SOME MIXED BOUNDARY VALUE PROBLEMS IN ELASTODYNAMICS

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CHAPTER - III

Mixed Boundary Value Problems in Viscoelastic Media

INTRODUCTION

The study of wave and vibration phenomena has a distinguished history of hundred years. The first mathematician to describe the vibrations of pendulums, the resonance phenomenon and the vibrations of strings was Galileo. Some pioneer workers in the field of wave propagation are Cauchy, Poisson, Ostrogradsky, Green, Lame, Stokes, Navier, Clebsch and Christoffel.

Some of the major developments in the area of wave propagation are given below in chronological order :

1678 : Robert Hooke formulated the law of proportionality between stress and strain for elastic bodies. This law forms the basis for the static and dynamic theory of elasticity.

1821 : Navier investigated the general equations of equilibrium and vibration of elastic solids. Although not all of the developments of the work met with complete acceptance, it represented one of the most important developments in mechanics.

1822 : Cauchy developed most of the aspects of the pure theory of elasticity including the dynamical equations of motion for a solid.

1828 : Poisson investigated the propagation of waves through an elastic solid. He found that the two wave types, longitudinal and transverse, could exist.

1862 : Clebsch founded the general theory for the free vibration of solid bodies using normal modes.

1872 : J. Hopkinson performed the first experiments on plastic waves propagation in wires.

1883 : Saint Venant summarized the work on impact of earlier investigators and presented his results on transverse impact.

1887 : Rayleigh investigated the propagation of surface waves on a solid.

1904 : Lamb made the first investigation of pulse propagation in a semi-infinite solid.

1911 : Love developed the theory of waves in a thin layer overlying a solid and showed that such waves accounted for certain anomalies in seismogram records.

1949 : Davies published an extensive theoritical and experimental study on waves in bars.

1955 : Pekeris presented the solution to Lamb's problem of pulse propagation in a semi-infinite solid.

During the first three decades of this century the subject was not given so much importance by Mathematicians or Physicists. But in the later part of the 19th century interest in the study of waves in elastic solids attracted the researchers because of applications in the field of geophysics. Since that time in seismology the wave propagation has remained an interesting area because of the need for details information on earthquake phenomena, prospecting techniques and the detection of nuclear explosions. Bullen (1963), Ewing et al (1957), Cagniard (1962) and Pilant (1979) have discussed about seismic waves in their books.

During last 30-40 years the developement of theory of wave propagation in elasticity has been characterized by a detailed investigation of the classical methods of mathematical analysis and the trends to obtain specific results.
The solutions of many of the problems in elastodynamics, which are frequently encountered in practice, have made a significant contribution to the development of the theory of wave propagation as a whole. While earlier investigations in the theory of elasticity were essentially reduced to the construction of particular solutions, the invention of computer technology has led to the developement of general and quite universal methods of solving the problems of this theory, namely, the boundary value problems and initial boundary value problems for systems of differential equations having partial derivatives of a definite structure.

In an unbounded isotropic solid, two types of elastic waves may be propagated. These are dilatational wave and distortional wave. When a solid medium is deformed, both distortional and dilatational waves will normally be produced, and when a wave of either type impinges on a boundary of the solid, waves of both types are generated. In addition to these two types of wave which can travel through an extended solid medium, elastic waves may be propagated along the surface of a solid; these are known as Rayleigh waves, and the disturbances associated with them decay exponentially with depth. Since these waves spread only in two dimensions, they fall off more slowly with distance than the other types of elastic wave. They are of importance in seismic phenomena.

The propagation of waves in solids may be divided roughly into three categories. The first is elastic waves, where the stresses in the material obey Hooke's law. The two other main categories, visco-elastic waves, where viscous as well as elastic stresses act, and plastic waves in which the yield stress of the material is exceeded.

With regard to other works specially dealing with the propagation of waves in elastic solids we mention the books by Kolsky (1963), Brekhovskikh (1960), Achenbach (1975), Graff (1973) and Hudson (1980).

In recent years problems of diffraction of elastic wavs are of considereable importance in view of their application in Seismology and Geophysics. These types of problems can mainly be classifed into two categories. Firsty, diffraction of waves by semi-infinite plane barriers or cracks that are present in the medium and secondly, the diffraction in the presence of inclusions like wedges rigid strips, cones, cylinders, spheres, spheroids, ellipsoids or obstacle of any arbitrary shape. In bonding two materials with different mechanical elastic properties, very often it is not possible to obtain a homogeneous perfect bond due to the existence of entraped imperfections, for example, in the joints involving ceramics and metals used in manufacturing electronic devices and variety of reinforced composites. In nature the stratification of the earth is another example of bodies consisting of layered strucure. Indeed in geophysical stratifications, faults occur at the interface while in manufactured laminates imperfections occur at the interface of the adjoining layers.

The study of diffraction problems are associated with mixed boundary value problems.

We describe briefly the background of mixed boundary value problems below :

We consider a deformed elastic body occupying an open region D, whose boundary surface is S. It is assumed that S is piecewise smooth and the closure of D is $\overline{D} = D + S$. The surface S is usually considered to be closed and bounded, having the region D internal or external to it. Also S may be taken as open and extended to infinity or lying entirely at infinity.

The deformation and the state of stress within D and on S characterise the solution of the statical problem of elasticity. We can obtain an elasticity field $E = (u,e,T)$ where the elements in the parenthesis are the displancement field, the strain field and the stress field respectively. To ensure the uniqueness of the solution, we have to prescribe on the surface S one from each group of the following :

$$(u_1, \overset{n}{T}_1)\ ,\ (u_2, \overset{n}{T}_2)\ ,\ (u_3, \overset{n}{T}_3)$$

where u_i (i=1,2,3,) are components of the displacement u, $\overset{n}{T}_i$(i = 1,2,3) are the components of the stress vectar $\overset{n}{T}$ and n is the outward unit normal at an elementary area of the surface S. Thus we have a class of problems called boundary value problems in the theory of elasticity (Knops and Payne ,1971).

The boundary value problems generally can be classified into three types .

A. Traction boundary value problems.

Values of the stress components are prescribed on the boundary surface S.

B. Displacement boundary value problems.

values of the displacement components are prescribed on the boundary suface S

C. Mixed boundary value problems.

various types of mixed boundary conditions may be specifed on S or on different parts of S. Some possible combinations of these conditions are as follows :

(i) Tangential component of the stress and the normal component of the displacement are prescribed on S.

(ii) Displacement field is prescribed on a portion S_1 of the surface S and the stress is prescribed on the remaining part S_2 of S where $S_1 U S_2$ = S.

(iii) Tangential component of displacement and the normal component of the stress are prescribed on S.

(iv) The linear combination $\tau_{ij}n_j + \beta u_i$ is prescribed on S, where τ_{ij} are the components of the stress tensor and β is a non-nagative function prescribed on S. This is the case of so-called elastic support condition.

(v) A mixed-mixed type boundary value problem may be formulated under the conditions :
the shear stress component and the normal displacement component being prescribed on S_1 and the normal stress component and the shear stress component being prescribed on S_2.

For problems on the half spaces, besides the conditions (i) - (v) one has to impose the conditions at infinity as follows :

(a) that the difference of any two stress distributions is bounded therein, together with the condition - (i)

(b) that the difference of any two displacement fields is bounded therein, together with the conditions (ii)-(v).

The boundary value problems stated above correspond to the elastostatic case. In elastodynamics it is also required to specify the displacement and the velocity of the points throughout the region D at time $t = t_o$ i.e., at the commencement of straining. This additional requirement together with the conditions A,B or C constitute initial boundary value problems.

Boundary value problems are intimately connected with the theory of existence and uniqueness of solutions consistent with the laws of elasticity and geometrical configuration of the region D.

We now discuss a certain type of mixed boundary value problems which are known as contact problems in the theory of elasticity. The contact problem is formulated as a problem about the influence of a rigid body or an elastic body. As a rule, the initial contact takes place at one point and a contact surface is formed only when contacting bodies become nearer to each other. Generally, this contact surface increases in size. Therefore, we naturally introduce a restriction having a physical meaning : the stresses along the contour encircling the contact surface are finite.

Let us assume that a surface bounding a rigid or an elastic body is piecewise smooth. In this case, the size of the contact surface may increase only within the limits of the smooth region, right up to its edges. Consequently, for a sufficiently large value of compressive force, we can find the contact surface and this leads to the mixed boundary value problem. Naturally, the values of stress at such pints of the contact surface lying on the edge may be unbounded. In all cases with the sole exception of the case of complete coupling, it should be remembered that the contact pressure must be compressive. Otherwise, a cavity is formed between an elastic and a rigid body, leading to quite apparent modifications in the formulation of the problem.

There are two distinct classes of problems relating to indentation by a frictionless punch. In the first kind of indentation, called complete penetration, there is complete contact between the punch and the half-plane over a specified contact region, in the sense that the normal displacement of the half-plane at the boundary matches the profile of the punch. Such problems are charaterised by a contact pressure which has a singularity (square root) at the ends of the contact region. In the second kind, called incomplete penetration, the extent of the contact region, i.e. the extent of the region over which the normal displacement of the half plane matches the profile of the punch, is initially unknown. Cases of incomplete penetration are characterized by a contact pressure which is zero at the ends of the contact region.

Contact problems for the elastic half-plane, i.e. problems in which one body is a punch and the other an elastic half-plane, fall into the class of problems treated by the classical theory of elasticity. This theory was largely developed in the 19th centuryand is fully described with historical references in Love's (1944) treatise.

The literature on contact problems has been reviewed by a number of authors. Shtaerman (1949), Galin (1961), Ufliand (1965), Rvachev (1967) and Abramian (1971) are concerned mostly with the Soviet literature. Muskhelishvili's treatise (1953) is the basis of much of the Russian work, particularly that using complex variable methods.

There are number of areas of research which fall within the general scope of 'classical' contact problems. Some of these are listed below briefly.

1. Problems associated with loading over a region of the surface of the half-space which is neither circular nor elliptic. Foremost here is the work of Kalker (1972), (1977) on "elastic line" contact. This relates to loading over a long slender region, and has practical importance for rolling contact. Panek and Kalker (1977), in particular, present a simple approximate solution for the deformation produced by a narrow rectangular punch with rounded ends.
   This problem is related to those for a strip-shaped punch discussed by Borodachev (962) and Protsenko (1974).

2. The contact problem for a flat -faced punch of rectangular cross-section, with particular reference to the nature of the singularities on the edges and at the corners, has been studied by Borodachev (1976). Extensive analytical and numerical results for both frictionless and adhesive cases may be found in Brothers, Sinclair and Segedin (1977).

3. Contact problems for an elastic rectangle have been treated by Abramian (1957), Prasad and Chatterjee (1973), Dundurs, Kiattikomol and Keer (1974), Prasad and Dasgupta (1975).

4. Problems associated with the compression of a rigid or

elastic body between two half-planes, strips, half-spaces or slabs have been considered by Okubo (1951), Alblas (1974).

5. Contact problems for a sphere or a spherical shell. This is the subject of many research papers, among which we mention following. Abramian, Arutiunian and Babloian (1964), Goodman and Keer (1965).

6. Problems associated with a block or cylinder embedded in a semi-infinite elastic medium, for which the references are Poulos and Davis (1968), Dhaliwal, singh and Sneddon (1979).

Another type of contact problems that is encountered in practice is the study of the dynamic response of an elastic solid to moving loads or to oscillation of rigid punch and inclusions. The moving load or moving punch problems which have been studied may be put into three categories :

(i) Steady wave motion due to a load or punch moving with constant velocity for all time.

(ii) Transient wave motion due to a load or punch which begins to act at certain instant and then moves with constant velocity.

(iii) Transient wave motion due to a load or punch which begins to act at certain instant and then moves in some direction with nonunifom speed.

The steady motion of a line load on the surface of an elastic half-space was studied by Sneddon (1952), Cole and Huth (1958),

Adams (1978), J.M.Golden (1982), Sve and Keer (1969), Alblas and Kuipers (1971,1971), Suhubi (1972).
The transient problem of a line load, which suddenly appears on the surface and then moves with constant velocity is of type (ii) and has been studied by Ang (1960).
As a representative of the third kind of problem we refer to the study of Freund (1972). An analytic technique was developed by Freund (1972) which made it possible to obtain an exact solution of a particular problem in category (iii).

Vibratory motion of a body on an elastic half-plane was treated by Karasudhi, Keer and Lee (1968). They considered the vertical, horizontal and rocking vibrations of a body on the surface of an otherwise unloaded half-plane. The problems were formulated so that shearing stress vanishes over the entire surface, and on oscillating displacement is prescribed in the loaded region. The problems were mixed with respect to the prescribed displacement and the remaining stress. Each case led to a mixed boundary value problem represented by dual integral equations which were reduced to a single Fredholm integral equation.
Wickham (1977) studied the problem of the forced two dimensional oscillations of a rigid strip in smooth contact with a semi-infinite elastic solid. He reduced the mixed boundary value problem with the help of Green's function to Fredholm integral equation of the first kind involving displacement boundary conditions. Using Noble's (1962) method, this equation was reduced

to Fredholm integral equation of the second kind with a kernel which was small in the low frequency limit. Then applying the method of iteration, a simple explicit long-wave asymptotic formula for the normal stress in terms of the prescribed displacement and dimensionless wave number K was rigorously derived.

Rocking motion of a rigid strip on a semi-infinite elastic medium has been studied by Ghosh and Ghosh (1985) by using a different technique. The forced rocking of the strip about the horizontal axis has been reduced to a solution of a dual integral equation. Following Tranter's (1968) method the dual integral equation was solved for low frequency oscillations by reducing the equation to a system of linear algebraic equations.

In case of low frequency oscillations Noble's (1963) method of solving dual integral equations, Tranter's (1968) technique for solving dual integral equations, Matched Asymptotic Expassion, and variational principle are found to be very useful techniques.

Suppose that a mixed boundary value problem is formulated by suitable integral transform so as to be governed by a set of dual integral equations of the form

$$\int_0^\infty x^{-1}\left[1+K(x)\right]S(x)J_\nu(rx)\,dx = f(r) \quad , \quad 0\le r<a$$

$$\int_0^\infty S(x)J_\nu(rx)\,dx = g(r) \quad , \quad r>a$$

where the functions K(x), f(r) and g(r) are known.
According to Noble (1963), when $\nu > -\frac{1}{2}$

$$S(x) = \sqrt{\frac{2x}{\pi}}\left\{\int_0^a t^{1/2}\theta(t)J_{\nu-1/2}(xt)dt + \int_a^\infty t^{\nu+1/2}G(t)J_{\nu-1/2}(xt)dt\right\}$$

where $\theta(t)$ satisfies the Fredholm integral equation

$$\theta(t) + \frac{1}{\pi}\int_0^a M(\tau,t)\theta(\tau)d\tau = t^{-\nu}F(t) - H(t) \qquad (0 \le t < a) \qquad (1)$$

in which $M(\tau,t) = \pi\sqrt{\tau t}\int_0^\infty xK(x)J_{\nu-1/2}(\tau x)J_{\nu-1/2}(tx)dx$

$$F(t) = \frac{d}{dt}\int_0^t f(r)r^{\nu+1}\,(t^2-r^2)^{-1/2}\,dr$$

$$H(t) = t^{1/2}\int_0^\infty xK(x)J_{\nu-1/2}(xt)dx\int_a^\infty \xi^{\nu+1/2}G(\xi)J_{\nu-1/2}(x\xi)d\xi$$

$$G(\xi) = \int_\xi^\infty g(r)r^{-\nu+1}\,(r^2-\xi^2)^{-1/2}\,dr\ .$$

The integral equation (1) can be solved for $\theta(t)$ and consequently S(x) can be determined.

The problem of diffraction of normally incident plane acoustic wave by two coplanar, infinite, parallel, perfectly soft or rigid strips was considered by Jain and Kanwal (1972). The problem was solved by integral equation methods. The problem for the soft strips led to a Fredholm integral equation of first kind while that of rigid strips gave an integro-differential equation. Those equations were solved by the regular perturbation technique (Kanwal, 1971).

At the same time Jain and Kanwal (1972) solved the problem of diffraction of normally incident longitudinal and antiplane shear elastic waves by two parallel and coplanar rigid strips of finite width embedded in an infinite, isotropic and homogeneous elastic medium. The mixed boundary value problem was reduced to a set of dual integral equations with trigonometrical kernel. The solutions were obtained by using an integral equation perturbation technique (Kanwal, 1971) and Hilbert transform (Srivastava and Lowengrub, 1968). Using the theorem (Tricomi, 1951),

if $p \in L_2(a,b)$, then the equation

$$\frac{1}{\pi}\int_a^b \frac{h(x)}{x-y}\,dx = p(y) \quad , \quad y \in (a,b)$$

has the solution

$$h(x) = -\frac{1}{\pi}\left(\frac{x-a}{b-x}\right)^{1/2}\int_a^b \left(\frac{b-y}{y-a}\right)^{1/2}\frac{p(y)}{y-x}\,dy + \frac{C}{\sqrt{(x-a)(b-x)}}$$

where C is an arbitrary constant and the first term belongs to the class $L_2(a,b)$, Srivastava and Lowengrub (1968) found that the solution of the integral equation

$$\frac{1}{\pi}\int_a^b \frac{2th(t^2)}{t^2-y^2}\,dt = p(y) \quad , \quad y \in (a,b)$$

(provided that p satisfies the conditions of the above theorem) is given by

$$h(t^2) = -\frac{1}{\pi}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\int_a^b\left(\frac{b^2-y^2}{y^2-a^2}\right)^{1/2}\frac{2yp(y)}{y^2-t^2}\,dy + \frac{D}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$

where D is an arbitrary constant.

Tait and Moodie (1981) have studied the problem of dynamic response of an elastic strip and that of pair of punches moving along the lateral boundaries of the strip and opening a crack along the mid surface. The problems were sloved in closed form by complex variable methods.

Diffraction of elastic waves by a rigid circular disc was considered by Mal, Ang and Knopoff (1968).
Low frequency diffraction by a elliptic disc have been studied by Sleeman (1967) and Roy and Sabina (1983).
Stallybrass and Scherer (1976) considered the problem of forced vertical vibration of a rigid frictionless elliptical disc on the surface of an elastic half-space. The mixed boundary value problem was reduced to a (two-dimensional) itegral equation and an approximation was obtained for the displacement of the disc by using variational procedure.

Arobinda Roy (1968) studied the dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. Both vertical and horizontal vibrations were treated. Now we discuss the contact between rigid axisymmetric punch and an elastic half-space. Such contact problems may be classifed according to the type of punch, i.e. whether it is circular or annulur and whether its face is flat or curved; the type of indentation, i.e., whether it is a rotation or a translation about one or other of the axes, the type of contact, i.e., whether it is frictionless, adhesive, or in limiting friction, and if frictionless whether the contact is complete or incomplete.

Following Gladwell (1968) it may be assumed that the type of indentation, specified bythe displacement of the punch, is either

1. a rotation about Z-axis
2. a translation in the Z-direction
3. a rotation about the Y-axis
4. a translation in the X-direction.

Each of cases (2)-(4) gives rise to two distinct extreme problems in which the contact is assumed to be either completely adhesive or completely frictionless. The frictionless version of (1) is trivial in the sense that the half-space is not deformed at all.

England (1961) considered the axially symmetric indentation of a transversely isotropic layer resting on a rigid foundation. The problem of oscillations of a semi-infinite elastic solid by a smooth rigid circular disc on the free surface, performing small oscillations normal to its plane, without loosing contact with the surface of the solid has been studied by S.K. Bose (1967). The method of solution consists in introducing Hankel transforms and reduction to dual integral equations which have been solved by Tranter's method. Using Noble's (1963) method, Gladwell (1968) solved the problem of tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid.

All the axisymmetric contact problems may be solved by using Hankel transforms and they then reduce to the solution of a number of sets (or pairs) of dual integral equations. To solve these dual integral equations there are various methods one of which is Tranter's method. Here we discuss briefly the method of Tranter (1986) method.

The solution of certain physical problems involving axisymmetric contact can be reduced to the determination of F(p) from so called dual integral equations of the form

$$\int_0^\infty G(p)F(p)J_\nu(rp)dp = f(r) \quad , \quad 0<r<1$$

$$\int_0^\infty pF(p)J_\nu(rp)dp = 0 \quad , \quad 1<r<\infty \tag{2}$$

where G(p) and f(r) are known functions.

A solution F(p) of the above integral equations as a series of Bessel functions can be found by setting

$$F(p) = p^{-k}\sum_{m=0}^{\infty} a_m J_{\nu+2m+k}(p) \tag{3}$$

where k is at present an arbitrary parameter, and proceeding as follows.

Substituting from (3) and changing the order of integration and summation, one gets

$$\int_0^\infty pF(p)J_\nu(rp)dp = \sum_{m=0}^\infty a_m \int_0^\infty p^{1-k} J_\nu(rp)J_{\nu+2m+k}(p)dp \qquad (4)$$

Provided $\nu>-1$ and $k>0$, the formula

$$I(\nu,\mu,\lambda,a,b) = \int_0^\infty \frac{J_\nu(at)J_\mu(bt)}{t^\lambda} dt = \frac{b^\mu \Gamma(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2})}{2^\lambda a^{\mu-\lambda+1}\Gamma(\mu+1)\Gamma(\frac{\lambda}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2})}$$

$$\times \; {}_2F_1\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{\mu-\lambda-\nu+1}{2}; \mu+1; \frac{b^2}{a^2}\right)$$

shows that all the integrals on the right of (4) vanish when $r>1$ (because of the factor $\Gamma(-m)$ in the denominator of the term multiplying the hypergeometric function) and hence the series in (3) automatically satisfies the second of the dual equations (2). The coefficients a_m have now to be chosen so that the series in (3) satisfies the first of the dual equations (2). For this purpose we need the result

$$p^{-k} J_{\nu+2n+k}(p) = \frac{\Gamma(\nu+n+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(n+k)} \int_0^1 r^{\nu+1}(1-r^2)^{k-1}F_n(k+\nu,\nu+1,r^2) \times$$

$$\times J_\nu(pr)dr \qquad (5)$$

where n is a positive integer or zero and

$$F_n(\alpha,\gamma,x) = {}_2F_1(-n,\alpha+n;\gamma;x) \tag{6}$$

is Jacobi's polynomial.

Substituting from (3) in the first of (2), multiplication by

$$r^{\nu+1}(1-r^2)^{k-1}F_n(k+\nu,\nu+1,r^2),$$

integration with respect to r between 0 and 1, interchange of the order of integrations and use of (5) give

$$\sum_{m=0}^{\infty} a_m \int_0^{\infty} G(p)p^{-2k} \; J_{\nu+2m+k}(p)J_{\nu+2n+k}(p)dp = E(\nu,n,k)$$

where

$$E(\nu,n,k) = \frac{\Gamma(\nu+n+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(n+k)} \int_0^1 f(r)r^{\nu+1}(1-r^2)^{k-1}F_n(k+\nu,\nu+1,r^2)dr \tag{8}$$

Equation (7) with n=0,1,2,3,....... gives a set of simultaneous equations for the determination of the coefficients a_m. These simultaneous equations can be rewritten in a more convenient form by making use of the formula

$$\int_0^{\infty} p^{-1} \; J_{\nu+2m+k}(p)J_{\nu+2n+k}(p)dp = \begin{cases} 0 \;, & m\neq n \\ (2\nu+4n+2k)^{-1} \;, & m=n \end{cases} \tag{9}$$

this being the form taken by equation

$$\int_0^{\infty} \frac{J_\nu(at)J_\mu(at)}{t}\,dt = \frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2})}{2\Gamma(1+\frac{\nu}{2}-\frac{\mu}{2})\Gamma(1+\frac{\nu}{2}+\frac{\mu}{2})\Gamma(1-\frac{\nu}{2}+\frac{\mu}{2})}$$

$$= \frac{2}{\pi}\,\frac{\sin\frac{1}{2}(\mu-\nu)\pi}{\mu^2-\nu^2} \tag{10}$$

when μ and ν are replaced respectively by $\nu+2n+k$, $\nu+2m+k$ and when 'at' is replaced by p. We find in this way

$$a_n + \sum_{m=0}^{\infty} L_{m,n}\, a_m = (2\nu+4n+2k)E(\nu,n,k) \tag{11}$$

where

$$L_{m,n} = (2\nu+4n+2k)\int_0^{\infty} \left[G(p)p^{1-2k} - 1\right]p^{-1}\, J_{\nu+2m+k}(p)J_{\nu+2n+k}(p)dp \tag{12}$$

The iterative solution of the simultaneous equations (11) is

$$a_n = E_n - E_n' + E_n'' - \ldots\ldots. \tag{13}$$

where

$$E_n = (2\nu+4n+2k)E(\nu,n,k)$$

$$E_n' = \sum_{m=0}^{\infty} L_{m,n}\, E_m \quad , \quad E_n'' = \sum_{m=0}^{\infty} L_{m,n}\, E_m' \tag{14}$$

and so on.

Equations (3), (13), (14), (12) and (8) provide a theoretical solution of the dual integral equations. For a practical solution it is necessary to be able to choose the parameter k so that the expression $\left[G(p)p^{1-2k} - 1\right]$, which occurs in the formula (12) for $L_{m,n}$, is fairly small.

Now we consider another axisymmetric contact problem involving annular punch and torsion of an elastic half-space. These type of problems are three part boundary value problems. Triple integral equation method may be used to solve these problems. Some references are Tranter (1960).Cooke (1963), W.E.Williams (1963) and sneddon (1966). Gubenko and Mossakovskii (1960) solved the annular punch problem by using different technique. Olesiak (1965) attempted to solve the punch problem by reducing it to a series of dual integral equations and solved by successive approximations.

B.M. Singh, T.B. Moodie and J.B. Haddow (1980) considered also the problem of torsion by an annular disk of an elastic cylinder embedded in and bonded to an elastic half-spce. The problem was reduced to the solution of a Fredholm integral equation which was then analyzed by the method of Williams (1963). D.P. Thomas (1965) discussed the problem of diffraction of a general acoustic wave by a soft annular disc. Torsional oscillations of an elastic half-spce due to an annular disk has been studied by Jain and Kanwal (1970). Following Williams (1963) the problem was converted to a set of integral equations which were solved by iterative schemes when the outer radius of the disk is much larger than the inner radius.

A general formulation was given for the first time to the nonaxisymmetric annular punch problem by V.I. Fabrikant (1991). The problem was reduced to a two dimensional Fredholm integral equation with an elementary kernel which was solved numerically.

We describe here briefly the solution of a general class of boundary value problem by Williams's (1963) method.

Consider the general type of integral equation

$$\int_b^a f(t)\left[\, K(\rho,t) + K_1(\rho,t) \,\right] dt = \phi(\rho) \quad , \quad b<\rho<a \tag{15}$$

Integral equations of the general form of equation (15) occur in boundary value problems which can in some sense be regarded as perturbations on the electrostatic problem for the annulus. Examples of such problems are the diffraction of an acoustic wave by a soft annulus and the electrostatic problem for an annulus in a circular cylinder.

Assuming $\qquad \phi(\rho) = \sum_{n=-\infty}^{\infty} a_n \rho^n$

we can write

$$\phi_1(\rho) = \sum_{n=0}^{\infty} a_n \rho^n \quad , \quad \phi_2(\rho) = \sum_{n=-1}^{-\infty} a_n \rho^n$$

The integral equation (15) is thus equivalent to the pair of equations

$$\int_0^\infty f_1(t)K(\rho,t)dt = \phi_1(\rho) - \int_0^\infty f_1(t)K_1(\rho,t)dt \quad ; \quad 0<\rho<a \tag{16}$$

$$\int_0^\infty f_2(t)K(\rho,t)dt = \phi_2(\rho) - \int_0^\infty f_2(t)K_1(\rho,t)dt \quad , \quad \rho>b \tag{17}$$

where $\qquad f(\rho) = f_1(\rho) + f_2(\rho) \quad , \quad b<\rho<a$

and $\qquad f_1 + f_2 = 0 \quad , \quad 0<\rho<b \quad , \quad \rho>a \qquad (18)$

Further the integral equations (16) and (17) can be reduced to

$$\rho^{2n} S_1(\rho) = g_1(\rho) - \frac{1}{4}\int_0^{\infty} S_1(t)M_1(\rho,t)dt \quad , \quad 0<\rho<a \qquad (19)$$

$$\rho^{-2n} S_2(\rho) = g_2(\rho) - \frac{1}{4}\int_0^{\infty} S_2(t)M_2(\rho,t)dt \quad , \quad \rho>b \qquad (20)$$

where S_1 and S_2 are defined by

$$S_1(\rho) = \int_{\rho}^{\infty} \frac{t^{-n} f_1(t)dt}{(t^2-\rho^2)^{1/2}} \quad , \quad 0<\rho<a$$

$$S_2(\rho) = \int_0^{\rho} \frac{t^{n} f_2(t)dt}{(\rho^2-t^2)^{1/2}} \quad , \quad \rho>b$$

The functions M_1 and M_2 are defined by the relationships

$$K_1(\rho,t) = \frac{1}{(\rho t)^n}\int_0^{\rho}\int_0^{t} \frac{M_1(u,v)dudv}{(\rho^2-u^2)^{1/2}(t^2-v^2)^{1/2}} \qquad (21)$$

$$K_2(\rho,t) = (\rho t)^n\int_{\rho}^{\infty}\int_t^{\infty} \frac{M_2(u,v)dudv}{(u^2-\rho^2)^{1/2}(v^2-t^2)^{1/2}} \qquad (22)$$

Next, assuming two new functions $h_1(\rho)$ and $h_2(\rho)$ as

$$h_1(\rho) = \int_{\rho}^{\infty} \frac{t^{-n} f_2(t)dt}{(t^2-\rho^2)^{1/2}} \quad , \quad \rho>a$$

$$h_2(\rho) = \int_0^{\rho} \frac{t^{n} f_1(t)dt}{(\rho^2-t^2)^{1/2}} \quad , \quad 0<\rho<b$$

and using (18) the equations (19) and (20) finally can be written as

$$\rho^{2n} S_1(\rho) = g_1(\rho) - \frac{1}{4}\int_0^a S_1(t)M_1(\rho,t)dt + \frac{1}{4}\int_a^\infty h_1(t)M_1(\rho,t)dt \quad , \quad 0<\rho<a \qquad (23)$$

$$\rho^{-2n} S_2(\rho) = g_2(\rho) + \frac{1}{4}\int_0^b h_2(t)M_2(\rho,t)dt - \frac{1}{4}\int_b^\infty S_2(t)M_2(\rho,t)dt \quad , \quad \rho>b \qquad (24)$$

It can be shown that h_1 and h_2 will satisfy the equations

$$h_1(\rho) = \frac{n!}{\sqrt{\pi}\,\Gamma(n+\frac{3}{2})}\left[\rho^{-2n}\int_0^b \frac{wh_2(w)}{(\rho^2-w^2)} F\left(\frac{1}{2}, n, n+\frac{3}{2}, \frac{w^2}{\rho^2}\right) dw + \right.$$

$$\left. + \frac{d}{d\rho}\left\{\rho\int_b^\infty \frac{wg_2(w)}{(\rho^2+w^2)^{n+1}} F\left(\frac{n+1}{2}, \frac{n}{2}+1, n+\frac{3}{2}, \frac{4\rho^2 w^2}{(\rho^2+w^2)^2}\right) dw\right\}\right] \quad , \quad \rho>a \qquad (25)$$

and

$$h_2(\rho) = \frac{n!\rho^{2n+1}}{\sqrt{\pi}\,\Gamma(n+\frac{3}{2})}\left[\int_a^\infty \frac{h_1(w)}{(w^2-\rho^2)} F\left(\frac{1}{2}, n, n+\frac{3}{2}, \frac{\rho^2}{w^2}\right) dw - \right.$$

$$\left. - \frac{d}{d\rho}\left\{\rho\int_0^a \frac{w^{2n}\, g_1(w)}{(\rho^2+w^2)^{n+1}} F\left(\frac{n+1}{2}, \frac{n}{2}+1, n+\frac{3}{2}, \frac{4\rho^2 w^2}{(\rho^2+w^2)^2}\right) dw\right\}\right] \quad , \quad 0<\rho<b \qquad (26)$$

respectively provided g_1 and g_2 in these equations are now replaced by S_1 and S_2 respectively. These equations together with equations (23) and (24) thus give a set of four Fredholm integral equations for the functions h_1, h_2, S_1, S_2.

Now we pass on to the most interesting branch of Elastodynamics i.e. the diffraction of elastic waves by cracks which is of recent interest. Cracks are present in essentially all structural materials, either as natural defects or as a result of fabrication processes. In many cases, the cracks are sufficiently small so that their presence does not significantly reduce the strength of the material. In other cases, however, the cracks are large enough, or theymay become large enough through fatigue, stress corrosion cracking, etc., so that they must be taken into account in determining the strength. The body of knowledge which has been developed for the analysis of stresses in cracked solids is known generally as fracture mechanics. In recent years problems of diffraction of elastic waves by cracks are of considerable importance in view of their application in seismology and geophysics. Indeed in geophisical stratifications, faults occur at the interfaces while in manufactured laminates imperfections occur at the interface of the adjoining layers. This stress singularity near the edge of finite crack at the bimaterial interface is important in view of its practical application. Also the results of research on dynamic crack propagation are relevant in numerous applications. For example, a primary objective in engineering structures is to avoid a running fracture, or at least to arrest a running crack once it is initiated. Indeed there are several kinds of large engineering structures in which rapid crack growth is a definite possibility. In earth science, study of the elastic field near about the propagating fault is also important from the stand point of earthquake engineering.

Within the framework of a continuum model, such as the homogeneous, isotropic linearly elastic continuum, the classic analytical problem of fracture mechanics consists of the computation of the fields of stress and deformation in the vicinity of the tip of a crack, together with the application of a fracture criterion. In a conventional analysis inertia (or dynamic) effects are neglected and the analytical work is quasi-static in nature.

Because of loading conditions and material characteristics, however, there are many fracture mechanics problems which cannot be viewed as being quasi-static and for which the inertia of the material must be taken into account. Also inertia effects become of importance if the propagation of the crack is so fast, as for example in essentially brittle fracture, that rapid motions are generated in the medium. The label "dynamic loading" is attached to the effects of inertia on fracture due to repidly applied loads.

There are two broad classes of fracture mechanics problems that have to be treated as dynamic problems. These are concerned with

1. cracked bodies subjected to rapidly varying loads,
2. bodies containing rapidly propagating cracks.

In both cases the crack tip is an environment disturbed by wave motions.

Inpact and vibration problems fall into the first class of dynamic problems. In the analysis of such problems it is often found that at inhomogeneities in a body the dynamic stresses are higher than the stresses computed from the corresponding problem of static equilibrium. This effect occurs when a propagating mechanical disturbance strikes a cavity. The dynamic stress "overshoot" is especially pronounced if the cavity contains a sharp edge. For a crack the intensity of the stress field in the vicinity of the crack tip can be significantly affected by dynamic effects. In view of the dynamic amplification, it is conceivable that there are cases for which fracture at a crack tip does not occur under a gradually applied system of loads, but where a crack does indeed propagate when the same system of loads is rapidly applied, and gives rise to waves which strike the crack tip.

The second class of problems is equally important. Indeed, there are several kinds of large engineering structures in which rapid crack growth is a definite possibility. Examples are gas transmission pipelines, ship hulls, aircraft fuselages and nuclear reactor components. Dynamic effects affect the stress fields near rapidly propagating cracks, and hence the conditions for further unstable crack propagation or for crack arrest. Another area to which the analysis of rapidly propagating cracks is relevant is the study of earthquake mechanisms.

There have been a number of comprehensive review articles in the general area of elastodynamic fracture mechanics. Some references are Achenbach (1972), Freund (1975), Achenbach (1976), Freund (1976) and Kanninen (1978).

At present, dynamic fracture mechanics solutions are largely confined to conditions where linear elastic fracture mechanics (LEFM) is valid. These are appropriate when the plastic deformation attending the crack tip is small enough to be dominated by the elastic field surrounding it. Problems of crack growth initiation under impact loads and of rapid unstable crack propagation and arrest can be treated with LEFM by using dynamically computed fields of stress and deformation. Engineering structures requiring protection against the possibility of large-scale catastrophic crack propagation are, however, generally constructed of ductile, tough materials. For the initiation of crack growth, LEFM procedures can give only approximately correct predictions for such materials. The elastic-plastic treatments required to give precise results have not yet been developed in a completely acceptable manner, even under static conditions.
The shapes of the cracks which have been studied uptil now are as follows :

(i) Semi-infinite plane cracks.
(ii) Finite Griffith cracks.
(iii) Penny shaped and annular cracks.
(iv) Non-planar cracks.

A transient problem involving the sudden appearance of a semi-infinite crack in a stretched elastic plate was considered by Maue (1954). Baker (1962) studied the problem of a semi-infinite crack suddenly appearing and growing at constant velocity in a stretched elastic body. The mixed boundary value problem is solved by transform methods including the Weiner-Hopf and Cagniard techniques. Atkinson and List (1978) considered the steady state semi-infinite crack propagation into media with spatially varying elastic properties. The quasi-static problem of an infinite elastic medium, weakened by an infinite number of semi-infinite, rectilinear, parallel and equally spaced cracks which are subjected to identic loads satisfying the conditions of antiplane state of strain was solved by Matczynski (1973). Sarkar, Ghosh and Mandal (1991) studied the problem of scattering of horizontally polarized shear wave by a semi-infinite crack running with uniform velocity along the interface of two dissimilar semi-infinite elastic media.

The powerful technique to solve the diffraction problem of semi-infinite crack is the Wiener-Hopf (1958) technique.

The in-plane problem of finite Griffith crack propagating at a constant velocity under a uniform load was first solved by Yoffe (1951). Sih (1968) has also provided a Riemann-Hilbert formulation of the same problem where both in-plane extensional and antiplane shear loads were considered.
Other reference treating elastodynamic problem involving a single finite Griffith crack are Loeber and Sih (1967), Ang and Knopoff (1964), Mal (1970, 1972) Chang (1971), Kassir and Bandyopadhyay (1983), Kassir and Tse (1983). Loeber and Sih (1967) solved the

problem of diffraction of antiplane shear waves by a finite crack by using integral transform method. Kassir and Bandyopadhyay (1983) considered the problem of impact response of a cracked orthotropic medium. Laplace and Fourier transforms were employed to reduce the transient problem to the solution of standard integral equation in the Laplace transform plane and was solved by Laplace inversion technique (Krylov et al, 1957); Miller and Gyuy, 1966).

The problems of finite Griffith crack lying at the interface of two dissimilar elastic media have been studied by Srivastava, Palaiya and Karaulia (1980), Nishida, Shindo and Atsumi (1984) and Bostrom (1987). Bostrom (1987) used the method of Krenk and Schmidt (1982) to solve the two-dimensional scalar problem of scattering of elastic waves under antiplane strain from an interface crack between two elastic half-spaces. Sih and Chen (1980) analyzed the dynamic response of a layered composite containing a Griffith crack under normal and shear impact.

The Problems of diffraction of elastic waves become more complicated when boundaries are present in the medium. Chen (1978) considered the problem of dynamic response of a central crack in a finite elastic strip. The crack was assumed to appear suddely when the strip is being stretched at its two ends. The problem was solved by Laplace and Fourier transform technique. Some other references are Srivastava, Gupta and Palaiya (1981), Srivastava, Palaya and Karaulia (1983), Shindo, Nozaki and Higaki (1986), De and Patra (1990).

Different techniques have been applied by many authors to tackle these type of crack problems. From these stand point, these problems may be divided into two categories : one for low frequency oscillation of the source or long wave scattering or transmission and the other for high frequency oscillation or short wave scattering or transmission in the medium. The term long and short are used in comparison to the region of the source of disturbance or the size of the crack or strip etc. inside the medium to the wave length of disturbance. The useful techniques for low frequency scattering due to Noble (1963) at Tranter (1968) have been discussed earlier. In case of high frequency oscillations Wiener-Hopf (Noble, 1958) technique and Keller's (1958) geometrical theory are found to be most suitable. Here we briefly discuss the methods.

Wiener-Hopf Method :

The typical problem obtained by applying Fourier transforms to partial differential equations is the following. One shall have to find unknown functions $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ satisfying

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\Psi_-(\alpha) + C(\alpha) = 0 \tag{27}$$

Where this equation holds in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ of the complex α-plane, Φ_+ is regular in the half-plane $\tau > \tau_-$,

$\Psi_-(\alpha)$ is regular in $\tau < \tau_+$, and certain information which will be specified later is available regarding the behaviour of these functions as α tends to infinity in appropriate half-planes. The functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ are given function of α, regular in the strip. For simplicity let us assume that A, B are also non-zero in the strip.

The fundamental step in the Wiener-Hopf procedure for solution of this equaion is to find $K_+(\alpha)$ regular and non-zero in $\tau > \tau_-$, $K_-(\alpha)$ regular and non-zero in $\tau < \tau_+$, such that

$$A(\alpha)/B(\alpha) = K_+(\alpha)/K_-(\alpha) \tag{28}$$

Use (28) to rearrange (27) as

$$K_+(\alpha)\Phi_+(\alpha) + K_-(\alpha)\Psi_-(\alpha) + K_-(\alpha)C(\alpha)/B(\alpha) = 0 \tag{29}$$

Decompose $K_-(\alpha)C(\alpha)/B(\alpha)$ in the form

$$K_-(\alpha)C(\alpha)/B(\alpha) = C_+(\alpha) + C_-(\alpha) \tag{30}$$

where $C_+(\alpha)$ is regular in $\tau>\tau_-$, $C_-(\alpha)$ is regular in $\tau<\tau_+$.

With the help of (30) rearrange (29) so as to define a function $J(\alpha)$ by

$$J(\alpha) = K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha) = - K_-(\alpha)\Psi_-(\alpha) - C_-(\alpha) \tag{31}$$

So far this equation defines $J(\alpha)$ only in the strip $\tau_- < \tau < \tau_+$. But the second part of the equation is defined and is regular in $\tau > \tau_-$, and the third part is defined and is regular in $\tau < \tau_+$. Hence by analytic continuation $J(\alpha)$ must be regular in the whole α-plane. Then by the extended form of Liouville's theorem $J(\alpha)$ is a polynomial $p(\alpha)$

$$K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha) = p(\alpha)$$

$$K_-(\alpha)\Psi_-(\alpha) + C_-(\alpha) = -\,p(\alpha) \tag{32}$$

These equations determine $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ to within the arbitrary polynomial $p(\alpha)$, i.e. to within a finite number of arbitrary constants which must be determined otherwise.

<u>Keller's geometrical method</u> :

Keller's theory of geometrical diffraction applied to elastodynamics states that the two conical surfaces of diffracted rays are generated when an incident ray strikes an edge. The surface of the inner cone consists of rays of longitudinal motion, while the surface of the outer cone is composed of rays of transverse motion. The half-angles of the cones are related by Snell's law. Fig.1 shows the cones generated by an incident longitudinal ray. For this case the diffracted longitudinal rays make the same angle ϕ_L with the tangent to the edge as the incident ray, and the diffracted rays of transverse motion are under an angle ϕ_T with the edge, where $C_L \cos\phi_T = C_T \cos\phi_L$.
For a straight diffracting edge, and an incident longitudinal ray, the diffracted displacement fields are related quantitatively to the incident field by

$$\vec{u}^L_d = e^{i\omega S_1/C_L} \left[S_1 (1+S_1/R_i) \right]^{-1/2} D_L \, \hat{i}^d_L \, A \, e^{i\omega (S_0/C_L - t)}$$

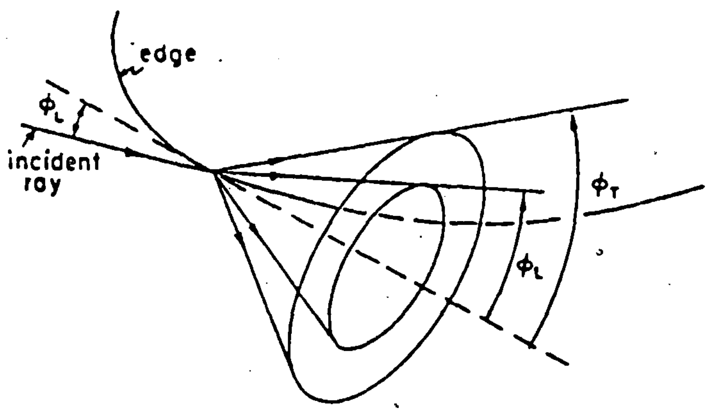

FIG. 1. Cones of diffracted longitudinal and transverse rays for an incident longitudinal ray.

$$\vec{u}_d^T = e^{i\omega S_2/C_T} \left[S_2(1+S_2/R_d)\right]^{-1/2} D_T \hat{i}_T^d A e^{i\omega(S_0/C_L - t)}$$

Here Aexp $[i\omega(S_o/C_L-t)]$ defines the amplitude and the phase of the incident field at the point of diffraction, and D_L and D_T are diffraction coefficents which relate the diffracted field to the incident field. Also S_1 and S_2 are the smaller of the principal radii of curvature of the diffracted wave front, or equivalently the distances along the diffracted rays from the points of diffraction to the observation point. The unit vectors $\hat{i}_L^d$ and $\hat{i}_T^d$ relate the directions of displacement of the diffracted field to the direction of displacement of the incident field. For a straight diffracting edge R_i is the radius of curvature at the point of diffraction of the curve formed by the intersection of the incident wave front and the plane which contains the incident ray and the edge, and

$$R_d = R_i \frac{\sin\phi_T \ \tan\phi_T}{\sin\phi_L \ \tan\phi_L}.$$

Papers involving the diffraction of elastic waves by two coplanar Griffith cracks are very few. Researches have been restricted to those of a single crack, because of the severe mathematical complexity encountered in finding solutions for two or more cracks. At first Jain and Kanwal (1972) overcame the difficulty and presented the solution for the diffraction problem of normally incident longitudinal and antiplane shear waves by two symmetrical coplanar Griffith cracks located in an infinite, isotropic and homogeneous elastic medium. However, they presented an approximate solution which is valid for low-frequency. Itou (1978) also studied the dynamic problem for an infinite elastic medium weakened by two coplanar Griffith cracks in which a self-equilibrated system of pressure is varied harmonically with time. To solve this problem, the author has expanded the surface displacement in a series of functions which is automatically zero outside the cracks and has used the Schmidt method. Itou (1980,1980) also solved two different problems involving two finite cracks. The problem of determining the transient stress distribution in an infinite elastic medium weakened by two coplanar Griffith cracks has been reduced to the following integral equation

$$\sum_{n=1}^{\infty} c_n(s)\left[-\frac{4c_L^3}{k^2 s^2 b}\int_0^{\infty} g(s,\xi)\sin\left(\frac{a+b}{2}\,\xi-\frac{n\pi}{2}\right)J_n\left(\frac{b-a}{2}\,\xi\right)\cos(\xi x)\,d\xi\right]$$

$$= -\,Pc_L(bs)\,,\qquad a<x<b \tag{33}$$

with

$$g(s,\xi) = \frac{\left[\,\xi^2+k^2s^2/(2c_L^2)\right]^2-\xi^2\gamma_1\gamma_2}{\xi\gamma_1} \tag{34}$$

where locations of the cracks are $a\le|x|\le b$, $|y|<\infty$, $z=0$,

$c_L = \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2}$, $c_T = \left(\frac{\mu}{\rho}\right)^{1/2}$, $k = c_L/c_T$ and $c_n(s)$ are the unknown coefficients.

To determine the coefficients $c_n(s)$ by Schmidt's method (1958) equation (33) can be rewritten as

$$\sum_{n=1}^{\infty} c_n(s)E_n(s,x) = - u(s,x) \quad , \quad a<|x|<b \tag{35}$$

where $E_n(s,x)$ and $u(s,x)$ are known functions and the coefficients $c_n(s)$ are unknown.
A set of functions $P_n(s,x)$ which satisfy the orthogonality condition

$$\int_o^b P_m(s,x)P_n(s,x)dx = N_n \, \delta_{mn} \quad , \quad N_n = \int_a^b P_n^2(s,x)dx \tag{36}$$

can be constructed from the function, $E_n(s,x)$, such that

$$P_n(s,x) = \sum_{n=1}^{\infty} \frac{M_{in}}{M_{nn}} E_i(s,x) \tag{37}$$

where M_{in} is the cofactor of the element d_{in} of D_n , which is defined as

$$D_n = \begin{vmatrix} d_{11} & d_{12} & \ldots\ldots & d_{1n} \\ d_{21} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ d_{n1} & \ldots\ldots\ldots\ldots & & d_{nn} \end{vmatrix} \tag{38}$$

$$d_{in} = \int_a^b E_i(s,x)E_n(s,x) \; dx \; .$$

Using equations (35) and (36) one can obtain

$$c_n(s) = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \tag{39}$$

with
$$q_j = \frac{-1}{N_j} \int_a^b u(s,x) P_j(s,x)\, dx \tag{40}$$

Recently, Verma and Jain (1990) have studied the problem of diffraction of obliquely incident longitudinal waves by two equal, parallel, and coplanar Griffith cracks located in an infinite, isotropic and homogeneous elastic medium. Using Green's functions the solution of this problem was first reduced to that of a pair of similar Fredholm integral equations of the first kind. When the wavelengths are large compared to the distance between the two outer edges of the two cracks, each integral equation of this pair was transformed to a set of Fredholm integral equations of the first kind with a simple kernel, which was solved by the inversion formula (Lowengrub and Srivastava, (1968).

Problems involving more than two finite cracks i.e. periodic array of coplanar finite cracks have been studied by Angel and Achenbach (1985), De Sarkar (1983), Garg (1981), Parihar and Lalitha (1987), Parton and Morozov (1978).

Another type of crack called cruciform crack has been studied by Brock and Deng (1985), Ong and Srivastava (1985).

Now we discuss the diffraction problem due to penny-shaped or annular cracks.

The transient stress and displacement fields around an embedded crack in the shape of a circle were first investigated by Embley and Sih (1971) for extensional impact and by Sih and Embley (1971) for torsional impact. Their method of solution involves isolating the singular portion of the dynamic stresses in the Laplace transform domain such that the dynamic stress intensity factor can be obtained by direct application of the Laplace inversion theorem. A penny shaped crack with its plane normal to the stretched direction of the elastic solid expanding at a constant velocity was considered by a number of investigators, namely Craggs (1966), Kostrov (1964) and Atkinson (1968). Sih and Loeber (1969) solved the problem of normal compression and radial shear waves impinging on a penny shaped crack. Other references are Mal (1970), Krenk and Shmidt (1982), Arin and Erdogan (1971), Ueda, Shindo and Atsumi (1983).

Krenk and Schmidt (1982) solved the problem of scattering of waves by a circular crack in an elastic medium by a direct integral equation method. The solution method was based on expansion of stresses and displacements on the crack surface in terms of trigonometric functions and orthogonal polynomials.

The general problem of two semi-infinite elastic media with different properties bonded to each other along a plane and containing a series of concentric ring shaped flat cavities was considered by Erdogan (1965), Using Green's functions for the semi-infinite plane, the problem was formulated as a system of simultaneous singular integral equations having cauychy type singularities. Shindo (1979,1981,1981) has studied different types of problem in elastodynamics involving flat annular crack. The problem of diffraction of normally incident torsional waves by a flat annular crack embedded in an infinite, isotropic, and homogeneous elastic medium was studied by Shindo (1979). The problem was reduced to that of solving the following triple integral equation :

$$\int_0^\infty \alpha^2 A(\alpha) J_1(\alpha r)\,d\alpha = \int_0^\infty \alpha g(\alpha) A(\alpha) J_1(\alpha r)\,d\alpha + (P_{30}/\mu)(r/b) \quad , \quad a<r<b$$

$$\int_0^\infty \alpha A(\alpha) J_1(\alpha r)\,d\alpha = 0 \quad , \quad 0\le r\le a \;,\; b\le r \tag{41}$$

Where $g(\alpha) = \alpha - \gamma(\alpha)$ in which the function $g(\alpha)$ has the order α^{-1} for large α. It is convenient to write the integral transform $A(\alpha)$ in terms of the finite integral given by

$$\alpha A(\alpha) = - \int_a^b t\phi(t) J_2(\alpha t)\,dt \tag{42}$$

where J_2 is the second-order Bessel function of the first kind. On substitution of (42) in (41) yields the following singular integral equation of the first kind :

$$\frac{1}{\pi}\int_a^b \frac{1}{t}\,\phi(t)\left[\frac{b}{t-r} - \frac{3b}{2r}\log\left|\frac{2(t-r)}{(1-a_o)b}\right| + m_o(r,t) + m_1(r,t)\right]dt$$

$$= -P_{30}/\mu \;, \quad a<r<b \tag{43}$$

in which the Fredholm Kernels $m_o(r,t)$ and $m_1(r,t)$ are bounded in closed interval $a \le r,\ t \ge b$ and are given by :

$$m_o(r,t) = b\left[\left(\frac{(r/t)}{r+t} - \frac{2}{r}\right)E(t/r) + \frac{(r/t)E(t/r) - 1}{t-r} + \frac{4K(t/r)}{r} + \right.$$

$$\left. + \frac{3}{2r}\log\left|\frac{2(r-t)}{(1-a_o)b}\right|\right] , \quad t < r$$

$$= b\left[\left(\frac{1}{t+r} - \frac{2t}{r^2}\right)E(r/t) + \frac{E(r/t) - 1}{t-r} + \frac{2tK(t/r)}{r^2} + \right.$$

$$\left. + \frac{3}{2r}\log\left|\frac{2(r-t)}{(1-a_o)b}\right|\right] , \quad t > r \tag{44}$$

$$m_1(r,t) = -\frac{\pi t^2 b}{r}\int_0^\infty g(\alpha)J_2(\alpha t)J_1(\alpha r)d\alpha \tag{45}$$

Here K and E are the complete elliptic integrals of the first and second kind respectively, and $a_o = a/b$ is the radius ratio of the annular crack. From the second equation of (41) and the definition (42) it is clear that the integral equation must be solved under the following single valuedness condition :

$$\int_a^b \frac{1}{t}\,\phi(t)dt = 0 \tag{46}$$

Substituting

$$R = r/b = \frac{1}{2}(1-a_o)s + \frac{1}{2}(1 + a_o)$$

$$T = t/b = \frac{1}{2}(1-a_o)r + \frac{1}{2}(1 + a_o)$$

$$P = bp/cT \tag{47}$$

$$\Phi(t) = \phi(t)/\left[(t/b)(p_{30}/\mu)\right]$$

the revised singular integral equation to the first kind (43) and single valuedness condition (46) are shown as

$$\frac{1}{\pi}\int_{-1}^{1} \Phi(\tau)\left[\frac{1}{\tau-s} - \frac{3(1-a_o)}{4R}\log|\tau-s| + \frac{1-a_o}{2}\left(M_o(s,\tau) + M_1(s,\tau)\right)\right]d\tau$$

$$= -1^{\circ} \tag{48}$$

$$\int_{-1}^{1} \Phi(\tau)\, d\tau = 0 \tag{49}$$

in which the Fredholm Kernels $M_o(s,\tau)$ and $M_1(s,\tau)$ are obtain by substituting (47) in (44) and (45) respectively. Thus the problem was reduced to the solution of the singular integral equation (48) under additional condition (49).

Following Erdogan (1973, 1963) the equations (48) and (49) has been solved by assuming

$$\Phi(\tau) = \frac{1}{(1-\tau^2)^{1/2}}\left[A_o + \sum_{n=1}^{\infty} A_n T_n(\tau)\right] \tag{50}$$

where $T_n(\tau)$ are Chebyshev polynomial of the first kind and A_n(n=0,

1, 2, ...) are unknown constants.
Bostrom and Olsson (1987) treated the problem of scattering of elastic waves by non-planar cracks. The method employed was a modification of the null field approach (T matrix method) where a fictitious surface was added to the surface of the crack so as to obtain a closed surface that should preferably be as sphere-like as possible.

Like elastic waves, diffraction of viscoelastic waves by crack or by inclusions are of considerable importance in view of their application of Seismology and Geophysics. Also the problems involving the motion of a punch on the surface of a viscoelastic half-space or on the free boundaries of long strips are extremely important in view of their application in road construction technology. Considerable studies had been made in the case of homogeneous media. But natural or artificial materieals are generally inhomogeneous. In addition, if the materials be dissipative, that effect can well be taken into account by considering the material to be viscoelastic.

Interest in the propagation of mechanical disturbances in viscoelastic media is comparatively recent in origin. The behaviour of extended anelastic structures under conditions of dynamic stressing is a development of the last fifty years. This situation contrasts with that obtained in the related field of linear elasticity in which technological requirements (e.g. the behaviour of brigdes under moving loads, the stresses involved in reciprocating mechanisms and the response of metals to shock loading) has stimulated much research into dynamical problems throughout the last hundred years. Perhaps the most important reason for the failure of a formal mathematical theory of viscoelasticity to develope stems from the lack of just such a practical stimulus. Traditionally, engineering design has made use of materials whose properties are adequately described in the working range by the laws of classical elasticity. However, with the gradual introduction of engineering compnents fabricated from the new synthetic plastic materials, it seems probable that the study of dynamic viscoelasticity will become a subject of increasing importance.

Other pertinent reasons contributing to the lack of a formal theory of dynamic viscoelasticity are the relative complexity of the equations describing the fundamental mechanical properties of anelastic solids and a lack of knowledge of these properties for many of the common materials. Fortunately, with the introduction of integral transform techniques by Gross (1947), a major simplification has been effected in handling the mathematical aspects of the subject. At the same time much experimental work, mostly dating from the publication of a paper by Alexandrov and Lazurkin (1940) on the mechanical properties of rubber, has been devoted to elucidating and classifying the behaviour of many of the rubber and polymer type materials. Here we give some references of papers on experimental investigation of some solids. Nolle (1949) (work on rubbers), Zener (1948) (metals), Sherby and Dorn (1958) (perpex) and Leadermann (1943)(silk, rayon, nylon). A generalized viscoelastic solid is specified by the existence of a functiuonal equation of state connecting stress (σ), strain (ε), time (t) and temperature (T).

$$F(\sigma,\varepsilon,t,T) = 0.$$

Presupposition of the existence of such a relation which may include time differential and integral operators of arbitrary order, immediately excludes problems associated with the plastic deformation of metals and the fracture of solids.

The simplest examples of linear viscoelastic solids are well known, e.g. Voigt solid, Maxwell solid, standard linear solid, Burgers solid, Newtonian fluid, etc. Diagrams of some models are shown in figure 2.

Recently and extensive study on boundary value problems in linear viscoelasticity has been made by Golden and Graham (1988) in their book. The problem of a rigid cylinder rolling on the surface of a viscoelastic half-space has been solved by Hunter (1961). The contact problem of rigid cylinder rolling slowly on a thin viscoelastic layer has been treated by Alblas and Kuipers (1970) assuming that the layer thickness is small compared to the width of the contact region of the cylinder. some contact punch problems in viscoelastic medium have been studied by Golden (1977, 1979,

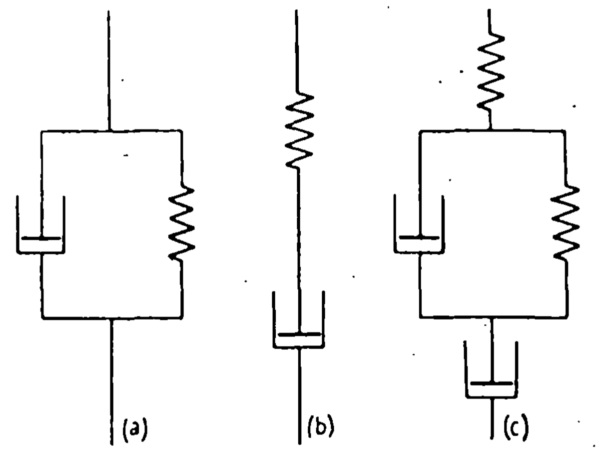

FIG. 2. Models of visco-elastic solids. (*a*) Voigt solid; (*b*) Maxwell solid; (*c*) more general solid.

1982). Crack propagation in viscoelastic medium has been studied by Willis (1972), Atkinson and List (1972), coussy (1987) and others. Willis (1972) considered steady-state Mode III crack propagation for a standard linear solid under general type of loading on the crack surfaces. Atkinson and List (1972) studied nonsteady SH-wave type crack propagation starting at t = 0 and moving with a constant velocity in the 'Maxwell solid' or using the viscoelastic model suggested by Achenbach and Chao. Sills and Benveniste (1981) and Coussy (1987) studied steady state crack propagation of SH-type at the interface between two visco-elastic media.

Recently, the transient elastodynamic stress intensity factor was determined for a cracked linearly viscoelastic body under impact by Georgiadis, Theocaris and Mouskos (1991). The body was considered to be infinite containing a finite crack. The solution was obtained by correspondence principle and the use of the Dubner-Abate-Crump Laplace-transform inversion technique.

In the thesis presented here we have studied some mixed boundary value problems in elastodynamics involving punches, inclusions and cracks. The work has been presented in three chapters. The first two chapters I and II deals with diffraction problems in elastic medium and the third chapter deals with diffraction problems in viscoelastic medium. Here we give the summeryof the thesis chapter wise.

In chapter-I, problem-1 contains vertical vibration of two rigid strips in smooth contact with a semi-infinite elastic medium. It is assumed that motion is forced by prescribed displacement distribution $v_o e^{-i\omega t}$ normal to the two strips located in the region $-a \le x \le -b$, $b \le x \le a$, $y = 0$, $|Z| < \infty$, where v_o is constant. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro-differential equation. Finally iterative solution valid for low frequency has been obtained. The integral equation was solved in a manner similar to that employed by Lowengrub and Srivastave (1968) in solving static problems for two caplanar cracks in an infinite elastic medium. From the solution of the integral equation, we have found out stresses just below the strips and also the vertical displacement at point outside the strips on the free surface. Low frequency solution due to antiplane motion of two strips on a semi-infinite elastic medium has also been derived.

In paper-2, we have considered the problem of diffraction of elastic waves by a apair of coplanar rigid strips between two homogeneous elastic half spaces for the case of antiplane strain. The resulting triple integral equation has been reduced to the solution of an integro differential equation and approximate solution has been obtained. These solutions have been used to obtain approximate values of the displacement field and also the stress intensity factors at the edges of the strips.

Making the distance between the inner edges of the strips tend to zero, the diffraction problem for a single rigid strip has been obtained. Even this result of the limiting case appears to have been presented here for the first time.

In third paper of this chapter we have studied the two dimensional problems of diffraction of elastic waves by four coplanar parallel rigid strips moving steadily on the free surface of a semi-infinite isotropic elastic medium. By Fourier transform the five part mixed boundary value problem has been reduced to the solution of a set of four integral equations. Following the technique, developed by Srivastava and Lowengrub (1970), the quadruple integral equations have been solved. The normal stress under the strips and displacement outside the strips are derived in closed form. The effect of stress intensity factors at the edges of the strips is shown by means of graphs. Also letting the strip velocity tend to zero the results for statical problem have been presented in this paper as a particular case.

In the last problem, i.e., paper-4 of chapter-I, we investigated the diffraction of torsional wave by a rigid annular disc at the interface of two bonded dissimilar elastic media. Here we have assumed that an antiplane shear wave given by $\Omega_2 r e^{ik_2(z-c_2 t)}$, where Ω_2 is a constant, $k_2=\omega/c_2$ and $c_2=\surd(\mu_2/\rho_2)$, the shear wave velocity in medium 2, be incident normally on the annular rigid disc of inner and outer radii b and a respectively. Applying the method developed by Williams (1963) and used subsequently by Thomas (1965) and Jain et al (1970), the three part mixed boundary value problem has been reduced to the solution of a set of integral equations. The solutions of these integral equations are obtained iteratively for low frequency and small values of the ratio of the inner and outer radii of the disc. These solutions are used to determine the jump in stresses across the annular disc and stress intensity factors at both the edges of the disc. Torque and far field amplitudes in both the media have also been deduced. The effect of normalized frequency, material properties and geometric parameters in stress intensity factors and far field amplitude are shown graphically.

First problem of chapter-II deals with the interaction of normally incident time harmonic elastic waves with a periodic array of coplanar Griffith cracks in an infinite orthotropic medium. Due to geometrical symmetry the problem has been reduced to the solution of the problem of a single crack in a strip whose boundaries are shear free and constrained in a way not to permit normal displacement. Fourier transform has been used to reduce the problem to the solution of dual integral equations. By the application of Abel's integral the dual integral equations finally has been converted to a Fredholm integral equation. Stress intensity factor at the tip of the crack and crack opening displacement have been derived in closed form. To display the influence of the material orthotropy numberical values of stress intensity factor and crack opening displacement have been derived in closed form. To display the influence of the material orthotropy numerical values of stress intensity factor and crack opening displacement have been found out after solving the Fredholm integral equation numerically and plotted against dimensionaless frequency, distance respectively for three sets of orthotropic materials.

In the second paper of chapter II we have studied the diffraction of normally incident SH-waves by a Griffith crack situated in an infinitely long inhomogeneous elastic strip. The shear modulus (μ) and the density (ρ) of the material have been assumed to vary both in horizontal and vertical directions. Applying Fourier transform the mixed boundary value problems has been converted to the solution of dual integral equations. The dual integral equations finally has been ruduced to a Fredholm integral equation of second kind by applying Abel transform. Expressions for stress intensity factor and crack opening displacement have been derived. The numerical values of stress intensity factor and crack opening displacement have been depicted by means of graphs to show the effect of material inhomogeneity.

In chapter-III, first paper deals with the analysis of the stress and displacement field produced by a long punch moving on the boundary of a semi-infintie viscoelastic medium and producing Horizontal Shear waves. Two types of viscoelastic models viz. Maxwell Solid and Standard Linear Solid have been considered and loading in assumed to be such that Mode III conditions prevail. The mathematical technique which is employed here consists of the application of integral transforms and the solution of the resulting Wiener-Hopf equations for the transformed unknown variables. Both the steady and nonsteady solutions of the problem have been derived. Displacement and stress on the free surface and at points below the punch have been derived analytically and the nature of their variations with the velocity of the moving punch has been shown by means of graphs.

The last paper of the this chapter contains the analysis of steady and nonsteady cases of Mode III crack propagation in an inhomogeneous viscoelastic medium. Two types of viscoelastic models, namely Maxwell solid and Standard Linear solid have been considered. Material properties have been assumed to vary exponentially in the direction perpendicular to the direction of crack propagation. The problem has been solved by using Wiener-Hopf technique. We have studied how the material inhomogeneity affects the stress intensity factor and also the crack opening displacement when a Mode III type crack propagates through the inhomogeneous viscoelastic medium.

With this much of introduction, we now present the thesis chapterwise. References given in the thesis do not include all the previous workers in this line. But attempt has been made to include most of them.

CHAPTER - I

SOME CONTACT PROBLEMS IN ELASTODYNAMICS

Paper - 1 : Forced vertical vibration of two rigid strips on a semi-infinite elastic solid.

Paper - 2 : Diffraction of antiplane shear wave by a pair of parallel rigid strips at the interface of two bonded dissimilar elastic media.

Paper - 3 : On steady motion of four rigid strips on the surface of a semi-infinite elastic medium.

Paper - 4 : Diffraction of torsional elastic waves by a rigid annular disc at the bimaterial interface.

FORCED VERTICAL VIBRATION OF TWO RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

1. INTRODUCTION

The study of the effect of vibrating source of pressure in different forms on the surface of an elastic medium is almost classical. Reissner (1937) and later Millar and Pursey (1954) treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. The problem of most physical interest occurs when displacement corresponding to indentation by a rigid body are prescribed over a given region and the surface tractions outside the region are zero. Analytical treatment on the dynamical response of footing and soil structure interaction are usually available in the literature only for circular and elliptical footings and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, also in the study of the vibration of dams and large structures subjected to earthquakes. Awojobi and Grootenhuis (1965), Robertson (1966), Gladwell (1968) and others have considered the problem of circular footing in detail. Roy (1986) considered the dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. For low frequency, both vertical and horizontal vibration were

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treated. Low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium has been obtained by Karasudhi, Keer and Lee (1968) by reducing the governing dual integral equations into a single inhomogeneous Fredholm equation of the second kind. Wickham (1977) however removed the flaws occuring in the above paper and worked out in detail the problem of forced two dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium using a different technique. To improve the dynamic models of buildings and other structures, it will be fruitful to have analytical results for foundations of more complicated nature. In this paper we have considered the problem of vertical vibration of two rigid strips in smooth contact with a semi-infinite elastic medium. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by the running wheels on a railway track. The resulting mixed boundary value problem is reduced to the solution of a triple integral equation which has further been reduced to the solution of an integro differential equation. Finally iterative solution valid for low frequency has been obtained. The integral equation was solved in a manner similar to that employed by Lowengrub and Srivastava (1968) in solving static problems for two coplanar cracks in an infinite elastic medium. Jain and Kanwal (1972,1972) also used the same technique to solve the problem of diffraction of elastic waves by two coplanar Griffith cracks and also by two coplanar rigid strips in an infinite elastic medium. It may be mentioned in this connection that recently Itu (1980) has also solved the problem of diffraction of SH-waves by two coplanar Griffith cracks in an infinite elastic medium using a different technique.

From the solution of the integral equation, we have found out stresses just below the strips and also the vertical displacement at point outside the strip on the free surface. Finally, making the distance between the strips tend to zero, we have found our results becoming idetical with the results given by Wickham (1977) for the vertical vibration of a single strip on a semi-infinite elastic medium. Low frequency solution due to antiplane motion of two strips on a semi-infinite elastic medium has also been derived.

2. FORMULATION OF THE INPLANE PROBLEM

Let us consider the normal vibration of frequency ω of two rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty<X<\infty$, $Y\geq 0$, $-\infty<Z<\infty$. It is assumed that motion is forced by prescribed displacement distribution $v_o e^{-i\omega t}$ normal to the two infinite strips located in the region $-a\leq X\leq -b$, $b\leq X\leq a$, $Y=0$, $|Z|<\infty$ (Fig.1), where v_o is constant. Normalizing all lengths with respect to a and putting $b/a=c$, we find that the rigid strips are defined by $c\leq|x|\leq 1$, $y=0$, $|z|<\infty$.

Suppressing the time factor $e^{-i\omega t}$ throughout the analysis, the displacement components can be written as

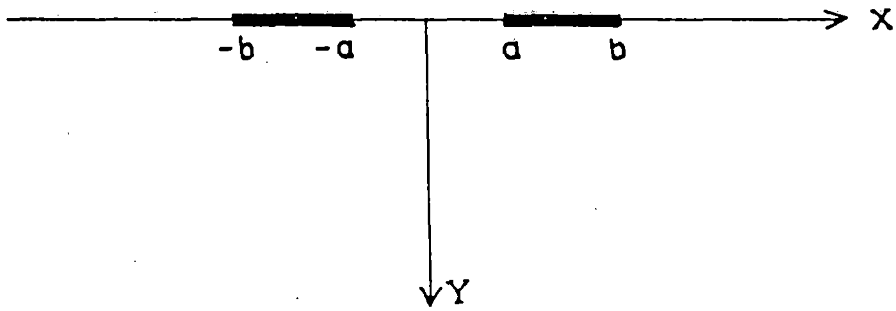

Fig 1. Geometry of the strips .

$$u(x,y) = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}$$

$$v(x,y) = \frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x} \qquad (1)$$

$$w(x,y) = 0$$

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k_1^2\phi = 0$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k_2^2\psi = 0 \qquad (2)$$

where $k_1^2 = \omega^2 a^2/c_1^2$ and $k_2^2 = \omega^2 a^2/c_2^2$.

Consequently, the values of stress components τ_{xy} , τ_{yy} and τ_{yz} are

$$\tau_{xy} = \mu\left[2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} \right]$$

$$\tau_{yy} = -\mu\left[\left(k_2^2 + 2\frac{\partial^2}{\partial x^2} \right)\phi - 2\frac{\partial^2\psi}{\partial x\partial y} \right] \qquad (3)$$

$$\tau_{zy} = 0$$

The boundary conditions are

$$v(x,0) = v_0 \quad , \qquad c\le|x|\le 1$$

$$\tau_{yy}(x,0) = 0 \quad , \qquad |x|<c \ , \ |x|>1 \qquad (4)$$

$$\tau_{xy}(x,0) = 0 \quad , \qquad -\infty<x<\infty$$

Solution of the Helmholtz equation given by (2) can be written as

$$\phi = \int_{-\infty}^{\infty} A(\xi)\ \exp(i\xi x - \gamma_1 y)\ d\xi$$

$$\psi = \int_{-\infty}^{\infty} B(\xi)\ \exp(i\xi x - \gamma_2 y)\ d\xi \tag{5}$$

where
$$\gamma_j = (\xi^2 - k_j^2)^{1/2}, \quad |\xi| \geq k_j,$$
$$= -i(k_j^2 - \xi^2)^{1/2}, \quad |\xi| \leq k_j, \qquad j=1,2 \tag{6}$$

and $A(\xi)$, $B(\xi)$ are unknown functions to be determined from the boundary conditions.

Using the last boundary condition of (4) it can be shown that

$$B(\xi) = \frac{-2i\xi\gamma_1}{\xi^2 + \gamma_2^2}\ A(\xi).$$

Now the displacements and stresses given by (1) and (3) become

$$u(x,y) = \int_{-\infty}^{\infty} i\xi \left[\exp(-\gamma_1 y) - \frac{2\gamma_1\gamma_2}{\xi^2 + \gamma_2^2}\exp(-\gamma_2 y)\right] A(\xi)\exp(i\xi x)d\xi \tag{7}$$

$$v(x,y) = -\int_{-\infty}^{\infty} \gamma_1 \left[\exp(-\gamma_1 y) - \frac{2\xi^2}{\xi^2 + \gamma_2^2}\exp(-\gamma_2 y)\right] A(\xi)\exp(i\xi x)d\xi \tag{8}$$

$$\tau_{yy}(x,y)=-\mu\int_{-\infty}^{\infty}\left[(k_2^2-2\xi^2)\exp(-\gamma_1 y)+\frac{4\xi^2\gamma_1\gamma_2}{\xi^2+\gamma_2^2}\exp(-\gamma_2 y)\right]A(\xi)\exp(i\xi x)d\xi \tag{9}$$

$$\tau_{xy}(x,y) = \mu\int_{-\infty}^{\infty}2i\xi\gamma_1\left[-\exp(-\gamma_1 y) + \exp(-\gamma_2 y)\right]A(\xi)\exp(i\xi x)d\xi \tag{10}$$

Next using the fact that $A(\xi)$ is an even function of ξ, and putting

$$P(\xi) = \frac{(2\xi^2-k_2^2)^2 - 4\xi^2\gamma_1\gamma_2}{2\xi^2-k_2^2}A(\xi) \tag{11}$$

the first and second boundary conditions given by (4) lead to the following dual integral equations in $P(\xi)$:

$$\int_0^{\infty}P(\xi)\cos\xi x\ d\xi = 0 \quad , \qquad |x|<c \ , \ |x|>1 \tag{12}$$

$$\int_0^{\infty}\frac{\gamma_1 k_2^2}{(2\xi^2-k_2^2)^2 - 4\xi^2\gamma_1\gamma_2}P(\xi)\cos\xi x\ d\xi = \frac{1}{2}v_0 \quad , \quad c\le|x|\le1 \tag{13}$$

3. SOLUTION OF THE INPLANE PROBLEM

Let us consider the solution of the integral equations (12) and (13) in the form

$$P(\xi) = \int_c^1 x_1 f(x_1^2) \cos\xi x_1 \; dx_1 \tag{14}$$

where $f(x_1^2)$ is an unknown function which will be determined. The relation (12) is therefore satisfied automatically and the equation (13) becomes

$$\int_c^1 x_1 f(x_1^2) \int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} \cos\xi x_1 \cos\xi x \; d\xi \, dx_1 = \frac{1}{2} v_o$$

$$, \quad c \le |x| \le 1 \tag{15}$$

Using the relation

$$\frac{\sin\xi x \; \sin\xi x_1}{\xi^2} = \int_0^x \int_0^{x_1} \frac{wv J_o(\xi w) J_o(\xi v) dw dv}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}}$$

the above equation converts to the form

$$\frac{d}{dx} \int_c^1 x_1 f(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv \; L_1(v,w) \; dw dv dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}} = \frac{1}{2} v_o \; , \quad c \le |x| \le 1 \tag{16}$$

where

$$L_1(v,w) = \int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} J_o(\xi w) J_o(\xi v) \; d\xi \tag{17}$$

By a simple contour integration technique done in the appendix-A, $L_1(v,w)$ can be written as

$$L_1(v,w) = -i\tau^2\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2 \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta \; -$$

$$-4i\tau^2\int_0^\tau \frac{\eta^2(\eta^2-1)(\tau^2-\eta^2)^{1/2} \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta \; +$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2} \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0} \quad , \qquad (w>v)$$

$$= \frac{-i\tau^2}{16(1-\tau^2)}\left[\sum_{j=0}^{2} p_j\int_0^1 \frac{(1-\eta^2)^{1/2} \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{\eta^2-\tau_j^2}\,d\eta \; +\right.$$

$$\left. + \sum_{j=0}^{2} s_j\int_0^\tau \frac{(\tau^2-\eta^2)^{1/2} \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{\eta^2-\tau_j^2}\,d\eta\right] +$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2} \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0} \quad , \qquad (w>v)$$

(18)

where $\tau=k_2/k_1=c_1/c_2$, $Q_0(\eta) = (2\eta^2-\tau^2)^2-4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$, and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta)=0$. τ_1, τ_2 are the roots of the equation $(2\eta^2-\tau^2)^2+4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}=0$. $Q_0'(\eta)$ denote the derivative of $Q_0(\eta)$ with respect to η and

$$p_j = \frac{(2\tau_j^2-\tau^2)}{\prod_i(\tau_j^2-\tau_i^2)} \quad , \quad s_j = \frac{4\tau_j^2(\tau_j^2-1)}{\prod_i(\tau_j^2-\tau_i^2)} \quad , \quad (i,j=0,1,2 \text{ and } i\neq j).$$

The corresponding expression of $L_1(v,w)$ for $w<v$ follows from (18) by interchanging w and v.

For poission ratio $\sigma=1/4$, the values of τ, τ_0, τ_1 and τ_2 are given by

$$\tau^2 = \frac{2(1-\sigma)}{1-2\sigma} = 3 \;,\quad \tau_0^2 = \frac{3}{(0.9194)^2} \;,\quad \tau_1^2 = \frac{3}{(2+2/\sqrt{3})} \quad \text{and} \quad \tau_2^2=3/4.$$

Hence in this case $\tau_2<\tau_1<1<\tau<\tau_0$.

Substituting the series expansion of $J_0(\ \ )$ and $H_0^{(1)}(\ \ )$ and evaluating the integrals arising in (18), we find after some algebraic manipulation details of which are given in the appendix-B

$$L_1(v,w) = \frac{2}{\pi}\tau^2\left[\left(\gamma+\log(k_1 w/2) - \frac{\pi i}{2}\right)M + N - \frac{P(w^2+v^2)}{4}k_1^2\log k_1\right] + o(k_1^2) \quad , \; w>v$$

$$= \frac{2}{\pi}\tau^2\left[\left(\gamma+\log(k_1 v/2) - \frac{\pi i}{2}\right)M + N - \frac{P(w^2+v^2)}{4}k_1^2\log k_1\right] + o(k_1^2) \quad , \; w<v \qquad (19)$$

where γ = 0.5772157......... is Euler's constant,

$$M = -\frac{\pi}{4(1-\tau^2)} \qquad (20)$$

$$N = \frac{\pi}{32(1-\tau^2)}\left[4\log(4/\tau) + \sum_{j=1}^{2} p_j \frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j} - \right.$$

$$- p_o \frac{\sqrt{(\tau_o^2 - 1)}}{\tau_o} \log\left\{\tau_o + \sqrt{(\tau_o^2 - 1)}\right\} + \sum_{j=1}^{2} s_j \frac{\sqrt{(\tau^2 - \tau_j^2)}}{\tau_j} \tan^{-1} \frac{\sqrt{(\tau^2 - \tau_j^2)}}{\tau_j} -$$

$$- s_o \frac{\sqrt{(\tau_o^2 - \tau^2)}}{\tau_o} \log\left\{ \frac{\tau_o + \sqrt{(\tau_o^2 - \tau^2)}}{\tau} \right\}\Bigg] , \tag{21}$$

and
$$P = \frac{\pi}{32(1-\tau^2)} \left[\sum_{j=o}^{2} p_j \left(\frac{1}{2} - \tau_j^2\right) + \sum_{j=o}^{2} s_j \left(\frac{\tau^2}{2} - \tau_j^2\right) \right] . \tag{22}$$

Now differentiating both sides of the relation (15) with respet to x we obtain

$$\int_c^1 x_1 f(x_1^2) \int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} \xi \cos\xi x_1 \sin\xi x \, d\xi \, dx_1 = 0 \quad , \quad c \le |x| \le 1 .$$

Following similar procedure as done for deriving equation (16), we obtain

$$x \int_c^1 \frac{x_1 f(x_1^2)}{x^2 - x_1^2} dx_1 = \int_c^1 x_1 f(x_1^2) \frac{\partial}{\partial x_1} \int_0^x \int_0^{x_1} \frac{wv \, L_2(v,w) \, dw dv dx_1}{(x^2 - w^2)^{1/2} (x_1^2 - v^2)^{1/2}}$$

$$, \quad c \le |x| \le 1 \tag{23}$$

where

$$L_2(v,w) = \int_0^\infty \left[\xi - \frac{2\gamma_1 \xi^2 (k_1^2 - k_2^2)}{(2\xi^2 - k_2^2)^2 - 4\xi^2 \gamma_1 \gamma_2} \right] J_0(\xi w) J_0(\xi v) \, d\xi \tag{24}$$

For small values of k_1 and k_2 such that $k_1=o(k_2)$, we use the contour integration technique mentioned above and get

$$L_2(v,w)= 2ik_1^2(1-\tau^2)\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\eta^2\; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta\; +$$

$$+\; 4ik_1^2(1-\tau^2)\int_0^\tau \frac{2\eta^4(\eta^2-1)(\tau^2-\eta^2)^{1/2}\; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta\; -$$

$$-\; 2\pi ik_2^2(1-\tau^2)\left[\frac{\eta^2(\eta^2-1)^{1/2}\; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}$$

$$,\quad w>v \tag{25}$$

By the procedure similar to one which led to the equation (19), (25) can be written as

$$L_2(v,w) = -\frac{4P}{\pi}(1-\tau^2)k_1^2\log k_1 + o(k_1^2) \tag{26}$$

where P is given by (22).

Now let us consider

$$f(x_1^2) = f_0(x_1^2) + k_1^2\log k_1 f_1(x_1^2) + o(k_1^2) \tag{27}$$

Putting the above expansion of $f(x_1^2)$ and the value of $L_2(v,w)$ given by (26) in the equation (23) and equating the coefficients of equal powers of k_1 we get

$$\int_c^1 \frac{x_1 f_0(x_1^2)}{x^2 - x_1^2}\, dx_1 = 0 \qquad , \quad c \le |x| \le 1 \tag{28}$$

and $$\int_c^1 \frac{x_1 f_1(x_1^2)}{x^2 - x_1^2}\, dx_1 = -\frac{4P}{\pi}(1-\tau^2)\int_c^1 x_1 f_0(x_1^2)\,dx_1 \quad , \quad c \le |x| \le 1 \tag{29}$$

Following Srivastava and Lowengrub (1968) the solutions of the above integral equations are

$$f_0(x_1^2) = \frac{D}{(1-x_1^2)^{1/2}(x_1^2-c^2)^{1/2}} \tag{30}$$

and

$$f_1(x_1^2) = \frac{4}{\pi}\, PD(1-\tau^2)\left[\frac{x_1^2-c^2}{1-x_1^2}\right]^{1/2} + \frac{B}{(1-x_1^2)^{1/2}(x_1^2-c^2)^{1/2}} \tag{31}$$

where D and B are constants which can be calculated as follows. We substitute the value of $L_1(v,w)$ from (19) as well as the expansion of $f(x_1^2)$ obtained from (27) in the equation (16). When the coefficients of like powers of k_1 from both sides of the resulting equation are equated we get the following results :

$$\frac{d}{dx}\int_c^1 x_1 f_0(x_1^2)\frac{\partial}{\partial x_1}\int_0^x \int_0^{x_1} \frac{wv}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}\,\frac{2}{\pi}\,\tau^2\Big[\Big(\gamma+\log(k_1 v/2) - $$

$$- \frac{\pi i}{2}\Big)M + N\Big]dw\,dv\,dx_1 = \frac{1}{2}\,v_0 \quad , \; v>w \; , \; c \le |x| \le 1 \tag{32}$$

and

$$\frac{d}{dx}\int_c^1 x_1 f_1(x_1^2)\frac{\partial}{\partial x_1}\int_0^x \int_0^{x_1} \frac{wv}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}} \frac{2}{\pi} \tau^2\left[\left\{\gamma+\log(k_1 w/2) - \right.\right.$$

$$\left.\left. - \frac{\pi i}{2}\right\}M+N\right]dwdvdx_1 - \frac{\tau^2}{2\pi} P \frac{d}{dx}\int_c^1 x_1 f_0(x_1^2)\frac{\partial}{\partial x_1}\int_0^x \int_0^{x_1} \frac{wv(w^2+v^2)dwdvdx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}$$

$$= 0 \quad , \quad v>w \ , \ c\leq|x|\leq 1 \qquad (33)$$

Next, putting the values of $f_0(x_1^2)$ and $f_1(x_1^2)$ from (30) and (31) in the above two equations and integrating, the values of D and B can be obtained as :

$$D = \frac{v_0}{2\tau^2\left[\left\{\gamma+\log(k_1/2) - \frac{\pi i}{2} + \log(1-c^2)^{1/2}\right\}M + N\right]} \qquad (34)$$

and $$B = \frac{2\tau^2 D^2 P}{v_0}\left[\frac{1}{2}(1+c^2) - \frac{(1-c^2)(1-\tau^2)v_0}{\pi\tau^2 D}\right] \qquad (35)$$

We can now obtain the values of the vertical displacement in the plane y=0 from equations (8), (11) and (14) as

$$v(x,0) = 2\int_c^1 x_1 f(x_1^2)\int_0^\infty \frac{\gamma_1 k_2^2 \cos\xi x_1 \cos\xi x}{(2\xi^2-k_2^2)^2 - 4\xi^2\gamma_1\gamma_2} d\xi\, dx_1$$

$$= 2\frac{d}{dx}\int_c^1 x_1 f(x_1^2)\frac{\partial}{\partial x_1}\int_0^x \int_0^{x_1} \frac{wv\ L_1(v,w)\ dwdvdx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}$$

Next putting the values of $L_1(v,w)$ and $f(x_1^2)$ from (19) and (27) in the above equation, we obtain,

$$v(x,0) = \frac{4}{\pi}\,\tau^2\left[\left\{\left(\gamma+\log(k_1/2) - \frac{\pi i}{2}\right)M + N\right\}\frac{d}{dx}\int_c^1 x_1 f_o(x_1^2)\;\times\right.$$

$$\times\;\frac{\partial}{\partial x_1}\int_o^x\int_o^{x_1}\frac{vw\;dwdvdx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}} +$$

$$\left. + \; M\,\frac{d}{dx}\int_c^1 x_1 f_o(x_1^2)\;\frac{\partial}{\partial x_1}\int_o^x\int_o^{x_1}\frac{wv\begin{cases}\log v\;, & v>w\\ \log w\;, & v<w\end{cases}}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}\;dwdvdx_1\right] +$$

$$+\;\frac{4\tau^2k_1^2\log k_1}{\pi}\left[\left\{\left(\gamma+\log(k_1/2) - \frac{\pi i}{2}\right)M + N\right\}\frac{d}{dx}\int_c^1 x_1 f_1(x_1^2)\;\times\right.$$

$$\times\;\frac{\partial}{\partial x_1}\int_o^x\int_o^{x_1}\frac{vw\;dwdvdx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}} +$$

$$\left. + \; M\,\frac{d}{dx}\int_c^1 x_1 f_1(x_1^2)\;\frac{\partial}{\partial x_1}\int_o^x\int_o^{x_1}\frac{wv\begin{cases}\log v\;, & v>w\\ \log w\;, & v<w\end{cases}}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}\;dwdvdx_1\right] -$$

$$-\;\frac{\tau^2}{2\pi}\,P\,\frac{d}{dx}\int_c^1 x_1 f_o(x_1^2)\frac{\partial}{\partial x_1}\int_o^x\int_o^{x_1}\frac{wv\;(w^2+v^2)\;dwdvdx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}} \qquad (36)$$

Substituting the values of $f_o(x_1)$ and $f_1(x_1)$ from (30) and (31) in (36) and integrating, the equation (36) yields after some algebraic manipulation

$$v(x,0) = v_o + 2M\tau^2\left[D+k_1^2\log k_1\left\{B+\frac{2}{\pi}(1-\tau^2)(1-c^2)PD\right\}\right]\sinh^{-1}\left(\frac{c^2-x^2}{1-c^2}\right)^{1/2} -$$

$$-\frac{4\tau^2 MPD(1-\tau^2)}{\pi}\,k_1^2\log k_1\left[(1-x^2)(c^2-x^2)\right]^{1/2} + o(k_1^2) \quad , \quad |x|<c$$

$$= v_o \quad , \qquad c\le|x|\le 1$$

$$= v_o + 2M\tau^2\left[D+k_1^2\log k_1\left\{B+\frac{2}{\pi}(1-\tau^2)(1-c^2)PD\right\}\right]\sinh^{-1}\left(\frac{x^2-1}{1-c^2}\right)^{1/2} +$$

$$+\frac{4\tau^2 MPD(1-\tau^2)}{\pi}\,k_1^2\log k_1\left[(x^2-1)(x^2-c^2)\right]^{1/2} + o(k_1^2) \quad , \quad |x|>1$$

(37)

The normal stress $\tau_{yy}(x,y)$ in the plane y=0 just below the strips can be found from the relation (9), (11) and (14) as

$$\tau_{yy}(x,0) = 2\mu\int_c^1 x_1 f(x_1^2)\,dx_1\int_0^\infty \cos\xi x\cos\xi x_1\,d\xi$$

Using the result

$$\int_0^\infty \cos\xi x\cos\xi x_1\,d\xi = \frac{\pi}{2}\,\delta(|x|-x_1)$$

and the value of $f(x_1^2)$ given by (27), the expression for stress becomes

$$\tau_{yy}(x,0) = \pi\mu\int_c^1 x_1\left[f_o(x_1^2)+k_1^2\log k_1 f_1(x_1^2)+o(k_1^2)\right]\delta(|x|-x_1)\,dx_1 .$$

Substituting the values of $f_0(x_1^2)$ and $f_1(x_1^2)$ from (30) and (31) respectively and integrating we finally obtain,

$$\tau_{yy}(x,0) = \frac{\pi\mu|x|}{[(1-x^2)(x^2-c^2)]^{1/2}}\left(D+Bk_1^2\log k_1\right)+4\mu|x|DP(1-\tau^2)\left(\frac{x^2-c^2}{1-x^2}\right)^{1/2}$$

$$\times\ k_1^2\log k_1 \quad + o(k_1^2)\ , \quad c\leq|x|\leq 1 \qquad (38)$$

Now putting c=0 in (38) we can obtain the normal stress for a single strip, $|x|\leq 1$, $y=0$, $-\infty<z<\infty$ as

$$\tau_{yy}(x,0) = \frac{\pi\mu D}{(1-x^2)^{1/2}} + \frac{\mu}{(1-x^2)^{1/2}}\,k_1^2\log k_1\left[4P(1-\tau^2)Dx^2+\pi B\right] + o(k_1^2)$$

where
$$D = \frac{v_0}{2\tau^2\left[\left(\gamma+\log(k_1/2) - \frac{\pi i}{2}\right)M + N\right]}$$

and
$$B = \frac{2\tau^2D^2P}{v_0}\left[\frac{1}{2} - \frac{(1-\tau^2)v_0}{\pi\tau^2D}\right]$$

Substituting $\Delta_0=v_0/\pi^2D$, $\beta_0= -\tau^2/2\pi(1-\tau^2)$ and $\beta_2= -P/\pi^2$ as done by Wickham (1977), we get

$$\tau_{yy}(x,0) = \frac{\mu v_0}{\pi\Delta_0(1-x^2)^{1/2}}\left\{1-\beta_2k_2^2\log k_2\left[\frac{1}{\Delta_0} + \frac{(1-2x^2)}{\beta_0}\right]\right\} + o(k_2^2)$$

which coincides with the result obtained by G.R.Wickham (1977).

4 FORMULATION AND SOLUTION OF THE ANTIPLANE PROBLEM

For SH-wave the displacement and stress are

$$w(x,y,t) = w(x,y)e^{-i\omega t}$$

and
$$\tau_{yz}(x,y) = \mu\frac{\partial w}{\partial y}$$

As in the previous case, we write the above expressions as

$$w(x,y) = \int_{-\infty}^{\infty} \frac{Q(\xi)}{\gamma_2} \exp(i\xi x-\gamma_2 y)d\xi \tag{39}$$

and
$$\tau_{yz}(x,y) = -\mu\int_{-\infty}^{\infty} Q(\xi)\exp(i\xi x-\gamma_2 y)d\xi \tag{40}$$

where $Q(\xi)$ is an unknown function to be determined from the boundary conditions

$$w(x,0) = w_o \quad , \qquad c\leq|x|\leq1 \tag{41}$$

$$\tau_{yz}(x,0) = 0 \quad , \qquad |x|<c \;,\; |x|>1 \tag{42}$$

where w_o is constant.

