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Displacement Produced in an Elastic Half-space by the Impulsive Torsional Motion of a Circular Ring Source

By MUKTIMOY GHOSH[1])

Summary – In this paper the problem of disturbance in an elastic semi-infinite medium due to the torsional motion of a circular ring source on the free surface of a medium are studied. Two cases, when the medium is either homogeneous or inhomogeneous, are treated. In order to solve the problem, the Laplace transform and the Hankel transform and the Laplace inversion by Cagniard's method as modified by DE HOOP (1959) are applied. Finally, the integrals for displacement are evaluated numerically. The displacement on the free surface as a function of time is shown by means of graphs, in the case of both a homogeneous and an inhomogeneous medium, indicating clearly the variation in displacement due to the presence of an inhomogeneity.

Key words: Theoretical seismology; Torsional ring source; Cagniard–de Hoop transformation.

1. Introduction

At present much attention has been given to problems concerned with wave propagation in homogeneous as well as in inhomogeneous, isotropic, elastic media. Much of this work has been connected with problems of seismological interest, involving wave propagation. The normal loading problem of an elastic half-space was first investigated by LAMB (1904). This type of problem was then investigated by EASON (1964), MITRA (1964), CHAKRABORTY and DE (1971) and many others. In fact a class of elastic half-space problems involving an axisymmetric, normally applied, surface load ~~is~~ was investigated by GAKENHEIMER (1971). He assumed that loads suddenly emanate from a point on the surface and expand radially at a constant rate. He used Cagniard's method to evaluate the inverse transforms. This paper has a particular reference to the work by GHOSH (1971) where techniques similar to those adopted here, are used. Many recent studies on elastic wave propagation are due to the work of CAGNIARD (1962), who developed a particular technique of finding the Laplace inversion, that has been found to be extremely useful in dealing with problems of this type.

The type of disturbing force considered in this paper is impulsive in time and acts over the circumference of a circular region of constant radius on the free surface of a semi-infinite, isotropic, elastic half-space. The effect of the inhomogeneity

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of the medium on the disturbance produced is determined in the integral form, whereas the displacement in the case of a homogeneous medium is determined exactly. The displacement at any point on the free surface is evaluated numerically and the graphs are drawn to show how the vibration of a point in the medium is affected due to the inhomogeneity of the medium, which enters into the expression for displacement through the factor ε.

Case I: Homogeneous medium

2. Formulation of the problem

Let (r, θ, z) be the cylindrical polar co-ordinates, z-axis being directed into the isotropic elastic medium, the plane boundary being $z = 0$ with the origin at the centre of the ring source $r = a$, $z = 0$.

The displacement is calculated at points inside and on the free surface of the medium, subject to the condition that the half-space is initially at rest and that the displacement remains bounded even for large values of z. For torsional motion of the ring all quantities depend on r, z and the time t, the only non-zero component of the displacement vector is the component v along the direction of θ increasing. The relevant non-vanishing stress components are

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \tag{1}$$

and

$$\tau_{\theta z} = \mu \frac{\partial v}{\partial z} \tag{2}$$

where μ is Lame's constant. The only non-zero equation of motion is

$$\frac{\partial}{\partial r}(\tau_{r\theta}) + \frac{\partial}{\partial z}(\tau_{\theta z}) + 2\frac{\tau_{r\theta}}{r} = \rho\frac{\partial^2 v}{\partial t^2} \tag{3}$$

where ρ is the density of the material, assumed constant. The boundary condition is

$$\tau_{\theta z} = P\delta(r - a)\delta(t) \tag{4}$$

where P is a constant, a is the radius of the ring source and $\delta(t)$ is Dirac's delta-function.

Using (1) and (2) the equation (3) can be written in the form

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} \tag{5}$$

where $\beta = \sqrt{(\mu/\rho)}$ is the shear wave velocity.

3. *Method of solution*

We define for all positive real values of s the Laplace transform $f_1(r, z, s)$ of a function $f(r, z, t)$ by the relation

$$f_1(r, z, s) = \int_0^\infty f(r, z, t)\, e^{-st}\, dt. \tag{6}$$

Applying the Laplace transform (6) to the equation (5) we obtain

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right) + \frac{\partial^2 v_1}{\partial z^2} = \frac{s^2 v_1}{\beta^2}. \tag{7}$$

Define the Hankel transform $v_2(\xi, z, s)$ of $v_1(r, z, s)$ by the equation

$$v_2(\xi, z, s) = \int_0^\infty r J_1(\xi r) v_1(r, z, s)\, dr, \tag{8}$$

where J_1 is a Bessel function.

Multiplying the equation (7) by $rJ_1\ (\xi r)$ and integrating with respect to r from 0 to ∞ we get,

$$\frac{d^2 v_2}{dz^2} = \left(\xi^2 + \frac{s^2}{\beta^2}\right) v_2. \tag{9}$$

The general solution of this equation which remains bounded as $z \to +\infty$ is

$$v_2 = A \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right], \tag{10}$$

where A is to be determined from the boundary conditions, $\tau_{\theta z_1} = P\delta(r - a)$, where $\tau_{\theta z_1}$ is the Laplace transform of $\tau_{\theta z}$. From the Hankel transform $(\tau_{\theta z_1})_2$ of $\tau_{\theta z_1}$, we obtain by using (2)

$$(\tau_{\theta z_1})_2 = \mu \frac{dv_2}{dz} = PaJ_1(\xi a)$$

on $z = 0$, $v_2 = A$ and $dv_2/dz = -A(\xi^2 + s^2/\beta^2)^{1/2}$.

Using these relations in equation (10) we get

$$A = -\frac{Pa}{\mu} \frac{J_1(\xi a)}{(\xi^2 + s^2/\beta^2)^{1/2}}.$$

Substituting the value of A in (10) and inverting the Hankel transform (8), we obtain

$$v_1(r, z, s) = -\frac{Pa}{\mu} \int_0^\infty \frac{\xi J_1(\xi a) J_1(\xi r)}{(\xi^2 + s^2/\beta^2)^{1/2}} \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right] d\xi. \tag{11}$$

From a well-known result (WATSON (1966), p. 358)

$$J_1(\xi r)J_1(\xi a) = \frac{1}{\pi}\int_0^{\pi} J_0(\xi R)\cos\phi\, d\phi,$$

and

$$J_0(\xi R) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi R\sin\psi}\, d\psi \qquad \text{(ERDELYI (1953), p. 14)}$$

where $R = \sqrt{(r^2 + a^2 - 2ar\cos\phi)}$, we obtain

$$\frac{2\pi^2\mu v_1}{Pa} = -\int_0^{\pi} I_1 \cos\phi\, d\phi \tag{12}$$

where

$$I_1 = \int_0^{2\pi}\int_0^{\infty} \frac{\xi \exp\left[-z(\xi^2 + s^2/\beta^2)^{1/2} + i\xi R\sin\psi\right]}{(\xi^2 + s^2/\beta^2)^{1/2}}\, d\xi\, d\psi.$$

If we put $p = \xi\sin\psi$ and $q = \xi\cos\psi$ in I_1, then

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\left[-z(p^2 + q^2 + s^2/\beta^2)^{1/2} + iRp\right]}{(p^2 + q^2 + s^2/\beta^2)^{1/2}}\, dp\, dq. \tag{13}$$

To find the inversion of I_1, we adopt Cagniard's technique as modified by DE HOOP (1959). Accordingly in (13), we put $p = ms$ *and* $q = ns$, then

$$I_1 = 2\int_0^{\infty} dn \int_{-\infty}^{\infty} \frac{s\exp\{-s[z(m^2 + n^2 + 1/\beta^2)^{1/2} - iRm]\}}{(m^2 + n^2 + 1/\beta^2)^{1/2}}\, dm. \tag{14}$$

In the above integral the path of integration with respect to m is the real axis (Fig. 1) which is deformed in such a way that $-iRm + z(m^2 + n^2 + 1/\beta^2)^{1/2} = t$, where t is real and positive. The deformed path of integration is the branch Γ of a hyperbola whose equation is

$$m = \frac{iRt \pm z[t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)]^{1/2}}{z^2 + R^2}, \qquad \{(z^2 + R^2)(n^2 + 1/\beta^2)\}^{1/2} < t < \infty.$$

In the course of deformation of the path of integration it is essential to know

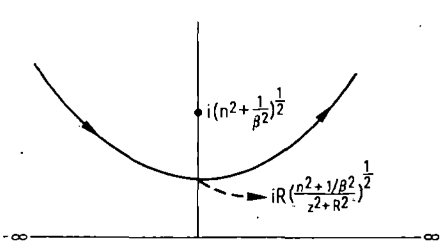

Figure 1
Paths of integration in the complex m-plane.

the singularities of the function $s/(m^2 + n^2 + 1/\beta^2)^{1/2}$ in the m-plane, which are the branch points $\pm i(n^2 + 1/\beta^2)^{1/2}$.

Since the hyperbolic path Γ does not cross any of the singularities during its deformation, it is possible by virtue of Cauchy's theorem and Jordon's lemma, to replace the integration along the real m-axis by an integration along the hyperbolic path Γ.

We assume

$$m_+ = \frac{iRt + z[t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)]^{1/2}}{z^2 + R^2}$$

and

$$m_- = \frac{iRt - z[t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)]^{1/2}}{z^2 + R^2}.$$

The point where Γ cuts the imaginary axis is given by

$$t = \{(z^2 + R^2)(n^2 + 1/\beta^2)\}^{1/2}$$

and the point is

$$m = \frac{iR(n^2 + 1/\beta^2)^{1/2}}{(z^2 + R^2)^{1/2}}$$

which is below the branch point $i(n^2 + 1/\beta^2)^{1/2}$. Hence (14) can be written as

$$I_1 = 2\int_0^\infty dn \int_{\{(z^2 + R^2)(n^2 + 1/\beta^2)\}^{1/2}}^\infty s\, e^{-st}\left[\frac{1}{(m_+^2 + n^2 + 1/\beta^2)^{1/2}}\frac{dm_+}{dt} - \frac{1}{(m_-^2 + n^2 + 1/\beta^2)^{1/2}}\frac{dm_-}{dt}\right] dt. \tag{15}$$

Now using the fact that $m_- = -\bar{m}_+$ and $dm_-/dt = -(d\bar{m}_+/dt)$ where $\bar{m}$ is the complex conjugate of m, (15) can be written as

$$I_1 = 4\int_{(z^2 + R^2)^{1/2}/\beta}^\infty s\, e^{-st}\, dt \int_0^{[t^2/(z^2 + R^2) - 1/\beta^2]^{1/2}} Rl\left[\frac{dm_+/dt}{(m_+^2 + n^2 + 1/\beta^2)^{1/2}}\right] dn. \tag{16}$$

Now,

$$Rl\left[\frac{(dm_+/dt)}{(m_+^2 + n^2 + 1/\beta^2)^{1/2}}\right] = \frac{1}{\{t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)\}^{1/2}}.$$

Substituting this result in (16), we obtain

$$I_1 = \frac{2\pi}{(z^2 + R^2)^{1/2}}\int_{[(z^2 + R^2)^{1/2}]/\beta}^\infty s\, e^{-st}\, dt.$$

Hence the Laplace inversion of I_1 is

$$I = \frac{2\pi}{(z^2 + R^2)^{1/2}}\frac{d}{dt}\left[H\left\{t - \frac{(z^2 + R^2)^{1/2}}{\beta}\right\}\right]$$

$$= \frac{2\pi}{(z^2 + R^2)^{1/2}}\,\delta\left[t - \frac{(z^2 + R^2)^{1/2}}{\beta}\right]. \tag{17}$$

Therefore the Laplace inversion of (12) by using the Laplace inversion of I_1 as given in (17) is

$$v(r, z, t) = -\frac{Pa}{\pi\mu}\int_0^\pi \frac{\delta\left[t - \frac{(z^2 + r^2 + a^2 - 2ra\cos\phi)^{1/2}}{\beta}\right]}{(z^2 + r^2 + a^2 - 2ra\cos\phi)^{1/2}} \cos\phi \, d\phi. \quad (18)$$

To evaluate the above integral we put

$$(z^2 + r^2 + a^2 - 2ra\cos\phi)^{1/2} = \beta\theta,$$

then

$$v(r, z, t) = \frac{P\beta}{\pi\mu r}\frac{\beta^2 t^2 - z^2 - r^2 - a^2}{\{2(r^2 + a^2)(\beta^2 t^2 - z^2) - (r^2 - a^2)^2 - (\beta^2 t^2 - z^2)^2\}^{1/2}}$$

for

$$\frac{\{z^2 + (r - a)^2\}^{1/2}}{\beta} < t < \frac{\{z^2 + (r + a)^2\}^{1/2}}{\beta}. \quad (19)$$

Case II: Inhomogeneous Medium

4. Formulation of the problem

In this case the same problem of torsional motion of a semi-infinite elastic medium due to the presence of a ring source $r = a$, on the free surface $z = 0$ as in Case I is considered. The only difference is that the medium under consideration is inhomogeneous in nature, the coefficient of rigidity and the density of the medium are assumed to be

$$\mu = \mu_0(1 + \varepsilon z)^2 \qquad \text{and} \qquad \rho = \rho_0(1 + \varepsilon z)^2. \quad (20)$$

Here also the non-vanishing stress components and the non-zero equations of motion are the same as in Case I, given by the equations (1), (2) and (3).

5. Method of solution

Firstly we put $\bar{v} = (1 + \varepsilon z)v$ in the equations (1), (2) and (3). The transformed equations are

$$\left.\begin{aligned} \tau_{r\theta} &= \mu_0(1 + \varepsilon z)\left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right), \\ \tau_{\theta z} &= \mu_0\left\{(1 + \varepsilon z)\frac{\partial \bar{v}}{\partial z} - \varepsilon \bar{v}\right\} \end{aligned}\right\} \quad (21)$$

and

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) + \frac{\partial^2 \bar{v}}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 \bar{v}}{\partial t^2} \quad (22)$$

where $\beta = \sqrt{(\mu_0/\rho_0)}$.

Taking the Laplace transform of the equation with respect to t, we obtain

$$\frac{\partial^2 \bar{v}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}_1}{\partial r} - \left(\frac{1}{r^2} + \frac{s^2}{\beta^2}\right)\bar{v}_1 + \frac{\partial^2 \bar{v}_1}{\partial z^2} = 0 \tag{23}$$

where s is the Laplace transform parameter which is real and positive. Taking the Hankel transform of the equation (23) we have

$$\frac{d^2 \bar{v}_2}{dz^2} = \left(\xi^2 + \frac{s^2}{\beta^2}\right)\bar{v}_2. \tag{24}$$

The general solution of this equation which remains bounded for large values of z is

$$\bar{v}_2 = B \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]. \tag{25}$$

Applying the Hankel transform and the Laplace transform on the boundary condition

$$\tau_{\theta z} = \mu_0\left[(1 + \varepsilon z)\frac{\partial \bar{v}}{\partial z} - \varepsilon \bar{v}\right] = P\delta(r - a)\delta(t)$$

and using (25), the value of B is found to be

$$B = -\frac{PaJ_1(\xi a)}{\mu_0\{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}\}}.$$

Substituting this value of B in (25), it follows that

$$\bar{v}_2 = -\frac{PaJ_1(\xi a)}{\mu_0\{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}\}} \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]. \tag{26}$$

Taking the Hankel inversion of (26), we have

$$\bar{v}_1 = -\frac{Pa}{\mu_0}\int_0^\infty \frac{\xi J_1(\xi a)J_1(\xi r)}{\{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}\}} \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right] d\xi. \tag{27}$$

Now,

$$\int_0^\infty \exp\left[-k\{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}\}\right] dk = \frac{1}{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}}.$$

Using the above result, (27) is written as

$$\bar{v}_1 = -\frac{Pa}{\mu_0}\int_0^\infty e^{-\varepsilon k}\, dk \int_0^\infty \xi J_1(\xi a)J_1(\xi r) \exp\left[-(z + k)\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right] d\xi. \tag{28}$$

We now replace $J_1(\xi a)J_1(\xi r)$ of (28) by the integral, which was used to modify equation (11). Finally we get

$$\bar{v}_1 = -\frac{Pa}{2\pi^2\mu_0}\int_0^\infty e^{-\varepsilon k}\, dk \int_0^\pi I_2 \cos\phi\, d\phi, \tag{29}$$

where

$$I_2 = \int_0^{2\pi} d\psi \int_0^{\infty} \exp\left[-(z+k)\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2} + i\xi R \sin\psi\right] d\xi.$$

Assuming $p = \xi \sin\psi$ and $q = \xi \cos\psi$, it follows that

$$I_2 = 2\int_0^{\infty} dn \int_{-\infty}^{\infty} s^2 \exp\left[-s\left\{(z+k)\left(m^2 + n^2 + \frac{1}{\beta^2}\right)^{1/2} - iRm\right\}\right] dm, \quad (30)$$

where, $p = ms$ and $q = ns$.

