_0^2} \frac{d^2 F}{dt^2} \cdot W = 0 \qquad \ldots (57)$$

where $f(t) = F^2(t)$ and $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

A solution of (57) is possible if

$$\frac{\nabla^4 W}{W} = K^4 \quad \text{and} \quad \frac{\nabla^2 W}{W} = -K^2 \qquad \ldots (58)$$

From (58)

$$\nabla^2 W + K^2 W = 0 \qquad \ldots (59)$$

To solve (59) let us put

$$W = \sum_{m=1,3,\cdots}^{\infty} R_m \sin m\theta \qquad \ldots (60)$$

R_m being the function of r only.

Substituting (60) in (59) and solving we get

$$R_m = A_m J_m(Kr) \qquad \ldots (61)$$

where A_m is a constant and J_m is the Bessel function of order m.

From (57) equation to determine $F(t)$ reduces to

$$\frac{d^2F}{dt^2} + \frac{h^2 c_p^2 k^4}{12} F + \frac{\alpha^2 h^2 c_p^2 k^2}{12} F^3 = 0 \qquad \ldots(62)$$

Solution of (62) is given in the form as in the previous case,

$$F(t) = c_n(\omega_s t, \lambda) \qquad \ldots(63)$$

where

$$\left.\begin{aligned} \omega_s^2 &= \frac{\alpha^2 c_p^2 k^4}{12}\left(1+\frac{\alpha^2}{k^2}\right) \\ \lambda^2 &= \frac{1}{2\left(1+\frac{k^2}{\alpha^2}\right)} \end{aligned}\right\} \qquad \ldots(64)$$

Thus

$$w = \sum_{m=1,3,\ldots}^{\infty} A_m J_m(kr) \sin m\theta . c_n(\omega_s t, \lambda) \qquad \ldots(65)$$

is known.

For the fundamental mode,

$$w = A_1 J_1(kr) \sin\theta \, c_n(\omega_s t, \lambda) \qquad \ldots(66)$$

w to vanish on the boundary $r = a$

K must be root of $J_1(Ka) = 0$ ...(67)

To determine α let us consider the equation (7) which in polar co-ordinates takes the form

$$\frac{\alpha^2 h^2}{12} f(t)$$

$$= \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad \dots (68)$$

Let us assume

$$\left.\begin{aligned} u &= \sum U(r) \cos m\theta \,.\, F^2(t) \\ v &= \sum V(r) \sin m\theta \,.\, F^2(t) \end{aligned}\right\} \qquad \dots (69)$$

Combining equations (68), (69), (65) and (63), multiplying (68) by $r\,dr\,d\theta$ and integrating w.r.t. r between the limits 0 to a and w.r.t. θ between the limits 0 to π we get the following equation to determine α

$$A_m^2 \Big[-\frac{k^2 a^2}{2} J_m^2(ka) + m J_m^2(ka) + \frac{1}{2} k^2 a^2 J_{m+1}^2(ka)$$

$$+ k(m+1) J_m(ka) \,.\, J_{m+1}(ka) \Big]$$

$$= \frac{\alpha^2 h^2 a^2}{6} \qquad \dots (70)$$

For fundamental mode of vibration,

$$\alpha^2 = \frac{3k^2 J_2^2(ka) \,.\, A_1^2}{h^2} \qquad \dots (71)$$

using $J_1(ka) = 0$

Considering equations (63), (64) and (71) we have

$$T_3 = \frac{4K\sqrt{12}}{h c_0 K^2\left(1+\frac{3J_2^2(ka)A_1^2}{h^2}\right)^{1/2}} \qquad \ldots(72)$$

As in the previous case,

$$T_4 = \frac{2\pi}{w_6}, \quad \text{where} \quad w_6^2 = \frac{K^4 h^2 c_0^2}{12} \qquad \ldots(73)$$

Hence

$$T_3/T_4 = \frac{\frac{2K}{\pi}}{\left(1+\frac{3A_1^2 J_2^2(ka)}{h^2}\right)^{1/2}} \qquad \ldots(74)$$

__Numerical calculation.__

Since $J_1(ka) = 0$, $ka = 3{\cdot}81$

From table $J_2(ka) = J_2(3{\cdot}81) = {\cdot}409$

Let $a = 10$, $\alpha = {\cdot}1$

Putting all these values in (71)

$$\frac{A_1}{h} = {\cdot}58$$

Corresponding T_3/T_4 is found from (74) in the form

$$\frac{T_3}{T_4} = {\cdot}71$$

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