From equations (39) to (42) the following integral equations can be derived

$$\int_0^{\infty} \frac{Q(\xi)}{\gamma_2} \cos\xi x \; d\xi = \frac{w_o}{2} \quad , \qquad c\leq|x|\leq1 \tag{43}$$

$$\int_0^{\infty} Q(\xi) \cos\xi x \; d\xi = 0 \quad , \qquad |x|<c \;,\; |x|>1 \tag{44}$$

Substitution of

$$Q(\xi) = \int_c^1 x_1 g(x_1^2)\cos\xi x_1 dx_1 \qquad (45)$$

satisfies equation (44) automatically and equation (43) then yields

$$\int_c^1\int_0^\infty \frac{x_1 g(x_1^2)}{\gamma_2} \cos\xi x\cos\xi x_1 dx_1 = \frac{1}{2} w_0 , \quad c\le|x|\le 1 \qquad (46)$$

which can be written as

$$\frac{d}{dx}\int_c^1 x_1 g(x_1^2)\frac{\partial}{\partial x_1}\int_0^x \int_0^{x_1} \frac{wv\ L_3(v,w)\ dwdvdx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}} = \frac{1}{2} w_0 , \quad c\le|x|\le 1 \qquad (47)$$

where

$$L_3(v,w) = \int_0^\infty \frac{J_0(\xi w)J_0(\xi v)}{\gamma_2} d\xi$$

Integrating along the contour as shown in fig.2 and putting $\xi=k_2\eta$ the above infinite integral can be reduced to the finite integral given by

$$L_3(v,w) = i\int_0^1 \frac{J_0(k_2\eta v)H_0^{(1)}(k_2\eta w)}{(1-\eta^2)^{1/2}} d\eta , \quad w>v$$

$$= i\int_0^1 \frac{J_0(k_2\eta w)H_0^{(1)}(k_2\eta v)}{(1-\eta^2)^{1/2}} d\eta , \quad w<v.$$

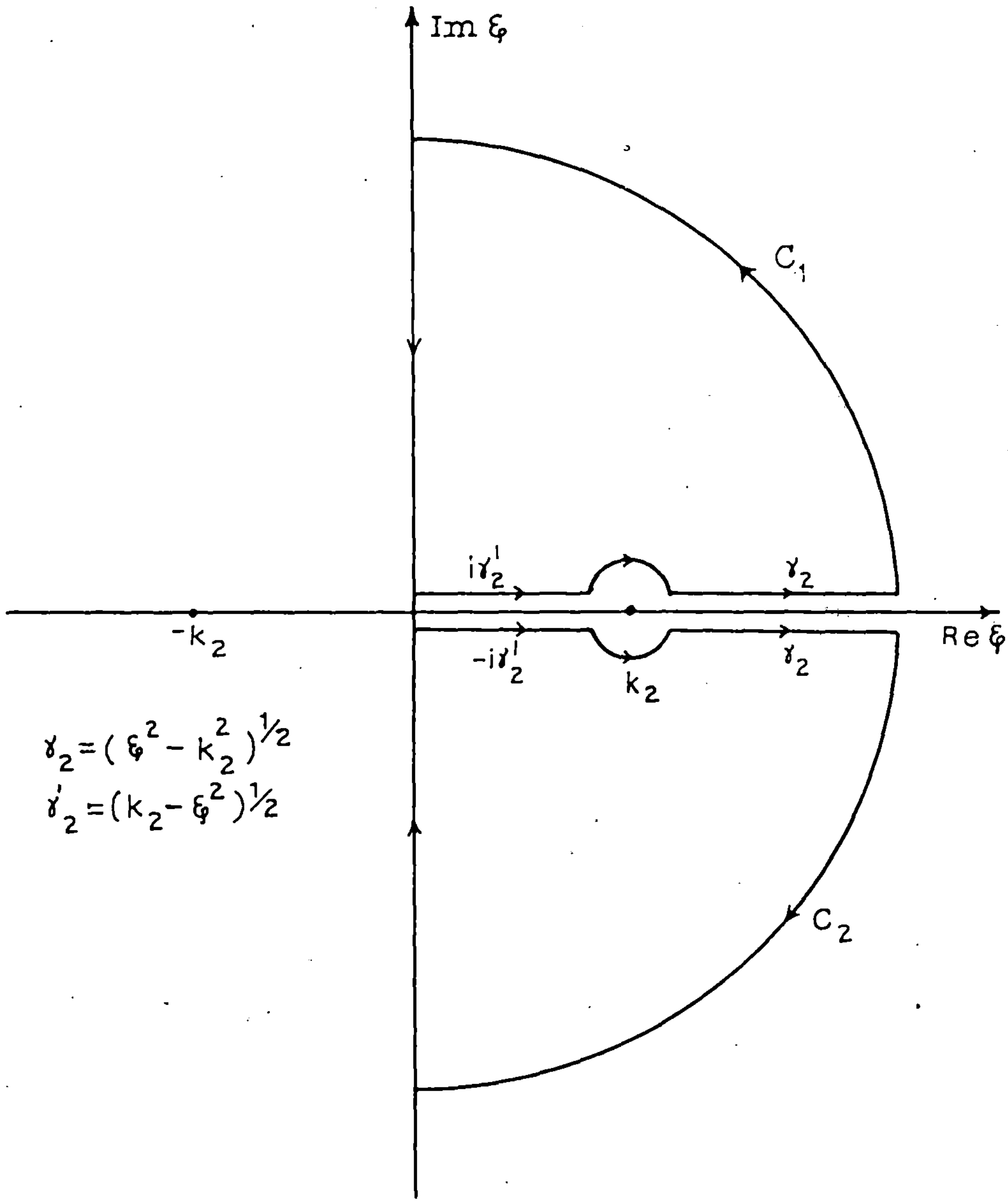

Fig. 2. The contour of integration to evaluate $L_3(v, w)$

For low frequency, $k_2\eta w$ and $k_2\eta v$ are small. So expanding both the Hankel function and Bessel function for small values of their arguments and integrating term by term, we obtain finally,

$$L_3(v,w) = \left[\frac{\pi i}{2} - \gamma - \log(k_2 w/4)\right] + \frac{1}{8}(w^2+v^2)k_2^2\log k_2 + o(k_2^2) \quad , \quad w>v$$

$$= \left[\frac{\pi i}{2} - \gamma - \log(k_2 v/4)\right] + \frac{1}{8}(w^2+v^2)k_2^2\log k_2 + o(k_2^2) \quad , \quad w<v$$

(48)

Next, Differentiating equation (46) with respect to x and proceeding as the previous case we get

$$x\int_c^1 \frac{x_1 g(x_1^2)}{x^2-x_1^2}\,dx_1 = \int_c^1 x_1 g(x_1^2)\frac{\partial}{\partial x_1}\int_0^x \int_0^{x_1} \frac{wv\ L_4(v,w)\ dwdvdx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}$$

$$, \quad c\le|x|\le 1 \qquad (49)$$

where $\quad L_4(v,w) = \int_0^\infty \left(\xi - \frac{\xi^2}{\gamma_2}\right)J_0(\xi w)J_0(\xi v)\,d\xi$

By the procedure similar to the one which led to the derivation of equation (48) from the corresponding infinite integral, we obtain

$$L_4(v,w) = -ik_2^2\int_0^1 \frac{\eta^2 J_0(k_2\eta v)H_0^{(1)}(k_2\eta w)}{(1-\eta^2)^{1/2}}\,d\eta$$

$$= \frac{1}{2}k_2^2\log k_2 + o(k_2^2) \tag{50}$$

Now considering the expansion for $g(x_1^2)$ as

$$g(x_1^2) = g_0(x_1^2) + k_2^2\log k_2\, g_1(x_1^2) + o(k_2^2) \tag{51}$$

and substituting $g(x_2^2)$ and $L_4(v,w)$ as given by (51) and (50) respectively in (49) it can be shown that ,

$$g_0(x_1^2) = \frac{D_1}{(1-x_1^2)^{1/2}(x_1^2-c^2)^{1/2}} \tag{52}$$

and $$g_1(x_1^2) = -\frac{\tau^2 D_1}{2}\left[\frac{x_1^2-c^2}{1-x_1^2}\right]^{1/2} + \frac{B_1}{(1-x_1^2)^{1/2}(x_1^2-c^2)^{1/2}} \tag{53}$$

where the constants D_1 and B_1 obtained from the equations (47) and (48) are found to be equal to

$$D_1 = \frac{-w_0}{\pi\left[\gamma+\log(k_2/4) - \frac{\pi i}{2} + \log(1-c^2)^{1/2}\right]} \tag{54}$$

and $$B_1 = -\frac{\pi D_1^2}{4w_o}\left[(1+c^2) - \frac{(1-c^2)w_o}{\pi D_1}\right] \tag{55}$$

The values of the stress $\tau_{yz}(x,y)$ and displacement $w(x,y)$ in the plane $y=0$ can be found from equations (39), (40), (45), (51) to (55) and are

$$\tau_{yz}(x,0) = \frac{-\pi\mu|x|}{\left[(1-x^2)(x^2-c^2)\right]^{1/2}}\left(D_1+B_1k_2^2\log k_2\right) + \frac{\pi\mu|x|D_1}{2}\left(\frac{x^2-c^2}{1-x^2}\right)^{1/2} \times$$

$$\times\ k_2^2\log k_2 + o(k_2^2)\ , \quad c\le|x|\le1 \tag{56}$$

and

$$w(x,0) = w_o - \pi\left[D_1 + k_2^2\log k_2\left\{B_1 - D_1(1-c^2)/4\right\}\right]\sinh^{-1}\left(\frac{c^2-x^2}{1-c^2}\right)^{1/2} -$$

$$- \frac{\pi D_1}{4}k_2^2\log k_2\left[(1-x^2)(c^2-x^2)\right]^{1/2} + o(k_2^2)\quad , \quad |x|<c$$

$$= w_o\quad , \qquad c\le|x|\le1$$

$$= w_o - \pi\left[D_1 + k_2^2\log k_2\left\{B_1 - D_1(1-c^2)/4\right\}\right]\sinh^{-1}\left(\frac{x^2-1}{1-c^2}\right)^{1/2} +$$

$$+ \frac{\pi D_1}{4}k_2^2\log k_2\left[(x^2-1)(x^2-c^2)\right]^{1/2} + o(k_2^2)\quad , \quad |x|>1$$

(57)

5. NUMERICAL RESULTS

The vertical and the transverse displacement fields for the inplane and antiplane problems respectively for points near about the rigid strips have been depicted by means of graphs (fig.3,4) for the Poisson Solid(τ^2=3). It is interesting to note from the graphs that the real part of the displacements decrease with the increase in the value of k_2 in both the cases.
Further the graphs (fig.5,6) of the stress factors

$$\tau_1^* = \mathrm{Re}\left[\frac{\tau_{yy}\left((1-x^2)(x^2-c^2)\right)^{1/2}}{\mu v_0} \right] \quad \text{and} \quad \tau_2^* = \mathrm{Re}\left[\frac{\tau_{yz}\left((1-x^2)(x^2-c^2)\right)^{1/2}}{\mu w_0} \right]$$

versus dimensionless distance x for the inplane and the antiplane problem respectively have been plotted for points just below the rigid strips. In both the cases the magnitude of the stress factor are found to increase as one proceeds from the inner to the outer edge of the strips.

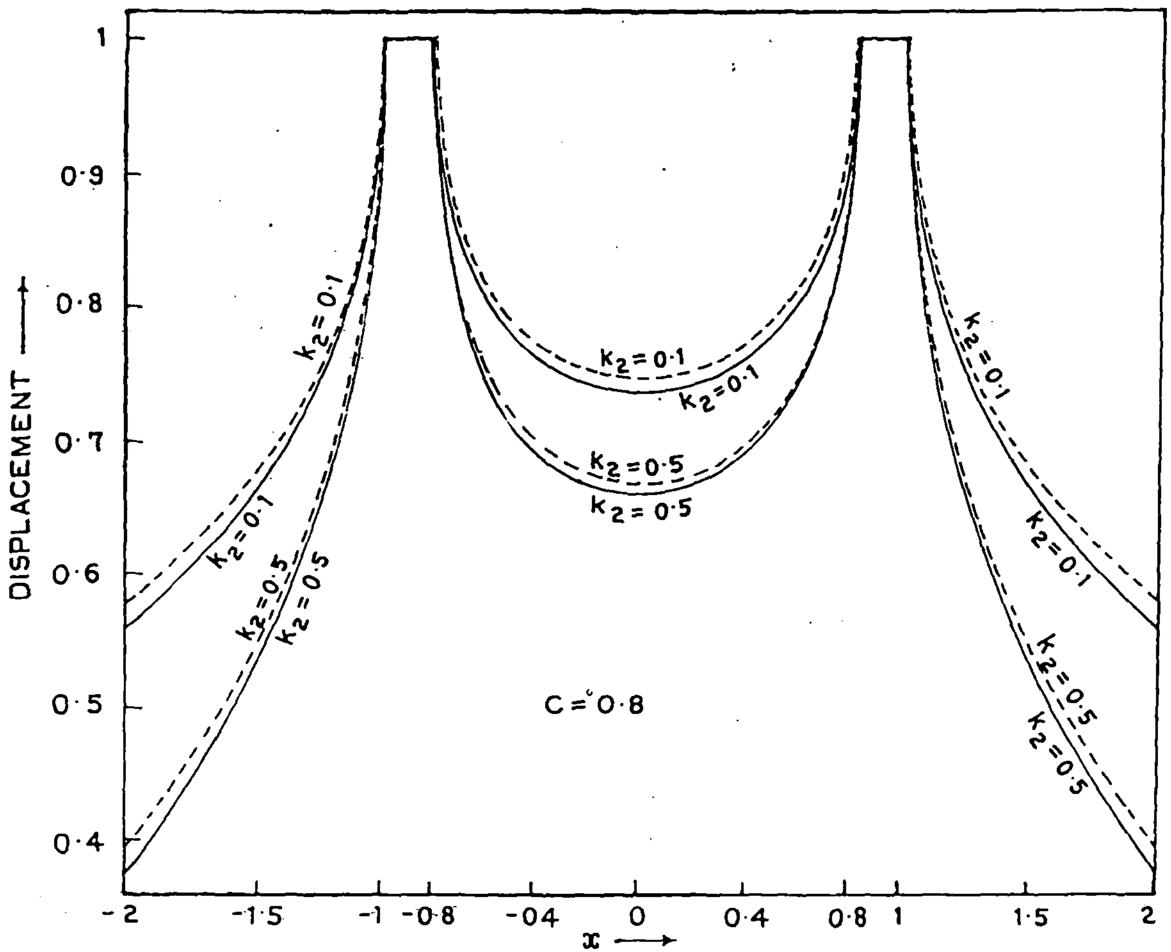

Fig. 3. Displacement vs. distance

—— $\mathrm{Re}\{v(x,0)\}$ for inplane problem, - - - - $\mathrm{Re}\{w(x,0)\}$ for antiplane problem.

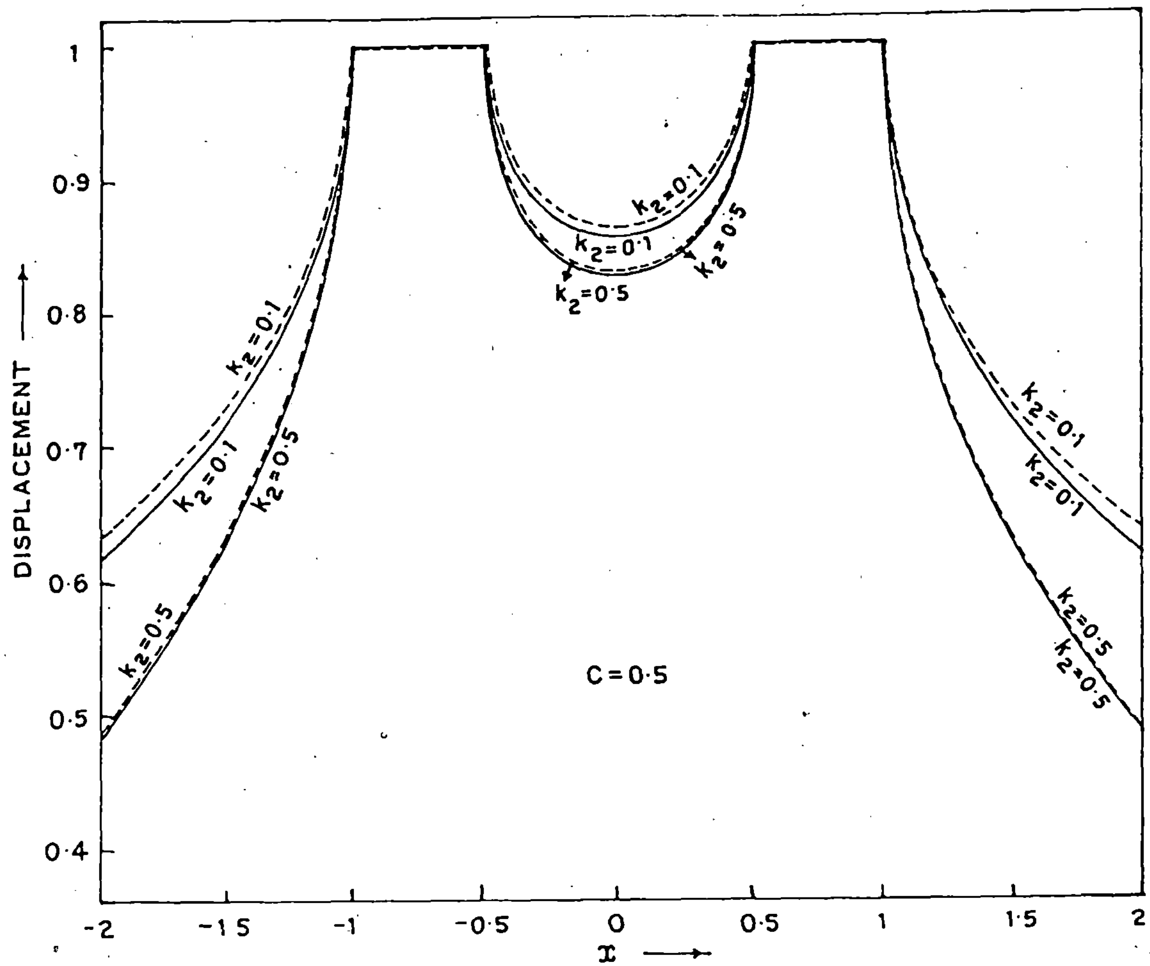

Fig. 4. Displacement vs. distance

—— Re$\{v(x,0)\}$ for inplane problem, - - - - Re$\{w(x,0)\}$ for antiplane problem.

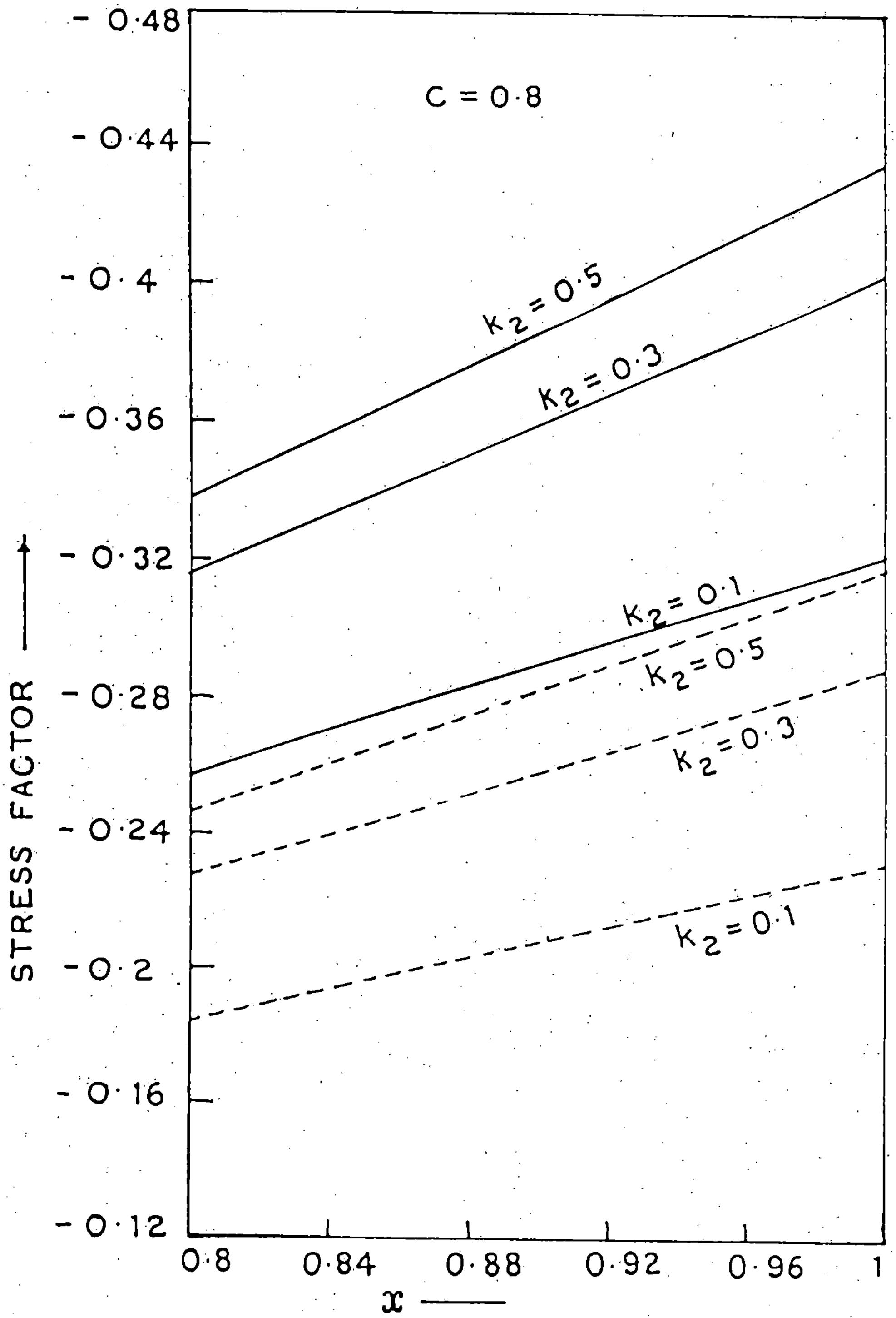

Fig. 5. Stress factor vs. displacement

—— τ_1^* for inplane problem, ---- τ_2^* for antiplane problem

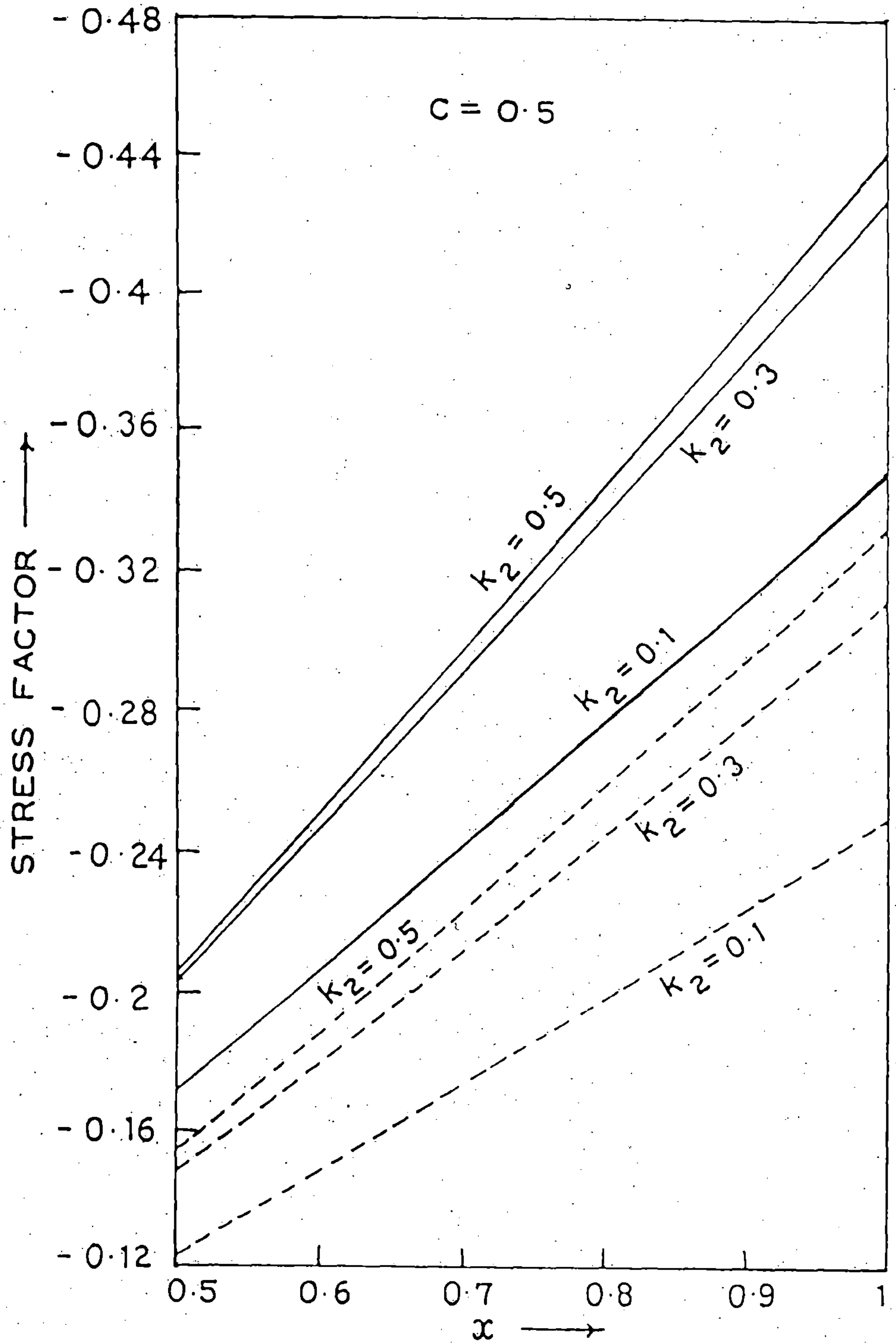

Fig.6. Stress factor vs. displacement

——— τ_1^* for inplane problem,

- - - - τ_2^* for antiplane problem

APPENDIX - A

CONTOUR INTEGRATION TO EVALUATE $L_1(v,w)$:

The kernel $L_1(v,w)$ given by equation (17) is

$$L_1(v,w) = \int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2-k_2^2)^2 - 4\xi^2\gamma_1\gamma_2} J_0(\xi w) J_0(\xi v)\, d\xi \tag{A1}$$

Let us consider the following two integrals

$$I_1 = \int_{C_1} \frac{\gamma_1 k_2^2}{(2\xi^2-k_2^2)^2 - 4\xi^2\gamma_1\gamma_2} H_0^{(1)}(\xi w) J_0(\xi v)\, d\xi \quad , \quad w>v \tag{A2}$$

$$I_2 = \int_{C_2} \frac{\gamma_1 k_2^2}{(2\xi^2-k_2^2)^2 - 4\xi^2\gamma_1\gamma_2} H_0^{(2)}(\xi w) J_0(\xi v)\, d\xi \quad , \quad w>v \tag{A3}$$

where C_1 and C_2 are the contours of integration of I_1 and I_2 respectively as shown in fig.7.

Branch points corresponding to γ_1 and γ_2 are $\xi = \pm k_1$ and $\xi = \pm k_2$ and the pole k_R is the root of the Rayleigh wave equation

$$(2\xi^2-k_2^2)^2 - 4\xi^2\gamma_1\gamma_2 = 0$$

Integrating I_1 and I_2 along the contours C_1 and C_2 respectively we get,

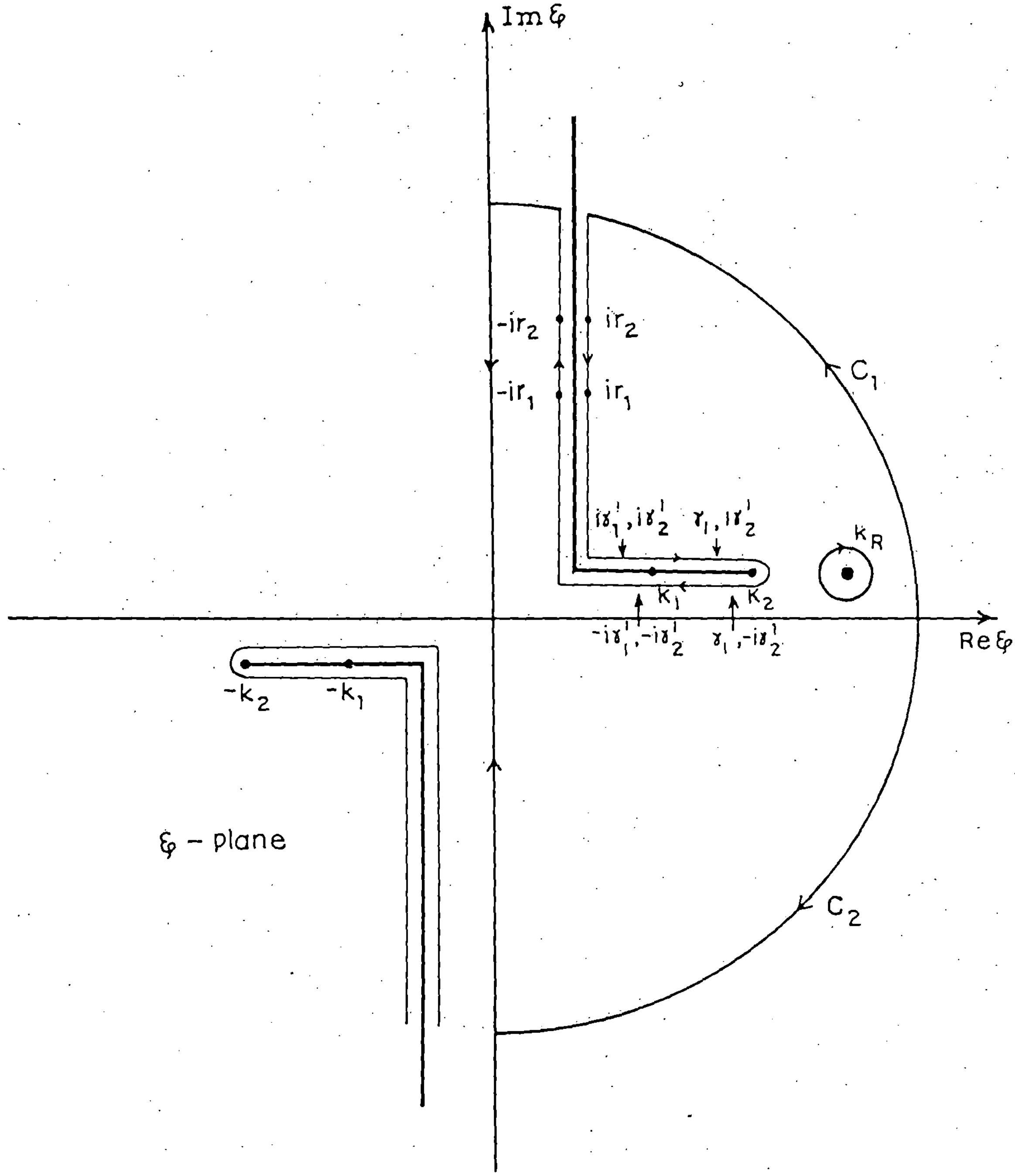

Fig. 7. The contour of integration to evaluate $L_1(v,w)$.

for contour C_1 :

$$\int_0^\infty \frac{\gamma_1 k_2^2\ H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2-k_2^2)^2\ -\ 4\xi^2\gamma_1\gamma_2}\,d\xi\ +\ i\int_\infty^0 \frac{i r_1 k_2^2\ H_0^{(1)}(i\eta w) J_0(i\eta v)}{(2\eta^2+k_2^2)^2\ -\ 4\xi^2 r_1 r_2}\,d\eta\ +$$

$$+\int_0^{k_1} \frac{i\gamma_1' k_2^2\ H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2-k_2^2)^2\ +\ 4\xi^2\gamma_1'\gamma_2'}\,d\xi\ +\ \int_{k_1}^{k_2} \frac{\gamma_1 k_2^2\ H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2-k_2^2)^2\ -\ 4i\xi^2\gamma_1\gamma_2'}\,d\xi\ +$$

$$+\ i\int_0^{k_1} \frac{\gamma_1' k_2^2\ H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2-k_2^2)^2\ +\ 4\xi^2\gamma_1'\gamma_2'}\,d\xi\ -\ \int_{k_1}^{k_2} \frac{\gamma_1 k_2^2\ H_0^{(1)}(\xi w) J_0(\xi v)}{(2\xi^2-k_2^2)^2\ +\ 4i\xi^2\gamma_1\gamma_2'}\,d\xi$$

$$=\ 2\pi i\ \frac{\gamma_1(k_R)\ \ k_2^2\ H_0^{(1)}(k_R w) J_0(k_R v)}{\phi'(k_R)} \qquad \text{(A4)}$$

and for contour C_2 :

$$\int_0^\infty \frac{\gamma_1 k_2^2\ H_0^{(2)}(\xi w) J_0(\xi v)}{(2\xi^2-k_2^2)^2\ -\ 4\xi^2\gamma_1\gamma_2}\,d\xi\ -\ \int_\infty^0 \frac{r_1 k_2^2\ H_0^{(2)}(-i\eta w) J_0(-i\eta v)}{(2\eta^2+k_2^2)^2\ -\ 4\eta^2 r_1 r_2}\,d\eta\ =\ 0$$

$$\text{(A5)}$$

where $\quad \gamma_1' = (k_1^2-\xi^2)^{1/2} \quad , \quad \gamma_2' = (k_2^2-\xi^2)^{1/2}$

$$r_1 = (\eta^2+k_1^2)^{1/2} \quad , \quad r_2 = (\eta^2+k_2^2)^{1/2}$$

$$\phi(\xi) = (2\xi^2-k_2^2)^2 - 4\xi^2\gamma_1\gamma_2$$

where k_R is the root of the equation $\phi(\xi) = 0$.

Now, using the results

$$H_0^{(2)}(-i\eta w) = -H_0^{(1)}(i\eta w) \quad \text{and} \quad J_0(-i\eta v) = J_0(i\eta v)$$

in (A5) and then addition with (A4) yields

$$L_1(v,w) = -i\int_0^{k_1} \frac{(k_1^2-\xi^2)^{1/2}k_2^2\, H_0^{(1)}(\xi w)J_0(\xi v)}{(2\xi^2-k_2^2)^2+4\xi^2(k_1^2-\xi^2)^{1/2}(k_2^2-\xi^2)^{1/2}}\,d\xi\; -$$

$$-4i\int_{k_1}^{k_2} \frac{k_2^2\xi^2(\xi^2-k_1^2)(k_2^2-\xi^2)^{1/2}\, H_0^{(1)}(\xi w)J_0(\xi v)}{(2\xi^2-k_2^2)^4+16\xi^4(\xi^2-k_1^2)(k_2^2-\xi^2)}\,d\xi\; +$$

$$+\pi i\,\frac{(k_R^2-k_1^2)^{1/2}k_2^2\, H_0^{(1)}(k_R w)J_0(k_R v)}{\phi'(k_R)} \qquad \text{(A6)}$$

Putting $\xi = k_1\eta$, $\tau = k_2/k_1 = c_1/c_2$ and after simple calculation, (A6) takes the form

$$L_1(v,w) = -i\tau^2\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2 \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta \; -$$

$$-4i\tau^2\int_0^\tau \frac{\eta^2(\eta^2-1)(\tau^2-\eta^2)^{1/2} \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta \; +$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2} \; H_0^{(1)}(k_1\eta w)J_0(k_1\eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0} \quad , \qquad (w>v)$$

(A7)

where $Q_0(\eta) = (2\eta^2-\tau^2)^2 - 4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$ and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta) = 0$. $Q_0'(\eta)$ denote the derivative of $Q_0(\eta)$ with respect to η.

Next taking

$$\Delta_0(\eta) = (2\eta^2-\tau^2)^4 + 16\eta^4(\eta^2-1)(\tau^2-\eta^2) = 16(1-\tau^2)(\eta^2-\tau_0^2)(\eta^2-\tau_1^2)(\eta^2-\tau_2^2),$$

τ_1, τ_2 being the roots of the equation

$$(2\eta^2-\tau^2)^2 + 4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2} = 0$$

and

$$\frac{(2\eta^2-\tau^2)^2}{(\eta^2-\tau_0^2)(\eta^2-\tau_1^2)(\eta^2-\tau_2^2)} = \frac{p_0}{\eta^2-\tau_0^2} + \frac{p_1}{\eta^2-\tau_1^2} + \frac{p_2}{\eta^2-\tau_2^2}$$

$$\frac{4\eta^2(\eta^2-1)}{(\eta^2-\tau_o^2)(\eta^2-\tau_1^2)(\eta^2-\tau_2^2)} = \frac{s_o}{\eta^2-\tau_o^2} + \frac{s_1}{\eta^2-\tau_1^2} + \frac{s_2}{\eta^2-\tau_2^2}$$

so that $\quad \sum_{j=o}^{2} p_j = \sum_{j=o}^{2} s_j = 4 \quad$ and

$$p_j = \frac{(2\tau_j^2-\tau^2)}{\prod_i(\tau_j^2-\tau_i^2)} \quad , \quad s_j = \frac{4\tau_j^2(\tau_j^2-1)}{\prod_i(\tau_j^2-\tau_i^2)} \quad , \quad (i,j=0,1,2 \text{ and } i\neq j),$$

the equation (A7) can be written as

$$L_1(v,w) = \frac{-i\tau^2}{16(1-\tau^2)}\left[\sum_{j=o}^{2} p_j\int_o^1 \frac{(1-\eta^2)^{1/2}\; H_o^{(1)}(k_1\eta w)J_o(k_1\eta v)}{\eta^2-\tau_j^2}\,d\eta\; + \right.$$

$$\left. + \sum_{j=o}^{2} s_j\int_o^{\tau} \frac{(\tau^2-\eta^2)^{1/2}\; H_o^{(1)}(k_1\eta w)J_o(k_1\eta v)}{\eta^2-\tau_j^2}\,d\eta\right] +$$

$$+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}\; H_o^{(1)}(k_1\eta w)J_o(k_1\eta v)}{Q_o'(\eta)}\right]_{\eta=\tau_o} \quad , \qquad (w>v) \tag{A8}$$

APPENDIX - B

EVALUATION OF THE INTEGRALS ARISING IN THE EXPRESSION OF $L_1(v,w)$ GIVEN BY EQUATION (18) :

Experssion of $L_1(v,w)$ is

$$L_1(v,w) = \frac{-i\tau^2}{16(1-\tau^2)} \left[\sum_{j=0}^{2} p_j \int_0^1 \frac{(1-\eta^2)^{1/2}\, H_0^{(1)}(k_1\eta w) J_0(k_1\eta v)}{\eta^2-\tau_j^2}\, d\eta \; + \right.$$

$$\left. + \sum_{j=0}^{2} s_j \int_0^{\tau} \frac{(\tau^2-\eta^2)^{1/2}\, H_0^{(1)}(k_1\eta w) J_0(k_1\eta v)}{\eta^2-\tau_j^2}\, d\eta \right] +$$

$$+\pi i\tau^2 \left[\frac{(\eta^2-1)^{1/2}\, H_0^{(1)}(k_1\eta w) J_0(k_1\eta v)}{Q_0'(\eta)} \right]_{\eta=\tau_0} , \qquad (w>v)$$

(B1)

For low frequency i.e., for small values of the arguments $k_1\eta w$ and $k_1\eta v$ of $H_0^{(1)}(\ )$ and $J_0(\ )$ we get the following result

$$H_0^{(1)}(k_1\eta w) J_0(k_1\eta v) = 1 + \frac{2i}{\pi}\left[\log(k_1\eta w/2) + \gamma\right] - \frac{i}{2\pi}\eta^2(w^2+v^2)k_1^2\log k_1 +$$

$$+ o(k_1^2) \qquad \text{(B2)}$$

where γ = 0.5772157........ is Euler's constant.

Substitution of this result in (B1) yields

$$L_1(v,w) = \frac{2}{\pi}\tau^2\left[\left(\gamma+\log(k_1 w/2) - \frac{\pi i}{2}\right)M + N - \frac{P(w^2+v^2)}{4}k_1^2\log k_1\right] + o(k_1^2)$$

$$, \quad w>v \qquad \text{(B3)}$$

where

$$M = \frac{1}{16(1-\tau^2)}\left[\sum_{j=0}^{2} p_j \int_0^1 \frac{(1-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)}\,d\eta + \sum_{j=0}^{2} s_j \int_0^{\tau} \frac{(\tau^2-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)}\,d\eta\right] -$$

$$- \pi\left[\frac{(\eta^2-1)^{1/2}}{Q_0'(\eta)}\right]_{\eta=\tau_0} \qquad \text{(B4)}$$

$$N = \frac{1}{16(1-\tau^2)}\left[\sum_{j=0}^{2} p_j \int_0^1 \frac{(1-\eta^2)^{1/2}\log\eta}{(\eta^2-\tau_j^2)}\,d\eta +\right.$$

$$\left. + \sum_{j=0}^{2} s_j \int_0^{\tau} \frac{(\tau^2-\eta^2)^{1/2}\log\eta}{(\eta^2-\tau_j^2)}\,d\eta\right] - \pi\left[\frac{(\eta^2-1)^{1/2}\log\eta}{Q_0'(\eta)}\right]_{\eta=\tau_0} \qquad \text{(B5)}$$

and

$$P = \frac{1}{16(1-\tau^2)}\left[\sum_{j=0}^{2} p_j \int_0^1 \frac{\eta^2(1-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)}\,d\eta +\right.$$

$$\left. + \sum_{j=0}^{2} s_j \int_0^{\tau} \frac{\eta^2(\tau^2-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)}\,d\eta\right] - \pi\left[\frac{\eta^2(\eta^2-1)^{1/2}}{Q_0'(\eta)}\right]_{\eta=\tau_0} \qquad \text{(B6)}$$

The first integral of M can be written as

$$\sum_{j=o}^{2} p_j \int_o^1 \frac{(1-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)} d\eta = \sum_{j=1}^{2} p_j \fint_o^1 \frac{(1-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)} d\eta - p_o \int_o^1 \frac{(1-\eta^2)^{1/2}}{(\tau_o^2-\eta^2)} d\eta \tag{B7}$$

where $\tau_2 < \tau_1 < 1 < \tau < \tau_o$ and $\fint$ means the principal value of the integral.

Using the results

$$\int_o^1 \frac{(1-z^2)^{1/2}}{z^2-a^2} dz = -\frac{\pi}{2} \qquad (a<1)$$

and

$$\int_o^1 \frac{(1-z^2)^{1/2}}{a^2-z^2} dz = \frac{\pi}{2} - \frac{(a^2-1)^{1/2}}{a}\frac{\pi}{2} \qquad (a>1)$$

in (B7) we get,

$$\sum_{j=o}^{2} p_j \int_o^1 \frac{(1-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)} d\eta = -\frac{\pi}{2}\left[4 - \frac{p_o(\tau_o^2-1)^{1/2}}{\tau_o}\right] \tag{B8}$$

Similarly, the second integral of M is found to be evaluated as

$$\sum_{j=o}^{2} s_j \int_o^{\tau} \frac{(\tau^2-\eta^2)^{1/2}}{(\eta^2-\tau_j^2)} d\eta = -\frac{\pi}{2}\left[4 - \frac{s_o(\tau_o^2-\tau^2)^{1/2}}{\tau_o}\right] \tag{B9}$$

Now, third term (denoted by D_o) of M given in (B4) can be written as

$$D_o = -\pi\left[\frac{(\eta^2-1)^{1/2}}{Q_o'(\eta)}\right]_{\eta=\tau_o}$$

$$= -\pi\left[\frac{\left((2\eta^2-\tau^2)^2+4\eta^2(\eta^2-\tau^2)^{1/2}(\eta^2-1)^{1/2}\right)(\eta^2-1)^{1/2}}{\Delta_o'(\eta)}\right]_{\eta=\tau_o}$$

(B10)

where $\Delta_o'(\eta)$ is the derivative of $\Delta_o(\eta)$ with respect to η.
Let us consider

$$\frac{(\eta^2-1)^{1/2}(2\eta^2-\tau^2)^2}{\Delta_o(\eta)} = \frac{(\eta^2-1)^{1/2}(2\eta^2-\tau^2)^2}{16(1-\tau^2)(\eta^2-\tau_o^2)(\eta^2-\tau_1^2)(\eta^2-\tau_2^2)}$$

$$= \frac{(\eta^2-1)^{1/2}}{16(1-\tau^2)}\left[\frac{p_o}{\eta^2-\tau_o^2}+\frac{p_1}{\eta^2-\tau_1^2}+\frac{p_2}{\eta^2-\tau_2^2}\right]$$

Multiplying both sides of the above equation by $(\eta^2-\tau_o^2)/2\eta$ and putting $\eta = \tau_o$, we finally obtain,

$$\left[\frac{(\eta^2-1)^{1/2}(2\eta^2-\tau^2)^{1/2}}{\Delta_o'(\eta)}\right]_{\eta=\tau_o} = \frac{1}{32(1-\tau^2)}\left[\frac{p_o(\tau_o^2-1)^{1/2}}{\tau_o}\right] \quad (B11)$$

Similarly,

$$\left[\frac{4\eta^2(\eta^2-\tau^2)^{1/2}(\eta^2-1)}{\Delta_o'(\eta)} \right]_{\eta=\tau_o} = \frac{1}{32(1-\tau^2)}\left[\frac{s_o(\tau_o^2-\tau^2)^{1/2}}{\tau_o} \right] \qquad (B12)$$

Adding (B11) and (B12) D_o can be obtained as

$$D_o = \frac{-\pi}{32(1-\tau^2)}\left[\frac{p_o(\tau_o^2-1)^{1/2}}{\tau_o} + \frac{s_o(\tau_o^2-\tau^2)^{1/2}}{\tau_o} \right] \qquad (B13)$$

Using (B8), (B9) and (B13) in (B4), M can be written as

$$M = -\frac{\pi}{4(1-\tau^2)} \qquad (B14)$$

Following the same procedure and using the results

$$\int_0^1 \frac{(1-z^2)^{1/2}\log z}{z^2-a^2}\,dz = \frac{\pi}{2}\left[\frac{(1-a^2)^{1/2}}{a}\tan^{-1}\frac{(1-a^2)^{1/2}}{a} + \log 2 \right], \quad (a<1)$$

$$\int_0^1 \frac{(1-z^2)^{1/2}\log z}{a^2-z^2}\,dz = \frac{\pi}{2}\left[\frac{(a^2-1)^{1/2}}{a}\log\left\{ 1+\frac{(a^2-1)^{1/2}}{a} \right\} - \log 2 \right], \quad (a>1)$$

$$\int_0^1 \frac{z^2(1-z^2)^{1/2}}{z^2-a^2}\,dz = \frac{\pi}{2}\left(\frac{1}{2} - a^2 \right), \quad (a<1)$$

$$\int_0^1 \frac{z^2(1-z^2)^{1/2}}{a^2-z^2}\,dz = -\frac{\pi}{2}\left(\frac{1}{2} - a^2 + a(a^2-1)^{1/2}\right), \quad (a>1)$$

$$\pi\left[\frac{(\eta^2-1)^{1/2}\log\eta}{Q_o'(\eta)}\right]_{\eta=\tau_o} = \frac{\pi}{32(1-\tau^2)}\left[\frac{p_o(\tau_o^2-1)^{1/2}}{\tau_o} + \frac{s_o(\tau_o^2-\tau^2)^{1/2}}{\tau_o}\right]\log\tau_o$$

$$\pi\left[\frac{\eta^2(\eta^2-1)^{1/2}}{Q_o'(\eta)}\right]_{\eta=\tau_o} = \frac{\pi}{32(1-\tau^2)}\left[p_o\tau_o(\tau_o^2-1)^{1/2} + s_o\tau_o(\tau_o^2-\tau^2)^{1/2}\right]$$

the values of N and P from (B5) and (B6) are evaluated and are given by

$$N = \frac{\pi}{32(1-\tau^2)}\left[4\log(4/\tau) + \sum_{j=1}^{2} p_j \frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j} - \right.$$

$$- p_o\frac{\sqrt{(\tau_o^2-1)}}{\tau_o}\log\left\{\tau_o + \sqrt{(\tau_o^2-1)}\right\} + \sum_{j=1}^{2} s_j \frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j^2}\tan^{-1}\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j^2} -$$

$$\left. - s_o\frac{\sqrt{(\tau_o^2-\tau^2)}}{\tau_o}\log\left\{\frac{\tau_o + \sqrt{(\tau_o^2-\tau^2)}}{\tau}\right\}\right], \qquad \text{(B15)}$$

and $$P = \frac{\pi}{32(1-\tau^2)}\left[\sum_{j=0}^{2} p_j\left(\frac{1}{2} - \tau_j^2\right) + \sum_{j=0}^{2} s_j\left(\frac{\tau^2}{2} - \tau_j^2\right)\right] \qquad \text{(B16)}$$

DIFFRACTION OF ANTIPLANE SHEAR WAVE BY A PAIR OF PARALLEL RIGID STRIPS AT THE INTERFACE OF TWO BONDED DISSIMILAR ELASTIC MEDIA

1. INTRODUCTION

The problems of diffraction of elastic waves by a cracks or by inclusions are of considerable importance in view of their application in Seismology and Geophysics. The study becomes more relevant if the cracks or inclusions are located at the interface of layered media. Following Mal (1970) low frequency solution of the interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar half spaces has been derived by Srivastava et al (1980). Bostrom (1987) adopted a different technique to solve the same problem. He followed a procedure similar to that of Krenk and Schmidt (1982) and reduced the problem to the solution of a Fredholm integral equation of first kind with the crack opening displacement as the unknown. The corresponding problem of diffraction of antiplane shear wave by a finite rigid strip at the interface has been treated by Palaiya and Majumder (1981). As regards the dynamic crack or strip problems, research has mainly been confined to the case of a single crack or a strip because of the severe mathematical complexity encountered in solutions for two or more cracks or

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strips. Jain and Kanwal (1972,1972) however presented low frequency solution for the diffraction of elastic waves by coplanar Griffith cracks and also by two coplanar rigid strips located in an infinite isotropic homogeneous elastic medium. Recently Itu (1980) reconsidered the elastodynamic problem involving diffraction by two Griffith cracks in an infinite elastic medium and used a different technique to solve the problem. He expands the surface displacement in a series of functions which is automatically zero outside of the cracks and uses the Schmidt (1982) method to solve the resulting integral equation.

In the present paper we have considered the problem of diffraction of elastic waves by a pair of coplanar rigid strips between two homogeneous elastic half spaces for the case of antiplane strain. The resulting triple integral equation has been reduced to the solution of an integro differential equation, approximate solution of which has been obtained following the method of Lowengrub and Srivastava (1968). These solutions have been used to obtain approximate values of the displacement field and also the stress intensity factors at the edges of the strips. Making the distance between the inner edges of the strips tend to zero, the diffraction problem for a single rigid strip has been obtained. Even this result of the limiting case appears to have been presented here for the first time.

2. FORMULATION OF THE PROBLEM

We consider the problem of low frequency scattering of antiplane shear waves by two rigid strips situated parallel to each other, at the interface of two bonded dissimilar elastic half spaces. The strips are located in the region $-a\le X\le -b$, $b\le X\le a$, $Z=0$, $|Y|<\infty$ (fig.1). Normalizing all lengths with respect to a and putting $b/a=c$, we find that the rigid strips are defined by $c\le|x|\le 1$, $|y|<\infty$, $z=0$ at the interface of the half-spaces $z\le 0$ and $z\ge 0$. Let an antiplane shear wave given by $v_o'\exp(im_2 z-\omega t)$, where $m_2=\omega a/c_2$ and v_o' a constant, be incident normally on the strips. Henceforth the time factor $e^{-i\omega t}$ will be suppressed throughout the analysis. The non-vanishing components of displacement and stresses are

$$v = v(x,z)$$

$$\tau_{xy} = \tau_{xy}(x,z) \tag{1}$$

and $$\tau_{yz} = \tau_{yz}(x,z)$$

The boundary conditions are

$$v_1(x,0) = v_2(x,0) = -v_o \quad , \qquad c\le|x|\le 1 \tag{2}$$

$$v_1(x,0) = v_2(x,0) \quad , \qquad |x|<c, \ |x|>1 \tag{3}$$

$$\tau_{yz}^{(1)}(x,0) = \tau_{yz}^{(2)}(x,0) \quad , \qquad |x|<c, \ |x|>1 \tag{4}$$

where $v_o = 2\mu_2 m_2 v_o'/(\mu_1 m_1+\mu_2 m_2)$.

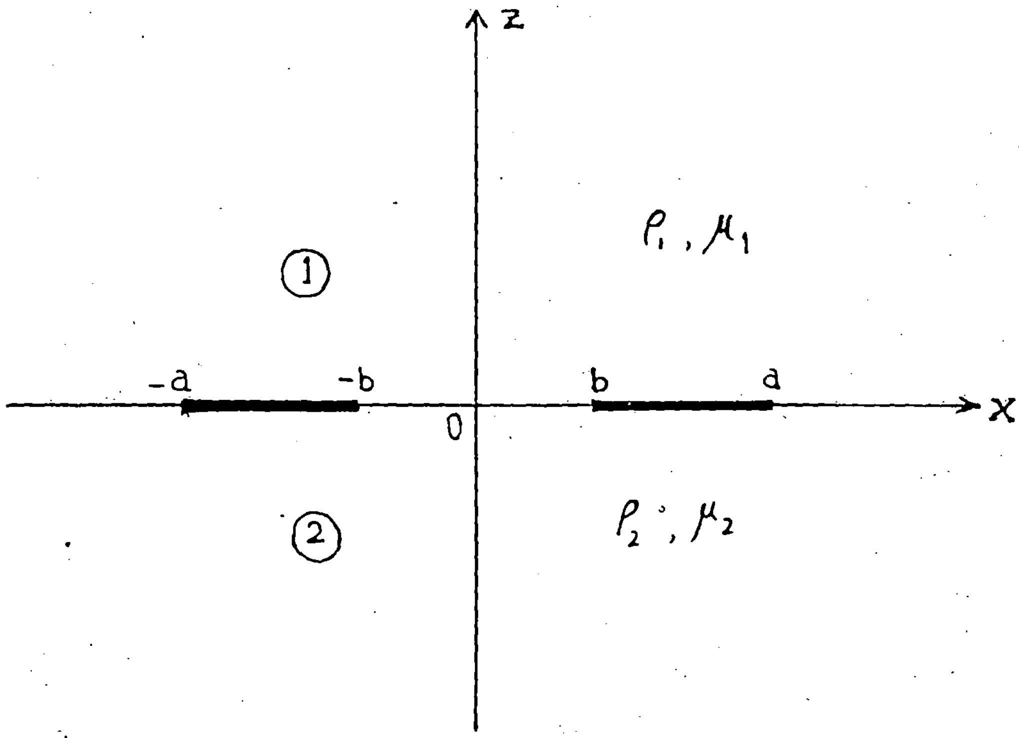

Fig. 1. The geometry of the strips .

The displacement v_j satisfies the equation

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial y^2} + m_j^2 v_j = 0 \tag{5}$$

where $m_j = \omega a / c_j$ $(j=1,2)$, c_j being the shear wave velocity. The suffices 1 and 2 are used to denote the values of the corresponding quantities in the upper and lower half-spaces respectively. Without any loss of generality we assume that $c_2 > c_1$. The solution of the equation (5) can be written as

$$v_j(x,z) = \int_0^\infty A_j(\xi) \exp(-\beta_j |z|) \cos\xi x \, d\xi \tag{6}$$

where
$$\begin{aligned} \beta_j &= (\xi^2 - m_j^2)^{1/2}, \quad \xi > m_j \\ &= -i(m_j^2 - \xi^2)^{1/2}, \quad \xi < m_j \quad (j=1,2), \end{aligned} \tag{7}$$

$A_1(\xi)$ and $A_2(\xi)$ are unknown functions to be determined from the boundary conditions.

Now the stress component τ_{yz} is given by

$$\tau_{yz}^{(j)}(x,z) = (-1)^j \mu_j \int_0^\infty \beta_j A_j(\xi) \exp(-\beta_j |z|) \cos\xi x \, d\xi \tag{8}$$

3. DERIVATION OF THE INTEGRAL EQUATIONS

The boundary conditions (2) and (3) imply that $v_1(x,0) = v_2(x,0)$ for all values of x and thus from (6) we get

$$A_1(\xi) = A_2(\xi) \tag{9}$$

Again the boundary conditions (2) and (4) lead to the following integral equations :

$$\int_0^\infty A_1(\xi)\cos\xi x \, d\xi = -v_0 \quad , \qquad c \le |x| \le 1 \tag{10}$$

$$\int_0^\infty (\mu_1\beta_1+\mu_2\beta_2)A_1(\xi)\cos\xi x \, d\xi = 0 \quad , \quad |x|<c, \ |x|>1 \tag{11}$$

Putting $\quad (\mu_1\beta_1+\mu_2\beta_2)A_1(\xi) = P(\xi)$ (12)

the equations (10) and (11) transform into the following set of equations involving $P(\xi)$:

$$\int_0^\infty \frac{P(\xi)}{(\mu_1\beta_1+\mu_2\beta_2)} \cos\xi x \, d\xi = -v_0 \quad , \quad c \le |x| \le 1 \tag{13}$$

$$\int_0^\infty P(\xi)\cos\xi x \, d\xi = 0 \quad , \qquad |x|<c, \ |x|>1 \tag{14}$$

4. SOLUTION OF THE PROBLEM

Let us consider the solution of the integral equations (13) and (14) in the form

$$P(\xi) = \int_c^1 t f(t^2) \cos\xi t \, dt \tag{15}$$

where $f(t^2)$ is an unknown function to be determined.

The relation (14) is therefore satisfied automatically and the equation (13) becomes

$$\int_c^1 t f(t^2) \int_0^\infty \frac{\cos\xi x \, \cos\xi t}{(\mu_1\beta_1 + \mu_2\beta_2)} \, d\xi \, dt = -v_o \quad , \quad c \le |x| \le 1 \tag{16}$$

Using the relation

$$\frac{\sin\xi x \, \sin\xi t}{\xi^2} = \int_0^x \int_0^t \frac{wvJ_o(\xi w) J_o(\xi v) dv dw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \tag{17}$$

the above equation converts to the form

$$\frac{d}{dx}\int_c^1 t f(t^2) \frac{\partial}{\partial t} \int_0^x \int_0^t \frac{wvL_1(v,w) dv dw dt}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} = -v_o \quad , \quad c \le |x| \le 1 \tag{18}$$

where $$L_1(v,w) = \int_0^\infty \frac{J_o(v\xi) J_o(w\xi)}{(\mu_1\beta_1 + \mu_2\beta_2)} \, d\xi \tag{19}$$

By a simple contour integration technique (Srivastava et al, 1980)

$$L_1(v,w) = \frac{i}{\mu_1}\left[\int_0^1 \frac{J_o(m_2\eta v) H_o^{(1)}(m_2\eta w)}{(\tau^2-\eta^2)^{1/2} + \mu(1-\eta^2)^{1/2}} \, d\eta \; + \right.$$

$$\left. + \int_1^\tau \frac{(\tau^2-\eta^2)^{1/2} J_o(m_2\eta v) H_o^{(1)}(m_2\eta w)}{\mu^2(\eta^2-1) + (\tau^2-\eta^2)} \, d\eta \right] \quad , \quad w > v \tag{20}$$

where $\tau = m_1/m_2$, $\mu = \mu_2/\mu_1$.

Substituting the series expansion of $J_0(\ )$ and $H_0^{(1)}(\ )$ the integrals arising in (20) have been evaluated assuming that $1<\tau<\mu$. We find after some algebraic manipulation,

$$L_1(v,w) = \frac{2}{\pi(\mu_1+\mu_2)}\left[\left\{\gamma+\log(m_2 w/2)-\frac{i\pi}{2}\right\}M + N - \frac{(w^2+v^2)}{4}\ Rm_2^2\log m_2\right] +$$

$$+\ o(m_2^2)\ ,\quad w>v$$

$$= \frac{2}{\pi(\mu_1+\mu_2)}\left[\left\{\gamma+\log(m_2 v/2)-\frac{i\pi}{2}\right\}M + N - \frac{(w^2+v^2)}{4}\ Rm_2^2\log m_2\right] +$$

$$+\ o(m^2)\ ,\quad w<v \tag{21}$$

where

$$M = -(1+\mu)\left[\int_0^1 \frac{d\xi}{(\tau^2-\xi^2)^{1/2}+\mu(1-\xi^2)^{1/2}} + \int_1^\tau \frac{(\tau^2-\xi^2)^{1/2}d\xi}{\mu^2(\xi^2-1)+(\tau^2-\xi^2)}\right] \tag{22}$$

$$N = -(1+\mu)\left[\int_0^1 \frac{\log\xi\ d\xi}{(\tau^2-\xi^2)^{1/2}+\mu(1-\xi^2)^{1/2}} + \int_1^\tau \frac{(\tau^2-\xi^2)^{1/2}\ \log\xi\ d\xi}{\mu^2(\xi^2-1)+(\tau^2-\xi^2)}\right] \tag{23}$$

$$R = -(1+\mu)\left[\int_0^1 \frac{\xi^2 d\xi}{(\tau^2-\xi^2)^{1/2}+\mu(1-\xi^2)^{1/2}} + \int_1^\tau \frac{\xi^2(\tau^2-\xi^2)^{1/2}d\xi}{\mu^2(\xi^2-1)+(\tau^2-\xi^2)}\right] \tag{24}$$

Now M can be written as

$$M = -(1+\mu)\left[-\mu\int_0^1 \frac{(1-\xi^2)^{1/2}\ d\xi}{\mu^2(\xi^2-1)+(\tau^2-\xi^2)} + \int_0^\tau \frac{(\tau^2-\xi^2)^{1/2}d\xi}{\mu^2(\xi^2-1)+(\tau^2-\xi^2)}\right]$$

$$= -(1+\mu)\left[- \frac{\mu}{(1-\mu^2)} \int_0^1 \frac{(1-\xi^2)^{1/2}\, d\xi}{\frac{\tau^2-\mu^2}{1-\mu^2} - \xi^2} + \frac{1}{(1-\mu^2)} \int_0^1 \frac{(1-\eta^2)^{1/2}\, d\eta}{\frac{\tau^2-\mu^2}{\tau^2(1-\mu^2)} - \eta^2} \right]$$

Using the result

$$\int_0^1 \frac{(1-z^2)^{1/2}\, dz}{a^2-z^2} = \frac{\pi}{2}\left[1 - \frac{(a^2-1)^{1/2}}{a} \right] \qquad (a>1)$$

M can be finally expressed as

$$M = -\frac{\pi}{2} \tag{25}$$

Similarly, using the results

$$\int_0^1 \frac{(1-z^2)^{1/2}\log z}{a^2-z^2}\, dz = \frac{\pi}{2}\left[\frac{(a^2-1)^{1/2}}{a} \log\left(1 + \frac{(a^2-1)^{1/2}}{a}\right) - \log 2 \right] \quad , \quad (a>1)$$

$$\int_0^1 \frac{z^2(1-z^2)^{1/2}\, dz}{a^2-z^2} = -\frac{\pi}{2}\left[\frac{1}{2} - a^2 + a(a^2-1)^{1/2} \right] \quad , \quad (a>1)$$

in (23) and (24), N and R can be obtained as

$$N = \frac{-\pi}{2(\mu-1)}\left[\mu\sqrt{\frac{(\tau^2-1)}{(\mu^2-\tau^2)}}\ \tan^{-1}\left\{\frac{(\tau^2-1)^{1/2}(\mu^2-\tau^2)^{1/2}}{(\tau^2+\mu)}\right\} - \log\tau - (\mu-1)\log 2 \right] \tag{26}$$

$$R = -\frac{\pi(\tau^2+\mu)}{4(1+\mu)} \tag{27}$$

and $\gamma = 0.5772157........$ is Euler's constant.

Now differentiating both sides of the equation (16) with respect to x and making some rearrangements and finally using the result (17), we obtain,

$$x\int_c^1 \frac{tf(t^2)dt}{x^2-t^2} = \int_c^1 tf(t^2)\frac{\partial}{\partial t}\int_0^x\int_0^t \frac{wvL_2(v,w)dvdwdt}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} ,$$

$$c\leq|x|\leq 1 \qquad (28)$$

where $$L_2(v,w) = \int_0^\infty \xi H(\xi)J_0(\xi v)J_0(\xi w)\ d\xi \qquad (29)$$

and $$H(\xi) = \frac{\mu_1(\beta_1-\xi) + \mu_2(\beta_2-\xi)}{\mu_1\beta_1 + \mu_2\beta_2} \qquad (30)$$

For small values of m_1 and m_2, we use the contour integration technique mentioned above to evaluate the integral given by (29) as :

$$L_2(v,w) = -i(1+\mu)m_2^2\left[\int_0^1 \frac{\eta^2 J_0(m_2\eta v)H_0^{(1)}(m_2\eta w)}{(\tau^2-\eta^2)^{1/2}+\mu(1-\eta^2)^{1/2}}\,d\eta\ +\right.$$

$$\left. +\int_1^\tau \frac{\eta^2(\tau^2-\eta^2)^{1/2}J_0(m_2\eta v)H_0^{(1)}(m_2\eta w)}{\mu^2(\eta^2-1)+(\tau^2-\eta^2)}\,d\eta\right] ,\ w>v , \qquad (31)$$

Following the similar process as done to derive the relation (21), (31) can be written as

$$L_2(v,w) = -\frac{2}{\pi}Rm_2^2\log m_2 + o(m_2^2) \qquad (32)$$

where R is given by (27).

Let us consider the solution of (28) as

$$f(t^2) = f_o(t^2) + m_z^2 \log m_z f_1(t^2) + o(m_z^2) \tag{33}$$

Substituting the above expression of $f(t^2)$ and the value of $L_z(v,w)$ in (28) and equating the coefficients of equal powers of m_z we get

$$\int_c^1 \frac{tf_o(t^2)}{x^2-t^2}\,dt = 0 \quad , \qquad c\le|x|\le1 \tag{34}$$

and
$$\int_c^1 \frac{tf_1(t^2)}{x^2-t^2}\,dt = -\frac{2}{\pi}\,R\int_c^1 tf_o(t^2)dt \quad , \qquad c\le|x|\le1 \tag{35}$$

From the paper of Srivastava and Lowengrub (1968) we know that the solution to the integral equation

$$\frac{2}{\pi}\int_a^b \frac{th(t^2)}{t^2-y^2}\,dt = p(y) \quad , \qquad y\in(a,b)$$

is given by

$$h(t^2) = -\frac{1}{\pi}\left[\frac{t^2-a^2}{b^2-t^2}\right]^{1/2}\int_a^b\left[\frac{b^2-y^2}{y^2-a^2}\right]^{1/2}\frac{2yp(y)\,dy}{y^2-t^2} +$$
$$+\frac{C}{(t^2-a^2)^{1/2}(b^2-t^2)^{1/2}}$$

where C is an arbitrary constant.

Applying the above result in the integral equations (34) and (35), the solutions $f_o(t^2)$ and $f_1(t^2)$ are obtained as

$$f_0(t^2) = \frac{D_1}{(1-t^2)^{1/2}(t^2-c^2)^{1/2}} \tag{36}$$

and $$f_1(t^2) = \frac{2}{\pi} RD_1\left(\frac{t^2-c^2}{1-t^2}\right)^{1/2} + \frac{D_2}{(1-t^2)^{1/2}(t^2-c^2)^{1/2}} \tag{37}$$

where D_1 and D_2 are constants.

The constants are determined by putting the value of $L_1(v,w)$ from (21) and the value of $f(t^2)$ obtained from (33), (36) and (37) in the equation (18). Equating the coefficients of like powers of m_2 from both sides of the resulting equation we obtain

$$D_1 = \frac{-(\mu_1+\mu_2)v_0}{\left[\left(\gamma+\log(m_2/2) - \frac{\pi i}{2}\right)M + N + M\log(1-c^2)^{1/2}\right]} \tag{38}$$

and $$D_2 = -\frac{D_1^2 R}{(\mu_1+\mu_2)v_0}\left[\frac{1}{2}(c^2+1) + \frac{(\mu_1+\mu_2)(1-c^2)v_0}{\pi D_1}\right] \tag{39}$$

Now, the displacement $v_1=v_2=v(x,z)$ in the plane $z=0$ is obtained from (6), (12) and (15) as

$$v(x,0) = \int_0^\infty A_1(\xi)\cos(\xi x)\, d\xi \quad , \quad |x|>1 \ , \ |x|<c$$

$$= \frac{d}{dx}\int_c^1 tf(t^2)\frac{\partial}{\partial t}\int_0^x\int_0^t \frac{wvL_1(v,w)dvdwdt}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \quad , \quad |x|>1 \ , \ |x|<c$$

Substituting the values of $L_1(v,w)$ and $f(t^2)$ from (21) and (33) in the above exrpession we get

$$v(x,0) = \frac{2}{\pi(\mu_1+\mu_2)} \frac{d}{dx}\int_c^1 tf_o(t^2)\frac{\partial}{\partial t}\int_o^x\int_o^t \frac{vw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \times$$

$$\times \left\{\left[\gamma+\log(m_2 w/2)-\frac{i\pi}{2}\right]M + N\right\} dvdwdt +$$

$$+ \frac{2}{\pi(\mu_1+\mu_2)} m_2^2 \log m_2 \left[\frac{d}{dx}\int_c^1 tf_1(t^2)\frac{\partial}{\partial t}\int_o^x\int_o^t \frac{vw}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \times\right.$$

$$\times \left\{\left[\gamma+\log(m_2 w/2)-\frac{i\pi}{2}\right]M + N\right\} dvdwdt -$$

$$\left. - \frac{R}{4} \frac{d}{dx}\int_c^1 tf_o(t^2)\frac{\partial}{\partial t}\int_o^x\int_o^t \frac{vw(v^2+w^2)dwdvdt}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \right] + o(m_2^2)$$

Using (36) and (37) in the above equation and integrating term by term v(x,0) can be finally obtained as

$$v(x,0) = -v_o + \frac{M}{(\mu_1+\mu_2)}\left[D_1 + m_2^2 \log m_2\left\{D_2 + \frac{(1-c^2)RD_1}{\pi}\right\}\right]\sinh^{-1}\left(\frac{c^2-x^2}{1-c^2}\right)^{1/2} -$$

$$- \frac{MRD_1 m_2^2 \log m_2}{\pi(\mu_1+\mu_1)}\left\{(c^2-x^2)(1-x^2)\right\}^{1/2} + o(m_2^2) \quad , \quad |x|<c$$

$$= -v_o \quad , \quad c\leq|x|\leq 1$$

$$= -v_o + \frac{M}{(\mu_1+\mu_2)}\left[D_1 + m_2^2 \log m_2\left\{D_2 + \frac{(1-c^2)RD_1}{\pi}\right\}\right]\sinh^{-1}\left(\frac{x^2-1}{1-c^2}\right)^{1/2} +$$

$$+ \frac{MRD_1 m_2^2 \log m_2}{\pi(\mu_1+\mu_1)}\left\{(x^2-c^2)(x^2-1)\right\}^{1/2} + o(m_2^2) \quad , \quad |x|>1$$

(40)

The difference of the stress components on the lower and upper surfaces of the strips is found from equations (8), (12) and (15) and is given by

$$\tau_{yz}^{(2)}(x,0) - \tau_{yz}^{(1)}(x,0) = \int_0^{\infty} (\mu_1\beta_1+\mu_2\beta_2)A_1(\xi)\cos(\xi x)d\xi$$

$$= \int_c^1 tf(t^2)\int_0^{\infty} \cos(\xi x)\cos(\xi t)d\xi\, dt$$

$$= \frac{\pi}{2}\int_c^1 tf(t^2)\delta(x-t)dt$$

$$= \frac{\pi}{2}\, xf(x^2) \quad , \quad c\le|x|\le 1$$

After putting the values of $f(x^2)$ from (33), (36), (37) and integrating, the difference of the stress components finally becomes

$$\tau_{yz}^{(2)}(x,0) - \tau_{yz}^{(1)}(x,0) = \frac{\pi x}{2\left[(1-x^2)(x^2-c^2)\right]^{1/2}}\Big[D_1 + m_2^2\log m_2 \times$$

$$\times \left\{ \frac{2}{\pi}RD_1(x^2-c^2)+D_2\right\}\Big] \tag{41}$$

Now, putting $\mu_1=\mu_2=\mu_0$, $v_0=1$ and omiting $m_2^2\log m_2$ order term we get from (40) and (41) the displacement and stress components for single medium as

$$v(x,0) = -1 - \frac{1}{2\left[q_2 + \log(1-c^2)^{1/2}\right]} \begin{cases} \sinh^{-1}\left(\dfrac{c^2-x^2}{1-c^2}\right)^{1/2}, & |x|<c \\ 0, & c\le|x|\le 1 \\ \sinh^{-1}\left(\dfrac{x^2-1}{1-c^2}\right)^{1/2}, & |x|>1 \end{cases}$$

and

$$\tau_{yz}^{(j)}(x,0) = \frac{\mp\mu_0|x|}{\left[q_2+\log(1-c^2)^{1/2}\right]\left[(1-x^2)(x^2-c^2)\right]^{1/2}} + o(m_2^2), \quad c\le|x|\le 1$$

where $q_2 = \gamma+\log(m_2/4) - \frac{\pi i}{2}$,

which coincide with the results obtained by Jain and Kanwal(1972). Now defining the stress intensity factors by the relations

$$K_1 = \underset{x\to 1-}{\mathrm{Lt}} (1-x)^{1/2}\left[\frac{\tau_{yz}^{(2)}(x,0) - \tau_{yz}^{(1)}(x,0)}{(\mu_1+\mu_2)v_0}\right], \quad c|x|<1$$

$$K_c = \underset{x\to c+}{\mathrm{Lt}} (x-c)^{1/2}\left[\frac{\tau_{yz}^{(2)}(x,0) - \tau_{yz}^{(1)}(x,0)}{(\mu_1+\mu_2)v_0}\right], \quad c|x|<1$$

we obtain from (41)

$$K_1 = \frac{\pi}{2\sqrt{2}(1-c^2)^{1/2}}\left[D_1+m_2^2\log m_2\left\{\frac{2}{\pi}RD_1(1-c^2)+D_2\right\}\right] \tag{42}$$

and $$K_c = \frac{\pi\sqrt{c}}{2\sqrt{2}(1-c^2)^{1/2}}\left[D_1+m_2^2\log m_2 D_2\right] \tag{43}$$

Putting c=0 we get the displacement $v^{o}(x,0)$ and stress intensity factor K_{1}^{o} from (40) and (41) for the single strip as :

$$v^{o}(x,0) = -v_{o} + \frac{M}{(\mu_{1}+\mu_{2})}\left[D_{1}^{o}+m_{2}^{2}\log m_{2}\left\{ D_{2}^{o}+\frac{RD_{1}^{o}}{\pi}\right\}\right]\sinh^{-1}(x^{2}-1)^{1/2} +$$

$$+ \frac{|x|MRD_{1}^{o}m_{2}^{2}\log m_{2}}{\pi(\mu_{1}+\mu_{2})}(x^{2}-1)^{1/2} + o(m_{2}^{2}) \quad , \quad |x|>1$$

$$= -v_{o} \quad , \quad |x|\leq 1 \tag{44}$$

and $$K_{1}^{o} = \frac{\pi}{2\sqrt{2}}\left[D_{1}^{o}+ m_{2}^{2}\log m_{2}\left\{ \frac{2}{\pi}RD_{1}^{o} + D_{2}^{o}\right\}\right] \tag{45}$$

where $$D_{1}^{o} = \frac{-(\mu_{1}+\mu_{2})v_{o}}{\left[\left(\gamma+\log(m_{2}/2) - \frac{\pi i}{2}\right)M + N\right]} \tag{46}$$

and $$D_{2}^{o} = -\frac{(D_{1}^{o})^{2}R}{(\mu_{1}+\mu_{2})v_{o}}\left[\frac{1}{2} + \frac{(\mu_{1}+\mu_{2})v_{o}}{\pi D_{1}^{o}}\right] \tag{47}$$

5. NUMERICAL RESULTS

While calculating numerical results, the displacement and the stress intensity factors have been obtained for the following set of materials :

Aluminium $\rho_1=2.7\text{gm/cm}^3$, $\mu_1=2.63\times10^{11}\text{dyne/cm}^2$

Wrought iron $\rho_2=7.8\text{gm/cm}^3$, $\mu_2=7.7\times10^{11}\text{dyne/cm}^2$

The displacement field in the interface near about the rigid strips has been depicted by means of graphs. It is interesting to note that the magnitude of the real part of the displacement increases with the increase in the value of the wave number m_2. The variation of the displacement with c, the separatong distance between the strips has also been showen by means of graphs. Further the graphs of the real part of the stress intensity factors at both the edges of the strips versus dimensionless wave number m_2 for several values of c have been plotted to show the nature of the variation of the stress intensity factors with different parameters.

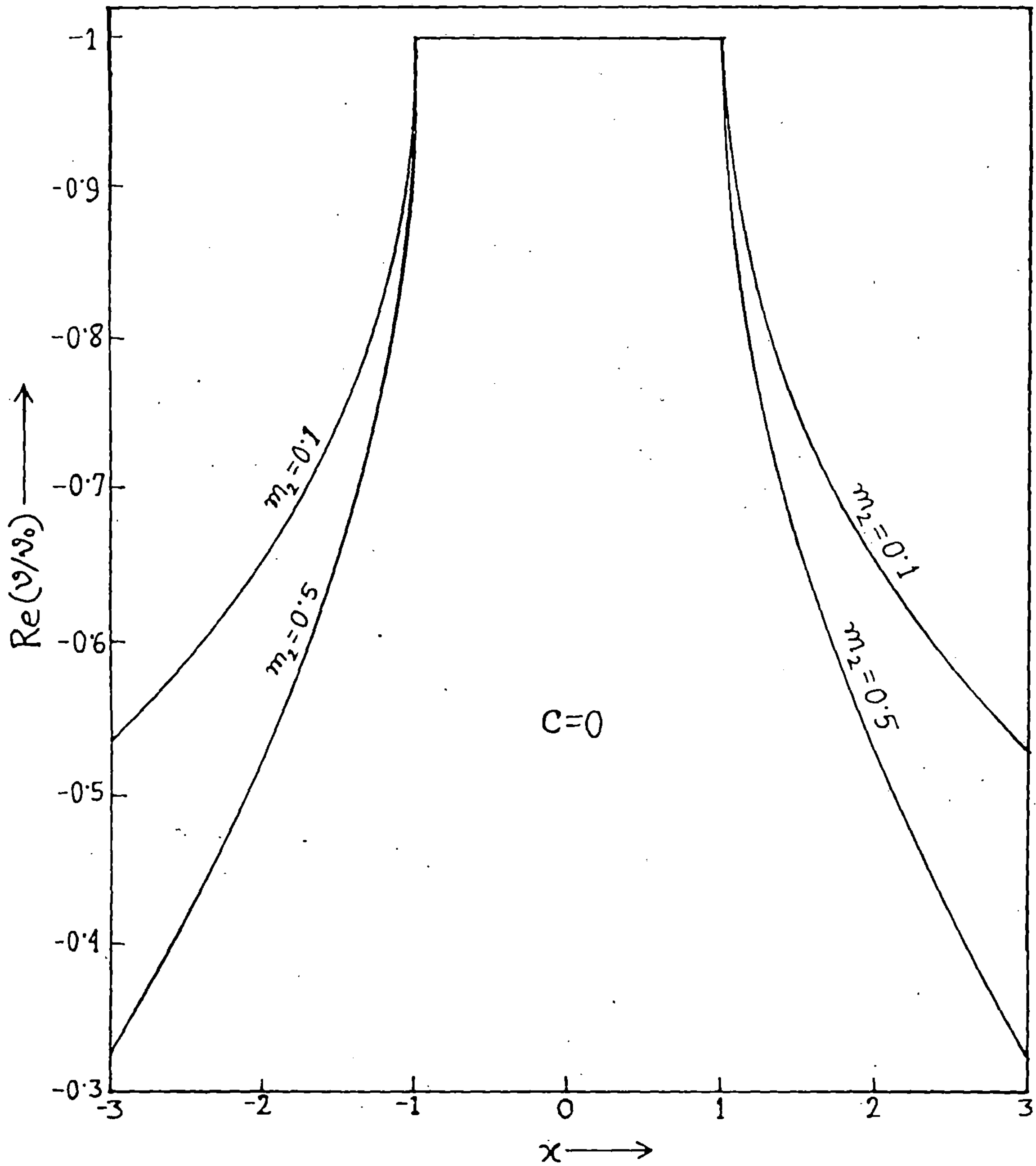

Fig. 2. Re(v/v_0) versus distance for $c=0$ (single strip).

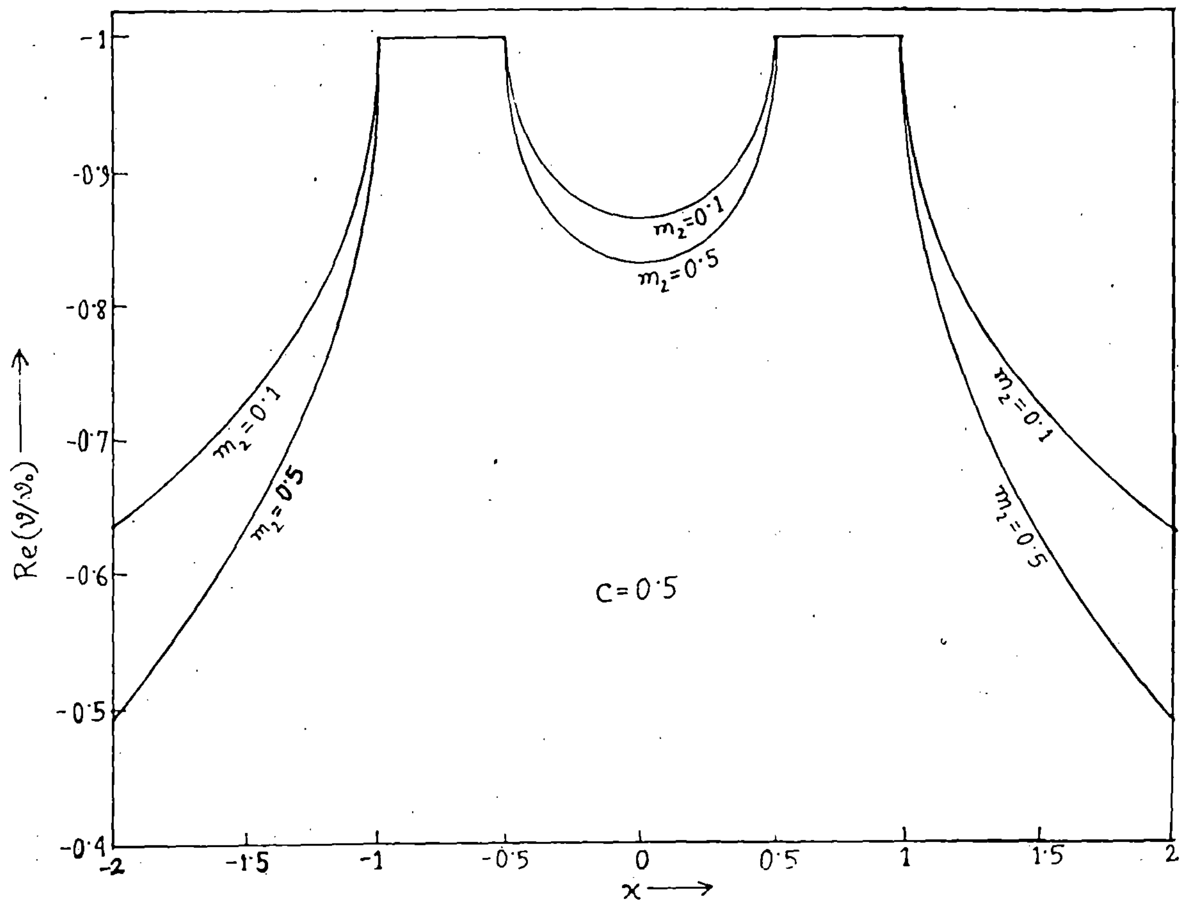

Fig. 3 Re(v/v_o) versus distance for c = 0 5

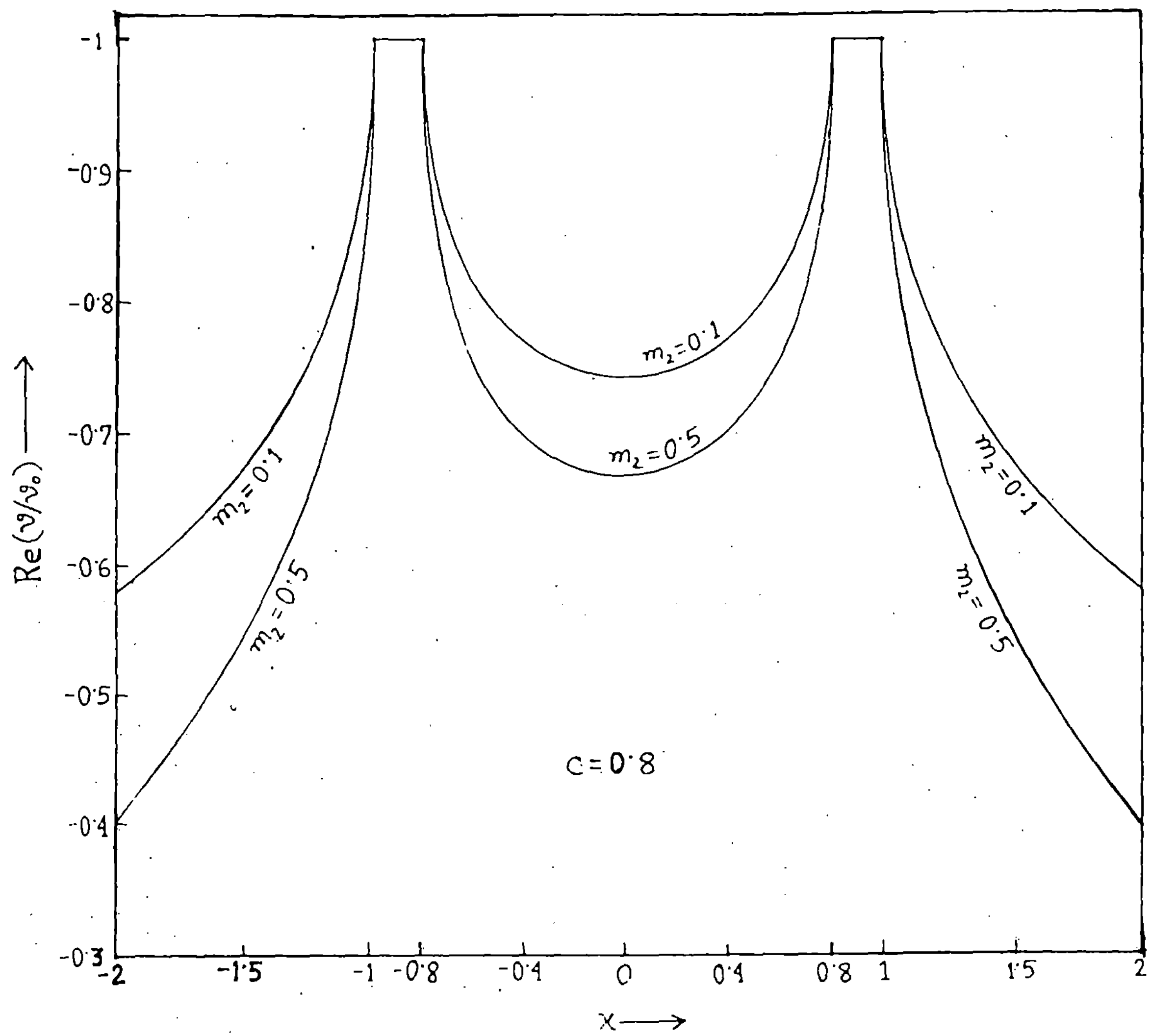

Fig 4 Re (v/v_o) versus distance for $c=0.8$

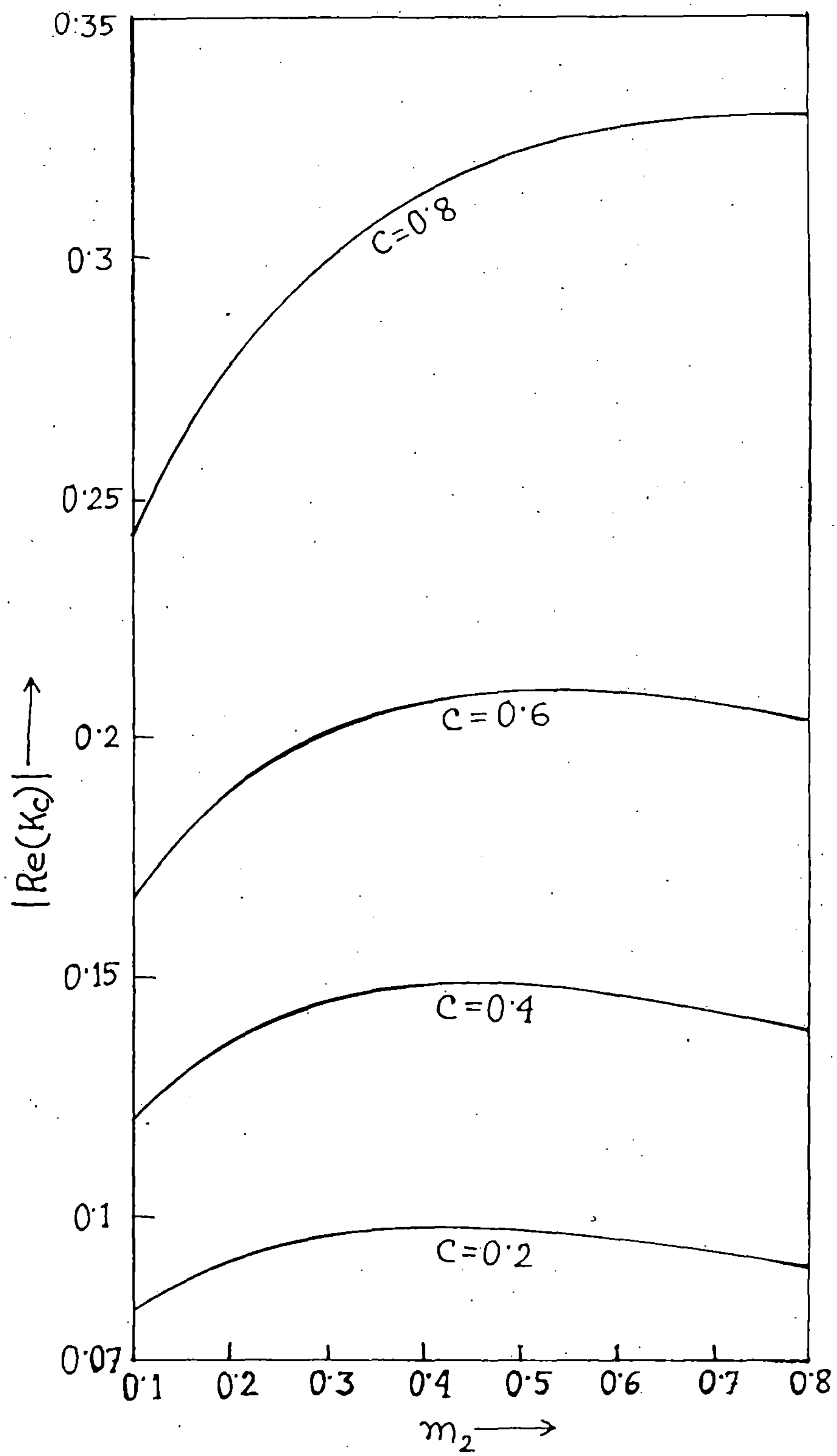

Fig. 5. $|Re(K_c)|$ versus m_2, the nondimensional wave number.

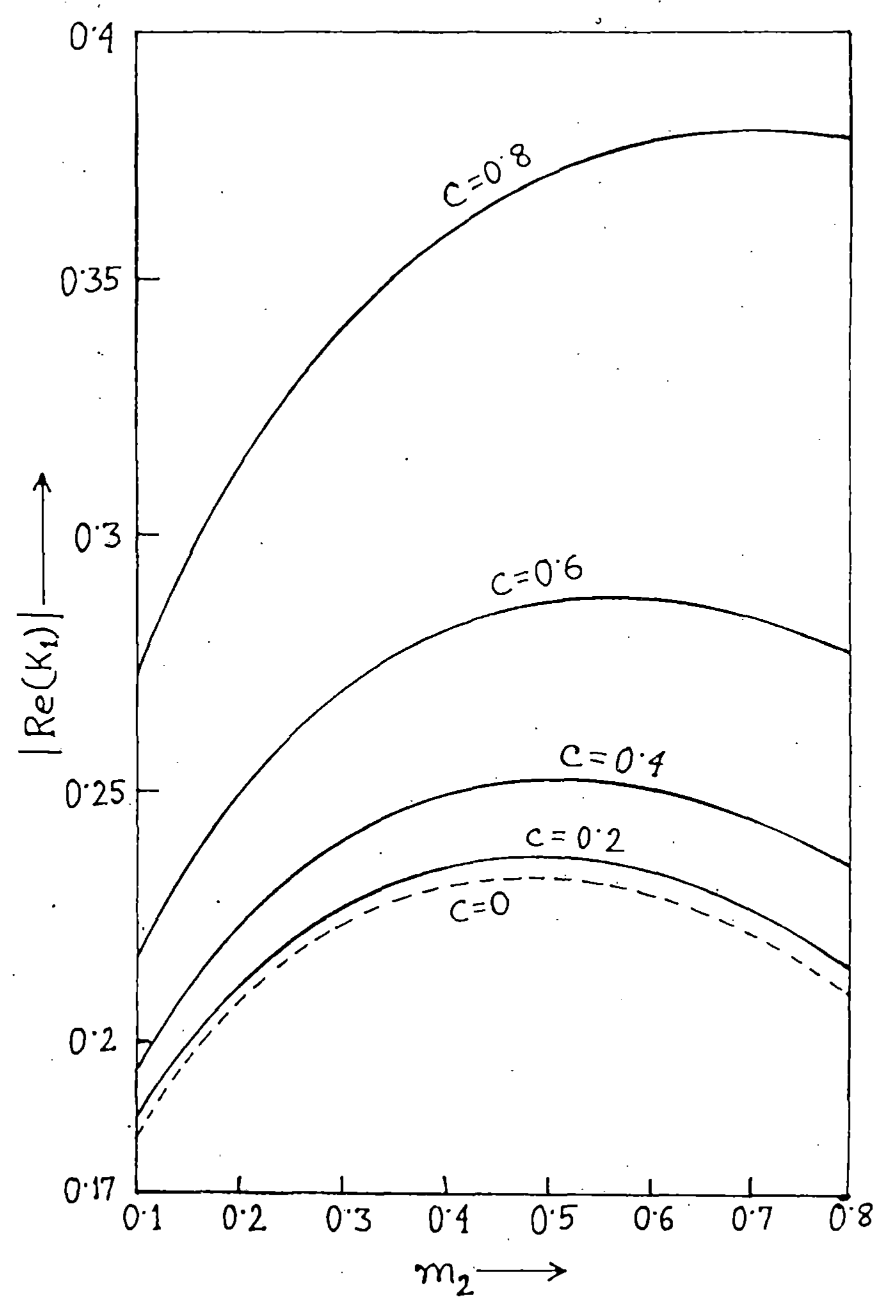

Fig. 6. $|\mathrm{Re}(K_1)|$ versus m_2, the nondimensional wave number.

ON STEADY MOTION OF FOUR RIGID STRIPS ON THE SURFACE OF A SEMI-INFINITE ELASTIC MEDIUM

1. INTRODUCTION

Recently, the problems of diffraction of elastic waves by cracks or inclusions have aroused attention in the field of fracture mechanics in view of their application in Seismology and Geophysics. Study of a single Griffith crak as well as two parallel and coplanar Griffith cracks have been made by Mal (1970) and Jain et al (1972), Itou (1980). The corresponding problems of diffraction by a single and two parallel rigid strips have been solved by Wickham (1977), Palaiya et al (1981) and Jain et al (1972), Mandal et al (1992) respectively. In most of the cases the problems have been solved by the integral equation technique. But the solution of interesting problems involving the scattering of elastic waves by more than two coplanar Griffith cracks or strips are still lacking. The statical problem of three coplanar cracks in an infinite transversely isotropic medium has been studied by Dhawan et al (1978). Using integral equation method and Hilbert transform the stress distribution and displacement have been derived in closed form. The interesting problem of interaction between a Griffith crack and two rigid inclusions has been discussed by Matysiak et al (1986). They considered that the crack had been opened out to the prescribed shape and the normal stresses at the crack tips and at the ends of the inclusions have been analysed for the crack openings in the shape of an ellipse and that of two symmetrical parabolas.

In our case, we have considered the two dimensional problems of diffraction of elastic waves by four coplanar parallel rigid strips moving steadily on the free surface of a semi-infinite isotropic elastic medium. By Fourier transform the five part mixed boundary value problem has been reduced to the solution of a set of four integral equations. Following the technique, developed by Srivastava and Lowengrub (1970), the quadruple integral equations have been solved. The normal stress under the strips and displacement outside the strips are derived in closed form. The effect of stress intensity factors at the edges of the strips is shown by means of graphs. Also letting the strip velocity tend to zero the results for statical problem have been prescribed in this paper as a particular case.