As in Case I, here also the path of integration with respect to m which is the real axis is deformed such that $-iRm + (z+k)(m^2 + n^2 + 1/\beta^2)^{1/2} = t$, where t is real and positive. The deformed path is a branch Γ_1 of a hyperbola the equation of which is

$$m = \frac{iRt \pm (z+k)[t^2 - \{(z+k)^2 + R^2\}(n^2 + 1/\beta^2)]^{1/2}}{(z+k)^2 + R^2},$$

$$\{(z+k)^2 + R^2\}^{1/2}\left(n^2 + \frac{1}{\beta^2}\right)^{1/2} < t < \infty.$$

Noting that the point where Γ_1 cuts the imaginary axis is

$$m = \frac{iR(n^2 + 1/\beta^2)^{1/2}}{\{(z+k)^2 + R^2\}^{1/2}}$$

when

$$t = \{(z+k)^2 + R^2\}^{1/2}\left(n^2 + \frac{1}{\beta^2}\right)^{1/2},$$

one gets from the equation (30)

$$I_2 = 4\int_0^{\infty} dn \int_{\{(z+k)^2 + R^2\}^{1/2}(n^2 + 1/\beta^2)^{1/2}}^{\infty} s^2 e^{-st} Rl\left(\frac{dm_+}{dt}\right) dt$$

$$= \frac{2\pi(z+k)}{\{(z+k)^2 + R^2\}^{3/2}} \int_{[\{(z+k)^2 + R^2\}^{1/2}]/\beta}^{\infty} ts^2 e^{-st}\, dt. \quad (31)$$

Hence the Laplace inversion of (31) is

$$I = \frac{2\pi(z+k)}{\{(z+k)^2 + R^2\}^{3/2}}\left\{2\delta\left[t - \left(\frac{(z+k)^2 + R^2}{\beta^2}\right)^{1/2}\right] + t\delta'\left[t - \left(\frac{(z+k)^2 + R^2}{\beta^2}\right)^{1/2}\right]\right\}.$$

On substitution of this value of I in (29), it is found that

$$\bar{v} = -\frac{Pa}{\pi\mu_0}\int_0^{\infty} (z+k) J e^{-\varepsilon k}\, dk. \quad (32)$$

where

$$J = \int_0^\pi \left\{2\delta\left[t - \left(\frac{(z+k)^2 + R^2}{\beta^2}\right)^{1/2}\right] + t\delta'\left[t - \left(\frac{(z+k)^2 + R^2}{\beta^2}\right)^{1/2}\right]\right\} \times \frac{\cos\phi}{\{(z+k)^2 + R^2\}^{3/2}}\, d\phi.$$

To evaluate the above integral we put

$$l = \frac{1}{\beta}\{(z+k)^2 + R^2\}^{1/2},$$

then

$$J = \int_{(1/\beta)\{(z+k)^2+(r-a)^2\}^{1/2}}^{(1/\beta)\{(z+k)^2+(r+a)^2\}^{1/2}} \{2\delta(t-l) + t\delta'(t-l)\}\frac{\cos\phi}{l^3\beta^3}\frac{d\phi}{dl}\, dl,$$

where

$$\frac{d\phi}{dl} = \frac{\beta^2 l}{ra\sin\phi} \quad \text{and} \quad \cos\phi = \frac{(z+k)^2 + r^2 + a^2 - \beta^2 l^2}{2ra}.$$

Substituting these values, we get

$$J = \frac{1}{ra\beta}\int_{(1/\beta)\{(z+k)^2+(r-a)^2\}^{1/2}}^{(1/\beta)\{(z+k)^2+(r+a)^2\}^{1/2}} f(l,k)[2\delta(t-l) + t\delta'(t-l)]\, dl, \tag{33}$$

where

$$f(l,k) = \frac{(z+k)^2 + r^2 + a^2 - \beta^2 l^2}{l^2[2(r^2+a^2)\{\beta^2 l^2 - (z+k)^2\} - (r^2-a^2)^2 - \{\beta^2 l^2 - (z+k)^2\}^2]^{1/2}}$$

and it is to be remembered that δ' is the derivative of the Dirac's δ-function with respect to t. Integrating (33), we obtain

$$J = \frac{1}{ra\beta}[2f(t,k) - tf(l_1,k)\delta(t-l_1) + tf(l_2,k)\delta(t-l_2) + tf'(t,k)] \tag{34}$$

where

$$l_1 = \frac{1}{\beta}\{(z+k)^2 + (r+a)^2\}^{1/2}, \quad l_2 = \frac{1}{\beta}\{(z+k)^2 + (r-a)^2\}^{1/2} \quad l_2 < t < l_1.$$

It is to be noted that if t does not belong to (l_2, l_1) then the integrand in (33) is zero, consequently $J = 0$.

Substituting the value of J in (32), we get

$$\bar{v} = -\frac{P}{\pi\mu_0\beta r}\int_0^\infty (z+k)\, e^{-\varepsilon k}[2f(t,k) - tf(l_1,k)\delta(t-l_1) + tf(l_2,k)\delta(t-l_2) + tf'(t,k)]\, dk. \tag{35}$$

Now, $l_2 < t < l_1$ implies that

$$\{\beta^2 t^2 - (r+a)^2\}^{1/2} - z \leq k \leq \{\beta^2 t^2 - (r-a)^2\}^{1/2} - z. \tag{36}$$

In evaluating the integral (35), the following sub-cases are to be considered, keeping in mind that k satisfies (36) and that k is positive.

i) If $\{\beta^2 t^2 - (r - a)^2\}^{1/2} - z < 0$, that is, if $\beta t < \{z^2 + (r - a)^2\}^{1/2}$ then, t does not belong to (l_2, l_1), so $J = 0$. Consequently $\bar{v} = 0$. This is in accordance with the physical condition of the problem because a disturbance cannot reach a point Q (Fig. 2) before the time $(1/\beta)\{z^2 + (r - a)^2\}^{1/2}$, which is the time of arrival of the disturbance at the point Q from the nearest point of the ring source.

ii) $\{\beta^2 t^2 - (r + a)^2\}^{1/2} - z < 0 < \{\beta^2 t^2 - (r - a)^2\}^{1/2} - z$, that is,

$$\{z^2 + (r - a)^2\}^{1/2} < \beta t < \{z^2 + (r + a)^2\}^{1/2}.$$

In this case (35) takes the form

$$\bar{v} = -\frac{P}{\pi\mu_0\beta r}\int_0^{\{\beta^2 t^2 - (r-a)^2\}^{1/2} - z} (z + k)\, e^{-\varepsilon k}[2f(t, k) - tf(l_1, k)\delta(t - l_1) + tf(l_2, k)\delta(t - l_2) + tf'(t, k)]\, dk. \tag{37}$$

The integrand of (37) is considered as a generalized function, so the finite part of the integral (37) is retained (JONES (1966), p. 89) and we get

$$\bar{v} = \frac{P\beta}{\pi r\mu_0}\frac{\beta^2 t^2 - z^2 - r^2 - a^2}{[2(r^2 + a^2)(\beta^2 t^2 - z^2) - (r^2 - a^2)^2 - (\beta^2 t^2 - z^2)^2]^{1/2}}$$
$$+ \frac{P\beta\varepsilon}{\pi r\mu_0}\int_0^{\{\beta^2 t^2 - (r-a)^2\}^{1/2} - z}$$
$$\times \frac{\{(z + k)^2 + r^2 + a^2 - \beta^2 t^2\}\, e^{-\varepsilon k}\, dk}{[2(r^2 + a^2)\{\beta^2 t^2 - (z + k)^2\} - (r^2 - a^2)^2 - \{\beta^2 t^2 - (z + k)^2\}^2]^{1/2}}$$

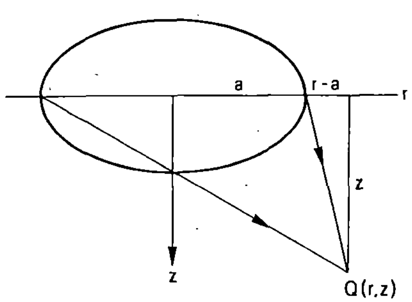

Figure 2
Arrival of the direct wave to Q from the nearest and the farthest point of the source.

Hence

$$v = \frac{P\beta}{\pi r \mu_0 (1 + \varepsilon z)} \frac{\beta^2 t^2 - z^2 - r^2 - a^2}{[2(r^2 + a^2)(\beta^2 t^2 - z^2) - (r^2 - a^2)^2 - (\beta^2 t^2 - z^2)^2]^{1/2}}$$

$$+ \frac{P\beta\varepsilon}{\pi r \mu_0 (1 + \varepsilon z)} \int_0^{\{\beta^2 t^2 - (r-a)^2\}^{1/2} - z}$$

$$\times \frac{\{(z+k)^2 + r^2 + a^2 - \beta^2 t^2\} e^{-\varepsilon k} \, dk}{[2(r^2 + a^2)\{\beta^2 t^2 - (z+k)^2\} - (r^2 - a^2)^2 - \{\beta^2 t^2 - (z+k)^2\}^2]^{1/2}}. \tag{38}$$

In (38) if we put $\varepsilon = 0$, we get the same result that we have determined in (19) of Case I.

iii) If $\{\beta^2 t^2 - (r + a)^2\}^{1/2} - z > 0$, that is if $\beta t > \{z^2 + (r + a)^2\}^{1/2}$ then

$$v = \frac{P\beta\varepsilon}{\pi r \mu_0 (1 + \varepsilon z)} \int_{\{\beta^2 t^2 - (r-a)^2\}^{1/2} - z}^{\{\beta^2 t^2 - (r-a)^2\}^{1/2} - z}$$

$$\times \frac{\{(z+k)^2 + r^2 + a^2 - \beta^2 t^2\} e^{-\varepsilon k} \, dk}{[2(r^2 + a^2)\{\beta^2 t^2 - (z+k)^2\} - (r^2 - a^2)^2 - \{\beta^2 t^2 - (z+k)^2\}^2]^{1/2}}. \tag{39}$$

It is interesting to note that in the case of a homogeneous medium there is no displacement at a point Q (Fig. 2) after the time $t = (1/\beta)\{z^2 + (r + a)^2\}^{1/2}$, which is the time required by the disturbance to reach the point Q directly from the farthest point on the ring source from the point Q. But in the case of an inhomogeneous medium the disturbance reaches a point Q even after the time $t = (1/\beta)\{z^2 + (r + a)^2\}^{1/2}$ which is the maximum time required by a direct wave to reach the point Q from the farthest point on the source from the point Q. This is due to the fact that in the case of an inhomogeneous medium the region $z > 0$ may be considered as an assembly of an infinite number of thin layers of material of infinitesimal thickness of continuously varying density and coefficient of rigidity. That is why the disturbance, which reaches the point Q after successive reflection and refraction in different layers of the medium, arrives at Q after the time $\beta t = \{z^2 + (r + a)^2\}^{1/2}$. The disturbance comes continuously after the time $\beta t = \{z^2 + (r + a)^2\}^{1/2}$ with decreasing intensity.

6. *Numerical solution on the free surface $z = 0$*

In order to obtain the displacement on the free surface we make the substitution

$$[2(r^2 + a^2)(\beta^2 t^2 - k^2) - (r^2 - a^2)^2 - (\beta^2 t^2 - k^2)^2]^{1/2} = 2ra \sin\theta.$$

which transforms the equations (38) and (39) to the forms given by

$$\frac{v \pi \mu_0 a}{P\beta} = d = d_1 + d_2$$

where

$$d_1 = \frac{a}{r}\frac{\dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1}{\left[2\left(\dfrac{r^2}{a^2}+1\right)\dfrac{\beta^2 t^2}{a^2} - \left(\dfrac{r^2}{a^2}-1\right)^2 - \dfrac{\beta^4 t^4}{a^4}\right]^{1/2}},$$

$$d_2 = \varepsilon a \int_0^{\cos^{-1} A} \frac{\cos\theta \exp\left\{-\varepsilon a\left[2\dfrac{r}{a}\cos\theta + \dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1\right]^{1/2}\right\}}{\left[2\dfrac{r}{a}\cos\theta + \dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1\right]^{1/2}}\, d\theta, \tag{40}$$

$$A = \frac{r^2 + a^2 - \beta^2 t^2}{2ra}, \qquad r - a < \beta t < r + a,$$

and

$$\frac{v\pi\mu_0 a}{P\beta} = d' = \varepsilon a \int_0^{\pi} \frac{\cos\theta \exp\left\{-\varepsilon a\left[2\dfrac{r}{a}\cos\theta + \dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1\right]^{1/2}\right\}}{\left[2\dfrac{r}{a}\cos\theta + \dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1\right]^{1/2}}\, d\theta, \tag{41}$$

for $\beta t > r + a$ respectively.

If $\varepsilon a = 0$, then from (40) it follows that $d = d_1$, which corresponds to the displacement inhomogeneous medium. The integrals in (40) and (41) giving the displacements d and d' have been numerically evaluated for different values of εa at different points on the free surface and are presented in Tables 1–4 for different values of $\beta t/a$.

Concluding remarks

From Tables 1–4 it is found that the difference in the values of the displacement at any point corresponding to $\varepsilon a = 0$ and $\varepsilon a = 10$ gradually diminishes with the

Table 1

$r/a = 2$, $(r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\varepsilon a = 0$	d when $\varepsilon a = 1$	d when $\varepsilon a = 10$
1.2	−0.97596	−0.32841	−0.50851
1.4	−0.58468	−0.08456	−0.41435
1.6	−0.38490	0.00497	−0.31149
1.8	−0.24498	0.05256	−0.21268
2.0	−0.12909	0.08585	−0.11623
2.2	−0.02001	0.11644	−0.01716
2.4	0.09676	0.15276	0.09355
2.6	0.24498	0.20795	0.23612
2.8	0.50411	0.32902	0.48230

Table 2

$r/a = 10, (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\varepsilon a = 0$	d when $\varepsilon a = 1$	d when $\varepsilon a = 10$
9.2	−0.14221	−0.00782	−0.13509
9.4	−0.08155	−0.00314	−0.08070
9.6	−0.04927	−0.00063	−0.04911
9.8	−0.02559	0.00324	−0.02556
10.0	−0.00500	0.00868	−0.00500
10.2	0.01537	0.01588	0.01537
10.4	0.03834	0.02557	0.03833
10.6	0.06901	0.03990	0.06900
10.8	0.12546	0.06799	0.12542

Table 3

$r/a = 50, (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\varepsilon a = 0$	d when $\varepsilon a = 1$	d when $\varepsilon a = 10$
49.2	−0.02700	−0.00822	−0.02699
49.4	−0.01525	−0.00693	−0.01525
49.6	−0.00894	−0.00474	−0.00894
49.8	−0.00428	−0.00232	−0.00428
50.0	−0.00020	0.00028	−0.00020
50.2	0.00387	0.00318	0.00387
50.4	0.00851	0.00665	0.00851
50.6	0.01475	0.01130	0.01475
50.8	0.02633	0.01944	0.02633

Table 4

$r/a = 2, (\beta t/a) > (r/a) + 1$

$\beta t/a$	d' when $\varepsilon a = 1$	d' when $\varepsilon a = 10$
3.2	−0.17211	
3.4	−0.07793	
3.6	−0.04250	
3.8	−0.02533	
4.0	−0.01593	d' is of the order of 10^{-7}
4.2	−0.01040	
4.4	−0.00697	
4.6	−0.00477	
4.8	−0.00332	

When $r = 10a$ or $\varepsilon a = 10$, d' is very small.

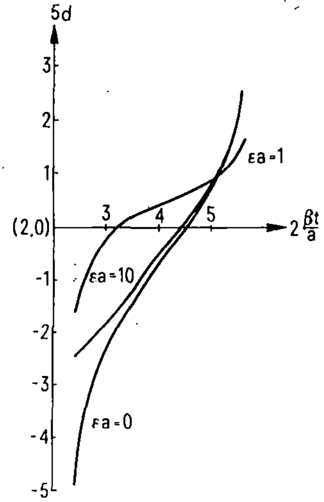

Figure 3
$r = 2a$, variation in displacement near the source for $\varepsilon a = 0, 1, 10$.

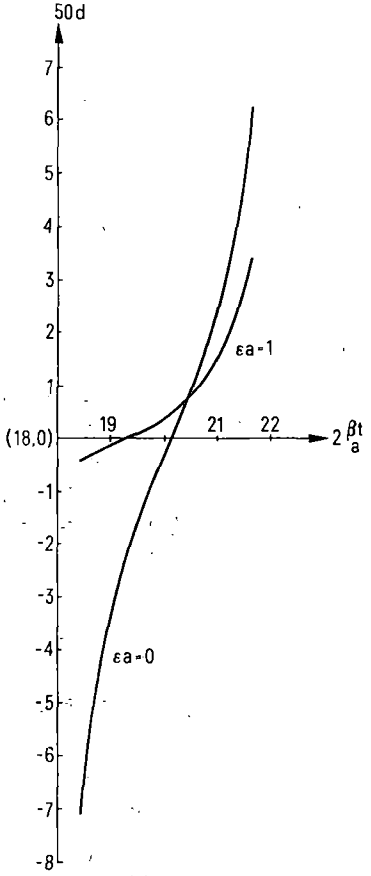

Figure 4
$r = 10a$, variation in displacement at a moderate distance from the source for $\varepsilon a = 0, 1$.

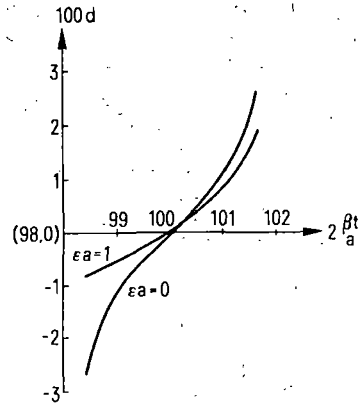

Figure 5
$r = 50a$, variation in displacement at a large distance from the source for $\varepsilon a = 0, 1$.

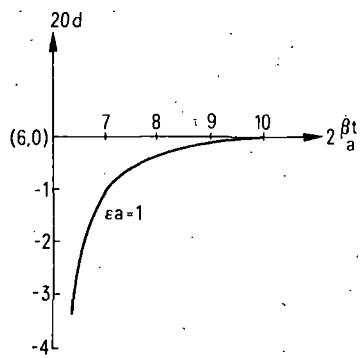

Figure 6
$r = 2a$, variation in displacement after the maximum time required by a direct wave to arrive from the farthest point of the source when $\varepsilon a = 1$.

increase in the value of r/a. This is also apparent from the expression for d_2 in (40) because the exponential term

$$\exp\left\{-\varepsilon a\left[2\frac{r}{a}\cos\theta + \frac{\beta^2 t^2}{a^2} - \frac{r^2}{a^2} - 1\right]^{1/2}\right\}$$

in the integrand for large values of r/a decreases rapidly with the increase in value of εa.

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SH-WAVES IN AN ELASTIC HALF SPACE DUE TO A RING SOURCE OF INCREASING RADIUS.

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ABSTRACT : It is assumed that the radius of a ring source on free surface is increasing with constant velocity c which is less than the shear wave velocity. Following Cagniard's method as modified by De-Hoop, the displacement produced at any point has been determined in the integral form, from which the displacement at any point just after the arrival of the disturbance has been evaluated. The displacement at any point has also been calculated after sufficiently large time.

1. **INTRODUCTION :** The torsional vibration of an elastic half space due to a surface force which is periodic in time was first considered by Reissner (1937). Reissner and Sagoci (1944) determined the distribution of the stresses in the interior of a semi-infinite, homogenous isotropic elastic material due to a periodic shear stresses applied in an axially symmetric manner to a circular area of the plane surface by means of a rigid disk, the torsional displacement being prescribed under the disk. Verma (1957) discussed the static distribution of stresses and displacement when shearing stress is prescribed on the circumference of a circle on the plane boundary. Datta (1961) discussed [the corresponding problem when shearing stress decreases exponentially with time. Ghosh (1964) exactly evaluated the displacement at any point of the medium when a twisting moment in the form $M_0\delta(t)$ is applied to the disk by following Cagniard (1939) and Dix (1954). Ghosh (1971) also discussed the axisymmetric problem of propagation of a stress discontinuity over a circular region by using Cagniard's (1939) method as modified by De-Hoop (1959). In the present paper the author determines the displacement in the integral form due to a

ring source which increases steadily when the twisting impulse is prescribed by $P\delta(r-ct)\,H(t)$, where δ, H are two dimensional delta function and Heaviside function respectively, and then the exact evaluation of the displacement is determined after the first arrival of the shear wave and, the displacement at any point for large values of the time t.

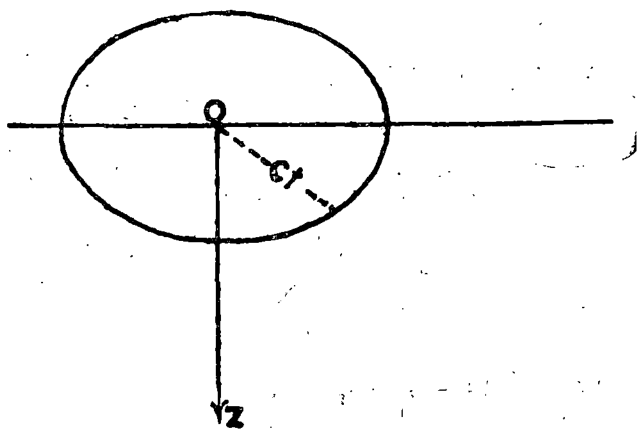

Fig. 1. Co-ordinates system in the medium.

2. FORMULATION OF THE PROBLEM :

The isotropic, elastic, semi-infinite medium is supposed to occupy the region $z \geqslant 0$. We choose cylindrical polar co-ordinates (r, θ, z) with the z-axis directed into the medium, the plane boundary being $z=0$ with origin at the centre of the source. The displacement is calculated at points inside the medium assuming that the half space is, initially, at rest and that the displacement remains bounded even as $z \to +\infty$. Since the motion is symmetrical about z-axis for torsional motion of the ring source, all quantities depend on r, z and the time t. The only non-vanishing component of the displacement vector is the component v along the direction of θ increasing. Hence the non-vanishing stresses components are

$$\tau_{r\theta}=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) \quad \text{and} \quad \tau_{\theta z}=\mu\frac{\partial v}{\partial z} \tag{1}$$

where μ is the coefficient of rigidity. The only non-zeros equation of motion is

$$\frac{\partial}{\partial r}(\tau_{r\theta})+\frac{\partial}{\partial z}(\tau_{\theta z})+2\frac{\tau_{r\theta}}{r}=\rho\frac{\partial^2 v}{\partial t^2} \tag{2}$$

where ρ is the density of the medium, assumed constant. The boundary condition is

$$\tau_{\theta z}=P\,\delta\,(r-ct)\,H\,(t) \qquad \text{at } z=0 \tag{3}$$

c, P being constant H is the Heaviside function and δ is the two dimensional delta function given by

$$2\pi\int_0^{\infty}\delta\,(r)\,r dr=1.$$

3. **SOLUTION ;** We define for all positive real values of s, the Lapiaee transform f_1 (r, z, s) of a function f (r, z, t) by

$$f_1\,(r, z, s)=\int_0^{\infty} e^{-st}\,f\,(r, z, t)\,dt \tag{4}$$

Substituting the values of $\tau_{r\theta}$ and $\tau_{\theta z}$ in equation (2) and then applying the L-alap ce transform (4), we obtain

$$\frac{\partial^2 v_1}{\partial r^2}+\frac{1}{r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\frac{\partial^2 v_1}{\partial z^2}=\frac{1}{\beta^2}\,s^2\,v_1 \tag{5}$$

where $\beta=\sqrt{(\mu/\rho)}$ is the shear wave velocity.

Defining v_2 by the equation.

$$v_2\,(\xi, z, s)=\int_0^{\infty} r\,J_1\,(\xi r)\,v_1\,(r, z, s)\,dr \tag{6}$$

and then multiplying the equation (5) by $rJ_1\,(\xi r)$ and integrating with respect to r from o to ∞, we get

$$\frac{d^2 v_2}{dz^2}=\left(\xi^2+s^2/\beta^2\right)\,v_2 \tag{7}$$

Taking ξ real, the general solution of the equation (7) which remains bounded for large values of z, is

$$v_2=A\,\exp\left[-z\,(\xi^2+s^2/\beta^2)^{1/2}\right] \tag{8}$$

The Laplace transform of $\tau_{\theta z}$ is

$$(\tau_{\theta z})_1=P\int_0^{\infty} e^{-st}\,\delta\,(r-ct)\,H\,(t)\,dt$$

$$=\frac{P}{2\pi cr}\,e^{-sr/c}$$

It's Hankel transform is

$$(\tau_{\theta z})_2 = -\frac{P}{2\pi c}\int_0^\infty e^{-sr/c} J_1(\xi r)\, dr$$

$$= \frac{P}{2\pi\xi c}\left[1-\frac{s}{c}\left(\xi^2+\frac{s^2}{c}\right)^{-1/2}\right] \text{ [See Erdelyietal 1964, p19]}$$

Noting that on $z=0$,

$\frac{dv_2}{dz} = -A\,(\xi^2+s^2/\beta^2)^{1/2}$ and using the boundary condition,

we get

$$A = -\frac{P}{2\pi\mu\xi c}\,\frac{\left[1-\frac{s}{c}\left(\xi^2+\frac{s^2}{c^2}\right)^{-1/2}\right]}{(\xi^2+s^2/\beta^2)^{1/2}}$$

Substituting this value of A in (8) and inverting the Hankel transform (6), we obtain

$$v_1 = -\frac{P}{2\pi\mu c}\int_0^\infty \frac{1-\frac{s}{c}\left(\xi^2+\frac{s^2}{c^2}\right)^{-1/2}}{(\xi^2+s^2/\beta^2)^{1/2}} J_1(\xi r)\, e^{-z(\xi^2+s^2/\beta^2)^{1/2}}\, d\xi \tag{9}$$

Now,

$$J_1(\xi r) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi r\sin\psi}(\cos\psi - i\sin\psi)\, d\psi$$

(See Erdelyietal 1953, p.14)

Substituting this value of $J_1(\xi r)$ in (9) and putting

$p=\xi\sin\psi$ and $q=\xi\cos\psi$, we get

$$v_1 = -\frac{P}{4\pi^2\mu c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(q-ip)\left\{\left(p^2+q^2+\frac{s^2}{c^2}\right)^{1/2}-\frac{s}{c}\right\}}{(p^2+q^2+(s^2/c^2))^{1/2}\,(p^2+q^2+(s^2/\beta^2))^{1/2}} \times \frac{e^{-z(p^2+q^2+s^2/\beta^2)^{1/2}+irp}}{(p^2+q^2)}\, dp\, dq.$$

To find the inversion of v_1, we put

$p=ms$ and $q=ns$ in the above integral, then we have

$$v_1 = \frac{iP}{2\pi^2\mu c}\int_0^\infty dn\int_{-\infty}^{\infty}\frac{m\left\{(m^2+n^2+1/c^2)^{1/2}-1/c\right\} e^{-s\{z(m^2+n^2+1/\beta^2)^{1/2}-irm\}}}{(m^2+n^2+1/c^2)^{1/2}\,(m^2+n^2+1/\beta^2)^{1/2}\,(m^2+n^2)}\, dm \tag{10}$$

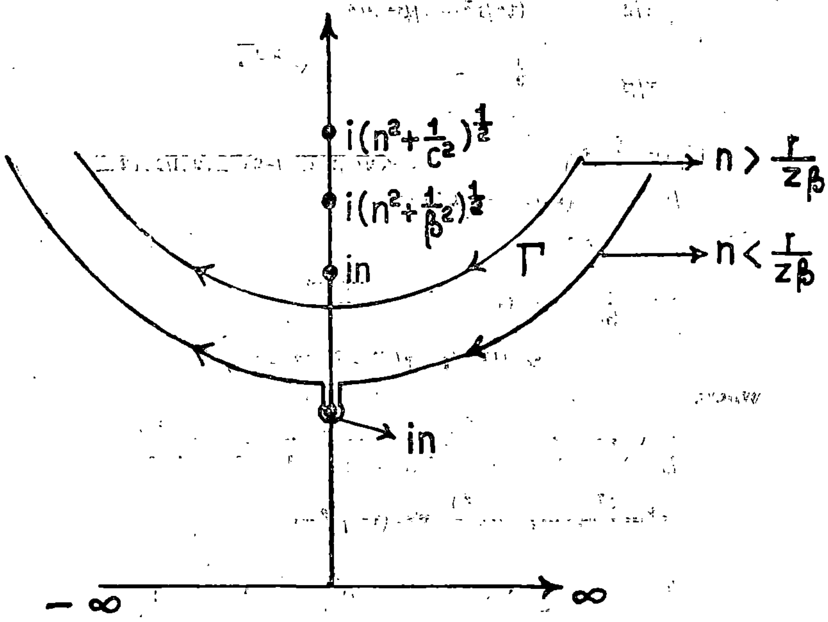

Fig--- 2

Fig 2. Path of integration in the complex m-plane.

Using the usual Cagniard-De-Hoop (1959) transformation given by

$$-imr+z\,(m^2+n^2+1/\beta^2)^{1/2}=t \quad \ldots \quad (11)$$

where t is real and positive we obtain finally the expression of v_1 in the form

$$v_1=\frac{-P}{\pi^2\mu c}\int_0^{\infty} dn \int^{\infty} \frac{e^{-st}\ \mathrm{Im}\left[K\,(m_+,n)\,\frac{dm_+}{dt}\right] dt.}{\{(z^2+r^2)\,(n^2+1/\beta^2)\}^{1/2}}$$

where $m_+=\dfrac{irt+z\,[t^2-(z^2+r^2)\,(n^2+1/\beta^2)]^{1/2}}{z^2+r^2}$

Next change in order of integration leads to

$$v_1=\frac{-P}{\pi^2\mu c}\int_{\beta^{-1}(z^2+r^2)^{1/2}}^{\infty}{}^{-st}\,dt.\int_0^{\sqrt{\gamma}}\mathrm{Im}\left[K(m_+,n)\ \frac{dm_+}{dt}\right]dn.$$

where $\gamma=t^2/(z^2+r^2)-\beta^{-2}$

Taking the Laplace inversion, we get

$$v=\frac{-P}{\pi^2\mu c} H\left[t-\beta^{-1}(z^2+r^2)^{1/2}\right]\int_0^{\sqrt{\gamma}} \mathrm{Im}\left[K(m_+, n)\frac{dm_+}{dt}\right. dn. \qquad (12)$$

4. APPROXIMATE EVALUATION OF THE DISPLACEMENT :

Case 1. Displacement after the first arrival.

To integrate $\int_0^{\sqrt{\gamma}} \mathrm{Im}\left[K(m_+, n)\frac{dm_+}{dt}\right] dn.$ (13)

we put $n=\sqrt{\gamma}\sin\alpha$ and $t_\beta=\beta^{-1}(z^2+r^2)^{1/2}$,

which is the time taken by the shear wave to reach the point (r, θ, z).

The integral (13) after the substitution takes the form

$$\int_0^{\pi/2} \mathrm{Im}\left[K(m_+, n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right] d\alpha. \qquad (14)$$

Now, $\mathrm{Im}\left[K(m_+, n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right]=\frac{\beta}{r}\left[\frac{(z^2+r^2)^{1/2}}{\{\beta^2(z^2+r^2)-c^2r^2\}^{1/2}}-1\right]$

as $t\to t\beta$.

Hence from (12), we obtain

$$v=\frac{-P\beta}{2\pi\mu cr}\left\{\frac{\beta(z^2+r^2)^{1/2}}{\{\beta^2(z^2+r^2)-c^2r^2)^{1/2}}-1\right\} H(t-t_\beta),$$

which is the displacement at any point (r, z) just after the arrival of the disturbance. It is interesting to note that the displacement due to the first arrival of the disturbance at any point of the z-axis is zero which is also expected from the physical stand point. It is to be noted that the displacement at any point on the free surface z=o, varies inversely as r.

Case 2. Displacement after sufficiently large time when z≠o.

In this case, $\mathrm{Im}\left[K(m_+, n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right]$

$$=\frac{(z^2+r^2)^{1/2}}{t}\frac{r\{z^2\sin^2\alpha-(r^2+z^2)\cos^2\alpha\}}{(z^2+r^2\cos^2\alpha)^2}-\frac{(z^2+r^2)^{3/2}}{ct^2}.$$

$$-\frac{(z^2+r^2)^{3/2}}{ct^2}\cdot\frac{rz\{z^2-3(r^2+z^2)\cos^2\alpha+r^2\cos^4\alpha\}}{(z^2+r^2\cos^2\alpha)^3}.$$

The terms containing $1/t^3$ and higher orders are neglected. After the above substitution (14) takes the following form

$$\frac{r\,(z^2+r^2)^{1/2}}{t}\int_0^{\pi/2}\frac{z^2\sin^2\alpha-(r^2+z^2)\cos^2\alpha}{(z^2+r^2\cos^2\alpha)^2}\,d\alpha-$$

$$-\frac{rz(z^2+r^2)^{3/2}}{ct^2}\int_0^{\pi/2}\frac{z^2-3\,(r^2+z^2)\cos^2\alpha+r^2\cos^4\alpha}{(z^2+r^2\cos^2\alpha)^3}\,d\alpha \tag{15}$$

The first integral of (15) is zero, hence for the large value of the time t the displacement is given by

$$v=-Pr\,(4z^2+5r^2)/4\pi\mu\,c^2\,t^2\,z^2.$$

In this case the displacement at any point varies inversely as t^2. Also this is to be noted that the displacement increases with the increase of r when t is very large, which is in conformity with the physical condition because the radius of the ring source after large time t is infinitely large.

Case 3. Displacement at the free surface.

In this case taking z=o, we obtain from Eq. (10)

$$v_1=\frac{iP}{2\pi^2\mu c}\left[\int_0^{\infty}dn\int_{-\infty}^{\infty}\frac{me^{isrm}}{(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}-\right.$$

$$\left.-\frac{1}{c}\int_0^{\infty}dn\int_{-\infty}^{\infty}\frac{me^{isrm}}{(m^2+n^2+1/c^2)^{1/2}\,(m^2+n^2\;1/\beta^2)^{1/2}\,(m^2+n^2)}\right]dm. \tag{16}$$

The path pf integration in the complex m-plane is the real axis, which is deformed in such a way that

$-irm=t$. where t is real and positive. Taking the integral over the deformed path we get

$$v_1=\frac{Pr}{\pi^2\mu c}\left[\int_0^{\infty}dn\int_{r(n^2+1/\beta^2)^{1/2}}^{\infty}\frac{te^{-st}}{\{t^2-r^2\,(n^2+1/\beta^2)\}^{1/2}\,(t^2-r^2\,n^2)}dt\right.$$

$$\left.+\frac{r}{c}\int_0^{\infty}dn\int_{r\,(n^2+1/\beta^2)^{1/2}}^{r\,(n^2+1/c^2)^{1/2}}\frac{te^{-st}}{\{r^2(n^2+1/c^2)-t^2\}\{t^2-r^2(n^2+1/\beta^2)\}^{1/2}(t^2-r^2n^2)}dt\right]$$

Changing the order of integration. we obtain

$$v_1=\frac{Pr}{\pi^2\mu c}\left[\int_{r/\beta}^{\infty}te^{-st}\,dt\int_0^{(t^2/r^2-1/\beta^2)^{1/2}}\frac{dn}{\{t^2-r^2\,(n^2+1/\beta^2)\}^{1/2}\,(t^2-r^2\,n^2)}\right.$$

$$+\frac{r}{c}\int_{r/\beta}^{r/c} te^{-st}\,dt\int_0^{(t^2/r^2-1/\beta^2)^{1/2}}\frac{dn}{\{r^2(n^2+1/c^2)-t^2\}^{1/2}\{t^2-r^2(n+1/\beta^2)\}^{1/2}(t^2-r^2n^2)}$$

$$+\int_{r/c}^{\infty} te^{-st}\,dt\int_{(t^2/r^2-1/c^2)^{1/2}}^{(t^2/r^2-1/\beta^2)^{1/2}}\frac{dn}{\{r^2(n^2+1/c^2)-t^2\}^{1/2}\{t^2-r^2(n^2+1/c^2)\}\{(t^2-r^2n^2)}\Bigg\}\Bigg]$$

Taking Laplace inversion of the above integral, we finally obtain,

$$v=\frac{P\beta}{2\pi\mu cr}H(t-r/\beta)+\frac{Pt\,\beta^3}{\pi^2\mu cr^2(\beta^2-c^2)^{1/2}}$$

$$\times\{[H(t-r/\beta)-H(t-r/c)\,\Pi(R,R)+H(t-r/c).\,\Pi(R,R,\Phi)\}.$$

where.

$\Pi(R, R)$ is the complete elliptic integral of 3rd kind,

$\Pi(R, R, \Phi)$ is the elliptic integral of 3rd kind,

$$R^2=\frac{c^2(t^2\beta^2-r^2)}{r^2(\beta^2-c^2)}\quad n=(t^2\beta^2-r^2)/r^2,$$

$$\Phi=\text{Cos}^{-1}\left[\beta/c\,\{(c^2t^2-r^2)/(\beta^2t^2-r^2)\}^{1/2}\right].$$

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Torsional Response of an Elastic Half Space to a Nonuniformly Expanding Ring Source

Es werden exakte Ausdrücke für die Verschiebung in einem homogenen isotropen elastischen Halbraum in integraler Form bestimmt, der impulsförmigen Torsionskräften ausgesetzt ist, die sich über den Rand einer sich ungleichförmig ausdehnenden Schallquelle auf einer freien Oberfläche ausbreiten. Es werden sowohl positiv als auch negativ beschleunigte Ausbreitungen der Schallquelle betrachtet. Mit Hilfe des Verfahrens von Cagniard De-Hoop wird die analytische Lösung in integraler Form bestimmt. Es werden unterschiedliche Wellenfrontflächen und ihr Existenzbereich dargestellt. Die Reaktionen der ersten Bewegung beim Eintreffen verschiedener Wellen werden durch einen Grenzwertprozeß bestimmt. Die Verschiebungen auf der freien Oberfläche werden für verschiedene Positionen der Schallquelle auch numerisch berechnet und grafisch dargestellt.