2. FORMULATION OF THE PROBLEM

Consider a semi-infinite elastic medium on which four rigid strips are moving steadily in the X - direction with constant velocity v. Strips are assumed to be in smooth contact with the semi-infinite medium and the vertical displacement just under the strips are assumed to be prescribed. In terms of the displacement potentials, non vanishing displacement components are given by

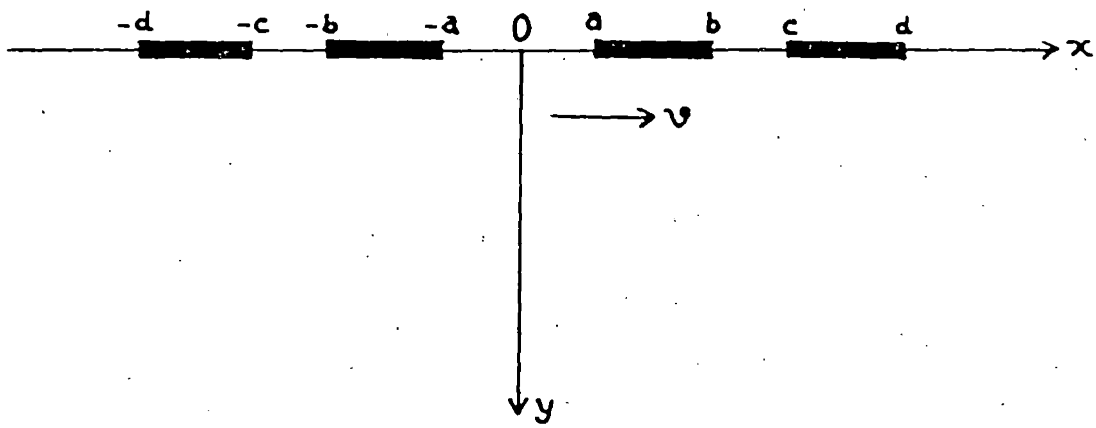

Fig. 1. Geometry of the strips .

$$u_1 = \frac{\partial \phi}{\partial X} - \frac{\partial \psi}{\partial Y} \quad , \quad u_2 = \frac{\partial \phi}{\partial Y} + \frac{\partial \psi}{\partial X} \tag{1}$$

where ϕ and ψ satisfy the following equations

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} = \frac{1}{c_1^2} \frac{\partial^2 \phi}{\partial t^2}$$

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} = \frac{1}{c_2^2} \frac{\partial^2 \psi}{\partial t^2} \tag{2}$$

where $c_1^2 = \dfrac{\lambda + 2\mu}{\rho}$, $c_2^2 = \dfrac{\mu}{\rho}$.

It is convenient to shift the origin of co-ordinate at X=vt. New co-ordinate axes (x,y) are parallel to the fixed ones (X,Y) (fig.1).

Therefore putting x=X-vt, y=Y we obtain from (1) to (2)

$$u_1 = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \quad , \quad u_2 = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \tag{3}$$

and

$$\beta_1^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\beta_2^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{4}$$

where $\beta_1^2 = 1 - v^2/c_1^2$ and $\beta_2^2 = 1 - v^2/c_2^2$.

The location of the strips referred to moving system of co-ordinates are $a \le |x| \le b$, $c \le |x| \le d$, $y=0$, $|z| < \infty$.

In terms of ϕ and ψ the non vanishing stress components are

$$\tau_{xy}(x,y) = \mu\left[2\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right]$$

$$\tau_{yy}(x,y) = -\mu \left\{ (1+\beta_2^2)\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 \psi}{\partial x \partial y} \right\} \qquad (5)$$

The boundary conditions are

$$u_2 = v_o , \quad a \le |x| \le b , \quad c \le |x| \le d , \quad y=0 \qquad (6)$$

$$\tau_{xy} = 0 , \quad -\infty < x < \infty , \quad y=0 \qquad (7)$$

$$\tau_{yy} = 0 , \quad |x| < a , \quad b < |x| < c , \quad |x| > d \qquad (8)$$

where v_o is constant.

Solutions of the equations (4) are given by

$$\phi = \int_0^\infty A_1(\xi) e^{-\beta_1 \xi y} \cos\xi x \, d\xi$$

$$\psi = \int_0^\infty A_2(\xi) e^{-\beta_2 \xi y} \sin\xi x \, d\xi \qquad (9)$$

where $A_1(\xi)$ and $A_2(\xi)$ are unknown functions to be determined from the boundary conditions.

From the boundary condition (7) we get

$$A_2(\xi) = \frac{2\beta_1}{(1+\beta_2^2)} A_1(\xi) \tag{10}$$

Now the displacement and stress components are

$$u_1(x,y) = -\int_0^{\infty} \xi \left[e^{-\beta_1 \xi y} - \frac{2\beta_1\beta_2}{(1+\beta_2^2)} e^{-\beta_2 \xi y} \right] A_1(\xi) \sin\xi x \; d\xi \tag{11}$$

$$u_2(x,y) = \int_0^{\infty} \xi \left[-\beta_1 e^{-\beta_1 \xi y} + \frac{2\beta_1}{(1+\beta_2^2)} e^{-\beta_2 \xi y} \right] A_1(\xi) \cos\xi x \; d\xi \tag{12}$$

$$\tau_{xy}(x,y) = \mu \int_0^{\infty} 2\beta_1 \xi^2 \left[e^{-\beta_1 \xi y} - e^{-\beta_2 \xi y} \right] A_1(\xi) \sin\xi x \; d\xi \tag{13}$$

$$\tau_{yy}(x,y) = -\mu \int_0^{\infty} \xi^2 \left[-(1+\beta_2^2) e^{-\beta_1 \xi y} + \frac{4\beta_1\beta_2}{(1+\beta_2^2)} e^{-\beta_2 \xi y} \right] A_1(\xi) \cos\xi x \; d\xi \tag{14}$$

Putting $$A(\xi) = \xi^2 A_1(\xi) \tag{15}$$

the boundary conditions (6) and (8) lead to the following quadruple integral equations (assuming that $v \neq v_R$,where v_R is the Rayleigh wave velocity).

$$\int_0^\infty A(\xi)\cos\xi x \, d\xi = 0 \quad , \quad |x|<a \tag{15}$$

$$\int_0^\infty \frac{A(\xi)}{\xi} \cos\xi x \, d\xi = p_o \quad , \quad a\leq|x|\leq b \tag{16}$$

$$\int_0^\infty A(\xi)\cos\xi x \, d\xi = 0 \quad , \quad b<|x|<c \quad , \quad |x|>d \tag{17}$$

$$\int_0^\infty \frac{A(\xi)}{\xi} \cos\xi x \, d\xi = p_o \quad , \quad c\leq|x|\leq d \tag{18}$$

where

$$p_o = \frac{1+\beta_2^2}{\beta_1(1-\beta_2^2)} v_o \tag{19}$$

3. SOLUTION OF THE QUADRUPLE INTEGRAL EQUATIONS

Let us consider the solutions of the integral equations (15) - (18) in the form

$$A(\xi) = \int_a^b \frac{h(t^2)}{t} \left\{1 - \cos\xi t\right\} dt + \int_c^d \frac{g(u^2)}{u} \left\{1 - \cos\xi u\right\} du \tag{20}$$

where $h(t^2)$ and $g(u^2)$ are unknown functions.

This choice of $A(\xi)$ automatically satisfies the equations (15) and (17). Substituting the value of $A(\xi)$ from (20) into (16) and using the relation (Gradshteyn and Ryzhik, 1965)

$$\int_0^\infty \frac{\cos\xi y(1-\cos\xi u)}{\xi}\, d\xi = \frac{1}{2}\log\left|1-\frac{u^2}{y^2}\right| \tag{21}$$

we obtain

$$\frac{1}{2}\int_a^b \frac{h(t^2)}{t}\log\left|1-\frac{t^2}{x^2}\right|dt + \frac{1}{2}\int_c^d \frac{g(u^2)}{u}\log\left|1-\frac{u^2}{x^2}\right|du = p_0 \ , \quad a\le|x|\le b \tag{22}$$

On differentiation with respect to x, the above equation yields

$$\int_a^b \frac{th(t^2)}{x^2-t^2}\, dt = -\int_c^d \frac{ug(u^2)}{x^2-u^2}\, du \quad , \qquad a\le|x|\le b$$

from which applying Hilbert transform (Srivastava et al, 1970) we get $h(t^2)$ as

$$h(t^2) = -\frac{2}{\pi}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\int_c^d \frac{ug(u^2)}{u^2-t^2}\left(\frac{u^2-b^2}{u^2-a^2}\right)^{1/2}du + \frac{D_1}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$

$$, \quad a\le t\le b \tag{23}$$

where D_1 is an arbitrary constant to be determined.
Next, using the relations (20) and (21) in (18) we obtain

$$\frac{1}{2}\int_a^b \frac{h(t^2)}{t}\log\left|1-\frac{t^2}{x^2}\right|dt + \frac{1}{2}\int_c^d \frac{g(u^2)}{u}\log\left|1-\frac{u^2}{x^2}\right|du = p_0 \ , \quad c\le|x|\le d \tag{24}$$

Differentiating both sides of the above equation with respect to x and substituting the value of $h(t^2)$ from (23) , we obtain

$$\int_c^d \frac{ug(u^2)}{x^2-u^2}\,du - \frac{2}{\pi}\int_a^b \frac{t}{x^2-t^2}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\int_c^d \frac{ug(u^2)}{u^2-t^2}\left(\frac{u^2-b^2}{u^2-a^2}\right)^{1/2} du\,dt\;+$$

$$+\,D_1\int_a^b \frac{t\;dt}{(x^2-t^2)\sqrt{(t^2-a^2)(b^2-t^2)}} = 0 \quad , \quad c\le x\le d$$

Using the results

$$\int_a^b \frac{t\;dt}{(x^2-t^2)\sqrt{(t^2-a^2)(b^2-t^2)}} = \frac{\pi}{2}\,\frac{1}{\sqrt{(x^2-a^2)(x^2-b^2)}}$$

and
$$\int_a^b \frac{t}{(u^2-t^2)(x^2-t^2)}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} dt$$

$$= \frac{\pi}{2}\,\frac{1}{(x^2-u^2)}\left[-\sqrt{\frac{(x^2-a^2)}{(x^2-b^2)}} + \sqrt{\frac{(u^2-a^2)}{(u^2-b^2)}}\right]$$

the above equation can be written as

$$\int_c^d \frac{ug(u^2)}{u^2-x^2}\left(\frac{u^2-b^2}{u^2-a^2}\right)^{1/2} du = \frac{\pi}{2}\,\frac{D_1}{(x^2-a^2)} \quad , \quad c\le u\le d$$

which after Hilbert transform and on use of the result

$$\int_c^d \frac{x}{(x^2-u^2)(x^2-a^2)}\left(\frac{d^2-x^2}{x^2-c^2}\right)^{1/2} dx = -\,\frac{\pi}{2}\,\frac{1}{(u^2-a^2)}\sqrt{\frac{(d^2-a^2)}{(c^2-a^2)}}\;,$$

yields

$$g(u^2)=D_1\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{u^2-c^2}{d^2-u^2}\right)^{1/2}\frac{1}{\sqrt{(u^2-b^2)(u^2-a^2)}}+$$

$$+\frac{D_2}{\sqrt{(u^2-c^2)(d^2-u^2)}}\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\quad,\quad c\le u\le d \qquad (25)$$

where D_2 is another arbitrary constant.

Substituting the value of $g(u^2)$ from (25) in (23), $h(t^2)$ takes the form

$$h(t^2)= -\frac{2}{\pi}D_1\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}\int_c^d\frac{u}{(u^2-t^2)(u^2-a^2)}\left(\frac{u^2-c^2}{d^2-u^2}\right)^{1/2}du\ -$$

$$-\frac{2}{\pi}D_2\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\int_c^d\frac{u\,du}{(u^2-t^2)\sqrt{(u^2-c^2)(d^2-u^2)}}+\frac{D_1}{\sqrt{(t^2-a^2)(b^2-t^2)}}$$

$$a\le t\le b$$

Again, using the results

$$\int_c^d\frac{u}{(u^2-t^2)(u^2-a^2)}\left(\frac{u^2-c^2}{d^2-u^2}\right)^{1/2}du$$

$$=\frac{\pi}{2}\frac{1}{(t^2-a^2)}\left[-\sqrt{\frac{(c^2-t^2)}{(d^2-t^2)}}+\sqrt{\frac{(c^2-a^2)}{(d^2-a^2)}}\right]$$

and
$$\int_c^d\frac{u\,du}{(u^2-t^2)\sqrt{(u^2-c^2)(d^2-u^2)}}=\frac{\pi}{2}\frac{1}{\sqrt{(d^2-t^2)(c^2-t^2)}}$$

in the above expression, $h(t^2)$ can be found out in the form

$$h(t^2)=D_1\left[\frac{d^2-a^2}{c^2-a^2}\right]^{1/2}\left[\frac{c^2-t^2}{d^2-t^2}\right]^{1/2}\frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} -$$

$$-\frac{D_2}{\sqrt{(d^2-t^2)(c^2-t^2)}}\left[\frac{t^2-a^2}{b^2-t^2}\right]^{1/2}, \quad a\le t\le b \tag{26}$$

Now in order to determine the unknown constants D_1 and D_2, occuring in the expression of $g(u^2)$ and $h(t^2)$ given by (25) and (26) respectively, we multiply the equation (22) by $\frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}$ and integrate with respect to x from a to b and then using the result

$$\int_a^b \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\log\left|1-\frac{z^2}{x^2}\right|dx = \begin{cases}\pi\log\left[\dfrac{\sqrt{(a^2-z^2)}+\sqrt{(b^2-z^2)}}{a+b}\right], & 0<z<a\\[2mm] \dfrac{\pi}{2}\log\left[\dfrac{b-a}{b+a}\right], & a\le z\le b\\[2mm] \pi\log\left[\dfrac{\sqrt{(z^2-a^2)}+\sqrt{(z^2-b^2)}}{a+b}\right], & z>b\end{cases} \tag{27}$$

we obtain,

$$\frac{\pi}{2}\log\left[\frac{b-a}{b+a}\right]\int_a^b \frac{h(t^2)}{t}\,dt + \pi\int_c^d \frac{g(u^2)}{u}\log\left[\frac{\sqrt{(u^2-a^2)}+\sqrt{(u^2-b^2)}}{a+b}\right]du = \pi p_o$$

which after substituting the values of $h(t^2)$ and $g(u^2)$ from (26) and (25) finally takes the form

$$D_1X_1 + D_2X_2 = p_o \tag{28}$$

where

$$X_1 = \frac{1}{2}\log\left(\frac{b-a}{b+a}\right)\frac{(c^2-b^2)}{b^2(c^2-a^2)}\left(\frac{d^2-a^2}{d^2-b^2}\right)^{1/2}\Pi\left(\frac{\pi}{2},\frac{c^2(b^2-a^2)}{b^2(c^2-a^2)},r\right) + \left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}M_1 \tag{29}$$

$$X_2 = -\frac{1}{2}\log\left(\frac{b-a}{b+a}\right)\frac{1}{\sqrt{(d^2-b^2)(c^2-a^2)}}\left\{\left(1-\frac{a^2}{c^2}\right)F\left(\frac{\pi}{2},r\right) - \frac{a^2(c^2-b^2)}{b^2c^2}\times\right.$$

$$\left.\times\,\Pi\left(\frac{\pi}{2}\,,\,\frac{c^2(b^2-a^2)}{b^2(c^2-a^2)}\,,\,r\right)\right\} + M_2 \tag{30}$$

$$M_1 = \int_c^d \frac{1}{u}\log\left[\frac{\sqrt{(u^2-a^2)}+\sqrt{(u^2-b^2)}}{a+b}\right]\left(\frac{u^2-c^2}{d^2-u^2}\right)^{1/2}\frac{du}{\sqrt{(u^2-b^2)(u^2-a^2)}} \tag{31}$$

$$M_2 = \int_c^d \frac{1}{u}\log\left[\frac{\sqrt{(u^2-a^2)}+\sqrt{(u^2-b^2)}}{a+b}\right]\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\frac{du}{\sqrt{(u^2-c^2)(d^2-u^2)}} \tag{32}$$

F and Π are elliptic integrals of first and third kind respectively

and $$r = \sqrt{\frac{(d^2-c^2)(b^2-a^2)}{(d^2-b^2)(c^2-a^2)}}\,.$$

Next multiplying the equation (24) by $\dfrac{x}{\sqrt{(d^2-x^2)(x^2-c^2)}}$ and integrating with respect to x from c to d and then following the same procedure as that while deriving equation (28), we obtain

$$D_1 X_3 + D_2 X_4 = p_0 \tag{33}$$

where

$$X_3 = \frac{1}{2}\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2} \log\left(\frac{d-c}{d+c}\right) \frac{(c^2-b^2)}{b^2\sqrt{(d^2-b^2)(c^2-a^2)}} \left\{\Pi\left(\frac{\pi}{2}, \frac{b^2(d^2-c^2)}{c^2(d^2-b^2)}, r\right) - \right.$$

$$\left. - F\left(\frac{\pi}{2}, r\right)\right\} + \left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2} M_3 \tag{34}$$

$$X_4 = \frac{1}{2}\log\left(\frac{d-c}{d+c}\right) \frac{1}{b^2\sqrt{(d^2-b^2)(c^2-a^2)}} \left\{(b^2-a^2)F\left(\frac{\pi}{2}, r\right) + \frac{a^2(c^2-b^2)}{c^2} \times \right.$$

$$\left. \times \Pi\left(\frac{\pi}{2}, \frac{b^2(d^2-c^2)}{c^2(d^2-b^2)}, r\right)\right\} - M_4 \tag{35}$$

$$M_3 = \int_a^b \frac{1}{t}\log\left[\frac{\sqrt{(c^2-t^2)}+\sqrt{(d^2-t^2)}}{d + c}\right] \left(\frac{c^2-t^2}{d^2-t^2}\right)^{1/2} \frac{dt}{\sqrt{(t^2-a^2)(b^2-t^2)}} \tag{36}$$

$$M_4 = \int_a^b \frac{1}{t}\log\left[\frac{\sqrt{(c^2-t^2)}+\sqrt{(d^2-t^2)}}{d + c}\right] \left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} \frac{dt}{\sqrt{(d^2-t^2)(c^2-t^2)}} \tag{37}$$

From equations (28) and (33) , D_1 and D_2 can be found to be

$$D_1 = \frac{p_0(X_4 - X_2)}{(X_1X_4 - X_2X_3)} \quad , \quad D_2 = \frac{p_0(X_3 - X_1)}{(X_2X_3 - X_1X_4)} \tag{38}$$

4. STRESS INTENSITY FACTORS AND DISPLACEMENT

The normal stress $\tau_{yy}(x,y)$ in the plane y=0 just below the strips can be found from the relation (14), (15) and (20) as

$$\tau_{yy}(x,0) = \mu \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{1+\beta_2^2} \frac{d}{dx}\Bigg[\int_a^b \frac{h(t^2)}{t} \int_0^\infty \frac{\sin\xi x}{\xi} \Big\{1 - \cos\xi t\Big\} d\xi\, dt +$$

$$+ \int_c^d \frac{g(u^2)}{u} \int_0^\infty \frac{\sin\xi x}{\xi} \Big\{1 - \cos\xi u\Big\} d\xi\, du \Bigg] \quad , \quad a \le x \le b \ , \ c \le x \le d$$

$$= 2\mu \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{1+\beta_2^2} \frac{d}{dx}\Bigg[\int_a^b \frac{h(t^2)}{t} \int_0^\infty \frac{\sin(\xi x)\sin^2(\xi t/2)}{\xi} d\xi\, dt +$$

$$+ \int_c^d \frac{g(u^2)}{u} \int_0^\infty \frac{\sin(\xi x)\sin^2(\xi u/2)}{\xi} d\xi\, du \Bigg] \quad , \quad a \le x \le b \ , \ c \le x \le d$$

Using the result (Gradsteyn and Ryzhik, 1965)

$$\int_0^\infty \frac{\sin^2(ax)\sin(bx)}{x} = \frac{\pi}{2} \quad , \quad 0<b<2a$$

$$= \frac{\pi}{8} \quad , \quad b=2a$$

$$= 0 \quad , \quad b>2a \quad ,$$

$\tau_{yy}(x,0)$ can be found to be given by

$$\tau_{yy}(x,0) = -\frac{\mu\pi}{2}\,\frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{1+\beta_2^2}\,\frac{h(x^2)}{x}\,, \quad a\le x\le b$$

$$= -\frac{\mu\pi}{2}\,\frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{1+\beta_2^2}\,\frac{g(x^2)}{x}\,, \quad c\le x\le d$$

Putting the values of $h(x^2)$ and $g(x^2)$ from (26) and (25), the normal stress $\tau_{yy}(x,0)$ finally can be obtained as

$$\tau_{yy}(x,0) = -\frac{(1+\beta_2^2)^2-4\beta_1\beta_2}{(1+\beta_2^2)}\,\frac{\pi\mu}{2x\sqrt{(x^2-a^2)(b^2-x^2)}}\left\{D_1\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}\times\right.$$

$$\left.\times\left(\frac{c^2-x^2}{d^2-x^2}\right)^{1/2} - \frac{D_2(x^2-a^2)}{\sqrt{(d^2-x^2)(c^2-x^2)}}\right\}, \quad a\le|x|\le b \qquad (39)$$

and

$$\tau_{yy}(x,0) = -\frac{(1+\beta_2^2)^2-4\beta_1\beta_2}{(1+\beta_2^2)}\,\frac{\pi\mu}{2x\sqrt{(x^2-c^2)(d^2-x^2)}}\left\{D_1\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}\times\right.$$

$$\left.\times\frac{(x^2-c^2)}{\sqrt{(x^2-b^2)(x^2-a^2)}} + D_2\left(\frac{x^2-a^2}{x^2-b^2}\right)^{1/2}\right\}, \quad c\le|x|\le d \qquad (40)$$

Now the stress intensity factors at the edges of the strips can be defined as

$$K_a = \underset{x \to a+0}{Lt} \left| \frac{d\ \tau_{yy}(x,0)}{\pi\mu v_o} \left(\frac{x-a}{d}\right)^{1/2} \right|$$

$$K_b = \underset{x \to b-0}{Lt} \left| \frac{d\ \tau_{yy}(x,0)}{\pi\mu v_o} \left(\frac{b-x}{d}\right)^{1/2} \right|$$

$$K_c = \underset{x \to c+0}{Lt} \left| \frac{d\ \tau_{yy}(x,0)}{\pi\mu v_o} \left(\frac{x-c}{d}\right)^{1/2} \right|$$

and $$K_d = \underset{x \to d-0}{Lt} \left| \frac{d\ \tau_{yy}(x,0)}{\pi\mu v_o} \left(\frac{d-x}{d}\right)^{1/2} \right|$$

which can be expressed as

$$K_a = \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{\beta_1(1-\beta_2^2)} \frac{(d)^{1/2}}{(2a)^{3/2}(b^2-a^2)^{1/2}} \frac{(X_4-X_2)}{(X_1X_4-X_2X_3)} \tag{41}$$

$$K_b = \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{\beta_1(1-\beta_2^2)} \frac{(d)^{1/2}}{(2b)^{3/2}(b^2-a^2)^{1/2}} \left\{ \left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{c^2-b^2}{d^2-b^2}\right)^{1/2} (X_4-X_2) + \right.$$

$$\left. + \frac{(b^2-a^2)(X_3-X_1)}{\sqrt{(d^2-b^2)(c^2-b^2)}} \right\} \frac{1}{(X_1X_4-X_2X_3)} \tag{42}$$

$$K_c = \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{\beta_1(1-\beta_2^2)} \frac{(d)^{1/2}}{(2c)^{3/2}(d^2-c^2)^{1/2}} \left(\frac{c^2-a^2}{c^2-b^2}\right)^{1/2} \frac{(X_3-X_1)}{(X_2X_3-X_4X_1)} \tag{43}$$

$$K_d = \frac{(1+\beta_2^2)^2 - 4\beta_1\beta_2}{\beta_1(1-\beta_2^2)} \frac{(d)^{1/2}}{(2d)^{3/2}(d^2-c^2)^{1/2}} \left\{ \frac{(d^2-c^2)(X_4-X_2)}{\sqrt{(d^2-b^2)(c^2-a^2)}} - \right.$$

$$\left. - \left(\frac{d^2-a^2}{d^2-b^2}\right)^{1/2} (X_3-X_1) \right\} \frac{1}{(X_1X_4-X_2X_3)} \tag{44}$$

The vertical displacement $u_2(x,y)$ in the plane $y=0$ outside the strips is obtained from (12), (15), (20) and (26) and is given by the expression

$$u_2(x,0) = \frac{\beta_1}{2}\left(\frac{1-\beta_2^2}{1+\beta_2^2} \right) \left[\int_a^b \frac{1}{t}\log\left|1-\frac{t^2}{x^2}\right| \left\{ D_1 \left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{c^2-t^2}{d^2-t^2} \right)^{1/2} \times \right.\right.$$

$$\left. \times \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} - \frac{D_2}{\sqrt{(d^2-t^2)(c^2-t^2)}} \left(\frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \right\} dt +$$

$$+ \int_c^d \frac{1}{u}\log\left|1-\frac{u^2}{x^2}\right| \left\{ D_1 \left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{u^2-c^2}{d^2-u^2}\right)^{1/2} \frac{1}{\sqrt{(u^2-b^2)(u^2-a^2)}} + \right.$$

$$\left.\left. + \frac{D_2}{\sqrt{(u^2-c^2)(d^2-u^2)}} \left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2} \right\} du \right], \quad 0<x<a, \; b<x<c, \; x>d \tag{45}$$

Now letting $v \to 0$ we can obtain expressions of normal stress $\tau_{yy}^{(0)}(x,0)$ and displacement $u_2^{(0)}(x,0)$ from (39), (40) and (45) corresponding to the statical case as

$$\tau_{yy}^{(0)}(x,0) = -\frac{\pi\mu}{x}\left(\frac{c_1^2}{c_2^2}-1\right)\frac{1}{\sqrt{(x^2-a^2)(b^2-x^2)}}\left\{C_1\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}\times\right.$$

$$\left.\times\left(\frac{c^2-x^2}{d^2-x^2}\right)^{1/2}-\frac{C_2(x^2-a^2)}{\sqrt{(d^2-x^2)(c^2-x^2)}}\right\}, \qquad a\le|x|\le b$$

$$= -\frac{\pi\mu}{x}\left(\frac{c_1^2}{c_2^2}-1\right)\frac{1}{\sqrt{(x^2-c^2)(d^2-x^2)}}\left\{C_1\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}\times\right.$$

$$\left.\times\frac{(x^2-c^2)}{\sqrt{(x^2-b^2)(x^2-a^2)}}+C_2\left(\frac{x^2-a^2}{x^2-b^2}\right)^{1/2}\right\}, \qquad c\le|x|\le d \tag{46}$$

$$u_2^{(0)}(x,0) = -\frac{c_1^2}{2c_2^2}\left[\int_a^b\frac{1}{t}\log\left|1-\frac{t^2}{x^2}\right|\left\{C_1\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-t^2}{d^2-t^2}\right)^{1/2}\times\right.\right.$$

$$\left.\times\frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}}-\frac{C_2}{\sqrt{(d^2-t^2)(c^2-t^2)}}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\right\}dt+$$

$$+ \int_c^d \frac{1}{u}\log\left|1-\frac{u^2}{x^2}\right|\left\{C_1\left(\frac{d^2-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{u^2-c^2}{d^2-u^2}\right)^{1/2}\frac{1}{\sqrt{(u^2-b^2)(u^2-a^2)}}\right. +$$

$$\left.+\frac{C_2}{\sqrt{(u^2-c^2)(d^2-u^2)}}\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\right\} du\Bigg], \quad 0<x<a,\ b<x<c,\ x>d \qquad (47)$$

where C_1 and C_2 are given by D_1 and D_2 replacing p_0 by $-(c_2^2/c_1^2)v_0$.

5. NUMERICAL RESULTS AND DISCUSSIONS

Stress intensity factors at the edges of the strips have been evaluated numerically and have been depicted by means of graphs. Accordingly, all the lengths have been made dimensionless with respect to d. Substituting $\frac{a}{d} = d_1$, $\frac{b}{d} = d_2$ and $\frac{c}{d} = d_3$, the stress intensity factors at the four edges of the strips viz. K_a, K_b, K_c and K_d have been plotted against v/c_2 for various values of the strip length parameters. It is found that whatever be the lengths of the strips, stress intensity factors at the four edges of the strips decrease with the increase in the value of v/c_2. From the graphs, it may be noted further that with the decrease in length of the inner strip which might be done either by increasing d_1 or by decreasing the value of d_2, the stress intensity factor K_a at the innermost edge gradually decreases whereas the stress intensity factor at the other edges show just the opposite character (fig.2 - fig.9).

Also the decrease in the value of the length of the outer strip, which might be done by increasing the value of d_3, causes a decrease in the value of the stress intensity factor K_a and increase in the values of the stress intensity factor K_b,K_c and K_d (fig.10 - fig.13) from which an interesting conclusion might be drawn that the presence of the inner strip supresses the stress intensity factors at both the edges of the outer strip whereas the presence of the outer strip suppresses the stress intensity factor at the outer edge of the inner strip but increases the stress intensity factor at its inner edge.

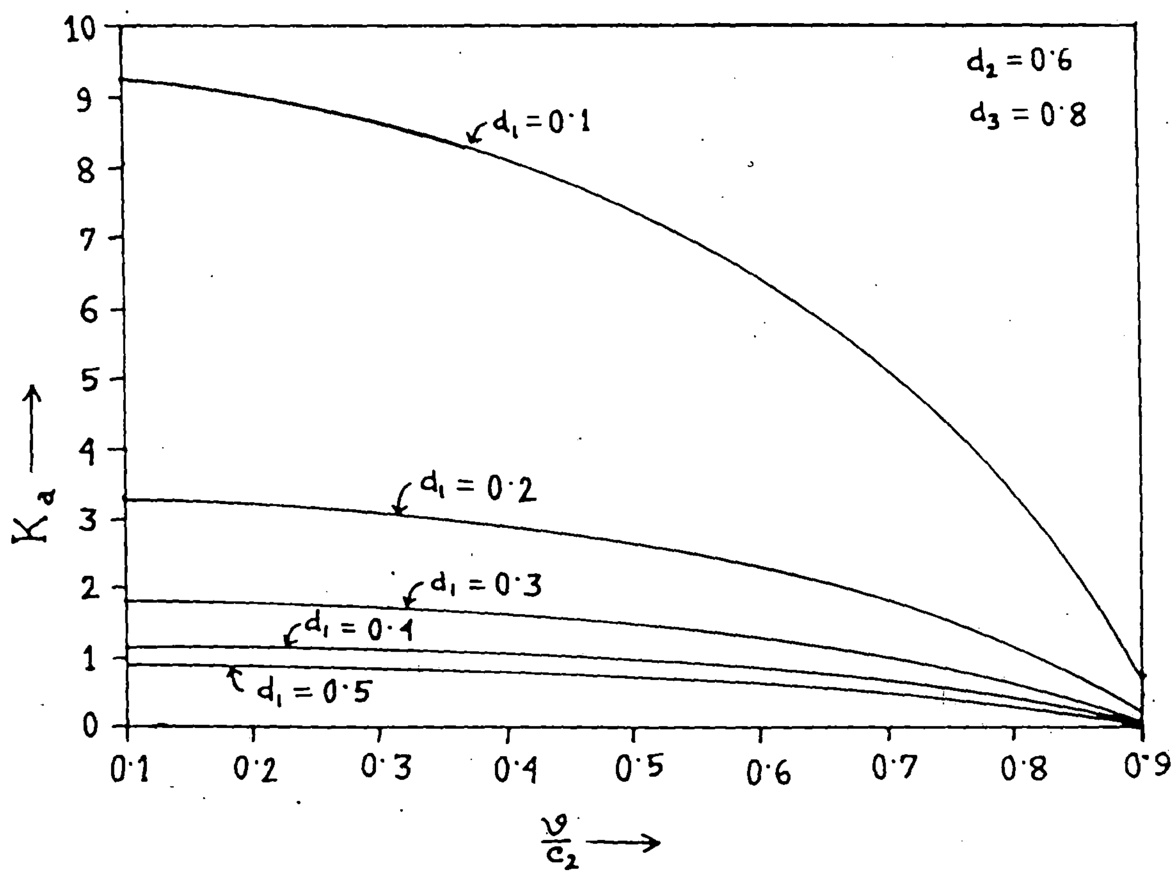

Fig. 2. Stress intensity factor K_a vs. v/c_2 ($d_2 = 0.6$, $d_3 = 0.8$).

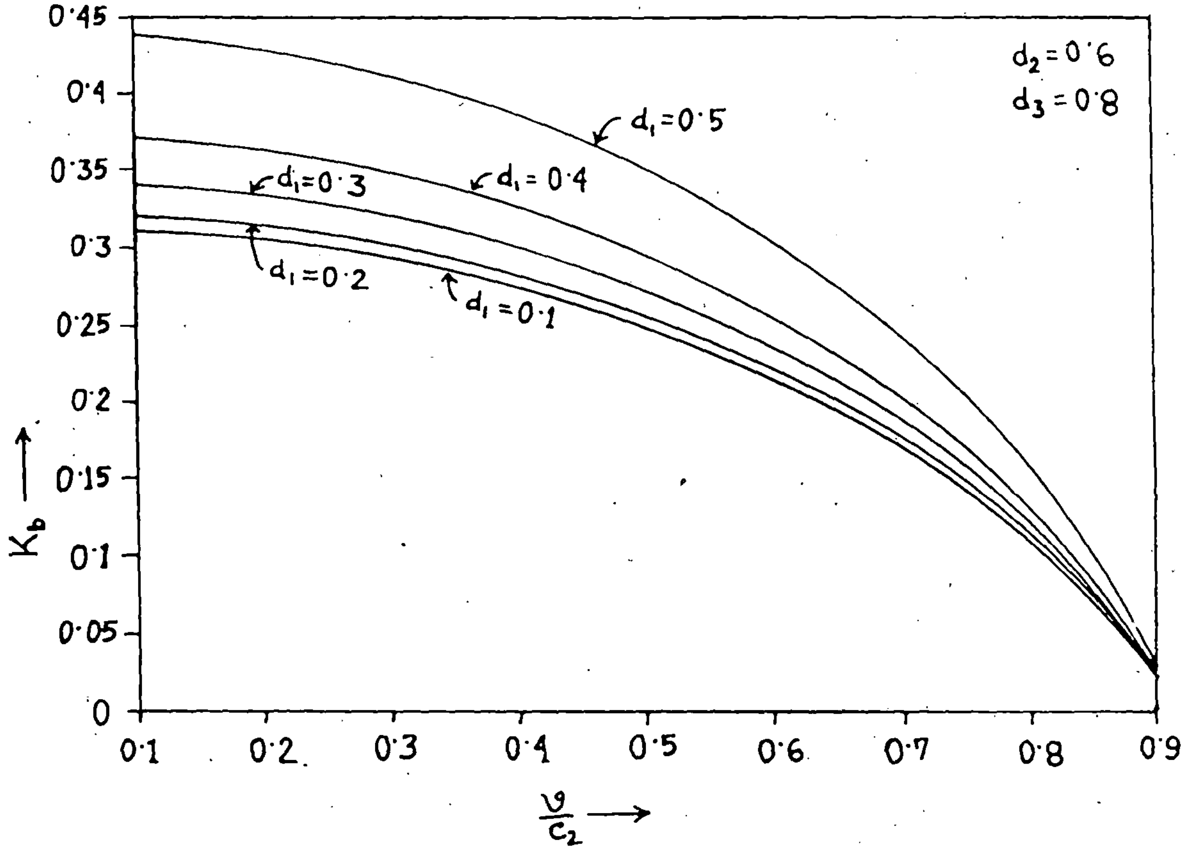

Fig. 3 Stress intensity factor K_b vs. v/c_2 (d_2=0.6, d_3=0.8).

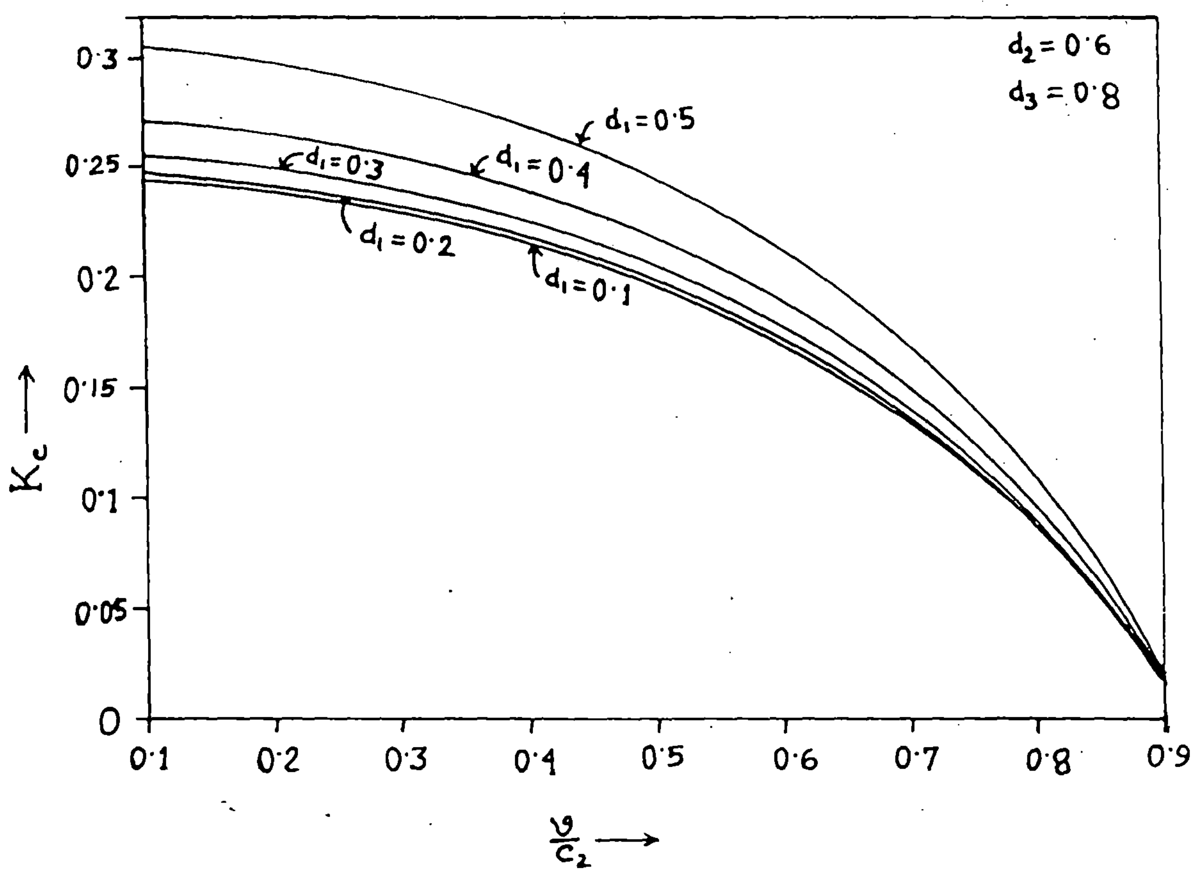

Fig. 4 Stress intensity factor K_c vs. v/c_2 ($d_2=0{\cdot}6$, $d_3=0{\cdot}8$).

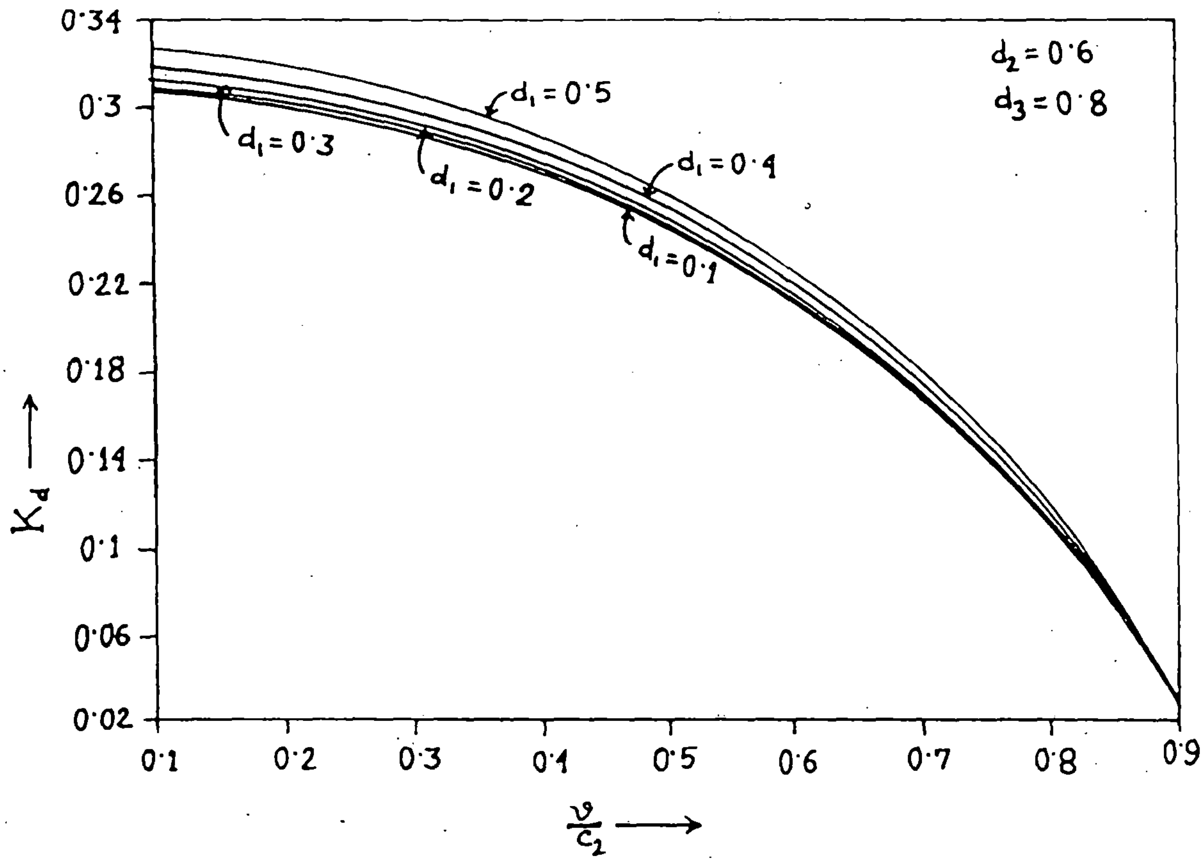

Fig. 5. Stress intensity factor K_d vs. v/c_2 ($d_2 = 0.6, d_3 = 0.8$).

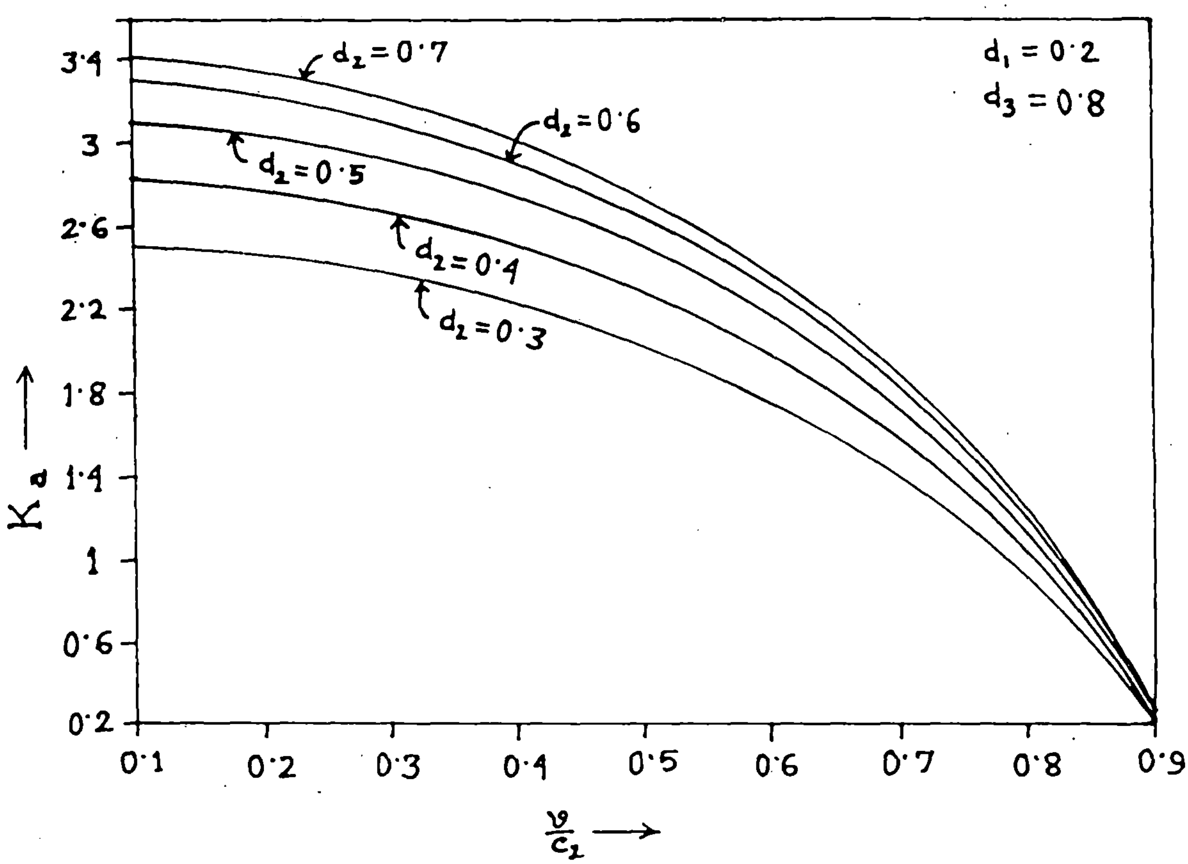

Fig. 6. Stress intensity factor K_d vs. v/c_2 ($d_1 = 0.2$, $d_3 = 0.8$)

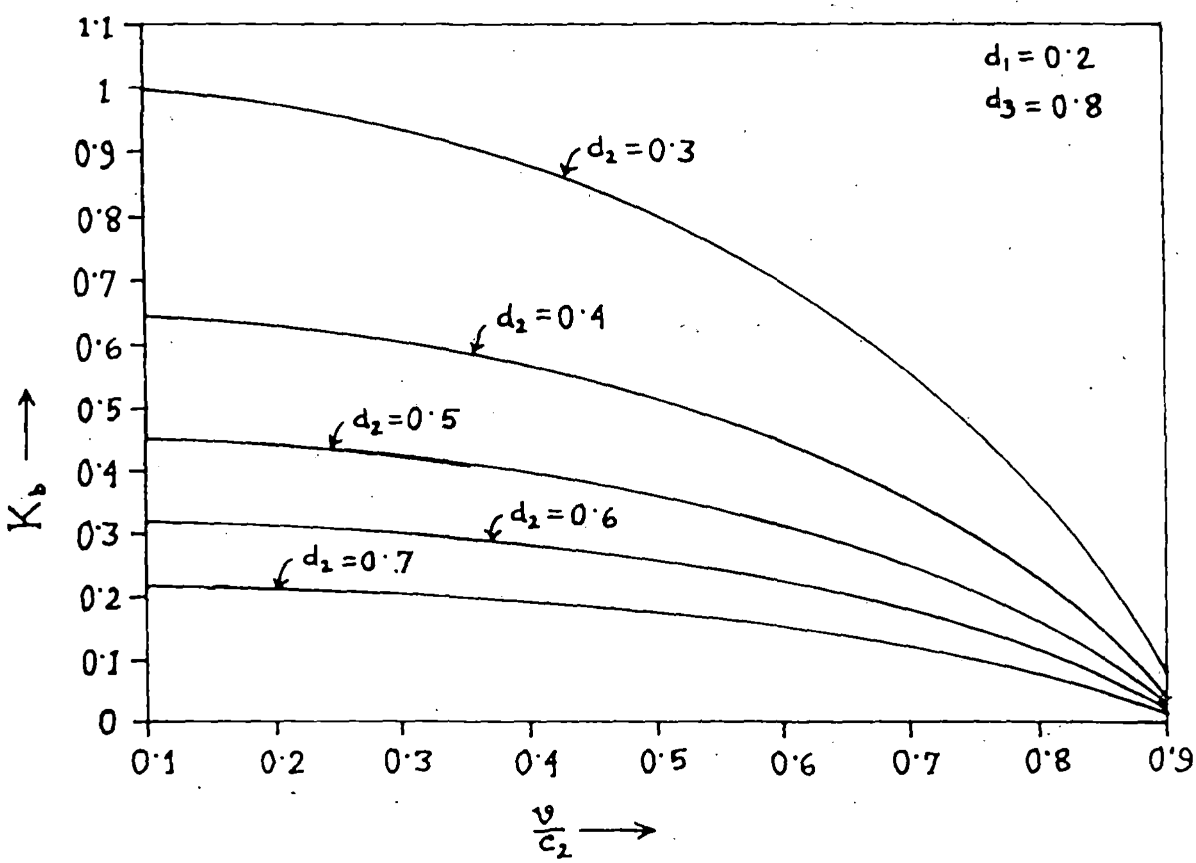

Fig. 7. Stress intensity factor K_b vs. v/c_2 ($d_1 = 0.2$, $d_3 = 0.8$).

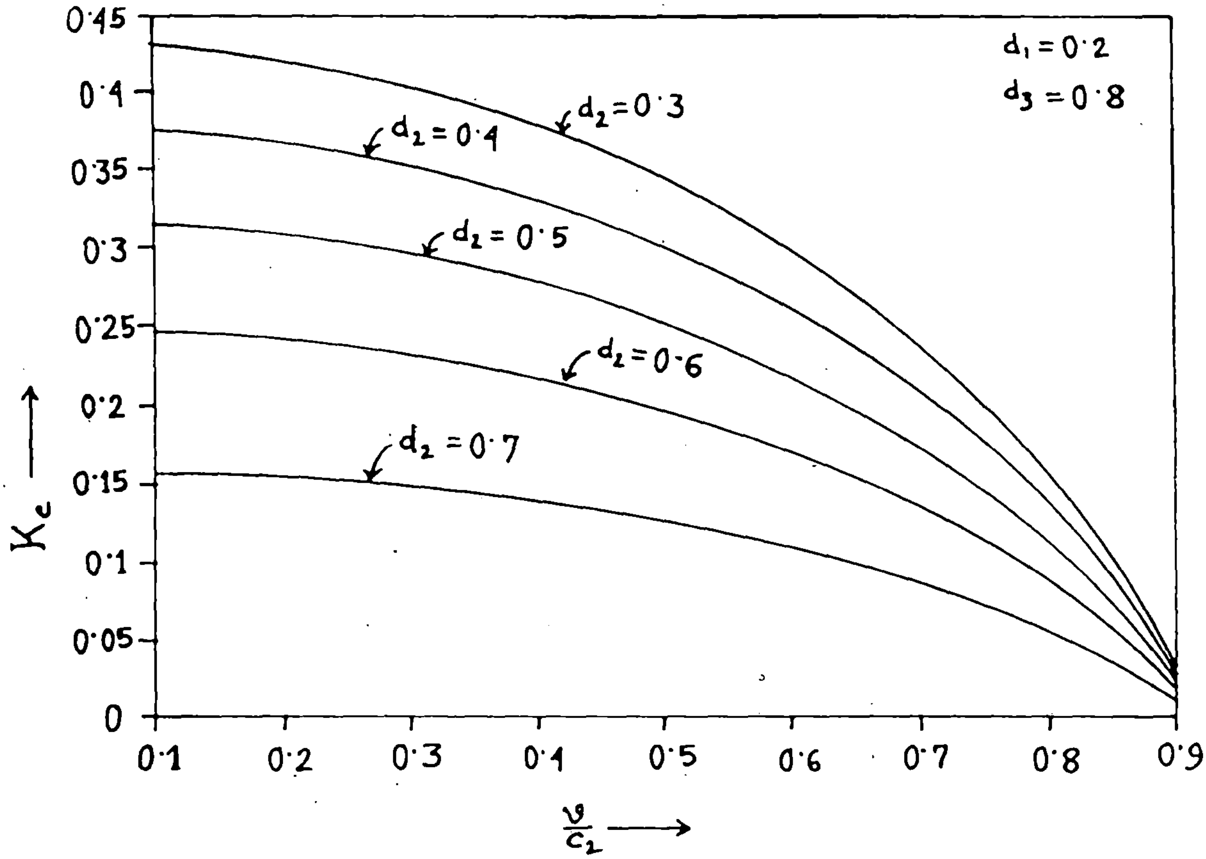

Fig. 8. Stress intensity factor K_c vs. v/c_2 ($d_1=0.2, d_3=0.8$).

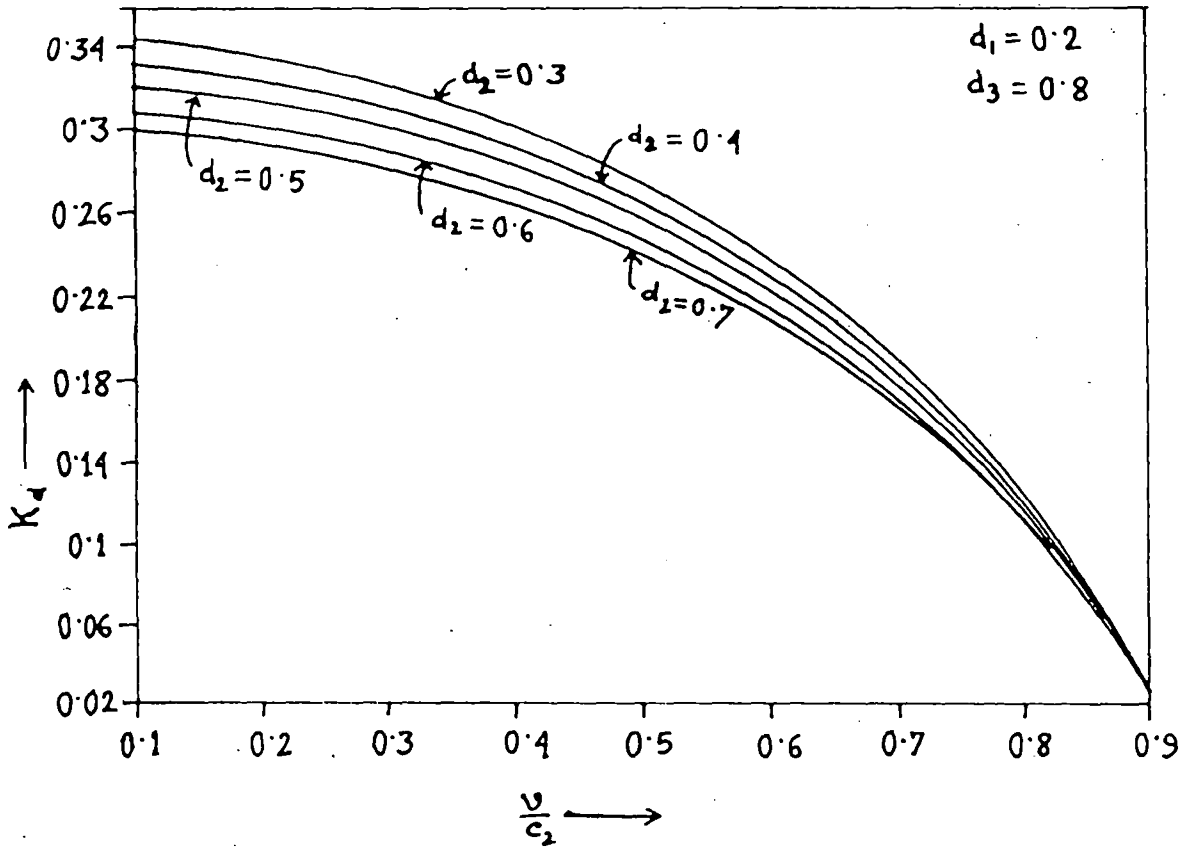

Fig. 9. Stress intensity factor K_d vs. v/c_2 (d_1=0.2, d_3 = 0.8).

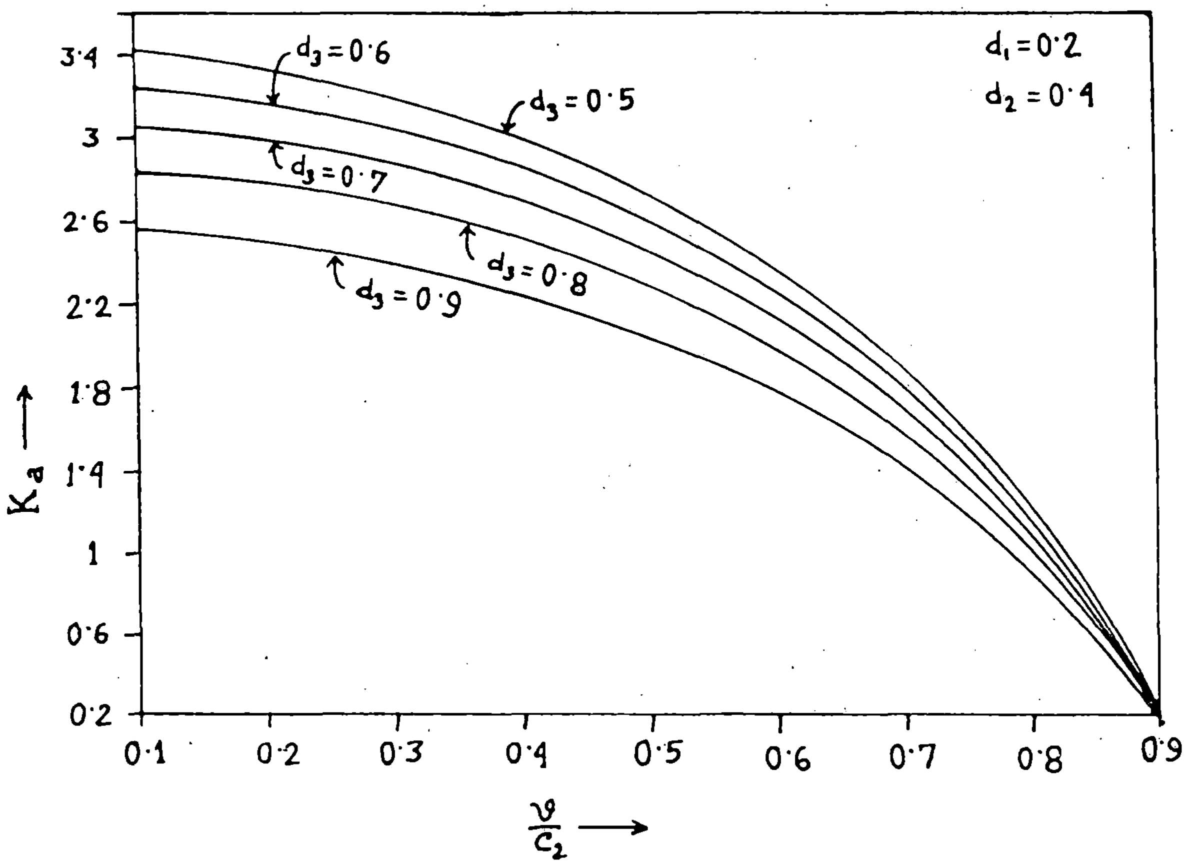

Fig. 10. Stress intensity factor K_a vs. v/c_2 ($d_1=0.2$, $d_2=0.4$).

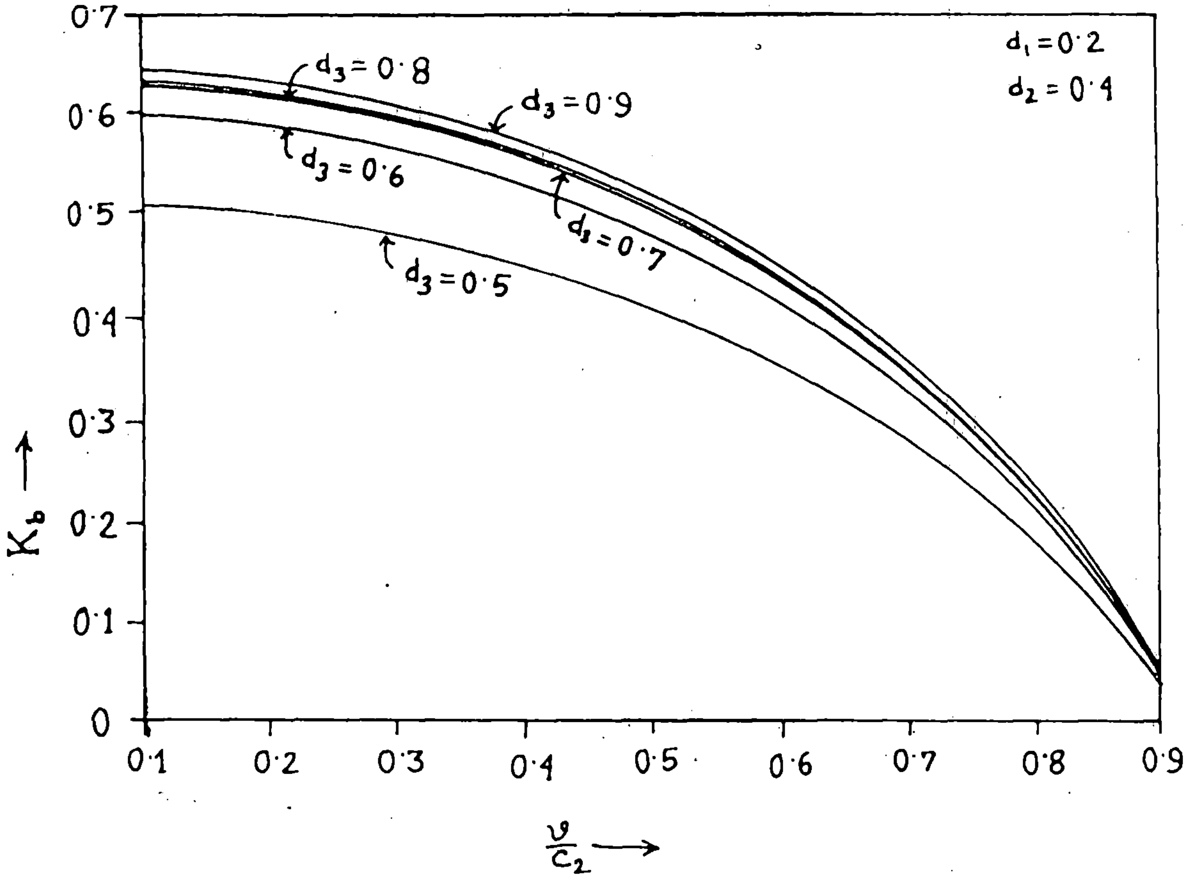

Fig. 11. Stress intensity factor K_b vs. v/c_2 (d_1= 0.2, d_2=0.4)

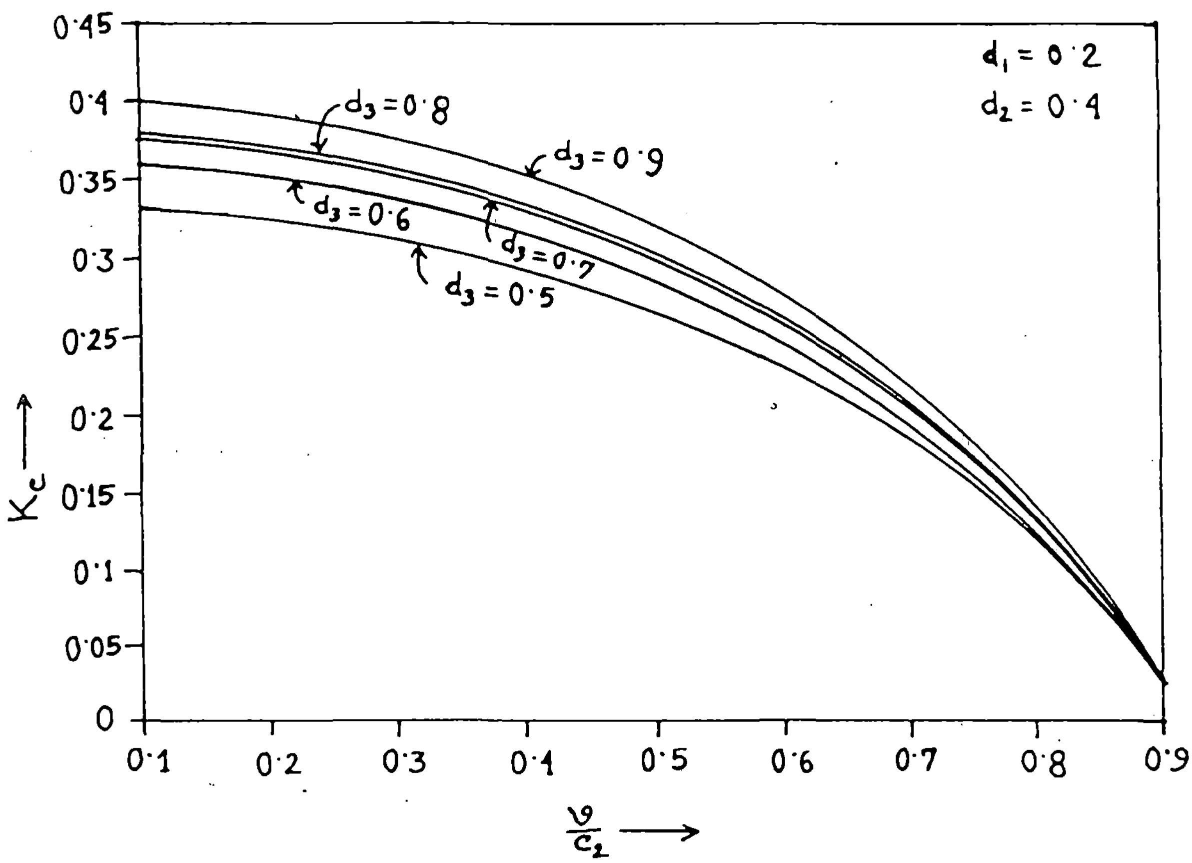

Fig. 12. Stress intensity factor K_c vs. v/c_2 ($d_1 = 0.2$, $d_2 = 0.4$).

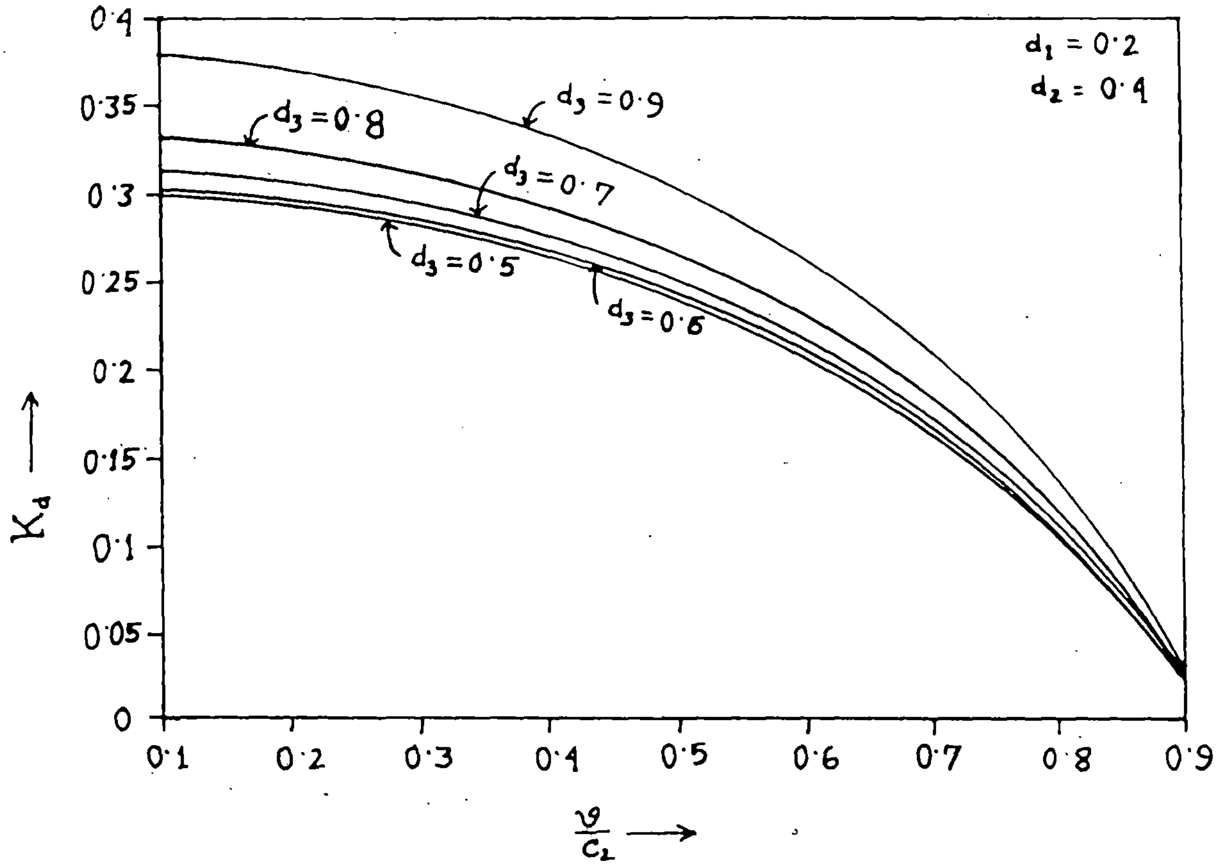

Fig. 13. Stress intensity factor K_d vs. v/c_2 (d_1=0.2, d_2=0.4).

DIFFRACTION OF TORSIONAL ELASTIC WAVES BY A RIGID ANNULAR DISC AT THE BIMATERIAL INTERFACE

1. INTRODUCTION

The study of the problems involving diffraction of elastic waves by cracks or inclutions are of considerable importance in view of their extensive applications in mechanical engineering and also in seismology and geophysics. If the cracks or inclutions are located at the interface of layered media, the study becomes more relevant. The extensive use of composite materials in modern technology has evoked interest in the wave propagation problems in layered media with interfacial discontinuities. Onder et al (1975) studied the diffraction of plane SH-wave obliquely incident on a rigid half plane lying at the interface of two dissimilar semi-infinite elastic media. Following Mal (1970), problem of interaction of antiplane shear wave by a Griffith crack at the interface of two bonded dissimilar elastic half spaces has been treated by Srivastava et al (1980). Bostrom (1987) also treated the same problem following a procedure similar to that of Krenk and Schmidt (1982). The corresponding problem of diffraction of antiplane shear wave by a finite rigid strip at the bimaterial interface has been treated by Palaiya and Majumder (1981). The problem of diffraction of transient torsional shear waves by a penny shaped crack at the interface of two bonded dissimilar elastic half spaces has been investigated by Ueda et al (1983). As regards the dynamic crack or strip problems, research has mainly been confined to the case of a single crack or strip of finite width or of circular in shape. These are two part mixed boundary value problems which are usually reduced to solutions of dual integral equations. But the solution of interesting problems involving the diffraction of elastic waves by annular discs or cracks at the bimaterial interface which give rise to three part mixed boundary value problems are still lacking.

However recently the problems involving the diffraction of torsional waves by flat annular crack in an infinite elastic medium have been studied by Shindo (1979,1981), the problems are reduced to that of solving singular integral equation of first kind which were later solved by following the technique of Erdogan (1965,1969). The problem of diffraction of acoustic wave by a soft annular disc was studied by Thomas (1965). Following the method of Williams (1963) the three part mixed boundary value problem was reduced to a set of integral equation which was solved by an iterative procedure for low frequency. The same technique was followed by Jain and Kanwal (1970) to study the problem of torsional oscillations of an elastic half space due to annular disc. In this paper we have discussed the problem of diffraction of torsional wave by a rigid annular disc at the interface of two bonded dissimilar elastic media. Applying the method developed by Williams (1963) and used subsequently by Thomas (1965) and Jain et al (1970), the three part mixed boundary value problem has been reduced to the solution of a set of integral equations. The solutions of these integral equations are obtained iteratively for low frequency and small values of the ratio of the inner and outer radii of the disc. These solutions are used to determine the jump in stresses across the annular disc and stress intensity factors at both the edges of the disc. Torque and Far field amplitudes in both the media have also been deduced. The effect of normalised frequency, material properties and geometric parameters in stress intensity factors and far field amplitude are shown graphically.

2. FORMULATION OF THE PROBLEM

Let us consider the torsional vibration of frequency ω of an annular rigid disc of inner and outer radii b and a respectively lying at the interface of two bonded dissimilar elastic half spaces. The region occupied by the annular disc is defined by z=0 and $b\le r\le a$ in a cylindrical polar co-ordinate system (r,θ,z) as shown in the fig.1. Let an antiplane shear wave given by $\Omega_2 r e^{ik_2(z-c_2 t)}$, where Ω_2 is a constant, $k_2=\omega/c_2$, and $c_2=\sqrt{(\mu_2/\rho_2)}$, the shear wave velocity in medium 2, be incident normally on the disc. Henceforth the time factor $e^{-i\omega t}$ will be suppressed throught the analysis.

The only non-vanishing θ-component of the displacement V_j and the non- vanishing stresses $\tau_{r\theta}^{(j)}$, $\tau_{z\theta}^{(j)}$ (j=1,2) due to the scattered field are independent of θ and are given by

$$V_j=V_j(r,z,t)=v_j(r,z)e^{-i\omega t} \tag{1}$$

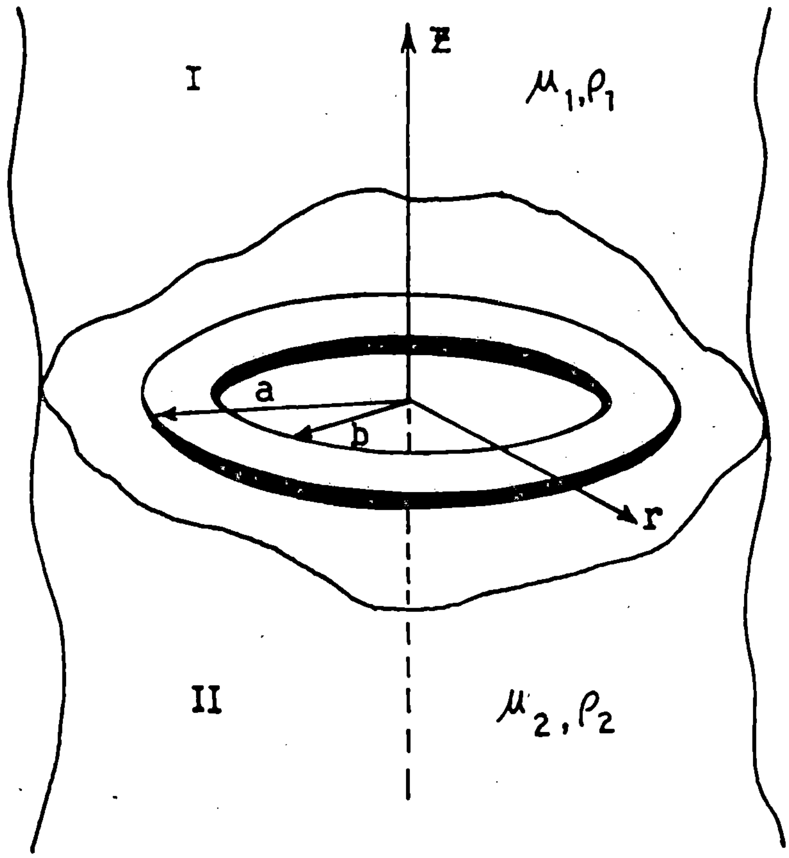

Fig. 1. Geometry of the annular disc.

$$\tau^{(j)}_{r\theta}=\tau^{(j)}_{r\theta}(r,z)=\mu_j\left(\frac{\partial v_j}{\partial r}-\frac{v_j}{r}\right)$$

$$\tau^{(j)}_{z\theta}=\tau^{(j)}_{z\theta}(r,z)=\mu_j\ \frac{\partial v_j}{\partial z} \tag{2}$$

where μ_j($j=1,2$) are the shear modulus of the elastic materials. The suffices 1 and 2 are used to denote the values of the corresponding quantities in the upper and lower half spaces respectively. Without any loss of generality we assume that $c_2>c_1$. The displacement V_j satisfies the equation

$$\frac{\partial^2 V_j}{\partial r^2}+\frac{1}{r}\frac{\partial V_j}{\partial r}-\frac{V_j}{r^2}+\frac{\partial^2 V_j}{\partial z^2}=\frac{\rho_j}{\mu_j}\frac{\partial^2 V_j}{\partial t^2} \tag{3}$$

where ρ_j ($j=1,2$) are the density of the elastic materials. Putting $V_j=v_j(r,z)e^{-i\omega t}$, equation (2.3) and the boundary conditions at the interface $z=0$, take the form

$$\frac{\partial^2 v_j}{\partial r^2}+\frac{1}{r}\frac{\partial v_j}{\partial r}-\frac{v_j}{r^2}+\frac{\partial^2 v_j}{\partial z^2}+k_j^2 v_j=0 \tag{4}$$

$$v_1(r,0)=v_2(r,0)=-\Omega r\quad,\quad b\le r\le a \tag{5}$$

$$v_1(r,0)=v_2(r,0)\quad,\quad 0\le r<b\ ,\ a<r<\infty \tag{6}$$

$$\tau^{(1)}_{z\theta}(r,0)=\tau^{(2)}_{z\theta}(r,0)\quad,\quad 0\le r<b\ ,\ a<r<\infty \tag{7}$$

where $k_j=\omega/c_j$, $c_j=\sqrt{(\mu_j/\rho_j)}$ and $\Omega=2\Omega_2\mu_2k_2/(\mu_1k_1+\mu_2k_2)$.

The solution of the equation (4) can be written as

$$v_j(r,z) = \int_0^\infty A_j(\xi)\exp(-\gamma_j|z|)J_1(\xi r)d\xi \tag{8}$$

where
$$\gamma_j = (\xi^2 - k_j^2)^{1/2} \quad , \ \xi > k_j$$
$$= -i(k_j^2 - \xi^2)^{1/2} \quad , \ \xi < k_j$$

and $A_j(\xi)$ $(j=1,2)$ are functions of ξ to be determined from the boundary conditions.

Therefore, the stress components are

$$\tau_{z\theta}^{(1)}(r,z) = -\mu_1\int_0^\infty \gamma_1 A_1(\xi)\exp(-\gamma_1|z|)J_1(\xi r)d\xi \quad , \ z\geq 0$$

(9)

$$\tau_{z\theta}^{(2)}(r,z) = \mu_2\int_0^\infty \gamma_2 A_2(\xi)\exp(-\gamma_2|z|)J_1(\xi r)d\xi \quad , \ z\leq 0$$

Now using the boundary conditions (2.5), (2.6) and (2.7) and assuming that $\tau_{z\theta}^{(2)}(r,0) - \tau_{z\theta}^{(1)}(r,0) = f(r)$, $b\leq r\leq a$, we obtain the integral equation

$$\int_b^a tf(t)dt\int_0^\infty \frac{\xi}{(\mu_1\gamma_1+\mu_2\gamma_2)} J_1(\xi r)J_1(\xi t)d\xi = -\Omega r \ , \ b\leq r\leq a \tag{10}$$

where $$A_1(\xi) = A_2(\xi) = \frac{\xi}{(\mu_1\gamma_1+\mu_2\gamma_2)}\int_b^a tf(t)J_1(\xi t)dt \tag{11}$$

3. METHOD OF SOLUTION

In order to solve the integral equation (10), we apply the technique developed by Williams (1963) for solving integral equations arising in three part boundary value problems. The same technique was also applied by Thomas (1965) and Jain et al (1970) in order to solve scattering problem by annular disc. Following Kanwal (1971), the kernal of the integral equation (10) is split into two kernals as follows:

$$(\mu_1+\mu_2)\int_0^\infty \frac{\xi}{(\mu_1\gamma_1+\mu_2\gamma_2)} J_1(\xi r)J_1(\xi t)d\xi = K_1(r,t) + K_2(r,t) \tag{12}$$

where
$$K_1(r,t) = \int_0^\infty J_1(\xi r)J_1(\xi t)d\xi \tag{13}$$

$$K_2(r,t) = \int_0^\infty M(\xi,\gamma_1,\gamma_2)J_1(\xi r)J_1(\xi t)d\xi \tag{14}$$

$$M(\xi,\gamma_1,\gamma_2) = \frac{\mu_1(\xi-\gamma_1)+\mu_2(\xi-\gamma_2)}{\mu_1\gamma_1+\mu_2\gamma_2} \tag{15}$$

The equation (2.10) then takes the form

$$\int_b^a tf(t)K_1(r,t)dt = -(\mu_1+\mu_2)\Omega r - \int_b^a tf(t)K_2(r,t)dt\ , \quad b\le r\le a \tag{16}$$

Next, consider two functions $f_1(r)$ and $f_2(r)$ such that

$$f_1(r)+f_2(r) = \begin{cases} 0 , & 0\le r<b \\ f(r) , & b\le r\le a \\ 0 , & a<r<\infty \end{cases} \tag{17}$$

As a result, the equation (5) reduces to two integral equations given by

$$\int_0^\infty tf_1(t)K_1(r,t)dt = -(\mu_1+\mu_2)\Omega r - \int_0^\infty tf_1(t)K_2(r,t)dt , \quad 0<r<a \tag{18}$$

and $$\int_0^\infty tf_2(t)K_1(r,t)dt = - \int_0^\infty tf_2(t)K_2(r,t)dt , \quad b<r<\infty \tag{19}$$

The procedure adopted by Williams (1963) and Thomas (1965) is followed to solve these integral equations. Using the results

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} \frac{1}{r^n}\int_0^r \frac{J_{n-1/2}(pw)\, w^{n+1/2}}{(r^2-w^2)^{1/2}}\, dw$$

$$= \left(\frac{2p}{\pi}\right)^{1/2} r^n\int_r^\infty \frac{J_{n+1/2}(pw)\, w^{-(n-1/2)}}{(w^2-r^2)^{1/2}}\, dw$$

and $$\int_0^\infty pJ_\mu(pw)J_\mu(pv)dp = \delta(w-v)/(wv)^{1/2}$$

we have the following relations

$$\int_0^\infty K_1(t,r)tf(t)dt = \frac{2}{\pi r}\int_0^r \frac{w^2 dw}{(r^2-w^2)^{1/2}}\int_w^\infty \frac{f(t)dt}{(t^2-w^2)^{1/2}} \quad , \quad 0<r<a$$

$$= \frac{2r}{\pi}\int_r^\infty \frac{w^{-2}dw}{(w^2-r^2)^{1/2}}\int_0^w \frac{t^2 f(t)dt}{(w^2-t^2)^{1/2}} \quad , \quad b<r<\infty \qquad (20)$$

and $$K_2(t,r) = \frac{2}{\pi tr}\int_0^r\int_0^t \frac{L_1(v,w)wv\ dvdw}{(r^2-w^2)^{1/2}(t^2-v^2)^{1/2}} \quad , \quad 0<r<a$$

$$= \frac{2tr}{\pi}\int_r^\infty\int_t^\infty \frac{L_2(v,w)\ dvdw}{wv(w^2-r^2)^{1/2}(v^2-t^2)^{1/2}} \quad , \quad b<r<\infty \qquad (21)$$

where

$$L_1(v,r) = (vr)^{1/2}\int_0^\infty \xi M(\xi,\gamma_1,\gamma_2)J_{1/2}(\xi v)J_{1/2}(\xi r)d\xi \qquad (22)$$

and $$L_2(v,r) = (vr)^{1/2}\int_0^\infty \xi M(\xi,\gamma_1,\gamma_2)J_{3/2}(\xi v)J_{3/2}(\xi r)d\xi \qquad (23)$$

Substituting the relations (20) and (21) in (18) we get

$$\frac{2}{\pi r}\int_0^r \frac{wdw}{(r^2-w^2)^{1/2}}\ w\int_w^\infty \frac{f_1(t)dt}{(t^2-w^2)^{1/2}} = -(\mu_1+\mu_2)\Omega r\ -$$

$$-\ \frac{2}{\pi r}\int_0^r \frac{wdw}{(r^2-w^2)^{1/2}}\int_0^\infty f_1(t)dt\int_0^t \frac{vL_1(v,w)dv}{(t^2-v^2)^{1/2}} \ , \quad 0<r<a$$

which after changing the order of integration can be written as

$$\frac{2}{\pi r}\int_0^r \frac{w\,dw}{(r^2-w^2)^{1/2}}\; w\int_w^\infty \frac{f_1(t)dt}{(t^2-w^2)^{1/2}} = -(\mu_1+\mu_2)\Omega r -$$

$$- \frac{2}{\pi r}\int_0^r \frac{w\,dw}{(r^2-w^2)^{1/2}} \int_0^\infty L_1(v,w)dv\; v\int_v^\infty \frac{f_1(t)\,dt}{(t^2-v^2)^{1/2}}\;, \quad 0<r<a \tag{24}$$

In view of the above equation, we assume

$$r\int_r^\infty \frac{f_1(t)dt}{(t^2-r^2)^{1/2}} = \begin{cases} S_1(r) & , \quad 0<r<a \\ -T_1(r) & , \quad a<r<\infty \end{cases} \tag{25}$$

Use of the relation (25) in (24) yields

$$\frac{2}{\pi r}\int_0^r \frac{wS_1(w)dw}{(r^2-w^2)^{1/2}} = -(\mu_1+\mu_2)\Omega r - \frac{2}{\pi r}\int_0^r \frac{wG(w)dw}{(r^2-w^2)^{1/2}} \tag{26}$$

where $$G(w) = \int_0^a S_1(v)L_1(v,w)dv - \int_a^\infty T_1(v)L_1(v,w)dv \tag{27}$$

In order to apply Abel's transform the equation (26) can be written as

$$\int_0^r \frac{w\left[S_1(w)+G(w)\right]}{(r^2-w^2)^{1/2}}\, dw = -\frac{\pi}{2}(\mu_1+\mu_2)\Omega r^2$$

and after taking Abel's transform and substituting the value of $G(w)$ from (27) we obtain the following integral equation :

$$S_1(r) + \int_0^a L_1(v,r)S_1(v)dv = -2\Omega(\mu_1+\mu_2)r + \int_a^\infty L_1(v,r)T_1(v)dv \;, \quad 0<r<a \tag{28}$$

Again substituting the relations (20) and (21) in (19) and following the same procedure as is done to deduce the integral equation (28), another integral equation can be derived as

$$S_2(r) + \int_b^\infty L_2(v,r)S_2(v)dv = \int_0^b L_2(v,r)T_2(v)dv \quad , \quad b<r<\infty \tag{29}$$

where it is assumed that

$$\frac{1}{r}\int_0^r \frac{t^2 f_2(t)dt}{(r^2-t^2)^{1/2}} = \begin{cases} -\,T_2(r) & , \quad 0<r<b \\ S_2(r) & , \quad b<r<\infty \end{cases} \tag{30}$$

Now, using Abel's transform in (25) and (30), the functions $f_1(t)$ and $f_2(t)$ are found to be

$$f_1(t) = -\frac{2}{\pi}\frac{d}{dt}\left[\int_t^a \frac{S_1(u)du}{(u^2-t^2)^{1/2}} - \int_a^\infty \frac{T_1(u)du}{(u^2-t^2)^{1/2}}\right] , \quad 0<t<a \tag{31}$$

and

$$f_2(t) = \frac{2}{\pi t^2}\frac{d}{dt}\left[-\int_0^b \frac{u^2T_2(u)du}{(t^2-u^2)^{1/2}} + \int_b^t \frac{u^2S_2(u)du}{(t^2-u^2)^{1/2}}\right] , \quad t>b \tag{32}$$

Further, by the help of the relation (17), $T_1(r)$ and $T_2(r)$ can be written from (25) and (30) as

$$T_1(r) = r\int_r^\infty \frac{f_2(t)dt}{(t^2-r^2)^{1/2}} \quad , \quad a<r<\infty$$

$$T_2(r) = \frac{1}{r}\int_0^r \frac{t^2 f_1(t)dt}{(r^2-t^2)^{1/2}} \quad , \quad 0<r<b$$

in which putting the values of $f_1(t)$ and $f_2(t)$ from (31) and (32) and using the results

$$\int_r^\infty \frac{dt}{t(t^2-r^2)^{1/2}(t^2-u^2)^{3/2}} = \frac{\sqrt{\pi}}{2r^2\Gamma(5/2)(r^2-u^2)}\ {}_2F_1(1/2,\ 1;\ 5/2;\ u^2/r^2)$$

$$, \ u<r \qquad (33)$$

$$\int_0^r \frac{t^3dt}{(r^2-t^2)^{1/2}(u^2-t^2)^{3/2}} = \frac{\sqrt{\pi}\ r^3}{2\Gamma(5/2)u(u^2-r^2)}\ {}_2F_1(1/2,\ 1;\ 5/2;\ r^2/u^2)$$

$$, \ u>r \qquad (34)$$

we get the following two integral equations

$$T_1(r) = l_2(r) + \frac{1}{\sqrt{\pi}r\Gamma(5/2)}\int_0^b \frac{u^2T_2(u)\ {}_2F_1(1/2,1;5/2;u^2/r^2)}{(r^2-u^2)}du$$

$$a<r<\infty \qquad (35)$$

$$T_2(r) = l_1(r) + \frac{r^2}{\sqrt{\pi}\Gamma(5/2)}\int_a^\infty \frac{T_1(u)\ {}_2F_1(1/2,1;5/2;r^2/u^2)}{u(u^2-r^2)}du$$

$$0<r<b \qquad (36)$$

where

$$l_1(r) = -\frac{2}{\pi r}\int_0^r \frac{t^2dt}{(r^2-t^2)^{1/2}}\frac{d}{dt}\int_t^a \frac{S_1(u)du}{(u^2-t^2)^{1/2}} \quad , \quad 0<r<b \qquad (37)$$

$$l_2(r) = \frac{2r}{\pi}\int_r^{\infty} \frac{dt}{t^2(t^2-r^2)^{1/2}}\,\frac{d}{dt}\int_b^t \frac{u^2 S_2(u)\,du}{(t^2-u^2)^{1/2}}\;, \qquad a<r<\infty \tag{38}$$

Assuming that $\alpha=k_2 a$, $\beta=k_2 b$ and $\lambda=b/a$ are small, the unknown functions $S_1(r)$, $S_2(r)$, $T_1(r)$, $T_2(r)$ which are solutions of integral equations (28),(29),(35) and (36) are obtained approximately following iterative process. Using the result that

$${}_2F_1(1/2,\ 1;\ 5/2;\ r^2/u^2) = \frac{3u}{4r^3}\left\{2ur - (u^2-r^2)\log\left(\frac{u+r}{u-r}\right)\right\}\;, \quad r<u$$

equations (35) and (36) become

$$T_1(ar)=l_2(ar)+\frac{1}{\pi}\int_0^1 T_2(bu)\left\{\frac{2\lambda r}{(r^2-\lambda^2u^2)} - \frac{1}{u}\log\left(\frac{r+\lambda u}{r-\lambda u}\right)\right\} du \quad , \; 1<r<\infty \tag{39}$$

and

$$T_2(br)=l_1(br)+\frac{1}{\pi\lambda r}\int_1^{\infty} T_1(au)\left\{\frac{2\lambda ur}{(u^2-\lambda^2r^2)} - \log\left(\frac{u+\lambda r}{u-\lambda r}\right)\right\} du \quad , \; 0<r<1 \tag{40}$$

Next, we assume that $\alpha=o(\lambda)$ so that $\beta=\alpha\lambda=o(\alpha^2)$.

In order to solve the equation (28), we rewrite it as

$$S_1(ar) + a\int_0^1 L_1(av,ar)S_1(av)dv = -2\Omega(\mu_1+\mu_2)ar + a\int_1^{\infty} L_1(av,ar)T_1(av)dv$$

$$0<r<1 \tag{41}$$

Now we put $S_1(ar) = X(ar) + Y(ar)$ (42)

so that equation (41) yields a pair of integral equations given by

$$X(ar) = -2\Omega(\mu_1+\mu_2)ar - a\int_0^1 L_1(av,ar)X(av)dv \ , \quad 0<r<1 \tag{43}$$

and

$$Y(ar) = a\int_1^\infty L_1(av,ar)T_1(av)dv - a\int_0^1 L_1(av,ar)Y(av)dv \ , \quad 0<r<1 \tag{44}$$

The kernal $L_1(av,ar)$ given by (22) can be converted to an expression involving finite integrals by the application of the contour integration technique followed by Srivastava et al (1980) and is given by

$$aL_1(av,ar) = i(1+\mu)\alpha^2(vr)^{1/2}\left[\int_0^1 \frac{\eta^2 J_{1/2}(\alpha\eta r)H^{(1)}_{1/2}(\alpha\eta v)}{\mu(1-\eta^2)^{1/2}+(\sigma^2-\eta^2)^{1/2}}\,d\eta \; + \right.$$

$$\left. + \int_1^\sigma \frac{\eta^2(\sigma^2-\eta^2)^{1/2} J_{1/2}(\alpha\eta r)H^{(1)}_{1/2}(\alpha\eta v)}{\mu^2(\eta^2-1)+(\sigma^2-\eta^2)}\,d\eta\right], \quad v>r$$

where $\sigma=k_1/k_2$, $\mu=\mu_2/\mu_1$. For $v<r$, v and r are to be interchanged. Next, expanding the Bessel and Hankel functions in series for small values of their arguments and integrating, assuming that $\mu>\sigma>1$, the above expression can be written as (details are given in appendix - A)

$$aL_1(av,ar) = \alpha^2 rM_1 + i\alpha^3 rvM_2 - \frac{\alpha^4(3v^2r+r^3)}{6}M_3 - \frac{i\alpha^5(v^3r+r^3v)}{6}M_4 \; +$$

$$+ \frac{\alpha^6(5v^4r+10r^3v^2+r^5)}{120}M_5 + \frac{i\alpha^7(3v^5r+10r^3v^3+3r^5v)}{360}M_6 \; +$$

$$+ o(\alpha^8), \quad v>r$$

$$= \alpha^2 v M_1 + i\alpha^3 r v M_2 - \frac{\alpha^4(3r^2 v + v^3)}{6} M_3 - \frac{i\alpha^5(r^3 v + v^3 r)}{6} M_4 +$$

$$+ \frac{\alpha^6(5r^4 v + 10v^3 r^2 + v^5)}{120} M_5 + \frac{i\alpha^7(3r^5 v + 10v^3 r^3 + 3v^5 r)}{360} M_6 +$$

$$+ o(\alpha^8), \quad v<r \qquad (45)$$

where $$M_1 = \frac{\sigma^2+\mu}{2(\mu+1)} \qquad (46)$$

$$M_2 = \frac{2}{\pi(\mu-1)}\left[\frac{\sigma^3-\mu}{3} + \frac{\mu^2-\sigma^2}{\mu^2-1}\left\{(\mu-\sigma) + \mu\sqrt{\frac{\sigma^2-1}{\mu^2-1}}\log\left(\frac{\sigma\sqrt{\mu^2-1}+\mu\sqrt{\sigma^2-1}}{\sqrt{\mu^2-1}+\sqrt{\sigma^2-1}}\right)\right\}\right] \qquad (47)$$

$$M_3 = \frac{1}{(\mu-1)}\left[\frac{1}{8}(\sigma^4-\mu) + \left(\frac{\mu^2-\sigma^2}{\mu^2-1}\right)\left\{\frac{\sigma^2-\mu}{2} + \frac{\mu^2-\sigma^2}{\mu+1}\right\}\right] \qquad (48)$$

$$M_4 = \frac{2}{\pi(\mu-1)}\left[\frac{2(\sigma^5-\mu)}{15} + \frac{\mu^2-\sigma^2}{\mu^2-1}\left(\frac{\sigma^3-\mu}{3} + \frac{\mu^2-\sigma^2}{\mu^2-1}\left\{(\mu-\sigma) + \mu\sqrt{\frac{\sigma^2-1}{\mu^2-1}} \times\right.\right.\right.$$

$$\left.\left.\left.\times \log\left(\frac{\sigma\sqrt{\mu^2-1}+\mu\sqrt{\sigma^2-1}}{\sqrt{\mu^2-1}+\sqrt{\sigma^2-1}}\right)\right\}\right)\right] \qquad (49)$$

$$M_5 = \frac{1}{(\mu-1)}\left[\frac{\sigma^6-\mu}{16} + \left(\frac{\mu^2-\sigma^2}{\mu^2-1}\right)\left\{\frac{1}{8}(\sigma^4-\mu) + \left(\frac{\mu^2-\sigma^2}{\mu^2-1}\right)\left(\frac{\sigma^2-\mu}{2} + \frac{\mu^2-\sigma^2}{\mu+1}\right)\right\}\right] \qquad (50)$$

$$M_\sigma = \frac{2}{\pi(\mu-1)}\left[\frac{8}{105}(\sigma^7-\mu) + \frac{\mu^2-\sigma^2}{\mu^2-1}\left\{\frac{2(\sigma^5-\mu)}{15} + \frac{\mu^2-\sigma^2}{\mu^2-1}\left(\frac{\sigma^3-\mu}{3} + \frac{\mu^2-\sigma^2}{\mu^2-1}\times\right.\right.\right.$$

$$\left.\left.\left.\times\left\{(\mu-\sigma) + \mu\sqrt{\frac{\sigma^2-1}{\mu^2-1}}\,\log\left(\frac{\sigma\sqrt{\mu^2-1}+\mu\sqrt{\sigma^2-1}}{\sqrt{\mu^2-1}+\sqrt{\sigma^2-1}}\right)\right\}\right)\right\}\right] \tag{51}$$

Substituting the value of $L_1(av,ar)$ given by (45) in (43) and using iterative method, an approximate value of $X(ar)$ for low frequency can be derived as follows :

$aL_1(ar,av)$ given by (45) is rewritten as

$$aL_1(ar,av) = \alpha^2 L_{12} + \alpha^3 L_{13} + \alpha^4 L_{14} + \alpha^5 L_{15} + \alpha^6 L_{16} + \alpha^7 L_{17} \tag{52}$$

where

$$L_{12} = M_1 r \quad , \quad v>r$$

$$\quad\;\; = M_1 v \quad , \quad v<r$$

$$L_{13} = iM_2 rv$$

$$L_{14} = -\frac{(3v^2r+r^3)}{6}M_3 \quad , \quad v>r$$

$$\quad\;\; = -\frac{(3r^2v+v^3)}{6}M_3 \quad , \quad v<r$$

$$L_{15} = -\frac{i(v^3r+r^3v)}{6}M_4$$

$$L_{16} = \frac{5v^4r+10v^2r^3+r^5}{120} M_5 \quad , \quad v>r$$

$$= \frac{5r^4v+10r^2v^3+v^5}{120} M_5 \quad , \quad v<r$$

$$L_{17} = \frac{i(3v^5r+10v^3r^3+3r^5v)}{360} M_6$$

Also let $X(ar) = X_0(ar)+\alpha X_1(ar)+\alpha^2 X_2(ar)+\alpha^3 X_3(ar)+\alpha^4 X_4(ar)+$

$$+\alpha^5 X_5(ar)+\alpha^6 X_6(ar)+\alpha^7 X_7(ar) + o(\alpha^8) \qquad (53)$$

where $X_i(ar)$, $(i=0,1,.....,7)$ are to be determined.