Exact expressions for displacement in a homogeneous isotropic elastic half-space subjected to an impulsive torsional force spreading over the rim of a nonuniformly expanding ring source on the free surface are obtained in integral form. Both accelerating and decelerating expansion of the source have been considered. The analytic solution, in integral form, is obtained by the Cagniard De-Hoop technique. Different wave front surfaces with their region of existence have been shown. The first motion responses near different wave arrivals have been determined by a limiting process. The displacements on the free surface for different positions of the source have also been evaluated numerically and have been shown by graphs.

Для перемещений в однородном изотропном упругом полупространстве под действием удераной силы кручения, распределённой за краями неравномерно распространяющегося кольцевого источника на свободной поверхности, получены точные выражения в интегральной форме. Рассмотрено ускоренное и замедленное распространение источника. Аналитическое решение в интегральной форме получено методом Каниарда Де-Хупа. Показаны различные поверхности фронта волн и области их существования. Реакции первого движения вблизи различных волн определены предельным переходом. Перемещения на свободной поверхности для различных положений источника подсчитаны численно и представлены графически.

1. Introduction

The study of the dynamic behaviour of an elastic solid under various forms of moving loads and torsional pressure has been gaining importance day by day. This is because of their importance in seismology, structural design and underground exploration.

Gakenheimer [1] in one of his papers presented in details the problem of a load emanating from a point on the surface and then expanding radially at a constant rate. He considered the cases when the loads are disk-shaped or ring-shaped and the expanding rates are super-seismic, transeismic and sub-seismic. Almost at the same time Ghosh [2] also considered the problem of propagation of a stress discontinuity over an expanding circular region with a constant velocity which is less than the shear wave velocity of the medium. Freund [3] considered the non uniformly moving line load as well as point load. Stronge [4] discussed the problem of an accelerating line load in an acoustic half space. The non uniform pressure distribution problem applied to an elastic half space over a circular zone are discussed by Brock [5] and by Roy [6]. Almost a same type of problem has been considered by Aggarwal and Ablow [7]. There it was assumed that circularly symmetric load spreads out from a point on an acoustic half-space with decelerating speed. Ghosh [8] determined exactly the displacement produced by SH-type of waves when a torsional force is prescribed over a circular region on the free surface of a homogeneous isotropic medium and that in the integral form in case of a non homogeneous medium.

In the present paper, the displacement at any point (r, z) in the semi-infinite medium is determined in the integral form by prescribing a time dependent torsional force over the rim of a circular zone. The ring is assumed to expand in an arbitrary manner with time. It is found that the displacement field contains besides the usual SH-waves, contribution from conical waves which arise due to the motion of the source. The region of conical waves which depend on the nature of the motion of the source and the initial speed of expansion of the source are investigated in details. Different wave front surfaces are located and first motion responses near different wave arrivals have been obtained.

Finally numerical evaluation of the displacement on the free surface has been made for a decelerating ring source whose radius at time t is of the form $h(t) = At^{1/2}$. Displacements at points on the free surface for different position of the source have been shown by means of graphs.

2. Formulation of the Problem

Consider a homogeneous isotropic elastic half space on the free surface of which a ring source producing SH-type of waves is expanding with non-uniform velocity. (r, θ, z) are the cylindrical polar co-ordinates, z-axis being directed into the medium and the plane boundary being $z = 0$. The origin of co-ordinates is at the centre of the ring $r = h(t)$, $z = 0$. The ring is assumed to expand with uniform acceleration or with deceleration and an impulsive torque applied to the ring is prescribed.

The displacement is determined in the integral form at any point inside and on the free surface of the medium, subject to the condition that the half-space is initially at rest and that the displacements remain bounded for large values of z. For torsional motion of the ring all quantities depend on r, z and the time t. We assume that $h(t)$ is non negative and monotone increasing function. The only non-zero component of the displacement vector is the component v along the direction of θ increasing. The relevant non vanishing stress components are

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \quad \text{and} \quad \tau_{\theta z} = \mu\frac{\partial v}{\partial z}, \tag{1 a, b}$$

where μ is the Lamé's constant. The nonzero equation of the displacement field is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} \tag{2}$$

where β is the shear wave velocity. The boundary condition of the motion is

$$\tau_{\theta z} = -P\delta[h(t) - r]\, H(t)\,, \qquad z = 0 \tag{3}$$

where P is a constant, $\delta(\ )$ is Dirac's delta function, $H(\ )$ is the Heaviside step function and $h(t)$ is the radius of the ring at time t. Initial conditions of motion are given by

$$h(t) = 0\,, \qquad t = 0 \quad \text{and} \quad \dot{h}(t) > 0\,, \qquad t > 0 \tag{4}$$

where dot denotes the time derivative.

3. Method of solution

We define Laplace transform $f_1(r, z, p)$ of the function $f(r, z, t)$ by

$$f_1(r, z, p) = \int_0^\infty \exp(-pt)\, f(r, z, t)\, dt \tag{5}$$

where p is real and positive and Hankel transform $f_2(\xi, z, p)$ of $f_1(r, z, p)$ by

$$f_2(\xi, z, p) = \int_0^\infty rJ_1(\xi r)\, f_1(r, z, p)\, dr \tag{6}$$

where J_n is the Bessel function of the first kind of order n.

Applying Laplace and Hankel transforms, to the equation (2) successively we obtain

$$\frac{d^2 v_2}{dz^2} - k^2 v_2 = 0 \tag{7}$$

where $k^2 = \xi^2 + (p^2/\beta^2)$.

The solution of the equation (7) which remains bounded as $z \to +\infty$ is

$$v_2 = K \exp(-kz)\,. \tag{8}$$

The value of the constant K is determined, by using the condition (4), the equation (8) and the Hankel transform of the Laplace transform of the equation (3). It is found to be

$$K = \frac{P}{\mu}\int_0^\infty \frac{1}{k}\, h(\tau)\, J_1(\xi h(\tau))\, \exp(-p\tau)\, d\tau\,. \tag{9}$$

Substituting the value of K in (8) and then taking Hankel's inversion one gets

$$v_1 = \frac{P}{\mu}\int_0^\infty h(\tau)\, \exp(-p\tau) \int_0^\infty \frac{\xi}{k}\, J_1(\xi r)\, J_1(\xi h(\tau))\, \exp(-kz)\, d\xi\, d\tau\,. \tag{10}$$

4. Laplace Inversion

In this section the Laplace inverse transform is evaluated by Cagniard's technique.

We make use of the following results

$$J_1(\xi h(\tau))\, J_1(\xi r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} J_0(\xi S)\cos\varphi\, d\varphi\,, \qquad J_0(\xi S) = \frac{1}{2\pi}\int_0^{2\pi} \exp(i\xi S\cos u)\, du$$

where

$$S = (r^2 + h^2(\tau) - 2rh(\tau)\cos\varphi)^{1/2},$$

to obtain the equation (10) as

$$v_1 = \frac{P}{4\pi^2\mu}\int_0^\infty h(\tau)\exp(-p\tau)\int_{-\pi}^{\pi} I\cos\varphi\,d\varphi\,d\tau, \tag{11}$$

where

$$I = \int_0^\infty\int_0^{2\pi}\frac{\xi}{k}\exp(i\xi S\cos(\psi - u) - kz)\,d\xi\,du \tag{12}$$

and ψ is any constant angle.

In (12), we put

$$\alpha' = \xi\cos u, \qquad \beta' = \xi\sin u,$$

then substitute $\alpha' = w\cos\psi - q\sin\psi$ and $\beta' = w\sin\psi + q\cos\psi$ and finally replace w by wp and q by qp to obtain (12) in the form

$$I = p\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp[-p\{-iwS + (w^2 + q^2 + 1/\beta^2)^{1/2}z\}]}{(w^2 + q^2 + 1/\beta^2)^{1/2}}\,dw\,dq. \tag{13}$$

Equation (13) is the well known form for determining the Laplace inversion of a function by applying Cagniard's technique as modified by De-Hoop. Substituting $t = -iwS + z(w^2 + q^2 + 1/\beta^2)^{1/2}$ in (13) where t is real and positive, the Laplace inversion of (13) is found to be equal to

$$G(t) = 4\frac{d}{dt}\left\{H[t - \varrho/\beta]\int_0^{(t^2/\varrho^2 - 1/\beta^2)^{1/2}} Rl\left[\frac{1}{(w_+^2 + q^2 + 1/\beta^2)^{1/2}}\frac{dw_+}{dt}\right]dq\right\} \tag{14}$$

where $\varrho^2 = z^2 + S^2$ and

$$w_+ = \frac{iSt + z\{t^2 - \varrho^2(q^2 + 1/\beta^2)\}^{1/2}}{\varrho^2}.$$

Applying the convolution theorem on (11), the Laplace inversion of v_1 is obtained in the form

$$v = \frac{P}{4\pi^2\mu}\int_0^\infty h(\tau)\,d\tau\int_{-\pi}^{\pi}\cos\varphi\,d\varphi\int_0^t\delta(u - \tau)\,G(t - u)\,du,$$

which when simplified takes the form

$$v = \frac{P}{\pi\mu}\int_0^t h(\tau)\,d\tau\int_0^\pi\frac{\cos\varphi}{\varrho}\delta(t - \tau - \varrho/\beta)\,d\varphi. \tag{15}$$

Integrating over φ, we obtain

$$v = \frac{P\beta}{\pi r\mu}\int_0^t\left\{H\left[t - \tau - \frac{\sqrt{z^2 + (r - h(\tau))^2}}{\beta}\right] - H\left[t - \tau - \frac{\sqrt{z^2 + (r + h(\tau))^2}}{\beta}\right]\right\}Q(\tau)\,d\tau \tag{16}$$

where

$$Q(\tau) = \frac{z^2 + r^2 + h^2(\tau) - \beta^2(t - \tau)^2}{[\{z^2 + (r + h(\tau))^2 - \beta^2(t - \tau)^2\}\{\beta^2(t - \tau)^2 - z^2 - (r - h(\tau))^2\}]^{1/2}}.$$

To facilitate our discussion, equation (16) is written in an alternative form,

$$\begin{aligned} v = \frac{P\beta}{\pi r\mu}\int_0^{t - z/\beta}\{&H[r - h(\tau) + \sqrt{\beta^2(t - \tau)^2 - z^2}]\,H[h(\tau) - \sqrt{\beta^2(t - \tau)^2 - z^2}] + \\ &+ H[r + h(\tau) - \sqrt{\beta^2(t - \tau)^2 - z^2}]\,H[-h(\tau) + \sqrt{\beta^2(t - \tau)^2 - z^2}] - \\ &- H[r - h(\tau) - \sqrt{\beta^2(t - \tau)^2 - z^2}]\}\,Q(\tau)\,d\tau. \end{aligned} \tag{17}$$

The region of support for τ-integration is bounded by the curves:

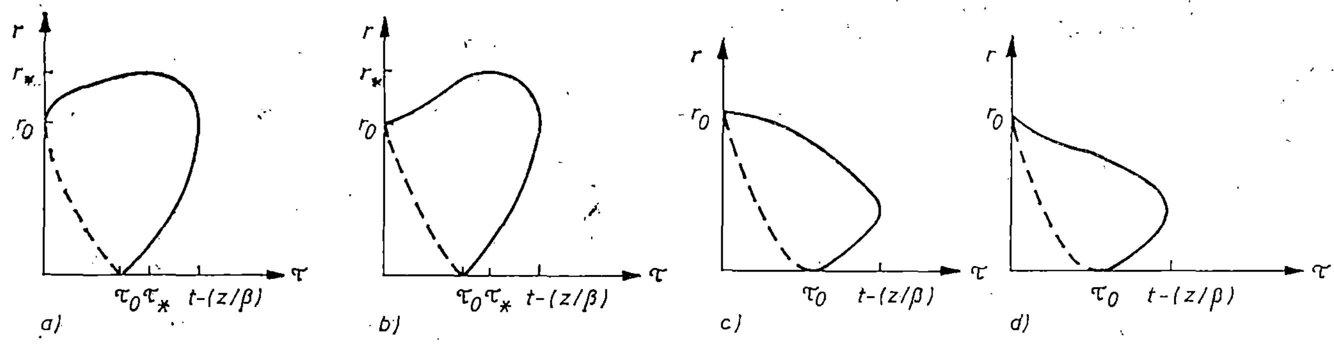
$$\text{I:}\quad r = h(\tau) + \sqrt{\beta^2(t - \tau)^2 - z^2}; \qquad 0 < \tau < t - z/\beta, \tag{18}$$

$$\text{II:}\quad r = h(\tau) - \sqrt{\beta^2(t - \tau)^2 - z^2}; \qquad \tau_0 < \tau < t - z/\beta, \tag{19}$$

$$\text{III:}\quad r = -h(\tau) + \sqrt{\beta^2(t - \tau)^2 - z^2}; \qquad 0 < \tau < \tau_0. \tag{20}$$

The region of τ integration for $Q(\tau)$ bounded by the curves I, II and III are shown in the figs. 1(a—f) and the following remarks can be made about them. It is to be noted that the curves II and III are monotone increasing and decreasing in their respective region of existence viz. $(\tau_0, t - z/\beta)$ and $(0, \tau_0)$ where

$$h(\tau_0) = \{\beta^2(t - \tau_0)^2 - z^2\}^{1/2} .$$



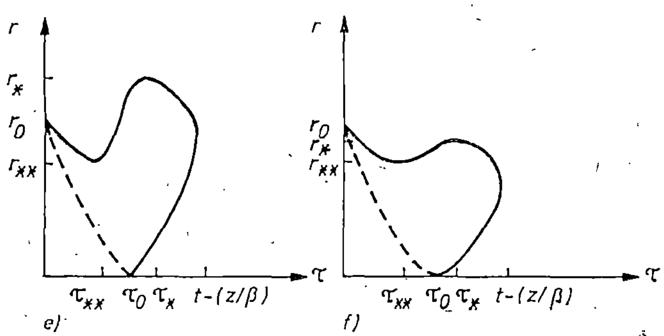


Fig. 1. Region of support for τ-integration for fixed z and t. $(a - b)$ when a single maximum exists i.e. the starting velocity $\dot{h}(0)$ is such that $(\partial r/\partial\tau) > 0$; $(c - d)$ the case when no extremum exists; (e—f) when a maximum(r_*) and a minimum (r_{**}) both exist

The curve I has an extremum where

$$\frac{\partial r}{\partial\tau} = \dot{h}(\tau) - \frac{\beta^2(t - \tau)}{\{\beta^2(t - \tau)^2 - z^2\}^{1/2}} \tag{21}$$

vanishes and

$$\frac{\partial^2 r}{\partial\tau^2} = \ddot{h}(\tau) - \frac{\beta^2 z^2}{\{\beta^2(t - \tau)^2 - z^2\}^{3/2}} \tag{22}$$

does not vanish.

We consider the different cases that arise due to non uniform increase of the ring source. Let the source increase with uniform acceleration $\ddot{h}(\tau) > 0$. In this case, if the initial velocity $\dot{h}(0)$ of the source be such that $\left(\frac{\partial r}{\partial\tau}\right)_0 > 0$, then since $\left(\frac{\partial r}{\partial\tau}\right)_0 > 0$ and $\left(\frac{\partial r}{\partial\tau}\right)_{t-z/\beta} < 0$, the curve I has only one maximum at $r = r_*$ because $\frac{\partial^2 r}{\partial\tau^2}$ either changes sign once from positive to negative or remains negative throughout in $(0, t - z/\beta)$. The corresponding cases are shown in Fig. 1 (a—b). Next let the initial velocity $\dot{h}(0)$ of the source be such that $\left(\frac{\partial r}{\partial\tau}\right)_0 < 0$. In this case, if $\left(\frac{\partial^2 r}{\partial\tau^2}\right)_0 < 0$, $\left(\frac{\partial^2 r}{\partial\tau^2}\right)$ will be negative throughout the interval $(0, t - z/\beta)$; the curve I then corresponds to Fig. 1 (c), since both $\left(\frac{\partial r}{\partial\tau}\right)_0$ and $\left(\frac{\partial r}{\partial\tau}\right)_{t-z/\beta}$ are negative. But if $\left(\frac{\partial^2 r}{\partial\tau^2}\right)_0$ be positive, then $\frac{\partial^2 r}{\partial\tau^2}$ changes sign once from positive to negative in the interval $(0, t - z/\beta)$. Hence in this case the curve I has either no extremum which corresponds to Fig. 1 (d) or there is a maximum preceded by a minimum which is shown in Fig. 1 (e, f). Finally, in case of decelerating motion of the source i.e. when $\ddot{h}(\tau)$ (not necessarily a constant) < 0, throughout the interval, the curve has either only one maximum if $\left(\frac{\partial r}{\partial\tau}\right)_0 > 0$ as in Fig. 1 (a) or no extremum as in Fig. 1 (c) when $\left(\frac{\partial r}{\partial\tau}\right)_0 < 0$.

We consider the curves I and II together. Their combined equation is

$$\left(r - h(\tau)\right)^2 = \beta^2(t - \tau)^2 - z^2 . \tag{23}$$

For figures 1 (c, d), τ is a single valued function of r. For the figs. 1 (a, b), τ may be a double valued function whereas for figs. 1 (e, f), τ may be triple valued function of r. Taking the equations (18) and (19) together, the values of τ are designated as $\tau = \tau_1$, $\tau = (\tau_1, \tau_2)$ and $\tau = (\tau_1, \tau_2, \tau_3)$ where $\tau_1 > \tau_2 > \tau_3$ depending on whether τ is single, double or triple valued function of r. In (20) r is a monotone decreasing function of τ, so the corresponding value of τ is designated as $\tau = \tau_4$.

With the above values of the roots of the equations (18)—(20) and from a close examination of the different figs. 1 (a—f), the displacement produced by the SH-type of waves is given by $v = v^1 + v^2 + v^3$, where

$$\left.\begin{aligned} v^1 &= BH(r_0 - r)\, I\big(Q(\tau); \tau_4, \tau_1\big), \\ v^2 &= B[H(r - r_0) - G(r - \max(r_*, r_0))]\, I\big(Q(\tau); \tau_2, \tau_1\big), \\ v^3 &= B[G(r - \min(r_*, r_0)) - G(r - \min(r_{**}, r_0))]\, I\big(Q(\tau); \tau_3, \tau_2\big) \end{aligned}\right\} \quad (24)$$

and

$$G\big(r - \max(r_*, r_0)\big) = \begin{cases} H(r - r_*) & \text{if} \quad r_* = \max(r_*, r_0) \\ H(r - r_0) & \text{if} \quad r_0 = \max(r_*, r_0) \quad \text{or } r_* \text{ does not exist}. \end{cases}$$

$$G\big(r - \min(r_*, r_0)\big) = \begin{cases} H(r - r_*) & \text{if} \quad r_* = \min(r_*, r_0) \\ H(r - r_0) & \text{if} \quad r_0 = \min(r_*, r_0) \quad \text{or } r_* \text{ does not exist}. \end{cases}$$

Similar meaning is attached to the symbol

$G\big(r - \min(r_{**}, r_0)\big)$. B has been written for $\dfrac{P\beta}{\pi r\mu}$. $r_0 = \sqrt{\beta^2 t^2 - z^2}$ is the value of r at $\tau = 0$ and

$$I\big(F(\tau); a, b\big) = \int_a^b F(\tau)\, d\tau .$$

5. Wave Front Analysis

In this section we locate and analyse the nature of the wave fronts.

It is known that wave front is a surface $\varphi(r, z, t) = 0$ which is a characteristic of the differential equation (2) and also satisfy the eikonal equation $\varphi_r^2 + \varphi_z^2 - \beta^{-2}\varphi_t^2 = 0$ [9].

The nature of the wave front changes due to non-uniform expansion of the source and also it depends on the initial velocity $\dot{h}(0)$ $(= u_0)$ of expansion of the source. We consider decelerating and accelerating expansion of the source for different initial velocities.

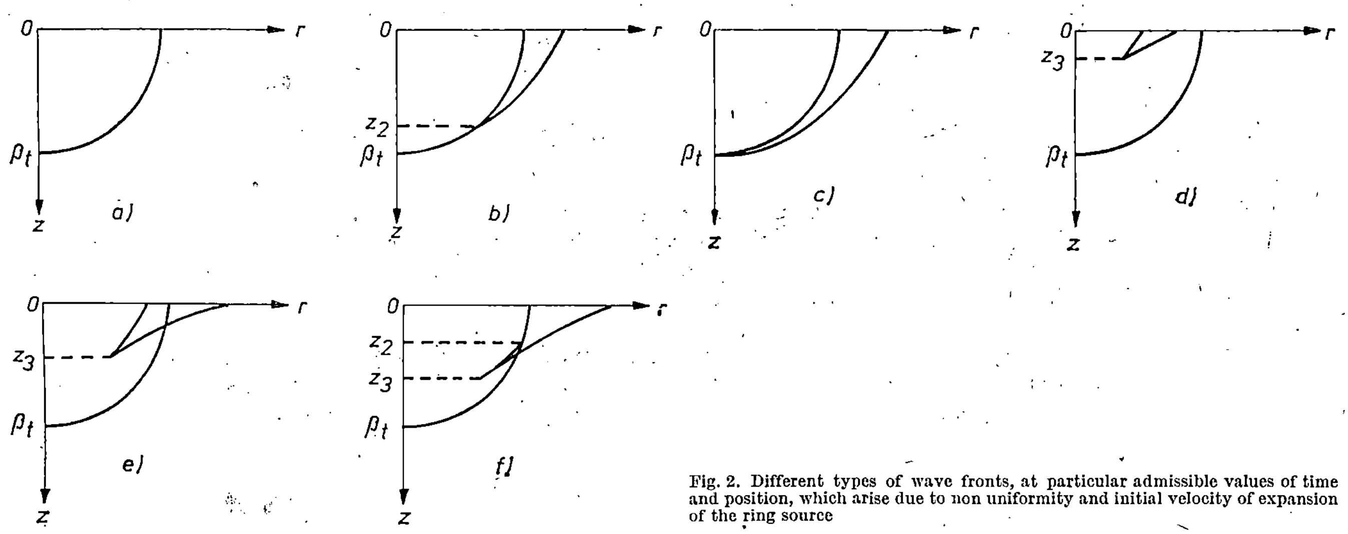


Fig. 2. Different types of wave fronts, at particular admissible values of time and position, which arise due to non uniformity and initial velocity of expansion of the ring source

Case of Deceleration

i) Let $\dot{h}(0) = u_0 < \beta$.

From (21) and (22), $\left(\dfrac{\partial r}{\partial \tau}\right)_0$ is negative for all z and $\dfrac{\partial^2 r}{\partial \tau^2}$ is also negative as $\ddot{h}(0)$ is negative. So the curve I in $(0, t - z/\beta)$ is such that r decreases with the increase of τ. This corresponds to the region of integration as depicted in fig. 1 (d) and consequently the wave front is of the form as shown in fig. 2 (a).

ii) u_0 $(> \beta)$ is finite.

It follows from (21), $\left(\dfrac{\partial r}{\partial \tau}\right)_0$ is positive for $0 < z < z_2$ and negative for $z > z_2$ where z_2 is obtained from $z_2 = \beta t(1 - \beta^2/u_0^2)^{1/2}$, therefore the region of integrations for $0 < z < z_2$ and for $z > z_2$ correspond to the regions shown in the figs. 1 (a) and 1 (c) respectively and consequently the wave front is given by the fig. 2 (b).

iii) u_0 is infinitely large.

From (21) and (22), it follows that $\left(\frac{\partial r}{\partial \tau}\right)_0$ is positive for all z and $\left(\frac{\partial^2 r}{\partial \tau^2}\right)$ is negative for all z and for all τ. Hence the region of integration is fig. 1 (a) and the corresponding wave front is as shown in fig. 2 (c).

Case of Acceleration

i) We assume that the ring source expands with uniform acceleration f and starts with the velocity u_0 $(= \dot{h}(0))$. First let $u_0 < \beta$; then $(\partial r/\partial \tau)_0$ is negative for all z and $(\partial^2 r/\partial \tau^2)_0$ is positive for $0 < z < z_1$ and negative for $z > z_1$, where z_1 is to be determined from the condition $(\partial^2 r/\partial \tau^2)_0 = 0$. For $z > z_1$, $(\partial^2 r/\partial \tau^2)$ is negative for τ in $(0, t - z/\beta)$. Consequently the region of integration is the fig. 1 (c). On the other hand if z lies in $(0, z_1)$ then $(\partial^2 r/\partial \tau^2)$ is first positive and then negative as τ increases in $(0, t - z/\beta)$, so in this case the region of integration is either fig. 1 (d) or fig. 1 (e or f).

By using (22), z_1 is determined from the equation

$$f = \beta^2 z_1^2/(\beta^2 t^2 - z_1^2)^{3/2} . \tag{25}$$

It is to be noted that $z_1 = 0$ when $f = 0$ and z_1 is a monotone increasing function of f. Further, in $(0, z_1)$, $(\partial r/\partial \tau)$ may have two zeroes or there is no zero in the region $0 < \tau < t - z/\beta$, depending on the value of z. The condition that $(\partial r/\partial \tau)$ may have two zeroes is $0 < z < z_3$, where

$$z_3 = \begin{cases} \left\{\beta\left(\frac{u_0 + ft}{f}\right)\left\{1 - \frac{\beta^{2/3}}{(u_0 + ft)^{2/3}}\right\}\right\}^{3/2} & \text{for} \quad u_0 + ft > \beta \\ 0 & \text{for} \quad u_0 + ft \leqq \beta . \end{cases}$$

It can be shown further that $z_3 < z_1$. Hence for $0 < z < z_3$ the region of integration is fig. 1 (e or f) and for $z_3 < z < z_1$, the region of integration is fig. 1 (d). Therefore for accelerating source with initial velocity $u_0 < \beta$, the wave front is of the form as shown in fig. 2 (a) if the observation time be such that $(u_0 + ft) \leqq \beta$ and for $(u_0 + ft) > \beta$ the wave front is like the figures as in 2 (d) or 2 (e) according as the position of the source at the observation time is inside or outside the characteristic surface $r^2 + z^2 = \beta^2 t^2$.

ii) Next let $u_0 > \beta$; from (21) we have $(\partial r/\partial \tau)_0$ is positive for $0 < z < z_2$ and is negative for $z > z_2$ where z_2 is given by

$$z_2 = \beta t(1 - \beta^2/u_0^2)^{1/2} . \tag{26}$$

Also $(\partial^2 r/\partial \tau^2)_0$ is positive for $0 < z < z_1$ and is negative for $z > z_1$, where z_1 is given by (25). So for $0 < z < z_1$, $(\partial^2 r/\partial \tau^2)$ is first positive and then negative in $0 \leqq \tau < (t - z/\beta)$. We consider the case for $z_1 < z_2$ first. In this case

$$\beta^2 z_1^2/(\beta^2 t^2 - z_1^2)^{3/2} < \beta^2 z_2^2/(\beta^2 t^2 - z_2^2)^{3/2} ,$$

since $\beta^2 z^2/(\beta^2 t^2 - z^2)^{3/2}$ is a monotone increasing function of z. Using (25) and (26), we obtain

$$\beta^2(u_0 + ft)/u_0^3 < 1 . \tag{27}$$

Under the condition obtained in (27), the region of integration is like that of the fig. 1 (b) in the range $0 < z < z_1$ and for $z_1 < z < z_2$, the region of integration is of the type as shown in fig. 1 (a). For $z_2 < z < \beta t$, the region of integration is shown in fig. 1 (c). Therefore for $u_0 > \beta$ and for the relation given in (27), it follows that the nature of the wave front is of the type as shown in fig. 2 (b).

Finally, we study the case when $z_1 > z_2$ i.e. when $\beta^2(u_0 + ft)/u_0^3 > 1$.

Here for $0 < z < z_2$ the region of integration is as in fig. 1 (b). Since z_3 is always less than z_1, so for $z_2 < z < z_3$ the region of integration is like fig. 1 (e or f) and for $z_3 < z < z_1$ the region of integration is like that as shown in fig. 1 (d). Fig. 1 (c) represents the region of integration for $z_1 < z < \beta t$. Accordingly the wave front takes the shape of the fig. 2 (f).

6. First Motion Responses

The expression for the displacement as given in (24) is in the form of integrals over finite ranges. As such, computation of displacement for a given model can be done with the high power computer. However some idea about the nature of displacement at the time of the first arrival of wave fronts can be obtained by a limiting process following Stronge [4].

The displacement field just after arrival time of the characteristic surface $r = r_*$ is from (24),

$$v = BI(Q(\tau); \tau_2, \tau_1) \tag{28}$$

whereas just before the arrival time the displacement is given by $v = 0$.

To evaluate (28) near $r = r_*$, we put $r = r_* - \Delta r$ and $\tau = \tau_* + \theta$ in equations (18) and (19). Using Taylor's expansion in the neighbourhood of (τ_*, r_*) and by help of equations (21) and (22) we find the limits of integration of equation (28) in the new variable θ as

$$\theta_{1,2} = \pm \frac{\sqrt{2\Delta r}\ \{\beta^2(t - \tau_*)^2 - z^2\}^{3/4}}{[\beta^2 z^2 - \ddot{h}(\tau_*)\ \{\beta^2(t - \tau_*)^2 - z^2\}^{3/2}]^{1/2}} . \tag{29}$$

The same procedure is followed to determine in the neighbourhood of (τ_*, r_*), the value of Q which is found to be

$$Q(\tau_* + \theta) = \frac{[r_* h(\tau_*) \{\beta^2(t - \tau_*)^2 - z^2\}]^{1/2}}{[\theta^2\{\ddot{h}(\tau_*)\,(\beta^2(t - \tau_*)^2 - z^2)^{3/2} - \beta^2 z^2\} + 2\Delta r\{\beta^2(t - \tau_*)^2 - z^2\}^{3/2}]^{1/2}} \tag{30}$$

where the lowest terms in θ and Δr are retained. The value of the integral (28) after substituting the value of Q from (30) and the limits of integration for the new variable θ as obtained in (29) is found to be

$$\frac{P\beta}{r\mu}\left[\frac{r_* h(\tau_*)\,\{\beta^2(t - \tau_*)^2 - z^2\}}{\beta^2 z^2 - \ddot{h}(\tau_*)\,\{\beta^2(t - \tau_*)^2 - z^2\}^{3/2}}\right]^{1/2},$$

which is the displacement at the first arrival of the wave front given by $r = r_*$.

To find the displacement at the first arrival of the wave front given by $r = r_{**}$, we define $Q(\tau)$ in the neighbourhood of (τ_{**}, r_{**}) and outside the region of integration by

$$Q(\tau) = \frac{z^2 + r^2 + h^2(\tau) - \beta^2(t - \tau)^2}{\{z^2 + (r + h(\tau))^2 - \beta^2(t - \tau)^2\}^{1/2}\,|\{\beta^2(t - \tau)^2 - z^2 - (r - h(\tau))^2\}^{1/2}|} \tag{31}$$

and put $r = r_{**} + \Delta r$ and $\tau = \tau_{**} + \theta$. Following the same procedure as done in case of $r = r_*$, the displacement at the first arrival of the wave surface $r = r_{**}$ is found to be

$$\frac{P\beta}{r\mu}\left[\frac{r_{**} h(\tau_{**})\,\{\beta^2(t - \tau_{**})^2 - z^2\}}{\ddot{h}(\tau_{**})\,\{\beta^2(t - \tau_{**})^2 - z^2\}^{3/2} - \beta^2 z^2}\right]^{1/2}.$$

The displacement at a point due to the first arrival of the wave fronts $r = r_*$ and $r = r_{**}$ simultaneously, is also determined. At this point wave fronts $r = r_*$ and $r = r_{**}$ from a cusp (cf. fig. 2 (d, e, f). In this case this is to be noted that at the cusp $r = r_* = r_{**} = \bar{r}$ (say) and $(\partial r/\partial \tau) = (\partial^2 r/\partial \tau^2) = 0$ where as $(\partial^3 r/\partial \tau^3)) \neq 0$. Hence it follows from equation (24) that the displacement due to first arrival of this wave front at $r = \bar{r}$ is

$$\frac{P\beta}{\pi r\mu}[I(Q(\tau); \bar{\tau}, \tau_1) + I(Q(\tau); \tau_2, \bar{\tau})] \tag{32}$$

where $\bar{\tau} = \tau_* = \tau_{**}$ and τ_1, τ_2 are the two values of τ close to $\bar{\tau}$ and correspond to the points lying on either side of $(\bar{\tau}, \bar{r})$ on the curve and I and II together.

To evaluate the integrals in (32), $\tau = \bar{\tau} + \theta$ and $r = \bar{r} - \Delta r$ are put in the first integral where as $\tau = \bar{\tau} - \theta$ and $r = \bar{r} + \Delta r$ are put into the second integral of (32). Also this is to be remembered that outside the region of integration in the neighbourhood of $(\bar{\tau}, \bar{r})$, $Q(\tau)$ is defined as in (31).

After the above mentioned substitution in (28) and retaining the lowest order term of θ and Δr, one gets the displacement due to first arrival at $r = \bar{r}$ as

$$\frac{2P\beta}{\pi\bar{r}\mu}\left[\frac{3\bar{r}h(\bar{\tau})\,\{\beta^2(t - \bar{\tau})^2 - z^2\}^2}{3\beta^4 z^2(t - \bar{\tau}) - \ddot{h}(\bar{\tau})\,\{\beta^2(t - \bar{\tau})^2 - z^2\}^{5/2}}\right]^{1/2} \int_0^a \frac{d\theta}{\sqrt{(a^3 - \theta^3)}} \tag{33}$$

where

$$a^3 = \frac{6\Delta r(\bar{r} - h(\bar{\tau}))^5}{3\beta^4 z^2(t - \bar{\tau}) - \ddot{h}(\bar{\tau})\,(\bar{r} - h(\bar{\tau}))^5}.$$

By substituting

$$\theta^3 = a^3 \sin^2 \alpha\,,$$

the integral in (33) is evaluated. The displacement due to first arrival of the wave front $r = \bar{r}$ is found to be

$$\frac{2^{5/6}P\beta}{3^{2/3}\pi\bar{r}\mu(\Delta r)^{1/6}}\;\frac{\bar{r}^{1/2}h^{1/2}(\bar{\tau})\,\{\beta^2(t - \bar{\tau})^2 - z^2\}^{7/12}}{[3\beta^4 z^2(t - \bar{\tau}) - \ddot{h}(\bar{\tau})\,\{\beta^2(t - \bar{\tau})^2 - z^2\}^{5/2}]^{1/3}}\;B\left(\frac{1}{3}, \frac{1}{2}\right)$$

where $B(m, n)$ is the Beta function.

It is interesting to note that in this case the displacement due to first arrival at this point is infinitely large due to the presence of the factor $(\Delta r)^{1/6}$ in the denominator.