Now putting the values of $X(ar)$ and $aL_1(ar,av)$ given by (53) and (52) in (43) and equating the coefficient of like powers of α from both sides we obtain,

$$X_0(ar) = -2a\Omega(\mu_1+\mu_2)r \qquad (54)$$

$$X_1(ar) = 0 \qquad (55)$$

$$X_2(ar) = -\int_0^1 L_{12}X_0(av)dv \qquad (56)$$

$$X_3(ar) = -\int_0^1 \Big[L_{12}X_1(av)+L_{13}X_0(av)\Big]dv \qquad (57)$$

$$X_4(ar) = -\int_0^1 \Big[L_{12}X_2(av)+L_{13}X_1(av)+L_{14}X_0(av)\Big]dv \qquad (58)$$

$$X_5(ar) = -\int_0^1 \Big[L_{12}X_3(av)+L_{13}X_2(av)+L_{14}X_1(av)+L_{15}X_0(av)\Big]dv \quad (59)$$

$$X_6(ar) = -\int_0^1 \Big[L_{12}X_4(av)+L_{13}X_3(av)+L_{14}X_2(av)+L_{15}X_1(av) + $$
$$+ L_{16}X_0(av)\Big]dv \quad (60)$$

$$X_7(ar) = -\int_0^1 \Big[L_{12}X_5(av)+L_{13}X_4(av)+L_{14}X_3(av)+L_{15}X_2(av) + $$
$$+ L_{16}X_1(av)+L_{17}X_0(av)\Big]dv \quad (61)$$

Substituting the value of $X_0(ar)$ and L_{12} in (56) and integrating, $X_2(ar)$ is found to be

$$X_2(ar) = \frac{a\Omega}{3} M_1(\mu_1+\mu_2)(3r-r^3) \quad (62)$$

Similarly replacing the necessary unknowns by their corresponding iterated values in (57) to (61) and integrating we obtain

$$X_3(ar) = \frac{2}{3}\, ia\Omega(\mu_1+\mu_2)M_2 r \quad (63)$$

$$X_4(ar) = -\frac{a\Omega}{60}(\mu_1+\mu_2)\Big[(M_1^2-M_3)r^5-10(M_1^2-M_3)r^3-5(M_1^2-3M_3)r\Big] \quad (64)$$

$$X_5(ar) = \frac{ia\Omega}{180}(\mu_1+\mu_2)\Big[20(M_1M_2-M_4)r^3-12(9M_1M_2+M_4)r\Big] \quad (65)$$

$$X_6(ar) = \frac{a\Omega}{180}(\mu_1+\mu_2)\Big[\frac{1}{14}(2M_1M_3-M_1^2-M_5)r^7+\frac{3}{2}(M_1^2-2M_1M_3+M_5)r^5+\frac{5}{2}(M_1^3+$$

$$+2M_1M_3+3M_5)r^3+\frac{1}{2}(94M_1M_3-29M_1^3+80M_2^2+5M_5)r\Big] \tag{66}$$

$$X_7(ar) = \frac{ia\Omega}{180}(\mu_1+\mu_2)\Big[(M_1^2M_2-M_1M_4-M_2M_3+M_6)r^5+2(3M_1M_4-9M_1^2M_2+5M_2M_3+$$

$$+M_6)r^3+\frac{1}{7}(269M_1^2M_2+109M_1M_4+249M_2M_3+3M_6)r\Big] \tag{67}$$

The values of $X_i(ar)$, $i=0,1,\ldots\ldots,7$ from (54), (55), (62) - (67) are substituted in (52) and arranging the terms in ascending powers of r, X(ar) can be rewritten as

$$X(ar) = a\Omega(\mu_1+\mu_2)\left[p_1(\alpha)r+p_3(\alpha)r^3+p_5(\alpha)r^5+p_7(\alpha)r^7+o(\alpha^8)\right] \tag{68}$$

where

$$p_1(\alpha) = -2+M_1\alpha^2+\frac{2i}{3}M_2\alpha^3+\frac{1}{12}\left(M_1^2-3M_3\right)\alpha^4+\frac{1}{360}\left(94M_1M_3-29M_1^3+80M_2^2+5M_5\right)\alpha^6 -$$

$$-\frac{i}{15}\left(9M_1M_2+M_4\right)\alpha^5+\frac{i}{1260}\left(269M_1^2M_2+109M_1M_4+249M_2M_3+3M_6\right)\alpha^7$$

$$p_3(\alpha) = -\frac{1}{3}M_1\alpha^2+\frac{1}{6}\left(M_1^2-M_3\right)\alpha^4+\frac{i}{9}\left(M_1M_2-M_4\right)\alpha^5+\frac{1}{72}\left(M_1^3+2M_1M_3+3M_5\right)\alpha^6 +$$

$$+\frac{i}{90}\left(3M_1M_4-9M_1^2M_2+5M_2M_3+M_6\right)\alpha^7$$

$$p_5(\alpha) = -\frac{i}{60}(M_1^2-M_3)\alpha^4+\frac{1}{120}(M_1^3-2M_1M_3+M_5)\alpha^6+\frac{i}{180}(M_1^2M_2-M_1M_4-M_2M_3+M_6)\alpha^7$$

$$p_7(\alpha) = \frac{1}{2520}(2M_1M_3-M_1^3-M_5)\alpha^6 .$$

Next, replacing r by br, equation (29) can be written as

$$S_2(br)+b\int_1^\infty L_2(bv,br)S_2(bv)dv = b\int_0^1 L_2(bv,br)T_2(bv)dv , \quad 1<r<\infty \tag{69}$$

Following the same procedure as done for the evaluation of $L_1(av,ar)$, $L_2(bv,br)$ given by (23) can be evaluated to the form

$$bL_2(bv,br) = i(1+\mu)\beta^2(vr)^{1/2}\left[\int_0^1 \frac{\eta^2 J_{3/2}(\beta\eta r)H_{3/2}^{(1)}(\beta\eta v)}{\mu(1-\eta^2)^{1/2}+(\sigma^2-\eta^2)^{1/2}}\,d\eta \;+\right.$$

$$\left.+\int_1^\sigma \frac{\eta^2(\sigma^2-\eta^2)^{1/2}J_{3/2}(\beta\eta r)H_{3/2}^{(1)}(\beta\eta v)}{\mu^2(\eta^2-1)+(\sigma^2-\eta^2)}\,d\eta\right], \quad v>r$$

For v<r, v and r are to be interchanged.

For low frequency $bL_2(bv,br)$ is now reduced to the following form after using the series expansions of Bessel and Hankel functions.

$$bL_2(bv,br) = \alpha^2\lambda^2\left[\ \frac{1}{3}M_1\frac{r^2}{v} + o(\alpha^4)\ \right] , \quad v>r$$

$$= \alpha^2\lambda^2\left[\ \frac{1}{3}M_1\frac{v^2}{r} + o(\alpha^4)\ \right] , \quad v<r \tag{70}$$

The functions which occur in the integral equations (43), (28), (37), (36), (29), (38), (35) and (44) are calculated by iterative process in the following order

$$X\ ,\ S_1\ ,\ l_1\ ,\ T_2\ ,\ S_2\ ,\ l_2\ ,\ T_1\ ,\ Y\ ,\ S_1$$

Iterative procedure is followed in order to obtain the following results suffiently accurate correct upto the order of (α^7)

$$l_1(br) = \frac{8\Omega(\mu_1+\mu_2)ar^2\lambda^2}{3\pi}\left[-1+\frac{2}{3}M_1\alpha^2+\frac{1}{3}M_2\alpha^3-\frac{2}{5}\lambda^2r^2+o(\alpha^4)\right],\quad 0<r<1 \tag{71}$$

$$T_2(br) = l_1(br) + o(\alpha^7),\quad 0<r<1 \tag{72}$$

$$S_2(br) = -\frac{8\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^4}{45\pi}\left[\frac{1}{r} + o(\alpha^2)\right],\quad 1<r<\infty \tag{73}$$

$$l_2(ar) = -\frac{16\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^5}{45\pi^2}\left[\frac{1}{r} + o(\alpha^2)\right],\quad 1<r<\infty \tag{74}$$

$$T_1(ar) = \frac{16\Omega(\mu_1+\mu_2)a\lambda^5}{45\pi^2}\left[-\frac{1}{r}M_1\alpha^2+2\left\{-1+\frac{2}{3}M_1\alpha^2-\frac{2}{7}\lambda^2\right\}\frac{1}{r^3} - \frac{12\lambda^2}{7}\frac{1}{r^5} + o(\alpha^3)\right],\quad 1<r<\infty \tag{75}$$

$$Y(ar) = -\frac{16\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^5}{45\pi^2}\left[r + o(\alpha)\right],\quad 0<r<1 \tag{76}$$

$$S_1(ar) = X(ar) - \frac{16\Omega(\mu_1+\mu_2)aM_1 r\alpha^2\lambda^5}{45\pi^2} + o(\alpha^8)\ , \qquad 0<r<1 \tag{77}$$

Detailed derivation of the above expressions have been done in appendix-B.

4. STRESS DIFFERENCE ACROSS THE ANNULAR DISC, TORQUE AND FAR FIELD AMPLITUDE

The jump of the stresses at the annular disc is given by

$$\tau(r,0,t)=\tau(r)e^{-i\omega t} = \tau_{z\theta}^{(2)}(r,0,t)-\tau_{z\theta}^{(1)}(r,0,t) = f(r)e^{-i\omega t}$$

$$, \quad b\le r\le a\ ,\ r=0$$

$$= f_1(r) + f_2(r) \qquad (\text{supressing } e^{-i\omega t}).$$

Putting the values of $f_1(r)$ and $f_2(r)$ from (31) and (32) in the above expression we obtain

$$\tau(r) = \frac{2}{\pi}\left[\frac{d}{dr}\left\{-\int_r^a \frac{S_1(u)du}{(u^2-r^2)^{1/2}} + \int_a^\infty \frac{T_1(u)du}{(u^2-r^2)^{1/2}}\right\} + \right.$$

$$\left. + \frac{1}{r^2}\frac{d}{dr}\left\{-\int_0^b \frac{u^2T_2(u)du}{(r^2-u^2)^{1/2}} + \int_b^r \frac{u^2S_2(u)du}{(r^2-u^2)^{1/2}}\right\}\right]\ , \quad b\le r\le a \tag{78}$$

Finally, substitution of the values of $S_1(u)$, $T_1(u)$, $T_2(u)$ and

$S_2(u)$ from (77), (75), (72) and (73) respectively in (78), yields after integration,

$$\tau(r) = \frac{2(\mu_1+\mu_2)\Omega}{\pi}\left[\frac{\nu_1}{(1-\nu_1^2)^{1/2}}\left\{-2+\left(\frac{2}{3}M_1(1-\nu_1^2)+\frac{2}{3}M_1\right)\alpha^2+\frac{2i}{3}M_2\alpha^3+\right.\right.$$

$$+\left(\frac{1}{30}(7M_1^2-12M_3)-\frac{4}{15}(M_1^2-M_3)(1-\nu_1^2)-\frac{2}{45}(M_1^2-M_3)(1-\nu_1^2)^2\right)\alpha^4-$$

$$-i\left(\frac{2}{45}(11M_1M_2+4M_4)+\frac{2}{9}(M_1M_2-M_4)(1-\nu_1^2)\right)\alpha^5+\left(\frac{1}{1260}(386M_1M_3-74M_1^3\ +\right.$$

$$+280M_2^2+101M_5)-\frac{1}{630}(37M_1^3+80M_1M_3+114M_5)(1-\nu_1^2)+\frac{1}{1260}(24M_1^3+78M_1M_3+$$

$$\left.+87M_5)(1-\nu_1^2)^2-\frac{2}{1575}(2M_1M_3-M_1^3-M_5)(1-\nu_1^2)^3\right)\alpha^6+i\left(\frac{1}{630}(75M_1^2M_2+\right.$$

$$+72M_1M_4+156M_2M_3+12M_6)+\frac{2}{135}(M_1^2M_2-M_1M_4-M_2M_3+M_6)(1-\nu_1^2)^2-\frac{2}{45}(M_1M_4-$$

$$\left.\left.-4M_1^2M_2+2M_2M_3+M_6)(1-\nu_1^2)\right)\alpha^7-\frac{16M_1\alpha^2\lambda^5}{45\pi^2}\right\}+\frac{16\lambda^5}{45\pi^2}\left\{-\frac{M_1\alpha^2}{\nu_1}\left(-\frac{\sin^{-1}\nu_1}{\nu_1}+\right.\right.$$

$$\left.+(1-\nu_1^2)^{-1/2}\right)+\frac{1}{\nu_1^3}\left(-1+\frac{2}{3}M_1\alpha^2-\frac{2}{7}\lambda^2\right)\left(-\frac{3\sin^{-1}\nu_1}{\nu_1}+(1-\nu_1^2)^{1/2}+2(1-\nu_1^2)^{-1/2}\right)-$$

$$\left.-\frac{3\lambda^2}{14\nu_1^5}\left(-\frac{15\sin^{-1}\nu_1}{\nu_1}-2(1-\nu_1^2)^{3/2}+9(1-\nu_1^2)^{1/2}+8(1-\nu_1^2)^{-1/2}\right)\right\}-$$

$$-\frac{2\lambda}{3\pi}\left\{2\left[-1+\frac{2}{3}M_1\alpha^2+\frac{1}{3}M_2\alpha^3\right]\left[3\nu_2\sin^{-1}\left(\frac{1}{\nu_2}\right)-\frac{\left(\nu_2^2-1\right)^{1/2}}{\nu_2}-2\nu_2\left(\nu_2^2-1\right)^{-1/2}\right]-\right.$$

$$\left.-\frac{\lambda^2\nu_2^2}{5}\left[15\nu_2\sin^{-1}\left(\frac{1}{\nu_2}\right)+\frac{2\left(\nu_2^2-1\right)^{3/2}}{\nu_2^3}-\frac{9\left(\nu_2^2-1\right)^{1/2}}{\nu_2}-8\nu_2\left(\nu_2^2-1\right)^{-1/2}\right]\right\}-$$

$$\left.-\frac{8M_1\alpha^2\lambda^4}{45\pi\nu_1\left(\nu_2^2-1\right)^{1/2}}+o(\alpha^8)\right], \quad b\le r\le a \qquad (79)$$

where $\nu_1=r/a$ and $\nu_2=r/b$

Substituting $\alpha=0$ and $\lambda=0$ in (79) the jump in the statical stress across the rigid circular disc of radius a embedded at the bimaterial interface is easily found to be

$$\tau_0(r) = -\frac{4(\mu_1+\mu_2)\Omega}{\pi}\frac{\nu_1}{\left(1-\nu_1^2\right)^{1/2}}$$

so that the stress intensity factor at the edge of the circular disc in the statical case is

$$K_0 = \underset{r\to a-}{\mathrm{Lt}}\left[\left(1-\nu_1\right)^{1/2}\tau_0(r)\right] = -\frac{2\sqrt{2}(\mu_1+\mu_2)\Omega}{\pi} .$$

Therefore, in our dynamical problem involving annular disc, stress intensity factors at the outer and inner edges of the disc defined by

$$K_a^* = \frac{K_a}{K_0} = \underset{r\to a-}{Lt}\left[\frac{\tau(r)\left(1-\nu_1\right)^{1/2}}{K_0}\right]$$

and
$$K_b^* = \frac{K_b}{K_0} = \underset{r\to b+}{Lt}\left[\frac{\tau(r)\left(\nu_2-1\right)^{1/2}}{K_0}\right]$$

are given by

$$K_a^* = -\frac{1}{2}\left[\left\{-2+\frac{2}{3}M_1\alpha^2+\frac{1}{30}\left(7M_1^2-12M_3\right)\alpha^4+\frac{1}{1260}\left(386M_1M_3-74M_1^3+280M_2^2+101M_5\right)\alpha^6 - \right.\right.$$
$$\left.-\frac{32\lambda^5}{45\pi^2}\left(1+\frac{1}{3}M_1\alpha^2+\frac{8}{7}\lambda^2\right)\right\}+i\left\{\frac{2}{3}M_2\alpha^3-\frac{2}{45}\left(11M_1M_2+4M_4\right)\alpha^5+\frac{1}{630}\left(75M_1^2M_2+\right.\right.$$
$$\left.\left.\left.+72M_1M_4+156M_2M_3+12M_6\right)\alpha^7\right\}\right] \tag{80}$$

and
$$K_b^* = -\frac{1}{2}\left[\left\{\frac{8\lambda}{3\pi}\left(-1+\frac{2}{3}M_1\alpha^2\right)-\frac{16\lambda^3}{15\pi} - \frac{8M_1\alpha^2\lambda^3}{45\pi}\right\} + i\frac{8\lambda}{9\pi}M_2\alpha^3\right] \tag{81}$$

The torque of the shear stress on the annular disc is represented by the expression

$$T = 2\pi\int_b^a r^2\tau(r)dr \tag{82}$$

which can be written after putting the value of $\tau(r)$ given by (79) and integrating as follows.

$$T = \frac{4}{3}(\mu_1+\mu_2)\Omega a^3\Big[-4+\frac{8}{5}M_1\alpha^2+\frac{4i}{3}M_2\alpha^3+\frac{1}{105}(37M_1-72M_3)\alpha^4-\frac{4i}{15}(4M_1M_2+M_4)\alpha^5+$$

$$+\frac{1}{4725}(2704M_1M_3+2100M_2^2-650M_1^3+472M_5)\alpha^6+\frac{i}{1575}(491M_1^2M_2+328M_1M_4+$$

$$+720M_2M_3+36M_6)\alpha^7+\frac{64\lambda^5}{15\pi^2}(1-\frac{4}{3}M_1\alpha^2+\frac{4}{7}\lambda^2) + o(\alpha^8)\Big] \qquad (83)$$

Next, in order to deduce the far field amplitude of the displacement in both the media, we substitute the value of $A_j(\xi)$ in equation (8) and obtain

$$v_j(r,z) = \int_b^a tf(t)dt\int_0^\infty \frac{\xi}{(\mu_1\gamma_1+\mu_2\gamma_2)} J_1(\xi r)J_1(\xi t)\exp(-\gamma_j|z|)d\xi \qquad (84)$$

Evaluating the integral with respect to ξ by the method of steepest descent for large values of $\sqrt{(r^2+z^2)}$, we obtain finally for $z>0$

$$v_1(r,\theta,z) = F_1(\theta)\frac{e^{ik_1R_1}}{R_1} + O\left(\frac{1}{R_1^2}\right) \quad \text{as} \quad R_1\to\infty \qquad (85)$$

where $r=R_1\cos\theta$, $|z|=R_1\sin\theta$

$$F_1(\theta) = -\frac{i\sigma\sin\theta}{\sigma\mu_1\sin\theta + \mu_2\sqrt{(1-\sigma^2\cos^2\theta)}}G_1(\theta) , \quad \text{for} \quad |\cos\theta|<\frac{1}{\sigma}$$

$$= -\frac{\sigma\sin\theta[\mu_2\surd(\sigma^2\cos^2\theta-1)+i\sigma\mu_1\sin\theta]}{\sigma^2\mu_1^2\sin^2\theta+\mu_2^2(\sigma^2\cos^2\theta-1)}\,G_1(\theta)\,, \quad \text{for} \quad |\cos\theta|>\frac{1}{\sigma}$$

(86)

$$\text{and} \quad G_1(\theta) = \frac{2\Omega a^2(\mu_1+\mu_2)\sigma\alpha\cos\theta}{\pi}\Bigg[-\frac{2}{3}+\frac{4}{15}M_1\alpha^2+\frac{2i}{9}M_2\alpha^3+\frac{1}{630}\left(37M_1^2-72M_3\right)\alpha^4-$$

$$-\frac{2i}{45}\left(4M_1M_2+M_4\right)\alpha^5+\frac{1}{2835}\left(256M_1M_3-65M_1^3+210M_2^2+40M_5\right)\alpha^6+$$

$$+\frac{i}{9450}\left(491M_1^2M_2+328M_1M_4+720M_2M_3+36M_6\right)\alpha^7 - \frac{\sigma^2\alpha^2\cos^2\theta}{6}\Bigg\{-\frac{2}{5}+$$

$$+\frac{16}{105}M_1\alpha^2+\frac{2i}{15}M_2\alpha^3+\frac{1}{1890}\left(73M_1^2-136M_3\right)\alpha^4-\frac{2i}{1575}\left(82M_1M_2+23M_4\right)\alpha^5\Bigg\} +$$

$$+\frac{\sigma^4\alpha^4\cos^4\theta}{120}\left\{-\frac{2}{7}+\frac{20}{189}M_1\alpha^2+\frac{2i}{21}M_2\alpha^3\right\} + \frac{\sigma^6\alpha^6\cos^6\theta}{22680} +$$

$$+\frac{32\lambda^5}{45\pi^2}\left\{1-\frac{4}{3}M_1\alpha^2+\frac{4}{7}\lambda^2+\frac{1}{6}\sigma^2\alpha^2\cos^2\theta\right\}\Bigg] \tag{87}$$

Also for z<0,

$$v_2(r,\phi,z) = F_2(\phi)\,\frac{e^{ik_2R_2}}{R_2} + O\left(\frac{1}{R_2^2}\right) \quad \text{as} \quad R_2\to\infty \tag{88}$$

where $r=R_2\cos\phi$, $|z|=R_2\sin\phi$

$$F_2(\phi) = -\frac{i\sin\phi}{\mu_1\sqrt{(\sigma^2-\cos^2\phi)}+\mu_2\sin\phi}\, G_2(\phi) \tag{89}$$

and $G_2(\phi)$ is obtained by replacing θ by ϕ and also σ by 1 in $G_1(\phi)$.

5. NUMERICAL RESULTS AND DISCUSSION

Numerical results have been calculated to study the variations of the dynamic stress intensity factors with the normalized frequency α at both the outer and inner edges of the annular disc situated at the bimaterial interface for different values of the ratio of the inner and outer radii of the annular disc for the following two sets of materials.

First set :

Aluminium	$\rho_1=2.7\text{gm/cm}^3$	$\mu_1=2.63\times10^{11}\text{dyne/cm}^2$
Wrought iron	$\rho_2=7.8\text{gm/cm}^3$	$\mu_2=7.7\times10^{11}\text{dyne/cm}^2$

Second set :

Copper	$\rho_1=8.96\text{gm/cm}^3$	$\mu_1=4.5\times10^{11}\text{dyne/cm}^2$
Steel	$\rho_2=7.6\text{gm/cm}^3$	$\mu_2=8.32\times10^{11}\text{dyne/cm}^2$

The dynamic stress intensity factors are normalized by the static solution $K_o=-2\sqrt{2}(\mu_1+\mu_2)\Omega/\pi$ for the penny shaped rigid disc.
It is interesting to note that for both the two set of materials

stress intensity factor at the outer edge changes appreciably with the normalized frequency α and gradually decreases with the increase of α but in the case of inner edge, the stress intensity factor decrease very slowly with the increase in the values of the normalized frequency. It may further be noted that change in the values of the stress intensity factor with increase in the values of λ is more prominant at the inner edge than that at the outer edge. We also note from fig.2 and fig.3 that stress intensity factors for the two sets of materials are nearly the same for low frequency and increase gradually with the increase in frequency.

Far field amplitudes defined by $F_1(\theta)$ and $F_2(\phi)$ in the upper and lower medium $z>0$ and $z<0$ respectively for fixed R have been plotted in fig.4 - fig.7 against their arguments for different values of the normalized frequency α and λ, the ratio of the inner and outer radii of the annular disc for two different sets of materials.

It may be noted that both in the upper and lower medium for the two sets of materials, amplitudes $F_1(\theta)$ and $F_2(\phi)$ respectively increase gradually from θ and ϕ equal to zero, attain maximum values and then gradually decrease to zero at θ and ϕ equal to 90^o. The values of the angle at which maxima are attained are found to depend on the material properties and not on the values of the frequency and λ. On the other hand if the material properties are kept fixed, maximum values of the far field amplitude are found to depend on the normalized frequency α and λ, which is equal to the ratio of the inner and outer radii of the annular disc.

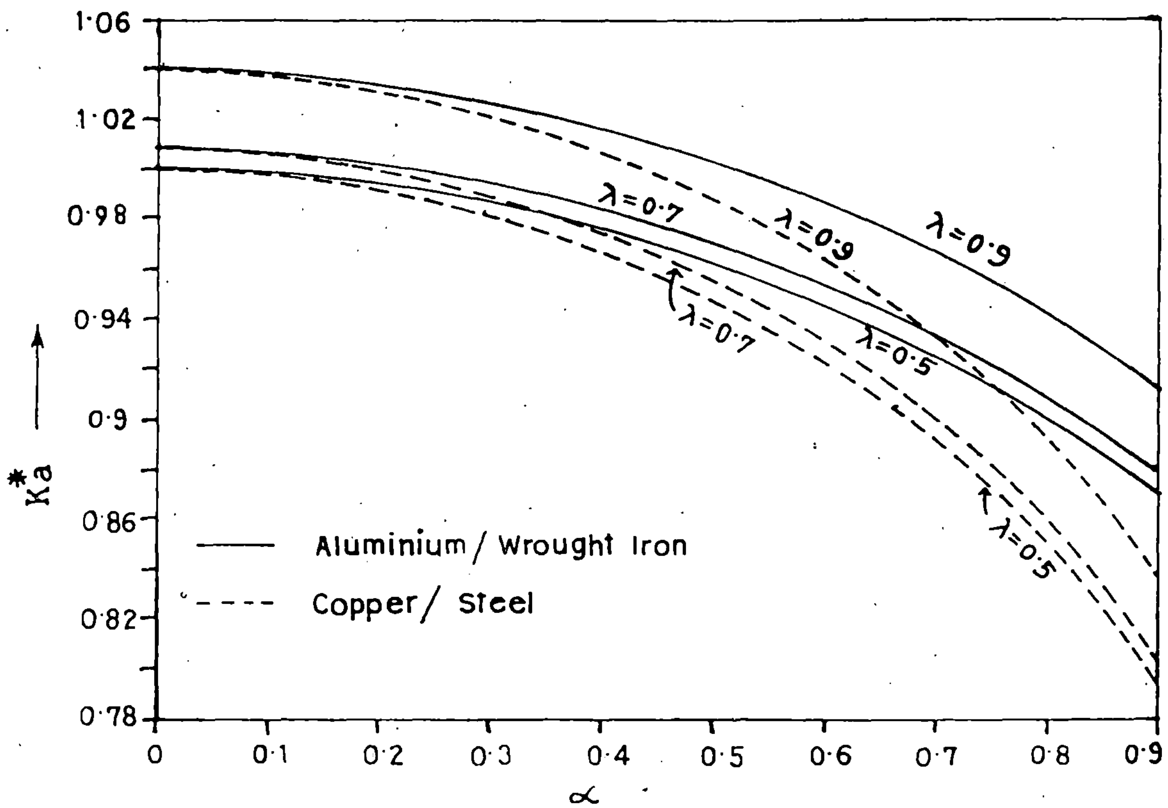

Fig. 2. Stress intensity factor vs. normalized frequency (outer edge)

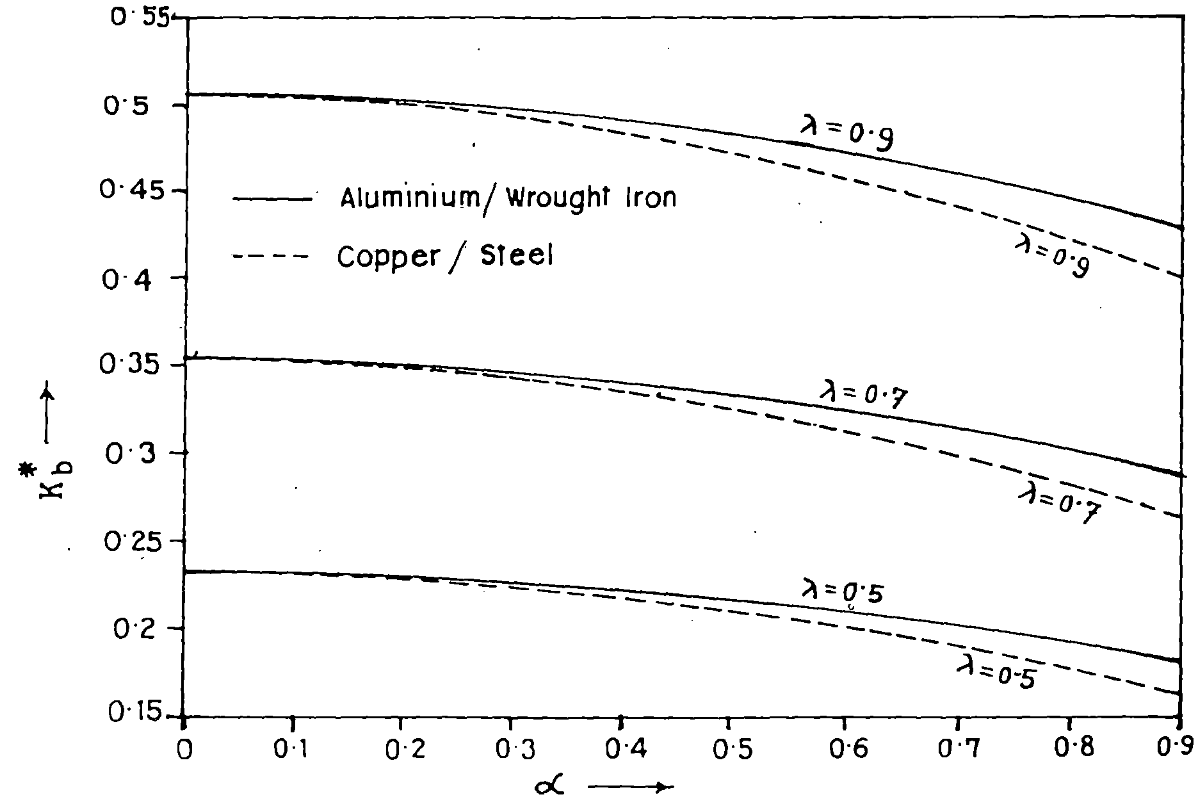

Fig. 3. Stress intensity factor vs. normalized frequency. (Inner edge)

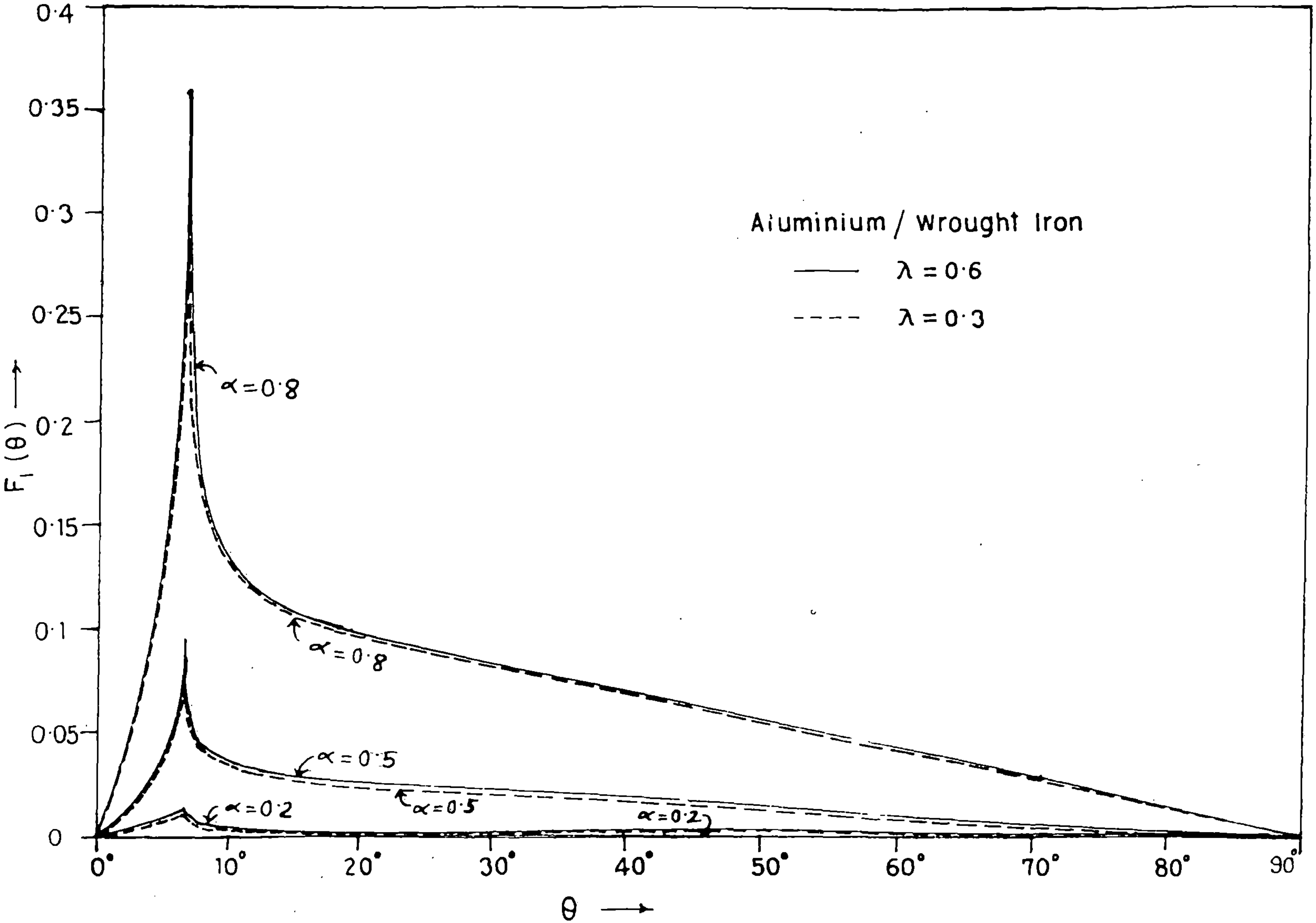

Fig. 4. Farfield amplitude $F_1(\theta)$ vs. argument θ of the amplitude for upper medium $(z > 0)$

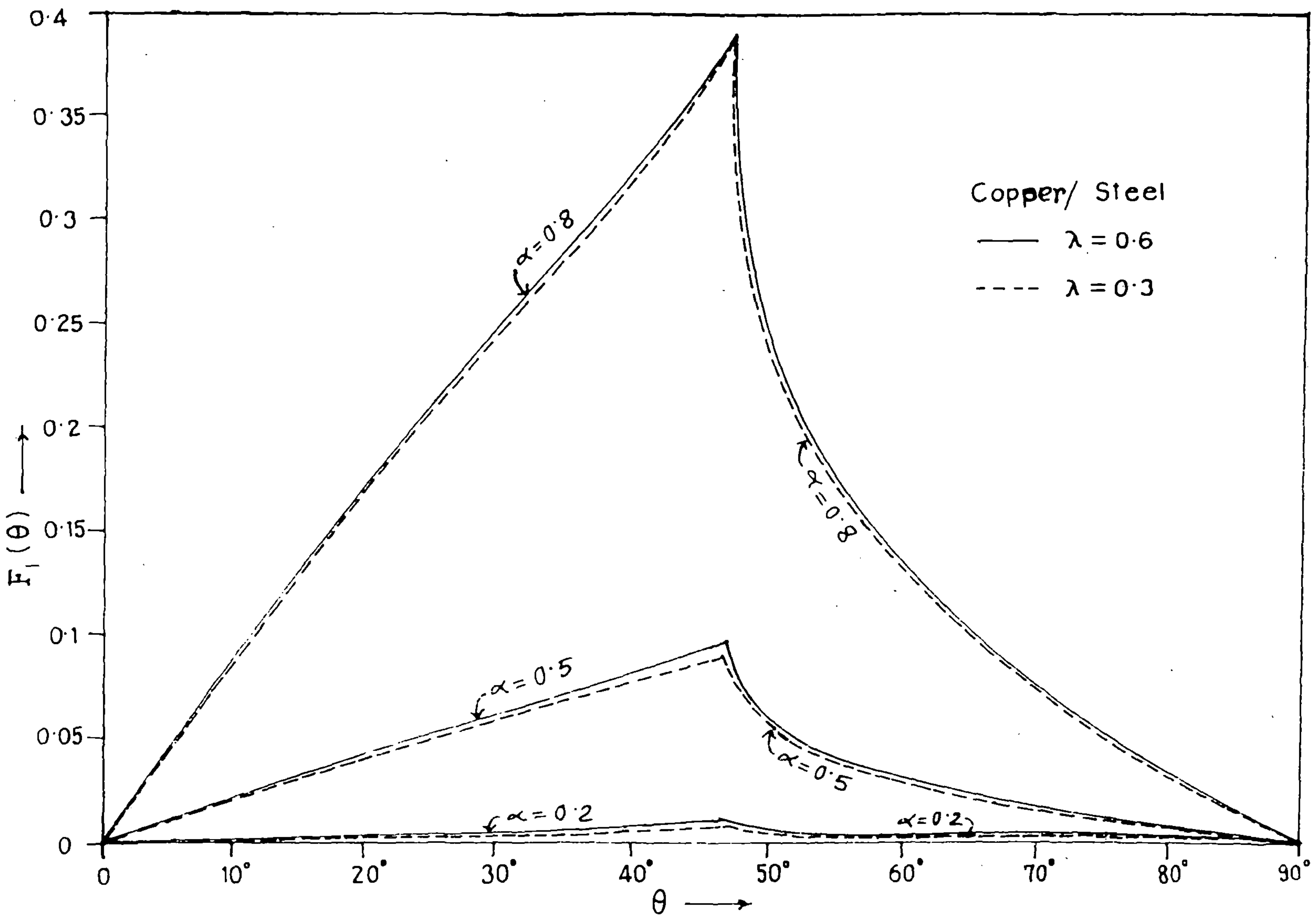

Fig. 5. Farfield amplitude $F_1(\theta)$ Vs. argument θ of the amplitude for upper medium ($z>0$)

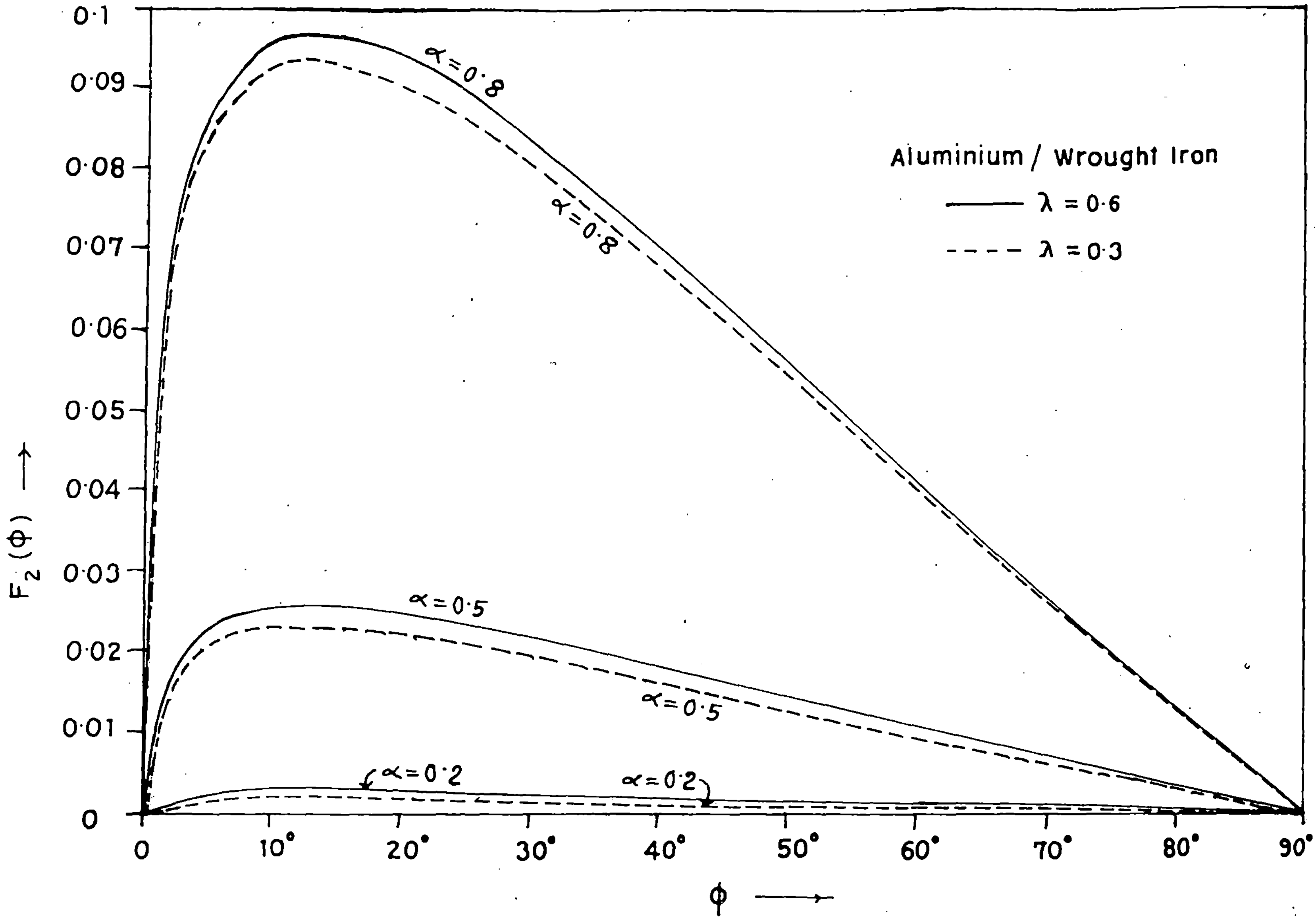

Fig. 6. Farfield amplitude $F_2(\phi)$ vs. argument ϕ of the amplitude for lower medium $(z<0)$

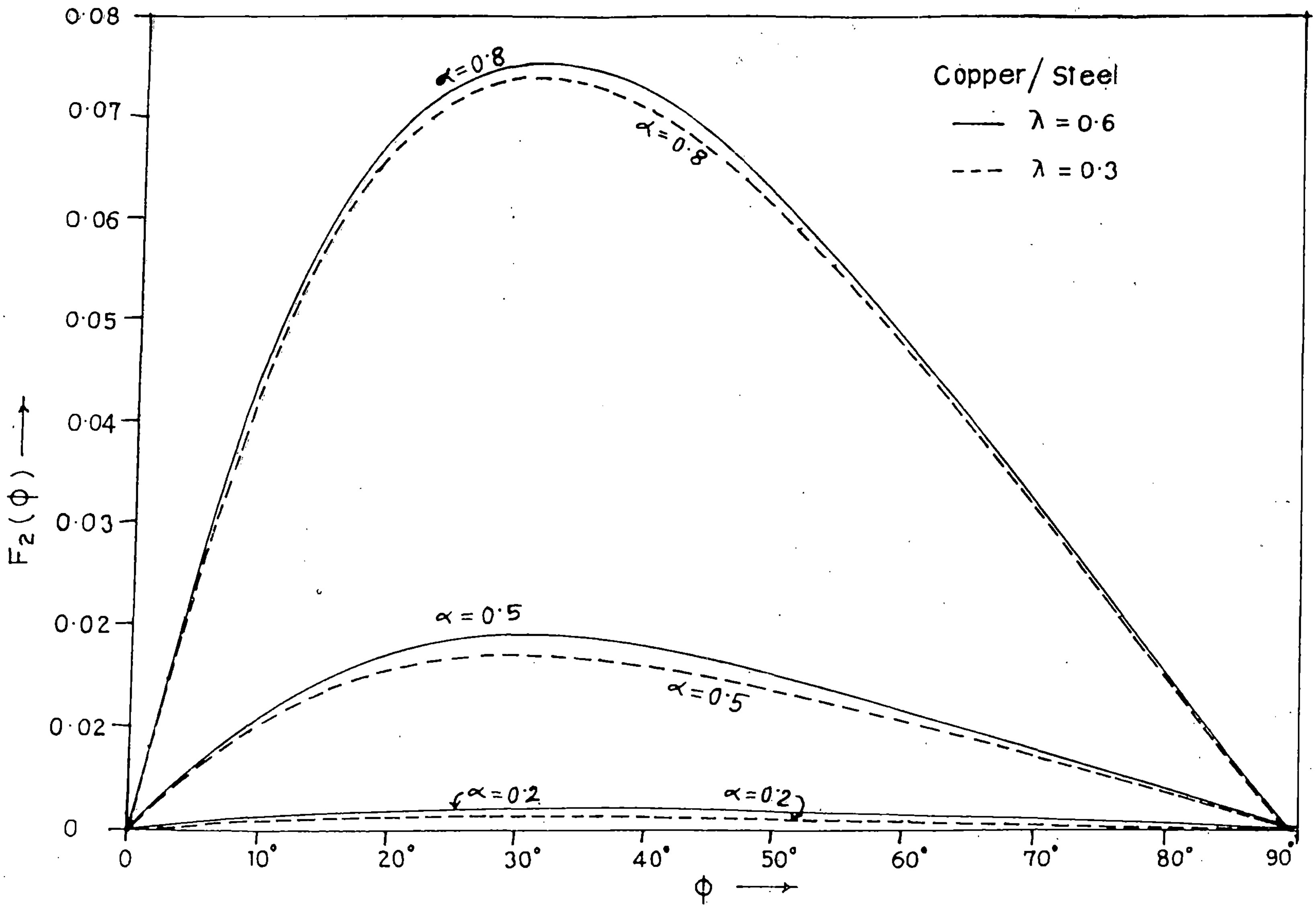

Fig. 7 Farfield amplitude $F_2(\phi)$ vs. argument ϕ of the amplitude for lower medium $(z < 0)$

APPENDIX - A

EVALUATION OF $aL_1(av, ar)$:

$$aL_1(av,ar) = i(1+\mu)\alpha^2(vr)^{1/2}\left[\int_0^1 \frac{\eta^2 J_{1/2}(\alpha\eta r)H^{(1)}_{1/2}(\alpha\eta v)}{\mu(1-\eta^2)^{1/2}+(\sigma^2-\eta^2)^{1/2}}\,d\eta\right. +$$

$$+\left.\int_1^\sigma \frac{\eta^2(\sigma^2-\eta^2)^{1/2} J_{1/2}(\alpha\eta r)H^{(1)}_{1/2}(\alpha\eta v)}{\mu^2(\eta^2-1)+(\sigma^2-\eta^2)}\,d\eta\right], \quad v>r \qquad \text{(A1)}$$

For small values of arguments expanding the Bessel and Hankel functions we get

$$J_{1/2}(\alpha\eta r)H^{(1)}_{1/2}(\alpha\eta v) = \frac{2}{\pi}\sqrt{\frac{r}{v}}\left[-i+\alpha\eta v + \frac{i\alpha^2\eta^2}{6}(3v^2+r^2) - \frac{\alpha^3\eta^3}{6}(v^2+r^2)v -\right.$$

$$\left. - \frac{i\alpha^4\eta^4}{120}(5v^4+10r^2v^2+r^4) + \frac{\alpha^5\eta^5}{360}(3v^5+10r^2v^3+3r^4v) + o(\alpha^6)\right]$$

Putting this expansion in (A1) $aL_1(av,ar)$ can be evaluated as

$$aL_1(av,ar) = \alpha^2 rM_1 + i\alpha^3 rvM_2 - \frac{\alpha^4(3v^2r+r^3)}{6}M_3 - \frac{i\alpha^5(v^3r+r^3v)}{6}M_4 +$$

$$+\frac{\alpha^6(5v^4r+10r^3v^2+r^5)}{120}M_5 + \frac{i\alpha^7(3v^5r+10r^3v^3+3r^5v)}{360}M_6 +$$

$$+ o(\alpha^8), \quad v>r$$

$$= \alpha^2 v M_1 + i\alpha^3 r v M_2 - \frac{\alpha^4(3r^2v+v^3)}{6} M_3 - \frac{i\alpha^5(r^3v+v^3r)}{6} M_4 +$$

$$+ \frac{\alpha^6(5r^4v+10v^3r^2+v^5)}{120} M_5 + \frac{i\alpha^7(3r^5v+10v^3r^3+3v^5r)}{360} M_6 +$$

$$+ o(\alpha^8), \quad v<r \qquad \text{(A2)}$$

where

$$M_i = \frac{2}{\pi}(1+\mu)\left[\int_0^{\sigma} \frac{\eta^{(i+1)}(\sigma^2-\eta^2)^{1/2}}{\mu^2(\eta^2-1) + (\sigma^2-\eta^2)}\,d\eta - \mu\int_0^{1} \frac{\eta^{(i+1)}(1-\eta^2)^{1/2}}{\mu^2(\eta^2-1) + (\sigma^2-\eta^2)}\,d\eta\right]$$

$$i=1,2,\ldots,6 \qquad \text{(A3)}$$

Now

$$M_1 = \frac{2}{\pi}(1+\mu)\left[\int_0^{\sigma} \frac{\eta^{2}(\sigma^2-\eta^2)^{1/2}}{\mu^2(\eta^2-1) + (\sigma^2-\eta^2)}\,d\eta - \mu\int_0^{1} \frac{\eta^{2}(1-\eta^2)^{1/2}}{\mu^2(\eta^2-1) + (\sigma^2-\eta^2)}\,d\eta\right]$$

$$\text{(A4)}$$

Without any loss of generality we assume $\mu>\tau>1$.

The first integral in the expression of M_1 is

$$\int_0^{\sigma} \frac{\eta^{2}(\sigma^2-\eta^2)^{1/2}}{\mu^2(\eta^2-1) + (\sigma^2-\eta^2)}\,d\eta$$

$$= \frac{1}{(\mu^2-1)}\int_0^{\sigma} (\sigma^2-\eta^2)^{1/2}d\eta + \frac{\mu^2-\sigma^2}{(\mu^2-1)^2}\int_0^{\sigma} \frac{(\sigma^2-\eta^2)^{1/2}}{\eta^2 - \dfrac{\mu^2-\sigma^2}{\mu^2-1}}\,d\eta$$

$$= \frac{\sigma^2}{2(\mu^2-1)}\,\frac{\pi}{2} - \frac{\mu^2-\sigma^2}{(\mu^2-1)^2}\,\frac{\pi}{2} \qquad \text{(A5)}$$

and the second integral is

$$\int_0^1 \frac{\eta^2(1-\eta^2)^{1/2}}{\mu^2(\eta^2-1)+(\sigma^2-\eta^2)}\,d\eta$$

$$= \frac{1}{(\mu^2-1)}\int_0^1 (1-\eta^2)^{1/2}d\eta + \frac{\mu^2-\sigma^2}{(\mu^2-1)^2}\int_0^1 \frac{(1-\eta^2)^{1/2}}{\eta^2-\dfrac{\mu^2-\sigma^2}{\mu^2-1}}\,d\eta$$

$$= \frac{1}{2(\mu^2-1)}\,\frac{\pi}{2} - \frac{\mu^2-\sigma^2}{(\mu^2-1)^2}\,\frac{\pi}{2} \tag{A6}$$

Putting the results (A5) and (A6) in (A4) and simplifying, M_1 can be obtained as

$$M_1 = \frac{\sigma^2+\mu}{2(\mu+1)} \tag{A7}$$

Similarly, $M_i\,(i=2,3,..,6)$ can be calculated and they are found to be given by

$$M_2 = \frac{2}{\pi(\mu-1)}\left[\frac{\sigma^3-\mu}{3}+\frac{\mu^2-\sigma^2}{\mu^2-1}\left\{(\mu-\sigma)+\mu\sqrt{\frac{\sigma^2-1}{\mu^2-1}}\,\log\left(\frac{\sigma\sqrt{\mu^2-1}+\mu\sqrt{\sigma^2-1}}{\sqrt{\mu^2-1}+\sqrt{\sigma^2-1}}\right)\right\}\right] \tag{A8}$$

$$M_3 = \frac{1}{(\mu-1)}\left[\frac{1}{8}(\sigma^4-\mu) + \left(\frac{\mu^2-\sigma^2}{\mu^2-1}\right)\left(\frac{\sigma^2-\mu}{2}+\frac{\mu^2-\sigma^2}{\mu+1}\right)\right] \tag{A9}$$

$$M_4 = \frac{2}{\pi(\mu-1)}\left[\frac{2(\sigma^5-\mu)}{15} + \frac{\mu^2-\sigma^2}{\mu^2-1} \left(\frac{\sigma^3-\mu}{3} + \frac{\mu^2-\sigma^2}{\mu^2-1} \left\{ (\mu-\sigma) + \mu\sqrt{\frac{\sigma^2-1}{\mu^2-1}} \times \right.\right.\right.$$

$$\left.\left.\left. \times \log\left(\frac{\sigma\sqrt{\mu^2-1} + \mu\sqrt{\sigma^2-1}}{\sqrt{\mu^2-1} + \sqrt{\sigma^2-1}} \right) \right\} \right) \right] \qquad \text{(A10)}$$

$$M_5 = \frac{1}{(\mu-1)} \left[\frac{\sigma^6-\mu}{16} + \left(\frac{\mu^2-\sigma^2}{\mu^2-1}\right)\left\{\frac{1}{8}(\sigma^4-\mu) + \left(\frac{\mu^2-\sigma^2}{\mu^2-1}\right)\left(\frac{\sigma^2-\mu}{2} + \frac{\mu^2-\sigma^2}{\mu+1}\right)\right\}\right] \qquad \text{(A11)}$$

$$M_6 = \frac{2}{\pi(\mu-1)}\left[\frac{8}{105}(\sigma^7-\mu) + \frac{\mu^2-\sigma^2}{\mu^2-1} \left\{ \frac{2(\sigma^5-\mu)}{15} + \frac{\mu^2-\sigma^2}{\mu^2-1} \left(\frac{\sigma^3-\mu}{3} + \frac{\mu^2-\sigma^2}{\mu^2-1} \times \right.\right.\right.$$

$$\left.\left.\left. \times \left\{ (\mu-\sigma) + \mu\sqrt{\frac{\sigma^2-1}{\mu^2-1}} \log\left(\frac{\sigma\sqrt{\mu^2-1} + \mu\sqrt{\sigma^2-1}}{\sqrt{\mu^2-1} + \sqrt{\sigma^2-1}} \right) \right\} \right) \right\} \right] \qquad \text{(A13)}$$

APPENDIX - B

$l_1(r)$ is given by

$$l_1(r) = -\frac{2}{\pi r}\int_0^r \frac{t^2 dt}{(r^2-t^2)^{1/2}} \frac{d}{dt}\int_t^a \frac{S_1(u)du}{(u^2-t^2)^{1/2}}, \qquad 0<r<b \tag{B1}$$

Taking $S_1(u) = X(u)$ as a first approximation from (42) (B1) can be written as

$$l_1(r) = -\frac{2}{\pi r}\int_0^r \frac{t^2 dt}{(r^2-t^2)^{1/2}} \frac{d}{dt}\int_{t/a}^1 \frac{X(ay)dy}{(y^2-t^2/a^2)^{1/2}}, \qquad 0<r<b \tag{B2}$$

Substituting $\quad X(ay) = a\Omega(\mu_1+\mu_2)\left[\, p_1(\alpha)y + p_3(\alpha)y^3 \,\right],$

(neglecting α^4 and higher powers of α)

where

$$p_1(\alpha) = -2+M_1\alpha^2+\frac{2i}{3} M_2\alpha^3 + o(\alpha^4)$$

$$p_3(\alpha) = -\frac{1}{3}M_1\alpha^2 + o(\alpha^4)$$

in (B2) and integrating we obtain

$$l_1(r) = \frac{2\Omega}{\pi r}(\mu_1+\mu_2)\left(-2 + \frac{4}{3} M_1\alpha^2 + \frac{2i}{3} M_2\alpha^3\right) \frac{r^3}{2a} \left(\frac{4}{3} + \frac{8r^2}{15a^2}\right) -$$

$$- \frac{4\Omega}{3\pi r a^2}(\mu_1+\mu_2)M_1\alpha^2 \frac{8r^5}{15a}, \qquad 0<r<b$$

Replacing r by br and putting $b/a=\lambda$ in the above expression we obtain,

$$l_1(br) = \frac{8\Omega(\mu_1+\mu_2)ar^2\lambda^2}{3\pi}\left[-1+\frac{2}{3}M_1\alpha^2+\frac{i}{3}M_2\alpha^3-\frac{2}{5}\lambda^2 r^2+o(\alpha^4)\right] \quad , \quad 0<r<1 \qquad (B3)$$

Next,

$$T_2(br)=l_1(br)+\frac{1}{\pi\lambda r}\int_1^\infty T_1(au)\left\{\frac{2\lambda ur}{(u^2-\lambda^2 r^2)} - \log\left(\frac{u+\lambda r}{u-\lambda r}\right)\right\} du \quad , \quad 0<r<1$$

Neglecting higher order terms of α, $T_2(br)$ is found to be

$$T_2(br) = l_1(br) + o(\alpha^7) \quad , \quad 0<r<1 \qquad (B4)$$

Replacing r by br, (29) can be rewritten as

$$S_2(br) + b\int_1^\infty L_2(bv,br)S_2(bv)dv = b\int_0^1 L_2(bv,br)T_2(bv)dv \quad , \quad 1<r<\infty \qquad (B5)$$

First approximation of the above integral equation yields

$$S_2(br) = b\int_0^1 L_2(bv,br)T_2(bv)dv \quad , \quad 1<r<\infty$$

in which substituting the value of $T_2(bv)$ from (B4) and

$$bL_2(bv,br) = \alpha^2\lambda^2\left[\frac{1}{3}M_1\frac{v^2}{r} + o(\alpha^4)\right] \quad , \quad v<r$$

we get

$$S_2(br) = -\frac{8\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^4}{45\pi}\left[\frac{1}{r} + o(\alpha^2)\right], \quad 1<r<\infty \qquad \text{(B6)}$$

Similarly, first approximation of other functions from their respective integral equations can be derived and are given by

$$l_2(ar) = -\frac{16\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^5}{45\pi^2}\left[\frac{1}{r} + o(\alpha^2)\right], \quad 1<r<\infty \qquad \text{(B7)}$$

$$T_1(ar) = \frac{16\Omega(\mu_1+\mu_2)a\lambda^5}{45\pi^2}\left[-\frac{1}{r}M_1\alpha^2+2\left\{-1+\frac{2}{3}M_1\alpha^2-\frac{2}{7}\lambda^2\right\}\frac{1}{r^3} - \frac{12\lambda^2}{7}\frac{1}{r^5} + \right.$$

$$\left. +o(\alpha^3)\right], \quad 1<r<\infty \qquad \text{(B8)}$$

$$Y(ar) = -\frac{16\Omega(\mu_1+\mu_2)aM_1\alpha^2\lambda^5}{45\pi^2}\left[r + o(\alpha)\right], \quad 0<r<1 \qquad \text{(B9)}$$

$$S_1(ar) = X(ar) - \frac{16\Omega(\mu_1+\mu_2)aM_1r\alpha^2\lambda^5}{45\pi^2} + o(\alpha^8), \quad 0<r<1 \qquad \text{(B10)}$$

where X(ar) is given by (68).

C H A P T E R - II

CRACK PROBLEMS IN ELASTODYNAMICS

Paper - 5 : Interaction of elastic waves with a Periodic array of coplanar Griffith cracks in an orthotropic elastic medium.

Paper - 6 : Diffraction of SH-waves by a Griffith crack in nonhomogeneous elastic strip.

INTERACTION OF ELASTIC WAVES WITH A PERIODIC ARRAY OF COPLANAR GRIFFITH CRACKS IN AN ORTHOTROPIC ELASTIC MEDIUM

1. INTRODUCTION

In recent years, with the increased usage of macroscopically anisotropic construction materials such as fibre-reinforced composites, the study of interaction of elastic waves with cracks or inclusions in composite materials has gained much importance. Recently, Kassir and Bandyopadhya (1983) have studied the elastodynamic response of an infinite orthotropic solid containing a crack under the action of impact loading and the elastodynamic problem of a finite Griffith crack in an orthotropic strip under normal impact was investigated by Shindo (1986). Problem involving a moving Griffith crack in an orthotropic strip has also been studied by De and Patra (1990). But perhaps, because of mathematical complexity, elastodynamic problems involving two or more Griffith cracks in anisotropic materials have not yet received much attention. The static problem of determining the stress distribution in an infinite transversely isotropic medium containing three coplanar cracks has been considered by Dhawan and Dhaliwal (1978). Static stress distribution near periodic cracks at the interface of two bonded dissimilar orthotropic half planes has been obtained by Garg (1981). Angel and Achenbach (1985) have studied the problem of reflection and transmission of elastic waves by a periodic array of cracks in an infinite isotropic medium. The steady state vibration of an infinite isotropic medium with a periodic system of coplanar cracks has been discussed by Parton and Morozov (1978) using the method of the finite Fourier transforms to reduce the relevant mixed boundary value problem to the solution of a pair of dual series relations.

In our problem, the interaction of normally incident time harmonic elastic waves with a periodic array of coplanar Griffith cracks in an infinite orthotropic medium has been considered. Due to geometrical symmetry the problem has been reduced to the solution of the problem of a single crack in a strip whose boundaries are shear free and constrained in a way not to permit normal displacement. Fourier transform has been used to reduce the problem to the solution of dual integral equations. By the application of Abel's integral the dual integral equations finally has been converted to a Fredholm integral equation. Stress intensity factor at the tip of the crack and crack opening displacement have been derived in closed form. To display the influence of the material orthotropy numerical values of stress intensity factor and crack opening displacement have been found out after solving the Fredholm integral equation numerically and plotted against dimensionless frequency and distance respectively for three sets of orthotropic materials.

2. FORMULATION OF THE PROBLEM

Consider an orthotropic, linearly elastic, unbounded solid weakened by a periodic array of cracks of length 2a as shown in fig.1. The period of the crack-array is $2h_1$. The cracks lie in the plane $x_2=0$ and extend to infinity in the x_3-direction which is perpendicular to the plane of the figure. Let E_i, μ_{ij} and $\nu_{ij}(i,j=1,2,3)$ denote the engineering elastic constants of the material where the subscripts 1,2,3 correspond to the x_1, x_2, x_3 directions which coincide with the axes of material orthotropy.

We normalize all lengths with respect to a so that $x_1/a=x$, $x_2/a=y$, $x_3/a=z$ and $h_1/a=h$. Let a time harmonic wave given by $u=0$ and $v=\exp\left[i\left(k_s y/\sqrt{c_{22}}-\omega t\right)\right]$ where $k_s=\omega a/c_s$ and $c_s=(\mu_{12}/\rho)^{1/2}$, travelling in the direction of positive y-axis be incident normally on the crack faces so that the conditions in the plane of the cracks($y=0$) due to the scattered field are $\tau_{xy}=0$, $\tau_{yy}=-\tau_o e^{-i\omega t}$, where $\tau_o=i\omega a\sqrt{c_{22}}/c_s$, on the crack faces and $v=0$ at points outside the cracks in the plane $y=0$.

By simple symmetry considerations, the displacement and stress distribution due to the scattered field in the entire xy-plane can be deduced by considering only the orthotropic strip $|x|\leq h$ with a central crack $|x|\leq a$, $y=0$; the boundaries of the strip $x=\pm h$ being shear free and constrained in a way not to permit normal displacement.

Therefore , substituting $u(x,y,t) = u(x,y)e^{-i\omega t}$ and $v(x,y,t) = v(x,y)e^{-i\omega t}$ our problem reduces to the solution of the equations

$$c_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1+c_{12})\frac{\partial^2 v}{\partial x \partial y} + \frac{a^2\omega^2}{c_s^2}u = 0 \tag{1}$$

and

$$c_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1+c_{12})\frac{\partial^2 u}{\partial x \partial y} + \frac{a^2\omega^2}{c_s^2}v = 0 \tag{2}$$

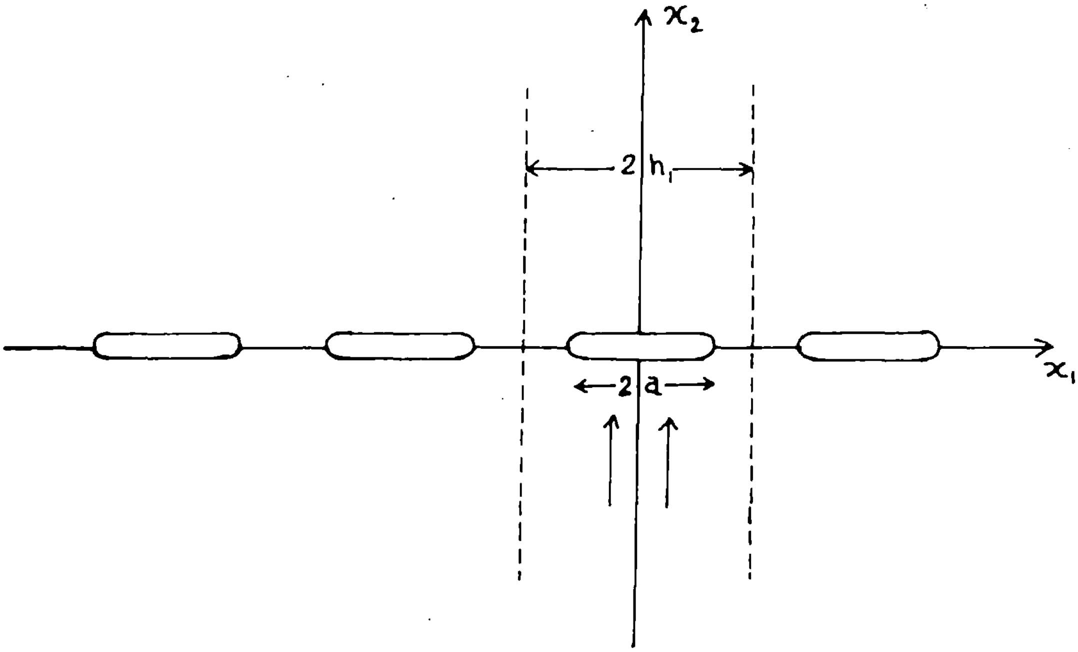

Fig. 1. Incidence of plane time-harmonic wave on a periodic array of cracks.

subject to the boundary conditions

$$\tau_{yy}(x,0) = -\tau_o e^{-i\omega t} \quad , \qquad |x|<1 \tag{3}$$

$$\tau_{xy}(x,0) = 0 \quad , \qquad |x|\le h \tag{4}$$

$$v(x,0) = 0 \quad , \qquad 1\le|x|\le h \tag{5}$$

$$\tau_{xy}(\pm h,y) = 0 \quad , \qquad |y|<\infty \tag{6}$$

$$u(\pm h,y) = 0 \quad , \qquad |y|<\infty \tag{7}$$

Henceforth the time factor $e^{-i\omega t}$ which is common to all field variables would be omitted in the sequel.

The nondimensional parameters c_{ij} arising in equations (1) and (2) are related to the elastic constants by the relations

$$\begin{aligned} c_{11} &= E_1/\mu_{12}(1-\nu_{12}^2E_2/E_1) \\ c_{22} &= E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1) = c_{11}E_2/E_1 \\ c_{12} &= \nu_{12}E_2/\mu_{12}(1-\nu_{12}^2E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11} \end{aligned} \tag{8}$$

for generalized plane stress, and by

$$\begin{aligned} c_{11} &= (E_1/\Delta\mu_{12})(1-\nu_{23}\nu_{32}) \\ c_{22} &= (E_2/\Delta\mu_{12})(1-\nu_{13}\nu_{31}) \\ c_{12} &= E_1(\nu_{21}+\nu_{13}\nu_{32}E_2/E_1)/\Delta\mu_{12} \\ &= E_2(\nu_{12}+\nu_{23}\nu_{31}E_1/E_2)/\Delta\mu_{12} \\ \Delta &= 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32} \end{aligned} \tag{9}$$

for plane strain. The constants E_i and ν_{ij} satisfy the Maxwell's relation $\nu_{ij}/E_i = \nu_{ji}/E_j$.

The stresses are related to the displacements by the equations

$$\tau_{xx}/\mu_{12} = c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y}$$

$$\tau_{yy}/\mu_{12} = c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y} \tag{10}$$

$$\tau_{xy}/\mu_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The solutions of equations (1) and (2) are taken as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left[A_1(\xi)e^{-\gamma_1 y} + A_2(\xi)e^{-\gamma_2 y}\right]\sin(\xi x)d\xi +$$

$$+ \frac{2}{\pi}\int_0^\infty \left[A_3(\zeta)\sinh(\gamma_3 x) + A_4(\zeta)\sinh(\gamma_4 x)\right]\cos(\zeta y)d\zeta$$

(11)

and $$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}\left[\alpha_1 A_1(\xi)e^{-\gamma_1 y} + \alpha_2 A_2(\xi)e^{-\gamma_2 y}\right]\cos(\xi x)d\xi +$$

$$+ \frac{2}{\pi}\int_0^\infty \frac{1}{\zeta}\left[\alpha_3 A_3(\zeta)\cosh(\gamma_3 x) + \alpha_4 A_4(\zeta)\cosh(\gamma_4 x)\right]\sin(\zeta y)d\zeta$$

where A_i (i=1-4) are the unknowns to be solved, γ_1^2 , γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\right\}\gamma^2 + (c_{11}\xi^2-k_s^2)(\xi^2-k_s^2) = 0 \tag{12}$$

and γ_3^2 , γ_4^2 are the roots of the quadratic

$$c_{11}\gamma^4 + \left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\zeta^2+(1+c_{11})k_s^2\right\}\gamma^2 + (c_{22}\zeta^2-k_s^2)(\zeta^2-k_s^2) = 0 \tag{13}$$

α_i(i=1-4) occuring in (11) are given by

$$\alpha_i = \frac{c_{11}\xi^2-k_s^2-\gamma_i^2}{(1+c_{12})\gamma_i} \qquad (i=1,2) \tag{14}$$

and

$$\alpha_i = \frac{\zeta^2-k_s^2-c_{11}\gamma_i^2}{(1+c_{12})\gamma_i} \qquad (i=3,4) \tag{15}$$

Therefore the stress components τ_{yy} and τ_{xy} are

$$\begin{aligned}\tau_{yy}/\mu_{12} = {} & \frac{2}{\pi}\int_0^\infty\left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)A_1(\xi)e^{-\gamma_1 y} + \left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)A_2(\xi)e^{-\gamma_2 y}\right]\times \\ & \times \cos(\xi x)d\xi \; + \\ & + \frac{2}{\pi}\int_0^\infty\left[(c_{12}\gamma_3+c_{22}\alpha_3)A_3(\zeta)\cosh(\gamma_3 x) + (c_{12}\gamma_4+c_{22}\alpha_4)A_4(\zeta) \times \right. \\ & \left. \times \cosh(\gamma_4 x)\right]\cos(\zeta y)d\zeta\end{aligned} \tag{16}$$

$$\begin{aligned}\tau_{xy}/\mu_{12} = {} & -\frac{2}{\pi}\int_0^\infty\left[(\gamma_1+\alpha_1)A_1(\xi)e^{-\gamma_1 y} + (\gamma_2+\alpha_2)A_2(\xi)e^{-\gamma_2 y}\right]\sin(\xi x)d\xi \; - \\ & - \frac{2}{\pi}\int_0^\infty\left[\left(\zeta - \frac{\alpha_3\gamma_3}{\zeta}\right)A_3(\zeta)\sinh(\gamma_3 x) + \left(\zeta - \frac{\alpha_4\gamma_4}{\zeta}\right)A_4(\zeta) \times \right. \\ & \left. \times \sinh(\gamma_4 x)\right]\sin(\zeta y)d\zeta\end{aligned} \tag{17}$$

3. DERIVATION OF INTEGRAL EQUATIONS

The boundary condition (4) implies

$$A_2(\xi) = -\beta A_1(\xi) \;, \quad \text{where} \quad \beta = \frac{\gamma_1+\alpha_1}{\gamma_2+\alpha_2} \tag{18}$$

From boundary conditions (3) and (5) , we obtain the following dual integral equations

$$\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right) - \beta\left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\right] A_1(\xi)\cos(\xi x)\,d\xi \;+$$

$$+\int_0^\infty \left[(c_{12}\gamma_3+c_{22}\alpha_3)A_3(\zeta)\cosh(\gamma_3 x) + (c_{12}\gamma_4+c_{22}\alpha_4)A_4(\zeta)\cosh(\gamma_4 x)\right]d\zeta$$

$$= -\frac{\pi\tau_0}{2\mu_{12}} \quad , \quad |x|<1 \tag{19}$$

and $$\int_0^\infty \frac{1}{\xi}(\alpha_1-\beta\alpha_2)A_1(\xi)\cos(\xi x)\,d\xi = 0 \quad , \quad 1\le|x|\le h \tag{20}$$

Assuming $$A(\xi) = \frac{\alpha_1-\beta\alpha_2}{\xi}\,A_1(\xi) = \frac{\beta\alpha_2-\alpha_1}{\beta\xi}\,A_2(\xi) \tag{21}$$

equations (19) and (20) can be rewritten as

$$\int_0^\infty \left[\frac{c_{12}\xi^2-c_{22}\alpha_1\gamma_1-\beta(c_{12}\xi^2-c_{22}\alpha_2\gamma_2)}{(\alpha_1-\beta\alpha_2)}\right]A(\xi)\cos(\xi x)\,d\xi \;+$$

$$+\int_0^\infty \left[(c_{12}\gamma_3+c_{22}\alpha_3)A_3(\zeta)\cosh(\gamma_3 x) + (c_{12}\gamma_4+c_{22}\alpha_4)A_4(\zeta)\cosh(\gamma_4 x)\right]d\zeta$$

$$= -\frac{\pi\tau_0}{2\mu_{12}} \quad , \quad |x|<1 \tag{22}$$

and $$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 \quad , \quad 1\le|x|\le h, \tag{23}$$

For large ξ the equation (12) takes the form

$$c_{22}\gamma^4 + (c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2\gamma^2 + c_{11}\xi^4 = 0 \tag{24}$$

Let the roots of the equation (23) be $\xi^2N_1^2$ and $\xi^2N_2^2$, where

$$N_1^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}+\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\} \tag{25}$$

$$N_2^2 = \frac{1}{2c_{22}}\left\{c_{11}c_{22}-c_{12}^2-2c_{12}-\left[(c_{12}^2+2c_{12}-c_{11}c_{22})^2-4c_{11}c_{22}\right]^{1/2}\right\} \tag{26}$$

Also for large ξ , $\alpha_i\,(i=1,2)$ and β given by (14) and (18) respectively become

$$\alpha_i = \xi\left[\frac{c_{11}-N_i^2}{(1+c_{12})N_i}\right] \quad , \quad (i=1,2) \tag{27}$$

and $$\beta = \frac{(c_{11}+c_{12}N_1^2)}{(c_{11}+c_{12}N_2^2)}\,\frac{N_2}{N_1} \tag{28}$$

Now, for large ξ

$$\frac{c_{12}\xi^2-c_{22}\alpha_1\gamma_1-\beta(c_{12}\xi^2-c_{22}\alpha_2\gamma_2)}{(\alpha_1-\beta\alpha_2)\xi}$$

$$= \frac{(c_{12}^2+c_{12}-c_{11}c_{22})(c_{12}N_1N_2-c_{11})-c_{22}\left[c_{12}N_1^2N_2^2+c_{11}(N_1^2+N_1N_2+N_2^2)\right]}{c_{11}(1+c_{12})(N_1+N_2)}$$

$$= \theta \text{ (say)} \tag{29}$$

Therefore we find that

$$H(\xi) = \frac{c_{12}\xi^2 - c_{22}\alpha_1\gamma_1 - \beta(c_{12}\xi^2 - c_{22}\alpha_2\gamma_2)}{(\alpha_1 - \beta\alpha_2)\xi\theta} - 1 \tag{30}$$

tends to zero as ξ tends to ∞.

Using (30) in equation (22) we finally obtain from (22) and (23) the following dual integral equations for the determination of the unknown function $A(\xi)$:

$$\int_0^\infty \xi\left[1+H(\xi)\right]A(\xi)\cos(\xi x)d\xi = p(x) \quad , \quad |x|<1 \tag{31}$$

$$\int_0^\infty A(\xi)\cos(\xi x)d\xi = 0 \quad , \quad 1\le|x|\le h \tag{32}$$

where $$p(x) = -\frac{\pi\tau_0}{2\mu_{12}\theta} - \frac{1}{\theta}\int_0^\infty\Big[(c_{12}\gamma_3 + c_{22}\alpha_3)A_3(\zeta)\cosh(\gamma_3 x) +$$

$$+(c_{12}\gamma_4 + c_{22}\alpha_4)A_4(\zeta)\cosh(\gamma_4 x)\Big]d\zeta \tag{33}$$

The boundary conditions (6) and (7) yields

$$\int_0^\infty \left[\left(\zeta - \frac{\alpha_3\gamma_3}{\zeta}\right)A_3(\zeta)\sinh(\gamma_3 h) + \left(\zeta - \frac{\alpha_4\gamma_4}{\zeta}\right)A_4(\zeta)\sinh(\gamma_4 h)\right]\sin(\zeta y)d\zeta$$

$$= \int_0^\infty \left[\frac{(\gamma_1+\alpha_1)\xi}{(\beta\alpha_2-\alpha_1)}e^{-\gamma_1 y} - \frac{(\gamma_2+\alpha_2)\xi\beta}{(\beta\alpha_2-\alpha_1)}e^{-\gamma_2 y}\right]A(\xi)\sin(\xi h)d\xi \tag{34}$$

and $$\int_0^\infty \left[A_3(\zeta)\sinh(\gamma_3 h) + A_4(\zeta)\sinh(\gamma_4 h)\right]\cos(\zeta y)d\zeta$$

$$= -\int_0^\infty \frac{\xi}{(\alpha_1-\beta\alpha_2)}\left[e^{-\gamma_1 y} - \beta e^{-\gamma_2 y}\right]A(\xi)\sin(\xi h)d\xi \tag{35}$$

Applying Fourier sine and cosine inverse transforms in equations (34) and (35) respectively, we get

$$\left(\zeta - \frac{\alpha_3\gamma_3}{\zeta}\right)A_3(\zeta)\sinh(\gamma_3 h) + \left(\zeta - \frac{\alpha_4\gamma_4}{\zeta}\right)A_4(\zeta)\sinh(\gamma_4 h) = \frac{2}{\pi}\int_0^\infty g_1(\xi,\zeta)A(\xi)d\xi \tag{36}$$

and $$A_3(\zeta)\sinh(\gamma_3 h) + A_4(\zeta)\sinh(\gamma_4 h) = \frac{2}{\pi}\int_0^\infty g_2(\xi,\zeta)A(\xi)d\xi \tag{37}$$

where $$g_1(\xi,\zeta) = \frac{\xi\zeta(\gamma_1+\alpha_1)}{\beta\alpha_2-\alpha_1}\left(\frac{1}{\zeta^2+\gamma_1^2} - \frac{1}{\zeta^2+\gamma_2^2}\right)\sin(\xi h) \tag{38}$$

$$g_2(\xi,\zeta) = \frac{\xi}{\beta\alpha_2-\alpha_1}\left(\frac{\gamma_1}{\zeta^2+\gamma_1^2} - \frac{\beta\gamma_2}{\zeta^2+\gamma_2^2}\right)\sin(\xi h) \tag{39}$$

Solving the equations (36) and (37), the unknown functions $A_3(\zeta)$ and $A_4(\zeta)$ can be found to be related to $A(\xi)$ as :

$$A_3(\zeta) = f_1(\zeta)\int_0^\infty g_1(\xi,\zeta)A(\xi)d\xi + f_2(\zeta)\int_0^\infty g_2(\xi,\zeta)A(\xi)d\xi$$

(40)

$$A_4(\zeta) = f_3(\zeta)\int_0^\infty g_1(\xi,\zeta)A(\xi)d\xi + f_4(\zeta)\int_0^\infty g_2(\xi,\zeta)A(\xi)d\xi$$

where $$f_1(\zeta) = \frac{2}{\pi D(\zeta)\sinh(\gamma_3 h)}$$

$$f_2(\zeta) = \frac{2}{\pi D(\zeta)\sinh(\gamma_3 h)}\left(\frac{\alpha_4\gamma_4}{\zeta} - \zeta\right)$$

$$f_3(\zeta) = \frac{-2}{\pi D(\zeta)\sinh(\gamma_4 h)}$$

$$f_4(\zeta) = \frac{-2}{\pi D(\zeta)\sinh(\gamma_4 h)}\left(\frac{\alpha_3\gamma_3}{\zeta} - \zeta \right)$$

$$D(\zeta) = \frac{\alpha_4\gamma_4 - \alpha_3\gamma_3}{\zeta} \tag{41}$$

Now, substituting the values of $\alpha_i (i=1,2)$ and β given by (14) and (18) and using the relations

$$\gamma_1^2+\gamma_2^2 = -\frac{1}{c_{22}}\left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\right\}$$

$$\gamma_1^2\gamma_2^2 = \frac{1}{c_{22}}(\xi^2-k_s^2)(c_{11}\xi^2-k_s^2) \tag{42}$$

$$(\zeta^2+\gamma_1^2)(\zeta^2+\gamma_2^2) = \frac{c_{11}}{c_{22}}(\xi^2+\gamma_3^2)(\xi^2+\gamma_4^2) \;,$$

$g_1(\xi,\zeta)$ and $g_2(\xi,\zeta)$ from (38) and (39) finally can be obtained as:

$$g_1(\xi,\zeta) = \xi\zeta\left[\frac{\gamma_3^2 q_1 - q_2}{\xi^2+\gamma_3^2} - \frac{\gamma_4^2 q_1 - q_2}{\xi^2+\gamma_4^2} \right]\frac{\sin(\xi h)}{\gamma_4^2-\gamma_3^2}$$

$$g_2(\xi,\zeta) = \xi\left[-\frac{\gamma_3^2 q_3 - q_4}{\xi^2+\gamma_3^2} + \frac{\gamma_4^2 q_3 - q_4}{\xi^2+\gamma_4^2} \right]\frac{\sin(\xi h)}{\gamma_4^2-\gamma_3^2} \tag{43}$$

where $$q_1 = \frac{c_{11}c_{22}-c_{12}^2}{c_{11}} \;,\quad q_2 = -\frac{k_s^2(c_{12}+c_{22})}{c_{11}}$$

$$q_3 = -\frac{c_{12}}{c_{11}} \;,\quad q_4 = \frac{c_{22}\zeta^2+c_{12}k_s^2}{c_{11}} \tag{44}$$

4. METHOD OF SOLUTION

In order to reduce the dual integral equations (31) and (32) to a single Fredholm integral equation, we assume

$$A(\xi) = -\frac{\pi\tau_o}{2\mu_{12}\theta}\int_o^\infty \eta\phi(\eta)J_o(\xi\eta)d\eta \tag{45}$$

so that equation (32) is automatically satisfied.

Next substituting the value of $A(\xi)$ from (45) in $A_3(\zeta)$ and $A_4(\zeta)$ given by (40) and using the result (Gradshteyn and Ryzhik, 1965)

$$\int_o^\infty \frac{\xi\sin(\xi h)J_o(\xi t)}{\xi^2+\gamma^2}d\xi = \frac{\pi}{2}e^{-h\gamma}\,I_o(t\gamma)$$

$A_3(\zeta)$ and $A_4(\zeta)$ can be written in terms of $\phi(t)$ as

$$A_3(\zeta) = -\frac{\pi^2\tau_o}{4\mu_{12}\theta(\gamma_4^2-\gamma_3^2)}\int_o^1\Big[\zeta f_1(\zeta)\Big\{(\gamma_3^2q_1-q_2)I_o(\gamma_3t)e^{-\gamma_3h}-(\gamma_4^2q_1-q_2)\times$$

$$\times I_o(\gamma_4t)e^{-\gamma_4h}\Big\} + f_2(\zeta)\Big\{-(\gamma_3^2q_3-q_4)I_o(\gamma_3t)e^{-\gamma_3h}+(\gamma_4^2q_3-q_4)\times$$

$$\times I_o(\gamma_4t)e^{-\gamma_4h}\Big\}\Big]t\phi(t)dt \tag{46}$$

and

$$A_4(\zeta) = -\frac{\pi^2\tau_o}{4\mu_{12}\theta(\gamma_4^2-\gamma_3^2)}\int_o^1\Big[\zeta f_3(\zeta)\Big\{(\gamma_3^2q_1-q_2)I_o(\gamma_3t)e^{-\gamma_3h}-(\gamma_4^2q_1-q_2)\times$$

$$\times I_o(\gamma_4t)e^{-\gamma_4h}\Big\} + f_4(\zeta)\Big\{-(\gamma_3^2q_3-q_4)I_o(\gamma_3t)e^{-\gamma_3h}+(\gamma_4^2q_3-q_4)\times$$

$$\times I_o(\gamma_4t)e^{-\gamma_4h}\Big\}\Big]t\phi(t)dt \tag{47}$$

Substitution of the values of $A(\xi)$, $A_3\ (\zeta)$ and $A_4\ (\zeta)$ in terms of $\phi(t)$ from equations (45), (46) and (47) respectively in (31) yields the following Fredholm integral equation of second kind for the determination of $\phi(\eta)$:

$$\phi(\eta) + \int_0^1 t\left\{K_1(\eta,t) + K_2(\eta,t)\right\} \phi(t)dt = 1 \tag{48}$$

The kernals $K_1(\eta,t)$ and $K_2(\eta,t)$ are given by

$$K_1(\eta,t) = \int_0^\infty \xi H(\xi) J_0(\xi t) J_0(\xi\eta) d\xi \tag{49}$$

and

$$K_2(\eta,t) = \int_0^\infty \Big\{S_1(\zeta) I_0(\gamma_3\eta) I_0(\gamma_3 t) + S_2(\zeta) I_0(\gamma_3\eta) I_0(\gamma_4 t) + $$
$$+ S_3(\zeta) I_0(\gamma_4\eta) I_0(\gamma_3 t) + S_4(\zeta) I_0(\gamma_4\eta) I_0(\gamma_4 t)\Big\} d\zeta \tag{50}$$

where

$$S_1(\zeta) = \frac{\pi(c_{12}\gamma_3 + c_{22}\alpha_3)}{2(\gamma_4^2-\gamma_3^2)\theta} \left\{\zeta f_1(\zeta)(\gamma_3^2 q_1 - q_2) - f_2(\zeta)(\gamma_3^2 q_3 - q_4)\right\} e^{-\gamma_3 h}$$

$$S_2(\zeta) = -\frac{\pi(c_{12}\gamma_3 + c_{22}\alpha_3)}{2(\gamma_4^2-\gamma_3^2)\theta} \left\{\zeta f_1(\zeta)(\gamma_4^2 q_1 - q_2) - f_2(\zeta)(\gamma_4^2 q_3 - q_4)\right\} e^{-\gamma_4 h}$$

(51)

$$S_3(\zeta) = \frac{\pi(c_{12}\gamma_4 + c_{22}\alpha_4)}{2(\gamma_4^2-\gamma_3^2)\theta} \left\{\zeta f_3(\zeta)(\gamma_3^2 q_1 - q_2) - f_4(\zeta)(\gamma_3^2 q_3 - q_4)\right\} e^{-\gamma_3 h}$$

$$S_4(\zeta) = -\frac{\pi(c_{12}\gamma_4 + c_{22}\alpha_4)}{2(\gamma_4^2-\gamma_3^2)\theta} \left\{\zeta f_3(\zeta)(\gamma_4^2 q_1 - q_2) - f_4(\zeta)(\gamma_4^2 q_3 - q_4)\right\} e^{-\gamma_4 h}$$

It is to be noted that the kernal $K_1(\eta,t)$ represented by the semi-infinite integral given by equation (49) has a slow rate of convergence. In order to make the numerical analysis easier, the semi-infinite integral has therefore been converted to finite integrals, details of which have been given in the appendix.

5. QUANTITIES OF PHYSICAL INTEREST

The normal stress $\tau_{yy}(x,y)$ in the plane y=0 in the neighbourhood of the crack can be found from equation (2.16) and is given by

$$\tau_{yy}(x,0) = \frac{2\mu_{12}\theta}{\pi}\int_0^\infty \xi A(\xi)\cos(\xi x)d\xi + o(1) \quad , \quad x>1$$

$$= -\tau_o \frac{d}{dx}\int_0^1 \frac{\eta\phi(\eta)d\eta}{(x^2-\eta^2)^{1/2}} + o(1) \quad , \quad x>1$$

$$= -\tau_o \frac{d}{dx}\left[-(x^2-1)^{1/2}\phi(1) + x\phi(0) + \int_0^1 \phi'(\eta)(x^2-\eta^2)^{1/2}d\eta\right] + $$
$$+ o(1) \quad , \quad x>1$$

$$= \frac{\tau_o x\phi(1)}{(x^2-1)^{1/2}} + o(1) \quad , \quad x>1 \tag{52}$$

Defining stress intensity factor by

$$K = \underset{x\to 1+}{\mathrm{Lt}}\left|\frac{\tau_{yy}(x,0)(x-1)^{1/2}}{\tau_o}\right|$$

it is found that

$$K = \left|\frac{\phi(1)}{\sqrt{2}}\right| \tag{53}$$

Next the crack opening displacement $\Delta v(x,0)=v(x,0+)-v(x,0-)$ on the surface of the crack has been obtained with the help of the equation (45) in the form

$$\Delta v(x,0) = \frac{4}{\pi}\int_0^\infty A(\xi)\cos(\xi x)d\xi$$

$$= -\frac{2\tau_0}{\mu_{12}\theta}\int_x^1 \frac{\eta\phi(\eta)}{(\eta^2-x^2)^{1/2}}\,d\eta \quad , \quad |x|<1 \tag{54}$$

6. NUMERICAL CALCULATIONS AND DISCUSSIONS

The method of Fox and Goodwin (1953) has been used to solve the integral equation (48) numerically for different values of dimensionless frequency k_s and h, the separating distance of the cracks. The integral in (48) has been represented by a quadrature formula involving values of the desired function ϕ at pivotal points in the range of integration which leads to a set of algebraic linear simultaneous equations. The solution of the set of linear algebraic equations gives a first approximation to the required pivotal values of ϕ which has been improved by the use of difference-correction technique. The kernal $K_1(\eta,t)$ has been transformed into two finite integrals and have been evaluated by using Gauss-quadrature integration formula. The second kernal $K_2(\eta,t)$ has been evaluated by simpson's method. After solving the integral equation (48) for different values of engineering elastic constants of several orthotropic materials listed in table

1, the stress intensity factor K given by (53) has been plotted against k_s for different values of h (Fig.2-Fig.4) and the crack opening displacement $\Delta v(x,0)\mu_{12}/\tau_0$ given in (54) has been plotted against dimensionless distance x ($0\le x\le 1$) (Fig.5-Fig.7).

The nature of crack opening displacements which have been plotted for three different orthotropic materials for different values of the separating distance h is found to show two maxima within the range ($0\le x\le 1$) one near x=0 and the other near x=1, the maximum value near x=0 being the greatest.

From fig.2-fig.4 it may be noted that the stress intensity factor increases with the increase in the values of frequency k_s, attains maximum value and then decreases for all orthotropic materials for $0<k_s\le 1$. The stress intensity factor is found to decrease with the increase in the values h, the separating distance of the cracks and also decreases sharply after the maximum value is attained.

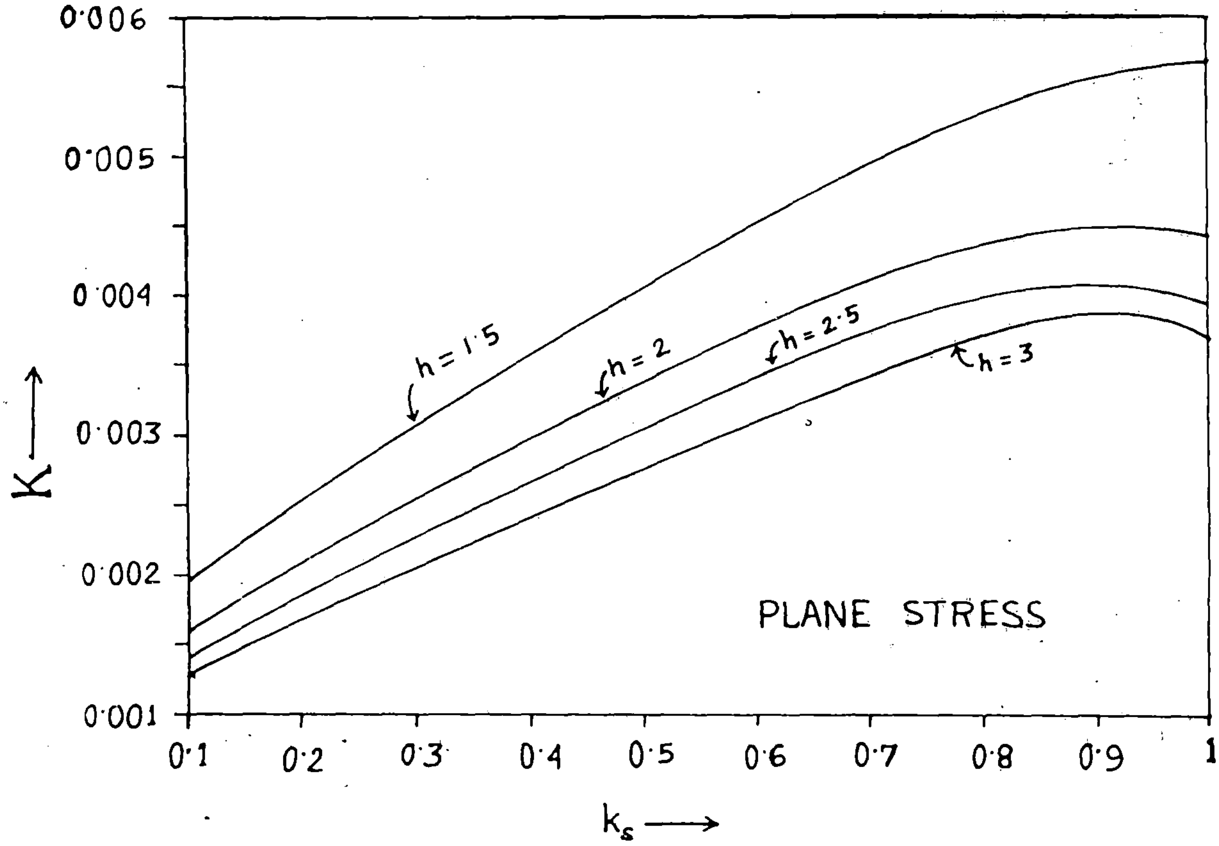

Fig. 2. Dynamic stress intensity factor K vs. dimensionless frequency k_s for Boron-Expoxi composite.