Finally we consider the characteristic surface $r^2 + z^2 = \beta^2 t^2$ which corresponds to a disturbance initiated at the origin when the torque is first applied at $\tau = 0$. This disturbance spreads out from the origin with a velocity equal to β. To find the displacement due to the first arrival of this surface, following Aggarwal and Ablow [7] let us consider the curve

$$\Gamma\colon\quad r = \{\beta^2(t - \tau)^2 - z^2\}^{1/2} \tag{34}$$

and the lines

$$l_1\colon\quad r = \sqrt{(\beta^2 t^2 - z^2)} - \varepsilon_2; \qquad l_2\colon\quad r = \sqrt{(\beta^2 t^2 - z^2)} + \varepsilon_1 \tag{35), (36}$$

where ε_1 and ε_2 are very small positive quantities. Then to the first order of ε_1 and ε_2,

$$(l_1 \times I) \equiv \frac{\varepsilon_2 \sqrt{\beta^2 t^2 - z^2}}{\beta^2 t} = \tau'(\text{say}),$$

$$(l_1 \times \text{III}) \equiv \frac{\varepsilon_2 \sqrt{\beta^2 t^2 - z^2}}{\dot{h}(0)\sqrt{(\beta^2 t^2 - z^2)} + \beta^2 t} = \tau_4(\text{say}),$$

$$(l_2 \times \text{I}) \equiv \frac{\varepsilon_1 \sqrt{\beta^2 t^2 - z^2}}{\dot{h}(0)\sqrt{(\beta^2 t^2 - z^2)} - \beta^2 t} = \tau_2(\text{say});$$

ε_1, ε_2 are such that $\tau_2 < \tau'$ and tends to zero as $t \to \varrho_0/\beta$, where $\varrho_0 = \sqrt{(r^2 + z^2)}$. Then it follows immediately that

$$I(Q(\tau); \tau_4, \tau') \to 0 \quad \text{and} \quad I(Q(\tau); \tau_2, \tau') \to 0 \quad \text{as} \quad t \to \varrho_0/\beta .$$

Also $I(Q(\tau); \tau', \tau_1') - I(Q(\tau); \tau', \tau_1) \to 0$ as $t \to \varrho_0/\beta$, where $\tau_1' \equiv (l_1 \times I)$ and $\tau_1 \equiv (l_2 \times I)$ are the values of τ which correspond to the points on the right of τ'. From this it follows that the displacement is continuous across the characteristic surface $\varrho_0 = \beta t$, showing that the displacement due to the first arrival of the characteristic surface $r^2 + z^2 = \beta^2 t^2$ is zero.

7. Surface Displacement

In this section surface displacement has been determined numerically for a particular type of nonuniformly moving surface. We consider a decelerating ring source whose radius $h(\tau)$ at any time τ is assumed to be $h(\tau) = A\tau^{1/2}$. The displacement at any point $(r, 0)$ at the time of observation t is determined.

According to the position of the source the following three possible cases are considered.

i) Radius $h(\tau)$ of the ring coinciding with the rim of the conical wave front and moving with it so that $\beta t < h(t)$.

ii) $\beta t < h(t) < r_*$.

iii) $h(t) < \beta t$.

To determine the displacement on the free surface, we put $z = 0$ in the function $Q(\tau)$ of equation (24) and the variable of integration τ is changed to T, by substituting $T = \tau/t$. $Q(\tau)$ is then obtained in the form

$$Q(Tt) = R(T) = \frac{\dfrac{r^2}{\beta^2 t^2} + \dfrac{A^2}{\beta^2 t} T - (1 - T)^2}{\left[\left\{\left(\dfrac{r}{\beta t} + \dfrac{A}{\beta}\sqrt{\dfrac{T}{t}}\right)^2 - (1 - T)^2\right\}\left\{(1 - T)^2 - \left(\dfrac{r}{\beta t} - \dfrac{A}{\beta}\sqrt{\dfrac{T}{t}}\right)^2\right\}\right]^{1/2}}$$

on a close examination of the regions of integration as shown in Fig. 1, the displacement v, in case of (i) is given by

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad 0 < r < \beta t,$$

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad \beta t < r < h(t).$$

The displacement in case of (ii) is given by

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad r < \beta t,$$

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad \beta t < r < h(t),$$

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_3) \quad \text{for} \quad h(t) < r < r_*$$

and the displacement in case (iii) is

$$\frac{\mu}{P} v = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad 0 < r < h(t),$$

$$\frac{\mu}{P} v = \frac{\beta t}{\pi r} I(R(T); T_1, T_3) \quad \text{for} \quad h(t) < r < \beta t,$$

$$\frac{\mu}{P} v = \frac{\beta t}{\pi r} I(R(T); T_1, T_3) \quad \text{for} \quad \beta t < r < r_*$$

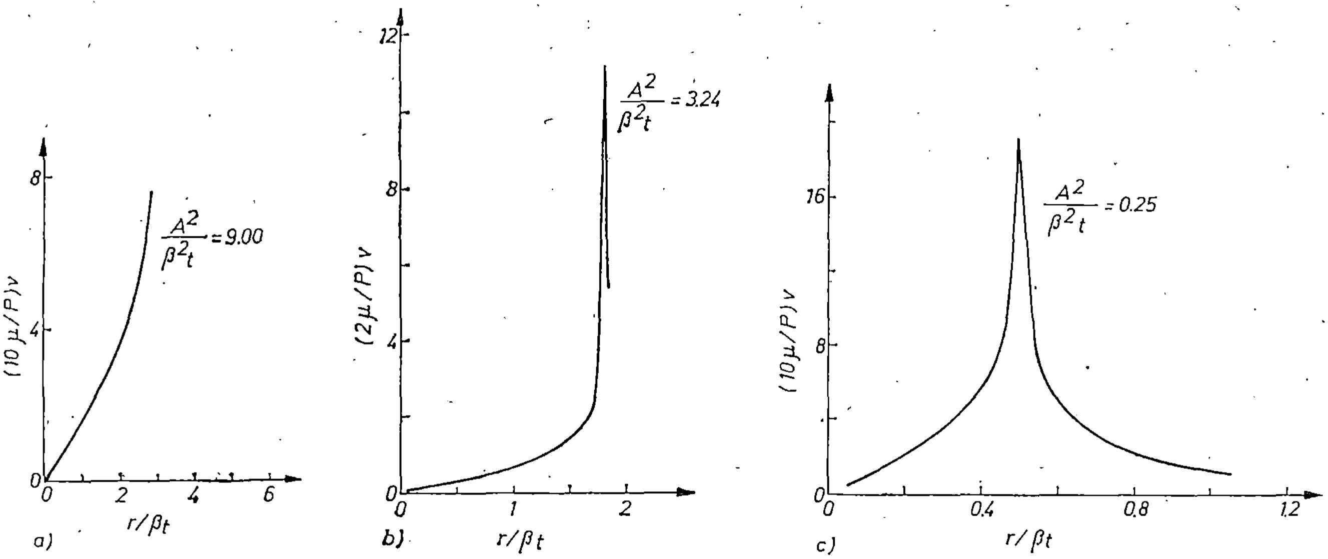


Fig. 3. Graphs showing $(\mu/P)\,v$ versus $(r/\beta t)$ when $z = 0$. (a), (b), (c) correspond to the cases (i), (ii) and (iii) respectively

where

$$T_1 = 1 - \frac{r}{\beta t} + \frac{1}{2}\frac{A^2}{\beta^2 t} - \frac{1}{2}\frac{A}{\beta\sqrt{t}}\sqrt{\frac{A^2}{\beta^2 t} + 4\left(1 - \frac{r}{\beta t}\right)},$$

$$T_2 = 1 + \frac{r}{\beta t} + \frac{1}{2}\frac{A^2}{\beta^2 t} - \frac{1}{2}\frac{A}{\beta\sqrt{t}}\sqrt{\frac{A^2}{\beta^2 t} + 4\left(1 + \frac{r}{\beta t}\right)},$$

$$T_3 = 1 - \frac{r}{\beta t} + \frac{1}{2}\frac{A^2}{\beta^2 t} + \frac{1}{2}\frac{A}{\beta\sqrt{t}}\sqrt{\frac{A^2}{\beta^2 t} + 4\left(1 - \frac{r}{\beta t}\right)}.$$

All the above integrals are numerically evaluated and the graphs are plotted by specifying admissible values of $(r/\beta t)$ against $(\mu/P)\,v$.

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Displacement Due to a Uniformly Moving Line Load over the Plane Boundary of an Inhomogeneous Elastic Half-Space

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(With 3 figures)

Summary

A concentrated line load originating at $t = 0$ at the origin of co-ordinates moves with uniform velocity along the boundary of an isotropic inhomogeneous medium. Following Cagniard's method as modified by De Hoop, the displacement components u and w are determined in the integral form. Finally, an approximate evaluation of the integrals is worked out near the first arrival of the wave fronts.

Zusammenfassung

Eine konzentrierte Linienbelastung, die zum Zeitpunkt $t = 0$ am Koordinatenursprung einsetzt, bewegt sich mit gleichförmiger Geschwindigkeit über die freie Oberfläche eines isotropen, inhomogenen Mediums. Mit Hilfe der Methode von Cagniard in ihrer De Hoopschen Abwandlung werden die Verrückungskomponenten u und w in ihrer Integraldarstellung bestimmt. Schließlich werden die Integrale für die Zeit um das erste Auftreffen der Wellenfronten näherungsweise berechnet.

1. Introduction

Since the publication of the classical paper by Lamb [6] the problem of line and point sources in homogeneous media has attracted the attention of many investigators. But the corresponding problems for inhomogeneous media have not been discussed by many authors as yet. The problem of wave propagation in an inhomogeneous medium is important to geophysicists, because any realistic model of the Earth must take into account the continuous change in the elastic properties of the material in the vertical direction. Since the mathematical treatment of a complicated model is extremely difficult and since the approximation to such a problem does not lead to any worth while solution, so some simplifying assumptions are usually made. Wilson [10] studied the propagation of surface waves in a semi-infinite medium, assuming the density to be constant and the coefficient of rigidity to be varying exponentially ith depth. Stoneley [9], however, considered the transmission of Rayleigh waves a heterogeneous medium in which the rigidity varies linearly with depth. The field e to a point source in an inhomogeneous isotropic medium in which density is stant but the bulk modulus λ varies with depth according to the law $\lambda = \lambda_0(1 + \varepsilon z)^2$ been considered by Singh [8].

n the present paper, considering an elastic medium in which the elastic parameters nd density ϱ vary according to the law $\lambda = \mu = \mu_0(1 + \varepsilon z)^2$ and $\varrho = \varrho_0(1 + \varepsilon z)^2$,

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the transient problem for a two-dimensional line load moving with uniform velocity v over the surface of the non-homogeneous semi-infinite medium is studied. The ground motion excited by the moving surface load occurs, for example, from nuclear blasts and from shock waves generated by supersonic aircrafts. These practical problems have been formulated mathematically by a two-dimensional normal line load which is suddenly created at $t = 0$ and moves subsequently with uniform velocity along the free surface. The method of solutions involves the use of the integral transform and CAGNIARD's [1] method as modified by DE HOOP [4]. The application of CAGNIARD's method in the solution of transient problems in inhomogeneous media does not seem to have been discussed earlier.

This steadily moving line load problem, where t varies from $-\infty$ to ∞, has been solved by CHAKRAVARTY and DE [2] following the method of COLE and HUTH [3]. Of course, the transient solution for a point load moving over the surface of a homogeneous isotropic half-space has been thoroughly discussed by GAKENHEIMER and MIKLOWITZ [5]. An exact solution of the buried uniformly moving line-source problem has also been obtained by MITRA [7].

2. Formulation of the problem

The inhomogeneous semi-infinite medium is supposed to occupy the region $z > 0$ as shown in Fig. 1. The x-axis is taken along the free surface, whereas the z-axis points vertically downwards into the medium. A concentrated line load, which is assumed to originate on the free surface at the origin at time $t = 0$, moves with uniform velocity v ($v < \alpha, \beta$) along the positive direction of the x-axis.

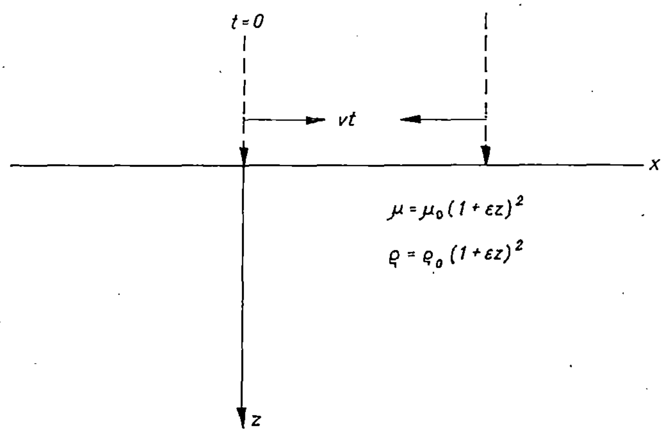

Fig. 1

The equations of motion for a non-homogeneous medium in the absence of bc forces are

$$\frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]=\varrho\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}\right]=\varrho\frac{\partial^2 w}{\partial t^2}.$$

u and w are the displacement components in the x- and z-directions, λ, μ are LAMÉ's constants and ϱ is the density of the medium. It is assumed that

$$\lambda = \mu = \mu_0(1 + \varepsilon z)^2 , \qquad \varrho = \varrho_0(1 + \varepsilon z)^2 , \tag{3}$$

such that the velocity of propagation is independent of z. The equations (1) and (2) have to be solved subject to the boundary conditions

$$\left.\begin{aligned} \tau_{xz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 && \text{at} \quad z = 0 , \\ \tau_{zz} &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} = -P\delta(x - vt) && \text{at} \quad z = 0 , \quad t > 0 . \end{aligned}\right\} \tag{4}$$

$\delta(x - vt)$ is DIRAC's delta function.

3. Formal solution

In order to solve the equations (1) and (2), we make the substitution

$$U = u(1 + \varepsilon z) \quad \text{and} \quad V = v(1 + \varepsilon z) . \tag{5}$$

This transforms the equations (1) and (2) into the forms

$$3\frac{\partial^2 U}{\partial x^2} + 2\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial z}\right) + \frac{\partial^2 U}{\partial z^2} = \frac{\varrho_0}{\mu_0}\frac{\partial^2 U}{\partial t^2} \tag{6}$$

and

$$\frac{\partial^2 W}{\partial x^2} + 2\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial z}\right) + 3\frac{\partial^2 W}{\partial z^2} = \frac{\varrho_0}{\mu_0}\frac{\partial^2 W}{\partial t^2} . \tag{7}$$

We introduce the FOURIER transform over x defined by

$$f_1(p, z, t) = \int_{-\infty}^{\infty} f(x, z, t)\, e^{ipx}\, dp$$

and then take the LAPLACE transform over t defined by

$$\bar{f}_1(p, z, s) = \int_0^{\infty} f_1(p, z, t)\, e^{-st}\, dt .$$

The equations (6) and (7) after these transformations take the forms

$$\left(\frac{d^2}{dz^2} - 3k_1^2\right)\bar{U}_1 = 2ip\frac{d\bar{W}_1}{dz} \tag{8}$$

and

$$\left(3\frac{d^2}{dz^2} - k_2^2\right)\bar{W}_1 = 2ip\frac{d\bar{U}_1}{dz} , \tag{9}$$

where

$$k_1^2 = p^2 + \frac{s^2}{\alpha^2}, \qquad k_2^2 = p^2 + \frac{s^2}{\beta^2}, \qquad \alpha^2 = 3\beta^2 = \frac{\lambda + 2\mu}{\varrho} .$$

Using the conditions that the displacement components vanish as z approaches ∞, the solutions of (8) and (9) are

$$\bar{U}_1 = A\, e^{-k_1 z} + B\, e^{-k_2 z}\,, \tag{10}$$

$$\bar{W}_1 = \frac{1}{ip}\left(Ak_1\, e^{-k_1 z} + \frac{p^2 B}{k_2}\, e^{-k_2 z}\right). \tag{11}$$

Using (5), the above equations become

$$\bar{u}_1 = \frac{1}{1+\varepsilon z}\,(A\, e^{-k_1 z} + B\, e^{-k_2 z}) \tag{12}$$

and

$$\bar{w}_1 = \frac{1}{ip(1+\varepsilon z)}\left(Ak_1\, e^{-k_1 z} + \frac{p^2 B}{k_2}\, e^{-k_2 z}\right). \tag{13}$$

A and B have to be determined from the conditions

$$\frac{d\bar{u}_1}{dz} = ip\bar{w}_1 \quad \text{and} \quad -ip\mu_0\bar{u}_1 + 3\mu_0\frac{d\bar{w}_1}{dz} = \frac{P}{ipv - s} \quad \text{on} \quad z = 0\,,$$

which are obtained by taking first the FOURIER and then the LAPLACE transform on both sides of equations (4). It is found that

$$A = \frac{ipP(\varepsilon k_2 + k_2^2 + p^2)}{\mu_0(ipv - s)\,(k_1k_2 - p^2)\,f(p)}\,, \qquad B = -\,\frac{ipP(2k_1k_2 + \varepsilon k_2)}{\mu_0(ipv - s)\,(k_1k_2 - p^2)\,f(p)}\,,$$

where

$$f(p) = (p^2 - 3k_1k_2) - 3\varepsilon(k_1 + k_2) - 3\varepsilon^2\,.$$

Substituting the values of A and B in (12) and (13) and taking FOURIER inversion, we get

$$\bar{u} = \frac{iP}{2\pi\mu_0(1+\varepsilon z)}\int_{-\infty}^{\infty}\frac{p(\varepsilon k_2 + k_2^2 + p^2)\, e^{-k_1 z} - pk_2(2k_1 + \varepsilon)\, e^{-k_2 z}}{(ipv - s)\,(k_1k_2 - p^2)\,f(p)}\, e^{-ipx}\, dp \tag{14}$$

and

$$\bar{w} = \frac{P}{2\pi\mu_0(1+\varepsilon z)}\int_{-\infty}^{\infty}\frac{k_1(\varepsilon k_2 + k_2^2 + p^2)\, e^{-k_1 z} - p^2(2k_1 + \varepsilon)\, e^{-k_2 z}}{(ipv - s)\,(k_1k_2 - p^2)\,f(p)}\, e^{-ipx}\, dp\,. \tag{15}$$

4. Laplace inversions

We assume

$$\bar{u} = \frac{P}{2\pi\mu_0(1+\varepsilon z)}\,(I_1 - I_2)\,, \tag{16}$$

where

$$I_1 = \int_{-\infty}^{\infty}\frac{ip(\varepsilon k_2 + k_2^2 + p^2)\, e^{-k_1 z - ipx}}{(ipv - s)\,(k_1k_2 - p^2)\,f(p)}\, dp$$

and

$$I_2 = \int_{-\infty}^{\infty} \frac{ipk_2(2k_1 + \varepsilon)\, e^{-k_2 z - ipx}}{(ipv - s)\,(k_1 k_2 - p^2)\, f(p)}\, dp\,.$$