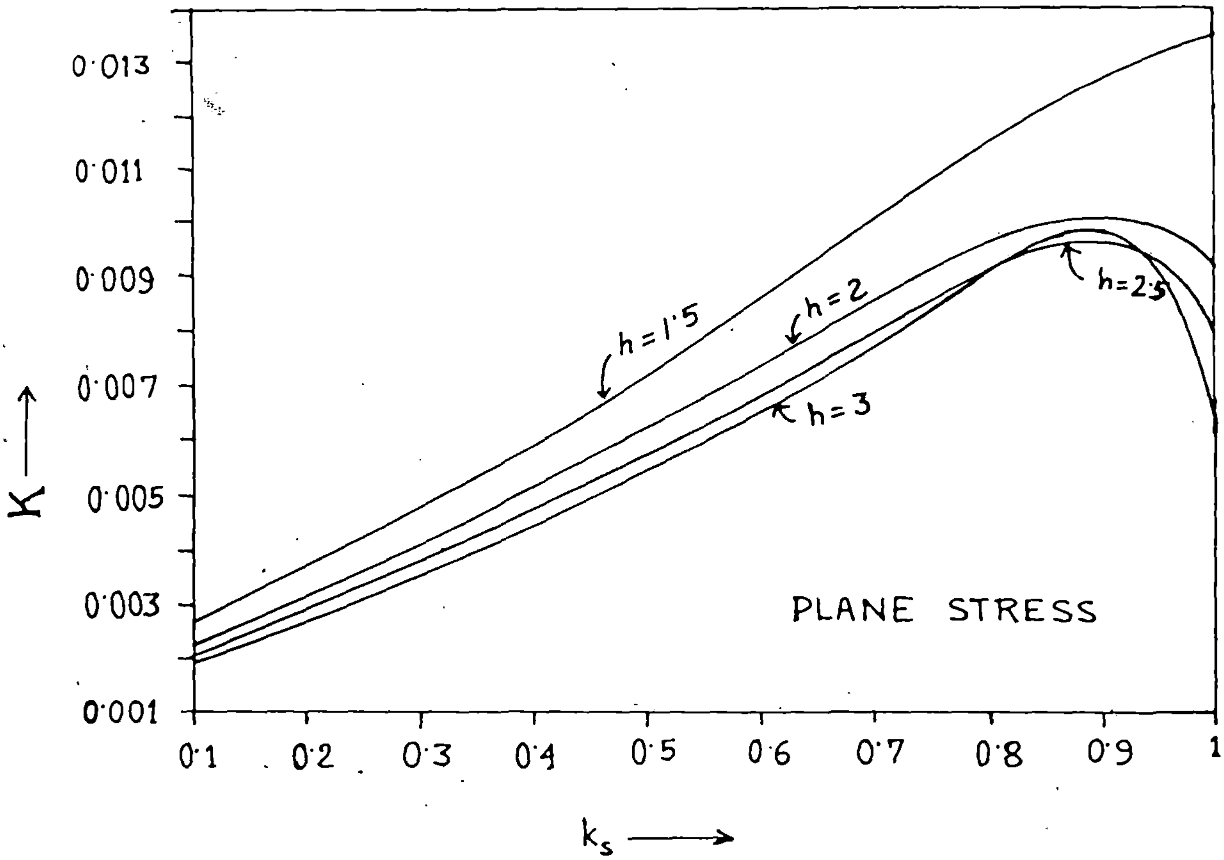

Fig. 3. Dynamic stress intensity factor K vs. dimensionless frequency k_s for Steel-Mylar composite.

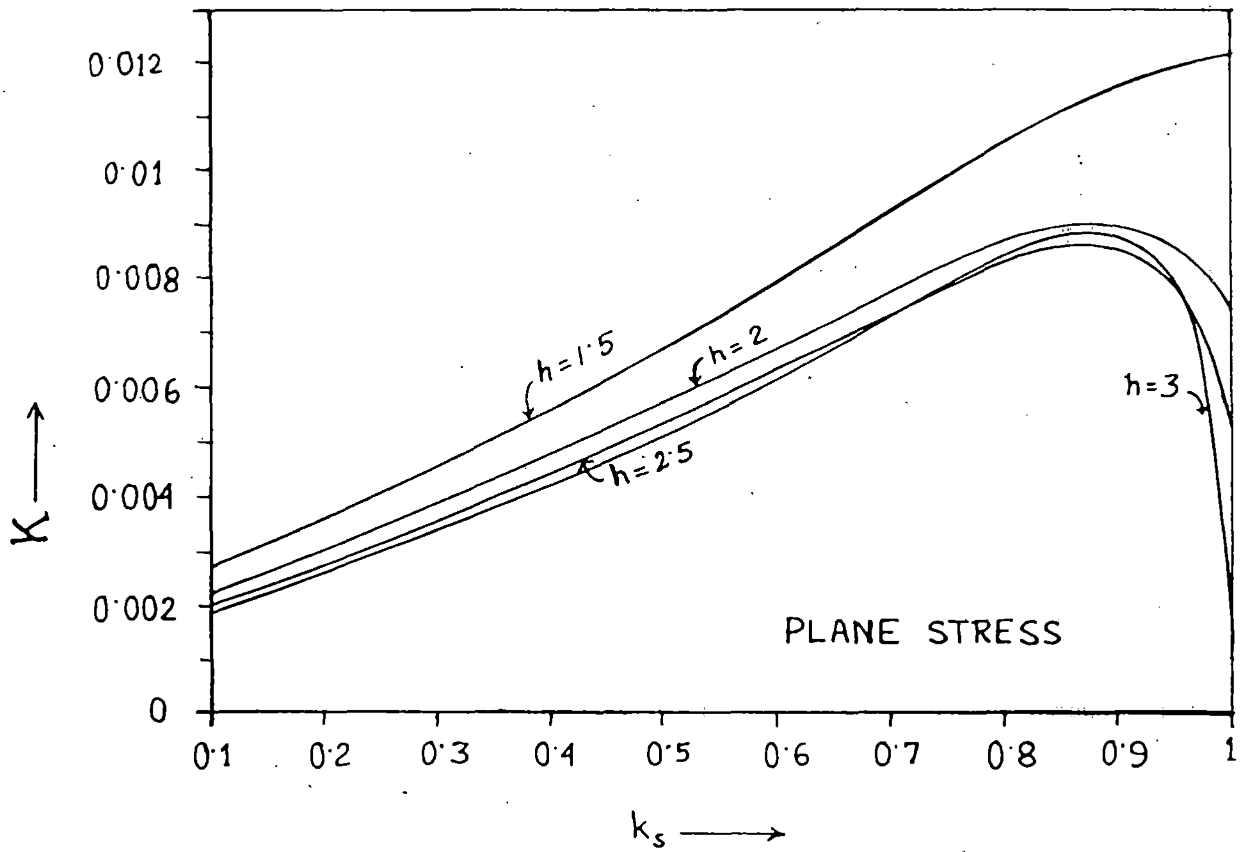

Fig. 4. Dynamic stress intensity factor K vs. dimensionless frequency k_s for Graphite-Epoxi composite.

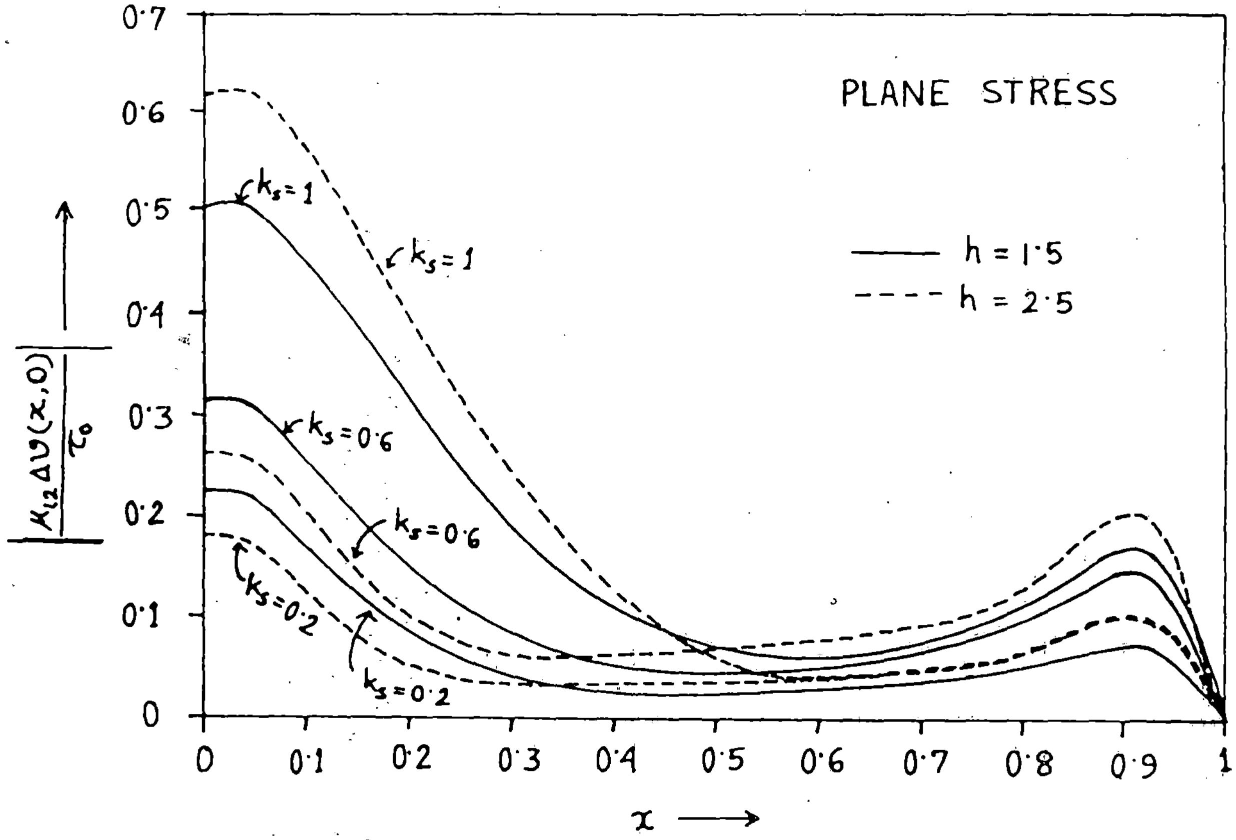

Fig. 5. Crack opening displacement vs. dimensionless distance for Graphite-Epoxi composite.

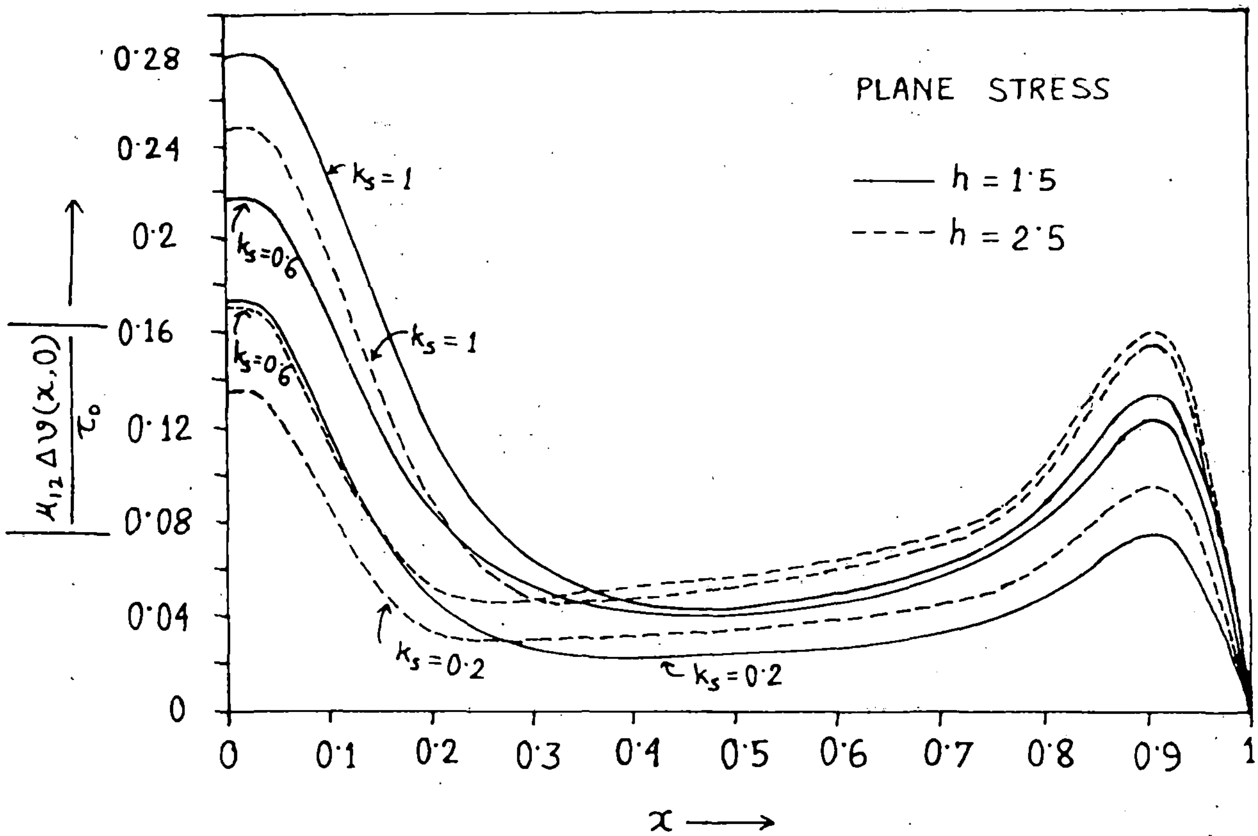

Fig. 6. Crack opening displacement vs. dimensionless distance for Boron-Epoxi composite.

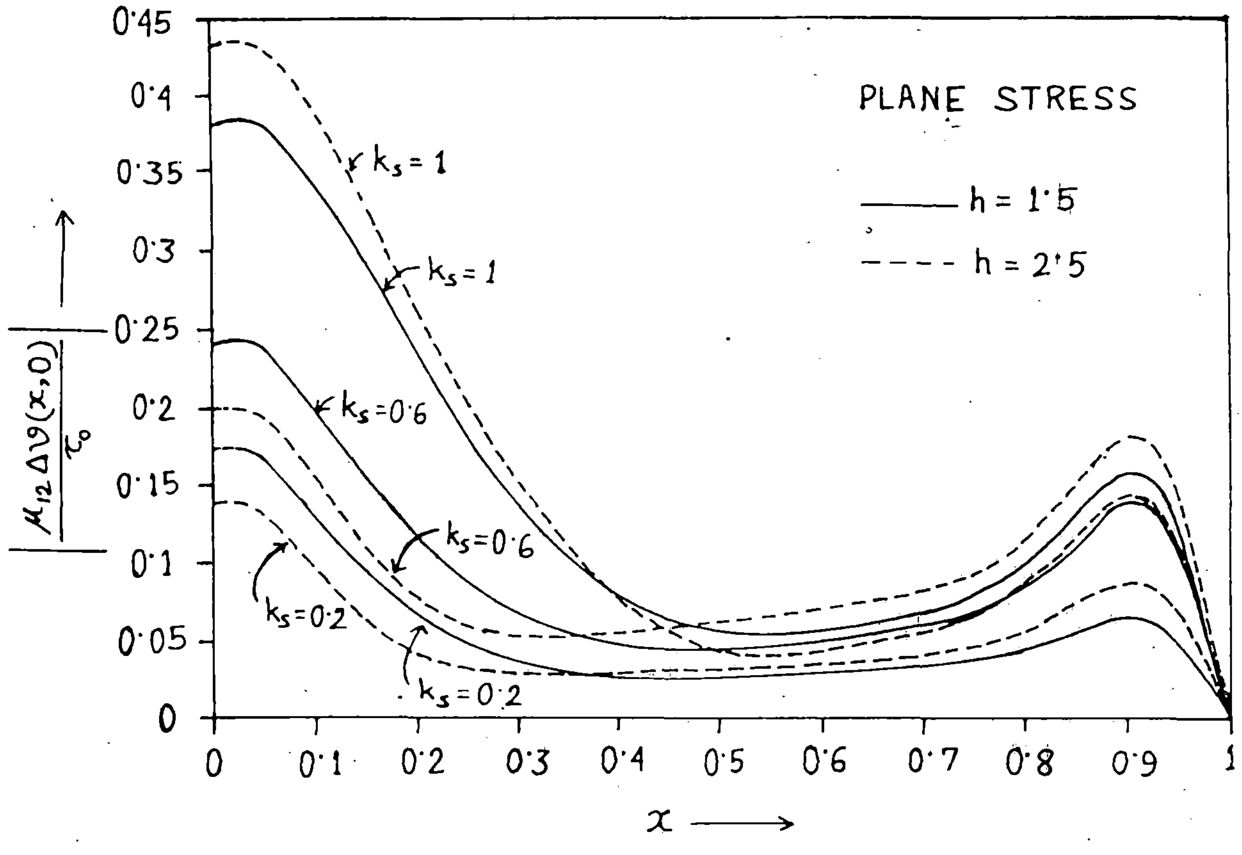

Fig. 7. Crack opening displacement vs. dimensionless distance for Steel-Mylar composite .

APPENDIX

EVALUATION OF THE KERNAL $K_1(\eta, t)$:

The kernal $K_1(\eta,t)$ given by equation (49) is

$$K_1(\eta,t) = \int_0^\infty \xi H(\xi) J_o(\xi t) J_o(\xi\eta) d\xi \tag{A1}$$

where $H(\xi)$ is given in equation (30)

To evaluate the integral (A1) we consider two contour integrals :

$$\begin{aligned} I_1 &= \int_{C_1} M(\xi,\gamma_1,\gamma_2) J_o(\xi t) H_o^{(1)}(\eta\xi) d\xi \ , \qquad \eta>t \\ I_2 &= \int_{C_2} M(\xi,\gamma_1,\gamma_2) J_o(\xi t) H_o^{(2)}(\eta\xi) d\xi \ , \qquad \eta>t \end{aligned} \tag{A2}$$

where

$$M(\xi,\gamma_1,\gamma_2) = \frac{c_{11}\xi^2-\alpha_1\gamma_1 c_{22} - \beta(c_{12}\xi^2-\alpha_2\gamma_2 c_{22})}{(\alpha_1-\beta\alpha_2)\theta} - \xi$$

$$\gamma_1 = \left[\frac{1}{2}\left\{-B_1+(B_1^2-4B_2)^{1/2}\right\}\right]^{1/2}$$

$$\gamma_2 = \left[\frac{1}{2}\left\{-B_1-(B_1^2-4B_2)^{1/2}\right\}\right]^{1/2}$$

$$B_1 = \frac{1}{c_{22}}\left\{(c_{12}^2+2c_{12}-c_{11}c_{22})\xi^2+(1+c_{22})k_s^2\right\}$$

$$B_2 = \frac{1}{c_{22}}(\xi^2-k_s^2)(c_{11}\xi^2-k_s^2)$$

and C_1, C_2 are the closed contours defined in Fig.8.

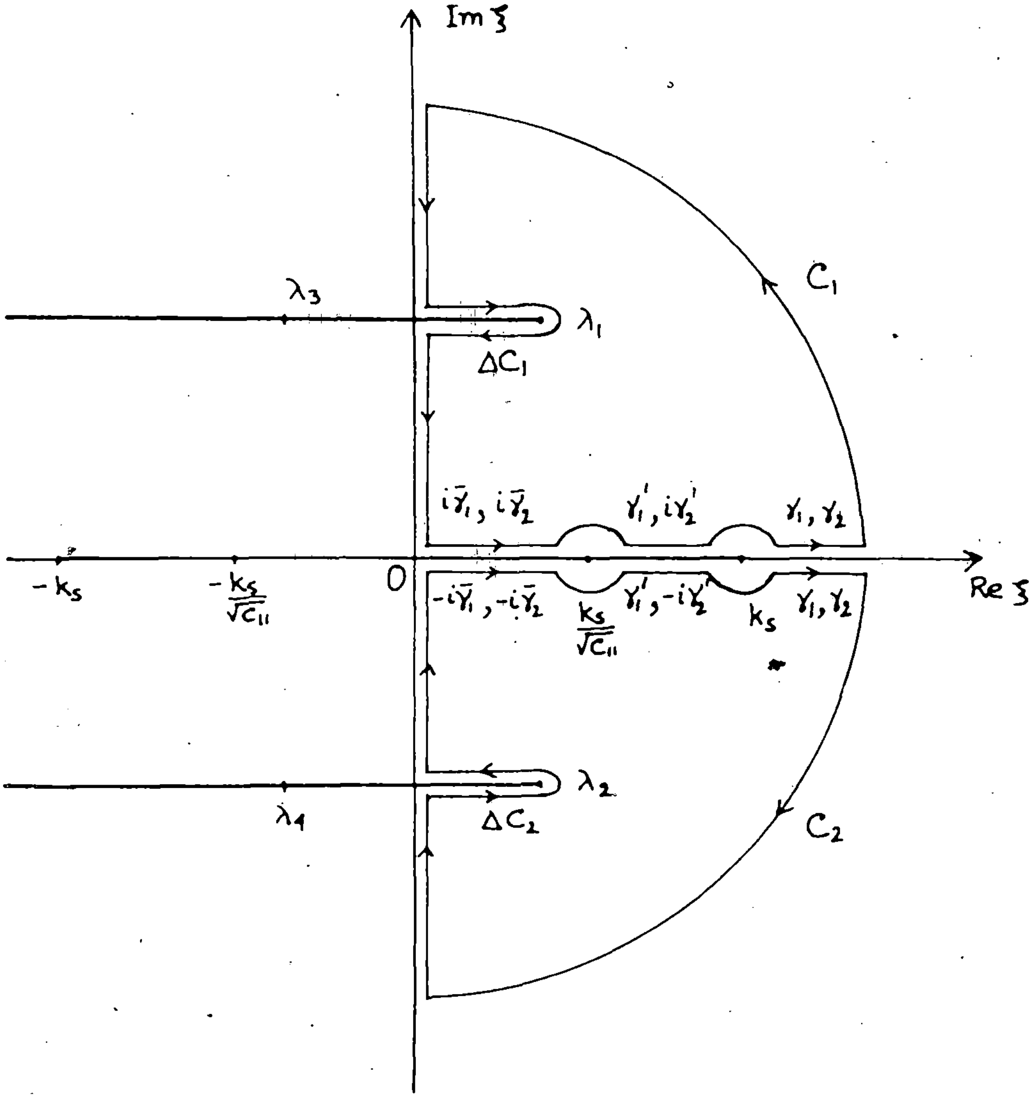

Fig. 8. Contours of integration for integral in equation (A1).

TABLE - 1

ENGINEERING ELASTIC CONSTANTS

Materials	E_1 (psi)	E_2 (psi)	μ_{12} (psi)	ν_{12}
Boron-Epoxi composite	32.5×10^6	1.84×10^6	0.642×10^6	0.256
Graphite-Epoxi composite	2.22×10^6	22.93×10^6	0.8×10^6	0.033
Steel-Mylar composite	26.28×10^6	4.1×10^6	0.9×10^6	0.44

Assuming the relation

$$\left\{\frac{(c_{12}^2+2c_{12}-c_{11}c_{22})(1+c_{22})}{c_{22}^2}+\frac{2(1+c_{11})}{c_{22}}\right\}^2-\left\{\frac{(c_{12}+2c_{12}-c_{11}c_{22})^2}{c_{22}^2}-\frac{4c_{11}}{c_{22}}\right\}\times$$

$$\times\left\{\frac{(1+c_{22})^2}{c_{22}^2}-\frac{4}{c_{22}}\right\}<0 \qquad \text{(A3)}$$

it is noted that the branch points $\xi=\lambda_i\,(i=1-4)$ corresponding to the roots of the equation $B_1^2-4B_2=0$ are always complex.

Now the branch points corresponding to the roots of the equations

$$-B_1+(B_1^2-4B_2)^{1/2}=0 \quad \text{and} \quad -B_1-(B_1^2-4B_2)^{1/2}=0$$

are $\xi=\pm k_s$ and $\xi=\pm k_s/\sqrt{c_{22}}$ respectively, where it is assumed that

$$c_{11}c_{22}-c_{12}^2-2c_{12} > 1+c_{22}$$

$$\text{(A4)}$$

and $$c_{12}^2+2c_{12}+c_{11} > 0$$

Most of the orthotropic materials satisfy the relations (A3) and (A4). Therefore under the above condition, $\xi=\pm k_s/\sqrt{c_{22}}$ and $\xi=\pm k_s$ are the branch points of γ_1 and γ_2 respectively.

The integrals in equation (A2) are found to be zero on the contours ΔC_1 and ΔC_2(fig.8) around the branch cuts from λ_1 and λ_2. Thus integrating the integrals I_1 and I_2 along the contours C_1 and C_2 (fig.8) we obtain for contour C_1 :

$$\int_0^{k_s/\sqrt{c_{11}}} M(\xi, i\bar{\gamma}_1, i\bar{\gamma}_2) J_o(\xi t) H_o^{(1)}(\eta\xi) d\xi \; +$$

$$+ \int_{k_s/\sqrt{c_{11}}}^{k_s} M(\xi, \gamma_1', i\gamma_2') J_o(\xi t) H_o^{(1)}(\eta\xi) d\xi \; +$$

$$+ \int_{k_s}^{\infty} M(\xi, \gamma_1, \gamma_2) J_o(\xi t) H_o^{(1)}(\eta\xi) d\xi \; +$$

$$+ i\int_{\infty}^{0} M(iy, i\bar{\gamma}_1', i\bar{\gamma}_2') J_o(iyt) H_o^{(1)}(iy\eta) dy = 0 \;, \quad \eta > t \qquad \text{(A5)}$$

and for the contour C_2 :

$$\int_0^{k_s/\sqrt{c_{11}}} M(\xi, -i\bar{\gamma}_1, -i\bar{\gamma}_2) J_o(\xi t) H_o^{(2)}(\eta\xi) d\xi \; +$$

$$+ \int_{k_s/\sqrt{c_{11}}}^{k_s} M(\xi, \gamma_1', -i\gamma_2') J_o(\xi t) H_o^{(2)}(\eta\xi) d\xi \; +$$

$$+ \int_{k_s}^{\infty} M(\xi, \gamma_1, \gamma_2) J_o(\xi t) H_o^{(2)}(\eta\xi) d\xi \; -$$

$$- i\int_{\infty}^{0} M(-iy, -i\bar{\gamma}_1', -i\bar{\gamma}_2') J_o(-iyt) H_o^{(2)}(-iy\eta) dy = 0 \;, \quad \eta > t \qquad \text{(A6)}$$

where

$$\bar{\gamma}_1 = \left[\frac{1}{2}\left\{B_1 - (B_1^2 - 4\bar{B}_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2 = \left[\frac{1}{2}\left\{B_1 + (B_1^2 - 4\bar{B}_2)^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_1' = \left[\frac{1}{2}\left\{\bar{B}_1' - (\bar{B}_1'^2 - 4\bar{B}_2')^{1/2}\right\}\right]^{1/2}$$

$$\bar{\gamma}_2' = \left[\frac{1}{2}\left\{\bar{B}_1' + (\bar{B}_1'^2 - 4\bar{B}_2')^{1/2}\right\}\right]^{1/2}$$

$$\gamma_1' = \left[\frac{1}{2}\left\{-B_1 + (B_1^2 + 4B_2')^{1/2}\right\}\right]^{1/2}$$

$$\gamma_2' = \left[\frac{1}{2}\left\{B_1 + (B_1^2 + 4B_2')^{1/2}\right\}\right]^{1/2}$$

$$\bar{B}_2 = \frac{c_{11}}{c_{22}}\left(k_s^2 - \xi^2\right)\left(\frac{k_s^2}{c_{11}} - \xi^2\right)$$

$$\bar{B}_1' = \frac{1}{c_{22}}\left\{-(c_{12}^2 + 2c_{12} - c_{11}c_{22})y^2 + (1 + c_{22})k_s^2\right\}$$

$$\bar{B}_2' = \frac{c_{11}}{c_{22}}\left(y^2 + k_s^2\right)\left(y^2 + \frac{k_s^2}{c_{11}}\right)$$

$$B_2' = \frac{c_{11}}{c_{22}}\left(k_s^2 - \xi^2\right)\left(\xi^2 - \frac{k_s^2}{c_{11}}\right)$$

Using the relations

$$J_o(-iyt) = J_o(iyt) \quad \text{and} \quad H_o^{(2)}(-iy\eta) = -H_o^{(1)}(iy\eta)$$

(A6) can be written as

$$2\int_0^\infty M(\xi,\gamma_1,\gamma_2)\; J_o(\xi t)J_o(\xi\eta)d\xi \; -$$

$$-\int_0^{k_s/\sqrt{c_{11}}} M(\xi,-i\bar{\gamma}_1,-i\bar{\gamma}_2)J_o(\xi t)H_o^{(1)}(\eta\xi)d\xi \; -$$

$$-\int_{k_s/\sqrt{c_{11}}}^{k_s} M(\xi,\gamma_1',-i\gamma_2')J_o(\xi t)H_o^{(1)}(\eta\xi)d\xi \; -$$

$$-\int_{k_s}^\infty M(\xi,\gamma_1,\gamma_2)J_o(\xi t)H_o^{(1)}(\eta\xi)d\xi \; +$$

$$+\; i\int_\infty^0 M(-iy,-i\bar{\gamma}_1',-i\bar{\gamma}_2')J_o(iyt)H_o^{(1)}(iy\eta)dy = 0 \;,\; \eta>t \qquad \text{(A7)}$$

Adding (A5) and (A7) and using

$$M(-iy,-i\bar{\gamma}_1',-i\bar{\gamma}_2') = -M(iy,i\bar{\gamma}_1',i\bar{\gamma}_2')$$

the kernal $K_1(\eta,t)$ for $\eta>t$ can be finally written as

$$K_1(\eta,t) = -i\left[\int_0^{k_s/\sqrt{c_{11}}} \left(\overline{X}_1\overline{\gamma}_1-\overline{X}_2\overline{\gamma}_2\right) J_0(\xi t)H_0^{(1)}(\xi\eta)\,d\xi\; + \right.$$

$$\left. + \int_{k_s/\sqrt{c_{11}}}^{k_s} \left(X_2'\gamma_2'\right) J_0(\xi t)H_0^{(1)}(\xi\eta)\,d\xi \right], \quad \eta>t \tag{A8}$$

where

$$\overline{X}_1 = \frac{\left[c_{12}(1+c_{12})\xi^2-c_{22}(c_{11}\xi^2-k_s^2+\overline{\gamma}_1^2)\right](c_{11}\xi^2-k_s^2-c_{12}\overline{\gamma}_2^2)}{(c_{11}\xi^2-k_s^2)(1+c_{12})(\overline{\gamma}_1^2-\overline{\gamma}_2^2)\theta} \tag{A9}$$

$$\overline{X}_2 = \frac{\left[c_{12}(1+c_{12})\xi^2-c_{22}(c_{11}\xi^2-k_s^2+\overline{\gamma}_2^2)\right](c_{11}\xi^2-k_s^2-c_{12}\overline{\gamma}_1^2)}{(c_{11}\xi^2-k_s^2)(1+c_{12})(\overline{\gamma}_1^2-\overline{\gamma}_2^2)\theta} \tag{A10}$$

$$X_2' = \frac{\left[c_{12}(1+c_{12})\xi^2-c_{22}(c_{11}\xi^2-k_s^2+\gamma_2'^2)\right](c_{11}\xi^2-k_s^2+c_{12}\gamma_1'^2)}{(c_{11}\xi^2-k_s^2)(1+c_{12})(\gamma_1'^2+\gamma_2'^2)\theta} \tag{A11}$$

The corresponding expression of $K_1(\eta,t)$ for $\eta<t$ follows from (A8) by interchanging η and t.

DIFFRACTION OF SH-WAVES BY A GRIFFITH CRACK IN NONHOMOGENEOUS ELASTIC STRIP

1. INTRODUCTION

The natural or artificial materials are usually inhomogeneous; so in recent years a great attention has been given in the study of diffraction of elastic waves by cracks or obstacles in inhomogeneous medium in view of their application in fracture mechanics. Many problems have been solved involving one or more cracks in an infinite homogeneous elastic medium. Loeber and Sih (1968) and Mal (1970) have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of finite crack at the interface of two elastic half spaces has been discussed by Srivastava et al (1980) and Bostrom (1987). Singh et al (1977, 1980) considered the problem of scattering of SH-wave by cracks or strips in nonhomogeneous infinite elastic medium. Papers involving crack or strip in infinitely long elastic strip are very few. The problem of an infinite elastic strip containing an arbitrary number of unequal size Griffith cracks, located parallel to its surfaces and opened by an arbitrary internal pressure has been treated by Adams (1980). Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by Chen (1978) for impact load and by Srivastava et al (1981) for normally incident waves. Recently Shindo et al (1986) considered the problem of impact response of a finite crack in an orthotropic strip.

In our paper, the diffraction of normally incident SH-waves by a Griffith crack situated in an infinitely long inhomogeneous elastic strip has been discussed. The shear modulus(μ) and the density(ρ) of the material have been assumed to vary both in horizontal and vertical directions. Applying Fourier transform the mixed boundary value problem has been converted to the solution of dual integral equations. The dual integral equations finally has been reduced to a Fredholm integral equation of second kind by applying Abel transform. Expressions for stress intensity factor and crack opening displacement have been derived. The numerical values of stress intensity factor and crack opening displacement have been depicted by means of graphs to show the effect of material inhomogeneity.

2. FORMULATION OF THE PROBLEM

Consider the problem of diffraction of SH-waves by a Griffith crack in an inhomogeneous elastic strip of width $2h_1$. The crack is located in the region $-d \le x_1 \le d$, $-\infty < y_1 < \infty$, $z_1 = 0$ (fig.1). Normalizing all the lengths with respect to d and putting $x_1/d = x$, $y_1/d = y$, $z_1/d = z$, $h_1/d = h$ it is found that the location of the crack is $-1 \le x \le 1$, $-\infty < y < \infty$, $z = 0$ referred to a cartesion co-ordinate system (x,y,z). Let a plane harmonic SH-wave originating at $z = -\infty$ impinge on the crack normally to the x-axis. The variation of the shear modulus μ and the density ρ is taken in both the vertical(z) and

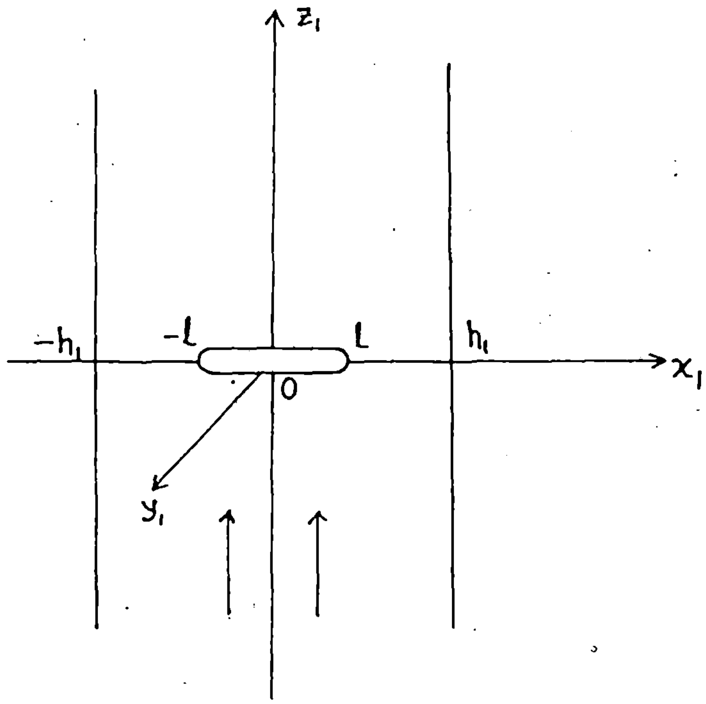

Fig. 1 Crack in the inhomogeneous strip .

horizontal(x) directions in such a manner that the shear velocity $(\mu/\rho)^{1/2}$ is constant.

The only non-vanishing y-component of the displacement which is independent of y is $v=v(x,z,t)$.

The equation of motion is given by

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) = \rho\frac{\partial^2 v}{\partial t^2} \tag{1}$$

If we consider $v(x,z,t)$ in the form

$$v(x,z,t) = \frac{V(x,y,t)}{\sqrt{\mu(x,z)}} \tag{2}$$

equation (1) takes the form

$$\mu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2}\right) + \frac{1}{2}\left[\frac{1}{2\mu}\left\{\left(\frac{\partial\mu}{\partial x}\right)^2 + \left(\frac{\partial\mu}{\partial z}\right)^2\right\} - \frac{\partial^2\mu}{\partial z^2}\right]V = \rho\frac{\partial^2 V}{\partial t^2} \tag{3}$$

Putting $V(x,z,t) = F(x)G(z)e^{-i\omega t}$ and $\mu(x,z)= \mu_o f(x)g(z)$, $\rho(x,z)= \rho_o f(x)g(z)$ in equation (3) where μ_o, ρ_o are constants, such that $(\mu_o/\rho_o)^{1/2}=c_2$ is the shear wave velocity, it is found that F(x) and G(z) satisfy the following equations

$$\frac{\partial^2 F}{\partial x^2} + n^2 F = 0 \tag{4}$$

$$\frac{\partial^2 G}{\partial z^2} + \left(\frac{d^2\omega^2}{c_2^2} - a^2-b^2-n^2\right)G = 0 \tag{5}$$

provided f(x) and g(z) are of the form

$$- \frac{1}{4}\left(\frac{\partial f}{\partial x} / f\right)^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} / f\right) = a^2 \tag{6}$$

$$- \frac{1}{4}\left(\frac{\partial g}{\partial z} / g\right)^2 + \frac{1}{2}\left(\frac{\partial^2 g}{\partial z^2} / g\right) = b^2 \tag{7}$$

where n, a and b are constants.

Equations (6) and (7) are of similar type. We rewrite the equation (6) as

$$\frac{d}{df}\left[\left(\frac{\partial f}{\partial x} \right)^2 / f \right] = 4a^2 \tag{8}$$

Integrating the above equation we obtain

$$\frac{df}{dx} = \pm\, 2(a^2 f^2 + a_1 f)^{1/2} \tag{9}$$

where a_1 is a constant.

Again integrating (9) it can be shown

$$x + a_2 = \pm \frac{1}{2} \int \frac{df}{(a^2 f^2 + a_1 f)^{1/2}} \tag{10}$$

where a_2 is another constant.

Now, we consider the following three possible cases :

Case - 1 : $a_1 = 0$.

In this case equation (10) takes the form

$$x + a_2 = \pm \frac{1}{2} \int \frac{df}{af}$$

which on integration yields

$$f(x) = C_1 e^{\pm 2ax} \tag{11}$$

where C_1 is a constant.

Case - 2 : $a = 0$.

Putting a=0 in equation (10) we get

$$x + a_2 = \pm \frac{1}{2} \int \frac{df}{\sqrt{(a_1 f)}}$$

from which we get

$$f(x) = C_2 \left(1 + \frac{x}{a_2} \right)^2 \tag{12}$$

where $C_2 = a_1 a_2^2$ is a constant.

Case - 3 : $a \neq 0$, $a_1 \neq 0$.

In this case (10) yields

$$f(x) = C_3 \cosh^2(ax + a_3) \tag{13}$$

where $C_3 = - a_1/a^2$, $a_3 = aa_2$.

Equation (13) gives the general solution of equation (8) when both the constants a and a_1 are non zero.

In view of the above results, we assume the forms of f(x) and g(z) as

$$f(x) = \cosh^2(ax) \quad \text{and} \quad g(z) = \cosh^2(bz) \tag{14}$$

so that equations (6) and (7) are automatically satisfied.

Now the shear modulus $\mu(x,z)$ and density of the medium $\rho(x,z)$ are

$$\mu=\mu_o\cosh^2(ax)\cosh^2(bz) \quad , \quad \rho=\rho_o\cosh^2(ax)\cosh^2(bz) \tag{15}$$

The displacement component $v^{(i)}(x,z,t)$ and stress $\tau^{(i)}(x,z,t)$ due to incident waves are given by

$$v^{(i)}(x,z,t) = \frac{A_o e^{i(k_2 z-\omega t)}}{\sqrt{\mu_o}\,\cosh(ax)\cosh(bz)} \tag{16}$$

and $\tau_{yz}^{(i)}(x,z,t) = A_o\sqrt{\mu_o}\,\cosh(ax)\left[ik\cosh(bz)-b\sinh(bz)\right]e^{i(k_2 z-\omega t)}$

(17)

where A_o is a constant and $k_2=\omega d/c_2$.
Henceforth the time factor $e^{-i\omega t}$ will be suppressed in the sequel.
Using (8) and (2), equation (1) takes the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial z^2} + k^2 V = 0 \quad , \qquad k^2=(k_2^2-a^2-b^2) \tag{18}$$

whose solution is

$$v(x,z) = \int_0^\infty B_1(\xi)e^{-\beta z}\cos(\xi x)d\xi + \int_0^\infty C_1(\zeta)\cosh(\alpha x)\sin(\zeta z)d\zeta \tag{19}$$

where $\alpha = (\zeta^2-k^2)^{1/2}$, $\zeta>k$, $\beta = (\xi^2-k^2)^{1/2}$, $\xi>k$

$=-i(k^2-\zeta^2)^{1/2}$, $\zeta<k$, $=-i(k^2-\xi^2)^{1/2}$, $\xi<k$

Now displacement $v(x,z)$ and stresses $\tau_{yz}(x,z)$, $\tau_{xy}(x,z)$ due to scattered field are

$$v(x,z)=\frac{1}{\cosh(ax)\cosh(bz)}\left[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x\,d\xi + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z\,d\zeta\right] \tag{20}$$

$$\tau_{yz}(x,z) = -\mu_o b\cosh(ax)\sinh(bz)\left[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x\,d\xi + \right.$$

$$\left. + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z\,d\zeta\right] + \mu_o\cosh(ax)\cosh(bz) \times$$

$$\times \left[-\int_0^\infty \beta B(\xi)e^{-\beta z}\cos\xi x\,d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)\cos\zeta z\,d\zeta\right] \tag{21}$$

$$\tau_{xy}(x,z) = -\mu_o a\sinh(ax)\cosh(bz)\left[\int_0^\infty B(\xi)e^{-\beta z}\cos\xi x\,d\xi + \right.$$

$$\left. + \int_0^\infty C(\zeta)\cosh(\alpha x)\sin\zeta z\,d\zeta\right] + \mu_o\cosh(ax)\cosh(bz) \times$$

$$\times \left[-\int_0^\infty \xi B(\xi)e^{-\beta z}\sin\xi x\,d\xi + \int_0^\infty \alpha C(\zeta)\sinh(\alpha x)\sin\zeta z\,d\zeta\right] \tag{22}$$

where $B(\xi) = \frac{1}{\sqrt{\mu_o}} B_1(\xi)$, $C(\zeta) = \frac{1}{\sqrt{\mu_o}} C_1(\zeta)$.

The boundary conditions are

$$\tau_{yz}(x,0) = -\tau_o\cosh(ax) \quad , \quad |x|\le 1 \tag{23}$$

$$v(x,0) = 0 \quad , \qquad 1 \le |x| \le h \tag{24}$$

and
$$\tau_{xy}(\pm h, z) = 0 \quad , \qquad |z| < \infty \tag{25}$$

where $\tau_o = ik_2\sqrt{\mu_o}$.

From the boundary condition (25) $C(\zeta)$ is found to be expressible in terms of $B(\xi)$ as follows:

$$C(\zeta) = \frac{2\zeta}{\pi F_2(\alpha, h, a)} \int_o^\infty \frac{F_1(\xi, h, a) B(\xi)}{\beta^2 + \zeta^2} \, d\xi \tag{26}$$

where
$$F_1(\xi, h, a) = a \sinh(ah) \cos(\xi h) + \xi \cosh(ah) \sin(\xi h) \tag{27}$$

and
$$F_2(\alpha, h, a) = \alpha \cosh(ah) \sinh(\alpha h) - a \sinh(ah) \cosh(\alpha h) \tag{28}$$

Next, the use of (26) in the boundary condition (23) and (24) yields the following dual integral equations from which the unknown function $B(\xi)$ is to be determined:

$$\int_o^\infty \xi \left[1 + M(\xi)\right] B(\xi) \cos(\xi x) d\xi = p(x) \quad , \qquad |x| \le 1 \tag{29}$$

and
$$\int_o^\infty B(\xi) \cos(\xi x) d\xi = 0 \quad , \qquad 1 \le |x| \le h \tag{30}$$

where
$$M(\xi) = \left[\frac{\beta}{\xi} - 1 \right] \tag{31}$$

$$p(x) = \frac{\tau_o}{\mu_o} + \frac{2}{\pi} \int_o^\infty \frac{\zeta^2 \cosh(\alpha x)}{F_2(\alpha, h, a)} \, d\zeta \int_o^\infty \frac{F_1(\xi, h, a) B(\xi)}{\beta^2 + \zeta^2} \, d\xi \tag{32}$$

3. METHOD OF SOLUTION

In order to solve the dual integral equations (29) and (30), $B(\xi)$ is taken in the form

$$B(\xi) = \frac{\tau_o}{\mu_o} \int_o^1 t\phi(t) J_o(\xi t)\, dt \tag{33}$$

so that equation (30) is automatically satisfied.

Substitution of the value of $B(\xi)$ from (33) in (29), yields a Fredholm integral equation of second kind

$$\phi(t) + \int_o^1 u\left[L_1(u,t) + L_2(u,t)\right]\phi(u)\, du = 1 \tag{34}$$

where $$L_1(u,t) = \int_o^\infty \xi M(\xi) J_o(\xi u) J_o(\xi t)\, d\xi \tag{35}$$

$$L_2(u,t) = -\frac{2}{\pi}\int_o^\infty \frac{I_o(\alpha t)\zeta^2 K(\zeta,u)}{F_2(\alpha,h,a)}\, d\zeta \tag{36}$$

$$K(\zeta,u) = \int_o^\infty \frac{F_1(\xi,h,a) J_o(\xi u)}{\beta^2+\zeta^2}\, d\xi \tag{37}$$

Using the results (Gradshteyn et al, 1965)

$$\int_o^\infty \frac{\xi \sin(\xi h) J_o(\xi u)}{\xi^2+\alpha^2}\, d\xi = \frac{\pi}{2}\, e^{-\alpha h}\, I_o(u\alpha)$$

$$\int_o^\infty \frac{\cos(\xi h) J_o(\xi u)}{\xi^2+\alpha^2}\, d\xi = \frac{\pi}{2\alpha}\, e^{-\alpha h}\, I_o(u\alpha)$$

$$\int_0^\infty \frac{\xi \sin(\xi h) J_o(\xi u)}{\xi^2-\alpha_1^2}\, d\xi = \frac{\pi}{2} \cos(\alpha_1 h) J_o(u\alpha_1)$$

and
$$\int_0^\infty \frac{\cos(\xi h) J_o(\xi u)}{\xi^2-\alpha_1^2}\, d\xi = -\frac{\pi}{2\alpha_1} \sin(\alpha_1 h) J_o(\alpha_1 u)$$

$K(\zeta,u)$ takes the form

$$K(\zeta,u) = \frac{\pi}{2}\left[\frac{a\sinh(ah)}{\alpha} + \cosh(ah)\right] e^{-\alpha h}\, I_o(u\alpha) \quad , \quad \zeta>k$$

$$= \frac{\pi}{2}\left[\cosh(ah)\cos(\alpha_1 h) - \frac{a\sinh(ah)\sin(\alpha_1 h)}{\alpha_1}\right] J_o(u\alpha_1) \quad , \quad \zeta<k \tag{38}$$

where $\alpha_1=(k^2-\zeta^2)^{1/2}$.

Using contour integration technique (Srivastava et al, 1980) , the infinite integral arising in the kernel $L_1(u,t)$ can be converted to finite integral and is given by

$$L_1(u,t) = -ik^2\int_0^1 (1-\eta^2)^{1/2} J_o(k\eta t) H_o^{(1)}(k\eta u)\, d\eta \quad , \quad u>t$$

$$= -ik^2\int_0^1 (1-\eta^2)^{1/2} J_o(k\eta u) H_o^{(1)}(k\eta t)\, d\eta \quad , \quad u<t \tag{39}$$

Now $$L_2(u,t) = -\frac{2}{\pi}\left[\int_0^k + \int_k^\infty\right] \frac{I_o(\alpha t)\zeta^2 K(\zeta,u)}{F_2(\alpha,h,a)}\, d\zeta \tag{40}$$

where

$$F_2(\alpha,h,a) = \alpha\cosh(ah)\sinh(\alpha h) - a\sinh(ah)\cosh(\alpha h) \quad , \qquad \zeta>k$$

$$= -\left[\alpha_1\cosh(ah)\sin(\alpha_1 h) + a\sinh(ah)\cos(\alpha_1 h)\right] \quad , \qquad \zeta<k$$

Using the value of $K(\zeta,u)$ from (38) and putting $\zeta^2=k^2(1-y^2)$; $\zeta^2=k^2(1+y^2)$ in first and second integrals of (40) respectively, it is found that

$$L_2(u,t) = k^2\left[\int_0^1 \frac{(1-y^2)^{1/2}[ky\cosh(ah)\cos(kyh)- a\sinh(ah)\sin(kyh)]}{ky\cosh(ah)\sin(kyh) + a\sinh(ah)\cos(kyh)} \times \right.$$

$$\times J_0(kyu)J_0(kyt)dy -$$

$$-\int_0^\infty \frac{(1+y^2)^{1/2}[a\sinh(ah) + ky\cosh(ah)]e^{-kyh}}{ky\cosh(ah)\sinh(kyh)-a\sinh(ah)\cosh(kyh)} \times$$

$$\left. \times I_0(kyu)I_0(kyt)dy \right] \tag{41}$$

4. STRESS INTENSITY FACTOR AND CRACK OPENING DISPLACEMENT

From (15) the stress τ_{yz} on the plane z=0 can be written as

$$\tau_{yz}(x,0) = \mu_0\cosh(ax)\left[-\int_0^\infty \beta B(\xi)\cos\xi x d\xi + \int_0^\infty \zeta C(\zeta)\cosh(\alpha x)d\zeta\right]$$

Substituting the value of $C(\zeta)$ and $B(\xi)$ from (20) and (27) the expression for the stress finally can be deduced as

$$\tau_{yz}(x,0) = \frac{\tau_o x\cosh(ax)}{(x^2-1)^{1/2}}\,\phi(1) + o(1) \quad , \quad |x|>1.$$

Defining the stress intensity factor T by

$$T = \underset{x\to 1+}{\mathrm{Lt}} \left| \frac{(x-1)^{1/2}\tau_{yz}(x,0)}{\tau_o} \right|$$

we obtain,

$$T = \frac{1}{\sqrt{2}}\cosh(a)\,|\phi(1)| \tag{42}$$

Now the crack opening displacement $\Delta v(x,0)=v(x,0+)-v(x,0-)$ can be obtained from (20) as

$$\Delta v(x,0) = \frac{2}{\cosh(ax)}\int_0^\infty B(\xi)\cos(\xi x)d\xi \quad , \quad |x|\le 1$$

which on substitution of the value of $B(\xi)$ from (33) takes the form

$$\Delta v(x,0) = \frac{2\tau_o}{\mu_o\cosh(ax)}\left[(1-x^2)^{1/2}\phi(1) - \int_x^1 (t^2-x^2)^{1/2}\phi'(t)dt\right]$$

$$, \quad |x|\le 1 \tag{43}$$

5. NUMERICAL RESULTS AND DISCUSSION

Using the method of Fox and Goodwin (1953) the Fredholm integral equation given by equation (34) has been solved numerically for different values of material inhomogeneity parameters. In this method the integral in (34) at first has been represented by a quadrature formula involving values of the desired function $\phi(t)$ at pivotal points inside the specified range of integration and then converted to a set of linear algebraic simultaneous equations, solving which the first approximation to the required pivotal values of $\phi(t)$ has been obtained. Applying difference-correction technique the first approximations has been improved. The kernel $L_1(u,t)$ given by (39) and the first integral of the kernel $L_2(u,t)$ given by (41) have been evaluated numerically by using Gauss quadrature integration formula. The second infinite integral of the kernal $L_2(u,t)$ has been evaluated by Simpson's method. After solving the integral equation (34) numerically, the stress intensity factor T and the crack opening displacement $2\mu_0 v(x,0)/\tau_0$ have been calculated numerically and plotted separately against dimensionless frequency k_2 $(0<k_2\le 1.5)$ and dimensionless distance x $(0\le x\le 1)$ respectively for different values of material inhomogeneity parameters a, b and strip width 2h.

In fig.2, the effect of the width of the strip on the stress intensity factor when the material is assumed to be homogeneous has been shown where as the effect of inhomogeneity of the material on the stress intensity factor for different width of the strip has been depicted in fig.3-fig.5.

It is found that the effect of the strip width increases prominently with the increase of the frequency where as the inhomogeneity parameters a and b have no remarkable effect on the stress intensity factor for a fixed value of the strip width 2h.

In fig.6-fig.9 the crack opening displacement against dimensionless distance x for different values of material inhomogeneity parameters a, b and the strip width 2h have been illustrated by means of graphs. In each case the maximum value of the crack opening displacement is found to occur at x=0.08 and then after a few oscillations it gradually decreases to zero at x=1. It is also to be noted that the magnitude of the crack opening displacements increases with the increase in the value of the dimensionless frequency k_2 ; the increase being more prominent in the case of the presence of lateral inhomogeneity.

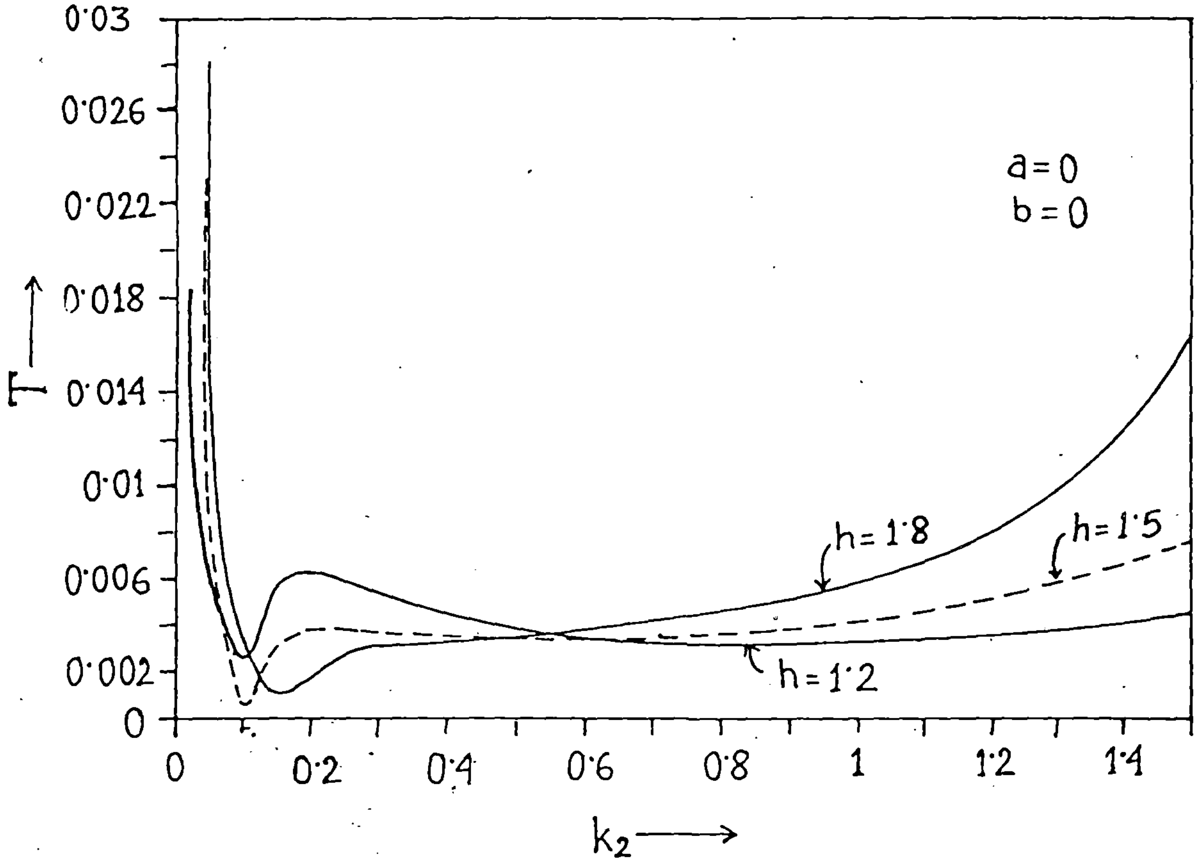

Fig. 2 Stress intensity factor T vs. dimensionless frequency k_2 for homogeneous medium (a=0, b=0).

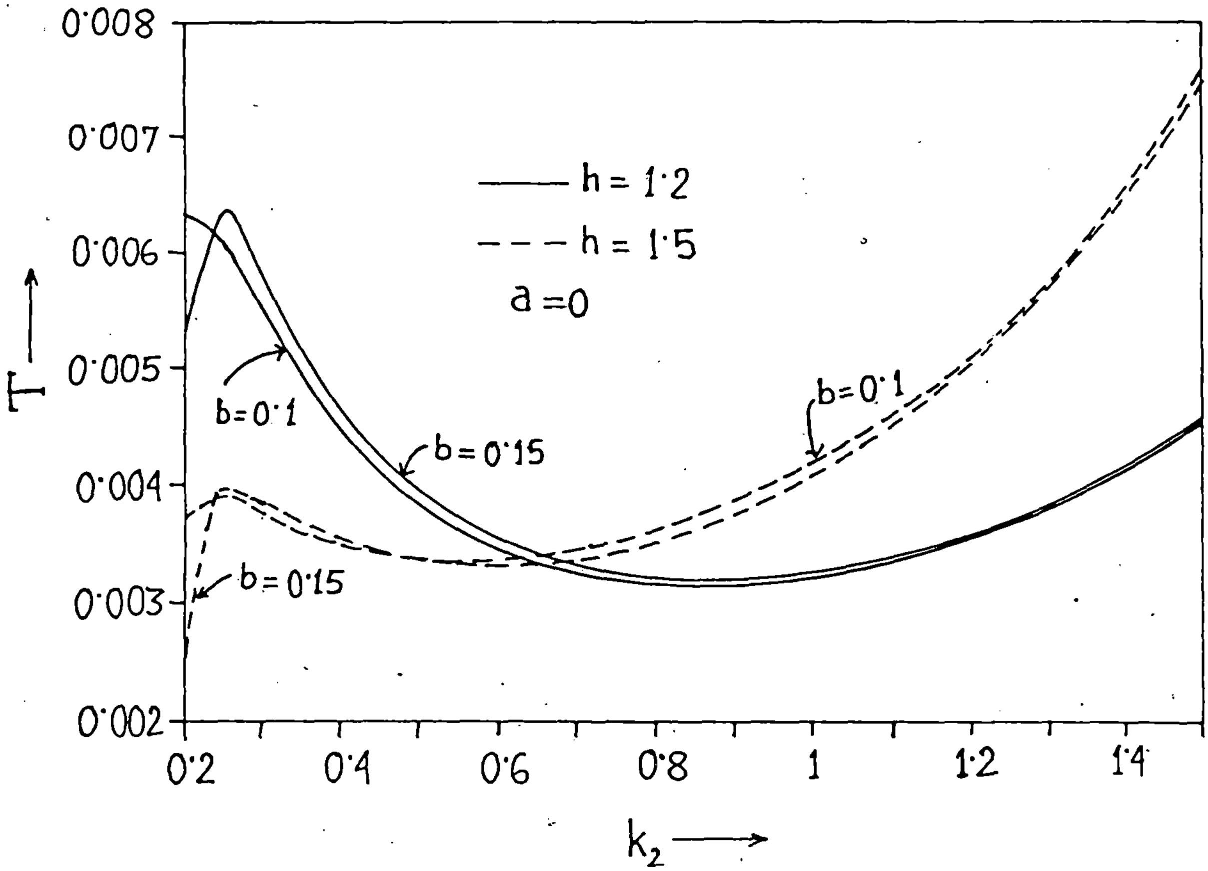

Fig. 3 Stress intensity factor T vs. dimensionless frequency k_2 .

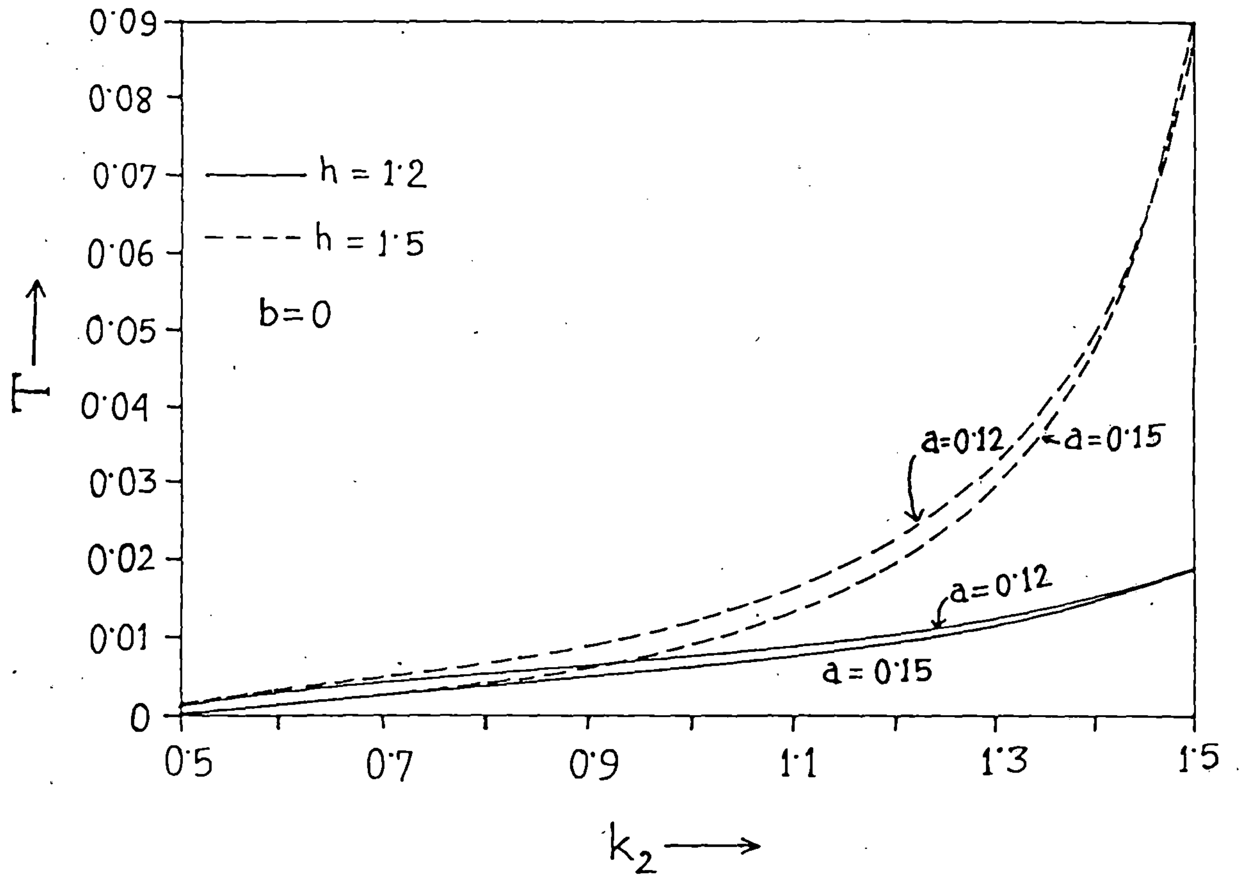

Fig. 4 Stress intensity factor T vs. dimensionless frequency k_2 .

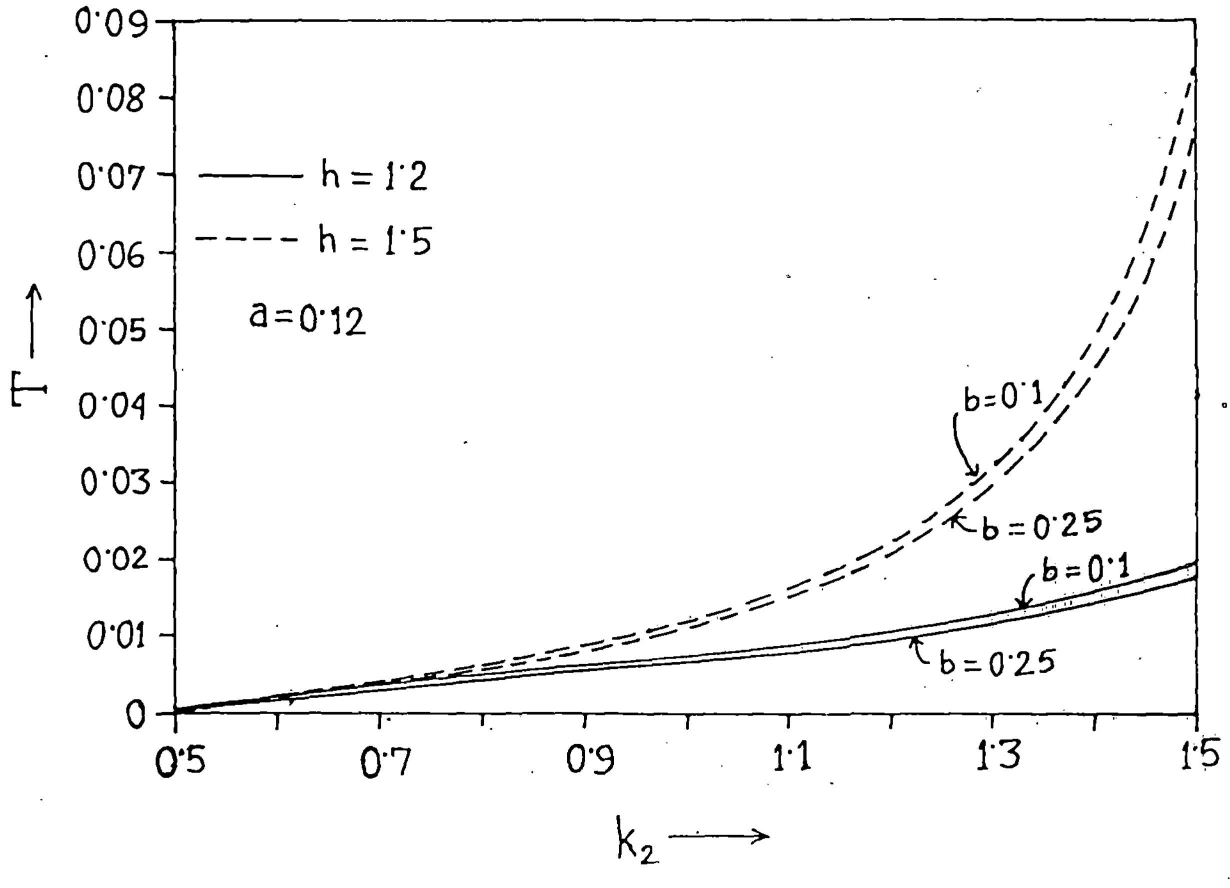

Fig. 5. Stress intensity factor T vs. dimensionless frequency k_2.

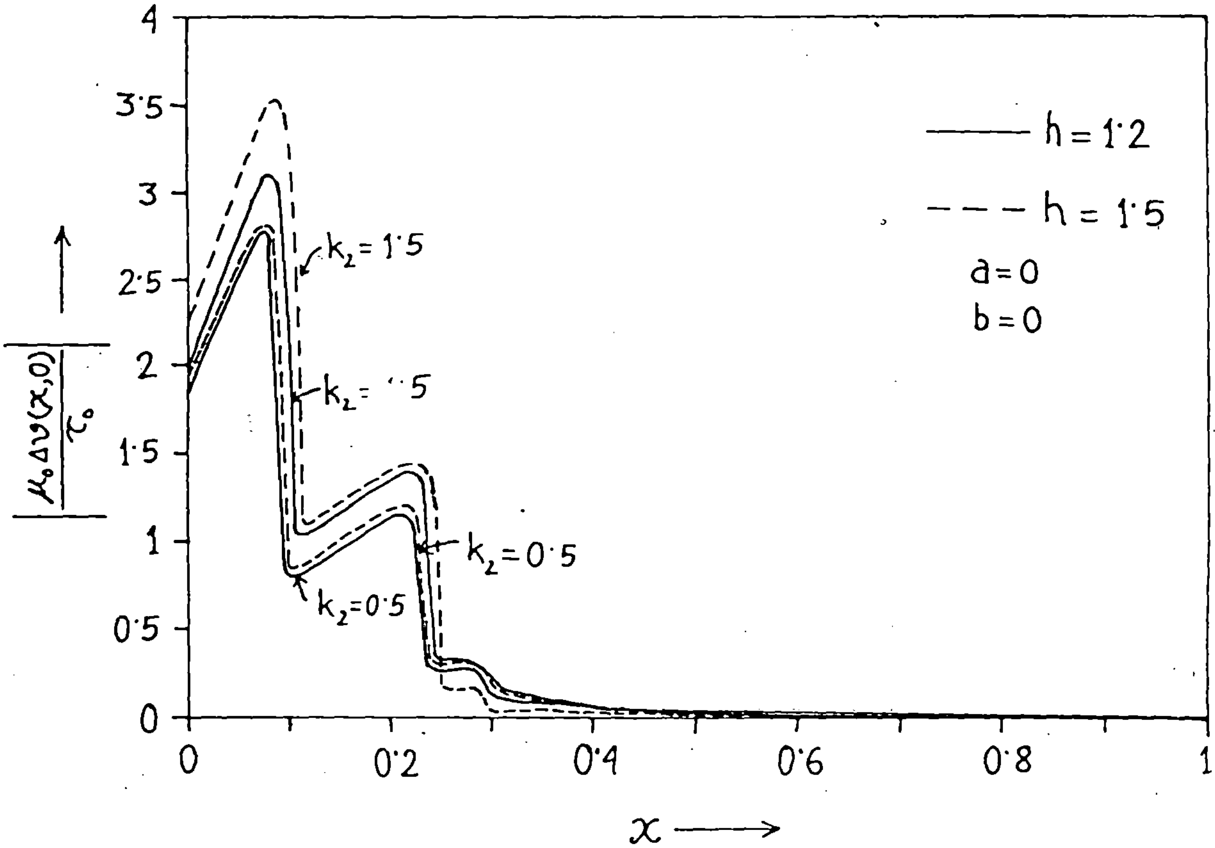

Fig. 6. Crack opening displacement vs. dimensionless distance x (a=0, b=0).

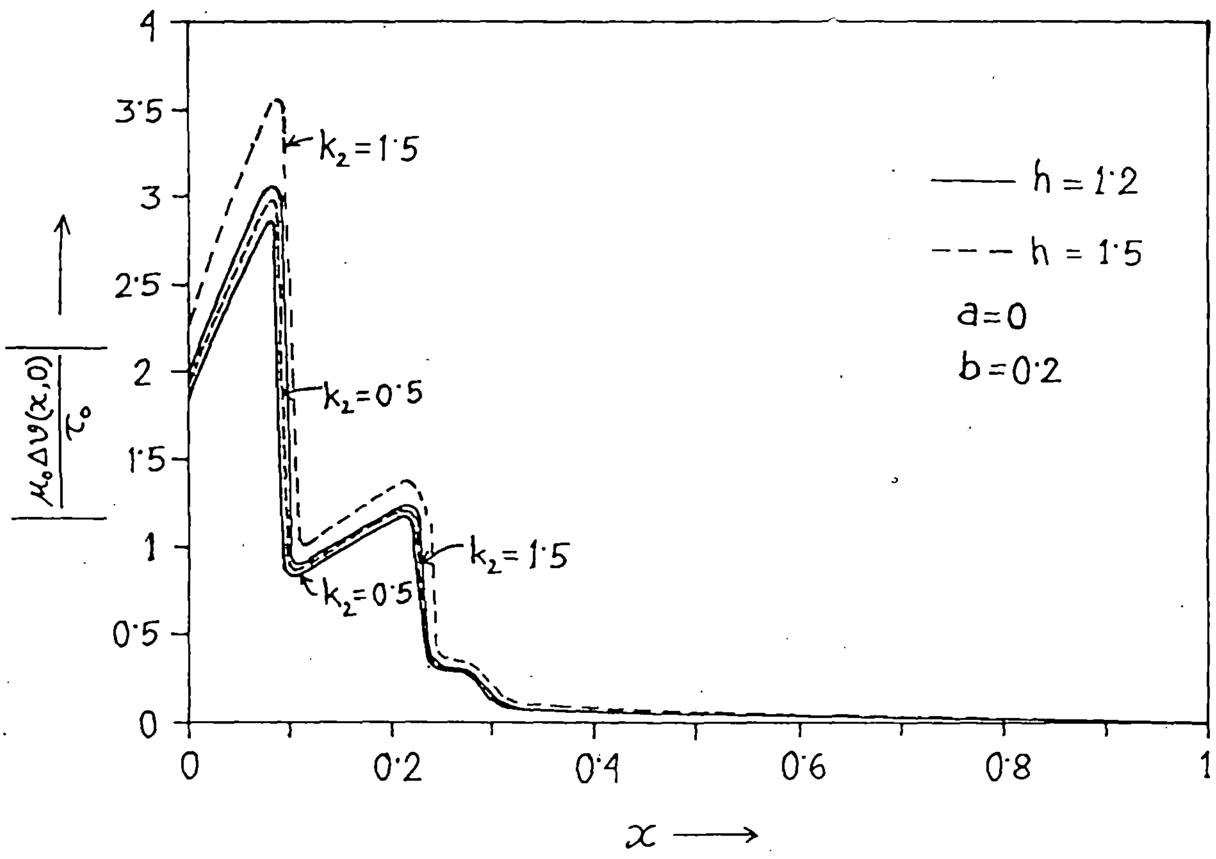

Fig. 7 Crack opening displacement vs. dimensionless distance x ($a = 0$, $b = 0.2$).

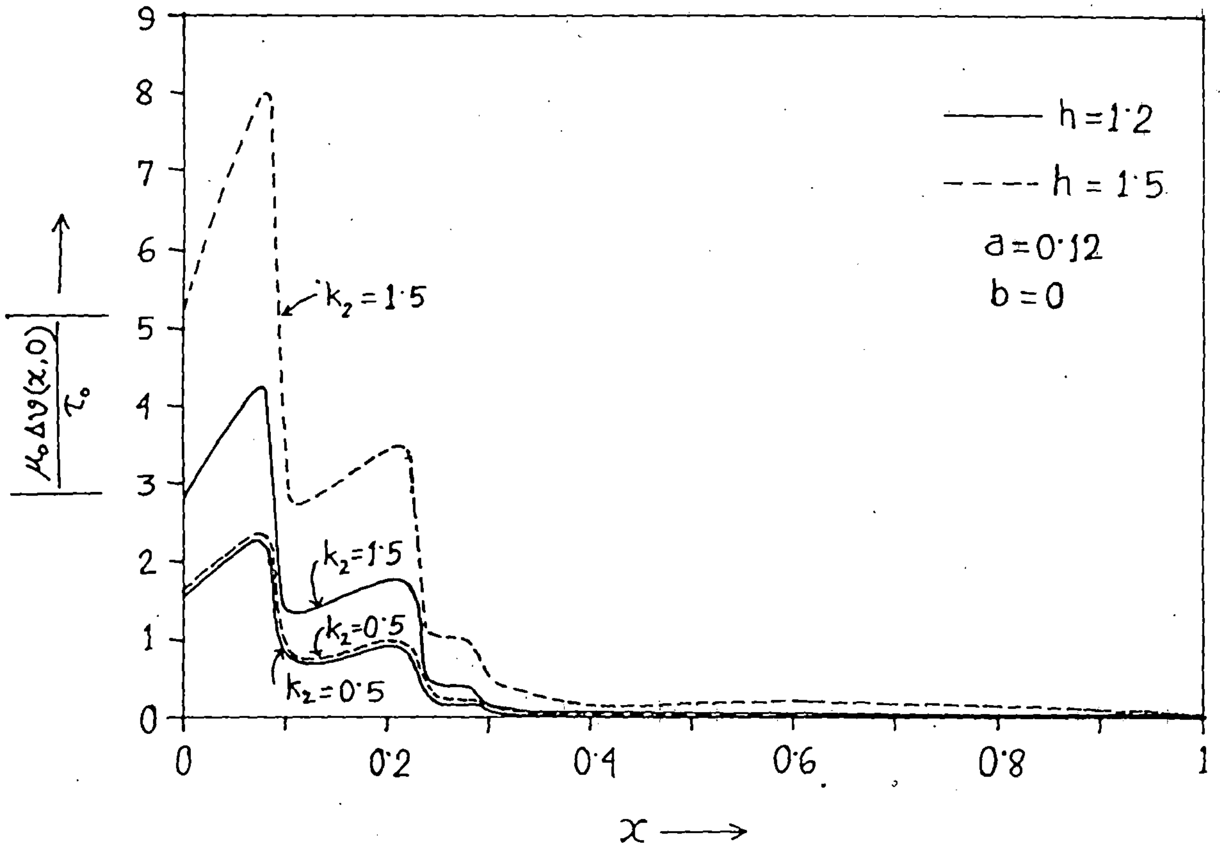

Fig. 8 Crack opening displacement vs. dimensionless distance x (a = 0, 12 , b = 0)

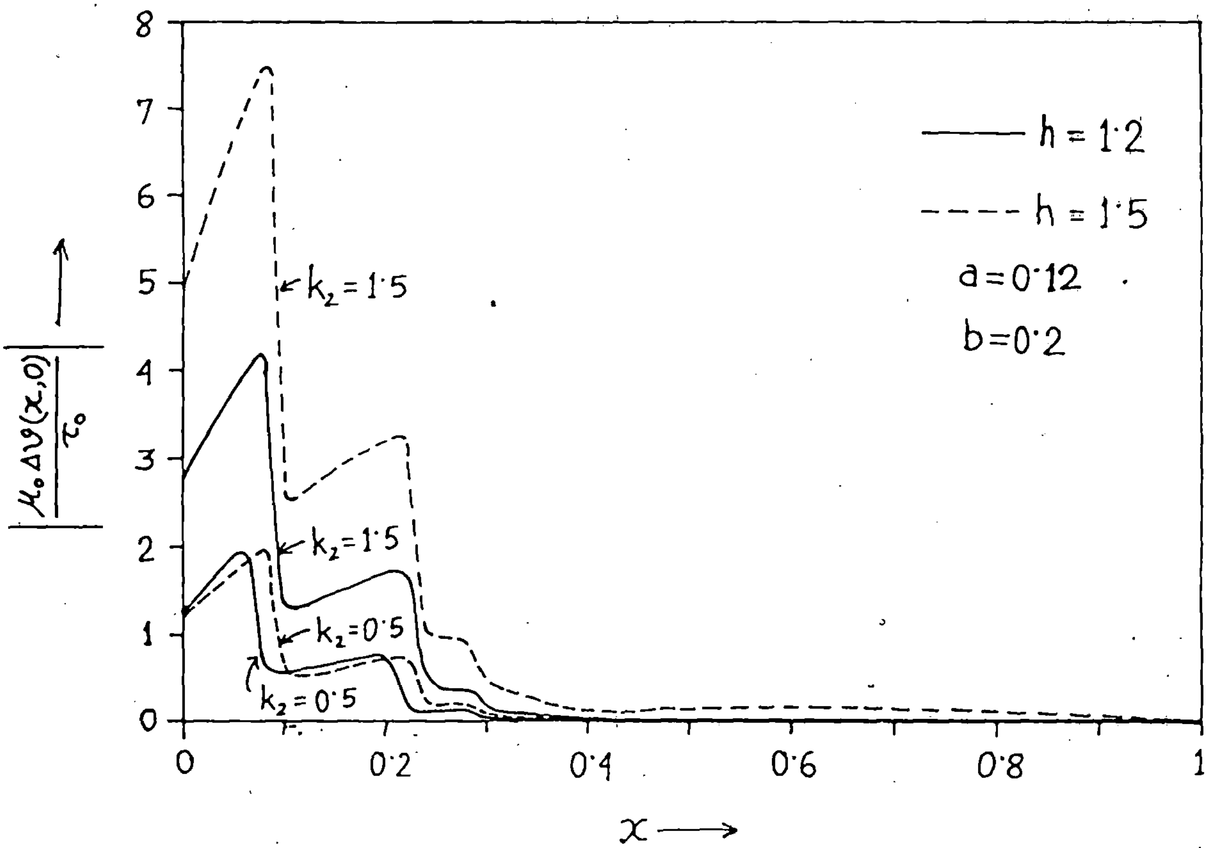

Fig. 9. Crack opening displacement vs. dimensionless distance x ($a = 0.12$, $b = 0.2$).

CHAPTER - III

MIXED BOUNDARY VALUE PROBLEMS IN VISCOELASTIC MEDIA

Paper - 7 : Moving punch on a viscoelastic semi-infinite medium.

Paper - 8 : Antiplane dynamic crack propagation in an inhomogeneous viscoelastic solid.

MOVING PUNCH ON A VISCOELASTIC SEMI-INFINITE MEDIUM

1. INTRODUCTION

Problems involving the motion of a punch on the surface of an elastic half-space or on the free boundaries of long strips are extremely important in view of their application in road construction technology and also in geophysical research. Punch problems within the classical theory of elasticity have been studied extensively by Galin (1961) and by Gladwell (1980) in their books. The motiom of a rough punch on an elastic half-space has been treated in detail by Suhubi (1972). Recently problems involving antiplane motion due to punches moving along the surfaces of an elastic strip have been solved by complex variable methods by Tait and Moodie (1981). An analytical solution to the problem of a long rigid punch moving rapidly on a strip of a highly orthotropic elastic layer has been solved by Georgiadis (1987) using integral transforms and the Wiener-Hopf techniques (1958).

However, natural or artificial materials have generally dissipative behaviour which often can be taken into account by viscoelastic models. Accordingly, problems involving the motion of a punch on a viscoelastic medium have drawn the attention of many

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scientists. The problem of a rigid cylinder rolling on the surface of a viscoelastic half space has been solved by Hunter (1961). The contact problem of rigid cylinder rolling slowly on a thin viscoelastic layer has been treated by Alblas and Kuipers (1970) assuming that the layer thickness is small compared to the width of the contact region of the cylinder. The problem of a plane punch sliding without friction on a viscoelastic half space has been considered by Golden (1977).

In the present paper, we have examined the stress and displacement field produced by a long punch moving on the boundary of a semi-infinite viscoelastic medium and producing Horizontal Shear waves. Two types of viscoelastic models viz. Maxwell Solid and Standard Linear Solid have been considered and loading is assumed to be such that Mode III conditions prevail. The mathematical technique which is employed here consists of the application of integral transforms and the solution of the resulting Wiener-Hopf equations for the transformed unknown variables. Both the steady and nonsteady solutions of the problem have been derived. Displacement and stress on the free surface and at points below the punch have been derived analytically and the nature of their variations with the velocity of the moving punch has been shown by means of graphs.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION FOR STEADY STATE MOTION.

Let us consider a semi-infinite viscoelastic medium which was set

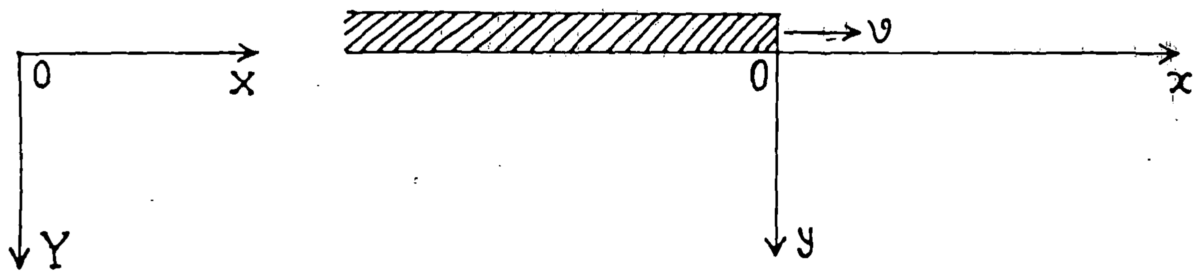

Fig. 1. The Geometry of the problem .

into motion by semi-infinite rigid punch moving with a constant velocity v in the direction of the x-axis. The y-axis is taken vertically downwards into the medium (Fig.1).

For horizontal shear waves, the displacements along X and Y directions are zero and only the displacement $W = W(X,Y,t)$ along Z-direction exists. The stresses under the punch are

$$\sigma_{13} = \sigma_{13}(X,Y,t) \quad \text{and} \quad \sigma_{23} = \sigma_{23}(X,Y,t) \tag{1}$$

The non-vanishing strains are

$$e_{13} = \frac{1}{2}\frac{\partial W}{\partial X} \quad \text{and} \quad e_{23} = \frac{1}{2}\frac{\partial W}{\partial Y} \tag{2}$$

Considering a 'Standard Linear Solid' as the viscoelastic model, the stress strain relations are

$$\frac{\partial \sigma_{i3}}{\partial t} + \beta\sigma_{i3} = 2\mu\left[\frac{\partial e_{i3}}{\partial t} + \alpha e_{i3}\right], \quad i=1,2 \tag{3}$$

where α, β are positive constants and μ is the instantaneous elastic modulus of rigidity of the material.

The equation of motion is

$$\frac{\partial \sigma_{13}}{\partial X} + \frac{\partial \sigma_{23}}{\partial Y} = \rho\frac{\partial^2 W}{\partial t^2} \tag{4}$$

where ρ is the density of the material.

The boundary conditions of the problem are

$$W(X,0,t) = w_o \quad , \quad X-vt<0$$
$$W(X,\infty,t) = 0 \quad , \quad -\infty<X<\infty \qquad (5)$$
$$\sigma_{23}(X,0,t) = 0 \quad , \quad X-vt>0$$

Since we are going to investigate the steady state propagation of a punch, it is convenient to define a moving co-ordinate system (x,y) whose origin coincides with the tip of the punch and whose axes are parallel to the fixed (X,Y) axes respectively (Fig.1). Hence putting $x = X-vt$, $y = Y$ equations (1) to (4) become respectively

$$\sigma_{13} = \sigma_{13}(x,y) \quad \text{and} \quad \sigma_{23} = \sigma_{23}(x,y) \qquad (6)$$

$$e_{13} = \frac{1}{2}\frac{\partial}{\partial x}W(x,y) \quad \text{and} \quad e_{23} = \frac{1}{2}\frac{\partial}{\partial y}W(x,y) \qquad (7)$$

$$-v\frac{\partial\sigma_{13}}{\partial x} + \beta\sigma_{13} = \mu\left[-v\frac{\partial^2 W}{\partial x^2} + \alpha\frac{\partial W}{\partial x}\right]$$
$$(8)$$
$$-v\frac{\partial\sigma_{23}}{\partial x} + \beta\sigma_{23} = \mu\left[-v\frac{\partial^2 W}{\partial x\partial y} + \alpha\frac{\partial W}{\partial y}\right]$$

and

$$\frac{\partial\sigma_{13}}{\partial x} + \frac{\partial\sigma_{23}}{\partial y} = \rho v^2\frac{\partial^2 W}{\partial x^2} \qquad (9)$$

The boundary conditions (5), now become

$$\begin{aligned} W(x,0) &= w_o \quad , \qquad x<0 \\ W(x,\infty) &= 0 \quad , \qquad -\infty<x<\infty \\ \sigma_{23}(x,0) &= 0 \quad , \qquad x>0 \end{aligned} \tag{10}$$

Now introduce Fourier transform

$$\overline{f}(\xi,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x,y)\exp(i\xi x)dx$$

so that $$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(\xi,y)\exp(-i\xi x)d\xi \tag{11}$$

Taking Fourier transform of (8) and (9) we get

$$(i\xi v + \beta)\overline{\sigma}_{13} = \mu(\xi^2 v - i\xi\alpha)\overline{W} \tag{12}$$

$$(i\xi v + \beta)\overline{\sigma}_{23} = \mu(i\xi v + \alpha)\frac{d\overline{W}}{dy} \tag{13}$$

and

$$i\xi\overline{\sigma}_{13} + \frac{d\overline{\sigma}_{23}}{dy} = -\rho v^2\xi^2\overline{W} \tag{14}$$

Eliminating $\overline{\sigma}_{13}$, $\overline{\sigma}_{23}$ from (12), (13) and (14)) we obtain,

$$\frac{d^2\overline{W}}{dy^2} - \gamma^2\overline{W} = 0 \tag{15}$$

where

$$\gamma^2 = \frac{\xi^2}{\left(\xi - \frac{i\alpha}{v}\right)}\left[\left(1 - \frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right] \quad , \quad c^2 = \frac{\mu}{\rho} \tag{16}$$

The branches of γ are so chosen that

$$\mathrm{Re}(\gamma)>0 \quad \text{for} \quad -a<\mathrm{Im}(\xi)<0$$

where

$$a = \left[\frac{v\beta}{c^2} - \frac{\alpha}{v} \right] \Big/ \left[1 - \frac{v^2}{c^2} \right] \tag{17}$$

Now the solution of equation (15) bounded as $y \to \infty$ is

$$\overline{W}(\xi,y) = B(\xi)e^{-\gamma y} \tag{18}$$

Let us consider

$$W(x,0) = w_o = W_o\ e^{\varepsilon x}, \quad x<0,\ \varepsilon>0 \text{ and } \varepsilon \text{ will be made to tend to zero finally}$$

$$= W_o p(x) \quad (\text{say}), \quad x>0 \tag{19}$$

$$\sigma_{23}(x,0) = 0 \quad , \quad x>0$$

$$= W_o t(x) \quad (\text{say}), \quad x<0 \tag{20}$$

where $p(x)$ and $t(x)$ are unknown functions such that

$$p(x) \sim O\left[e^{-k_1 x}\right] \quad \text{as} \quad x \to \infty, \quad k_1>0$$

$$t(x) \sim O\left[e^{+k_2 x}\right] \quad \text{as} \quad x \to -\infty, \quad k_2>0.$$

Taking Fourier transform of (19)

$$\overline{W}(\xi,0) = \frac{W_o}{\sqrt{(2\pi)}(\varepsilon+i\xi)} + \frac{W_o}{\sqrt{(2\pi)}} P_+(\xi) \tag{21}$$

where

$$P_+(\xi) = \int_0^{\infty} p(x)\exp(i\xi x)dx \ , \quad (\xi=\sigma+i\tau) \tag{22}$$

In (21) the first term on the right hand side is analytic in the lower half plane $Im(\xi) = \tau < \varepsilon$ and $P_+(\xi)$ is analytic in the upper half plane $\tau>-k_1$ ($k_1<a$, say).

Again taking Fourier transforms of (20)

$$\bar{\sigma}_{23}(\xi,0) = \frac{W_0}{\sqrt{(2\pi)}} T_-(\xi) \tag{23}$$

where $$T_-(\xi) = \int_{-\infty}^{0} t(x)\exp(i\xi x)dx \tag{24}$$

$T_-(\xi)$ is analytic in the lower half plane $\tau<k_2$. Therefore, $\bar{W}(\xi,0)$ is analytic for $-k_1<\tau<\varepsilon$ and $\bar{\sigma}_{23}(\xi,0)$ is analytic in the lower half plane $\tau<k_2$.

From (13),

$$\left[(i\xi v + \beta)\bar{\sigma}_{23} \right]_{y=0} = \left[\mu(i\xi v + \alpha) \frac{d\bar{W}}{dy} \right]_{y=0}$$

Using (18), (21) and (23) this becomes

$$T_-(\xi) = - H(\xi)\left[P_+(\xi) - \frac{i}{\xi - i\varepsilon} \right] \tag{25}$$

where

$$H(\xi) = \frac{\mu\xi\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \left[\left(1 - \frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \tag{26}$$

It may be noted that the problem has been reduced to a form suitable for the application of the Wiener-Hopf technique.

Now $H(\xi)$ can be written as

$$H(\xi) = H_+(\xi)H_-(\xi) \tag{27}$$

where

$$H_+(\xi) = \mu\left[\left(1 - \frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \tag{28}$$

and

$$H_-(\xi) = \frac{\xi\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \tag{29}$$

$H_+(\xi)$ is analytic in the upper half plane $\tau > -a$ and $H_-(\xi)$ is analytic in the lower half plane $\tau < 0$.

Introducing (27) in equation (25), we obtain after a little algebraic simplification,

$$R_-(\xi) - \frac{T_-(\xi)}{H_-(\xi)} = H_+(\xi)P_+(\xi) - R_+(\xi) \tag{30}$$

where

$$R_+(\xi) = \frac{i\left[H_+(\xi) - H_+(i\varepsilon)\right]}{\xi - i\varepsilon} \tag{31}$$

and
$$R_-(\xi) = \frac{i\,H_+(i\varepsilon)}{\xi - i\varepsilon} \tag{32}$$

The functions $R_+(\xi)$ and $R_-(\xi)$ are such that each is analytic and non-zero in some upper and lower half planes respectively.

The functions on the R.H.S. of (30) are analytic and non-zero in the upper half plane $\tau > -a$ and the functions on the L.H.S. are analytic and non-zero in the lower half plane $\tau < 0$.

Since, both the functions are analytic in the strip $-a<\tau<0$, the principle of analytic continuation states that each represents an entire function $M(\xi)$ in the whole ξ-plane.

Now near the tip of the punch,

$$\sigma_{23}(x,0) \sim o\left(-\frac{1}{x}\right)^{1/2} \text{ as } x \to 0^{-}, \text{ so } T_{-}(\xi) \sim o\left(\xi^{-1/2}\right), \text{ as } |\xi| \to \infty$$

and $R_{-}(\xi) \sim o\left(\xi^{-1}\right)$ as $|\xi| \to \infty$.

Thus the L.H.S. of (30) approaches zero as $|\xi| \to \infty$. It may be concluded by Liouville's theorem that $M(\xi) = 0$ and therefore

$$T_{-}(\xi) = R_{-}(\xi) H_{-}(\xi) \tag{33}$$

and

$$P_{+}(\xi) = \frac{R_{+}(\xi)}{H_{+}(\xi)} \tag{34}$$

Now,

$$T_{-}(\xi) = i\mu\left[i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \text{ as } \varepsilon \to 0.$$

So from (23)

$$\bar{\sigma}_{23}(\xi,0) = \frac{W_0}{\sqrt{(2\pi)}}\, i\mu\left[i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)}$$

Therefore for $x<0$

$$\sigma_{23}(x,0) = \frac{i\mu W_0}{2\pi}\left[i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right]^{1/2} \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \exp(-i\xi x)\,d\xi \tag{35}$$

Considering a branch cut along the positive imaginary axis starting from $\xi = \frac{i\alpha}{v}$ and changing the path of integration from real ξ-axis to the path around the branch cut as shown in Fig.2, it can easily be shown that the integral

$$I = \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \exp(-i\xi x)d\xi \qquad (\text{assuming } \beta>\alpha)$$

can be converted to the following integral

$$I = 2\, e^{\frac{\pi i}{4}}\, e^{-\frac{\alpha}{v}x_1} \fint_0^{\infty} \frac{\sqrt{u}\; e^{-ux_1}}{u - \left(\frac{\beta}{v} - \frac{\alpha}{v}\right)}\, du \tag{36}$$

where x has been replaced by $-x_1$, $\fint_0^{\infty}$ denotes the principal value of the integral.