To find the inversions of I_1 and I_2, we adopt CAGNIARD's technique as modified by DE HOOP [4]. Accordingly, we put $p = -sh$ in I_1, which then reduces to the form

$$I_1 = \int_{-\infty}^{\infty} \frac{ih(\varepsilon k_2' + s k_2'^2 + s h^2)\, e^{-s(k_1' z - ihx)}}{(ihv + 1)\, \Phi(h)\, \Psi(s, h)}\, dh\,, \tag{17}$$

where

$$k_1'^2 = h^2 + \frac{1}{\alpha^2}\,, \qquad k_2'^2 = h^2 + \frac{1}{\beta^2}\,,$$

$$\Phi(h) = (3k_1' k_2' - h^2)\,(h^2 - k_1' k_2') = -(h^4 - 4k_1' k_2' + 3k_1' k_2')\,.$$

It has to be noted that $\Phi(h) = 0$ is the RAYLEIGH wave velocity equation corresponding to the homogeneous medium with $\lambda = \mu_0 = \mu$ and

$$\Psi(s, h) = s^2 + \frac{3s\varepsilon(k_1' + k_2')}{3k_1' k_2' - h^2} + \frac{3\varepsilon^2}{3k_1' k_2' - h^2} = (s - \varepsilon m_1)\,(s - \varepsilon m_2)\,,$$

where

$$m_{1,2} = \frac{-3(k_1' + k_2') \pm [9(k_1' - k_2')^2 + 12h^2]^{1/2}}{2(3k_1' k_2' - h^2)}\,;$$

m_1 and m_2 are both negative. Breaking up $(\varepsilon k_2' + s k_2'^2 + s h^2)/\Psi(s, h)$ into partial fractions, the equation (17) can be written as

$$I_1 = \int_{-\infty}^{\infty} \frac{ih(1 - ihv)\, e^{-s(k_1' z - ihx)}}{(1 + h^2 v^2)\, \Phi(h)} \left(\frac{M}{s - \varepsilon m_1} + \frac{N}{s - \varepsilon m_2}\right) dh\,; \tag{18}$$

similarly,

$$I_2 = \int_{-\infty}^{\infty} \frac{ih(1 - ihv)\, e^{-s(k_2' z - ihx)}}{(1 + h^2 v^2)\, \Phi(h)} \left(\frac{S}{s - \varepsilon m_1} + \frac{T}{s - \varepsilon m_2}\right) dh\,. \tag{19}$$

In (18) and (19),

$$M = \frac{k_2' + m_1(k_2'^2 + h^2)}{m_1 - m_2}\,, \qquad N = \frac{k_2' + m_2(k_2'^2 + h^2)}{m_2 - m_1}$$

and

$$S = \frac{k_2' + 2m_1 k_1' k_2'}{m_1 - m_2}\,, \qquad T = \frac{k_2' + 2m_2 k_1' k_2'}{m_2 - m_1}\,.$$

First let us consider the integral

$$\int_{-\infty}^{\infty} \frac{ih(1 - ihv)\, e^{-s(k_1' z - ihx)}}{(1 + h^2 v^2)\, \Phi(h)} \frac{M}{s - \varepsilon m_1}\, dh\,, \tag{20}$$

which occurs in equation (18). In this integral the path of integration with respect to h, which is the real axis, is deformed in such a way that

$$z\left(h^2 + \frac{1}{\alpha^2}\right)^{1/2} - ihx = q\,,$$

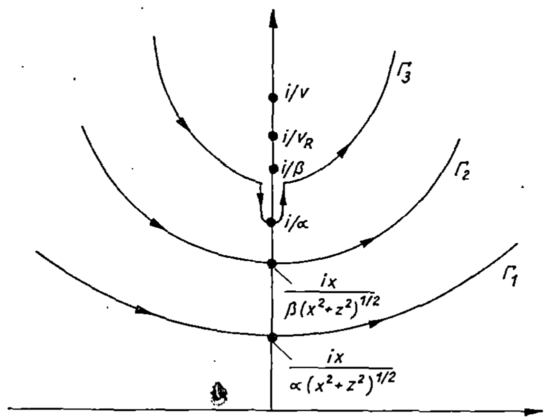

Fig. 2

where q is real and positive. The deformed path of integration is the branch Γ_1 (Fig. 2) of a hyperbola, whose equation is

$$h = \frac{iqx \pm z\left(q^2 - \dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2+z^2}\,, \qquad \frac{(x^2+z^2)^{1/2}}{\alpha} < q < \infty\,.$$

In the course of deformation of the path of integration it is essential to know all the singularities of $M/[(1 + h^2v^2)\,\Phi(h)]$ in the h-plane, which are the poles at $\pm(i/u)$, $\pm(i/v_R)$ and the branch points at $\pm(i/\alpha)$ and $\pm(i/\beta)$, where v_R is the Rayleigh wave velocity corresponding to the homogeneous medium when $\lambda = \mu = \mu_0$.

Since the hyperbolic path Γ_1 does not cross any of the singularities during its deformation, it is possible by virtue of Cauchy's theorem and Jordan's lemma to replace the integration along the real h-axis by an integration along the hyperbolic path Γ_1. We write

$$h_+ = \frac{iqx + z\left(q^2 - \dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2+z^2}\,, \qquad h_- = \frac{iqx - z\left(q^2 - \dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2+z^2}\,;$$

then

$$\frac{dh_\pm}{dq} = \frac{ix\left(q^2 - \dfrac{x^2+z^2}{\alpha^2}\right)^{1/2} \pm qz}{(x^2+z^2)\left(q^2 - \dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}}\,.$$

Using the facts that

$$h_- = -\bar{h}_+, \qquad \frac{dh_-}{dq} = -\left(\frac{d\bar{h}_+}{dq}\right), \qquad m_{1-} = \bar{m}_{1+}, \qquad M_- = \bar{M}_+,$$

where $\bar{h}$ is the complex conjugate of h, the expression (20) takes the form

$$\int\limits_{\frac{(x^2+z^2)^{1/2}}{\alpha}}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+M_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{e^{-sq}}{s-\varepsilon m_{1+}}\,\frac{dh_+}{dq}\right]dq\; +$$

$$+\int\limits_{\frac{(x^2+z^2)^{1/2}}{\alpha}}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2M_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{e^{-sq}}{s-\varepsilon m_{1+}}\,\frac{dh_+}{dq}\right]dq\,.$$

Using the convolution theorem, the LAPLACE inversion of the above integral is

$$\int\limits_0^t d\tau \int\limits_{t_\alpha}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+M_+\,e^{\varepsilon m_{1+}(t-\tau)}}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{dh_+}{dq}\right]\delta(\tau-q)\,dq\; +$$

$$+\int\limits_0^t d\tau \int\limits_{t_\alpha}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2M_+\,e^{\varepsilon m_{1+}(t-\tau)}}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{dh_+}{dq}\right]\delta(\tau-q)\,dq\,,$$

where $t_\alpha = (x^2+z^2)^{1/2}/\alpha$ is the arrival time of P-waves. By use of the properties of the δ-function, the above integrals can be written as

$$H(t-t_\alpha)\left[\int\limits_{t_\alpha}^{t} -2\,\mathrm{Im}\left\{\frac{h_+M_+\,e^{\varepsilon m_{1+}(t-\tau)}}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{dh_+}{d\tau}\right\}d\tau\; + \right.$$

$$\left. + \int\limits_{t_\alpha}^{t} 2v\,\mathrm{Re}\left\{\frac{h_+^2M_+\,e^{\varepsilon m_{1+}(t-\tau)}}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{dh_+}{d\tau}\right\}d\tau\right]. \tag{21}$$

It should be noted that in the integrand of the above integral q has been replaced by τ everywhere.

In a similar manner the LAPLACE inversion of the other part of I_1 in (18) can be determined. It is found to be a similar expression as the expression in (21) except that M_+ and m_{1+} have to be replaced by N_+ and m_{2+} respectively. Thus the LAPLACE inversion of I_1 is

$$H(t-t_\alpha)\int\limits_{t_\alpha}^{t} -2\,\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\left\{M_+\,e^{\varepsilon m_{1+}(t-\tau)} + N_+\,e^{\varepsilon m_{2+}(t-\tau)}\right\}\frac{dh_+}{d\tau}\right]d\tau\; +$$

$$+\,H(t-t_\alpha)\int\limits_{t_\alpha}^{t} 2v\,\mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\left\{M_+\,e^{\varepsilon m_{1+}(t-\tau)} + N_+\,e^{\varepsilon m_{2+}(t-\tau)}\right\}\frac{dh_+}{d\tau}\right]d\tau\,. \tag{22}$$

Next we shall calculate the LAPLACE inversion of I_2 that occurs in (19). As before here also we define

$$z\left(h^2 + \frac{1}{\beta^2}\right)^{1/2} - ihx = r\,,$$

where r is real and positive. So,

$$h_\pm = \frac{irx + z\left(r^2 - \dfrac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}\,. \tag{23}$$

Case 1: A path along which r is real and non-negative is the hyperbolic path Γ_2 (Fig. 2) represented parametrically by the above equation with $r > (x^2+z^2)^{1/2}/\beta$, provided the path where it cuts the imaginary axis, viz. $h = ix/\beta(x^2+z^2)^{1/2}$, lies below the branch point i/α, which occurs when $x < \beta z/(\alpha^2-\beta^2)^{1/2}$. In this case the path Γ_2 (Fig. 2) does not cross any of the singularities during the deformation. Following the same procedure as that done in case of I_1, the LAPLACE inversion of I_2 is found to be

$$H(t-t_\beta)\int\limits_{t_\beta}^{t} -2\,\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\left\{S_+\,e^{\varepsilon m_{1+}(t-\tau)} + T_+\,e^{\varepsilon m_{2+}(t-\tau)}\right\}\frac{dh_+}{d\tau}\right]d\tau\, +$$
$$+\,H(t-t_\beta)\int\limits_{t_\beta}^{t} 2v\,\mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\left\{S_+\,e^{\varepsilon m_{1+}(t-\tau)} + T_+\,e^{\varepsilon m_{2+}(t-\tau)}\right\}\frac{dh_+}{d\tau}\right]d\tau\,, \tag{24}$$

where $t_\beta = (x^2+z^2)^{1/2}/\beta$ is the arrival time of S-waves, and h_+ occurring in the above expression is obtained by replacing r by τ in the expression for h_+ as given in (23).

Case 2: If $x > \beta z/(\alpha^2-\beta^2)^{1/2}$, the point $ix/\beta(x^2+z^2)^{1/2}$ lies above the branch point i/α. Therefore, the path of integration in the h-plane has to be deformed to the path Γ_3 (Fig. 2) round the branch point i/α as shown in Fig. 2.

We consider the integral

$$\int\limits_{-\infty}^{\infty} \frac{ih(1-ihv)\,S}{(1+h^2v^2)\,\Phi(h)}\,\frac{e^{-s(k_2'z-ihx)}}{s-\varepsilon m_1}\,dh \tag{25}$$

occurring in I_2 of equation (19). Here too we put

$$z\left(h^2 + \frac{1}{\beta^2}\right)^{1/2} - ihx = r\,.$$

On the two finite straight line portions of the path Γ_3, h is given by

$$h_\pm = \pm\eta + \frac{i\left\{rx - z\left(\dfrac{x^2+z^2}{\beta^2} - r^2\right)^{1/2}\right\}}{x^2+z^2}\,,$$

where finally η should be made to tend to zero, and on the remaining portions of the path of Γ_3,

$$h_\pm = \frac{irx \pm z\left(r^2 - \dfrac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}\,.$$

On the straight line portions of the path Γ_3, r varies from $r = t_{\alpha\beta}$ to $r = t_\beta$, where $t_{\alpha\beta} = x/\alpha + z(1/\beta^2 - 1/\alpha^2)^{1/2}$ is the arrival time of PS-waves. The expression in (25) can then be written in the form

$$\int_{t_{\alpha\beta}}^{t_\beta}\left[\frac{ih_+(1 - ih_+v)\, S_+}{(1+h_+^2v^2)\,\Phi(h_+)\,(s-\varepsilon m_{1+})}\frac{dh_+}{dr} - \frac{ih_-(1 - ih_-v)}{(1+h_-^2v^2)\,\Phi(h_-)\,(s-\varepsilon m_{1-})}\frac{dh_-}{dr}\right]e^{-sr}\,dr +$$

$$+ \int_{t_\beta}^{\infty}\left[\frac{ih_+(1 - ih_+v)\, S_+}{(1+h_+^2v^2)\,\Phi(h_+)\,(s-\varepsilon m_{1+})}\frac{dh_+}{dr} - \frac{ih_-(1 - ih_-v)}{(1+h_-^2v^2)\,\Phi(h_-)\,(s-\varepsilon m_{1-})}\frac{dh_-}{dr}\right]e^{-sr}\,dr. \tag{26}$$

Noting that $h_- = -\bar{h}_+$, $dh_-/dr = -(d\bar{h}_+/dr)$ and $S_- = \bar{S}_+$ on the path Γ_3, the expression (26) takes the form

$$\int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\,\frac{h_+S_+}{(1 + h_+^2v^2)\,\Phi(h_+)}\,\frac{e^{-sr}}{s - \varepsilon m_{1+}}\,\frac{dh_+}{dr}\,dr + \int_{t_{\alpha\beta}}^{t_\beta} 2v\,\mathrm{Re}\,\frac{h_+^2S_+}{(1 + h_+^2v^2)\,\Phi(h_+)}\times$$

$$\times\frac{e^{-sr}}{s - \varepsilon m_{1+}}\,\frac{dh_+}{dr}\,dr + \int_{t_\beta}^{\infty} -2\,\mathrm{Im}\,\frac{h_+S_+}{(1 + h_+^2v^2)\,\Phi(h_+)}\,\frac{e^{-sr}}{s - \varepsilon m_{1+}}\,\frac{dh_+}{dr}\,dr +$$

$$+ \int_{t_\beta}^{\infty} 2v\,\mathrm{Re}\,\frac{h_+^2S_+}{(1 + h_+^2v^2)\,\Phi(h_+)}\,\frac{e^{-sr}}{s - \varepsilon m_{1+}}\,\frac{dh_+}{dr}\,dr\,. \tag{27}$$

To transform the other integral of I_2 occurring in (19), a similar procedure is applied, and finally I_2 in (19) takes the following form:

$$I_2 = \int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1 + h_+^2v^2)\ \Phi(h_+)}\left(\frac{S_+}{s - \varepsilon m_{1+}} + \frac{T_+}{s - \varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}dr +$$

$$+ \int_{t_\beta}^{\infty} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1 + h_+^2v^2)\ \Phi(h_+)}\left(\frac{S_+}{s - \varepsilon m_{1+}} + \frac{T_+}{s - \varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}dr +$$

$$+ \int_{t_{\alpha\beta}}^{t_\beta} 2v\,\mathrm{Re}\left\{\frac{h_+^2\,e^{-sr}}{(1 + h_+^2v^2)\ \Phi(h_+)}\left(\frac{S_+}{s - \varepsilon m_{1+}} + \frac{T_+}{s - \varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}dr +$$

$$+ \int_{t_\beta}^{\infty} 2v\,\mathrm{Re}\left\{\frac{h_+^2\,e^{-sr}}{(1 + h_+^2v^2)\ \Phi(h_+)}\left(\frac{S_+}{s - \varepsilon m_{1+}} + \frac{T_+}{s - \varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}dr\,. \tag{28}$$

It must be remembered that the value of h_+ when r lies in $[t_{\alpha\beta}, t_\beta]$ has to be taken as

$$h_+ = \frac{i\left\{rx - z\left(\dfrac{x^2 + z^2}{\beta^2} - r^2\right)^{1/2}\right\}}{x^2 + z^2},$$

and for r lying in $[t_\beta, \infty)$,

$$h_+ = \frac{irx + z\left(r^2 - \dfrac{x^2 + z^2}{\beta^2}\right)^{1/2}}{x^2 + z^2}.$$

Next the LAPLACE inversion of I_2 in (28) has to be calculated. By applying the convolution theorem, the LAPLACE inversion of the first integral of I_2 in (28) is found to be

$$\int_0^t d\tau \int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1+h_+^2v^2)\,\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}\delta(\tau-r)\,dr =$$

$$= \int_0^t d\tau \int_{t_{\alpha\beta}}^{\infty} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1+h_+^2v^2)\,\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}\delta(\tau-r)\,dr -$$

$$- \int_0^t d\tau \int_{t_\beta}^{\infty} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1+h_+^2v^2)\,\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}\delta(\tau-r)\,dr\,,$$

and it takes the following form when the δ-function property is used:

$$H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -2\,\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)})\,\frac{dh_+}{d\tau}\right\}d\tau -$$

$$- H(t-t_\beta)\int_{t_\beta}^{t} -2\,\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)})\,\frac{dh_+}{d\tau}\right\}d\tau\,. \tag{29}$$

It can be shown that the last term of (29) is cancelled with the LAPLACE inversion of the second integral in (28).