For large values of $\frac{\beta-\alpha}{v}\, x_1 = mx_1$, where $m = \frac{\beta-\alpha}{v}$, the integral (36) can be evaluated in the form

$$I = -2\, e^{\frac{\pi i}{4}}\, e^{-\frac{\alpha}{v}x_1} (x_1)^{-1/2} \left[\frac{\Gamma(3/2)}{mx_1} + \frac{\Gamma(5/2)}{m^2x_1^2} + \frac{\Gamma(7/2)}{m^3x_1^3} + \ldots\ldots \right] \tag{37}$$

and for small values of mx_1 it can be shown that

$$I = 2\, e^{\frac{\pi i}{4}}\, e^{-\frac{\alpha}{v}x_1} \sqrt{\frac{\pi}{x_1}} \tag{38}$$

The details of the evaluation of the integral I has been shown in the Appendix-1.

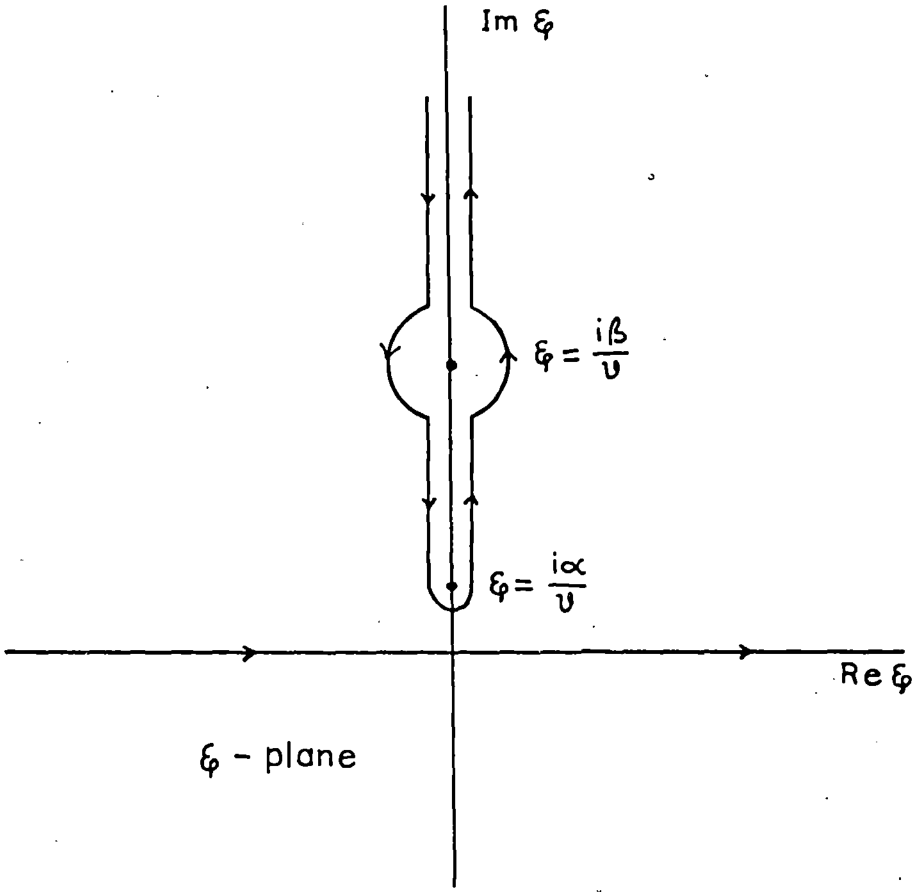

Fig.2. Path of Integration to evaluate I .

Therefore using (35) and (36) we obtain for $x<0$,

$$\sigma_{23}(x,0) = -\frac{\mu W_o}{\pi}\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)^{1/2} e^{-\frac{\alpha}{v}x_1}\int_0^{\infty}\frac{\sqrt{u}\,e^{-ux_1}}{u-\left(\frac{\beta}{v}-\frac{\alpha}{v}\right)}\,du \quad , \quad x<0$$

(39)

Using the value of the integral arising in (39) by (38), we get for small values of mx_1,

$$\sigma_{23}(x,0) = -\frac{\mu W_o}{\sqrt{\pi m x_1}}\left[m\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)\right]^{1/2} e^{-\frac{\alpha}{v}x_1} \quad , \quad x_1 \to 0^+ \qquad (40)$$

Also with the help of (39) and (37), for large values of mx_1 $(x<0)$ we have,

$$\sigma_{23}(x,0) = \frac{\mu W_o}{\pi\sqrt{m x_1}}\left[m\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)\right]^{1/2} e^{-\frac{\alpha}{v}x_1} \times$$

$$\times\left[\frac{\Gamma(3/2)}{mx_1}+\frac{\Gamma(5/2)}{m^2x_1^2}+\frac{\Gamma(7/2)}{m^3x_1^3}+\ldots\ldots\right] \qquad (41)$$

Now, from (28), (31) and (39)

$$P_+(\xi) = \frac{i}{\xi} - \frac{i\left[i\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)\right]^{1/2}}{\xi\left[\left(1-\frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right)\right]^{1/2}} \quad , \qquad \varepsilon \to 0.$$

Using this result in (21) we get

$$\overline{W}(\xi,0) = -\frac{iW_o}{2\pi}\sqrt{ia}\,\frac{1}{\xi\sqrt{(\xi+ia)}} \quad , \quad a = \left(\frac{v\beta}{c^2}-\frac{\alpha}{v}\right) \Big/ \left(1-\frac{v^2}{c^2}\right)$$

Taking inverse Fourier transform

$$W(x,0) = -\frac{iW_o}{2\pi}\sqrt{ia}\int_{-\infty-id}^{\infty-id}\frac{1}{\xi\sqrt{(\xi+ia)}}\exp(-i\xi x)d\xi\ , \quad x>0 \quad (0<d<a) \tag{42}$$

Transforming the integral in (42) to an integral along the contour around the branch cut from $-ia$ to $-i\infty$, it can be shown that

$$W(x,0) = -\frac{iW_o}{\pi}\sqrt{iax}\ e^{-ax}\ e^{\frac{\pi i}{4}}\int_0^{\infty}\frac{e^{-U}\ U^{-1/2}}{U+ax}\,dU \qquad (x>0)$$

which can be written as

$$W(x,0) = \frac{W_o}{\sqrt{\pi}}\ e^{-ax/2}\ (ax)^{-1/4}\ W_{-1/4,-1/4}(ax)\ , \quad (x>0) \tag{43}$$

where $W_{k,m}$ is the Whittaker function (1969).

Using the results that

$$W_{k,m}(z) \sim \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-k)}\ z^{\frac{1}{2}+m}\ e^{-z/2} + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}\ z^{\frac{1}{2}-m}\ e^{-z/2}$$

for small z,

and $\quad W_{k,m}(z) \sim e^{-z/2}\ (z)^{k}\quad$ for large z,

in (43) we obtain for small values of ax $(x>0)$

$$W(x,0) = W_o\ e^{-ax} - \frac{2W_o}{\sqrt{\pi}}\ e^{-ax}\sqrt{(ax)}\ , \quad x \to 0^{+} \tag{44}$$

and for large values of ax $(x>0)$

$$W(x,0) = \frac{W_o}{\sqrt{\pi}} \frac{e^{-ax}}{\sqrt{(ax)}} \quad , \quad x \to \infty \ (x>0) \tag{45}$$

3. STEADY STATE SOLUTION FOR MAXWELL SOLID

For 'Maxwell Solid' the stress strain relations obtained from (3) putting $\alpha = 0$ are

$$\frac{\partial \sigma_{i3}}{\partial t} + \beta \sigma_{i3} = 2\mu \frac{\partial e_{i3}}{\partial t} \quad , \quad i=1,2 \tag{46}$$

The stress can be found by putting $\alpha = 0$ in (39) as (for $x<0$, $y=0$)

$$\sigma_{23}(x,0) = -\frac{\mu W_o}{\pi}\left(\frac{v\beta}{c^2}\right)^{1/2} \int_0^{\infty} \frac{\sqrt{u}\ e^{-ux_1}}{u - \frac{\beta}{v}}\, du \tag{47}$$

For small values of $\frac{\beta}{v} x$, $x<0$, putting $\alpha = 0$ in (40) we get

$$\sigma_{23}(x,0) = -\frac{\mu W_o}{\sqrt{\pi x_1}}\left(\frac{v\beta}{c^2}\right)^{1/2} \quad , \quad x_1 \to 0^+ \quad , \quad (x_1 = -x) \tag{48}$$

Again for large values of $\beta x/v$, $(x<0)$, from (41)

$$\sigma_{23}(x,0) = \frac{\mu W_o}{\pi\sqrt{x_1}}\left(\frac{v\beta}{c^2}\right)^{1/2}\left[\frac{v}{\beta x_1}\Gamma(\tfrac{3}{2}) + \frac{v^2}{\beta^2 x_1^2}\Gamma(\tfrac{5}{2}) + \frac{v^3}{\beta^3 x_1^3}\Gamma(\tfrac{7}{2}) + \ldots\right] \tag{49}$$

Putting $\alpha = 0$ the displacement on the free surface ($y=0$, $x>0$) is obtained from (43) as

$$W(x,0) = \frac{W_o}{\sqrt{\pi}} e^{-kx/2} (kx)^{-1/4} W_{-1/4,-1/4}(kx) , \quad (x>0) \qquad (50)$$

where

$$k = \left(\frac{v\beta}{c^2} \right) / \left(1 - \frac{v^2}{c^2} \right)$$

which for small values of kx>0 becomes by help of (44)

$$W(x,0) = W_o \; e^{-kx} - \frac{2W_o}{\sqrt{\pi}} e^{-kx} \sqrt{(kx)} \quad , \quad x \to 0^+ \qquad (51)$$

and for large values of kx>0, using (45), we obtain

$$W(x,0) = \frac{W_o}{\sqrt{\pi}} \frac{e^{-kx}}{\sqrt{(kx)}} \quad , \quad x \to \infty \; (x>0) \qquad (52)$$

4. SOLUTION OF THE PROBLEM FOR NON STEADY STATE MOTION

In this case it is assumed that at time t=0 a semi-infinite punch starts to move with a constant velocity v at X=Y=0 on the surface of the semi-infinite viscoelastic medium.

The 'Standard Linear Solid' is taken as the viscoelastic model. Shifting the origin at X=vt and putting X-vt=x and Y=y so that

$\frac{\partial}{\partial X} = \frac{\partial}{\partial x}$, $\frac{\partial}{\partial Y} = \frac{\partial}{\partial y}$ and time derivative equal to $-v \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ the

stress displacement relations given by (8) become in this case

$$- v \frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{13}}{\partial t} + \beta \sigma_{13} = \mu \left[- v \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial t \partial x} + \alpha \frac{\partial W}{\partial x} \right]$$

(53)

$$- v \frac{\partial \sigma_{23}}{\partial x} + \frac{\partial \sigma_{23}}{\partial t} + \beta \sigma_{23} = \mu \left[- v \frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 W}{\partial t \partial y} + \alpha \frac{\partial W}{\partial y} \right]$$

Both these equations can be reduced to ordinary differential equations by the application of the Laplace transform over t and the Fourier transform over x.

Let us denote the Laplace transform by a single bar

$$\bar{f} \equiv \bar{f}(x,y,p) = \int_0^\infty f(x,y,t)\exp(-pt)dt \tag{54}$$

and Fourier transform by two bars

$$\bar{\bar{f}} \equiv \bar{\bar{f}}(\xi,y,p) = \int_{-\infty}^\infty \bar{f}(x,y,p)\exp(i\xi x)dx \tag{55}$$

Applying these transforms to (53) we get

$$(i\xi v + p + \beta)\bar{\bar{\sigma}}_{13} = \mu(\xi^2 v - i\xi p - i\xi\alpha)\bar{\bar{W}} \tag{56}$$

$$(i\xi v + p + \beta)\bar{\bar{\sigma}}_{23} = \mu(i\xi v + p + \alpha) \frac{d\bar{\bar{W}}}{dy} \tag{57}$$

Now the equation of motion given by (4) becomes

$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial y} = \rho\left[v^2 \frac{\partial^2 W}{\partial x^2} - 2v \frac{\partial^2 W}{\partial x \partial t} + \frac{\partial^2 W}{\partial t^2} \right]$$

which after taking Laplace and Fourier transforms takes the form

$$- i\xi\bar{\sigma}_{13} + \frac{d\bar{\sigma}_{23}}{dy} = -\rho(-v^2\xi^2 + 2vi\xi p + p^2)\bar{W} \tag{58}$$

Substituting for $\bar{\sigma}_{13}$ and $\bar{\sigma}_{23}$ from (56) and (57) in (58) we have

$$\frac{d^2\bar{W}}{dy^2} - \gamma^2 \bar{W} = 0 \tag{59}$$

where

$$\gamma^2 = \frac{1}{(vi\xi+p+\alpha)} \left[\xi^2(vi\xi+p+\alpha) + \frac{\rho}{\mu}(vi\xi+p)^2(vi\xi+p+\beta) \right] \tag{60}$$

The branches of γ are defined by $Re(\gamma)>0$.

Since the stresses are bounded as $y \to \infty$, $W(x,y,t)$ and hence also $\bar{W}(\xi,y,p)$ must remain bounded as $y \to \infty$.

Hence, the solution of the equation (59) is given by

$$\bar{W}(\xi,y,p) = A(\xi,p)\ e^{-\gamma y}.$$

Now the boundary conditions are

$$\begin{aligned} &W(x,0,t) = W_0H(t) , && x<0 \\ &W(x,\infty,t) = 0 , && -\infty<x<\infty \\ &\sigma_{23}(x,0,t) = 0. && \end{aligned} \tag{61}$$

Taking Laplace transform with respect to t, these boundary conditions become

$$\overline{W}(x,0,p) = \frac{W_o}{p} \quad , \quad x<0$$

(62)

$$\bar{\sigma}_{23}(x,0,p) = 0 \quad , \quad x>0$$

Let us consider

$$\overline{W}(x,0,p) = W_o p(x) \quad \text{(say)} \quad , \quad x>0$$

(63)

and $$\bar{\sigma}_{23}(x,0,p) = \mu W_o t(x) \quad \text{(say)} \quad , \quad x<0$$

The functions p(x) and t(x) are such that

$$p(x) \sim O\left[e^{-k_1 x} \right] \quad \text{as} \quad x \to \infty \, , \quad k_1>0$$

$$t(x) \sim O\left[e^{+k_2 x} \right] \quad \text{as} \quad x \to -\infty \, , \quad k_2>0.$$

Taking Fourier transform of (63) and (64) we obtain

$$\overline{\overline{W}}(\xi,0,p) = \frac{W_o}{i\xi p} + W_o P_+(\xi) \tag{64}$$

where $$P_+(\xi) = \int_0^\infty p(x)\exp(i\xi x)dx \quad , \quad (\xi=\sigma+i\tau)$$

and $$\bar{\bar{\sigma}}_{23}(\xi,0,p) = \mu W_o T_-(\xi) \tag{65}$$

where $$T_-(\xi) = \int_{-\infty}^0 t(x)\exp(i\xi x)dx \; .$$

The integral of $\overline{W}(\xi,0,p)$ over $(-\infty,0)$ converges if and only if $\mathrm{Im}(\xi) = \tau < 0$ and integral over $(0,\infty)$ converges if $\tau>-k_1$. $\bar{\bar{\sigma}}_{23}$ is analytic over $(-\infty,0)$ if $\tau<k_2$.

Now (57) becomes with the help of (61), (64) and (65)

$$-\frac{(vi\xi+p+\beta)\ T_-(\xi)}{(vi\xi+p+\alpha)\gamma} = \frac{1}{i\xi p} + P_+(\xi) \tag{66}$$

In this form of equation Wiener-Hopf technique can easily be applied.

5. NON STEADY STATE SOLUTION FOR MAXWELL SOLID

For general α and β, γ does not readily factorize. Expressions for the roots of $\gamma = 0$ can be obtained but these are difficult to handle. We discuss here the case of the Maxwell Solid, where $\alpha=0$. In this case γ^2 reduces to

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)\left[\xi^2 + \frac{vi(2p+\beta)}{c^2\left(1-\frac{v^2}{c^2}\right)}\xi + \frac{p(p+\beta)}{c^2\left(1-\frac{v^2}{c^2}\right)}\right] \quad , \quad c^2 = \frac{\mu}{\rho} \tag{67}$$

The quadratic within the second bracket equated to zero has the complex roots

$$\frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[-\frac{vi(2p+\beta)}{c^2} \pm \frac{2i}{c}\sqrt{p(p+\beta) + \frac{v^2\beta^2}{4c^2}}\right] \tag{68}$$

one positive and one negative for $v<c$.

Hence

$$\gamma = \left(1-\frac{v^2}{c^2}\right)^{1/2} (\xi+iX_1)^{1/2}\ (\xi-iX_2)^{1/2} \quad , \quad \mathrm{Re}\ X_1,\ X_2>0 \tag{69}$$

where

$$X_1 = \frac{1}{2\left(1 - \frac{v^2}{c^2}\right)} \left[\frac{v(2p+\beta)}{c^2} + \frac{2}{c} \sqrt{p(p+\beta) + \frac{v^2\beta^2}{4c^2}} \right]$$

$$X_2 = \frac{1}{2\left(1 - \frac{v^2}{c^2}\right)} \left[- \frac{v(2p+\beta)}{c^2} + \frac{2}{c} \sqrt{p(p+\beta) + \frac{v^2\beta^2}{4c^2}} \right] \qquad (70)$$

Branches are chosen so that $\gamma \to +\infty$ as $\xi \to \pm\infty$.

Thus for a Maxwell Solid, (66) can be written after simplication as

$$- \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \frac{\left(\xi - \frac{i(p+\beta)}{v}\right)}{\left(\xi - \frac{ip}{v}\right)} \frac{T_-(\xi)}{(\xi - iX_2)^{1/2}} - \frac{(iX_1)^{1/2}}{i\xi p}$$

$$= \frac{(\xi + iX_1)^{1/2} - (iX_1)^{1/2}}{i\xi p} + (\xi + iX_1)^{1/2} P_+(\xi) \qquad (71)$$

The function on the left hand side of (71) is analytic for $\tau<0$ and the function on the right hand side of (71) is analytic for $\tau>-k_1$. As they are both equal for $-k_1<\tau<0$ the principle of analytic continuation gives that either side represents the same entire function $M(\xi)$, say. Further we had previously noted that

$$T_-(\xi) \to 0 \quad , \quad |\xi| \to \infty \quad , \quad \tau<0$$

$$P_+(\xi) \to 0 \quad , \quad |\xi| \to \infty \quad , \quad \tau>-k_1.$$

Liouville's theorem gives that $M(\xi) = 0$ and thus

$$P_{+}(\xi) = \frac{(iX_{1})^{1/2}}{i\xi p(\xi + iX_{1})^{1/2}} - \frac{1}{i\xi p} \tag{72}$$

and

$$T_{-}(\xi) = -\left[1 - \frac{v^{2}}{c^{2}}\right]^{1/2} \frac{(iX_{1})^{1/2}}{i\xi p} \frac{\left[\xi - \frac{ip}{v}\right](\xi - iX_{2})^{1/2}}{\left[\xi - \frac{i(p+\beta)}{v}\right]} \tag{73}$$

Therefore, $\overline{W}(\xi,0,p)$ given in (64), with the help of (72), takes the form

$$\overline{W}(\xi,0,p) = \frac{W_{o}(iX_{1})^{1/2}}{i\xi p(\xi + iX_{1})^{1/2}} \tag{74}$$

Taking inverse transforms one get,

$$W(x,0,t) = \frac{W_{o}}{i} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(iX_{1})^{1/2}}{p} e^{pt}\, dp\, \frac{1}{2\pi} \int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}}{\xi(\xi + iX_{1})^{1/2}}\, d\xi$$

$$0<d<k_{1}, \quad x>0 \tag{75}$$

Taking the path of integration around the branch cut along negative imaginary axis from $-iX_{1}$ to $-i\infty$ the integral

$$I = \int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}}{\xi(\xi + iX_{1})^{1/2}}\, d\xi$$

can be converted to the integral

$$I = 2e^{\pi i/4}\ e^{-X_1 x}\ \sqrt{x} \int_0^\infty \frac{e^{-U}\ U^{-1/2}}{U + xX_1}\, dU$$

which is finally evaluated as

$$I = 2\sqrt{\pi}\ e^{\pi i/4}\ e^{-X_1 x/2}\ \sqrt{x}\ (xX_1)^{-3/4}\ W_{-1/4,-1/4}(xX_1).$$

Putting this value of the integral in (75) we obtain

$$W(x,0,t) = \frac{W_0}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}\ e^{-xX_1/2}}{p} (xX_1)^{-1/4}\ W_{-1/4,-1/4}(xX_1)\ dp$$

$$, \quad x>0 \qquad (76)$$

Now for small p,

$$X_1 = \left(\frac{v\beta}{c^2}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right) = k \text{ (say)} \qquad (77)$$

and for large p,

$$X_1 = \frac{p}{c - v},$$

therefore for large $\frac{px}{c - v}$,

$$W_{-1/4,-1/4}\left(\frac{px}{c - v}\right) \sim \exp\left(-\frac{1}{2}\frac{px}{c - v}\right)\left(\frac{px}{c - v}\right)^{-1/4} \qquad (78)$$

So in equation (76) putting the value of X_1 for small p given by (77), we obtain for large time t,

$$W(x,0,t) = \frac{W_0}{\sqrt{\pi}}\, e^{-kx/2}\ (kx)^{-1/4}\ W_{-1/4,-1/4}(kx) \qquad (79)$$

which is same as the result for the steady state case for all $x > 0$ given by (50).

For large p, i.e. for small time t and for all finite x such that $px/(c-v)$ is large, using (78) we obtain from (76)

$$W(x,0,t) = \frac{2W_o}{\pi}\sqrt{\frac{c-v}{x}}\left[t - \frac{x}{c-v}\right]^{1/2} H\left[t - \frac{x}{c-v}\right] \tag{80}$$

Now, using (73), (65) becomes

$$\bar{\sigma}_{23}(\xi,0,p) = -\mu W_o\left[1 - \frac{v^2}{c^2}\right]^{1/2} \frac{(iX_1)^{1/2}}{i\xi p} \frac{\left[\xi - \frac{ip}{v}\right](\xi - iX_2)^{1/2}}{\left[\xi - \frac{i(p+\beta)}{v}\right]}$$

After taking inverse transforms it converts to

$$\sigma_{23}(x,0,t) = -\frac{\mu W_o}{i}\left[1 - \frac{v^2}{c^2}\right]^{1/2} \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p}(iX_1)^{1/2}dp \quad \times$$

$$\times \frac{1}{2\pi}\int_{-\infty-id}^{\infty-id} \frac{\left[\xi - \frac{ip}{v}\right](\xi - iX_2)^{1/2}}{\xi\left[\xi - \frac{i(p+\beta)}{v}\right]} \exp(-i\xi x)d\xi \tag{81}$$

Reversing the order of integration

$$\sigma_{23}(x,0,t) = -\frac{\mu W_o}{i}\left[1 - \frac{v^2}{c^2}\right]^{1/2} \frac{1}{2\pi}\int_{-\infty-id}^{\infty-id} F(\xi)\exp(-i\xi x)d\xi \tag{82}$$

where

$$F(\xi) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{1/2}}{\xi\left(\xi - \frac{i(p+\beta)}{v}\right)} \frac{e^{pt}}{p} (iX_1)^{1/2} dp \tag{83}$$

For large ξ, (83) becomes

$$F(\xi) = \sqrt{\frac{i}{\xi}}\, B \tag{84}$$

where

$$B = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p} (iX_1)^{1/2} dp \tag{85}$$

Putting the value of $F(\xi)$ from (84) in (82) we get

$$\sigma_{23}(x,0,t) = -\frac{\mu W_0}{\sqrt{\pi x_1}}\left(1 - \frac{v^2}{c^2}\right)^{1/2} B \ , \quad \text{as } -x = x_1 \to 0^+ \text{ (for all } t>0) \tag{86}$$

The evaluation of B for all t ($t>0$) has been done in the Appendix-2.

For small p i.e., for large t, using from (77) that $X_1 \sim k$, we have

$$B = \sqrt{k} = \sqrt{\left(\frac{v\beta}{c^2}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right)} \tag{87}$$

Substituting the value of B given by (87) in (86) we obtain

$$\sigma_{23}(x,0,t) = -\frac{\mu W_0}{\sqrt{\pi x_1}}\left(\frac{v\beta}{c^2}\right)^{1/2} \ , \quad x_1 \to 0^+$$

which is same as the result for the steady state case given by (48).

The variation of the nondimensional values of B given by

$$B^{*} = \sqrt{\frac{c}{\beta}}\left(1 - \frac{v^2}{c^2}\right)^{1/2} B$$

has been plotted against nondimensional time $t_1 = \beta t$ for various values of v/c = 0.5, 0.7 and 0.9 and has been shown by means of graphs in Fig.3.

Now for all values of x i.e., for general value of ξ the integral

$$\frac{1}{2\pi i}\int_{-\infty-id}^{\infty-id} \frac{\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{1/2}}{\xi\left(\xi - \frac{i(p+\beta)}{v}\right)} \exp(-i\xi x)\,d\xi$$

appearing in equation (81) can be converted to the integral

$$I_1 = e^{3\pi i/4}\frac{p}{p+\beta}(X_1)^{1/2} + \frac{e^{\pi i/4}\,e^{X_2 x}}{\pi i}\int_0^{\infty}\frac{e^{ux}\sqrt{u}\left(u+X_2-\frac{p}{v}\right)}{(u+X_2)\left(u+X_2-\frac{p+\beta}{v}\right)}\,du\ ,$$

$$\text{since } \frac{p+\beta}{v} > X_2 \tag{88}$$

considering the path of integration around the branch cut along positive imaginary axis from iX_2 to $i\infty$ as shown in Fig.4.

So using (88), (81) becomes

$$\sigma_{23}(x,0,t) = -\frac{\mu W_0}{\pi}\left(1 - \frac{v^2}{c^2}\right)^{1/2}\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{pt}}{p}(X_1)^{1/2}e^{X_2 x}\,dp \times$$

$$\times \int_0^{\infty}\frac{e^{ux}\sqrt{u}\left(u+X_2-\frac{p}{v}\right)}{(u+X_2)\left(u+X_2-\frac{p+\beta}{v}\right)}\,du + \mu W_0\left(1 - \frac{v^2}{c^2}\right)^{1/2} \times$$

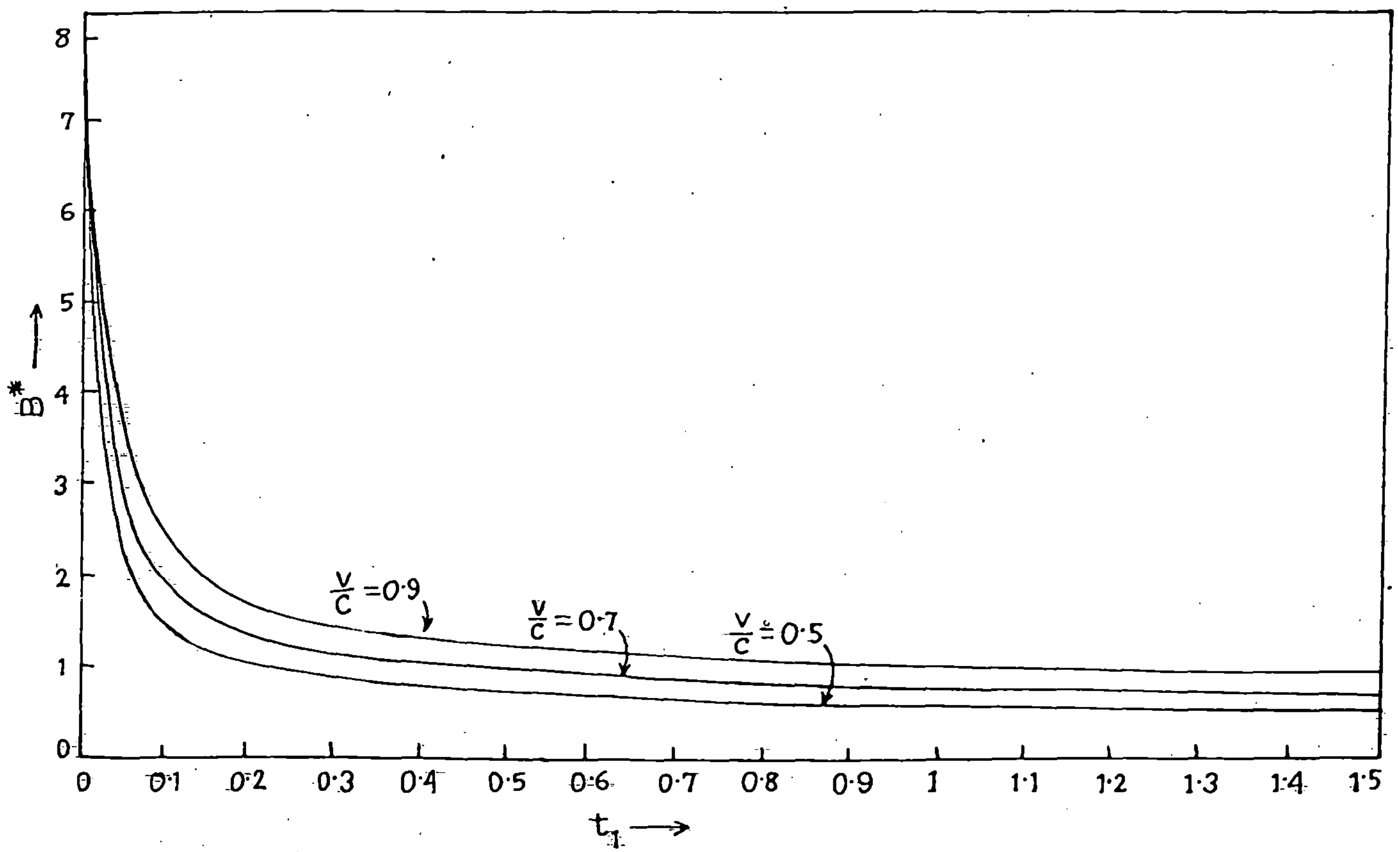

Fig. 3. Variation of B^* vs t_1 in the non steady state case of the Maxwell Solid

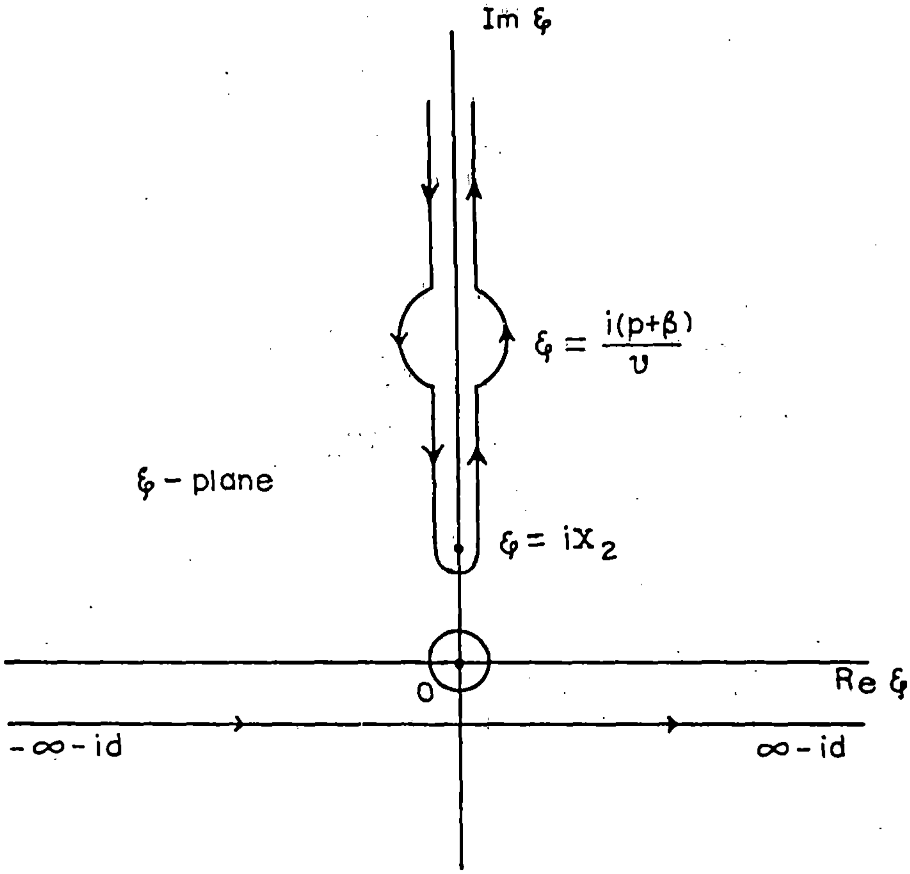

Fig 4. Path of Integration to evaluate I_1

$$\times \quad \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p+\beta} (X_1 X_2)^{1/2} \, dp \tag{89}$$

For large t (i.e., for small p using $X_1=k$, $X_2=0$) and for all x(x<0) we obtain from (89)

$$\sigma_{23}(x,0,t) = -\frac{\mu W_o}{\pi} \left(\frac{v\beta}{c^2} \right)^{1/2} \int_0^\infty \frac{e^{ux} \sqrt{u}}{u - \frac{\beta}{v}} \, du$$

where

$$k = \left(\frac{v\beta}{c^2} \right) \Big/ \left(1 - \frac{v^2}{c^2} \right)$$

which is same as the solution for the steady state case for all values of x>0 given by (47).

6. RESULTS AND DISCUSSION

The stress $\sigma_{23}(x,0)$ just below the punch (x<0) and the displacement W(x,0) on the free surface (y=0, x>0) have been computed numerically from equations (39) and (43) for different values of parameters v/c and α/β. The case $\alpha/\beta=0$ corresponds to Maxwell Solid. In Fig.5 non dimensional stress $\tau^* = \dfrac{\sigma_{23}(x,0)}{\mu W_o \beta/c}$ has been plotted against nondimensional distance $x_1^* = \beta x_1/c$ for values of the parameter v/c = 0.5, 0.9 and for values of the parameter α/β = 0 and 0.2.

For the same sets of the parameter values nondimensional displacement $W^* = W/W_o$ has been plotted versus nondimensional distance $x^* = \beta x/c$ in Fig.6. W^* varies from 1 to zero as x^* changes gradually from zero to ∞.

It may be noted from the graphs that variation of the values of W^* with x^* is rapid with the increase in the values of the parameter v/c. Further it is found that the graphs become steeper with the decrease in the values of the parameter α/β. From Fig.5 it is found that nondimensional stress τ^* changes rapidly with the decrease in the values of v/c where as for a fixed value of v/c graphs become flat with the increase in the values of α/β.

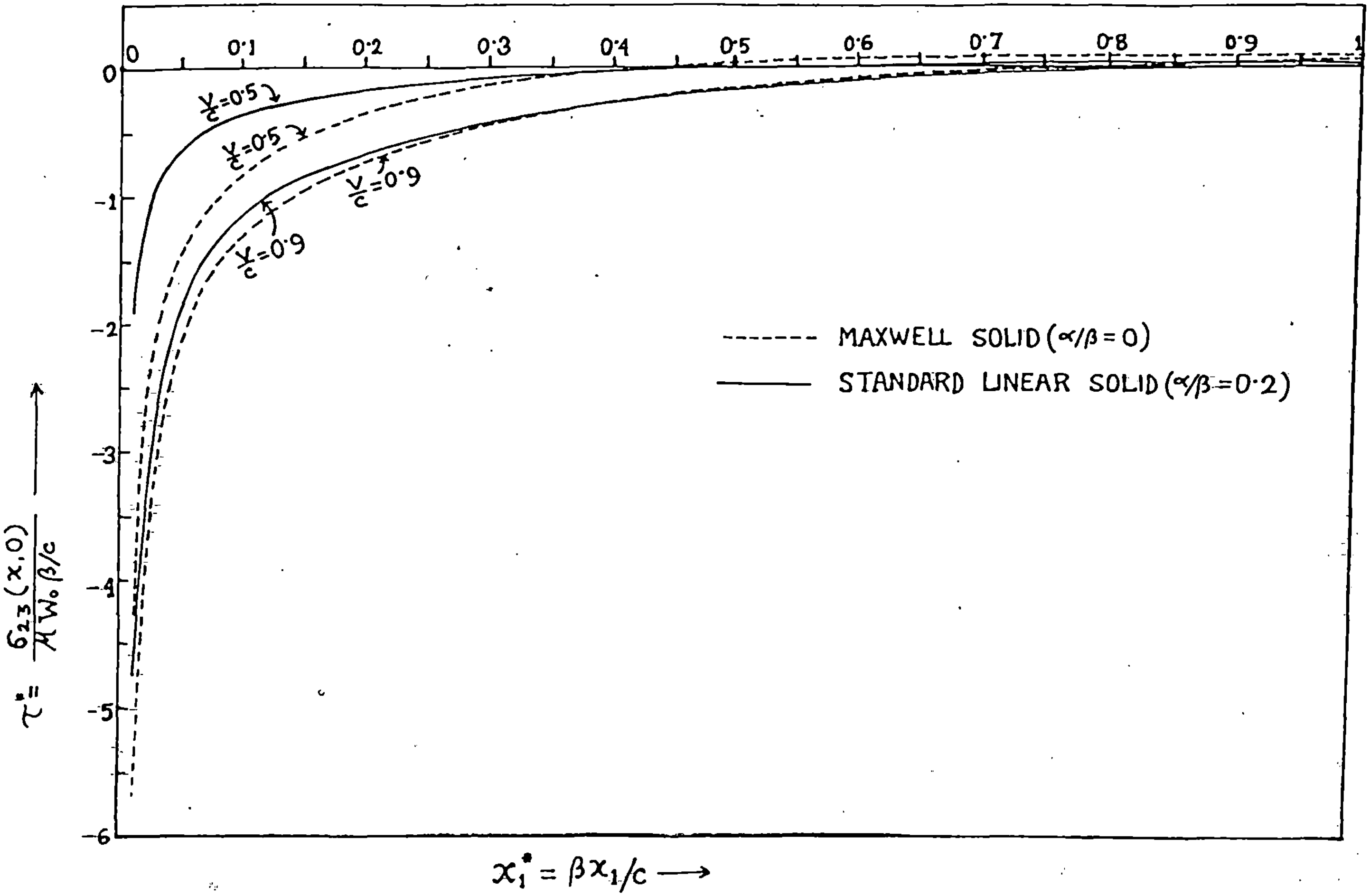

Fig. 5. Nondimentional stress τ^* vs. nondimentional distance x_1^* $(x<0)$ just below the punch in the steady state case

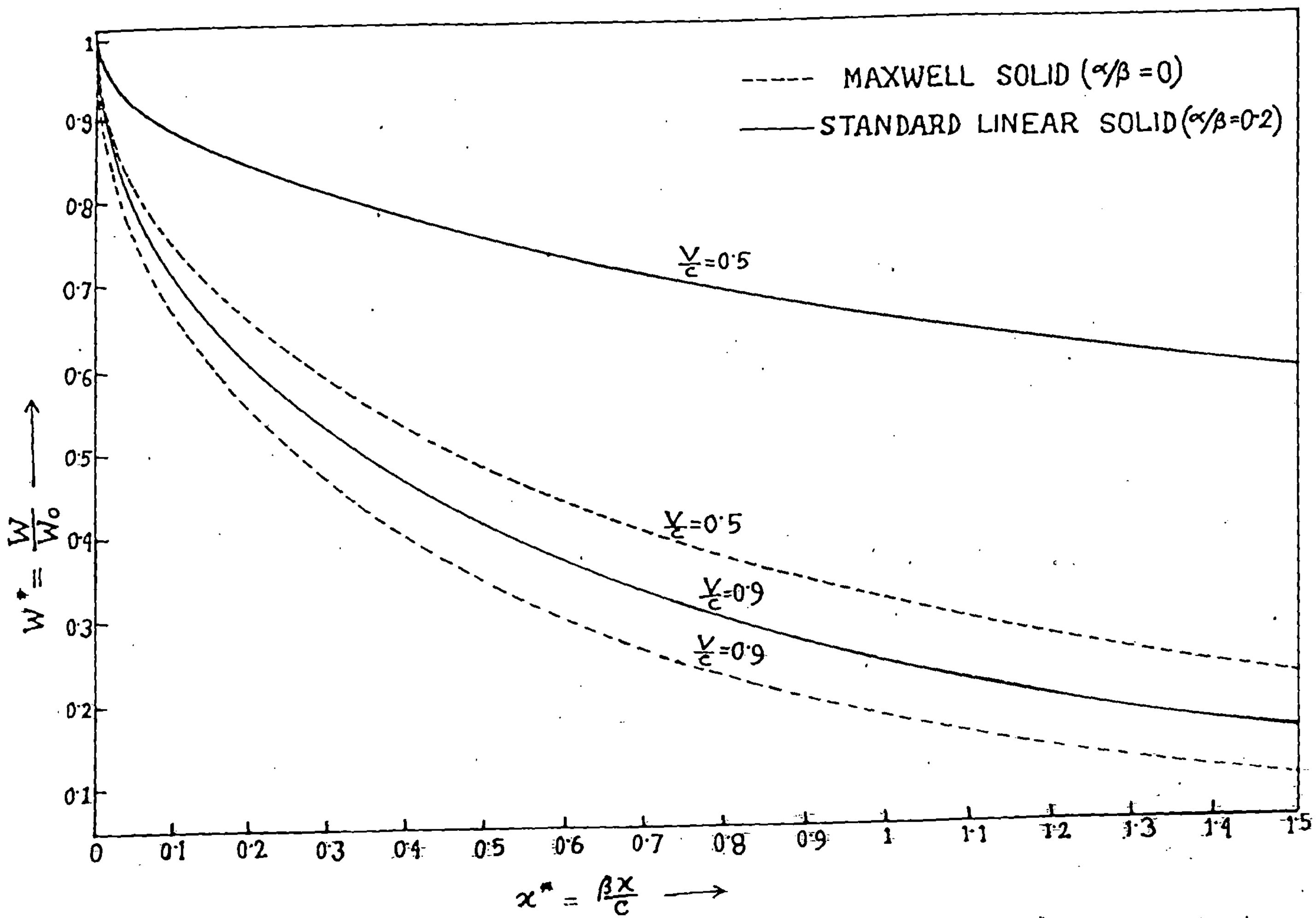

Fig. 6 Nondimensional displacement W^* vs nondimensional distance x^* ($x > 0, y = 0$) in the steady state case

APPENDIX - 1

Evaluation of the integral

$$I = \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{v}\right)^{1/2}}{\left(\xi - \frac{i\beta}{v}\right)} \exp(-i\xi x)\, d\xi$$

The contour of integration is shown in Fig.2.

Putting $\xi = \frac{i\alpha}{v} + iu$, $0<u<\infty$, the integral can be written as

$$I = 2\, e^{\frac{\pi i}{4}}\, e^{\frac{\alpha}{v}x} ⨍_{0}^{\infty} \frac{\sqrt{u}\; e^{-ux_1}}{u - \left(\frac{\beta}{v} - \frac{\alpha}{v}\right)}\, du \quad , \quad x=-x_1 \;,\; x<0$$

where $⨍$ means the principal value of the integral.

Now, for large $\left(\frac{\beta}{v} - \frac{\alpha}{v}\right)x_1$ $(x<0)$, putting $ux_1 = z$ the integral I becomes

$$I = \frac{2e^{\pi i/4}\, e^{\alpha x/v}}{\sqrt{x_1}} \int_{0}^{\infty} \frac{\sqrt{z}\; e^{-z}}{z - \left(\frac{\beta}{v} - \frac{\alpha}{v}\right)x_1}\, dz$$

$$= \frac{2e^{\pi i/4}\, e^{\alpha x/v}}{\sqrt{x_1}} \left[\int_{0}^{\infty} \frac{e^{-z}}{\sqrt{z}}\, dz - \int_{0}^{\infty} \frac{e^{-z}}{\sqrt{z}}\, dz \left\{ 1 + \frac{z}{\left(\frac{\beta-\alpha}{v}x_1\right)} + \frac{z^2}{\left(\frac{\beta-\alpha}{v}x_1\right)^2} + \ldots\ldots \right\} \right]$$

$$= - \frac{2e^{\pi i/4} e^{\alpha x/v}}{\sqrt{x_1}} \left[\frac{v}{(\beta-\alpha) x_1} \Gamma(\tfrac{3}{2}) + \frac{v^2}{(\beta-\alpha)^2 x_1^2} \Gamma(\tfrac{5}{2}) + \right.$$

$$\left. + \frac{v^3}{(\beta-\alpha)^3 x_1^3} \Gamma(\tfrac{7}{2}) + \ldots\ldots\ldots \right]$$

Again for small $\left(\frac{\beta}{v} - \frac{\alpha}{v}\right)x_1$, putting $u = \left(\frac{\beta}{v} - \frac{\alpha}{v}\right)z$, I can be converted to the integral

$$I = 2 e^{\frac{\pi i}{4}} e^{-\frac{\alpha}{v} x_1} \left(\frac{\beta-\alpha}{v}\right)^{1/2} \int_0^\infty \frac{\sqrt{z}}{z - 1} \exp\left[-z(\frac{\beta-\alpha}{v}) x_1\right] dz$$

$$= 2 e^{\frac{\pi i}{4}} e^{-\frac{\alpha}{v} x_1} \sqrt{\frac{\pi}{x_1}} + 2 e^{\frac{\pi i}{4}} e^{-\frac{\alpha}{v} x_1} \left(\frac{\beta-\alpha}{v}\right)^{1/2} \times$$

$$\times \int_0^\infty \frac{1}{t^2 - 1} \exp\left[-t^2(\frac{\beta-\alpha}{v}) x_1\right] dt$$

$$= 2 e^{\frac{\pi i}{4}} e^{-\frac{\alpha}{v} x_1} \sqrt{\frac{\pi}{x_1}} + O\left[(\frac{\beta-\alpha}{v}) x_1\right] \quad \text{where} \quad z = t^2$$

So,

$$I = 2 e^{\frac{\pi i}{4}} e^{-\frac{\alpha}{v} x_1} \sqrt{\frac{\pi}{x_1}} \qquad \text{as} \quad \left[(\frac{\beta-\alpha}{v}) x_1\right] \to 0.$$

APPENDIX - 2

EVALUATION OF THE INTEGRAL B.

The integral in (85)

$$B = \frac{1}{\sqrt{c}}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p} \sqrt{\frac{v}{c}\left(p + \frac{\beta}{2}\right) + \sqrt{p(p+\beta) + \frac{v^2\beta^2}{4c^2}}}\, dp$$

has a simple pole at p = 0 and branch points at $p = -\beta$,

$$p = \alpha_1 = \frac{\beta}{2}\left(-1 + \sqrt{1 - \frac{v^2}{c^2}}\right)$$

and

$$p = \alpha_2 = \frac{\beta}{2}\left(-1 - \sqrt{1 - \frac{v^2}{c^2}}\right) .$$

Taking the branch cut along the negative real axis from α_1 to $-\infty$ the integral can be considered as a contour integral around the path as shown in Fig.7.

Let,

$$I = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p} \sqrt{\frac{v}{c}\left(p + \frac{\beta}{2}\right) + \sqrt{(p-\alpha_1)(p-\alpha_2)}}\, dp$$

which can be written as

$I = \sqrt{\frac{v\beta}{c}} + I_1$ where $\sqrt{\frac{v\beta}{c}}$ is the contribution to the integral from pole at p = 0.

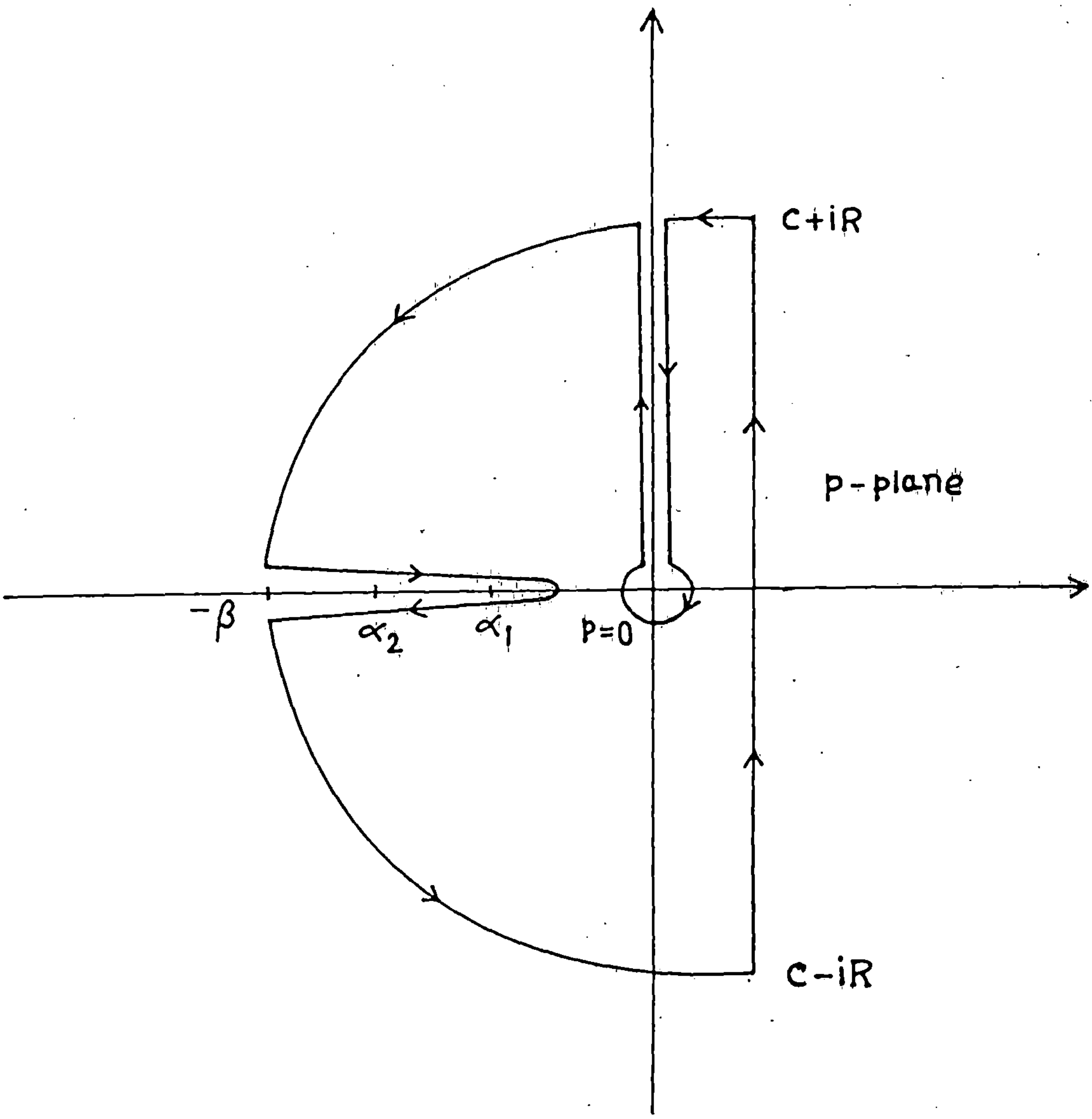

Fig. 7. The integration contour to evaluate B for the Maxwell Solid.

ANTIPLANE DYNAMIC CRACK PROPAGATION IN AN INHOMOGENEOUS VISCOELASTIC SOLID

1. INTRODUCTION

Until now many authors, Baker (1962), Cherepanov and Afanasev (1974) and others have investigated the dynamic crack propagation in a homogeneous elastic medium. This problem presents an interest for a better understanding of the brittle behaviour of the material. However, natural or artificial materials are usually inhomogeneous. There exist very few solutions to the problem of dynamic crack propagation in inhomogeneous elastic media. Atkinson and List (1978) and Atkinson (1977) considered steady-state crack propagation in different types of inhomogeneous elastic media. In addition, if the materials are dissipative, that effect can be taken into account by considering the material to be viscoelastic. Crack propagation in viscoelastic medium has been studied by Willis (1972), Atkinson and List (1972), Coussy (1987) and others. Willis (1972) considered steady-state Mode III crack propagation for a standard linear solid under general type of loading on the crack surfaces. Atkinson and List (1972) studied nonsteady SH-wave type crack propagation starting at t=0 and moving with a constant velocity in the "Maxwell Solid" or using the viscoelastic model suggested by Achenbach and Chao. Finally, Sills and Benveniste

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Let $I_1 = I_2 + I_3$ where I_2 is the value of the integral I_1 around the branch cut from α_1 to α_2 and I_3 is its value round the branch cut from α_2 to $-\infty$.

Now it can be shown that

$$I_2 = \frac{1}{\pi}\sqrt{\frac{\beta}{2}}\int_0^b \frac{\sqrt{(R^* - x^*)}\; e^{t_1(\alpha_1^* - r)}}{(\alpha_1^* - r)}\, dr$$

In the interval $(\alpha_2, -\infty)$

$$I_3 = \frac{\sqrt{\beta}}{\pi}\int_b^\infty \frac{\sqrt{(-x^{**})}\; e^{t_1(\alpha_1^* - r)}}{(\alpha_1^* - r)}\, dr$$

where $\alpha_1 = \beta\alpha_1^*$, $\alpha_2 = \beta\alpha_2^*$, $t_1 = \beta t$, $b = \sqrt{1 - \frac{v^2}{c^2}}$,

$$x^* = \frac{v}{c}\left(\frac{b}{2} - r\right), \qquad y^* = \sqrt{r(b-r)}$$

$$R^* = \sqrt{(x^*)^2 + (y^*)^2}$$

$$x^{**} = -\frac{v}{c}\left(r - \frac{b}{2}\right) - \sqrt{r(r-b)}$$

Finally, we obtain

$$B = \frac{1}{\sqrt{c}}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}\left[\sqrt{\frac{v\beta}{c}} + \frac{1}{\pi}\sqrt{\frac{\beta}{2}}\int_0^b \frac{\sqrt{(R^* - x^*)}\; e^{t_1(\alpha_1^* - r)}}{(\alpha_1^* - r)}\, dr + \right.$$

$$\left. + \frac{\sqrt{\beta}}{\pi}\int_b^\infty \frac{\sqrt{(-x^{**})}\; e^{t_1(\alpha_1^* - r)}}{(\alpha_1^* - r)}\, dr\right].$$

(1969) and Coussy (1987) studied steady state crack propagation of SH-type at the interface between two viscoelastic media.

In our case we have considered steady and nonsteady cases of Mode III crack propagation in an inhomogeneous viscoelastic medium. Two types of viscoelastic models, namely Maxwell Solid and Standard Linear Solid have been considered. Material properties have been assumed to vary exponentially in the direction perpendicular to the direction of crack propagation. We have studied how the material inhomogeneity affects the stress intensity factor and also the crack opening displacement when a Mode III type crack propagates through the inhomogeneous viscoelastic medium.

2. FORMULATION OF THE PROBLEM AND ITS SOLUTION FOR NONSTEADY CASE IN MAXWELL SOLID

Let us consider an inhomogeneous viscoelastic medium which was set in motion by a semi-infinite crack suddenly appearing at t=0 and moving with a constant velocity V in the direction of the X-axis. The Y-axis is taken perpendicular to the X-axis(fig.1). For SH-waves, the displacements along X and Y directions are zero and only the displacement $W = W(X,Y,t)$ along the Z-direction exists.

The shear modulus is

$$\mu(Y) = \mu_0 \exp(2\beta Y) \quad \text{and density} \quad \rho(Y) = \rho_0 \exp(2\beta Y),$$

where β, μ_0 and ρ_0 are constants.

The non-zero stresses are

$$\sigma_{YZ} = \sigma_{YZ}(X,Y,t) \quad \text{and} \quad \sigma_{XZ} = \sigma_{XZ}(X,Y,t) \qquad (1)$$

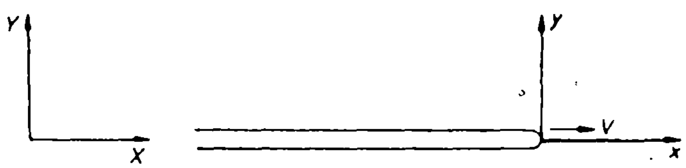

FIG. 1. The crack geometry.

and nonvanishing strains are

$$e_{XZ} = \frac{1}{2}\frac{\partial W}{\partial X} \quad , \quad e_{YZ} = \frac{1}{2}\frac{\partial W}{\partial Y} \tag{2}$$

Considering a Maxwell Solid as the viscoelastic model, the stress-strain relations are

$$\begin{aligned} \frac{\partial \sigma_{YZ}}{\partial t} + \beta_1 \sigma_{YZ} &= 2\mu(Y)\,\frac{\partial e_{YZ}}{\partial t} \\ \frac{\partial \sigma_{XZ}}{\partial t} + \beta_1 \sigma_{XZ} &= 2\mu(Y)\,\frac{\partial e_{XZ}}{\partial t} \end{aligned} \tag{3}$$

where β_1 is a positive constant.

The equation of motion has the form

$$\frac{\partial \sigma_{XZ}}{\partial X} + \frac{\partial \sigma_{YZ}}{\partial Y} = \rho(Y)\,\frac{\partial^2 W}{\partial t^2} \tag{4}$$

and the boundary conditions of the problem are

$$\begin{aligned} &W(X,0,t) = 0 \ , \quad X-Vt>0, \quad t>0 \\ &\sigma_{YZ}(X,0,t) = -\sigma H(t) \ , \quad X-Vt<0 \ , \quad t>0 \\ &\sigma_{YZ}(X,Y,t) \to 0 \ \text{ as } \ X^2+Y^2 \to \infty \end{aligned} \tag{5}$$

It is convenient to shift the origin of co-ordinates to the tip of the crack at X=Vt. New co-ordinate axes (x,y) are parallel to the respective fixed ones (X,Y).

Hence, putting x=X-Vt, y=Y, we obtain $\frac{\partial}{\partial X} = \frac{\partial}{\partial x}$, $\frac{\partial}{\partial Y} = \frac{\partial}{\partial y}$ and the time derivative transforms to $-V\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$. Equations (1), (2), (3) and (4) become

$$\sigma_{yz} = \sigma_{yz}(x,y,t) \quad \text{and} \quad \sigma_{xz} = \sigma_{xz}(x,y,t) \tag{6}$$

$$e_{xz} = \frac{1}{2}\frac{\partial}{\partial x} W(x,y,t) \quad , \quad e_{yz} = \frac{1}{2}\frac{\partial}{\partial y} W(x,y,t) \tag{7}$$

$$-V\frac{\partial\sigma_{yz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial t} + \beta_1\sigma_{yz} = \mu(y)\left[-V\frac{\partial^2 W}{\partial x\partial y} + \frac{\partial^2 W}{\partial t\partial y}\right]$$

$$-V\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{xz}}{\partial t} + \beta_1\sigma_{xz} = \mu(y)\left[-V\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial t\partial x}\right] \tag{8}$$

and

$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} = \rho(y)\left[V^2\frac{\partial^2 W}{\partial x^2} - 2V\frac{\partial^2 W}{\partial x\partial t} + \frac{\partial^2 W}{\partial t^2}\right] \tag{9}$$

The boundary conditions (5) now assume the form

$$W(x,0,t) = 0 \;, \quad x>0$$

$$\sigma_{yz}(x,0,t) = -\sigma H(t) \;, \quad x<0 \tag{10}$$

$$\sigma_{yz}(x,y,t) \to 0 \quad \text{as} \quad x^2+y^2 \to \infty$$

Let us denote the Laplace transform by a single bar

$$\bar{f} \equiv \bar{f}(x,y,p) = \int_0^\infty f(x,y,t)\exp(-pt)dt$$

and the Fourier transform by two bars

$$\bar{\bar{f}} \equiv \bar{\bar{f}}(\xi,y,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(x,y,p)\exp(i\xi x)dx$$

Applying these transforms to equations (8) and (9), we get

$$(i\xi V+p+\beta_1)\bar{\bar{\sigma}}_{yz} = \mu(y)(Vi\xi+p)\frac{d\bar{\bar{W}}}{dy} \tag{11}$$

$$(i\xi V+p+\beta_1)\bar{\bar{\sigma}}_{xz} = \mu(y)(V\xi^2-i\xi p)\bar{\bar{W}} \tag{12}$$

and
$$-i\xi\bar{\bar{\sigma}}_{xz} + \frac{d}{dy}\bar{\bar{\sigma}}_{yz} = \rho(y)(-V^2\xi^2+2Vi\xi p+p^2)\bar{\bar{W}} \tag{13}$$

Eliminating $\bar{\sigma}_{xz}$, $\bar{\sigma}_{yz}$ from equations (11), (12) and (13) we obtain

$$\frac{d^2\bar{W}}{dy^2} + 2\beta\frac{d\bar{W}}{dy} - \gamma^2\bar{W} = 0 \tag{14}$$

where

$$\gamma^2 = \xi^2 + \frac{(Vi\xi+p)(Vi\xi+p+\beta_1)}{c^2}$$

$$c^2 = \frac{\mu_0}{\rho_0} \tag{15}$$

The branches of γ are chosen so that $\mathrm{Re}(\gamma)>0$.

Since $\bar{W}$ must remain bounded as $y \to \pm\infty$, so solutions of (14) are

$$\bar{W}^{(1)} = A_1 \exp\left[-\left(\beta+\sqrt{\beta^2+\gamma^2}\right)y\right], \quad y>0 \tag{16}$$

and

$$\bar{W}^{(2)} = A_2 \exp\left[\left(-\beta+\sqrt{\beta^2+\gamma^2}\right)y\right], \quad y<0 \tag{17}$$

where $W^{(1)}$ and $W^{(2)}$ denote the displacement in the upper and lower half-plane respectively.

Let us consider on $y=0$

$$\begin{aligned}\bar{W}^{(1)} - \bar{W}^{(2)} &= h(x,p), \quad x<0\\ &= 0, \quad x>0\end{aligned} \tag{18}$$

where $h(x,p)$ is an unknown function such that

$$h(x,p) \sim o\left[\exp(k_1 x)\right] \quad \text{as} \quad x \to -\infty, \quad k_1>0.$$

Applying the Fourier transform to equation (18), we get

$$\begin{aligned}\bar{W}^{(1)} - \bar{W}^{(2)} = A_1 - A_2 &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x,p)\exp(i\xi x)\,dx\\ &= H_-(\xi,p)\end{aligned} \tag{19}$$

where $H_{-}(\xi,p)$ is an analytic function in the lower half-plane $\tau<k_1$ and $\xi=\sigma+i\tau$.

Now from equations (11), (16) and (17) we obtain

$$\bar{\bar{\sigma}}^{(1)}_{yz} = \mu(y)\,\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\,\frac{\partial}{\partial y}\,\bar{\bar{W}}^{(1)}$$

$$= -\mu(y)A_1\left(\beta+\sqrt{\beta^2+\gamma^2}\right)\exp\left[-\left(\beta+\sqrt{\beta^2+\gamma^2}\right)y\right]\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\ ,\quad y>0$$

$$\bar{\bar{\sigma}}^{(2)}_{yz} = \mu(y)A_2\left(-\beta+\sqrt{\beta^2+\gamma^2}\right)\exp\left[\left(-\beta+\sqrt{\beta^2+\gamma^2}\right)y\right]\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\ ,\quad y<0 \tag{20}$$

where $\sigma^{(1)}_{yz}$ and $\sigma^{(2)}_{yz}$ are the stresses on the upper and lower surfaces of the crack.

Since the stresses are continuous on y=0,

$$\bar{\bar{\sigma}}^{(1)}_{yz} = \bar{\bar{\sigma}}^{(2)}_{yz}$$

Using equations (20) we obtain

$$A_1 = -\frac{-\beta+\sqrt{(\beta^2+\gamma^2)}}{\beta+\sqrt{(\beta^2+\gamma^2)}}\,A_2 \tag{21}$$

Using equation (21), (19) becomes

$$H_{-}(\xi,p) = -\frac{2\sqrt{(\beta^2+\gamma^2)}}{\beta+\sqrt{(\beta^2+\gamma^2)}}\,A_2 \tag{22}$$

Again let us assume that on y=0

$$\bar{\sigma}_{yz} = \bar{\sigma}^{(1)}_{yz} = \bar{\sigma}^{(2)}_{yz} = -\frac{\sigma_0\exp(\lambda x)}{p}\ ,\quad x<0$$

$$= e(x)\ ,\quad x>0 \tag{23}$$

Here $e(x)$ is an unknown function such that

$$e(x) \sim o\left[\exp(-k_2 x)\right] \quad \text{as} \quad x \to \infty, \quad k_2>0.$$

Taking Fourier transforms of equation (23) we get

$$\bar{\sigma}_{yz}^{(2)} = \mu_0 A_2\left(-\beta+\sqrt{\beta^2+\gamma^2}\right)\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\bar{\sigma}_{yz}^{(2)}\exp(i\xi x)dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\bar{\sigma}_{yz}^{(2)}\exp(i\xi x)dx$$

$$= E_+(\xi,p) - \frac{\sigma_0}{\sqrt{2\pi}\,(\lambda+i\xi)p} \tag{24}$$

where

$$E_+(\xi,p) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\bar{\sigma}_{yz}^{(2)}\exp(i\xi x)dx \tag{25}$$

and is an analytic function in the upper half-plane $\tau>-k_2$ and $\dfrac{\sigma_0}{i\sqrt{2\pi}\,(\xi-i\lambda)p}$ is analytic in the lower half-plane $\tau<\lambda$.

From equations (22) and (24) we get

$$-\frac{\mu_0(Vi\xi+p)\gamma^2 H_-(\xi,p)}{2(Vi\xi+p+\beta_1)\sqrt{(\beta^2+\gamma^2)}} = E_+(\xi,p) - \frac{\sigma_0}{\sqrt{2\pi}\,(\lambda+i\xi)p} \tag{26}$$

It may be noted that the problem has been reduced to a form suitable for application of the Wiener-Hopf technique.

Now

$$\gamma^2 = \left(1-\frac{V^2}{c^2}\right)(\xi+iX_1)(\xi-iX_2) \tag{27}$$

where

$$X_1 = \frac{1}{2(1-V^2/c^2)}\left[\frac{(2p+\beta_1)V}{c^2} + \sqrt{\frac{(2p+\beta_1)^2V^2}{c^4} + \frac{4p(p+\beta_1)(1-V^2/c^2)}{c^2}}\right] \tag{28}$$

$$X_2 = \frac{1}{2(1-V^2/c^2)}\left[-\frac{(2p+\beta_1)V}{c^2} + \sqrt{\frac{(2p+\beta_1)^2V^2}{c^4} + \frac{4p(p+\beta_1)(1-V^2/c^2)}{c^2}}\right] \tag{29}$$

and $$\sqrt{(\beta^2+\gamma^2)} = (\xi+iY_1)^{1/2}(\xi-iY_2)^{1/2}(1-V^2/c^2)^{1/2} \tag{30}$$

where

$$Y_1 = \frac{1}{2(1-V^2/c^2)}\left[\frac{(2p+\beta_1)V}{c^2} + \sqrt{\frac{(2p+\beta_1)^2V^2}{c^4} + 4\left\{\frac{p(p+\beta_1)}{c^2} + \beta^2\right\}(1-V^2/c^2)}\right] \tag{31}$$

$$Y_2 = \frac{1}{2(1-V^2/c^2)}\left[-\frac{(2p+\beta_1)V}{c^2} + \sqrt{\frac{(2p+\beta_1)^2V^2}{c^4} + 4\left\{\frac{p(p+\beta_1)}{c^2} + \beta^2\right\}(1-V^2/c^2)}\right] \tag{32}$$

Using equations (27) and (30), (26) becomes

$$-\frac{\mu_0(1-V^2/c^2)^{1/2}(\xi-ip/V)(\xi-iX_2)H_-(\xi,p)}{2\left[\xi-i(p+\beta_1)/V\right](\xi-iY_2)^{1/2}} + \frac{\sigma_0(i\lambda+iY_1)^{1/2}}{\sqrt{2\pi}\; i(\xi-i\lambda)(i\lambda+iX_1)p}$$

$$= \frac{(\xi+iY_1)^{1/2}\,E_+(\xi;p)}{(\xi+iX_1)} - \frac{\sigma_0}{i\sqrt{2\pi}\;(\xi-i\lambda)p}\left[\frac{(\xi+iY_1)^{1/2}}{(\xi+iX_1)} - \frac{(i\lambda+iY_1)^{1/2}}{(i\lambda+iX_1)}\right] \tag{33}$$

The functions on the R.H.S. of (33) are analytic and non-zero in the upper half-plane $\tau > -k_2$, and functions on the L.H.S. are analytic and non-zero in the lower half-plane $\tau < \lambda$ $(\lambda < k_1)$. Since both the functions are analytic in the strip $-k_2 < \tau < \lambda$, the principle of an analytic continuation states that each of them represents an entire function $M(\xi)$ in the whole ξ-plane.

Now, the L.H.S. of (33) approaches zero as $|\xi| \to \infty$. It may then be concluded by Liouville's theorem that $M(\xi) = 0$ and therefore

$$H_-(\xi,p) = \frac{2\sigma_o(i\lambda+iY_1)^{1/2}(\xi-iY_2)^{1/2}[\xi-i(p+\beta_1)/V]}{\mu_o\sqrt{2\pi}\ ip(\xi-i\lambda)(i\lambda+iX_1)(\xi-iX_2)(\xi-ip/V)(1-V^2/c^2)^{1/2}} \tag{34}$$

and

$$E_+(\xi,p) = \frac{\sigma_o}{i\sqrt{2\pi}\ (\xi-i\lambda)p} - \frac{\sigma_o(i\lambda+iY_1)^{1/2}(\xi+iX_1)}{i\sqrt{2\pi}\ (\xi-i\lambda)p(i\lambda+iX_1)(\xi+iY_1)^{1/2}} \tag{35}$$

From equation (34) it follows that

$$H_-(\xi,p) = \frac{2\sigma_o(i\lambda+iY_1)^{1/2}}{\mu_o\sqrt{2\pi}\ ip(i\lambda+iX_1)(1-V^2/c^2)^{1/2}}\,\xi^{-3/2}, \quad \xi \to \infty.$$

Application of the Inverse Fourier transform yields

$$h(x,p) = \frac{4\sigma_o}{\mu_o}\sqrt{-\frac{x}{\pi}}\ \frac{1}{(1-V^2/c^2)^{1/2}}\ \frac{(\lambda+Y_1)^{1/2}}{(\lambda+X_1)p}, \quad x \to 0^- .$$

Again, taking the Inverse Laplace transform, the displacement jump across the surface of the crack near the crack tip is

$$W^{(1)} - W^{(2)} = \frac{4\sigma_o}{\mu_o}\sqrt{-\frac{x}{\pi}}\ \frac{1}{(1-V^2/c^2)^{1/2}}\ \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{(\lambda+Y_1)^{1/2}}{(\lambda+X_1)p}\,e^{pt}\,dp \tag{36}$$

From (35)

$$E_+(\xi,p) = -\frac{\sigma_o(i\lambda+iY_1)^{1/2}}{i\sqrt{2\pi}\ p(i\lambda+iX_1)}\,\xi^{-1/2}, \quad \xi \to \infty .$$

Taking the Inverse Fourier transform we obtain

$$e(x,p) = \frac{\sigma_0}{\sqrt{\pi x}} \frac{(\lambda+Y_1)^{1/2}}{p(\lambda+X_1)} , \quad x \to 0^+ .$$

Again, taking the inverse Laplace transform

$$\sigma_{yz} = \frac{\sigma_0}{\sqrt{\pi x}} \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda+Y_1)^{1/2}}{p(\lambda+X_1)} e^{pt} \, dp \qquad (37)$$

If ΔW is the displacement jump, then the crack opening displacement near the crack tip ia given by

$$\mu_0 \Delta W = 4\sigma_0 \sqrt{-\frac{x}{\pi}} \frac{1}{(1-V^2/c^2)^{1/2}} A , \quad (1 \ll x < 0) \qquad (38)$$

and the stress near the crack tip is

$$\sigma_{yz} = \frac{\sigma_0}{\sqrt{\pi x}} A , \quad (0 < x \ll 1) \qquad (39)$$

where

$$A = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda+Y_1)^{1/2}}{p(\lambda+X_1)} e^{pt} \, dp = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(Y_1)^{1/2}}{pX_1} e^{pt} \, dp , \quad \lambda \to 0$$

$$(40)$$

Evaluation of the integral A given by (40) corresponding to constant stress $-\sigma_0$ on the crack surfaces is presented in the appendix.

In the fracture mechanics, it is customary to write $\sigma_{yz} = (0^+,0,t)$ in the form $K/\sqrt{(2\pi x)}$, where K is stress intensity factor.

In our case

$$K = \sqrt{2}\, \sigma_0 A \qquad (41)$$

Putting $\beta=0$ in the expression for A, we obtain the stress intensity factor in a homogeneous viscoelastic medium as

$$K = \sqrt{2}\, \sigma_0 A_1$$

where

$$A_1 = \frac{1}{2\pi i}\sqrt{2(1-V^2/c^2)} \times$$

$$\times \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}\,dp}{p\left[(2p+\beta_1)V/c^2+\sqrt{(2p+\beta_1)^2V^2/c^4+4p(p+\beta_1)(1-V^2/c^2)/c^2}\right]}$$

which agrees with the results of Atkinson and List (1972).

3. STEADY STATE CASE FOR MAXWELL SOLID

Steady state solutions are the results of Sec.2 corresponding to the case of t approaching infinity. So for the steady state case, passing to the limit $p \to 0$ and using the Tauberian theorem we obtain from equation (34)

$$H_-(\xi,p) = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(\xi-iY_2)^{1/2}\ (\xi-i\beta_1/V)}{\mu_0\sqrt{2\pi}\ i(\xi-i\lambda)(i\lambda+iX_1)\xi^2(1-V^2/c^2)^{1/2}}$$

Applying the Inverse Fourier transform we obtain

$$W^{(1)} - W^{(2)} = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}}{\mu_0(i\lambda+iX_1)(1-V^2/c^2)^{1/2}} \times$$

$$\times \frac{1}{2\pi i}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{(\xi-iY_2)^{1/2}(\xi-i\beta_1/V)}{(\xi-i\lambda)\xi^2}\exp(-i\xi x)\ d\xi$$

$$= \frac{2\sigma_0(i\lambda+iY_1)^{1/2}}{2\pi i\mu_0(i\lambda+iX_1)(1-V^2/c^2)^{1/2}}\ I \tag{42}$$

where

$$I = \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{(\xi - iY_2)^{1/2}(\xi - i\beta_1/V)}{(\xi - i\lambda)\xi^2} \exp(-i\xi x)\ d\xi$$

For $x<0$, the above integral can be replaced with the integral taken along the positive imaginary ξ-axis round the branch point at $\xi = iY_2$, together with the contribution from the poles at $\xi=0$ and $\xi = i\lambda$ as shown in Fig.2.

Thus it can be shown that

$$I = \exp\left[-\left(\frac{\pi i}{4} + x_1 Y_2\right)\right]\left[\frac{2}{\lambda}\left(\frac{\beta_1}{\lambda V} - 1\right)\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{u+Y_2}\,du + \right.$$

$$\left. + \frac{2\beta_1}{\lambda V}\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{(u+Y_2)^2}\,du - \frac{2}{\lambda}\left(\frac{\beta_1}{\lambda V} - 1\right)\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{u+(Y_2-\lambda)}\,du\right] +$$

$$+ \frac{2\pi}{\lambda}\left[\left(\frac{\beta_1}{\lambda V} - 1\right)\sqrt{-iY_2} + \frac{\beta_1}{V}\left(x_1\sqrt{-iY_2} + \frac{1}{2\sqrt{-iY_2}}\right) - \right.$$

$$\left. - \left(\frac{\beta_1}{\lambda V} - 1\right)\sqrt{(i\lambda - iY_2)}\ \exp(-\lambda x_1)\right]$$

$$= \frac{1}{\lambda}\exp(-\pi i/4)\left[\left(\frac{\beta_1}{\lambda V} - 1\right)\sqrt{\frac{\pi}{x_1}}\ (x_1Y_2)^{-1/4}\exp(-x_1Y_2/2)\ \times\right.$$

$$\times\ W_{-3/4,1/4}(x_1Y_2) + \frac{\beta_1}{V}\sqrt{(\pi x_1)}\ (x_1Y_2)^{-3/4}\exp(-x_1Y_2/2)\ \times$$

$$\times\ W_{-5/4,-1/4}(x_1Y_2) + \left(1 - \frac{\beta_1}{\lambda V}\right)\sqrt{\frac{\pi}{x_1}}\left[x_1(Y_2-\lambda)\right]^{-1/4}\exp\left[-\frac{x_1(Y_2+\lambda)}{2}\right]\times$$

$$\left.\times\ W_{-3/4,1/4}\left[x_1(Y_2-\lambda)\right]\right] +$$

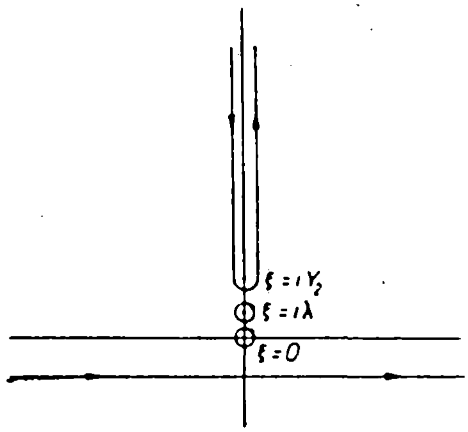

FIG. 2. The path of integration of the integral I.

$$+\ \frac{2\pi}{\lambda}\left[\left(\frac{\beta_1}{\lambda V} - 1\right)\sqrt{-iY_2} + \frac{i\beta_1}{V}\left(ix_1\sqrt{-iY_2} + \frac{1}{2\sqrt{-iY_2}}\right) - \right.$$

$$\left. - \left(\frac{\beta_1}{\lambda V} - 1\right)\sqrt{(i\lambda - iY_2)}\ \exp(-\lambda x_1)\right] \tag{43}$$

where $W_{k,m}$ is the Whittaker function (1969).

Therefore the displacement jump ΔW across the surface of the crack ($x<0$) is given by

$$\mu_0 \Delta W = -\frac{\sigma_0 (\lambda + Y_1)^{1/2}}{\pi(\lambda + X_1)(1-V^2/c^2)^{1/2}} \exp(\pi i/4)\ I \tag{44}$$

where I is given by equation (43).

Using the result that

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - m - k)} (z)^{1/2+m} \exp(-z/2) + \frac{\Gamma(2m)}{\Gamma(\frac{1}{2} + m + k)} (z)^{1/2-m} \exp(-z/2)$$

for small z,

we find that for small $(x_1 Y_2)$, equation (43) yields

$$I = -4\sqrt{(\pi x_1)}\ \exp(-\pi i/4) \tag{45}$$

Substituting the value of I from equation (45) in to equation (44) we get

$$\mu_0 \Delta W = -\frac{4\sigma_0}{(1-V^2/c^2)^{1/2}} \sqrt{-\frac{x}{\pi}}\ \frac{(\lambda + \alpha_1)^{1/2}}{(\lambda + \alpha_2)}, \quad -1 \ll x < 0 \tag{46}$$

where

$$\alpha_1 = \frac{1}{2(1-V^2/c^2)}\left[\frac{\beta_1 V}{c^2} + \sqrt{\frac{\beta_1^2 V^2}{c^4} + 4\beta^2(1-V^2/c^2)}\right] \tag{47}$$

and $$\alpha_2 = \frac{\beta_1 V}{(1-V^2/c^2)c^2} \tag{48}$$

Again, letting $p \to 0$ and using the Tauberian theorem we find from equations (35) and (24) that in the steady state case

$$\bar{\sigma}_{yz} = - \frac{\sigma_0 (i\lambda + iY_1)^{1/2} (\xi + iX_1)}{i\sqrt{2\pi}\ (\xi - i\lambda)(i\lambda + iX_1)(\xi + iY_1)^{1/2}} \tag{49}$$

Taking the inverse Fourier transform we obtain

$$\sigma_{yz} = - \frac{\sigma_0 (i\lambda + iY_1)^{1/2}}{2\pi i (i\lambda + iX_1)} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{(\xi + iX_1)\exp(-i\xi x)}{(\xi - i\lambda)(\xi + iY_1)^{1/2}}\, d\xi$$

$$= - \frac{\sigma_0 (i\lambda + iY_1)^{1/2}}{2\pi i (i\lambda + iX_1)}\, I_1 \tag{50}$$

where $\quad I_1 = \displaystyle\int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{(\xi + iX_1)\exp(-i\xi x)}{(\xi - i\lambda)(\xi + iY_1)^{1/2}}\, d\xi$

$$= 2\sqrt{\pi}\ \exp(-\pi i/4) \left[\frac{\exp(-Y_1 x)}{\sqrt{x}} - \frac{(\lambda + X_1)\sqrt{x}}{[x(\lambda + Y_1)]^{3/4}} \times \right.$$

$$\left. \times \exp\left(\frac{x(\lambda - Y_1)}{2} \right) W_{-1/4,-1/4}\left[x(\lambda + Y_1) \right] \right] \tag{51}$$

Thus the stress at y=0 for all x (x>0) is given by (50).

Now for small $(\lambda + Y_1)x$

$$I_1 = 2\sqrt{\frac{\pi}{x}}\ \exp(-\pi i/4) \tag{52}$$

so from (50) it follows that

$$\sigma_{yz} = \frac{\sigma_0}{\sqrt{\pi x}} \frac{(\lambda + \alpha_1)^{1/2}}{(\lambda + \alpha_2)}, \quad 0 < x \ll 1\ ,\ y = 0 \tag{53}$$

Stress intensity factor K is given by

$$K = \sqrt{2}\ \sigma_o B \tag{54}$$

where $$B = \frac{(\lambda+\alpha_1)^{1/2}}{(\lambda+\alpha_2)}$$

Now putting β_1=0 in the expression for α_1 and α_2 we get from (44) and (53) the displacement jump and stress intensity factor in an inhomogeneous elastic medium as

$$\mu_o \Delta W = - \frac{4\sigma_o}{\lambda(1-V^2/c^2)^{1/2}} \sqrt{-\frac{x}{\pi}} \left[\lambda + \frac{\beta}{(1-V^2/c^2)^{1/2}} \right]^{1/2} \tag{55}$$

and $$\sigma_{yz} = \frac{\sigma_o}{\lambda\sqrt{\pi x}} \left[\lambda + \frac{\beta}{(1-V^2/c^2)^{1/2}} \right]^{1/2} \tag{56}$$

which agree with the results derived by Atkinson (1977).

4. STEADY STATE SOLUTION FOR STANDARD LINEAR SOLID

In this case the stress strain relations are

$$\begin{aligned} \frac{\partial\sigma_{YZ}}{\partial t} + \beta_1\sigma_{YZ} &= 2\mu(Y)\left[\frac{\partial e_{YZ}}{\partial t} + \alpha e_{YZ} \right] \\ \frac{\partial\sigma_{XZ}}{\partial t} + \beta_1\sigma_{XZ} &= 2\mu(Y)\left[\frac{\partial e_{XZ}}{\partial t} + \alpha e_{XZ} \right] \end{aligned} \tag{57}$$

where β_1 and α are constants.

Equation of motion has the form

$$\frac{\partial\sigma_{XZ}}{\partial X} + \frac{\partial\sigma_{YZ}}{\partial Y} = \rho(Y)\ \frac{\partial^2 W}{\partial t^2} \tag{58}$$

Now, putting x=X-Vt and y=Y so that

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial x}\ ,\quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial t} = -V\frac{\partial}{\partial x}\ ,$$

equations (57) and (58) become

$$-V\frac{\partial\sigma_{yz}}{\partial x}+\beta_1\sigma_{yz}=\mu(y)\left[-V\frac{\partial^2 W}{\partial x\partial y}+\alpha\frac{\partial W}{\partial y}\right]$$

$$-V\frac{\partial\sigma_{xz}}{\partial x}+\beta_1\sigma_{xz}=\mu(y)\left[-V\frac{\partial^2 W}{\partial x^2}+\alpha\frac{\partial W}{\partial x}\right] \tag{59}$$

and

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}=\rho(y)V^2\frac{\partial^2 W}{\partial x^2} \tag{60}$$

Introducing the Fourier transform denoted by

$$\overline{f}\equiv\overline{f}(\xi,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x,y)\exp(i\xi x)dx \tag{61}$$

equations (59) and (60) can be transformed to

$$(i\xi V+\beta_1)\overline{\sigma}_{yz}=\mu(y)(Vi\xi+\alpha)\frac{d\overline{W}}{dy} \tag{62}$$

$$(i\xi V+\beta_1)\overline{\sigma}_{xz}=\mu(y)(V\xi^2-i\xi\alpha)\overline{W} \tag{63}$$

and

$$i\xi\overline{\sigma}_{xz}+\frac{d}{dy}\overline{\sigma}_{yz}=-\rho(y)V^2\xi^2\overline{W} \tag{64}$$

Eliminating $\overline{\sigma}_{xz}$, $\overline{\sigma}_{yz}$ from equations (11), (12) and (13) we obtain

$$\frac{d^2\overline{W}}{dy^2}+2\beta\frac{d\overline{W}}{dy}-\gamma^2\overline{W}=0 \tag{65}$$

where

$$\gamma^2=\frac{\xi^2\left[(1-V^2/c^2)\xi+i(V\beta_1/c^2-\alpha/V)\right]}{(\xi-i\alpha/V)} \tag{66}$$

Branches of γ are chosen so that $\mathrm{Re}(\gamma)>0$.

Since $\overline{W}$ must remain bounded as $y\to\pm\infty$, so solutions of (65) are

$$\overline{W}^{(1)} = A_1 \exp\left[-\left(\beta+\sqrt{\beta^2+\gamma^2}\right)y\right], \quad y>0$$

and (67)

$$\overline{W}^{(2)} = A_2 \exp\left[\left(-\beta+\sqrt{\beta^2+\gamma^2}\right)y\right], \quad y<0$$

where $W^{(1)}$ and $W^{(2)}$ denote the displacement in the upper and lower half-plane respectively.

Let us consider on y=0

$$W^{(1)} - W^{(2)} = h(x), \quad x<0$$
$$= 0, \quad x>0 \tag{68}$$

where h(x) is an unknown function such that

$$h(x) \sim o\left[\exp(k_1 x)\right] \quad \text{as} \quad x \to -\infty, \quad k_1>0$$

and

$$\sigma_{yz} = -\sigma_o \exp(\lambda x), \quad x<0,$$
$$= e(x), \quad x>0 \tag{69}$$

where e(x) is an unknown function satisfying the condition

$$e(x) \sim o\left[\exp(-k_2 x)\right] \quad \text{as} \quad x \to \infty, \quad k_2>0.$$

In this case equation (26) becomes

$$-\frac{\mu_o(Vi\xi+\alpha)\gamma^2 H_-(\xi)}{2(Vi\xi+\beta_1)\sqrt{(\beta^2+\gamma^2)}} = E_+(\xi) - \frac{\sigma_o}{\sqrt{2\pi}\; i(\xi-i\lambda)} \tag{70}$$

This equation holds in the region of regularity of the functions appearing in equation (71).

Owing to our former assumptions regarding the behaviour of e(x) and h(x) at infinity, this region is represented by the inequality $-k_2<\tau<\lambda<k_1$ where $\xi=\sigma+i\tau$.

Now equation (71) is suitable for the application of the Wiener-Hopf technique. Again,

$$\gamma^2 = \frac{\xi^2(1-V^2/c^2)(\xi+ia)}{(\xi - i\alpha/V)} \tag{71}$$

where

$$a = \frac{(V\beta_1/c^2 - \alpha/V)}{(1-V^2/c^2)}$$

and

$$\gamma^2 + \beta^2 = \frac{\left[\xi^3(1-V^2/c^2) + i(V\beta_1/c^2 - \alpha/V)\xi^2 + \beta^2(\xi - i\alpha/V)\right]}{(\xi - i\alpha/V)} \tag{72}$$

Since it is difficult to factorise $\sqrt{(\gamma^2+\beta^2)}$, i.e. to represent it as a product of two functions, one analytic in the upper half plane and other analytic in the lower half plane, we follow the approximate method of Koiter (1954) of solving Wiener-Hopf type equations. Accordingly, we write

$$P(\xi) = \sqrt{(\gamma^2+\beta^2)} \quad \text{in the form} \quad P(\xi) = \overline{P}(\xi)P_1(\xi) ,$$

where the function $\overline{P}(\xi)$ is required to behave at $|\xi| \to \infty$ and at $|\xi| \to 0$ in the same manner as $P(\xi)$. The auxiliary function $P_1(\xi)$ should be non-zero and should have no singularity within the strip $-k_2<-\tau_1<\tau<\tau_2<\lambda$; it has to be suitably chosen such that $P(\xi)$ is non-zero and possesses no singularity within the strip $-\tau_1<\tau<\tau_2$. Now we note that

$$P(\xi) = \sqrt{(\gamma^2+\beta^2)} \approx \left[(1-V^2/c^2)\xi^2 + i(V\beta_1/c^2 - \alpha/V)\xi\right]^{1/2} \quad \text{as} \quad |\xi| \to \infty$$

and $$\sqrt{(\tau^2+\beta^2)} \approx \beta \quad \text{as} \quad |\xi| \to 0.$$

Therfore we choose $\overline{P}(\xi)$ in the form

$$\overline{P}(\xi) = \left[(1-V^2/c^2)\xi^2 + i(V\beta_1/c^2 - \alpha/V)\xi + \beta^2\right]^{1/2} \tag{73}$$

which behaves in the same manner as $P(\xi)$ for $|\xi| \to \infty$ and $|\xi| \to 0$. Now $\overline{P}(\xi)$ can be written as

$$\overline{P}(\xi) = (1-V^2/c^2)^{1/2}(\xi - ia_2)^{1/2}(\xi + ia_1)^{1/2} \tag{74}$$

where

$$a_1 = \frac{1}{2(1-V^2/c^2)}\left[\left(\frac{V\beta_1}{c^2} - \frac{\alpha}{V}\right) + \sqrt{\left(\frac{V\beta_1}{c^2} - \frac{\alpha}{V}\right)^2 + 4\beta^2(1-V^2/c^2)}\right] \tag{75}$$

and

$$a_2 = \frac{1}{2(1-V^2/c^2)}\left[-\left(\frac{V\beta_1}{c^2} - \frac{\alpha}{V}\right) + \sqrt{\left(\frac{V\beta_1}{c^2} - \frac{\alpha}{V}\right)^2 + 4\beta^2(1-V^2/c^2)}\right] \tag{76}$$

It consequently follows that the assumptions concerning $P_1(\xi)$ are satisfied and in view of the fact that $P_1(\xi) \to 1$ in the strip $-\tau_1<\tau<\tau_2$ for $\xi \to \infty$, the function may be represented in the form

$$P_1(\xi) = P_1^+(\xi)P_1^-(\xi) \tag{77}$$

where

$$P_1^+(\xi) = \exp\left[\frac{1}{2\pi i}\int_{-\infty+id_2}^{\infty+id_2} \frac{\ln P_1(\eta)}{\eta-\xi}\,d\eta\right]$$

$$P_1^-(\xi) = \exp\left[-\frac{1}{2\pi i}\int_{-\infty+id_1}^{\infty+id_1} \frac{\ln P_1(\eta)}{\eta-\xi}\,d\eta\right] \tag{78}$$

where $-\tau_1<d_1<d_2<\tau_2$ and the functions $P_1^{\pm}(\xi)$ are regular in the respective half planes $\tau>-\tau_1$ and $\tau<\tau_2$.

It follows from (79) and from the fact $P_1(0) = P_1(\infty) = 1$ that these functions satisfy the additional condition $P_1^{\pm}(0) = P_1^{\pm}(\infty) = 1$ with the help of (71), (74), (77) and the relation

$$P(\xi) = \overline{P}(\xi)P_1(\xi),$$

equation (69) becomes

$$-\frac{\mu_o(1-V^2/c^2)^{1/2}H_-(\xi)\xi^2}{2(\xi-i\beta_1/V)(\xi-ia_2)^{1/2}P_1^-(\xi)} + \frac{\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\ i(\xi-i\lambda)(i\lambda+ia)} =$$

$$= \frac{P_1^+(\xi)(\xi+ia_1)^{1/2}E_+(\xi)}{(\xi+ia)} - \frac{\sigma_o}{\sqrt{(2\pi)}\ i(\xi-i\lambda)} \times$$

$$\times \left[\frac{P_1^+(\xi)(\xi+ia_1)^{1/2}}{(\xi+ia)} - \frac{P_1^+(i\lambda)(i\lambda+a_1)^{1/2}}{(i\lambda+ia)} \right] \tag{79}$$

Using the same arguments as in equation (33) we get

$$H_-(\xi) = -\frac{2\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}(\xi-ia_2)^{1/2}P_1^-(\xi)(\xi-i\beta_1/V)}{\mu_o\sqrt{(2\pi)}\ (\xi-i\lambda)(\lambda+a)(1-V^2/c^2)^{1/2}\xi^2}$$

$$= -\frac{2\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\mu_o\sqrt{(2\pi)}\ (\lambda+a)(1-V^2/c^2)^{1/2}}\,\xi^{-3/2} \quad \text{as} \quad \xi \to \infty \tag{80}$$

and

$$E_+(\xi) = \frac{\sigma_o}{\sqrt{(2\pi)}\ i(\xi-i\lambda)} - \frac{\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}(\xi+ia)}{\sqrt{(2\pi)}\ i(\xi-i\lambda)(i\lambda+ia)P_1^+(\xi)(\xi+ia_1)^{1/2}}$$

$$= -\frac{\sigma_o P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\ i(i\lambda+ia)}\,\xi^{-1/2} \quad \text{as} \quad \xi \to \infty \tag{81}$$

Now taking the Inverse Fourier transform we get from (80) and (81)

$$h(x) = \frac{4\sigma_o P_1^+(i\lambda)(\lambda+a_1)^{1/2}}{\sqrt{\pi}\ \mu_o(\lambda+a)(1-V^2/c^2)^{1/2}} (-x)^{1/2} , \quad -1\ll x<0 \tag{82}$$

and

$$e(x) = \frac{\sigma_o P_1^+(i\lambda)(\lambda+a_1)^{1/2}}{\sqrt{\pi}\ (\lambda+a)} (x)^{-1/2} , \quad 0<x\ll 1 \tag{83}$$

The corresponding results for the case of constant loading $\sigma_{yz} = -\sigma_o$ $(x<0)$ on the crack surface are obtained by putting $\lambda=0$ in the above equation. If ΔW is the displacement jump then the crack opening displacement in this case is given by

$$\mu_o \Delta W = \frac{4\sigma_o \sqrt{(-xa_1)}}{\sqrt{\pi}\ a(1-V^2/c^2)^{1/2}} , \quad -1\ll x<0 \tag{84}$$

and also the stress near the crack tip is

$$\sigma_{yz} = \frac{\sigma_o \sqrt{a_1}}{a\sqrt{\pi x}} , \quad 0<x\ll 1 \quad (\text{since } P_1^+(0)=1) \tag{85}$$

Therefore the stress intensity factor is equal to

$$K = \sqrt{2}\ \sigma_o\ B_1 \quad \text{where} \quad B_1 = \frac{\sqrt{a_1}}{a} \tag{86}$$

Now, putting $\alpha=0$ in equations (84) and (86) we get the crack opening displacement and stress intensity factor for the Maxwell Solid

$$\mu_o \Delta W = \frac{4\sigma_o \sqrt{(-x\alpha_1)}}{\sqrt{\pi}\ \alpha_2 (1-V^2/c^2)^{1/2}} , \quad -1\ll x<0 \tag{87}$$

and $$K = \sqrt{2}\,\sigma_0\,B \qquad \text{where} \quad B = \frac{\sqrt{\alpha_1}}{\alpha_2} \tag{88}$$

which agree with the results given by (53) and (54) in the Maxwell Solid corresponding to $\lambda=0$.

5. RESULTS AND DISCUSSION

5.1. The Maxwell Solid.

In this case time variation of the stress intensity factor is given by $K = \sqrt{2}\,\sigma_0 A$ where A is given by equation (40) and has been evaluated in the Appendix.

The dimensionless stress intensity factor $K^* = (K/\sigma_0)(\beta_1/c)^{1/2}$ has been plotted against $t_1 = \beta_1 t$ for the range of values of V/c = 0.1, 0.3, 0.5, 0.7 and 0.9 for different values of the inhomogeneity factor $\beta^* = 4\beta^2 c^2/\beta_1^2$.

It is interesting to note by inspecting the graphs given in Fig.3, Fig.4 and Fig.5 that the effect of inhomogeneity of the medium introduced through the factor β^* in the stress intensity factor K^* becomes more significant for small values of V/c, whereas for values of V/c differing slightly from unity, the effect of inhomogeneity of the medium on the stress intensity factor is negligible.

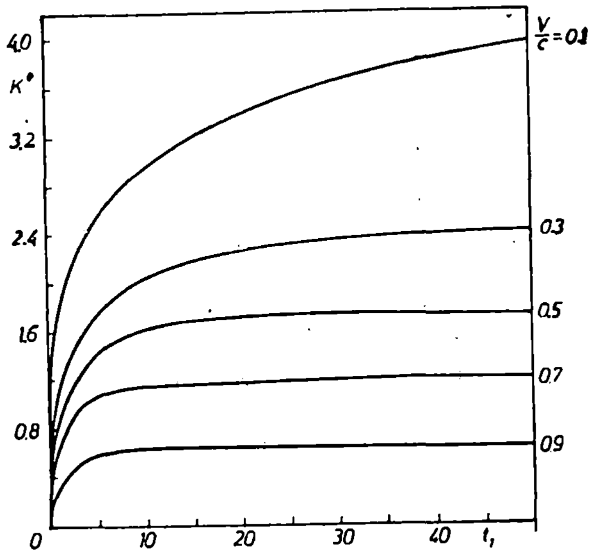

FIG. 3. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0$ (homogeneous medium).

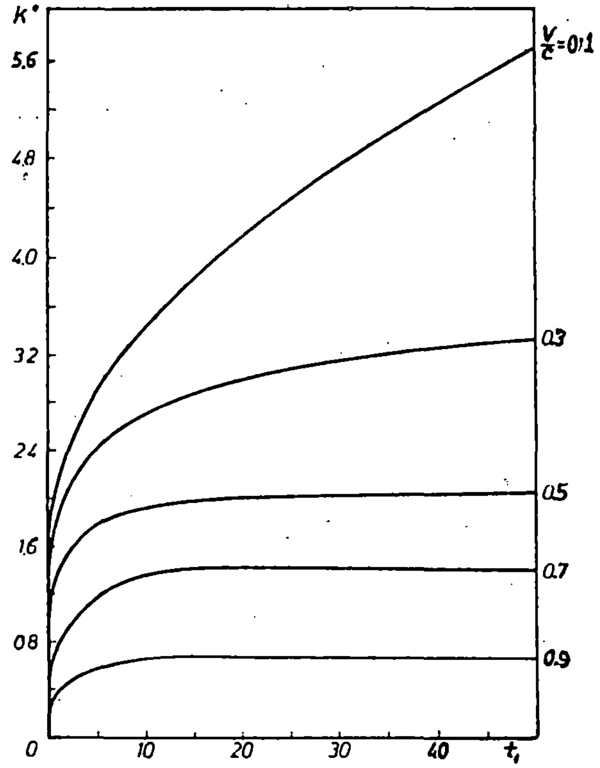

FIG. 4. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0.1$.

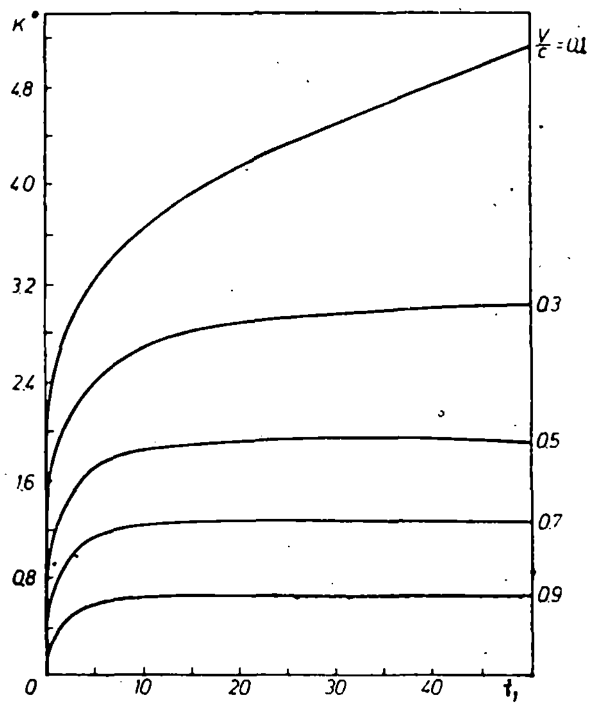

FIG. 5. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0.2$.

5.2. Standard Linear Solid.

In this case the stress intensity factor for the steadily propagating crack is given by $K = \sqrt{2}\,\sigma_0 B_1$, where B_1 is given by equation (86).

We have plotted also the stress intensity factor $K^* = (K/\sigma_0)(\beta_1/c)^{1/2}$ against β^* for various values of V/c, V/c=0.5, 0.6, 0.7, 0.8 and 0.9, and for different values of α/β_1 = 0, 0.1, 0.2. The case α/β_1=0 corresponds to the steady state values of K^* for the Maxwell solid. It is evident from the graphs given in Fig.6, Fig.7 and Fig.8 that at large values of α/β_1, values of K^* increase rapidly with the increase in values of β^* if V/c is very small. But for values of V/c close to unity the variation of K^* with the change in the value of β^* is small showing that the inhomogeneity effect is negligible in this case. This is also evident from the expressions (87) and (75).

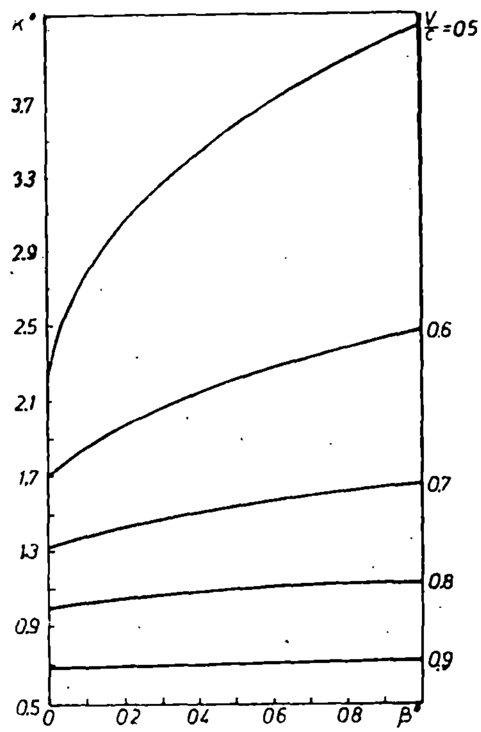

FIG. 6. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0$ (Maxwell solid).

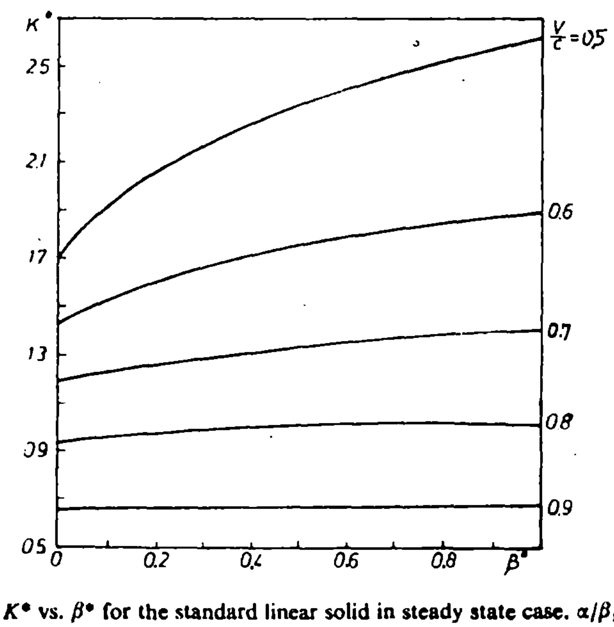

FIG. 7. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0.1$.

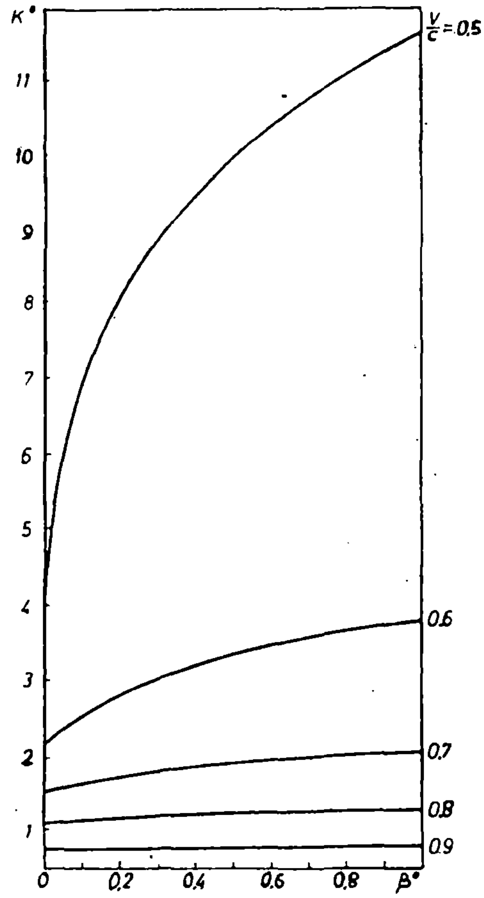

FIG. 8. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0.2$.

APPENDIX

EVALUATION OF THE INTEGRAL A IN EQUATION (40).

The integral

$$A = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{(Y_1)^{1/2}}{pX_1} e^{pt}\, dp$$

The integrand has poles at p=0 and also at $p= -\beta_1$ which correspond to the zero of X_1.

Further the integrand has branch points at

$$\delta_1 = \frac{\beta_1}{2}\left[-1 + \sqrt{(1-V^2/c^2)(1-4z)} \right]$$

$$\delta_2 = \frac{\beta_1}{2}\left[-1 - \sqrt{(1-V^2/c^2)(1-4z)} \right]$$

$$\delta_3 = \frac{\beta_1}{2}\left[-1 - \sqrt{(1-4z)} \right]$$

$$\delta_4 = \frac{\beta_1}{2}\left[-1 + \sqrt{(1-V^2/c^2)} \right]$$

$$\delta_5 = \frac{\beta_1}{2}\left[-1 - \sqrt{(1-V^2/c^2)} \right]$$

where $z = \beta^2 c^2/\beta_1^2$ which is assumed to be less than 1/4.

Evidently, $\delta_4 > \delta_1 > \delta_2 > \delta_5 > \delta_3$.

Now taking the branch cut along the negative real axis from δ_4 to

$-\infty$, the integral can be considered as a contour integral around the path shown in Fig.9.

Now,

$$A = \sqrt{2(1-V^2/c^2)} \times$$

$$\times \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{\left[(2p+\beta_1)V/c^2 + (2/c)\sqrt{(p-\delta_1)(p-\delta_2)}\right]^{1/2}}{p\left[(2p+\beta_1)V/c^2 + (2/c)\sqrt{(p-\delta_4)(p-\delta_5)}\right]} e^{pt}\, dp$$

It can be shown that

$$A = \sqrt{2(1-V^2/c^2)} \left[\frac{1}{2} \frac{c}{V} \sqrt{\frac{c}{\beta_1}} \sqrt{\frac{V}{c} + \sqrt{\frac{V^2}{c^2} + 4z(1-V^2/c^2)}} \right] +$$

$$+ \sqrt{2(1-V^2/c^2)}\, \frac{1}{\pi} \sqrt{\frac{c}{\beta_1}} \left(I_1 + I_2 - I_3 - I_4 \right)$$

where

$$I_1 = \int_0^{b_1} \frac{\sqrt{(x_1^{**})}\; y_1^*}{(\delta_4^* - r)R_1^*} \exp\left[(\delta_4^* - r)t_1\right] dr$$

$$I_2 = \int_0^{b_2} \frac{\sqrt{\left[R_2^{**} - (x_2^*)^3 + (y_2^{**})^2 x_2^* - 2x_2^* y_2^* y_2^{**} \right]/2}}{(\delta_1^* - r)R_2^*} \exp\left[(\delta_1^* - r)t_1\right] dr$$

$$I_3 = \int_{b_2}^{b_3} \frac{\sqrt{(x_3^{**})}\; x_3^*}{(\delta_1^* - r)R_3^*} \exp\left[(\delta_1^* - r)t_1\right] dr$$

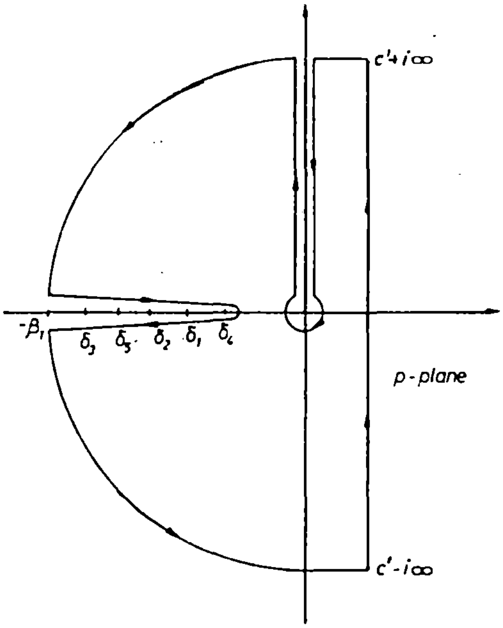

FIG. 9. The integration contour to evaluate A for the Maxwell solid.

$$I_4 = \int_{b_3}^{\infty} \frac{\sqrt{(x_3^{**})}}{(\delta_1^* - r)x_4^*} \exp\left[(\delta_1^* - r)t_1\right] dr$$

where $\delta_1 = \beta_1 \delta_1^*$, $\delta_2 = \beta_1 \delta_2^*$, $\delta_3 = \beta_1 \delta_3^*$, $\delta_4 = \beta_1 \delta_4^*$, $\delta_5 = \beta_1 \delta_5^*$, $t_1 = \beta_1 t$,

$$b_1 = \frac{1}{2} \sqrt{(1-V^2/c^2)} \left[1 - \sqrt{(1-4z)} \right]$$

$$b_2 = \sqrt{(1-V^2/c^2)(1-4z)}$$

$$b_3 = \frac{1}{2} \sqrt{(1-V^2/c^2)} \left[1 + \sqrt{(1-4z)} \right]$$

$$x_1^{**} = \left[\sqrt{(1-V^2/c^2)} - 2r \right] \frac{V}{c} + 2 \sqrt{ \left[r^2 - r\sqrt{(1-V^2/c^2)} + (1-V^2/c^2)z \right] }$$

$$x_1^* = \left[\sqrt{(1-V^2/c^2)} - 2r \right] \frac{V}{c}$$

$$y_1^* = 2 \sqrt{ r\sqrt{(1-V^2/c^2)} - r^2 }$$

$$R_1^* = (x_1^*)^2 + (y_1^*)^2$$

$$x_2^* = \left[\sqrt{(1-V^2/c^2)(1-4z)} - 2r \right] \frac{V}{c}$$

$$y_2^* = 2 \sqrt{ r\sqrt{(1-V^2/c^2)(1-4z)} - r^2 }$$

$$y_2^{**} = 2\sqrt{-r^2 + r\sqrt{(1-V^2/c^2)(1-4z)} + z(1-V^2/c^2)}$$

$$R_2^{**} = \left[(x_2^*)^2 + (y_2^{**})^2 \right]\left[(x_2^*)^2 + (y_2^*)^2 \right]^{1/2}$$

$$R_2^* = (x_2^*)^2 + (y_2^{**})^2$$

$$x_3^{**} = -\left[\sqrt{(1-V^2/c^2)(1-4z)} - 2r \right] \frac{V}{c} + 2\sqrt{r^2 - r\sqrt{(1-V^2/c^2)(1-4z)}}$$

$$x_3^* = \left[\sqrt{(1-V^2/c^2)(1-4z)} - 2r \right] \frac{V}{c}$$

$$y_3^* = 2\sqrt{z(1-V^2/c^2) + r\sqrt{(1-V^2/c^2)(1-4z)} - r^2}$$

$$R_3^* = (x_3^*)^2 + (y_3^*)^2$$

$$x_4^* = \left[\sqrt{(1-V^2/c^2)(1-4z)} - 2r \right] \frac{V}{c} -$$

$$- 2\sqrt{r^2 - z(1-V^2/c^2) - r\sqrt{(1-V^2/c^2)(1-4z)}} \,.$$

B I B L I O G R A P H Y

1. ABRAMIAN, B.L., ARUTIUNIAN, N.Kh. and BABLOOIAN, A.A. (1964) :
   On two contact problems for the elastic sphere. Prikladnaia matematika i mekhanika. Vol. 21, p. 622.
2. ABRAMIAN, B.L. (1957) :
   On the plane problem of the theory of elasticity for a rectangle. Prikladnaia matematika i mekhanika. Vol. 21, p. 89.
3. ACHENBACH, J.D. (1976) :
   Wave propagation, elastodynamic stress intensity factors and fracture. Theoritical and Applied Mechanics, ed. W.T. Koiter, Proc. 14th IUTAM congress, North-Holland publishing co., Amsterdam.
4. ACHENBACH, J.D. (1975) :
   Wave propagation in elastic solids. North-Holland publishing company, Amsterdam-Oxford. American Elsevier Publishing Company, INC., New York.
5. ACHENBACH, J.D. (1972) :
   Dynamic effects in brittle fracture. Mechanics Today Vol. 1. ed. S. Nemat-Nasser, Pergamon Press, New York.
6. ADAMS, G.G. (1978) :
   A steadily moving load on an elastic strip resting on a rigid foundation. Int. J. Engng. Sci. Vol. 16, p. 659.
7. ADAMS, G.G. (1980) :

Crack interaction in an infinite elastic strip. Int.J. Engng. Sci. Vol. 18, p. 455.

8. ALBLAS, J.B. and KUIPERS, M. (1970) :

The contact problem of a rigid cylinder rolling on a thin viscoelastic layer. Int. J. Engng. Sci. vol. 8, p. 363.

9. ALBLAS, J.B. (1974) :

On the two dimensional contact problem of a rigid cylinder pressed between two elastic half-planes. Mech. Res. communications. Vol. 1, p. 15.

10. ALBLAS, J.B. and KUIPERS, M. (1971) :

The two dimensional contact problem of a rough stamp sliding slowly on an elastic layer-I. General considerations and thick layer asymptotics. Int. J. Solids Structures. Vol. 7, p. 99.

11. ALBLAS, J.B. and KUIPERS, M. (1971) :

The two dimensional contact problem of a rough stamp sliding solwly on an elastic layer-II. Thin layer asymptotics. Int. J. Solid Structures. Vol. 7, p. 225.

12. ALEXANDROV, A. P. and LAZURKIN I. S. (1940) :

Acta Phyic Chimica U.R.S.S. Vol. 12, p. 647.

13. ANG, D.D. (1960) :

Transient motion of a line load on the surface of an elastic half space. Quart. Appl. Math. Vol. 18, p. 251.

14. ANG, D.D. and KNOPOFF, L. (1964) :

Diffraction of scalar elastic waves by a finite crack.

Proc. Nat. Acd. Sci. Vol. 51, p. 593.

15. ANGEL, Y.C. and ACHENBACH, J.D. (1985) :

Reflection and transmission of elastic waves by a periodical array of cracks. J. Appl. Mech. vol. 52, p. 33.

16. ARIN, K. and ERDOGAN, F. (1971) :

Penny-shaped crack in an elastic layer bonded to dissimilar half spaces. Int. J. Engng. Sci. Vol. 9, p. 213.

17. ATKINSON, C. and LIST, R.D. (1972) :

A moving crack problem in viscoelastic solid. Int. J. Engng. Sci. Vol. 10, p. 309.

18. ATKINSON, C. (1977) :

Cracks in media with spatially varying elastic moduli. Mechanics of Fracture. Vol. 4, Editor G. C. Sih, Noordhoff International Publishing, Leyden.

19. ATKINSON, C. (1968) :

On axially symmetric expanding boundary value problems in classical elasticity. Int. J. Engng. Sci. Vol. 6, p. 27.

20. ATKINSON, C. and LIST, R.D. (1978) :

Steady state crack propagation into media with spatially varying elastic properties. Int. j. Engng. Sci. Vol. 16, p. 717.

21. AWOJOBI, O.A. and GROOTENHUIS, P. (1965) :

Vibration of rigid bodies on semi-infinite elastic media. Proc. Roy. Soc. Vol. A223, p.521.

22. BAKER, B.R. (1962) :

Dynamic stress created by a moving crack. J. Appl. Mech. Vol 15, p. 449.

23. BORODACHEV, N.M. (1962) :

The plane contact problem for an elastic body of finite width. (in Russian). Akademiia nauk S.S.S.R. Izvestiia. Vol. 6, p. 629.

24. BORODACHEV, N.M. (1976) :

Contact problem for a stamp with a rectangular base. Prikladnaia matematika i mekhanika. Vol. 40, p. 554.

25. BREKHOVSKIKH, L.M. (1960) :

Waves in layered media. Academic Press, New York.

26. BOSE, S.K. (1968) :

The oscillatory indentation problem by a rigid circular disk. Bull. Cal. Math. Soc. Vol. 60, p. 87.

27. BOSTROM, A. (1987) :

Elastic wave scattering from an interface crack : Antiplane strain. J. Appl. Mech. Vol. 54, p. 503.

28. BOSTROM, A. and OLSSON, P. (1987) :

Scattering of elastic waves by non-planar cracks. Wave Motion. vol 9, p. 61.

29. BROCK, L.M. and DENG, Y.C. (1985) :

Dynamic studies of running half-plane and crusiform cracks. Int. J. Engng. Sci.. Vol 23, p. 163.

30. BROTHERS, P.W., SINCLAIR, G.B. and SEGEDIN, C.M. (1977) :

Uniform indentation of the elastic half-space by a rigid rectangular punch. Int. J. solids and structures. Vol. 13, p. 1059.

31. BULLEN, K.E. (1963) :

An introduction to the theory of seismology. University Press, Cambridge.

32. CAGNIARD, L. (1962) :

Rflection and refraction of progressive waves, translated and revised by E.A. Flinn and C.H.Dix. MeGraw-Hill Book Company, New York.

33. CEREPANOV, G.P. and AFANASEV, E.F. (1974) :

Some dynamic problems of the theory of elasticity-a review. Int. J. Engng. Sci. Vol. 12, p. 665.

34. CHANG, SHIH-JUNG. (1971) :

Diffraction of plane dilatational waves by a finite crack. Quart. J. Mech. Appl. Math.

35. CHEN, E.P. (1978) :

Sudden appearance of a crack in a stretched finite strip. J. Appl. Mech. vol. 45, p. 277.

36. COLE, J. and HUTH, J. (1958) :

Stress produced in a half-plane by miving loads. J. Alppl. Mech. Vol. 25, p. 43.

37. COOKE, J.C. (1963) :

Some further triple integral equation solutions. Proc. Camb. Phil. Soc. Vol. 13, p. 303.

38. COUSSY, O. (1987) :

A moving crack problem along the interface of two viscoelastic media. Int. J. Engng. Sci. Vol. 25, p. 609.

39. CRAGGS, J.W. (1966) :

The growth of a disk-shaped crack. Int. J. Engng. Sci. Vol. 4, p. 113.

40. DE, J. and PATRA, B. (1990) :

Mving Griffith crack in an orthotropic strip. Int. J. Engng. Sci. Vol. 28, p. 809.

41. DE SARKAR, P.K. (1983) :

On diffraction of SH-waves by cut in nonhomogeneous solids. Proc. Indian Acad. Science. (Math. Sci.). vol. 92, p. 41.

42. DHALIWAL, G. K. and DHALIWAL, R. S. (1978) :

On three coplanar cracks in an infinite transversely isotropic medium. Int. J. Engng. Sci. Vol. 16, p. 253.

43. DHALIWAL, R.S., SHING, B.M. and SNEDDON, I.N. (1979) :

A problem of Reissner-Sagoci type for an elastic cylinder embedded in an elastic half-space. Int. J. Engng. Sci. Vol. 16, p. 649.

44. DUNDURS, J., KIATTIKOMOL, K. and KEER, L. M. (1974) :

Contact between plates with sagged supports. American society of civil engineers. Engineering mechanics division, Journal. Vol. 100, p. 445.

45. EMBLEY, G.T. and SIH, G.C. (1971) :

Response of a penny-shaped crack to impact waves, Proceedings of the 12th Midwestern Mech. Conf. Vol. 6,

p. 473.

46. ENGLAND, A.H. (1961) :

A punch problem for a transversely isotropic layer. Proc. Camb. Phil. Soc.

47. ERDOGAN, F. (1965) :

Stress distribution in bonded dissimilar materials containing circular or ring-shaped cavities. J. Appl. Mech. Vol. 32, p. 829.

48. ERDOGAN, F. (1969) :

Approximate solutions of systems of singular integral equations. SIAM J. Appl. Mech. Vol. 17, p. 1041.

49. EWING, W.M., JARDETZKY, W.S. and PRESS, F. (1957) :

Elastic waves in layered media. McGraw-Hill Book company, New York.

50. FABRIKANT, V.I. (1991) :

Nonaxisymmetric annular punch problem. J. Appl. Mech. Vol. 58, p. 947.

51. FOX, L. and GOODWIN, E. T. (1953) :

The numerical solution of non-singular linear integral equations. Phil. Trans. Vol. A245, p.501.

52. FREUND, L.B. (1972) :

Wave motion in an elastic solid due to a non-uniformaly moving line load. Quart. Appl. Math. Vol. 30, p. 27.

53. FREUND, L.B. (1975) :

Dynamic crack propagation. The Mathematics of Fracture, ed. F. Erdogan, ASME AMD, Vol. 19, American Society of

Mechanical Engineers, New York.

54. FRUND, L.B. (1976) :

The analysis of elastodynamic crack tip stress fields. Mechanics Today, ed. S. Nemat- Nasser, Vol. III, Pergamon Press, New York.

55. GALIN, A.L. (1961) :

Contact problems in the theory of elasticity (ed. : I.N. Sneddon). North Carolina State University, Raleigh.

56. GARD, A.C. (1981) :

Stress distribution near periodic cracks at the interface of two bonded dissimilar orthotropic half-planes. Int. J. engng. Sci. Vol 19, p. 1101.

57. GEORGIADIS, H.G., THEOCARIS, P.S. and MOUSKOS, S.C. (1991) :

Plane impact of a cracked viscoelastic body. Int. J. Engng. Sci. Vol. 29, p. 171.

58. GHOSH, M. and GHOSH, M.L. (1985) :

Harmonic rocking of a rigid strip on a semi-infinite elastic medium. Indian J. Pure Appl. Math. Vol. 16, p. 938.

59. GLADWELL, G. M. L. (1968) :

Forced tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid. Int. J. Engng. Sci. Vol. 6, p. 591.

60. GLADWELL, G. M. L. (1980) :

Contact problems in the classical theory of elasticity. Sijthoff and Noordhoff, Alphen ann den Rijn.

61. GOLDEN, J.M. (1977) :

Hysteretic friction of a plane punch on a half-plane with arbitrary viscoelastic behaviour. Quart. J. Appl. Math. Vol. XXX, p. 23.

62. GOLDEN, J.M. (1979) :

The problem of a moving rigid punch on an unlubricated viscoelastic half-plane. Quart. J. Appl. Math. Vol. XXXII, p. 25.

63. GOLDEN, J. M. (1982) :

The problem of a rigid punch sliding on an elastic or a viscoelastic layer. Acta Mechanica. Vol. 43, p. 201.

64. GOLDEN, J. M. and GRAHAM, G. A. C. (1988) :

Boundary value problems in linear viscoelasticity. Spinger-Verlag, Berlin, Heidelberg, New York.

65. GOODMAN, L.E. and KEER, L.M. (1965) :

The contact stress problem for an elastic sphere indenting an elastic cavity. Int. J. of solid and Structures. Vol. 1, p. 407.

66. GRADSHTEYN, I. S. and RYZIK, I. M. (1965) :

Table of integrals, series and products. Academic Press, London, New York.

67. GRAFF, KARL F. (1973) :

Wave motion in elastic solids. Ohio State University Press.

68. GUBENKO, V.S. and MOSSAKOVSKII, V.I. (1960) :

Pressure of an axially symmetric circular die on an elastic half-space. Prikladnaia matematika i mekhanika. Vol. 24, p. 334.

69. HUDSON J. A. (1980) :
The excitation and propagation of elastic waves. Cambridge University Press, Cambridge.

70. HUNTER, S.C. (1961) :
The rolling contact of a rigid cylinder with a viscoelastic half-space. J. Appl. Mech. Vol. 4, p. 611.

71. ITOU, S. (1978) :
Dynamic stress concentration around two coplanar Griffith cracks in and infinite elastic medium. J. Appl. Mech. Vol. 4, p. 803.

72. ITOU, S. (1980) :
Diffraction of an antiplane shear wave by two coplanar Griffith cracks in an infinite elastic medium. Int. J. Solids Structures Vol. 16, p. 1142.

73. ITOU, S. (1980) :
Transient analysis of stress waves around two coplanar Griffith cracks under impact load. Engng. Fract. Mech. Vol. 13, p. 349.

74. JAIN, D.L. and KANWAL, R.P. (1972) :
Acustic diffraction of a Plane Wave by two coplanar parallel perfectly soft or rigid strips. Canadian Journal of Physics. Vol. 50, p. 928.

75. JAIN, D.L. and KANWAL, R.P. (1972) :
Dffraction of elastic waves by two coplanar and parallel

rigid strips. Int. J. Engng. Sci. Vol. 10, p. 925.

76. JAIN, D.L. and KANWAL, R.P. (1972) :
Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium. Int. J. solids Structures. Vol. 8, p. 961.

77. JAIN, D.L. and KANWAL, R.P. (1970) :
Torsional oscillations of an elastic half-space due to an annuular disk and related problems. Int. J. Engng. Sci. Vol. 8, p. 687.

78. KALKER, J.J. (1972) :
On elastic line contact, J. Appl. Mech. Vol. 39, p. 1125.

79. KALKER, J.J. (1977) :
A survey of the mechanics of contact between solid bodies. ZAMM. Vol. 57.

80. KANNINEN, M.F. (1978) :
A critical appraisal of solution techniques in dynamic fracture mechanics. Numerical Methods in Fracture Mechanics, ed. A.R. Luxmoore and D.R.J. Owen, University College of Swansea, U.K.

81. KANWAL, R. P. (1971) :
Linear integral equations, theory and techniques. Academic Press, New York.

82. KARASUDHI, P., KEER, L.M. and LEE, S.L. (1968) :
Vibratory motion of a body on an elastic half-plane. J. Appl. Mech. Vol. 35, p. 697.

83. KASSIR, M.K. and TSE, S. (1983) :

Moving Griffith crack in an orthotropic material. Int. J. Engng. Sci. Vol. 21, p. 315.

84. KASSIR, M.K. and BANDYOPADHYA, K.K. (1983) :
Impact response of a cracked orthotropic medium. J. Appl. Mech. Vol. 50, p. 630.

85. KELLER, J.B. (1958) :
"A geometrical theory of diffraction" in calculus of variations and its applications. Vol. VIII McGraw-Hill. Publishing company, New York.

86. KNOPS, R. and PAYNE, L.E. (1971) :
Uniqueness theorems in linear elasticity. Springer-verlag, New York.

87. KOITER, W.T. (1954) :
Approximate solutions of Wiener-Hopf type equations with applications, Proc. Kon. Ned. Acad. Wet. Vol. 57, p. 2.

88. KOLSKY, H. (1963) :
Stress waves in solids. Dover Publications, INC. New York.

89. KOSTROV, B.V. (1964) :
The axisymmetric problem of propagation of a tensile crack. Appl. Math. Mech. Vol. 28, p. 793.

90. KRENK, S. and SCHMIDT, H. (1982) :
Elastic wave scattering by a circular crack. Phil Trans. R. Soc. Lond. Vol. A 308, p. 167.

91. KRYLOV, V.I. and KANTOROVICH, L.V. (1957) :
Approximate methods of higher analysis. Interscience, New York.

92. LEADERMANN, H. (1943) :

The elastic and creep properties of filamentous materials and other high polymers. Textile Foundation, National Bureau of standareds, Washington.

93. LOEBER, J.F. and SIH, G.C. (1968) :

Dffraction of antiplane shear waves by finite crack. J. Accous. Soc. Am. Vol. 44, p. 90.

94. LOVE, A.E.H. (1944) :

The mathematical theory of elasticity, 4th ed. dover Publications, Inc. New York.

95. LOWENGRUB, L. and SRIVASTAVA, K. N. (1968) :

On two coplanar cracks in an infinite elastic medium. Int. J. Engng. Sci. Vol.6, p.359

96. MAL, A.K., ANG, D.D. and KNOPOFF, L. (1968) :

Diffraction of elastic waves by a rigid circular disc. Proc. Camb. Phil. Soc. Vol. 64, p. 237.

97. MAL, A.K. (1970) :

Interaction of elastic waves with a griffith crack. Int. J. Engng. Sci. Vol. 8, p. 763.

98. MAL, A.K. (1970) :

Interaction of elastic waves with a penny-shaped crack. Int. J. Engng. Sci. Vol. 8, p. 381.

99. MAL, A.K. (1972) :

A note on the low frequency diffraction of elastic waves by a Griffith crack. Int. J. Engng. Sci. Vol 10, p. 609.

100. MATCZYNSKI, M. (1973) :

Motion of a crack in antiplane strain of an elastic strip. Archives of Mechanics. Vol. 25, p. 823.

101. MATYSIAK, S. and PASZT, A. (1986) :

On interaction between a crack and rigid inclusions in an elastic space. Bull. Polish Acad. Sci, Tech. Sci. Vol. 34, p. 495.

102. MAUE, A.W. (1954) :

Die entspannuangswelle bei plotzlichen einschnitt eines gespannten elastischen korpers. ZAMM. Vol. 34, p. 1.

103 MILLAR, G.F. and PURSEY, H. (1954) :

The field and radiation impedence of mechanical radiators on the surface of semi-infinite isotropic solid.

104. MILLER, M.K. and GUY, W.T. (1966) :

Numerical inversion of the Laplace Transform by use of Jacobi polynomials. STAM J. Numerical Analysis. Vol. 3, p. 624.

105. MUSKHELISHVILI, N.I. (1953) :

Some basic problems of the mathematical theory of elasticity. English translation, J.R.M. Radok. Noordhoff international publishing, Leyden.

106. NISHIDA, Y., SHINDO, Y. and ATSUMI, A. (1984) :

Diffraction of horizontal shear waves by a moving interface creack. Acta Mechanica. Vol. 54, p. 23.

107. NOBLE, B. (1958) :

Methods based on the Wiener-Hopf Technique. Pergamon Press, New York.

108. NOBLE, B. (1962) :

Electromagnetic waves. Ed. R.E. Langer, University of Wisconsin Press.

109. NOBLE, E.B. (1963) :

Proc. Camb. Philos. Vol. 59, p. 351.

110. NOLLE, A.W. (1949) :

J.Polymer Sci. Vol. 5, p. 1.

111. OKUBO, H. (1951) :

On the two dimensional problem of a semi-infinite elastic body compressed by an elastic plane. Quqrt. J. Mech. Appl. Math. Vol. 4, p. 260.

112. OLESIAK, Z. (1965) :

Annular punch on elastic semi-space. Archives of mechanics. Vol. 17, p. 633.

113 ONDER, Y., SUHUBI, E. S. and IDEMEN, M. (1975) :

Diffraction of scalar elastic waves by a rigid half-plane between two semi-infinite media. Lett. Appl. Engng. Sci. Vol. 3 ,p. 15.

114. ONG, M.K. and SRIVASTAVA R.P. (1985) :

A crusiform crack opened by internal pressure varying with time. Int. J. Engng. Sci. Vol. 23, p. 999.

115. PALAIYA, R. M. and MUJUMDER, P. (1981) :

Interaction of antiplane shear waves by rigid strips lying at the interface of two bonded dissinilar half spaces. ZAMM. Vol. 16, p. 120.

116. PANEK, C. and KALKER, J.J. (1977) :
A solution for a narrow rectangular punch. J. Elasticity. Vol. 7, p. 213.

117. PARIHAR, K.S. and LALITHA, S. (1987) :
Cracks located on a single line in an orthotropic elastic medium in which body forces are acting. Int. J. Engng. Sci. Vol. 25, p. 735.

118. PARTON, V. Z. and MOROZOV, E. M. (1978) :
Elastic plastic fracture mechanics. Mir Publishers, Moscow.

119. PILANT, WALTER L. (1979) :
Elastic waves in the earth. Elsevier Scientific Publishing Company, Amsterdam-Oxford- New York.

120. POULOS, H.G. and DAVIS, E.H. (1968) :
The settlement behaviour of single axially-loaded piles and piers. Geotechnique. Vol. 18, p. 351.

121. PRASAD, S.N. and CHATTERJEE, S.N. (1973) :
Some mixed boundary value problems of elasticity in a rectangular domain. Int. J. Solids Structures. Vol. 9, p. 1193.

122. PRASAD, S.N. and DASGUPTA, S. (1975) :
Effect of slidding friction on contact stresses in a rectangle compressed by rigid planes. J. Appl. Mech. Vol. 42, p. 656.

123. PROTSENKO, V.S. (1974) :

The pressure on an elastic half-space due to a strip-shaped punch. Prikladnaia mekhanika. (Soviet applied Mechanics). Vol. 8, p. 45.

124. RESSNER, R. E. (1937) :

Freie und erzwungene Torsionsschwingungen den elastischen Halbraumes. Ing. Arch. Vol. 8, p.229.

125. ROBERTSON, I. A. (1966) :

Forced vertical vibration of a rigid circular disc on a semi-infinite elastic solid. Proc. Camb. Phill. Soc. Vol. 62, p. 547.

126. ROY, ARABINDA and SHABINA, F.J. (1983) :

Low frequency acoustic diffraction by a soft elliptic disk. J. Acoust. Soc. Am. Vol. 73, p. 1494.

127. ROY, ARABINDA. (1986) :

Dynamic response of elliptical footings. Int. J. Solids Structures. Vol. 22, p. 293.

128. RVACHEV, V.L. (1967) :

Studies on the contact problems of elasticity theory in the Ukraine. Prikladnaia mekhanika (Soviet applied mechanics). Vol. 3, p. 109.

129. SARKAR, J., GHOSH, M.L. and MANDAL, S.C. (1991) :

Scattering of antiplane shear wave by a propagating crack at the interface of two dissimilar elastic media. Proc. Indian Acad. Sci. (Math. Sci.). Vol. 101, p. 183.

130. SHERBY, O.D. and DORN, J.E. (1958) :

J. Mech. Phys. Solids. Vol. 6, p. 145.

131. SHINDO, Y. (1979) :

Diffraction of torsional waves by a flat annular crack in an infinite elastic medium. J. Appl. Mech. Vol. 46, p. 827.

132. SHINDO, Y. (1981) :

Sudden twisting of a flat annular crack. Int. J. Solids Structures. Vol. 17, p. 1103.

133. SHINDO, Y. (1981) :

Normal compression waves scattering at a flat annular crack in an infinite elastic solid. Quart. Appl. Math. Vol XXXIX, p. 305.

134. SHINDO, Y., NOZAKI, H. and HIGAKI, H. (1986) :

Impact response of a finite crack in an orthotropic strip. Acta Mechanica, Vol. 62, p. 87.

135. SIH, G.C. (1968) :

Some elastodynamic problems of cracks. Int. J. Fract. Mech. vol. 1, p. 51.

136. SIH, G.C. and LOEBER, J.F. (1969) :

Normal compression and radial shear waves scattering at a penny-shaped crack in an elastic solid. J. Acou. Soc. Am. vol. 46, p. 711.

137. SIH, G.C. and CHEN, E.P. (1980) :

Normal and shear impact of layered composite with a crack : Dynamic stress intensification. J. Appl. Mech. Vol. 47, p. 351.

138. SIH, G.C. and EMBLEY, G.T. (1971) :

Sudden twisting of a peny-shaped crack J. Appl. Mech. Vol. 39, p. 395.

139. SILLS, L.B. and BENVENSITE, Y. (1981) :

Steady state propagation of mode III interface crack between dissimilar viscoelastic media. Int. J. Engng. Sci. Vol. 19, p. 1255.

140. SINGH, B.M., MOODIE, T.B. and HADDOW, J.B. (1980) :

Torsion by an annular disk of an infinite cylinder embedded in an elastic half-space. Utilitas Mathematica. Vol. 18, p. 97.

141. SINGH, B. M. and DHALIWAL, R. S. (1977) :

Diffraction of SH-waves by Griffith cracks or rigid strips lying in a laterally and vertically nonhomogeneous infinite medium. Int. J. Engng. Sci. Vol. 15, p. 295.

142. SINGH, B. M., DHALIWAL, R. S. and ROKNE, J. G. (1980) :

Diffraction of antiplane shear waves by a Griffith crack or a rigid strip lying at the interface of homogeneous and nonhomogeneous elastic half-space. Acta Mechanica. Vol. 37, p. 137.

143. SLEEMAN, B.D. (1967) :

The low frequency scalar diffraction by an elliptic disc. Proc. Camb. Phil. Soc. Vol. 63, p. 1273.

144. SNEDDON, I.N. (1951) :

Fourier transform. McGraw-Hill Book company, New York.

145. SNEDDON, I.N. (1966) :

Mixed boundary value problems in potential theory. North Holland, Amsterdam.

146. SRIVASTAVA, K.N. and LOWENGRUB, M. (1968) :
Finite Hilbert transform technique for triple integral equations with trigonometrical kernels.

147. SRIVASTAVA, K.N., PALAIYA, R.M. and KARAULIA, D.S. (1980) :
Interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half-spaces. Int. J. Fracture. Vol. 16, p. 349.

148. SRIVASTAVA, K.N., GUPTA, O.P. and PALAIYA, R.M. (1981) :
Interaction of elastic waves with a Griffith crack situated in an infinitely long strip. ZAMM. Vol. 61 p. 583.

149. SRIVATAVA, K.N., PALAIYA, R.M. and KARAULIA, D.S. (1983) :
Interaction of shear waves with a Griffith crack situated in an infinitely long elastic strip. Int. J. Fracture. Vol. 21, p. 39.

150. STALLYBRASS, M.P. and SCHERER, S.E. (1976) :
Forced vertical vibration of a rigid elliptic disc on elastic half-space. Int. J. Engng. Sci. Vol 14, p. 511.

151. SUHUBI, E.S. (1972) :
Moving rough punch on an elastic half-space. Bull. Tech. Univ. Istm. vol. 25, p. 71.

152. SVE, C. and KEER, L.M. (1969) :
Indentations of an elastic layer by moving punches. Int. J. Solids and structures. Vol. 5, p. 795.

153. TAIT, R.J. and MOODIE, T.B. (1981) :

Complex variable methods and closed from solutions to dynamic crack and punch problems in the classical theory of elasticity. Int. J. Engng. Sci. Vol. 19, p. 221.

154. TRANTER, C.J. (1960) :

Some triple integral equations. Proc. Glasgo Math. Asso. Vol. 4, p. 200.

155. TRANTER, C.J. (1968) :

Bessel functions with some physical applications. The English University Press Ltd., London.

156. TRICOMI, F.G. (1951) :

Quart. J. Math. Vol. 2, p. 199.

157. THOMAS, D.P. (1965) :

Acoustic diffraction by an annular disc. Int. J. Engng. Sci. Vol. 3, p. 405.

158. UEDA, S., SHINDO, Y. and ATSUMI, A. (1983) :

Torsional impact response of a penny-shaped crack lying on a bimaterial interface. Engng. Fract. Mech. vol. 18, p. 1059.

159. UFLIAND, IA. S. (1965) :

Survay of Articles on the applications of integral transforms in the theory of elasticity. English translation, Sneddon, I.N. (ed.). North Corolina State University, Raleigh.

160. VERMA, S.K. and JAIN, D.L. (1990) :

Diffraction of obliquely incident P-waves by two parallel and coplanar Griffith cracks. J. Appl. Phys. Vol. 67, p. 6787.

161. WHITTAKER, E.T. and WATSON, G.N. (1969) :

A course of modern analysis. Cambridge University Press, London.

162. WICKHAM, G.R. (1977) :

The forced two dimensional oscillations of a rigid strip in smooth contact with a semi-infinite elastic solid. Math Proc. Camb. Phil. Soc. Vol. 81, p. 291.

163. WILLIS, J.R. (1972) :

Crack propagation in a viscoelastic media. J. Mech. Phys. Solids, Vol. 15, p. 229.

164. WILLIAMS, W.E. (1963) :

Integral equation formulation of some three part boundary value problems. Proc. Edin. Math. Soc. Vol. 13, p. 317.

165. YOFFE, E.H. (1951) :

The moving Griffith crack. Philasophical Magazine. Vol. 42, p. 739.

166. ZENER, C. (1948) :

The Elasticity and Anelasticity of Metals. Chicago University Press.

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FORCED VERTICAL VIBRATION OF TWO RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

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The problem of two-dimensional oscillations of a pair of parallel rigid strips, situated on a homogeneous isotropic semi-infinite elastic solid and forced by a specified normal component of the displacement, is considered. The resulting mixed boundary value problem is solved by the application of an integral equation method. The normal stress just below the strips and the normal displacement away from the strips are derived. By using a similar procedure, the antiplane problem due to the motion of two strips on a semi-infinite elastic medium has also been solved. Finally, graphs are presented which illustrate the salient features of the displacement and stresses in both the cases.

1. INTRODUCTION

The study of the effect of a vibrating source of pressure in different forms on the surface of an elastic medium is almost classical. Reissner [1], and later Millar and Pursey [2], treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. The problem of most physical interest occurs when a displacement corresponding to indentation by a rigid body is prescribed over a given region and the surface tractions outside the region are zero. Analytical treatment of the dynamical response of footings and soil–structure interaction are usually available in the literature only for circular and elliptical footings and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, and also in the study of the vibration of dams and large structures subjected to earthquakes. Awojobi and Grootenhuis [3], Robertson [4], Gladwell [5] and others have considered the problem of a circular footing in detail. Roy [6] considered the dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. For low frequencies, both vertical and horizontal vibration were treated. A low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium has been obtained by Karasudhi, Keer and Lee [7] by reducing the governing dual integral equations into a single inhomogeneous Fredholm equation of the second kind. Wickham [8], however, removed the flaws occurring in the above paper and worked out in detail the problem of forced two-dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium using a different technique.

To improve the dynamic models of buildings and other structures, it will be fruitful to have analytical results for foundations of more complicated nature. In what follows here the problem of vertical vibration of two rigid strips in smooth contact with a semi-infinite elastic medium is considered. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by running wheels on a railway track. The resulting mixed boundary value problem is reduced to the solution of a triple

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integral equation, which is further reduced to the solution of an integro-differential equation. Finally, an iterative solution valid for low frequency is obtained. The integral equation was solved in a manner similar to that employed by Lowengrub and Srivastav [9] in solving static problems for two coplanar cracks in an infinite elastic medium. Jain and Kanwal [10, 11] also used the same technique to solve the problem of diffraction of elastic waves by two coplanar Griffith cracks and also by two coplanar rigid strips in an infinite elastic medium. In this connection, recently Itou [12] has also solved the problem of diffraction of SH-waves by two coplanar Griffith cracks in an infinite elastic medium using a different technique.

From the solution of the integral equation, the stresses just below the strips and also the vertical displacement at points outside the strip on the free surface are found. Finally, in the limit as the distance between the strips tends to zero, the results are found to become identical with the results given by Wickham [8] for the vertical vibration of a single strip on a semi-infinite elastic medium. A low frequency solution due to anti-plane motion of two strips on a semi-infinite elastic medium is also derived.

2. FORMULATION OF THE IN-PLANE PROBLEM

Consider the normal vibration of frequency ω of two rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty<X<\infty$, $Y\geqslant 0$, $-\infty<Z<\infty$ (see Figure 1). It is assumed that the motion is forced by prescribed displacement distribution $v_0\,e^{-i\omega t}$ normal to the two infinite strips located in the region $-a\leqslant X\leqslant -b$, $b\leqslant X\leqslant a$, $Y=0$, $|Z|<\infty$, where v_0 is constant. Normalizing all lengths with respect to a and putting $b/a=c$, one finds that the rigid strips are defined by $c\leqslant|x|\leqslant 1$, $y=0$, $|z|<\infty$.

With the time factor $e^{-i\omega t}$ suppressed throughout the analysis, the displacement components can be written as

$$u(x,y)=\partial\phi/\partial x-\partial\psi/\partial y,\qquad v(x,y)=\partial\phi/\partial y+\partial\psi/\partial x,\qquad w(x,y)=0, \tag{2.1}$$

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+k_1^2\phi=0,\qquad \partial^2\psi/\partial x^2+\partial^2\psi/\partial y^2+k_2^2\psi=0, \tag{2.2}$$

in which $k_1^2=\omega^2a^2/c_1^2$ and $k_2^2=\omega^2a^2/c_2^2$. Consequently, the values of the stress components τ_{xy}, τ_{yy} and τ_{zy} are

$$\tau_{xy}=\mu[2\,\partial^2\phi/\partial x\,\partial y+\partial^2\psi/\partial x^2-\partial^2\psi/\partial y^2],$$

$$\tau_{yy}=-\mu[(k_2^2+2\,\partial^2/\partial x^2)\phi-2\,\partial^2\psi/\partial x\,\partial y],\qquad \tau_{zy}=0. \tag{2.3}$$

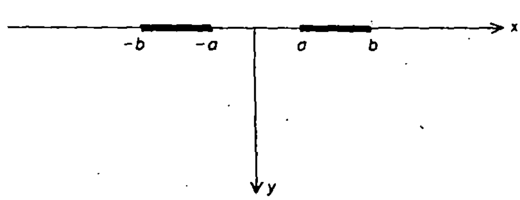

Figure 1. Geometry of the strips.

The boundary conditions are

$$v(x,0)=v_0, \qquad c\leqslant|x|\leqslant 1,$$

$$\tau_{yy}(x,0)=0, \quad |x|<c,\ |x|>1, \qquad \tau_{xy}(x,0)=0, \qquad -\infty<x<\infty. \tag{2.4}$$

The solution of the Helmholtz equation (2.2) can be written as

$$\phi=\int_{-\infty}^{\infty} A(\xi)\exp(\mathrm{i}\xi x-\gamma_1 y)\,\mathrm{d}\xi, \qquad \psi=\int_{-\infty}^{\infty} B(\xi)\exp(\mathrm{i}\xi x-\gamma_2 y)\,\mathrm{d}\xi, \tag{2.5}$$

where

$$\gamma_j=\begin{Bmatrix}(\xi^2-k_j^2)^{1/2}, & |\xi|\geqslant k_j\\ -\mathrm{i}(k_j^2-\xi^2)^{1/2}, & |\xi|\leqslant k_j\end{Bmatrix}, \qquad j=1,2, \tag{2.6}$$

and $A(\xi)$ and $B(\xi)$ are unknown functions, to be determined from the boundary conditions.

By using the last of the boundary conditions (2.4) it can be shown that

$$B(\xi)=-\{2\mathrm{i}\xi\gamma_1/(\xi^2+\gamma_2^2)\}A(\xi).$$

Then the displacements and stresses given by expressions (2.1) and (2.3) become

$$u(x,y)=\int_{-\infty}^{\infty}\mathrm{i}\xi\left[\exp(-\gamma_1 y)-\frac{2\gamma_1\gamma_2}{\xi^2+\gamma_2^2}\exp(-\gamma_2 y)\right]A(\xi)\exp(\mathrm{i}\xi x)\,\mathrm{d}\xi, \tag{2.7}$$

$$v(x,y)=\int_{-\infty}^{\infty}-\gamma_1\left[\exp(-\gamma_1 y)-\frac{2\xi^2}{\xi^2+\gamma_2^2}\exp(-\gamma_2 y)\right]A(\xi)\exp(\mathrm{i}\xi x)\,\mathrm{d}\xi, \tag{2.8}$$

$$\tau_{yy}(x,y)=-\mu\int_{-\infty}^{\infty}\left[(k_2^2-2\xi^2)\exp(-\gamma_1 y)+\frac{4\xi^2\gamma_1\gamma_2}{\xi^2+\gamma_2^2}\exp(-\gamma_2 y)\right]$$
$$\times A(\xi)\exp(\mathrm{i}\xi x)\,\mathrm{d}\xi, \tag{2.9}$$

$$\tau_{xy}(x,y)=\mu\int_{-\infty}^{\infty}2\mathrm{i}\xi\gamma_1[-\exp(-\gamma_1 y)+\exp(-\gamma_2 y)]A(\xi)\exp(\mathrm{i}\xi x)\,\mathrm{d}\xi. \tag{2.10}$$

Next, upon using the fact that $A(\xi)$ is an even function of ξ, and putting

$$P(\xi)=\{[(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2]/[2\xi^2-k_2^2]\}A(\xi),$$

the first and second of the boundary conditions (2.4) lead to the following dual integral equations in $P(\xi)$:

$$\int_0^{\infty}P(\xi)\cos\xi x\,\mathrm{d}\xi=0, \qquad |x|<c, \qquad |x|>1, \tag{2.11}$$

$$\int_0^{\infty}\frac{\gamma_1 k_2^2}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}P(\xi)\cos\xi x\,\mathrm{d}\xi=\tfrac{1}{2}v_0, \qquad c\leqslant|x|\leqslant 1. \tag{2.12}$$

3. SOLUTION OF THE IN-PLANE PROBLEM

Consider the solution of the integral equations (2.11) and (2.12) in the form

$$P(\xi)=\int_c^1 x_1 f(x_1^2)\cos \xi x_1 \, dx_1, \tag{3.1}$$

where $f(x_1^2)$ is an unknown function to be determined. The relation (2.11) is therefore satisfied automatically and equation (2.12) becomes

$$\int_c^1 x_1 f(x_1^2)\int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}\cos\xi x\cos\xi x_1 \, d\xi \, dx_1=\tfrac{1}{2}v_0, \qquad c\leqslant|x|\leqslant 1. \tag{3.2}$$

Using the relation

$$\frac{\sin\xi x\sin\xi x_1}{\xi^2}=\int_0^x\int_0^{x_1}\frac{wv\,J_0(\xi w)\,J_0(\xi v)\,dv\,dw}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}$$

converts equation (3.2) to the form

$$\frac{d}{dx}x_1 f(x_1^2)\frac{\partial}{\partial x_1}\int_0^x\int_0^{x_1}\frac{wvL_1(v,w)\,dv\,dw\,dx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}=\tfrac{1}{2}v_0, \qquad c\leqslant|x|\leqslant 1, \tag{3.3}$$

where

$$L_1(v,w)=\int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}J_0(\xi w)\,J_0(\xi v)\,d\xi. \tag{3.4}$$

By a simple contour integration technique [13], $L_1(v,w)$ can be written as

$$\begin{aligned}
L_1(v,w)=&-i\tau^2\int_0^1\frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta\\
&-4i\tau^2\int_0^\tau\frac{\eta^2(\eta^2-1)(\tau^2-\eta^2)^{1/2}\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta\\
&+\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \qquad w>v\\
=&\frac{-i\tau^2}{16(1-\tau^2)}\left[\sum_{j=0}^{2}p_j\int_0^1\frac{(1-\eta^2)^{1/2}\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{\eta^2-\tau_j^2}\,d\eta\right.\\
&\left.+\sum_{j=0}^{2}s_j\int_0^\tau\frac{(\tau^2-\eta^2)^{1/2}\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{\eta^2-\tau_j^2}\,d\eta\right]\\
&-\pi i\tau^2\left[\frac{(\eta^2-1)^{1/2}\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \qquad w>v,
\end{aligned} \tag{3.5}$$

where $\tau=k_2/k_1=c_1/c_2$, $Q_0(\eta)=(2\eta^2-\tau^2)^2-4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$ and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta)=0$. τ_1, τ_2 are the roots of the equation

$(2\eta^2-\tau^2)^2+4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}=0$. $Q_0'(\eta)$ denotes the derivative of $Q_0(\eta)$ with respect to η and

$$p_j=(2\tau_j^2-\tau^2)\Big/\prod_i(\tau_j^2-\tau_i^2), \qquad s_j=4\tau_j^2(\tau_j^2-1)\Big/\prod_i(\tau_j^2-\tau_i^2), \qquad i,j=0,1,2 \text{ and } i\neq j.$$

The corresponding expression for $L_1(v, w)$ for $w<v$ follows from equation (3.5) by interchanging w and v. For a Poisson ratio $\sigma=1/4$, the values of τ, τ_0, τ_1 and τ_2 are given by

$$\tau^2=\frac{2(1-\sigma)}{(1-2\sigma)}=3, \qquad \tau_0^2=\frac{3}{(0{\cdot}9194)^2}, \qquad \tau_1^2=\frac{3}{(2+2/\sqrt{3})} \quad\text{and}\quad \tau_2^2=3/4.$$

Hence in this case $\tau_2<\tau_1<1<\tau<\tau_0$.

By using the series expansions of J_0 and $H_0^{(1)}$ and evaluating the integrals arising in equation (3.5), one finds, after some algebraic manipulation,

$$L_1(v,w)=\begin{cases}(2/\pi)\tau^2[\{\gamma+\log(k_1w/2)-(\pi i/2)\}M+N-(P/4)(w^2+v^2)k_1^2\log k_1]+o(k_1^2), & w>v\\ (2/\pi)\tau^2[\{\gamma+\log(k_1v/2)-(\pi i/2)\}M+N-(P/4)(w^2+v^2)k_1^2\log k_1]+o(k_1^2), & w<v\end{cases}, \tag{3.6}$$

where $\gamma=0{\cdot}5772157\ldots$ is Euler's constant,

$$M=-\pi/4(1-\tau^2), \tag{3.7}$$

$$N=\frac{\pi}{32(1-\tau^2)}\left[4\log\frac{4}{\tau}+\sum_{j=1}^{2}p_j\frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j}-p_0\frac{\sqrt{(\tau_0^2-1)}}{\tau_0}\log\{\tau_0+\sqrt{(\tau_0^2-1)}\}\right.$$
$$\left.+\sum_{j=1}^{2}s_j\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j^2}\tan^{-1}\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j^2}-s_0\frac{\sqrt{(\tau_0^2-\tau^2)}}{\tau_0}\log\left\{\frac{\tau_0+\sqrt{(\tau_0^2-\tau^2)}}{\tau}\right\}\right], \tag{3.8}$$

$$P=\frac{\pi}{32(1-\tau^2)}\left[\sum_{j=0}^{2}p_j(\tfrac{1}{2}-\tau_j^2)+\sum_{j=0}^{2}s_j\left(\frac{\tau^2}{2}-\tau_j^2\right)\right]. \tag{3.9}$$

Next, differentiating both sides of the relation (3.2) with respect to x, we obtain

$$\int_c^1 x_1 f(x_1^2)\int_0^\infty\frac{\gamma_1k_2^2}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}\,\xi\sin\xi x\cos\xi x_1\,d\xi\,dx_1=0, \qquad c\leqslant|x|\leqslant1.$$

Following a procedure similar to that for deriving equation (3.3), one obtains

$$x\int_c^1\frac{x_1f(x_1^2)}{x^2-x_1^2}dx_1=\int_c^1 x_1f(x_1^2)\frac{\partial}{\partial x_1}\int_0^x\int_0^{x_1}\frac{wvL_2(v,w)\,dw\,dv\,dx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}, c\leqslant|x|\leqslant1, \tag{3.10}$$

where

$$L_2(v,w)=\int_0^\infty\left[\xi-\frac{2\gamma_1\xi^2(k_1^2-k_2^2)}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}\right]J_0(\xi w)\,J_0(\xi v)\,d\xi. \tag{3.11}$$

For small values of k_1 and k_2 such that $k_1 = o(k_2)$, one can use the contour integration technique mentioned above and obtain

$$L_2(v, w) = 2ik_1^2(1-\tau^2)\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\eta^2\, H_0^{(1)}(k_1\eta w)\, J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\, d\eta$$

$$+4ik_1^2(1-\tau^2)\int_0^\tau \frac{2\eta^4(\eta^2-1)(\tau^2-\eta^2)^{1/2}\, H_0^{(1)}(k_1\eta w)\, J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\, d\eta$$

$$-2\pi ik_1^2(1-\tau^2)\left[\frac{\eta^2(\eta^2-1)^{1/2}\, H_0^{(1)}(k_1\eta w)\, J_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \qquad w>v. \tag{3.12}$$

By a process similar to the one which led to equation (3.6), equation (3.12) can be written as

$$L_2(v, w) = -(4P/\pi)(1-\tau^2)k_1^2 \log k_1 + o(k_1^2), \tag{3.13}$$

where P is given by equation (3.9).

Now consider

$$f(x_1^2) = f_0(x_1^2) + k_1^2 \log k_1 f_1(x_1^2) + o(k_1^2). \tag{3.14}$$

Putting this expansion of $f(x_1^2)$ and the value of $L_2(v, w)$ given by equation (3.13) in equation (3.10) and equating the coefficients of equal powers of k_1 yields

$$\int_c^1 \frac{x_1 f_0(x_1^2)}{x^2-x_1^2}\, dx_1 = 0, \quad c \leqslant |x| \leqslant 1, \tag{3.15}$$

and

$$\int_c^1 \frac{x_1 f_1(x_1^2)}{x^2-x_1^2}\, dx_1 = -\frac{4P}{\pi}(1-\tau^2)\int_c^1 x_1 f_0(x_1^2)\, dx_1, \qquad c \leqslant |x| \leqslant 1. \tag{3.16}$$

Following Srivastava and Lowengrub [8] one finds the solutions of the integral equations (3.15) and (3.16) to be

$$f_0(x_1^2) = \frac{D}{(1-x_1^2)^{1/2}(x_1^2-c^2)^{1/2}}, \tag{3.17}$$

and

$$f_1(x_1^2) = \frac{4}{\pi} PD(1-\tau^2)\left[\frac{x_1^2-c^2}{1-x_1^2}\right]^{1/2} + \frac{B}{(x_1^2-c^2)^{1/2}(1-x_1^2)^{1/2}}, \tag{3.18}$$

where D and B are constants which can be calculated as follows. One substitutes the value of $L_1(v, w)$ from equation (3.6) as well as the expansion of $f(x_1^2)$ obtained from equations (3.14), (3.17) and (3.18) into equation (3.3). When the coefficients of like powers of k_1 on both sides of the resulting equation are equated the following results are obtained:

$$D = \frac{v_0}{2\tau^2[\{\gamma + \log(k_1/2) - (\pi i/2) + \log(1-c^2)^{1/2}\}M + N]}, \tag{3.19}$$

$$B = \frac{2\tau^2 D^2 P}{v_0}\left[\tfrac{1}{4}(2x^2+c^2+1) - \frac{M}{\pi}(1-\tau^2)(1-2x^2+c^2) - \frac{(1-c^2)(1-\tau^2)v_0}{\pi\tau^2 D}\right]. \tag{3.20}$$

One can now obtain the values of the vertical displacement in the plane $y=0$ from equations (2.8) and (3.1) as

$$v(x,0)=\begin{cases} v_0+2M\tau^2\left[D+k_1^2\log k_1\left\{B+\frac{2}{\pi}(1-\tau^2)(1-c^2)PD\right\}\right]\sinh^{-1}\left[\frac{c^2-x^2}{1-c^2}\right]^{1/2} \\ \qquad -\frac{4\tau^2MPD(1-\tau^2)}{\pi}k_1^2\log k_1\{(1-x^2)(c^2-x^2)\}^{1/2}+o(k_1^2), \quad |x|<c \\ v_0, \quad c\leqslant|x|\leqslant 1 \\ v_0+2M^2\left[D+k_1^2\log k_1\left\{B+\frac{2}{\pi}(1-\tau^2)(1-c^2)PD\right\}\right]\sinh^{-1}\left[\frac{x^2-1}{1-c^2}\right]^{1/2} \\ \qquad +\frac{4\tau^2MPD(1-\tau^2)}{\pi}k_1^2\log k_1\{(x^2-1)(x^2-c^2)\}^{1/2}+o(k_1^2), \quad |x|>1 \end{cases} \tag{3.21}$$

The normal stress $\tau_{yy}(x, y)$ in the plane $y=0$ just below the strips can be found from the relation (2.9) as

$$\tau_{yy}(x,0)=\frac{\pi\mu|x|}{(1-x^2)^{1/2}(x^2-c^2)^{1/2}}(D+Bk_1^2\log k_1)$$
$$+4\mu xDP(1-\tau^2)\left[\frac{x^2-c^2}{1-x^2}\right]^{1/2}k_1^2\log k_1+o(k_1^2), \qquad c\leqslant|x|\leqslant 1. \tag{3.22}$$

Now putting $c=0$ in (3.20) one can obtain the normal stress for a single strip, $|x|\leqslant 1$, $y=0$, $-\infty<z<\infty$ as

$$\tau_{yy}(x,0)=\frac{\pi\mu D}{(1-x^2)^{1/2}}+\frac{\mu}{(1-x^2)^{1/2}}k_1^2\log k_1[4P(1-\tau^2)Dx^2+\pi B]+o(k_1^2),$$

where

$$D=\frac{v_0}{2\tau^2[(\gamma+\log(k_1/2)-(\pi i/2))M+N]},$$

$$B=\frac{2\tau^2D^2P}{v_0}\left[\tfrac{1}{4}(2x^2+1)-\frac{M}{\pi}(1-\tau^2)(1-2x^2)-\frac{(1-\tau^2)v_0}{\pi\tau^2 D}\right].$$

Upon defining $\Delta_0=v_0/\pi^2D$, $\beta_0=-\tau^2/2\pi(1-\tau^2)$ and $\beta_2=-P/\pi^2$, as done by Wickham [8], one has

$$\tau_{yy}(x,0)=\frac{\mu v_0}{\pi\Delta_0(1-x^2)^{1/2}}\left\{1-\beta_2k_2^2\log k_2\left[\frac{1}{\Delta_0}+\frac{(1-2x^2)}{\beta_0}\right]\right\}+o(k_2^2),$$

which coincides with the result obtained by Wickham [8].

4. FORMULATION AND SOLUTION OF THE ANTI-PLANE PROBLEM

For an SH-wave the displacement and stress are $w(x, y, t) = w(x, y)\, e^{-i\omega t}$ and $\tau_{yz}(x, y) = \mu\, \partial w/\partial y$. As in the previous case one can write these expressions as

$$w(x, y) = \int_{-\infty}^{\infty} \frac{Q(\xi)}{\gamma_2} \exp(i\xi x - \gamma_2 y)\, d\xi, \tag{4.1}$$

$$\tau_{yz}(x, y) = -\mu \int_{-\infty}^{\infty} Q(\xi) \exp(i\xi x - \gamma_2 y)\, d\xi, \tag{4.2}$$

where $Q(\xi)$ is an unknown function to be determined from the boundary conditions, which are

$$w(x, 0) = w_0, \quad c \leqslant |x| \leqslant 1, \qquad \tau_{yz}(x, 0) = 0, \quad |x| < c,\ |x| > 1, \tag{4.3, 4.4}$$

where w_0 is a constant. By using a procedure similar to that followed for the solution of the in-plane problem, the values of stress $\tau_{yz}(x, y)$ and displacement $w(x, y)$ in the plane $y = 0$ can be found to be given by

$$\begin{aligned}\tau_{yz}(x, 0) = {} & \frac{-\pi\mu|x|}{(1-x^2)^{1/2}(x^2-c^2)^{1/2}} (D_1 + B_1 k_2^2 \log k_2) \\ & + \frac{\pi\mu|x|D_1}{2} k_2^2 \log k_2 \left[\frac{x^2-c^2}{1-x^2}\right]^{1/2} + o(k_2^2), \qquad c \leqslant |x| \leqslant 1,\end{aligned} \tag{4.5}$$

$$w(x, 0) = \left\{\begin{array}{l} w_0 - \pi[D_1 + k_2^2 \log k_2\{B_1 - D_1(1-c^2)/4\}]\sinh^{-1}\left[\dfrac{c^2-x^2}{1-c^2}\right]^{1/2} \\ \qquad - (\pi D_1/4)k_2^2 \log k_2[(1-x^2)(c^2-x^2)]^{1/2} + o(k_2^2), \qquad |x| < c \\ w_0, \qquad c \leqslant |x| \leqslant 1 \\ w_0 - \pi[D_1 + k_2^2 \log k_2\{B_1 - D_1(1-c^2)/4\}]\sinh^{-1}\left[\dfrac{x^2-1}{1-c^2}\right]^{1/2} \\ \qquad + (\pi D_1/4)k_2^2 \log k_2[(x^2-1)(x^2-c^2)]^{1/2} + o(k_2^2), \qquad |x| > 1 \end{array}\right\} \tag{4.6}$$

where

$$D_1 = \frac{w_0}{\pi[(\pi i/2) - \gamma - \log(k_2/4) - \log(1-c^2)^{1/2}]}, \tag{4.7}$$

$$B_1 = \frac{\pi D_1^2}{4w_0}\left[\frac{w_0(1-c^2)}{\pi D_1} - (1+c^2)\right]. \tag{4.8}$$

5. NUMERICAL RESULTS

The vertical and the transverse displacement fields for the in-plane and the anti-plane problems, respectively, for points near the rigid strips are illustrated graphically in Figures 2 and 3 for a Poisson solid ($\tau^2 = 3$). It is interesting to note from the graphs that the real parts of the displacements decrease with an increase in the value of k_2 in both cases.

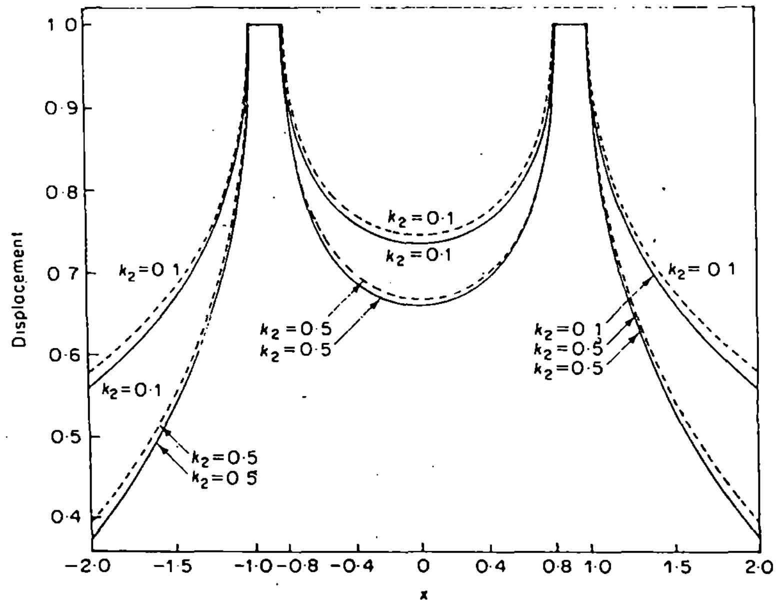

Figure 2. Displacement *vs.* distance. ——, Re $\{v(x,0)\}$ for in-plane problem; – – – –, Re $\{w(x,0)\}$ for anti-plane problem. $c=0{\cdot}8$.

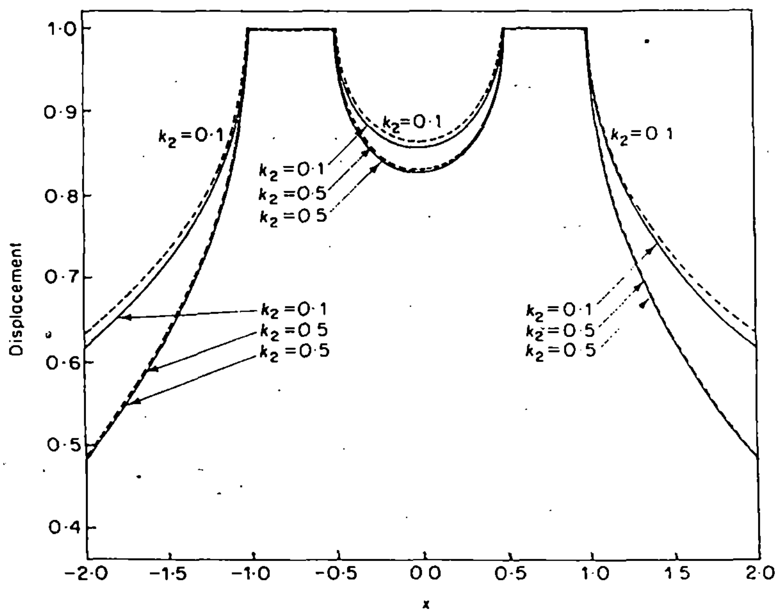

Figure 3. Displacement *vs.* distance. ——, Re $\{v(x,0)\}$ for in-plane problem; – – –, Re $\{w(x,0)\}$ for anti-plane problem. $c=0{\cdot}5$.

Graphs of the stress factors

$$\tau_1^*=\mathrm{Re}\left[\frac{\tau_{yy}\{(1-x^2)(x^2-c^2)\}^{1/2}}{\mu v_0}\right] \quad \text{and} \quad \tau_2^*=\mathrm{Re}\left[\frac{\tau_{yz}\{(1-x^2)(x^2-c^2)\}^{1/2}}{\mu w_0}\right]$$

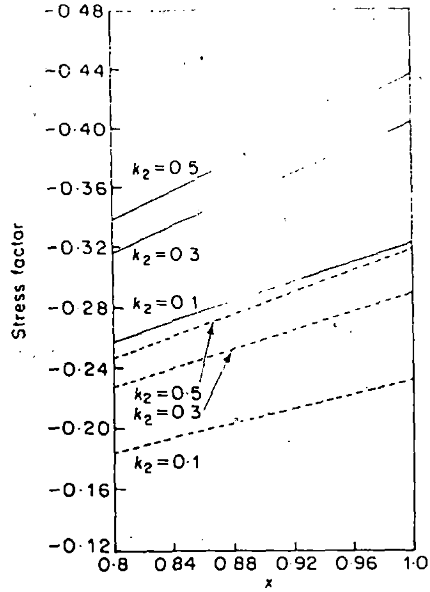

Figure 4. Stress factor *vs.* displacement. ——, τ_1^* for in-plane problem; - - -, τ_2^* for anti-plane problem. $c=0{\cdot}8$.

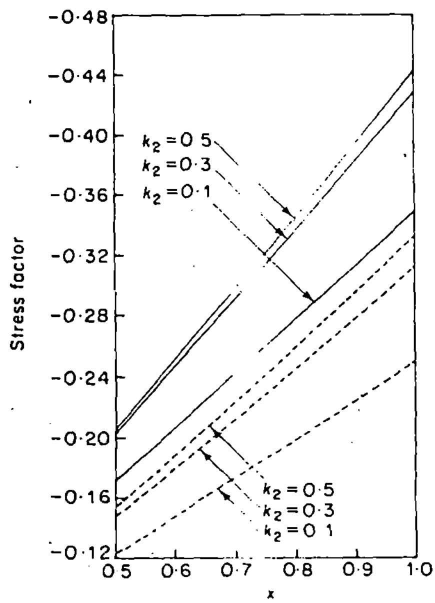

Figure 5. Stress factor *vs.* displacement. ——, τ_1^* for in-plane problem; - - -, τ_2^* for anti-plane problem. $c=0{\cdot}5$.

vs. dimensionless distance x for the in-plane and the anti-plane problems, respectively, are shown in Figures 4 and 5, plotted for points just below the rigid strips. In both the cases the magnitude of the stress factor is found to increase as one proceeds from the inner to the outer edge of the strips.

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REFERENCES

1. E. Reissner 1937 *Ingenien Archiv* **8**, 229–445. Freie und erzwungene Torsionsschwingungen den elastischen Halbraumes.
2. G. F. Miller and H. Pursey 1954 *Proceedings of the Royal Society, London* **A223**, 521–541. The field and radiation impedence of mechanical radiators on the surface of semi-infinite isotropic solid.
3. O. A. Awojobi and P. Grootenhuis 1965 *Proceedings of the Royal Society, London* **A287**, 27–63. Vibration of rigid bodies on semi-infinite elastic media.
4. I. A. Robertson 1966 *Proceedings of the Cambridge Philosophical Society* **62**, 547–553. Forced vertical vibration of a rigid circular disc on a semi-infinite elastic solid.
5. G. M. L. Gladwell 1968 *International Journal of Engineering Science* **6**, 591. Forced tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid.
6. A. Ray 1986 *International Journal of Solids and Structures* **22**, 293–309. Dynamic response of elliptical footing.
7. P. Karasudhi, L. M. Keer and S. L. Lee 1968 *Journal of Applied Mechanics* **4**, 697–705. Vibratory motion of a body on an elastic half plane.
8. G. R. Wickham 1977 *Proceedings of the Cambridge Philosophical Society* **81**, 291–311. The forced two dimensional oscillations of a rigid strip in a smooth contact with a semi-infinite elastic solid.
9. M. Lowengrub and K. N. Srivastava 1968 *International Journal of Engineering Science* **6**, 359. On two coplanar cracks in an infinite elastic medium.
10. D. L. Jain and R. P. Kanwal 1972 *International Journal of Solids and Structures* **8**, 961–975. Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium.
11. D. L. Jain and R. P. Kanwal 1972 *International Journal of Engineering Science* **10**, 961–975. Diffraction of elastic waves by two coplanar and parallel rigid strips.
12. S. Itou 1980 *International Journal of Solids and Structures* **16**, 1147–1153. Diffraction of an antiplane shear wave by two coplanar Griffith cracks in an infinite elastic medium.
13. M. Ghosh and M. L. Ghosh 1985 *Indian Journal of Pure and Applied Mathematics* **16**(8), 938–955. Harmonic rocking of a rigid strip on a semi-infinite elastic medium.

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MOVING PUNCH ON A VISCOELASTIC SEMI-INFINITE MEDIUM

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We study the problem of a semi-infinite punch moving on the free surface of a semi-infinite viscoelastic medium and producing horizontal shear waves. The mixed boundary value problem has been solved by the use of integral transforms and the Wiener-Hopf technique for both the steady and non-steady cases. Two types of viscoelastic models, viz., the Maxwell solid and the standard linear solid have been considered. Solutions have been derived in close form in both the cases and graphs have been presented to bring out the salient features of the problem.

1. INTRODUCTION

Problems involving the motion of a punch on the surface of an elastic half-space or on the free boundaries of long strips are extremely important in view of their application in road construction technology and also in geophysical research. Punch problems within the classical theory of elasticity have been studied extensively by Galin[1] and by Gladwell[2] in their books. The motion of a rough punch on an elastic half-space has been treated in detail by Suhubi[3]. Recently problems involving antiplane motion due to punches moving along the surfaces of an elastic strip have been solved by complex variable methods by Tait and Moodie[4]. An analytical solution to the problem of a long rigid punch moving rapidly on a strip of a highly orthotropic elastic layer has been solved by Georgiadis[5] using integral transforms and the Wiener-Hopf techniques[6].

However, natural or artificial materials have generally disipative behaviour which often can be taken into account by viscoelastic models. Accordingly, problems involving the motion of a punch on a viscoelastic medium have drawn the attention of many scientists. The problem of a rigid cylinder rolling on the surface of a viscoelastic half space has been solved by Hunter[7]. The contact problem of rigid cylinder rolling slowly on a thin viscoelastic layer has been treated by Alblas and Kuipers[8] assuming that the layer thickness is small compared to the width of the contact region of the cylinder. The problem of a plane punch sliding without friction on a viscoelastic half space has been considered by Golden[9].

In the present paper, we have examined the stress and displacement field produced by a long punch moving on the boundary of a semi-infinte viscoelastic medium and producing Horizontal Shear waves. Two types of viscoelastic models viz. Maxwell solid and Standard Linear Solid have been considered and loading is assumed to be such that Mode III conditions prevail. The mathematical technique which is employed here consists of the application of integral transforms and the solution of the resulting Wiener-Hopf equations for the transformed unknown variables. Both the steady and nonsteady solutions of the problem have been derived. Displacement and stress on the free surface and at points below the punch have been derived analytically and the nature of their variations with the velocity of the moving punch has been shown by means of graphs.

2. Formulation of the Problem and Its Solution for Steady State Motion

Let us consider a semi-infinite viscoelastic medium which was set into motion by a semi-infinite rigid punch moving with a constant velocity v in the direction of the x-axis. The y-axis is taken vertically downwards into the medium (Fig. 1).

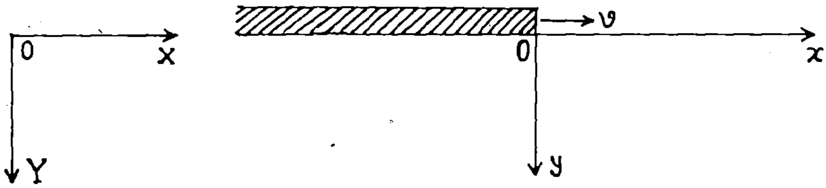

Fig. 1

For horizontal shear waves, the displacements along X and Y directions are zero and only the displacement $W = W(X, Y, t)$ along z-direction exists. The stresses under the punch are

$$\sigma_{13} = \sigma_{13}(X, Y, t) \text{ and } \sigma_{23} = \sigma_{23}(X, Y, t). \qquad \ldots(1)$$

The non-vanishing strains are

$$e_{13} = \frac{1}{2}\frac{\partial W}{\partial X} \text{ and } e_{23} = \frac{1}{2}\frac{\partial W}{\partial Y}. \qquad \ldots(2)$$

Considering a 'standard linear solid' as the viscoelastic model, the stress strain relations are

$$\frac{\partial \sigma_{i3}}{\partial t} + \beta\sigma_{i3} = 2\mu\left(\frac{\partial e_{i3}}{\partial t} + \alpha\, e_{i3}\right), \; i = 1, 2 \qquad \ldots(3)$$

where α, β are positive constants and μ is the instantaneous elastic modulus of rigidity of the material.

The equation of motion is

$$\frac{\partial \sigma_{13}}{\partial X} + \frac{\partial \sigma_{23}}{\partial Y} = \rho\,\frac{\partial^2 W}{\partial t^2} \qquad \ldots(4)$$

where ρ is the density of the material.

The boundary conditions of the problem are

$$\left.\begin{aligned} W(X, 0, t) &= w_0 \quad , \quad X - vt < 0 \\ W(X, \infty, t) &= 0 \quad , \quad -\infty < X < \infty \\ \sigma_{23}(X, 0, t) &= 0 \quad , \quad X - vt > 0 \end{aligned}\right\} \qquad \ldots(5)$$

Since we are going to investigate the steady state propagation of a punch, it is convenient to define a moving co-ordinate system (x, y) whose origin coincides with the tip of the punch and whose axes are parallel to the fixed (X, Y) axes, respectively (Fig. 1)

Hence putting $x = X - vt$, $y = Y$ eqns. (1) to (4) become respectively

$$\sigma_{13} = \sigma_{13}(x, y) \quad , \quad \sigma_{23} = \sigma_{23}(x, y) \qquad \ldots(6)$$

$$e_{13} = \frac{1}{2}\frac{\partial W}{\partial x}(x, y) \quad , \quad e_{23} = \frac{1}{2}\frac{\partial W}{\partial y}(x, y) \qquad \ldots(7)$$

$$\left.\begin{aligned} -v\frac{\partial \sigma_{13}}{\partial x} + \beta\sigma_{13} &= \mu\left(-v\frac{\partial^2 W}{\partial x^2} + \alpha\frac{\partial W}{\partial x}\right) \\ -v\frac{\partial \sigma_{23}}{\partial x} + \beta\sigma_{23} &= \mu\left(-v\frac{\partial^2 W}{\partial x \partial y} + \alpha\frac{\partial W}{\partial y}\right) \end{aligned}\right\} \qquad \ldots(8)$$

and

$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial y} = \rho v^2 \frac{\partial^2 W}{\partial x^2}. \qquad \ldots(9)$$

The boundary conditions (5), now become

$$\left.\begin{aligned} W(x, 0) &= w_0, \quad x < 0 \\ W(x, \infty) &= 0, \quad -\infty < x < \infty \\ \sigma_{23}(x, 0) &= 0, \quad x > 0. \end{aligned}\right\} \qquad \ldots(10)$$

Now introduce Fourier transform

$$\bar{f}(\xi, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x, y)\, e^{i\xi x}\, dx$$

so that $$f(x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(\xi, y)\, e^{-i\xi x}\, dx.$$

Taking Fourier transform of (8) and (9) we get

$$(i\xi v + \beta)\,\bar{\sigma}_{13} = \mu\,(\xi^2 v - i\xi\alpha)\,\bar{W} \qquad \ldots(12)$$

$$(i\xi v + \beta)\,\bar{\sigma}_{23} = \mu\,(i\xi v + \alpha)\,\frac{d\bar{W}}{dy} \qquad \ldots(13)$$

and

$$i\xi\bar{\sigma}_{13} + \frac{d\,\bar{\sigma}_{23}}{dy} = -\rho v^2\xi^2\,\overline{W}. \qquad ...(14)$$

Eliminating $\bar{\sigma}_{13}$, $\bar{\sigma}_{23}$ from (12), (13) and (14) we obtain,

$$\frac{d^2\overline{W}}{dy^2} - \gamma^2\overline{W} = 0 \qquad ...(15)$$

where

$$\gamma^2 = \frac{\xi^2}{\left(\xi - \frac{i\alpha}{v}\right)}\left[\left(1 - \frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right],\ c^2 = \frac{\mu}{\rho}. \qquad ...(16)$$

The branches of γ are so chosen that

$$\mathrm{Re}\,(\gamma) > 0 \quad \text{for} \quad -a < \mathrm{Im}\,(\xi) < 0$$

where $$a = \left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right). \qquad ...(17)$$

Now the solution of eqn. (15) bounded as $y \to \infty$ is

$$\overline{W}\,(\xi, y) = B\,(\xi)\,e^{-\gamma y} \qquad ...(18)$$

Let us consider

$$W\,(x, 0) = w_0 = W_0\,e^{\epsilon x},$$

$x < 0$, $\epsilon > 0$ and ϵ will be made to tend to zero finally ...(19)

$$= W_0\,p\,(x) \quad \text{(say)}\ ,\ x > 0 \qquad ...(19)$$

$$\sigma_{23}\,(x, 0) = 0\ ,\ x > 0$$

$$= W_0\,t\,(x) \quad \text{(say)}\ ,\ x < 0 \qquad ...(20)$$

where $p\,(x)$ and $t\,(x)$ are unknown functions such that

$$p\,(x) \sim O\,(e^{-k_1 x}) \quad \text{as} \quad x \to \infty,\ k_1 > 0$$

$$t\,(x) \sim O\,(e^{+k_2 x}) \quad \text{as} \quad x \to -\infty,\ k_2 > 0.$$

Taking Fourier transform of (19)

$$\overline{W}\,(\xi, 0) = \frac{W_0}{\sqrt{2\pi}\,(\epsilon + i\xi)} + \frac{W_0}{\sqrt{2\pi}}\,P_+\,(\xi) \qquad ...(21)$$

where

$$P_+(\xi) = \int_0^\infty p\,(x)\,e^{i\xi x}\,dx,\ (\xi = \sigma + i\,\tau). \qquad ...(22)$$

In (21) the first term on the right-hand side is analytic in the lower half plane $\mathrm{Im}\,(\xi) = \tau < \epsilon$ and $P_+(\xi)$ is analytic in the upper half plane $\tau > -k_1$ ($k_1 < a$, say). Again taking Fourier transforms of (20)

$$\bar{\sigma}_{23}(\xi, 0) = \frac{W_0}{\sqrt{2\pi}} T_-(\xi) \qquad \ldots(23)$$

where $$T_-(\xi) = \int_{-\infty}^{0} t(x)\, e^{i\xi x}\, dx. \qquad \ldots(24)$$

$T_-(\xi)$ is analytic in the lower half plane $\tau < k_2$. Therefore, $\bar{W}(\xi, 0)$ is analytic for $-k_1 < \tau < \epsilon$ and $\bar{\sigma}_{23}(\xi, 0)$ is analytic in the lower half plane $\tau < k_2$.

From (13), $$\left[(i\xi\nu + \beta)\,\bar{\sigma}_{23}\right]_{y=0} = \left[\mu\,(i\xi\nu + \alpha)\,\frac{d\bar{W}}{dy}\right]_{y=0}$$

Using (18), (21) and (23) this becomes

$$T_-(\xi) = -H(\xi)\left[P_+(\xi) - \frac{i}{\xi - i\epsilon}\right] \qquad \ldots(25)$$

where $$H(\xi) = \frac{\mu\left(\xi - \dfrac{i\alpha}{\nu}\right)^{1/2}}{\left(\xi - \dfrac{i\beta}{\nu}\right)}\,\xi.\left[\left(1 - \frac{\nu^2}{c^2}\right)\xi + i\left(\frac{\nu\beta}{c^2} - \frac{\alpha}{\nu}\right)\right]^{1/2}. \qquad \ldots(26)$$

It may be noted that the problem has been reduced to a form suitable for the application of the Wiener-Hopf technique. Now $H(\xi)$ can be written as

$$H(\xi) = H_+(\xi)\, H_-(\xi) \qquad \ldots(27)$$

where

$$H_+(\xi) = \mu\left[\left(1 - \frac{\nu^2}{c^2}\right)\xi + i\left(\frac{\nu\beta}{c^2} - \frac{\alpha}{\nu}\right)\right]^{1/2} \qquad \ldots(28)$$

and

$$H_-(\xi) = \frac{\left(\xi - \dfrac{i\alpha}{\nu}\right)^{1/2}\xi}{\left(\xi - \dfrac{i\beta}{\nu}\right)}. \qquad \ldots(29)$$

$H_+(\xi)$ is analytic in the upper half plane $\tau > -a$ and

$H_-(\xi)$ is analytic in the lower half plane $\tau < 0$.

Applying well known Wiener-Hopf technique we get,

$$T_-(\xi) = i\mu\left[i\,\frac{\nu\beta}{c^2} - \frac{\alpha}{\nu}\right]^{1/2}\frac{\left(\xi - \dfrac{i\alpha}{\nu}\right)^{1/2}}{\xi - \dfrac{i\beta}{\nu}} \quad \text{as } \epsilon \to 0.$$

So from (23)

$$\bar{\sigma}_{23}(\xi, 0) = \frac{W_0}{\sqrt{2\pi}} \cdot i\mu \left[i\left(\frac{\beta v}{c^2} - \frac{\alpha}{v} \right) \right]^{1/2} \cdot \frac{\left(\xi - \frac{i\alpha}{v} \right)^{1/2}}{\xi - \frac{i\beta}{v}} .$$

Therefore for $x < 0$

$$\sigma_{23}(x, 0) = \frac{iW_0\mu}{\sqrt{2\pi}} \left[i\left(\frac{\beta v}{c^2} - \frac{\alpha}{v} \right) \right]^{1/2} \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{v} \right)^{1/2}}{\left(\xi - \frac{i\beta}{v} \right)} e^{-i\xi x} \, d\xi, \qquad \text{...(30)}$$

Considering a branch cut along the positive imaginary axis starting from $\xi = i\alpha/v$ and changing the path of integration from real ξ–axis to the path around the branch cut it can easily be shown that the integral

$$I = \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{v} \right)}{\left(\xi - \frac{i\beta}{v} \right)} e^{-i\xi x} \, d\xi \qquad \text{(assuming } \beta > \alpha)$$

can be converted to the following integral

$$I = 2e^{\frac{\pi i}{4}} e^{-\frac{\alpha}{v} x_1} \,\fint_0^{\infty} \frac{u^{1/2} e^{-ux_1}}{u - \left(\frac{\beta}{v} - \frac{\alpha}{v} \right)} du \qquad \text{...(31)}$$

where x has been replaced by $-x_1$, $\fint_0^{\infty}$ denotes the principal value of the integral. For large values of $(\beta - \alpha)\, x_1/v = mx_1$, where $m = (\beta - \alpha)/v$ the integral (36) can be evaluated in the form

$$I = \frac{-2e^{\frac{\pi i}{4}} e^{-\frac{\alpha}{v} x_1}}{x_1^{1/2}} \left[\frac{\Gamma(3/2)}{mx_1} + \frac{\Gamma(5/2)}{m^2 x_1^2} + \frac{\Gamma(7/2)}{m^3 x_1^3} + \dots \right] \qquad \text{...(32)}$$

and for small values of mx_1 it can be shown that

$$I = 2e^{\pi i/4} e^{-\alpha x_1/v} \sqrt{\frac{\pi}{x_1}}. \qquad \text{...(33)}$$

Therefore using (30) and (31) we obtain for $x < 0$,

$$\sigma_{23}(x, 0) = -\frac{W_0\,\mu}{\pi} \left(\frac{\beta v}{c^2} - \frac{\alpha}{v} \right)^{1/2} e^{-\alpha x_1/v} \int_{\infty}^{\infty} \frac{u^{1/2} e^{-ux_1}}{\left| u - \left(\frac{\beta}{v} - \frac{\alpha}{v} \right) \right|} du, \; x < 0. \qquad \text{...(34)}$$

Using the value of the integral arising in (34) by (33), we get for small values of mx_1,

$$\sigma_{23}(x, 0) = \frac{-W_0 \mu}{\sqrt{\pi m x_1}} \left[m \left(\frac{\beta v}{c^2} - \frac{\alpha}{v} \right) \right]^{1/2} e^{-\alpha x_1 / v}, \; x_1 \to 0^+. \quad \text{...(35)}$$

Also with the help of (34) and (32), for large values of mx_1 $(x < 0)$

$$\sigma_{23}(x, 0) = \frac{W_0 \mu \, e^{-\alpha x_1 / v}}{\pi \sqrt{m x_1}} \left[m \left(\frac{\beta v}{c^2} - \frac{\alpha}{v} \right) \right]^{1/2}$$

$$\left[\frac{\Gamma(3/2)}{m x_1} + \frac{\Gamma(5/2)}{m^2 x_1^2} + \frac{\Gamma(7/2)}{m^3 x_1^3} + \dots \right]. \quad \text{...(36)}$$

Now from (21) we get as $\epsilon \to 0$

$$\bar{W}(\xi, 0) = \frac{-iW_0}{2\pi} \sqrt{ia} \, \frac{1}{\xi (\xi + ia)^{1/2}}, \; a = \left(\frac{v\beta}{c^2} - \frac{\alpha}{v} \right) / \left(1 - \frac{v^2}{c^2} \right).$$

Taking inverse Fourier transform

$$W(x, 0) = -\frac{iW_0}{2\pi} \sqrt{ia} \int_{-\infty - id}^{\infty - id} \frac{e^{-i\xi x}}{\xi (\xi + ia)^{1/2}} \, d\xi, \; x > 0 \; (0 < d < a). \quad \text{...(37)}$$

Transforming the integral in (37) to an integral along the contour around the branch cut from $-ia$ to $-i\infty$, it can be shown that

$$W(x, 0) = \frac{-iW_0}{\pi} \sqrt{ia} \, e^{-ax} e^{\pi i/4} x^{1/2} \int_0^\infty \frac{e^{-U} U^{-1/2}}{U + ax} \, dU \; (x > 0)$$

which can be written as

$$W(x, 0) = \frac{W_0}{\sqrt{\pi}} e^{-\frac{1}{2} ax} (ax)^{-1/4} W_{-1/4, -1/4}(ax), \; (x > 0) \quad \text{...(38)}$$

where $W_{k,m}$ is the Whittaker function[10].

Therefore, for small values of ax $(x > 0)$ we obtain from (38)

$$W(x, 0) = W_0 e^{-ax} - \frac{2W_0}{\sqrt{\pi}} e^{-ax} (ax)^{1/2}, \; x \to 0^+ \quad \text{...(39)}$$

and for large values of ax $(x > 0)$

$$W(x, 0) = \frac{W_0}{\sqrt{\pi}} \frac{e^{-ax}}{\sqrt{ax}}, \; x \to \infty \; (x > 0). \quad \text{...(40)}$$

3. Steady State Solution for Maxwell Solid

For 'Maxwell solid' the stress-strain relations obtained from (3) putting $\alpha = 0$ are

$$\frac{\partial \sigma_{i3}}{\partial t} + \beta \sigma_{i3} = 2\mu \frac{\partial e_{i3}}{\partial t}, \; i = 1, 2. \qquad \ldots(41)$$

The stress can be found by putting $\alpha = 0$ in (34) as (for $x < 0$, $y = 0$)

$$\sigma_{23}(x, 0) = -\frac{W_0 \mu}{\pi} \left(\frac{\beta}{c} \cdot \frac{v}{c}\right)^{1/2} \int_0^\infty \frac{e^{-ux_1} u^{1/2}}{(u - \beta/v)} du. \qquad \ldots(42)$$

For small values of $\frac{\beta}{v} x$, $x < 0$, putting $\alpha = 0$ in (35) we get

$$\sigma_{23}(x, 0) = \frac{-W_0 \mu}{\sqrt{\pi}} \left(\frac{\beta}{c} \cdot \frac{v}{c}\right)^{1/2} \frac{(\beta/v)^{1/2}}{\sqrt{x_1 \beta/v}} \qquad \ldots(43)$$

$$x_1 \to 0^+, \; (x_1 = -x).$$

Again for large values of $\beta x/v$, $(x < 0)$, from (36)

$$\sigma_{23}(x, 0) = \frac{W_0 \mu}{\sqrt{\pi}} \left(\frac{\beta}{c} \cdot \frac{v}{c}\right)^{1/2} \frac{(\beta/v)^{1/2}}{\sqrt{x_1 \beta/v}}$$

$$\times \left[\frac{v}{\beta x_1} \Gamma\left(\frac{3}{2}\right) + \frac{v^2}{\beta^2 x_1^2} \Gamma\left(\frac{5}{2}\right) + \frac{v^3}{\beta^3 x_1^3} \Gamma\left(\frac{7}{2}\right) + \ldots\right]. \qquad \ldots(44)$$

Putting $\alpha = 0$ the displacement on the free surface $(y = 0, x > 0)$ is obtained from (38) as

$$W(x, 0) = \frac{W_0}{\sqrt{\pi}} e^{-\frac{1}{2}kx} (kx)^{-\frac{1}{4}} \cdot W_{-\frac{1}{4}, -\frac{1}{4}}(kx), \; x > 0 \qquad \ldots(45)$$

where $k = \left(\frac{\beta}{c} \cdot \frac{v}{c}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right)$

which for small values of $kx > 0$ becomes by help of (39)

$$W(x, 0) = W_0 e^{-kx} - \frac{2W_0}{\sqrt{\pi}} e^{-kx} (kx)^{\frac{1}{2}}, \; x \to 0^+ \qquad \ldots(46)$$

and for large values of $kx > 0$, using (40) we obtain

$$W(x, 0) = \frac{W_0}{\sqrt{\pi}} \frac{e^{-kx}}{\sqrt{kx}}, \; x \to \infty, \; (x > 0). \qquad \ldots(47)$$

4. Solution of the Problem for Non Steady State Motion

In this case it is assumed that at time $t = 0$ a semi-infinite punch starts to move with a constant velocity v at $X = Y = 0$ on the surface of the semi-infinite viscoelastic medium.

The 'standard linear solid' is taken as the viscoelastic model. Shifting the origin at $X=vt$ and putting $X - vt=x$ and $Y=y$ so that $\frac{\partial}{\partial X} = \frac{\partial}{\partial x}$, $\frac{\partial}{\partial Y} = \frac{\partial}{\partial y}$ and time derivative equal to $-v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ the stress-displacement relations given by (8) become in this case

$$\left.\begin{aligned} -v\frac{\partial\sigma_{13}}{\partial x} + \frac{\partial\sigma_{13}}{\partial t} + \beta\sigma_{13} &= \mu\left(-v\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial t\,\partial x} + \alpha\frac{\partial W}{\partial x}\right) \\ -v\frac{\partial\sigma_{23}}{\partial x} + \frac{\partial\sigma_{23}}{\partial t} + \beta\sigma_{23} &= \mu\left(-v\frac{\partial^2 W}{\partial x\,\partial y} + \frac{\partial^2 W}{\partial t\,\partial y} + \alpha\frac{\partial W}{\partial y}\right)\end{aligned}\right\} \quad \ldots(48)$$

Both these equations can be reduced to ordinary differential equations by the application of the Laplace transform over t and the Fourier transform over x.

Let us denote the Laplace transform by a single bar

$$\bar{f} \equiv \bar{f}(x, y, p) = \int_0^\infty e^{-pt} f(x, y, t)\, dt \ldots(49)$$

and Fourier transform by two bars

$$\bar{\bar{f}} \equiv \bar{\bar{f}}(\xi, y, p) = \int_{-\infty}^{\infty} e^{i\xi t}\bar{f}(x, y, p)\, dx \ldots(50)$$

Applying these transforms to (48) we get

$$(i\xi v + p + \beta)\,\bar{\bar{\sigma}}_{13} = \mu\,(v\xi^2 - i\xi p - i\xi\alpha)\,\bar{\bar{W}} \quad \ldots(51)$$

$$(i\xi v + p + \beta)\,\bar{\bar{\sigma}}_{23} = \mu\,(vi\xi + p + \alpha)\frac{d\bar{\bar{W}}}{dy} \quad \ldots(52)$$

Now the equation of motion given by (4) becomes

$$\frac{\partial\sigma_{13}}{\partial x} + \frac{\partial\sigma_{23}}{\partial y} = \rho\left(v^2\frac{\partial^2 W}{\partial x^2} - 2v\frac{\partial^2 W}{\partial x\partial t} + \frac{\partial^2 W}{\partial t^2}\right)$$

which after taking Laplace and Fourier transforms takes the form

$$-i\xi\,\bar{\bar{\sigma}}_{13} + \frac{d\bar{\bar{\sigma}}_{23}}{dy} = \rho\,(-v^2\xi^2 + 2\,vi\xi p + p^2)\,\bar{\bar{W}}. \quad \ldots(53)$$

Substituting for $\bar{\bar{\sigma}}_{13}$ and $\bar{\bar{\sigma}}_{23}$ from (51) and (52) in (53) we have

$$\frac{d^2\bar{\bar{W}}}{dy^2} - \gamma^2\,\bar{\bar{W}} = 0 \quad \ldots(54)$$

where $$\gamma^2 = \frac{1}{(Vi\xi+p+\alpha)}\left\{\xi^2(Vi\xi+p+\alpha)+\frac{\rho}{\mu}(Vi\xi+p)^2(Vi\xi+p+\beta)\right\}. \quad ...(55)$$

The branches of γ are defined by Re $(\gamma) > 0$.
Since the stresses are bounded as $y \to \infty$, $W(x, y, t)$ and hence also $\bar{\bar{W}}(\xi, y, p)$ must remain bounded as $y \to \infty$. Hence, the solution of eqn. (54) is given by

$$\bar{\bar{W}}(\xi, y, p) = A(\xi, p)\, e^{-\gamma y}.$$

Now the boundary conditions are

$$\left.\begin{aligned} W(x, 0, t) &= W_0 H(t), && x < 0 \\ W(x, \infty, t) &= 0, && -\infty < x < \infty \\ \sigma_{23}(x, 0, t) &= 0, && x > 0. \end{aligned}\right\} \quad ...(56)$$

Taking Laplace transforms with respect to t, these conditions become

$$\left.\begin{aligned} \bar{W}(x, 0, p) &= \frac{W_0}{p}, \quad x < 0 \\ \bar{\sigma}_{23}(x, 0, p) &= 0, \quad x > 0. \end{aligned}\right\} \quad ...(57)$$

Let us consider

$$\left.\begin{aligned} \bar{W}(x, 0, p) &= W_0\, p(x) \quad \text{(say)}, x > 0 \\ \text{and} \quad \bar{\sigma}_{23}(x, 0, p) &= \mu W_0\, t(x) \text{ (say)}, x < 0. \end{aligned}\right\} \quad ...(58)$$

The functions $p(x)$ and $t(x)$ are such that

$$p(x) \sim O(e^{-k_1 x}) \text{ as } x \to \infty, \quad k_1 > 0$$

and $$p(x) \sim O(e^{-k_2 x}) \text{ as } x \to -\infty, \quad k_2 > 0.$$

Taking Fourier transform of (58) and (59) we obtain

$$\bar{\bar{W}}(\xi, 0, p) = \frac{W_0}{ip\xi} + W_0 P_+(\xi) \quad ...(59)$$

where $$P_+(\xi) = \int_0^\infty e^{i\xi x} p(x)\, dx, \quad (\xi = \sigma + i\tau)$$

and $$\bar{\sigma}_{23}(\xi, 0, p) = \mu W_0 T_-(\xi) \quad ...(60)$$

where $$T_-(\xi) = \int_{-\infty}^0 e^{i\xi x} t(x)\, dx.$$

The integral of $\bar{\bar{W}}(\xi, 0, p)$ over $(-\infty, 0)$ converges if and only if Im $(\xi) = \tau < 0$ and integral over $(0, \infty)$ converages if $\tau > -k_1$. $\bar{\sigma}_{23}$ is analytic over $(-\infty, 0)$ if $\tau < k_2$.

Now (52) becomes with the help of (56), (59) and (60)

$$\frac{(Vi\xi+p+\beta)\ T_-(\xi)}{(Vi\xi+p+\alpha)\ \gamma} = \frac{1}{ip\xi} + P_+(\xi). \qquad \ldots(61)$$

In this form of equation Wiener-Hopf technique can easily be applied.

5. Non Steady State Solution for Maxwell Solid

For general α and β, γ does not readily factorize. Expressions for the roots of $\gamma = 0$ can be obtained but these are difficult to handle. We discuss here the case of the Maxwell solid, where $\alpha = 0$.

In this case γ^2 reduces to

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)\left\{\xi^2 + \frac{vi\xi}{c^2} \cdot \frac{2p+\beta}{\left(1-\frac{v^2}{c^2}\right)} + \frac{p\,(p+\beta)}{c^2\left(1-\frac{v^2}{c^2}\right)}\right\},\ c^2 = \frac{\mu}{\rho}. \qquad \ldots(62)$$

$$\text{Hence}\quad \gamma = \sqrt{\left(1-\frac{v^2}{c^2}\right)}\,(\xi+iX_1)^{1/2}\,(\xi-iX_2)^{1/2},\ \text{Re}\ X_1,\ X_2 > 0. \qquad \ldots(63)$$

$$\text{where}\ X_1 = \frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[\frac{v\,(2p+\beta)}{c^2} + \frac{2}{c}\sqrt{p\,(p+\beta)+\frac{v^2\beta^2}{4c^2}}\right]$$

$$X_2 = \frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[\frac{-v\,(2p+\beta)}{c^2} + \frac{2}{c}\sqrt{p\,(p+\beta)+\frac{v^2\beta^2}{4c^2}}\right]. \qquad \ldots(64)$$

Branches are chosen so that $\gamma \to +\infty$ as $\xi \to \pm\infty$.

Thus for a Maxwell solid, (61) can be written after simplification as

$$-\left(1-\frac{v^2}{c^2}\right)^{-1/2}\frac{\left(\xi - i\left(\frac{p+\beta}{v}\right)\right)}{\left(\xi-\frac{ip}{v}\right)} \cdot \frac{T_-(\xi)}{(\xi-iX_2)^{1/2}} - \frac{(iX_1)^{1/2}}{i\xi p}$$

$$= \frac{(\xi+iX_1)^{1/2}-(iX_1)^{1/2}}{ip\xi} + (\xi+iX_1)^{1/2}\,P_+(\xi). \qquad \ldots(65)$$

Applying the Wiener-Hopf technique we get

$$P_+(\xi) = \frac{(iX_1)^{1/2}}{ip\xi\ (\xi+iX_1)^{1/2}} - \frac{1}{ip\xi} \qquad \ldots(66)$$

and $$T_-(\xi) = -\left(1-\frac{v^2}{c^2}\right)^{1/2} \frac{(iX_1)^{1/2}}{ip\xi} \cdot \frac{\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{1/2}}{\left(\xi - \frac{i\,(p+\beta)}{v}\right)}. \qquad \ldots(67)$$

Therefore, $\bar{\bar{W}}(\xi, 0, p)$ given in (59), with the help of (66), takes the form

$$\bar{\bar{W}}(\xi, 0, p) = \frac{W_0\,(iX_1)^{1/2}}{ip\xi\,(\xi+iX_1)^{1/2}}. \qquad \ldots(68)$$

Taking inverse transforms, one gets

$$W(x, 0, t) = \frac{1}{2\pi i}\,\frac{W_0}{i} \int_{c'-i\infty}^{c'+i\infty} \frac{(iX_1)^{1/2}\,e^{pt}}{p}\,dp \times$$

$$\times \frac{1}{2\pi} \int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}}{\xi\,(\xi+iX_1)^{1/2}}\,d\xi,$$

$$0 < d < k_1,\ x > 0. \qquad \ldots(69)$$

Taking the path of integration around the branch cut along negative imaginary axis from $-iX_1$ to $-i\infty$ the integral

$$I = \int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}}{\xi\,(\xi+iX_1)^{1/2}}\,d\xi$$

can be converted to the integral

$$I = 2e^{\pi i/4}\,e^{-X_1 x}\,(x)^{1/2} \int_0^{\infty} \frac{e^{-U}U^{-1/2}}{U+xX_1}\,dU$$

which is finally evaluated as

$$I = 2\sqrt{\pi}\,e^{\pi i/4}\,e^{-\frac{1}{2}xX_1}.(x)^{1/2}\,(xX_1)^{-3/4}\,W_{-\frac{1}{4},-\frac{1}{4}}(xX_1).$$

Putting this value of the integral in (69) we obtain

$$W(x, 0, t) = \frac{W_0}{\sqrt{\pi}} \cdot \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}\,e^{-\frac{1}{2}xX_1}}{p}\,(xX_1)^{-1/4}\,W_{-\frac{1}{4},-\frac{1}{4}}(xX_1)\,dp,\ x>0. \qquad \ldots(70)$$

Now for small p,

$$X_1 = \left(\frac{v}{c} \cdot \frac{\beta}{c}\right) / \left(1 - \frac{v^2}{c^2}\right) \doteq k \text{ (say)} \qquad \ldots(71)$$

and for large p, $X_1 = \dfrac{p}{c - V}$

therefore for large $\frac{px}{c-V}$, $W_{-\frac{1}{4},-\frac{1}{4}}\left(p\,\frac{x}{c-V}\right) \sim \exp\left(-\frac{1}{2}\,\frac{px}{c-V}\right)\left(\frac{px}{c-V}\right)^{-\frac{1}{4}}$.

...(72)

So in eqn. (70), putting the value of X_1 for small p given by (71), we obtain for large time t,

$$W(x, 0, t) = \frac{W_0}{\sqrt{\pi}}\, e^{-\frac{1}{2}kx}\, (kx)^{-\frac{1}{4}}\, W_{-\frac{1}{4},-\frac{1}{4}}(kx) \qquad ...(73)$$

which is same as the result for the steady state case for all $x > 0$ given by (45).

For large p, i.e. for small time t and for all finite x such that $px/(c-V)$ is large, using (72) we obtain from (70)

$$W(x, 0, t) = \frac{2W_0}{\pi}\sqrt{\frac{c-V}{x}}\left(t - \frac{x}{c-V}\right)^{\frac{1}{2}} H\left(t - \frac{x}{c-V}\right). \qquad ...(74)$$

Now, using (67), (60) becomes

$$\bar{\bar{\sigma}}_{23}(\xi, 0, p) = \frac{-\mu W_0\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} (iX_1)^{\frac{1}{2}}\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{\frac{1}{2}}}{i\xi p\left(\xi - \frac{i(p+\beta)}{v}\right)}.$$

After taking inverse transforms it converts to

$$\sigma_{23}(x, 0, t) = \frac{-\mu W_0\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p}(iX_1)^{\frac{1}{2}}\, dp.\, \frac{1}{2\pi i}$$

$$\times \int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{\frac{1}{2}}}{\xi\left(\xi - \frac{i(p+\beta)}{v}\right)}\, d\xi. \qquad ...(75)$$

Evaluating the integral with respect to ξ for small x, it can be shown that

$$\sigma_{23}(x, 0, t) = -\frac{\mu W_0 (1 - v^2/c^2)^{\frac{1}{2}}}{\sqrt{\pi x_1}} \cdot B, \text{ as } -x = x_1 \to 0^+ \qquad ...(76)$$

(for all $t > 0$)

where

$$B = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{(X_1)^{\frac{1}{2}} e^{pt}}{p}\, dp. \qquad ...(77)$$

The evaluation of B for all t ($t > 0$) has been done in the appendix.

For small p i.e., for large t, using from (71) the result that $X_1 \sim k$,

we have $$B = \sqrt{k} = \sqrt{\left(\frac{\beta}{c} \cdot \frac{v}{c}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right)}. \qquad \ldots(78)$$

Substituting the value of B given by (78) in (77) we obtain

$$\sigma_{23}(x, 0) = -\frac{\mu W_0 \left(\frac{\beta}{c} \cdot \frac{v}{c}\right)^{1/2}}{\sqrt{\pi x_1}}, \; x_1 \to 0^+$$

which is same as the result for the steady state case given by (43). The variation of the nondimensional values of B given by $B^* = (1 - v^2/c^2)^{1/2} \times B\sqrt{c/\beta}$ has been plotted against nondimensional time $t_1 = \beta t$ for various values of $v/c = 0.5, 0.7$ and 0.9 and has been shown by means of the following graphs (Fig. 2)

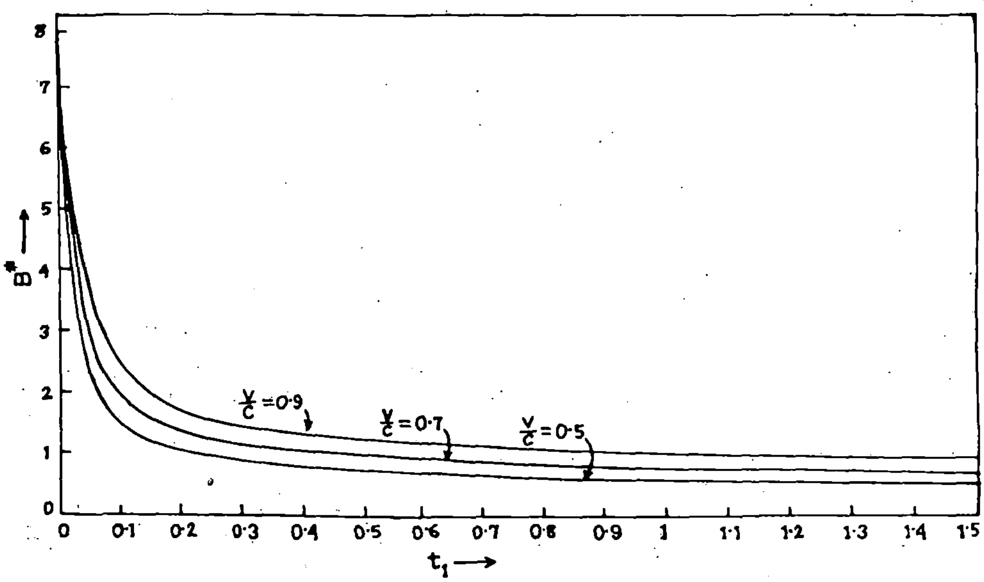

FIG. 2

Now for all values of x i.e., for general value of ξ the integral

$$\frac{1}{2\pi i} \int_{-\infty - id}^{\infty - id} \frac{e^{-i\xi x}\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{1/2}}{\xi\left[\xi - \frac{i(p+\beta)}{v}\right]} d\xi$$

appearing in eqn. (75) can be converted to the integral

$$I_1 = e^{\frac{3\pi i}{4}} \cdot \frac{p}{p+\beta} (X_2)^{1/2} + \frac{e^{\pi i/4} e^{X_2 x}}{\pi i} \int_0^{\infty} \frac{e^{ux} u^{1/2}\left(u + X_2 - \frac{p}{v}\right)}{(u+X_2)\left(u + X_2 - \frac{p+\beta}{v}\right)} du,$$

$$\left(\because \frac{p+\beta}{v} > X_2\right) \qquad \ldots(79)$$

considering the path of integration around the branch cut along positive imaginary axis from iX_2 to $i\infty$.

So using (79), (75) becomes

$$\sigma_{23}(x, 0, t) = -\frac{\mu W_0}{\pi}\left(1-\frac{v^2}{c^2}\right)^{1/2} \cdot \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p}(X_1)^{1/2} e^{X_2 x}$$

$$\times \int_0^\infty \frac{e^{ux}u^{1/2}\left(u+X_2-\dfrac{p}{v}\right)}{(u+X_2)\left(u+X_2-\dfrac{p+\beta}{v}\right)}\, du + \mu W_0\left(1-\frac{v^2}{c^2}\right)^{1/2} \cdot \frac{1}{2\pi i}$$

$$\int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p+\beta}(X_1X_2)^{1/2}\, dp.$$

For large t (i.e., for small p using $X_1 = k$, $X_2 = 0$) and for all x $(x < 0)$ we obtain from (80)

$$\sigma_{23}(x, 0, t) = -\frac{\mu W_0}{\pi}\left(\frac{\beta}{c}\cdot\frac{v}{c}\right)^{1/2}\int_0^\infty \frac{e^{ux}u^{1/2}}{\left(u-\dfrac{\beta}{v}\right)}\, du$$

$$k = \left(\frac{v}{c}\cdot\frac{\beta}{c}\right) / \left(1-\frac{v^2}{c^2}\right)$$

which is same as the solution for the steady case for all values of $x > 0$ given by (42).

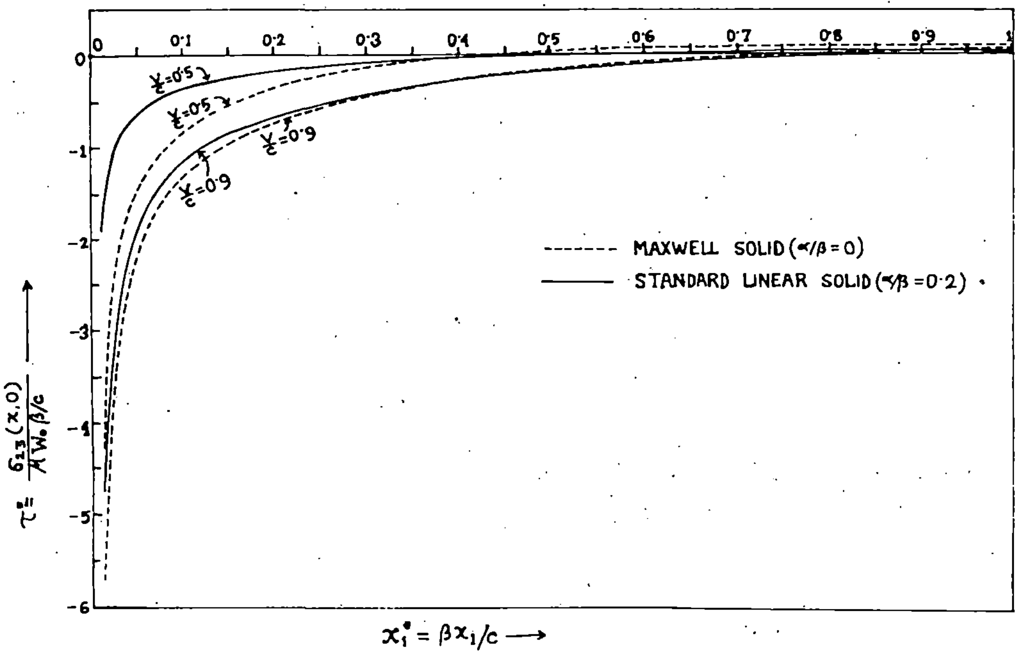

FIG. 3

6. Results and Discussion

The stress σ_{23} $(x, 0)$ just below the punch $(x < 0)$ and the displacement W $(x, 0)$ on the free surface $(y=0, x > 0)$ have been computed numerically from eqns. (39) and (43) for different values of parameters v/c and α/β. The case $\alpha/\beta = 0$ corresponds to Maxwell-Solid. In Fig. 3 non dimensional stress $\tau^* = \dfrac{\sigma_{23}(x, 0)}{\mu W_0 \beta/c}$ has been plotted against non-dimensional distance $x_1^* = \beta x_1/c$ for values of the paramter $v/c=0.5$, 0.9 and for values of the parameter $\alpha/\beta=0$ and 0.2.

For the same sets of the parameter values non dimensional displacement $W^* = W/W_0$ has been plotted versus non-dimensional distance $x^* = \beta x/c$ in Fig. 4. W^* varies from 1 to zero as x^* changes gradually from $x^* = 0$ to ∞.

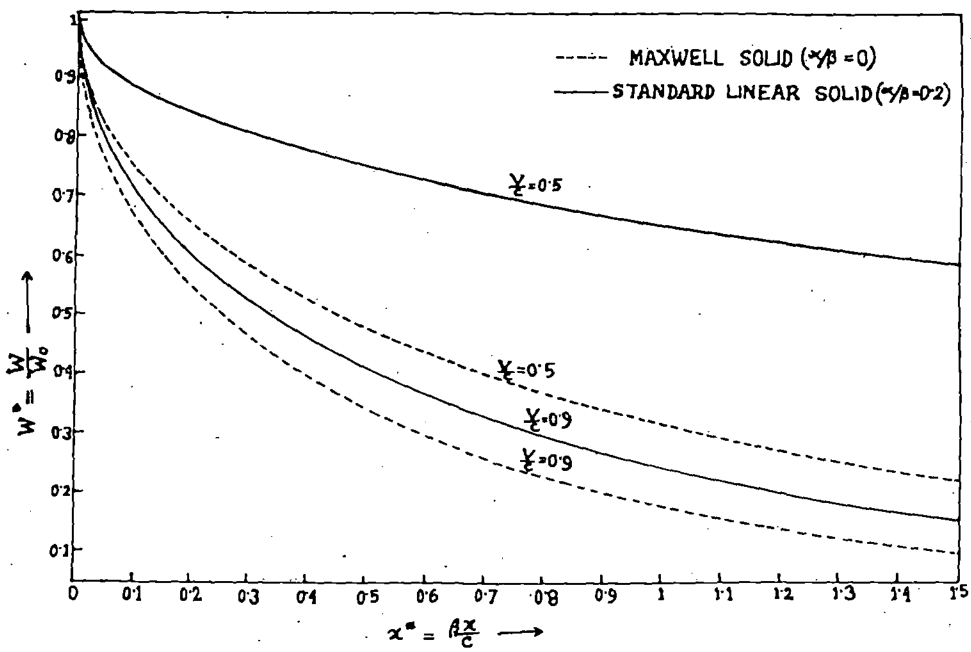

Fig. 4

It may be noted from the graphs that variation of the values of W^* with x^* is rapid with the increase in the values of the paramter v/c. Further it is found that the graphs become steeper with the decrease in the values of the paramter α/β. From Fig. 3 it is found that nondimensional stress τ^* changes rapidly with the decrease in the values of v/c where as for a fixed value of v/c graphs become flat with the increase in the values of α/β.

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References

1. A. L. Galin, *Contact Problems in the Theory of Elasticity* (ed.: I. N. Sneddon), North Carolina State University, Raleigh, 1961.
2. G. M. L. Gladwell, *Contact Problems in the Classical Theory of Elasticity*, Sijthoff and Noordhoff, Alphen ann den Rijn, 1980.
3. E. S. Suhibi, *Bull. Tech. Univ. of Istambul* **25** (1972), 72.
4. R. J. Tait and T. B. Moodie, *Int. J. Engeng. Sci.* **19** (1981), 221.
5. H. G. Georgiadis, *Acta Mechanica* **68** (1987), 193.
6. B. Noble, *Methods based on the Wiener-Hopf Technique*, Pergamon Press, New York, 1958, p. 13.
7. S. C. Hunter, *J. Appl. Mech.* 4 (1961), 611.
8. J. B. Alblas and M. Kuipers, *Int. J. Engng. Sci.* **8** (1970), 363
9. J. M. Golden, *Q. J. Mech. appl. Math.* **30** (1977), 23.
10. E. T. Whittacker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1969, p. 337.

Appendix

I. Evaluation of the Integral B

The integral in (77)

$$B = \frac{1}{2\pi i} \frac{1}{\sqrt{c\left(1-\frac{v^2}{c^2}\right)}} \int_{c'-i\infty}^{c'+i\infty} e^{pt} \frac{\sqrt{\frac{v}{c}\left(p+\frac{\beta}{2}\right)+\sqrt{p(p+\beta)+\frac{v^2\beta^2}{4c^2}}}}{p}\, dp$$

has a simple pole at $p = 0$ and branch points at $p = -\beta$

$$\alpha_1 = \frac{\beta}{2}\left(-1+\sqrt{1-\frac{v^2}{c^2}}\right)$$

$$\alpha_2 = \frac{\beta}{2}\left(-1-\sqrt{1-\frac{v^2}{c^2}}\right).$$

Taking the branch cut along the negative real axis from α_1 to $-\infty$ the integral can be considered as a contour integral around the path as shown in Fig. 5.

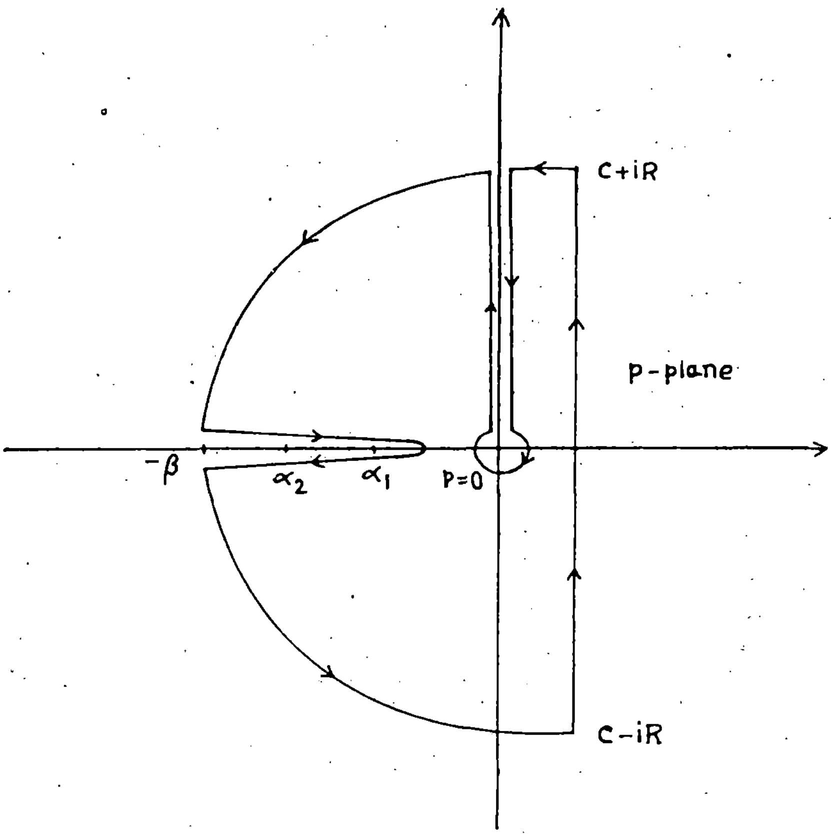

Fig. 5

Let $$I = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \epsilon^{pt} \frac{\sqrt{\frac{v}{c}\left(p+\frac{\beta}{2}\right) + \sqrt{(p-\alpha_1)(p-\alpha_2)}}}{p} dp$$

which can be written as $I = \sqrt{\frac{v\beta}{c}} + I_1$ where $\sqrt{\frac{v\beta}{c}}$ is the contribution to the integral from pole at $p=0$.

Let $I_1 = I_2 + I_3$ where I_2 is the value of the integral I_1 around the branch cut from α_1 to α_2 and I_3 is its value round the branch cut from α_2 to $-\infty$.
Now it can be shown that

$$I_2 = \frac{\sqrt{\beta}}{\pi} \int_0^{\sqrt{1-v^2/c^2}} \frac{e^{t_1(\alpha_1^*-r)}\sqrt{\frac{R^*-x^*}{2}}}{(\alpha_1^*-r)} dr.$$

In the interval $(\alpha_2, -\infty)$

$$I_3 = \frac{\sqrt{\beta}}{\pi} \int_0^{\sqrt{1-v^2/c^2}} \frac{e^{t_1(\alpha_1^*-r)}\sqrt{-x^{**}}}{(\alpha_1^*-r)} dr.$$

where $\alpha_1 = \beta\alpha_1^*$, $\alpha_2 = \beta\alpha_2^*$, $t_1 = \beta t$,

$$x^* = \frac{v}{c}\left(\frac{1}{2}\sqrt{1-\frac{v^2}{c^2}} - r\right), \quad y^* = \sqrt{r}\sqrt{\sqrt{1-\frac{v^2}{c^2}} - r}$$

$$R^* = \sqrt{(x^*)^2 + (y^*)^2}$$

$$x^{**} = -\frac{v}{c}\left(r - \frac{1}{2}\sqrt{1-\frac{v^2}{c^2}}\right) - \sqrt{r}\sqrt{r - \sqrt{\left(1-\frac{v^2}{c^2}\right)}}.$$

Finally, we obtain

$$B = \frac{1}{\sqrt{c\left(1-\frac{v^2}{c^2}\right)}}\left[\sqrt{\frac{v\beta}{c}} + \frac{\sqrt{\beta}}{\pi}\int_0^{\sqrt{1-v^2/c^2}} \frac{e^{t_1(\alpha_1^*-r)}\sqrt{\frac{R^*-x^*}{2}}}{(\alpha_1^*-r)} dr + \right.$$

$$\left. + \int_{\sqrt{1-v^2/c^2}}^{\infty} \frac{e^{t_1(\alpha_1^*-r)}\sqrt{-x^{**}}}{(\alpha_1^*-r)} dr\right].$$

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ANTIPLANE DYNAMIC CRACK PROPAGATION IN AN INHOMOGENEOUS VISCOELASTIC SOLID

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1. Introduction

Until now many authors, BAKER [1], CHEREPANOV and AFANASEV [2] and others have investigated the dynamic crack propagation in a homogeneous elastic medium. This problem presents an interest for a better understanding of the brittle behaviour of the material. However, natural or atificial materials are usually inhomogeneous. There exist very few solutions to the problem of dynamic crack propagation in inhomogeneous elastic media. ARKINSON and LIST [3] and ATKINSON [4] considered steady-state crack propagation in different types of inhomogeneous elastic media. In addition, if the materials are dissipative, that effect can be taken into account by considering the material to be viscoelastic. Crack propagation in viscoelastic medium has been studied by WILLIS [5], ATKINSON and LIST [6], COUSSY [7] and others. WILLIS [5] considered steady-state Mode III crack propagation for a standard linear solid under general type of loading on the crack surfaces. ATKINSON and LIST [6] studied nonsteady SH-wave type crack propagation starting at $t = 0$ and moving with a constant velocity in the "Maxwell Solid" or using the viscoelastic model suggested by Achenbach and Chao. Finally, SILLS and BENVENISTE [9] and COUSSY [7] studied steady state crack propagation of SH-type at the interface between two viscoelastic media.

In our case we have considered steady and nonsteady cases of Mode III crack propagation in an inhomogeneous viscoelastic medium. Two types of viscoelastic models, namely Maxwell Solid and Standard Linear Solid have been considered. Material properties have been assumed to vary exponentially in the direction perpendicular to the direction of crack propagation. We have studied how the material inhomogeneity affects the stress intensity factor and also the crack opening displacement when a Mode III type crack propagates through the inhomogeneous viscoelastic medium.

2. Formulation of the Problem and its Solution for Nonsteady Case in Maxwell Solid

Let us consider an inhomogeneous viscoelastic medium which was set in motion by a semi-infinite crack suddenly appearing at $t = 0$ and moving with a constant velocity V in the direction of the X-axis. The Y-axis is taken perpendicular to the X-axis (Fig. 1). For SH-waves, the displacements along X and Y directions are zero and only the displacement $W = W(X, Y, t)$ along the Z-direction exists.

The shear modulus is $\mu(Y) = \mu_0 \exp(2\beta Y)$ and density $\varrho(Y) = \varrho_0 \exp(2\beta Y)$, where β, μ_0, and ϱ_0 are constants.

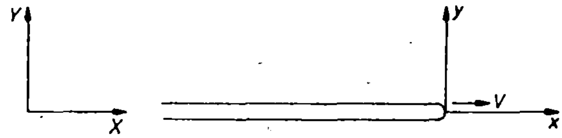

FIG. 1. The crack geometry.

The non-zero stresses are

(2.1) $$\sigma_{YZ} = \sigma_{YZ}(X, Y, t) \quad \text{and} \quad \sigma_{XZ} = \sigma_{XZ}(X, Y, t),$$

and the nonvanishing strains are

(2.2) $$e_{XZ} = {}^1/_2(\partial W/\partial X), \qquad e_{YZ} = {}^1/_2(\partial W/\partial Y).$$

Considering a Maxwell Solid as the viscoelastic model, the stress-strain relations are

(2.3) $$\begin{aligned}(\partial\sigma_{YZ}/\partial t)+\beta_1\sigma_{YZ} &= 2\mu(Y)(\partial e_{YZ}/\partial t),\\ (\partial\sigma_{XZ}/\partial t)+\beta_1\sigma_{XZ} &= 2\mu(Y)(\partial e_{XZ}/\partial t),\end{aligned}$$

where β_1 is a positive constant.

The equation of motion has the form

(2.4) $$(\partial\sigma_{XZ}/\partial X)+(\partial\sigma/\partial Y_{YZ}) = \varrho(Y)(\partial^2 W/\partial t^2),$$

and the boundary conditions of the problem are

(2.5) $$\begin{aligned}&W(X, 0, t) = 0, \quad X-Vt > 0, \quad t > 0,\\ &\sigma_{YZ}(X, 0, t) = -\sigma H(t), \quad X-Vt < 0, \quad t > 0,\\ &\sigma_{YZ}(X, Y, t) \to 0 \quad \text{as} \quad X^2+Y^2 \to \infty.\end{aligned}$$

It is convenient to shift the origin of coordinate to the tip of the crack at $X = Vt$. New coordinate axes (x, y) are parallel to the respective fixed ones (X, Y).

Hence, putting $x = X-Vt$, $y = Y$, we obtain $(\partial/\partial X) = (\partial/\partial x)$, $(\partial/\partial Y) = (\partial/\partial y)$ and the time derivative transforms to $-V(\partial/\partial x)+(\partial/\partial t)$. Equations (2.1), (2.2), (2.3) and (2.4) become

(2.6) $$\sigma_{xz} = \sigma_{xz}(x, y, t), \qquad \sigma_{yz} = \sigma_{yz}(x, y, t),$$

(2.7) $$e_{xz} = {}^1/_2[\partial W(x, y, t)/\partial x], \qquad e_{yz} = {}^1/_2[\partial W(x, y, t)/\partial y],$$

(2.8) $$\begin{aligned}-V(\partial\sigma_{yz}/\partial x)+(\partial\sigma_{yz}/\partial t)+\beta_1\sigma_{yz} &= \mu(y)[-V(\partial^2 W/\partial x\,\partial y)+(\partial^2 W/\partial t\,\partial y)],\\ -V(\partial\sigma_{xz}/\partial x)+(\partial\sigma_{xz}/\partial t)+\beta_1\sigma_{xz} &= \mu(y)[-V(\partial^2 W/\partial x^2)+(\partial^2 W/\partial t\,\partial x)]\end{aligned}$$

and

(2.9) $$(\partial\sigma_{xz}/\partial x)+(\partial\sigma_{xz}/\partial y) = \varrho(y)[V^2(\partial^2 W/\partial x^2)-2V(\partial^2 W/\partial x\,\partial t)+(\partial^2 W/\partial t^2)].$$

The boundary conditions (2.5) now assume the form

(2.10) $$\begin{aligned}&W(x, 0, t) = 0, \quad x > 0,\\ &\sigma_{yz}(X, 0, t) = -\sigma H(t), \quad x < 0,\\ &\sigma_{yz}(x, y, t) \to 0 \quad \text{as} \quad x^2+y^2 \to \infty.\end{aligned}$$

Let us denote the Laplace transform by a single bar

$$\bar{f} \equiv \bar{f}(x, y, p) = \int_0^\infty f(x, y, t)\exp(-pt)dt,$$

and the Fourier transform by two bars

$$\bar{\bar{f}} \equiv \bar{\bar{f}}(\xi, y, p) = 1/\sqrt{(2\pi)} \int_{-\infty}^{\infty} \bar{f}(x, y, p)\exp(i\xi x)dx.$$

Applying these transforms to Eqs. (2.8) and (2.9), we get

$$(i\xi V + p + \beta_1)\bar{\bar{\sigma}}_{yz} = \mu(y)(Vi\xi + p)(d\bar{\bar{W}}/dy), \tag{2.11}$$

$$(i\xi V + p + \beta_1)\bar{\bar{\sigma}}_{xz} = \mu(y)(V\xi^2 - i\xi p)\bar{\bar{W}} \tag{2.12}$$

and

$$-i\xi\bar{\bar{\sigma}}_{xz} + (d\bar{\bar{\sigma}}_{yz}/dy) = \varrho(y)(-V^2\xi^2 + 2Vi\xi p + p^2)\bar{\bar{W}}. \tag{2.13}$$

Eliminating $\bar{\bar{\sigma}}_{xz}$, $\bar{\bar{\sigma}}_{yz}$ from Eqs. (2.11), (2.12) and (2.13), we obtain

$$(d^2\bar{\bar{W}}/dy^2) + 2\beta(d\bar{\bar{W}}/dy) - \gamma^2\bar{\bar{W}} = 0, \tag{2.14}$$

where

$$\begin{aligned} \gamma^2 &= \xi^2 + (1/c^2)(Vi\xi + p)(Vi\xi + p + \beta_1), \\ c^2 &= \mu_0/\varrho_0. \end{aligned} \tag{2.15}$$

The branches of γ are chosen so that $\mathrm{Re}(\gamma) > 0$.

Since $\bar{\bar{W}}$ must remain bounded as $y \to \pm\infty$, so solutions of (2.14) are

$$\bar{\bar{W}}^{(1)} = B_1 \exp\left[-(\beta + \sqrt{(\beta^2 + \gamma^2)})y\right], \quad y > 0, \tag{2.16}$$

and

$$\bar{\bar{W}}^{(2)} = A_2 \exp\left[(-\beta + \sqrt{(\beta^2 + \gamma^2)})y\right], \quad y < 0, \tag{2.17}$$

where $W^{(1)}$, and $W^{(2)}$ denote the displacement in the upper and lower half-plane, respectively.

Let us consider the case when for $y = 0$

$$\begin{aligned} \bar{W}^{(1)} - \bar{W}^{(2)} &= h(x, p), \quad x < 0, \\ &= 0, \quad x > 0, \end{aligned} \tag{2.18}$$

where $h(x, p)$ is an unknown function such that

$$h(x, p) \sim 0\{\exp(k_1 x)\} \quad \text{as} \quad x \to -\infty, \quad k_1 > 0,$$

Applying the Fourier transform to Eq. (2.18), we get

$$\begin{aligned} \bar{\bar{W}}^{(1)} - \bar{\bar{W}}^{(2)} = B_1 - A_2 &= 1/\sqrt{(2\pi)} \int_{-\infty}^{\infty} h(x, p)\exp(i\xi x)dx, \\ &= H_-(\xi, p), \end{aligned} \tag{2.19}$$

where $H_-(\xi, p)$ is an analytic function in the lower half-plane $\tau < k_1$ and $\xi = \sigma + i\tau$.

Now from Eqs. (2.11), (2.16), and (2.17), we obtain

$$(2.20)\quad \bar{\sigma}_{yz}^{(1)} = \frac{(V_i\xi+p)}{(V_i\xi+p+\beta_1)}\,\mu(y)(\partial\bar{\bar{W}}^{(1)}/\partial y) = -\mu(y)B_1\left(\beta+\sqrt{(\beta^2+\gamma^2)}\right)\times$$

$$\times\exp\left[-(\beta+\sqrt{(\beta^2+\gamma^2)})y\right]\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}, \quad y>0,$$

$$\bar{\bar{\sigma}}_y^{(2)} = \frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\cdot\mu(y)A_2\left(-\beta+\sqrt{(\beta^2+\gamma^2)}\right)\exp\left[\left(-\beta+\sqrt{(\beta^2+\gamma^2)}\right)y\right],\ y<0,$$

where $\sigma_{yz}^{(1)}$ and $\sigma_{yz}^{(2)}$ are the stresses on the upper and lower surfaces of the crack.

Since the stresses are continuous for $y=0$,

$$\bar{\bar{\sigma}}_{yz}^{(1)} = \bar{\bar{\sigma}}_{yz}^{(2)}.$$

Using Eqs. (2.20), we obtain

$$(2.21)\qquad B_1 = -\frac{\left[-\beta+\sqrt{(\beta^2+\gamma^2)}\right]}{\left[\beta+\sqrt{(\beta^2+\gamma^2)}\right]}A_2.$$

Using Eq. (2.21), (2.19) becomes

$$(2.22)\qquad H_-(\xi,p) = -\frac{2\sqrt{(\beta^2+\gamma^2)}}{\left[\beta+\sqrt{(\beta^2+\gamma^2)}\right]}A_2.$$

Again let us assume that for $y=0$

$$(2.23)\qquad \begin{aligned}\bar{\sigma}_{yz} = \bar{\sigma}_{yz}^{(1)} = \bar{\sigma}_{yz}^{(2)} &= -[\sigma_0\exp(\lambda x)]/p, && x<0,\\ &= e(x), && x>0.\end{aligned}$$

Here $e(x)$ is an unknown function such that

$$e(x)\sim 0\{\exp(-k_2x)\},\quad x\to\infty,\quad k_2>0.$$

Taking Fourier transforms of Eq. (2.23) we get

$$(2.24)\quad \bar{\bar{\sigma}}_{yz}^{(2)} = \frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\,\mu_0A_2\left[-\beta+\sqrt{(\beta^2+\gamma^2)}\right]$$

$$= 1/\sqrt{(2\pi)}\int_0^\infty \bar{\sigma}_{yz}^{(2)}\exp(i\xi x)\,dx + 1/\sqrt{(2\pi)}\int_{-\infty}^0 \bar{\sigma}_{yz}^{(2)}\exp(i\xi x)\,dx =$$

$$= E_+(\xi,p)-\sigma_0/\left[\sqrt{(2\pi)}\,(\lambda+i\xi)p\right],$$

where

$$(2.25)\qquad E_+(\xi,p) = 1/\sqrt{(2\pi)}\int_0^\infty \bar{\sigma}_{yz}^{(2)}\exp(i\xi x)\,dx$$

and is an analytic function in the upper half-plane $\tau > -k_2$ and $\sigma_0/[i\sqrt{(2\pi)}\,(\xi-i\lambda)\,p]$ is analytic in the lower half-plane $\tau<\lambda$.

From Eqs. (2.22) and (2.24), we get

$$(2.26)\qquad -\frac{\mu_0(Vi\xi+p)\gamma^2 H^-(\xi,p)}{2(Vi\xi+p+\beta_1)\sqrt{(\beta^2+\gamma^2)}} = E_+(\xi,p)-\sigma_0/\left[\sqrt{(2\pi)}\,(\lambda+i\xi)p\right].$$

It may be noted that the problem has been reduced to a form suitable for application of the Wiener–Hopf technique.

Now

$$(2.27)\qquad \gamma^2 = (1-V^2/c^2)(\xi+iX_1)(\xi-iX_2),$$

where

$$(2.28)\qquad X_1 = {}^1/_2(1-V^2/c^2)^{-1}\left[(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2V^2/c^4+} \right.$$
$$\left. \overline{+4p(p+\beta_1)(1-V^2/c^2)/c^2\}}\right],$$

$$(2.29)\qquad X_2 = {}^1/_2(1-V^2/c^2)^{-1}\left[-(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2V^2/c^4+}\right.$$
$$\left.\overline{+4p(p+\beta_1)(1-V^2/c^2)/c^2\}}\right]$$

and

$$(2.30)\qquad \sqrt{(\beta^2+\gamma^2)} = (\xi+iY_1)^{1/2}(\xi-iY_2)^{1/2}(1-V^2/c^2)^{1/2},$$

where

$$(2.31)\qquad Y_1 = {}^1/_2(1-V^2/c^2)^{-1}\left[(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2V^2/c^4+}\right.$$
$$\left.\overline{+4\{p(p+\beta_1)/c^2+\beta^2\}(1-V^2/c^2)\}}\right],$$

$$(2.32)\qquad Y_2 = {}^1/_2(1-V^2/c^2)^{-1}\left[-(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2V^2/c^4+}\right.$$
$$\left.\overline{+4\{p(p+\beta_1)/c^2+\beta^2\}(1-V^2/(c^2)\}}\right].$$

Using Eqs. (2.27) and (2.30), (2.26) becomes

$$(2.33)\qquad -\frac{\mu_0(1-V^2/c^2)^{1/2}(\xi-ip/V)(\xi-iX_2)H_-(\xi,p)}{2[\xi-i(p+\beta_1)/V](\xi-iY_2)^{1/2}}+\frac{\sigma_0(i\lambda+iY_1)^{1/2}}{\sqrt{(2\pi)}\,i(\xi-i\lambda)(i\lambda+iX_1)p} =$$
$$= \frac{(\xi+iY_1)^{1/2}E_+(\xi,p)}{(\xi+iX_1)}-\frac{\sigma_0}{\sqrt{(2\pi)}\,ip(\xi-i\lambda)}\left[\frac{(\xi+iY_1)^{1/2}}{(\xi+iX_1)}-\frac{(i\lambda+iY_1)^{1/2}}{(i\lambda+iX_1)}\right].$$

The functions on the R.H.S. of Eq. (2.33) are analytic and non-zero in the upper half-plane $\tau > -k_2$, and functions on the L.H.S. are analytic and non-zero in the lower half-plane $\tau < \lambda$ ($\lambda < k_1$). Since both the functions are analytic in the strip $-k_2 < \tau < \lambda$, the principle of an analytic continuation states that each of them represents an entire function $M(\xi)$ in the whole ξ-plane.

Now the L.H.S. of (3.33) approaches zero as $|\xi| \to \infty$. It may then be concluded by Liouville's theorem that $M(\xi) = 0$, and therefore

$$(2.34)\qquad H_-(\xi,p) = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(\xi-iY_2)^{1/2}(1-V^2/c^2)^{-1/2}(\xi-i(p+\beta_1)/V)}{\mu_0\sqrt{(2\pi)}\,ip(\xi-i\lambda)(i\lambda+iX_1)(\xi-iX_2)(\xi-ip/V)}$$

and

$$E_+(\xi, p) = \frac{\sigma_0}{\sqrt{(2\pi)}\; ip(\xi - i\lambda)} - \frac{\sigma_0(i\lambda + iYy)^{1/2}(\xi + iX_1)}{\sqrt{(2\pi)}\; ip(\xi - i\lambda)(i\lambda + iX_1)(\xi + iY_1)^{1/2}} \tag{2.35}$$

From Eq. (2.34) it follows that

$$H_-(\xi, p) = \frac{2\sigma_0(1 - V^2/(c^2)^{-1/2}(i\lambda + iY_1)^{1/2}}{\mu_0\sqrt{(2\pi)}\; ip(i\lambda + iX_1)}\,\xi^{-3/2}, \quad \xi \to \infty.$$

Application of the inverse Fourier transform yields

$$h(x, p) = \frac{4\sigma_0}{\mu_0}(1 - V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2}\,\frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p}, \quad x \to 0^-.$$

Again, taking the inverse Laplace transform, the displacement jump across the surface of the crack near the crack tip is

$$W^{(1)} - W^{(2)} = (4\sigma_0/\mu_0)(1 - V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2} \times \tag{2.36}$$

$$\times (1/2\pi i)\int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p}\exp(pt)\,dp.$$

From Eq. (2.35)

$$E_+(\xi, p) = -\frac{\sigma_0(i\lambda + iY_1)^{1/2}}{\sqrt{(2\pi)}\; ip(i\lambda + iX_1)}\,\xi^{-1/2}, \quad \xi \to \infty.$$

Taking the inverse Fourier transform we obtain

$$e(x, p) = \sigma_0\pi^{-1/2}(x)^{-1/2}\frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p}, \quad x \to 0^+.$$

Again, taking the inverse Laplace transform

$$\sigma_{yz} = \sigma_0\pi^{-1/2}(x)^{-1/2}(1/2\pi i)\int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p}\exp(pt)\,dp. \tag{2.37}$$

If ΔW is the displacement jump, then the crack opening displacement near the crack tip is given by

$$\mu_0\Delta W = 4\sigma_0(1 - V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2}\cdot A, \quad (-1 \ll x < 0), \tag{2.38}$$

and the stress near the crack tip is

$$\sigma_{yz} = \sigma_0\pi^{-1/2}(x)^{-1/2}\cdot A, \quad (0 < x \ll 1), \tag{2.39}$$

where

$$A = (1/2\pi i)\int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p}\exp(pt)\,dp = \tag{2.40}$$

$$= (1/2\pi i)\int_{c'-i\infty}^{c'+i\infty} \frac{(Y_1)^{1/2}}{X_1 p}\exp(pt)\,dp, \quad \lambda \to 0.$$

Evaluation of the integral A given by Eq. (2.40) corresponding to constant stress $-\sigma_0$ on the crack surfaces is presented in the Appendix.

In the fracture mechanics, it is customary to write $\sigma_{yz} = (0^+, 0, t)$ in the form $K/\sqrt{(2\pi x)}$, where K is the stress intensity factor. In our case

$$K = \sqrt{2\sigma_0 A}. \tag{2.41}$$

Putting $\beta = 0$ in the expression for A, we obtain the stress intensity factor in a homogeneous viscoelastic medium as

$$K = \sqrt{2\sigma_0 A_1},$$

where

$$A_1 = (1/2\pi i)\sqrt{2(1-V^2/c^2)} \times$$
$$\times \int_{c'-i\infty}^{c'+i\infty} \frac{\exp(pt)dp}{p[(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2 V^2/c^4+4p(p+\beta_1)(1-V^2/c^2)/c^2\}}]},$$

which agrees with the results of ATKINSON and LIST [6]

3. Steady State Case for Maxwell Solid

Steady state solutions are the results of Sect. 2 corresponding to the case of t approaching infinity. So for the steady state case, passing to the limit $p \to 0$ and using the Tauberian theorem we obtain from Eq. (2.34)

$$H_-(\xi, p) = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(\xi-iY_2)^{1/2}(1-V^2/c^2)^{-1/2}(\xi-i\beta_1/V)}{\mu_0\sqrt{(2\pi)}\; i(\xi-i\lambda)(i\lambda+iX_1)\xi^2}.$$

Applying the inverse Fourier transform we obtain

$$W^{(1)}-W^{(2)} = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(1-V^2/c^2)^{-1/2}}{\mu_0 2\pi i(i\lambda+iX_1)} \times \tag{3.1}$$
$$\times \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{(\xi-iY_2)^{1/2}(\xi-i\beta_1/V)}{(\xi-i\lambda)\xi^2}\exp(-i\xi x)d\xi =$$
$$= \frac{2\sigma_0(i\lambda+iY_1)^{1/2}}{\mu_0 2\pi i(i\lambda+iX_1)}(1-V^2/c^2)^{-1/2}I,$$

where

$$I = \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{(\xi-iY_2)^{1/2}(\xi-i\beta_1/V)}{(\xi-i\lambda)\xi^2}\exp(-i\xi x)d\xi.$$

For $x < 0$, the above integral can be replaced with the integral taken along the positive imaginary ξ-axis and round the branch point at $\xi = iY_2$, together with the contribution from the poles at $\xi = 0$ and $\xi = i\lambda$ as shown in Fig. 2.

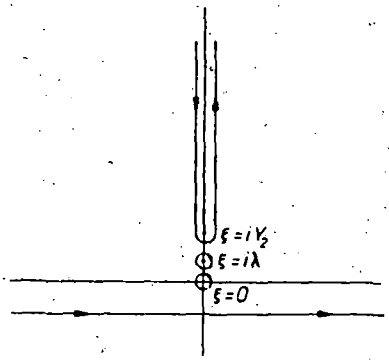

FIG. 2. The path of integration of the integral I.

Thus it can be shown that

$$(3.2) \qquad I = \exp\{-(\pi i/4 + x_1 Y_2)\}\Big[(2/\lambda)(\beta_1/\lambda V - 1)\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{u+Y_2}\,du + \\ +(2\beta_1/\lambda V)\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{(u+Y_2)^2}\,du + \\ -(2/\lambda)(\beta_1/\lambda V - 1)\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{u+(Y_2-\lambda)}\,du\Big] + \\ +(2\pi/\lambda)[(\beta_1/\lambda V - 1)(-iY_2)^{1/2} + (\beta_1/V)\{x(-iY_2)^{1/2} + (i/2)(-iY_2)^{-1/2}\} + \\ -(\beta_1/\lambda V - 1)(i\lambda - iY_2)^{1/2}\exp(-\lambda x_1)] = \\ = (1/\lambda)\exp(-\pi i/4)[(\beta_1/\lambda V - 1)(x_1)^{-1/2}(x_1 Y_2)^{-1/4}\sqrt{\pi}\,\exp(-{}^1/_2 x_1 Y_2)\times \\ \times W_{-3/4,\,1/4}(x_1 Y_2) + (\beta_1/V)\sqrt{(\pi x_1)}\,(x_1 Y_3)^{-3/4}\exp(-{}^1/_2 x_1 Y_2)\times \\ \times W_{-5/4,\,-1/4}(x_1 Y_2) + (1-\beta_1/\lambda V)(x_1)^{-1/2}(x_1(Y_2-\lambda))^{-1/4}\sqrt{\pi}\,\exp\{-{}^1/_2 x_1(Y_2+\lambda)\}\times \\ \times W_{-3/4,\,1/4}(x_1(Y_2-\lambda))] + (2\pi/\lambda)[(\beta_1/\lambda V - 1)(-iY_2)^{1/2} + \\ +(i\beta_1/V)\{{}^1/_2(-iY_2)^{-1/2} + ix_1(-iY_2)^{1/2}\} + \\ +(1-\beta_1/\lambda V)(i\lambda - iY_2)^{1/2}\exp(-\lambda x_1)],$$

where $W_{k,m}$ is the Whittaker function [9].

Therefore the displacement jump ΔW across the surface of the crack ($x < 0$) is given by

$$(3.3) \qquad \mu_0 \Delta W = -\frac{\sigma_0(\lambda + Y_1)^{1/2}(1 - V^2/c^2)^{-1/2}}{\pi(\lambda + X_1)}\exp(\pi i/4)\cdot I,$$

where I is given by Eq. (3.2).

Using the result that

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma({}^1/_2 - m - k)}(z)^{1/2+m}\exp(-z/2) + \frac{\Gamma(2m)}{\Gamma({}^1/^2 + m + k)}(z)^{1/2-m}\exp(-z/2)$$

for small z,

we find that for small $x_1 Y_2$, Eq. (3.2) yields

$$I = -4\sqrt{(\pi x_1)}\,\exp(-\pi i/4), \tag{3.4}$$

Substituting the value of I from Eq. (3.4) into Eq. (3.3) we get

$$\mu_0 \Delta W = 4\sigma_0(1-V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2}\frac{(\lambda+\alpha_1)^{1/2}}{(\lambda+\alpha^2)}, \quad -1 \leqslant x < 0, \tag{3.5}$$

where

$$\alpha_1 = {}^1/_2(1-V^2/c^2)^{-1}\left[(\beta_1 V/c^2)+\sqrt{\{(\beta_1^2 V^2/c^4)}+4\beta^2(1-V^2/c^2)\}\right] \tag{3.6}$$

and

$$\alpha_2 = \beta_1 V(1-V^2/c^2)^{-1}/c^2. \tag{3.7}$$

Again, letting $p \to 0$ and using the Tauberian theorem we find from Eqs. (2.35) and (2.24) that in the steady-state case

$$\bar{\sigma}_{yz} = -\frac{\sigma_0(i\lambda+iY_1)^{1/2}(\xi+iX_1)}{\sqrt{(2\pi)}\,i(\xi-i\lambda)(i\lambda+iX_1)(\xi+iY_1)^{1/2}}. \tag{3.8}$$

Taking the inverse Fourier transform we obtain

$$\sigma_{yz} = -\frac{\sigma_0(i\lambda+iY_1)^{1/2}}{2\pi i(i\lambda+iX_1)}\int_{-\infty-is}^{\infty-is}\frac{(\xi+iX_1)\exp(-i\xi x)}{(\xi-i\lambda)(\xi+iY_1)^{2/1}}\,d\xi = -\frac{\sigma_0(i\lambda+iY_1)^{1/2}}{2\pi i(i\lambda+iX_1)}I_1, \tag{3.9}$$

where

$$I_1 = \int_{-\infty-is}^{\infty-is}\frac{(\xi+iX_1)\exp(-i\xi x)}{(\xi-i\lambda)(\xi+iY_1)^{1/2}}\,d\xi = \tag{3.10}$$

$$= 2\sqrt{\pi}\,\exp(-\pi i/4)\left[(x)^{-1/2}\exp(-Y_1x)-(\lambda+X_1)(x)^{1/2}(x(\lambda+Y_1))^{-3/4}\right.$$
$$\left.\times\exp\{{}^1/_2x(\lambda-Y_1)\}W_{-1/4,-1/4}(x(\lambda+Y_1))\right].$$

Thus the stress at $y = 0$ for all x $(x > 0)$ is given by Eq. (3.9).

Now for small $(\lambda+Y_1)x$

$$I_1 = 2\sqrt{(\pi/x)}\,\exp(-\pi i/4), \tag{3.11}$$

so from Eq. (3.9) it follows that

$$\sigma_{yz} = \sigma_0(\pi x)^{-1/2}\frac{(\lambda+\alpha_1)^{1/2}}{(\lambda+\alpha_2)}, \quad (0 < x \leqslant 1), \quad y = 0. \tag{3.12}$$

Stress intensity factor K is given by

$$K = \sqrt{2\sigma_0 B}, \tag{3.13}$$

where

$$B = \frac{(\lambda+\alpha_1)^{1\,2}}{(\lambda+\alpha_2)}.$$

Now putting $\beta_1 = 0$ in the expression for α_1 and α_2 we get from Eqs. (3.5) and (3.12) the displacement jump and stress intensity factor in an inhomogeneous elastic medium,

(3.14) $$\mu_0 \Delta W = 4\sigma_0(1-V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2}\lambda^{-1}\left[\lambda+\beta/\sqrt{(1-V^2/c^2)}\right]^{1/2}$$

and

(3.15) $$\sigma_{yz} = \sigma_0(\pi x)^{-1/2}\lambda^{-1}\left[\lambda+\beta/\sqrt{(1-V^2/c^2)}\right]^{1/2},$$

which agree with the results derived by ATKINSON [4]

4. Steady State Solution for Standard Linear Solid

In this case the stress-strain relations are

(4.1) $$\begin{aligned}(\partial\sigma_{YZ}/\partial t)+\beta_1\sigma_{YZ} &= 2\mu(Y)[\partial e_{YZ}/\partial t)+\alpha e_{YZ}],\\ (\partial\sigma_{XZ}/\partial t)+\beta_1\sigma_{XZ} &= 2\mu(Y)[(\partial e_{XZ}/\partial t)+\alpha e_{XZ}],\end{aligned}$$

where β_1 and α are constants.

Equation of motion has the form

(4.2) $$(\partial\sigma_{XZ}/\partial X)+(\partial\sigma_{YZ}/\partial Y) = \varrho(Y)(\partial^2 W/\partial t^2).$$

Now, putting $x = X-Vt$ and $y = Y$ so that

$$(\partial/\partial X) = (\partial/\partial x), \quad (\partial/\partial Y) = (\partial/\partial y), \quad \text{and} \quad (\partial/\partial t) = -V(\partial/\partial x).$$

Equations (4.1) and (4.2) become

(4.3) $$\begin{aligned}-V(\partial\sigma_{yz}/\partial x)+\beta_1\sigma_{yz} &= \mu(y)[-V(\partial^2 W/\partial x\partial y)+\alpha(\partial W/\partial y)],\\ -V(\partial\sigma_{xz}/\partial x)+\beta_1\sigma_{xz} &= \mu(y)[-V(\partial^2 W/\partial x^2)+\alpha(\partial W/\partial x)]\end{aligned}$$

and

(4.4) $$(\partial\sigma_{xz}/\partial x)+(\partial\sigma_{yz}/\partial y) = \varrho(y)V^2(\partial^2 W/\partial x^2).$$

Introducing the Fourier transform denoted by

(4.5) $$\overline{f}(\xi, y) = 1/\sqrt{(2\pi)}\int_{-\infty}^{\infty} f(x, y)\exp(i\xi x)\,dx.$$

Equations (4.3) and (4.4) can be transformed to

(4.6) $$(i\xi V+\beta_1)\overline{\sigma}_{yz} = \mu(y)(i\xi V+\alpha)(d\overline{W}/dy),$$

(4.7) $$(i\xi V+\beta_1)\overline{\sigma}_{xz} = \mu(y)(\xi^2 V-i\xi\alpha)\overline{W}$$

and

(4.8) $$i\xi\overline{\sigma}_{xz} = (d\overline{\sigma}_{yz}/dy) = -\varrho(y)V^2\xi^2\overline{W}.$$

Eliminating $\overline{\sigma}_{yz}$ and $\overline{\sigma}_{xz}$ from Eqs. (4.6), (4.7) and (4.8) we get

(4.9) $$(d^2\overline{W}/dy^2)+2\beta(d\overline{W}/dy)-\gamma^2\overline{W} = 0,$$

where

(4.10) $$\gamma^2 = \xi^2[(1-V^2/c^2)\xi+i(V\beta_1/c^2-\alpha/V)]/(\xi-i\alpha/V).$$

Branches of γ are chosen so that $\mathrm{Re}(\gamma) > 0$.

Since $\overline{W}$ must remain bounded as $y \to \pm\infty$, solutions of Eq. (4.9) are

$$\text{(4.11)} \qquad \begin{aligned} \overline{W}^{(1)} &= B_1 \exp\left[-\{\beta+\sqrt{(\beta^2+\gamma^2)}\}y\right], \quad y > 0 \\ \overline{W}^{(2)} &= A_2 \exp\left[\{-\beta+\sqrt{(\beta^2+\gamma^2)}\}y\right], \quad y < 0 \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ denote the displacements in the upper and lower half-planes.

Let us consider the case when for $y = 0$

$$\text{(4.12)} \qquad \begin{aligned} W^{(1)} - W^{(2)} &= h(x), \quad x < 0, \\ &= 0, \quad x > 0, \end{aligned}$$

where $h(x)$ is an unknown function such that

$$h(x) \sim 0[\exp(k_1 x)] \quad \text{as} \quad x \to -\infty \quad \text{and} \quad k_1 > 0$$

and

$$\text{(4.13)} \qquad \begin{aligned} \sigma_{yz} &= -\sigma_0 \exp(\lambda x), \quad x < 0, \\ &= e(x), \quad x > 0, \end{aligned}$$

where $e(x)$ is an unknown function satisfying the condition

$$e(x) \sim 0[\exp(-k_2 x)] \quad \text{as} \quad x \to \infty \quad \text{and} \quad k_2 > 0.$$

In this case Eq. (2.26) becomes

$$\text{(4.14)} \qquad -\frac{\mu_0(Vi\xi+\alpha)\gamma^2 H_-(\xi)}{2(Vi\xi+\beta_1)\sqrt{(\beta^2+\gamma^2)}} = E_+(\xi) - \frac{\sigma_0}{\sqrt{(2\pi)}\, i(\xi-i\lambda)}.$$

This equation holds in the region of regularity of the functions appearing in Eq. (4.14). Owing to our former assumptions regarding the behaviour of $e(x)$ and $h(x)$ at infinity, this region is represented by the inequality $-k_2 < \tau < \lambda < k_1$ where $\xi = \sigma + i\tau$.

Now Eq. (4.14) is suitable for the application of the Wiener-Hopf technique. Again,

$$\text{(4.15)} \qquad \gamma^2 = \xi^2(1-V^2/c^2)(\xi+ia)/(\xi-i\alpha/V),$$

where

$$a = (V\beta_1/c^2 - \alpha/V)/(1-V^2/c^2)$$

and

$$\text{(4.16)} \qquad \tau^2+\beta^2 = [\xi^3(1-V^2/c^2)+i(V\beta_1/c_2-\alpha/V)\xi^2+\beta^2(\xi-i\alpha/V)]/(\xi-i\alpha/V).$$

Since it is difficult tofactorize $\sqrt{\gamma^2+\beta^2}$, i.e. to represent it as a product of two functions, one analytic in the upper half-plane and the other analytic in the lower half-plane, we follow the approximate method of KOITER [10] of solving Wiener–Hopf type equations. Accordingly, we write $P(\xi) = \sqrt{(\gamma^2+\beta^2)}$ in the form $P(\xi) = \overline{P}(\xi)P_1(\xi)$, where the function $\overline{P}(\xi)$ is required to behave at $|\xi| \to \infty$ and at $|\xi| \to 0$ in the same manner as $P(\xi)$. The auxiliary function $P_1(\xi)$ should be non-zero and should have no singularity within the strip $-k_2 < -\tau_1 < \tau < \tau_2 < \lambda$; it has to be suitably chosen such that $P(\xi)$ is non-zero and possesses no singularity within the strip $-\tau_1 < \tau < \tau_2$. Now we note that

$$P(\xi) = \sqrt{(\tau^2+\beta^2)} \approx [(1-V^2/c^2)\xi^2+i(V\beta_1/c^2-\alpha/V)\xi]^{1/2} \quad \text{as} \quad |\xi| \to \infty$$

and

$$\sqrt{(\tau^2+\beta^2)} \approx \beta \quad \text{as} \quad |\xi| \to 0.$$

Therefore we choose P in the form

$$\bar{P}(\xi) = [(1-V^2/c^2)\xi^2 + i(V\beta_1/c^2 - \alpha/V)\xi + \beta^2]^{1/2}, \tag{4.17}$$

which behaves in the same manner as $P(\xi)$ for $|\xi| \to \infty$ and $|\xi| \to 0$.

Now $\bar{P}(\xi)$ can be written as

$$\bar{P}(\xi) = (1-V^2/c^2)^{1/2}(\xi - ia_2)^{1/2}(\xi + ia_1)^{1/2}, \tag{4.18}$$

where

$$a_1 = {}^1/_2\left[(V\beta_1/c^2 - \alpha/V) + \sqrt{\{(V\beta_1/c^2 - \alpha/V)^2 + 4\alpha\beta^2(1-V^2/c^2)\}}\right]/(1-V^2/c^2) \tag{4.19}$$

and

$$a_2 = {}^1/_2\left[-(V\beta_1/c_2 - \alpha/V) + \sqrt{\{/V\beta_1/c^2 - \alpha/V)^2 + 4\beta^2(1-V^2/c^2)\}}\right]/(1-V^2/c^2). \tag{4.20}$$

It consequently follows that the assumptions concerning $P_1(\xi)$ are satisfied, and in view of the fact that $P_1(\xi) \to 1$ in the strip $-\tau_1 < \tau < \tau_2$ for $\xi \to \infty$, the function may be represented in the form

$$P_1(\xi) = P_1^+(\xi)P_1^-(\xi) \tag{4.21}$$

where

$$\begin{aligned} P_1^+(\xi) &= \exp\left[\frac{1}{2\pi i}\int_{-\infty+id_2}^{\infty+id_2} \frac{\ln P_1(\eta)}{\eta-\xi}\,d\eta\right], \\ P_1^-(\xi) &= \exp\left[-\frac{1}{2\pi i}\int_{-\infty+id_1}^{\infty+id_1} \frac{\ln P_1(\eta)}{\eta-\xi}\,d\eta\right], \end{aligned} \tag{4.22}$$

where $-\tau_1 < d_1 < d_2 < \tau_2$ and the functions $P_1^\pm(\xi)$ are regular in the respective half-planes $\tau > -\tau_1$ and $\tau < \tau_2$.

It follows from (4.22) and from the fact $P_1(0) = P_1(\infty) = 1$ that these functions satisfy the additional condition $P_1^\pm(0) = P_1^\pm(\infty) = 1$ with the help of (4.15), (4.18), (4.21) and the relation $P(\xi) = \bar{P}(\xi)P_1(\xi)$, Eq. (4.14) becomes

$$\begin{aligned} -\frac{\mu_0(1-V^2/c^2)^{1/2}H_-(\xi)\xi^2}{2(\xi - i\beta_1/V)(\xi - ia_2)^{1/2}P_1^-(\xi)} + \frac{\sigma_0 P_2^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\; i(\xi - i\lambda)(i\lambda + ia)} &= \\ = \frac{P_1^+(\xi)(\xi+ia_1)^{1/2}E_+(\xi)}{(\xi+ia)} - \frac{\sigma_0}{\sqrt{(2\pi)}\; i(\xi-i\lambda)} &\times \\ \times\left[\frac{P_1^+(\xi)(\xi+ia_1)^{1/2}}{(\xi+ia)} - \frac{P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{(i\lambda+ia)}\right]&. \end{aligned} \tag{4.23}$$

Using the same arguments as in Eq. (2.33) we get

$$(4.24)\quad H_-(\xi) = -\frac{2\sigma_0(1-V^2/c^2)^{-1/2}P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}(\xi-ia_2)^{1/2}}{\sqrt{(2\pi)}\;\mu_0(\xi-i\lambda)(\lambda+a)\xi}P_1^-(\xi)(\xi-i\beta_1/V) =$$

$$= -\frac{2\sigma_0(1-V^2/c^2)^{-1/2}P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\;\mu_0(\lambda+a)}\xi^{-3/2}\quad \text{as}\quad \xi\to\infty$$

and

$$(4.25)\quad E_+(\xi) = \frac{\sigma_0}{\sqrt{(2\pi)}\;i(\xi-i\lambda)} - \frac{\sigma_0 P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}(\xi+ia)}{\sqrt{(2\pi)}\;i(\xi-i\lambda)(i\lambda+ia)P_1^+(\xi)(\xi+ia_1)^{1/2}} =$$

$$= -\frac{\sigma_0 P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\;i(i\lambda+ia)}\xi^{-1/2}\quad \text{as}\quad \xi\to\infty.$$

Now taking the inverse Fourier transform we get from Eqs. (4.24) and (4.25)

$$(4.26)\quad h(x) = \frac{4\sigma_0(1-V^2/c^2)^{-1/2}P_1^+(i\lambda)(\lambda+a_1)^{1/2}}{\mu_0(\lambda+a)\sqrt{\pi}}(-x)^{1/2},\quad -1\leqslant x<0$$

and

$$(4.27)\quad e(x) = \frac{\sigma_0\pi^{-1/2}P_1^+(i\lambda)(\lambda+a_1)^{1/2}}{(\lambda+a)}(x)^{-1/2},\quad 0<x\leqslant 1.$$

The corresponding results for the case of constant loading $\sigma_{yz} = -\sigma_0$ $(x<0)$ on the crack surface are obtained by putting $\lambda = 0$ in the above equation. If ΔW is the displacement jump then the crack opening displacement in this case is given by

$$(4.28)\quad \mu_0\Delta W = [4\sigma_0(1-V^2/c^2)^{-1/2}\pi^{-1/2}(a_1)^{1/2}(-x)^{1/2}]/a,\quad -1\leqslant x<0$$

and also the stress near the crack tip is

$$(4.29)\quad \sigma_{yz} = [\sigma_0\pi^{-1/2}(a_1)^{1/2}(x)^{-1/2}]/a,\quad 0<x\leqslant 1\quad (\text{since}\quad P_1^+(0)=1).$$

Therefore the stress intensity factor is equal to

$$(4.30)\quad K = \sqrt{2\sigma_0}\,B_1\quad \text{where}\quad B_1 = (a_1)^{1/2}/a.$$

Now putting $\alpha = 0$ in Eqs. (4.28) and (4.30) we get the crack opening displacement and stress intensity factor for the Maxwell Solid

$$(4.31)\quad \mu_0\Delta W = [4\sigma_0(1-V^2/c^2)^{-1/2}\pi^{-1/2}(\alpha_1)^{1/2}(-x)^{1/2}]/\alpha_2$$

and

$$(4.32)\quad K = \sqrt{2\sigma_0}\,B\quad \text{where}\quad B = (\alpha_1)^{1/2}/\alpha_2$$

which agree with the results given by (3.12) and (3.13) in the Maxwell Solid corresponding to $\lambda = 0$.

5. Results and Discussion

5.1. The Maxwell solid. In this case time variation of the stress intensity factor is given by $K = \sqrt{2\sigma_0 A}$ where A is given by Eq. (2.40) and has been evaluated in the Appendix.

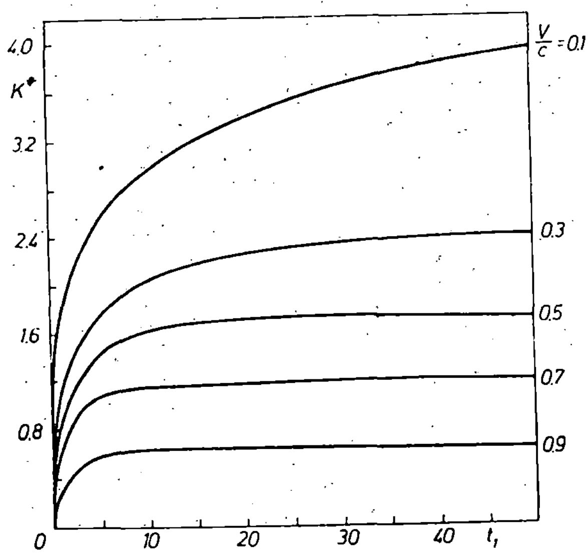

FIG. 3. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0$ (homogeneous medium).

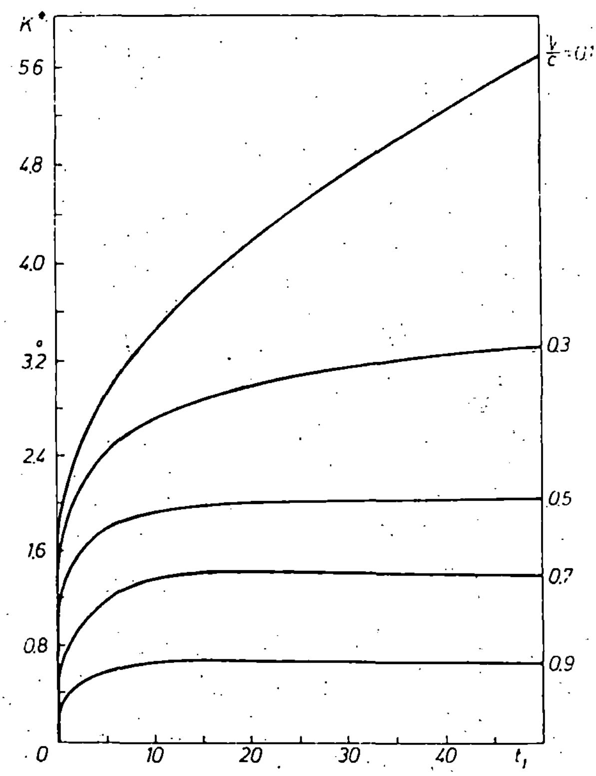

FIG. 4. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0.1$.

The dimensionless stress intensity factor $K^* = (K/\sigma_0)(\beta_1/c)^{1/2}$ has been plotted against $t_1 = \beta_1 t$ for the range of values of $V/c = 0.1, 0.3, 0.5, 0.7$ and 0.9 for different values of the inhomogeneity factor $\beta^* = 4\beta^2 c^2/\beta_1^2$.

It is interesting to note by inspecting the graphs given in Fig. 3, Fig. 4 and Fig. 5 that

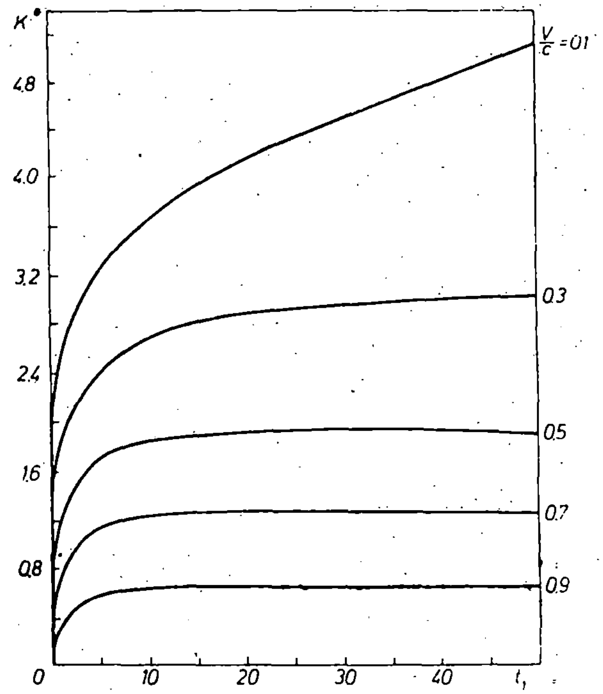

FIG. 5. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0.2$.

the effect of inhomogeneity of the medium introduced through the factor β^* in the stress intensity factor K^* becomes more significant for small values of V/c, whereas for values of V/c differing slightly from unity, the effect of inhomogeneity of the medium on the stress intensity factor is negligible

5.2. Standard linear solid. In this case the stress intensity factor for the steadily propagating crack is given by $K = \sqrt{2}\,\sigma_0 B_1$, where B_1 is given by Eq. (4.30).

We have plotted also the stress intensity factor $K^* = (K/\sigma_0)(\beta_1/c)^{1/2}$ against β^* for various values of V/c, $V/c = 0.5, 0.6, 0.7, 0.8$ and 0.9, and for different values of $\alpha/\beta_1 = 0$, 0.1, 0.2. The case $\alpha/\beta_1 = 0$ corresponds to the steady-state values of K^* for the Maxwell solid. It is evident from the graphs given in Fig. 6. Fig. 7 and Fig. 8 that at large values of α/β_1, values of K^* increase rapidly with the increase in values of β^* if V_0 V/c is very small. But for values of V/c close to unity the variation of K^* with the change in the value of β^* is small showing that the inhomogeneity effect is negligibly in this case. This is also evident from the expressions (4.31) and (4.19).

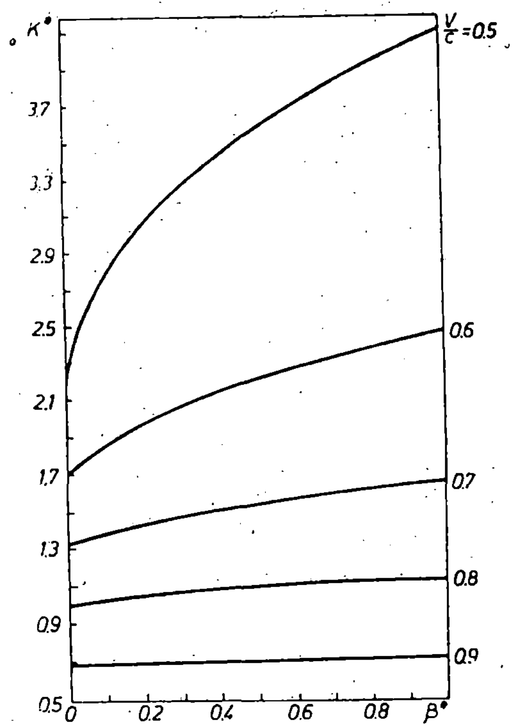

FIG. 6. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0$ (Maxwell solid).

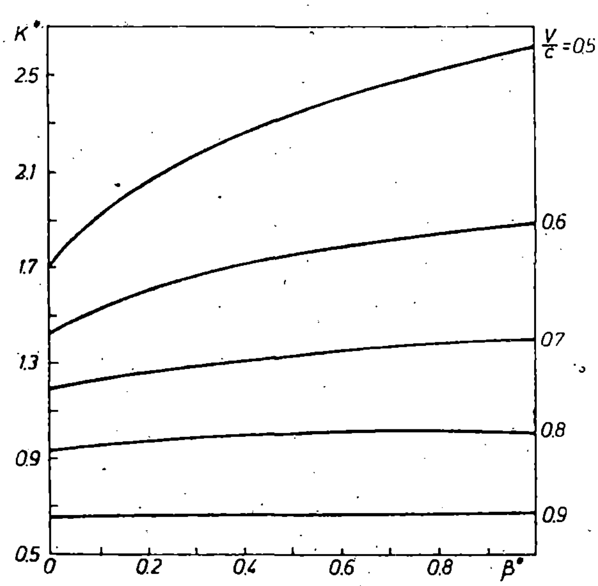

FIG. 7. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0.1$.

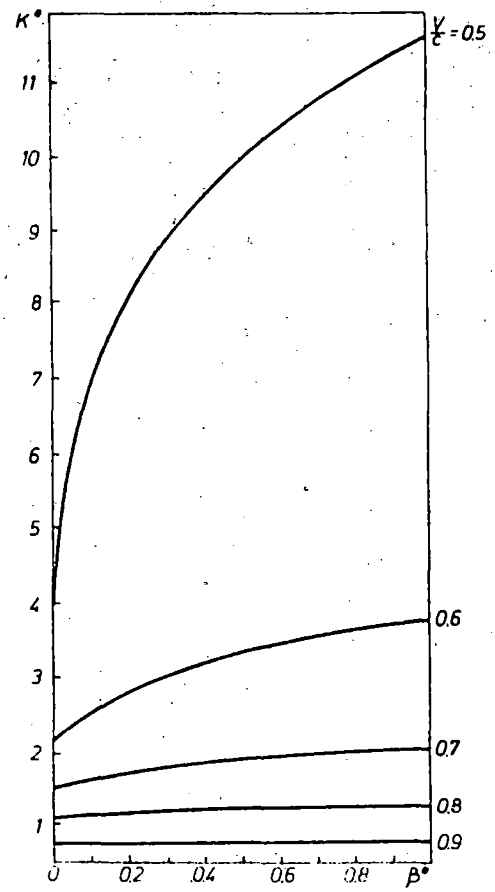

FIG. 8. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0.2$.

Appendix. Evaluation of the Integral A in Eq. (2.40).

The integral

$$A = (1/2\pi i) \int_{c'-i\infty}^{c'+i\infty} \frac{(Y_1)^{1/2}}{pX_1} \exp(pt)dp.$$

The integrand has poles at $p = 0$ and also at $p = -\beta_1$ which correspond to the zero of X_1. Further the integrand has branch points at

$$\delta_1 = (\beta_1/2)\left[-1+\sqrt{\{(1-V^2/c^2)(1-4z)\}}\right],$$
$$\delta_2 = (\beta_1/2)\left[-1-\sqrt{\{(1-V^2/c^2)(1-4z)\}}\right],$$
$$\delta_3 = (\beta_1/2)\left[-1-\sqrt{(1-4z)}\right],$$
$$\delta_4 = (\beta_1/2)\left[-1+\sqrt{(1-V^2/c^2)}\right],$$
$$\delta_5 = (\beta_1/2)\left[-1-\sqrt{(1-V^2(c^2)}\right],$$

where $z = \beta^2c^2/\beta_1^2$ which is assumed to be less than 1/4.

Evidently, $\delta_4 > \delta_1 > \delta_2 > \delta_5 > \delta_3$.

Now taking the branch cut along the negative real axis from δ_4 to $-\infty$, the integral can be considered as a contour integral around the path shown in Fig. 9.

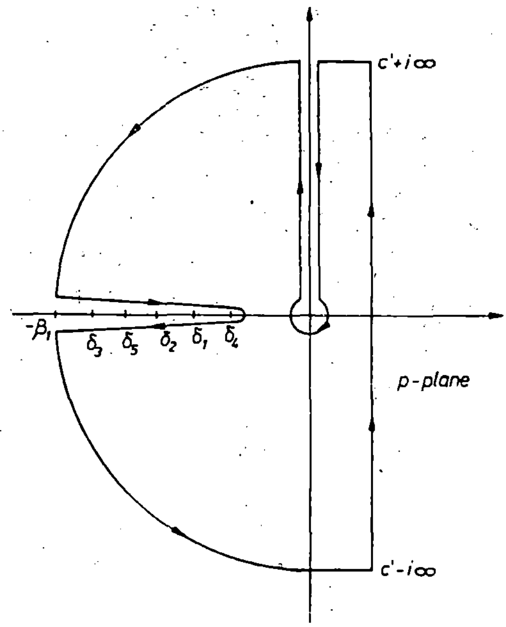

FIG. 9. The integration contour to evaluate A for the Maxwell solid.

Now

$$A = (1/2\pi i)\sqrt{2}\,(1-V^2/c^2)^{1/2}\times$$
$$\times\int_{c'-i\infty}^{c'+i\infty}\frac{[(2p+\beta_1)V/c^2+(2/c)\sqrt{\{(p-\delta_1)(p-\delta_2)\}}]^{1/2}}{p[(2p+\beta_1)V/c^2+(2/c)\sqrt{\{(p-\delta_4)(p-\delta_5)\}}]}\exp(pt)\,dp.$$

It can be shown that

$$A = \sqrt{2}\,(1-V^2/c^2)^{1/2}[{}^1/_2(c/V)\sqrt{(c/\beta_1)}\,\sqrt{\{V/c+\sqrt{(V^2/c^2+4z(1-V^2/c^2))}\}}]+$$
$$+\sqrt{2}\,(1-V^2/c^2)^{1/2}\{\sqrt{(c/\beta_1)}/\pi\}\,[I_1+I_2-I_3-I_4],$$

where

$$I_1 = \int_0^{b_1}\frac{\sqrt{(x_1^{**})}\;y_1^*}{(\delta_4^*-r)R_1^*}\exp[(\delta_4^*-r)t_1]\,dr,$$

$$I_2 = \int_0^{b_2}\frac{\sqrt{[\{R_2^{**}-(x_2^*)^3+(y_2^{**})^2x_2^*-2x_2^*y_2^*y_2^{**}\}/2]}}{(\delta_1^*-r)R_2^*}\exp[(\delta_1^*-r)t_1]\,dr,$$

$$I_3 = \int_{b_2}^{b_3} \frac{\sqrt{(x_3^{**})}\ x^*}{(\delta_1^* - r) R_3^*} \exp[(\delta_1^* - r) t_1)] dr,$$

$$I_4 = \int_{b_3}^{\infty} \frac{\sqrt{(x_3^{**})}}{(\delta_1^* - r) x_4^*} \exp[(\delta_1^* - r) t_1] dr,$$

where

$$\delta_1 = \beta_1 \delta_1^*, \quad \delta_2 = \beta_1 \delta_2^*, \quad \delta_3 = \beta_1 \delta_3^*, \quad \delta_4 = \beta_1 \delta_1^*, \quad \delta_5 = \beta_1 \delta_5^*, \quad t_1 = \beta_1 t,$$

$$b_1 = {}^1/_2 \sqrt{(1 - V^2/c^2)} \left[1 - \sqrt{(1-4z)}\right], \quad b_2 = \sqrt{[(1 - V^2/c^2)(1-4z)]},$$

$$b_3 = {}^1/_2 \sqrt{(1 - V^2/c^2)} \left[1 + \sqrt{(1-4z)}\right],$$

$$x_1^{**} = \left[\sqrt{(1 - V^2/c^2)} - 2r\right](V/c) + 2\sqrt{[r^2 - r\sqrt{(1 - V^2/c^2)} + (1 - V^2/c^2) z]},$$

$$x_1^* = \left[\sqrt{[(1 - V^2/c^2)} - 2r\right](V/c),$$

$$y_1^* = 2\sqrt{[r\sqrt{(1 - V^2/c^2)} - r^2]},$$

$$R_3^* = (x_1^*)^2 + (y_1^*)^2,$$

$$x_2^* = \left[\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - 2r\right](V/c),$$

$$y_2^* = 2\sqrt{[r\sqrt{(1 - V^2/c^2)(1-4z)} - r^2]},$$

$$y_2^{**} = 2\sqrt{[-r^2 + r\sqrt{\{(1 - V^2/c^2)(1-4z)\}} + z(1 - V^2/c^2)]},$$

$$R_2^{**} = [(x_2^*)^2 + (y_2^{**})^2][(x_2^*)^2 + (y_2^*)^2]^{1/2},$$

$$R_2^* = (x_2^*)^2 + (y_2^{**})^2,$$

$$x_3^{**} = -\left[\sqrt{[\{(1 - V^2/c^2)(1-4z)\}} - 2r\right](V/c) + 2\sqrt{[r^2 - r\sqrt{\{(1 - V^2/c^2)(1-4z)\}}]},$$

$$x_3^* = \left[\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - 2r\right](V/c),$$

$$y_3^* = 2\sqrt{[z(1 - V^2(c^2) + r\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - r^2]},$$

$$R_3^* = (x_3^*)^2 + (y_3^*)^2,$$

$$x_4^* = \left[\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - 2r\right](V/c) - 2\sqrt{[r^2 - r\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - z(1 - V^2/c^2)]}.$$

References

1. B. R. BAKER, *Dynamic stress created by a moving crack*, J. Appl. Mech., **15**, 449–459, 1962.
2. G. P. CEREPANOV and E. F. AFANASEV, *Some dynamic problems of the theory of elasticity—a review*, Int. J. Engng. Sci., **12**, 665 - 690, 1974.
3. C. ATKINSON and R. D. LIST, *Steady state crack propagation into media with spatially varying elastic properties*, Int. J. Engng. Sci., **16**, 717 - 730, 1978.
4. C. ATKINSON, *Cracks in media with spatially varying elastic moduli*, Mechanics of Fracture, Vol-4, [Ed.] G. C. SIH, Noordhoff International Publishing, Leyden 1977.
5. J. R. WILLIS, *Crack propagation in a viscoelastic media*, J. Mech. Phys. Solids, **15**, 229–240, 1972.

6. C. Atkinson and R. D. List, *A moving crack problem in a viscoelastic solid*, Int. J. Engng. Sci., **10**, 309-322, 1972.
7. O. Coussy, *A moving crack problem along the interface of two viscoelastic media*, Int. J. Engng. Sci., **25**, 609–620, 1987.
8. L. B. Sills and Y. Benveniste, *Steady state propagation of mode III interface crack between dissimilar viscoelastic media*, Int. J. Engng. Sci., 19, 1255-1268, 1981.
9. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge University Press, London 1969.
10. W. T. Koiter, *Approximate solutions of Wiener–Hopf type equations with applications*, Proc. Kon. Ned. Acad. Wet., **57**, 2, 1954.

Streszczenie

DYNAMICZNA PROPAGACJA ANTYPŁASKIEJ SZCZELINY W NIEJEDNORODNYM OŚRODKU LEPKOSPRĘŻYSTYM

Rozważono model półnieskończonej szczeliny w antypłaskim stanie odkształcenia, która w chwili $t = 0$ zaczyna propagować się ze stałą prędkością w niejednorodnym, liniowo-lepkosprężystym ośrodku. Rozważono makswellowski i standardowy model ciała lepkosprężystego oraz przyjęto, że stałe materiałowe zmieniają się wykładniczo w kierunku prostopadłym do kierunku propagacji. Zastosowanie transformacji całkowych pozwala zredukować zagadnienie do rozwiązania problemu Wienera-Hopfa. Zmienność czasowa współczynnika intensywności naprężenia jest funkcją prędkości propagacji szczeliny i stałych materiałowych ośrodka. Zmienność tę zilustrowano wykresami.

Резюме

ДИНАМИЧЕСКОЕ РАСПРОСТРАНЕНИЕ АНТИПЛОСКОЙ ТРЕЩИНЫ В НЕОДНОРОДНОЙ ВЯЗКОУПРУГОЙ СРЕДЕ

Рассмотрена модель полубесконечной трещины в антиплоском деформационном состоянии, которая в момент $t = 0$ начинает распросраняться с постоянной скоростью в неоднородной, линейно-вязкоупругой среде. Рассмотрены максвелловская и стендартная модели вязкоупругого тела и принято, что материальные постоянные изменяются экспоненциальным образом в перпендикулярном направлении к направлении распространения. Применение интегральных преобразований позволяет свести задачу к решению задачи Випера-Хопфа. Временная переменность коэффициента интенсивности напряжения является функцией скорости распространения трещины и материальных постоянных среды. Эта переменность иллюстрируется диаграммами.

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