Similarly, the LAPLACE inversion of the other integrals of (29) can be determined, and finally, after simplification, we get the LAPLACE inversion of I_2 as

$$H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -2\,\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)})\,\frac{dh_+}{d\tau}\right\}d\tau -$$

$$- H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} 2v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)})\,\frac{dh_+}{d\tau}\right\}d\tau\,. \tag{30}$$

Combining the results of the inverse LAPLACE transforms of I_1 and I_2 from (22) and (24) it follows that

$$u(x, z, t) = \frac{P}{\pi\mu_0(1+\varepsilon z)}\times$$

$$\times\Bigg[H(t-t_\alpha)\int_{t_\alpha}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(M_+\,e^{\varepsilon m_{1+}(t-\tau)}+N_+\,e^{\varepsilon m_{2+}(t-\tau)})\,\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+ H(t-t_\alpha)\int_{t_\alpha}^{t} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(M^+\,e^{\varepsilon m_{1+}(t-\tau)}+N_+\,e^{\varepsilon m_{2+}(t-\tau)})\,\frac{dh_+}{d\tau}\right\}d\tau -$$

$$- H(t - t_\beta) \int\limits_{t_\beta}^{t} -\mathrm{Im} \left\{ \frac{h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau -$$

$$- H(t - t_\beta) \int\limits_{t_\beta}^{t} v\, \mathrm{Re} \left\{ \frac{h_+^2}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau \Bigg]. \tag{31}$$

for $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, and when $x > \beta z/(\alpha^2 - \beta^2)^{1/2}$, from (22) and (30) it follows that

$$u(x, z, t) = \frac{P}{\pi \mu_0 (1 + \varepsilon z)} \times$$

$$\times \Bigg[H(t - t_\alpha) \int\limits_{t_\alpha}^{t} -\mathrm{Im} \left\{ \frac{h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_\alpha) \int\limits_{t_\alpha}^{t} v\, \mathrm{Re} \left\{ \frac{h_+^2}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau -$$

$$- H(t - t_{\alpha\beta}) \int\limits_{t_{\alpha\beta}}^{t} -\mathrm{Im} \left\{ \frac{h_+}{(1 + \quad {}^2 \quad \Phi(h_+)} \left(S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau -$$

$$- H(t - t_{\alpha\beta}) \int\limits_{t_{\alpha\beta}}^{t} v\, \mathrm{Re} \left\{ \frac{h_+^2}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau \Bigg]. \tag{32}$$

Carrying on a similar procedure as done for the evaluation of the displacement along the x-direction, the expression for the displacement along the z-direction can also be determined from (13) and is found to be equal to

$$w(x, z, t) = \frac{P}{\pi \mu_0 (1 + \varepsilon z)} \times$$

$$\times \Bigg[H(t - t_\alpha) \int\limits_{t_\alpha}^{t} \mathrm{Re} \left\{ \frac{k'_{1+}}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_\alpha) \int\limits_{t_\alpha}^{t} v\, \mathrm{Im} \left\{ \frac{k'_{1+} h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_\beta) \int\limits_{t_\beta}^{t} \mathrm{Re} \left\{ \frac{h_+^2}{k'_{2+} (1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_\beta) \int\limits_{t_\beta}^{t} v\, \mathrm{Im} \left\{ \frac{h_+^3}{k'_{2+} (1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau \Bigg] \tag{33}$$

for $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, and if $x > \beta z/(\alpha^2 - \beta^2)^{1/2}$:

$$w(x, z, t) = \frac{P}{\pi\mu_0(1 + \varepsilon z)} \times$$

$$\times \Bigg[H(t - t_\alpha) \int_{t_\alpha}^{t} \operatorname{Re} \left\{ \frac{k'_{1+}}{(1 + h_+^2 v^2)\, \Phi(h_+)} (M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_\alpha) \int_{t_\alpha}^{t} v \operatorname{Im} \left\{ \frac{k'_{1+} h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} (M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_{\alpha\beta}) \int_{t_{\alpha\beta}}^{t} \operatorname{Re} \left\{ \frac{h_+^2}{k'_{2+}(1 + h_+^2 + v^2)\, \Phi(h_+)} (S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_{\alpha\beta}) \int_{t_{\alpha\beta}}^{t} v \operatorname{Im} \left\{ \frac{h_+^3}{k'_{2+}(1 + h_+^2 v^2)\, \Phi(h_+)} (S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}) \frac{dh_+}{d\tau} \right\} d\tau \Bigg]. \tag{34}$$

It should be remembered that in the first two integrals of the equations (31), (32), (33) and (34)

$$h_+ = \frac{i\tau x + z\left(\tau^2 - \dfrac{x^2 + z^2}{\alpha^2}\right)^{1/2}}{x^2 + z^2}, \qquad t_\alpha \leqq \tau \leqq t\,,$$

and in the last two integrals of those equations

$$h_+ = \frac{i\tau x + z\left(\tau^2 - \dfrac{x^2 + z^2}{\beta^2}\right)^{1/2}}{x^2 + z^2}, \qquad t_\beta \leqq \tau \leqq t\,,$$

where as in the last two integrals of (32) and (34)

$$h_+ = \frac{i\left\{\tau x - z\left(\dfrac{x^2 + z^2}{\beta^2} - \tau^2\right)^{1/2}\right\}}{x^2 + z^2}, \qquad t_{\alpha\beta} \leqq \tau \leqq t_\beta\,.$$

5. Wave front expansion

The wave forms of the solutions given in (31) to (34) are evaluated by approximate estimation of the above integrals in the neighbourhood of the time of the first arrival of the different waves. To facilitate this evaluation we put $\tau = A + a$, where A is the lower limit of the integrals in question and a varies from 0 to $t - A$. Then when $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, from (31) we get

$$u(x, z, t) = \frac{P}{\pi\mu_0(1 + \varepsilon z)} \times$$

$$\times \Bigg[H(t - t_\alpha) \int_0^{t - t_\alpha} - \operatorname{Im} \left\{ \frac{h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} (M_+ e^{\varepsilon m_{1+}(t - t_\alpha - a)} + \right.$$

$$\left. + N_+ e^{\varepsilon m_{2+}(t - t_\alpha - a)}) \frac{dh_+}{da} \right\} da +$$

$$+H(t-t_\alpha)\int_0^{t-t_\alpha} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(M_+\,e^{\varepsilon m_{1+}(t-t_\alpha-a)}+\right.$$
$$\left.+N_+\,e^{\varepsilon m_{2+}(t-t_\alpha-a)})\,\frac{dh_+}{da}\right\}da+$$
$$+H(t-t_\beta)\int_0^{t-t_\beta} \mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-t_\beta-a)}+\right.$$
$$\left.+T_+\,e^{\varepsilon m_{2+}(t-t_\beta-a)})\,\frac{dh_+}{da}\right\}da+$$
$$+H(t-t_\beta)\int_0^{t-t_\beta} -v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-t_\beta-a)}+\right.$$
$$\left.\left.+T_+\,e^{\varepsilon m_{2+}(t-t_\beta-a)}\,\frac{dh_+}{da}\right\}da\right]. \tag{35}$$

For $x > \beta z/(\alpha^2-\beta^2)^{1/2}$,

$$u(x,z,t)=\frac{P}{\pi\mu_0(1+\varepsilon z)}\times$$
$$\times\left[H(t-t_\alpha)\int_0^{t-t_\alpha} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(M_+\,e^{\varepsilon m_{1+}(t-t_\alpha-a)}+\right.\right.$$
$$\left.N_+\,e^{\varepsilon m_{2+}(t-t_\alpha-a)})\,\frac{dh_+}{da}\right\}da+$$
$$+H(t-t_\alpha)\int_0^{t-t_\alpha} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(M_+\,e^{\varepsilon m_{1+}(t-t_\alpha-a)}+\right.$$
$$\left.N_+\,e^{\varepsilon m_{2+}(t-t_\alpha-a)})\,\frac{dh_+}{da}\right\}da+$$
$$+H(t-t_{\alpha\beta})\int_0^{t-t_{\alpha\beta}} \mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-t_{\alpha\beta}-a)}+\right.$$
$$\left.+T_+\,e^{\varepsilon m_{2+}(t-t_{\alpha\beta}-a)})\,\frac{dh_+}{da}\right\}da+$$
$$+H(t-t_{\alpha\beta})\int_0^{t-t_{\alpha\beta}} -v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-t_{\alpha\beta}-a)}+\right.$$
$$\left.\left.+T_+\,e^{\varepsilon m_{2+}(t-t_{\alpha\beta}-a)})\,\frac{dh_+}{da}\right\}da\right]. \tag{36}$$

A similar type of expressions for $w(x,z,t)$ can be written by substituting $\tau = A + a$ in the equations (33) and (34). For approximate evaluation of the integrals (35) and (36) just after the arrival of the corresponding wave fronts it has to be noted that

$$e^{\varepsilon m_{1+}(t-A-a)}\to 1\,,\qquad e^{\varepsilon m_{2+}(t-A-a)}\to 1\quad\text{and}\quad a\to 0\quad\text{as}\quad t\to A\,,$$

where A is the arrival time of a typical wave front. So, when $A = t_\alpha$, using the facts that

$$h_+ \to \frac{ix}{\alpha(x^2+z^2)^{1/2}}, \qquad h_+^2 \to \frac{-x^2}{\alpha^2(x^2+z^2)}$$

and

$$\frac{(M_+ + N_+)}{(1+h_+^2 v^2)\,\Phi(h_+)} \to$$
$$\to \frac{(x^2+z^2)^2\alpha^4\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\}}{\{x^2(\alpha^2-v^2)+\alpha^2z^2\}[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4\beta x^2z\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\}^{1/2}]}$$

as $a \to 0$ and that

$$\frac{dh_+}{da} = \frac{\left(h_+^2+\dfrac{1}{\alpha^2}\right)^{1/2}}{\left\{2\dfrac{(x^2+z^2)^{1/2}}{\alpha}+a\right\}^{1/2}}\frac{1}{a^{1/2}},$$

the first two integrals of the equations (35) and (36) just after the arrival of P-waves can approximately be evaluated to the form

$$u(x,z,t) =$$
$$= -\frac{\sqrt{2}PH(t-t_\alpha)\,xz\alpha^{3/2}(x^2+z^2)^{1/4}\{vx+\alpha(x^2+z^2)^{1/2}\}\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\}(t-t_\alpha)^{1/2}}{\pi\mu_0(1+\varepsilon z)\{x^2(\alpha^2-v^2)+\alpha^2z^2\}[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4x^2z\beta\times\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\}^{1/2}]}.$$

Similarly, the approximate value of w just after the arrival of P-waves is given by

$$w(x,z,t) =$$
$$= \frac{\sqrt{2}\,PH(t-t_\alpha)\,z^2\alpha^{3/2}(x^2+z^2)^{1/4}\{vx+\alpha(x^2+z^2)^{1/2}\}\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\}(t-t_\alpha)^{1/2}}{\pi\mu_0(1+\varepsilon z)\{x^2(\alpha^2-v^2)+\alpha^2z^2\}[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4x^2z\beta\times\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\}^{1/2}]}.$$

The same method is applied for approximate evaluation of u and w just after the arrival of S-waves. It should be remembered in this case that

$$A = t_\beta, \qquad h_+ = \frac{i\tau x + z\left(\tau^2-\dfrac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}, \qquad t_\beta < \tau < t.$$

The effects of u_1 and w_1 on the displacement components u and v due to S-waves just after their arrival are found to be

$$u_1(x,z,t) =$$
$$= \frac{2\sqrt{2}\,PH(t-t_\alpha)\,xz^2\alpha\beta^{3/2}(x^2+z^2)^{1/4}\{vx+\beta(x^2+z^2)^{1/2}\}\{\beta^2z^2-x^2(\alpha^2-\beta^2)\}^{1/2}(t-t_\beta)^{1/2}}{\pi\mu_0(1+\varepsilon z)\{x^2(\beta^2-v^2)+\beta^2z^2\}[\alpha^2x^2(3z^2-x^2)-3\beta^2z^2(x^2+z^2)-4x^2z\alpha\times\{\beta^2z^2-x^2(\alpha^2-\beta^2)\}^{1/2}]},$$

$$w_1(x,z,t) =$$
$$= -\frac{2\sqrt{2}PH(t-t_\beta)\,x^2z\alpha\beta^{3/2}(x^2+z^2)^{1/4}\{vx+\beta(x^2+z^2)^{1/2}\}\{\beta^2z^2-x^2(\alpha^2-\beta^2)\}^{1/2}(t-t_\beta)^{1/2}}{\pi\mu_0(1+\varepsilon z)\{x^2(\beta^2-v^2)+\beta^2z^2\}[\alpha^2x^2(3z^2-x^2)-3\beta^2z^2(x^2+z^2)-4x^2z\alpha\times\{\beta^2z^2-x^2(\alpha^2-\beta^2)\}^{1/2}]}$$

for $0 < x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, and in the region $x > \beta z/(\alpha^2 - \beta^2)^{1/2}$, PS-waves exist and arrive earlier than S-waves.

We approximately calculate the last two integrals of (36), which will give the effect of u_2 on the displacement components u due to PS-waves just after their arrival. In this case

$$A = t_{\alpha\beta}\,, \qquad h_+ = \frac{i\left\{\tau x - z\left(\frac{x^2+z^2}{\beta^2} - \tau^2\right)^{1/2}\right\}}{x^2+z^2}\,, \qquad t_{\alpha\beta} \leqq \tau \leqq t_\beta\,.$$

Then

$$h_+ \to i\left[\frac{1}{\alpha} + \frac{\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2}}{x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - \frac{z}{\alpha}}\,a\right],$$

$$\frac{S+T}{(1+h_+^2v^2)\,\Phi(h_+)} \to -\frac{i\,2\sqrt{2}\,\alpha^6\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{3/4} a^{1/2}}{(\alpha^2 - v^2)\left\{\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z\right\}^{1/2}}$$

and

$$\frac{dh_+}{da} \to \frac{i\alpha\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2}}{\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z}\,,$$

where in these expressions terms containing higher order of a are neglected because $a \to 0$ as $t \to t_{\alpha\beta}$, and we get

$$u_2(x, z, t) = \frac{4\sqrt{2}PH(t - t_{\alpha\beta})\,\alpha^5\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{5/4}}{3\pi\mu_0(1+\varepsilon z)\,(\alpha - v)\left\{\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z\right\}^{3/2}}\,(t - t_{\alpha\beta})^{3/2}\,.$$

Similarly, the effect on the displacement component w just after the arrival of PS-waves is given by

$$w_2(x, z, t) = -\frac{4\sqrt{2}\,PH(t - t_{\alpha\beta})\,\alpha^4\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{3/4}}{3\pi\mu_0(1+\varepsilon z)\,(\alpha - v)\left\{\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z\right\}^{3/2}}\,(t - t_{\alpha\beta})^{3/2}\,.$$

We now find out the effects of u_3 and w_3 on the displacement components u and w in the neighbourhood of the point C (Fig. 3), where S- and PS-waves arrive at the same time. In this case $t_\beta = t_{\alpha\beta}$ and

$$u_3(x, z, t) = -\frac{4PH(t - t_{\alpha\beta})}{3\pi\mu_0(1+\varepsilon z)}\,\frac{2^{3/4}\alpha^{1/4}\beta^{3/2}z^{5/2}}{x^{13/4}(\alpha - v)}\,(t - t_{\alpha\beta})^{3/4}\,,$$

$$w_3(x, z, t) = \frac{4PH(t - t_{\alpha\beta})}{3\pi\mu_0(1 + \varepsilon z)} \frac{2^{3/4}\alpha^{1/4}\beta^{3/2}z^{3/2}}{x^{9/4}(\alpha - v)} (t - t_{\alpha\beta})^{3/4} .$$

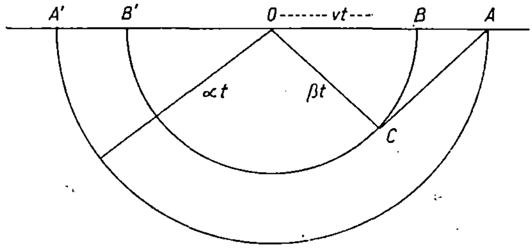

Fig. 3

6. Concluding remarks

It is found from the integrals (31) to (34) that the effect of inhomogeneity enters into the expressions for u and v through the factors $e^{\varepsilon m_1(t-\tau)}$ and $e^{\varepsilon m_2(t-\tau)}$ in the corresponding integrands. So, if these two factors are absent, and that is so if $\varepsilon = 0$, a parallel case for a homogeneous medium is obtained.

Also, it is interesting to note that in the neighbourhood of points just after the arrival of the different wave fronts the displacement components are independent of ε, i.e., at any point, the effect of the first arrival of wave fronts on the displacement components is the same for homogeneous as well as for inhomogeneous media. But as time goes on, ε occurring in the exponential terms of the integrals (31) to (34) for u and w will have its effect, and consequently, the amplitude of the wave fronts will decay exponentially with time due to inhomogeneity of the medium.

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RAYLEIGH WAVES DUE TO NONUNIFORMLY PROPAGATING DIP-SLIP FAULT

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It is assumed that a crack is developed suddenly along a horizontal line at a finite depth below the surface of the earth which is assumed to be an isotropic homogeneous elastic medium. The crack moves along a vertical plane upto the free surface. Assuming the motion to be two dimensional the surface displacement due to Rayleigh waves produced by nonuniformly moving crack has been determined by using Green's function representation theorem and following the technique developed by Knopoff and Gilbert (1959). For different types of fault propagation, the displacement components derived in integral form are numerically evaluated and are shown by means of graphs which may be of interest in earthquake engineering.

1. INTRODUCTION

The study of dynamic crack propagation is very important in geophysics and in earthquake engineering science. In geophysics it is desirable to formulate the earthquake source in terms of physical parameters and to study the long period waves over a large distance and for a long time. Also in structural engineering it is essential to know the nature of surface waves covering a large distance. At a particular place the ground motion produced by the earthquake is a very complicated function of the nature of propagation of the crack and the geological properties of the place as well. Most of the known solutions of the moving crack are restricted by the assumption of constant velocity of propagation, which is not in general expected. Mal (1972) discussed Rayleigh wave propagation by a finite fault moving with constant velocity. He represented the shear failure by a jump in the tangential components of displacement across the fault surface. Achenbach and Abo-Zeno (1972) analyzed the wave motions generated by a vertical strike slip fault on which motion is opposed by a frictional shear stress and which is assumed to increase linearly with depth. Freund (1973) discussed wave motions as expected in case of a nonuniformly expanding line load. Fossum and Freund (1975) considered a model in which a plane strain shear crack moves from rest at a nonuniform rate under the action of general loading. First motion response of an elastic half space due to a nonuniformly moving dislocation by Cagniard De-Hoop technique is determined by Roy (1978). In a recent paper Markenscoff and Clifton (1981) analyzed the motion of an edge dislocation starting from rest and moving thereafter nonuniformly on its slip plane by means of Laplace transform, where the inversion of the transform is accomplished by Cagniard De-Hoop method.

In the present paper an idealised earthquake model is considered. A fault break along a horizontal line at a finite depth below the free surface is assumed to appear suddenly and to move vertically upward with nonuniform motion upto the free surface. A discontinuity in components of displacement across the fault break is prescribed. The displacement components on the free surface due to Rayleigh waves are determined for nonuniform motion of the crack.

To find the solution of the problem the technique developed by Knopoff and Gilbert with appropriate modification is used. The technique is found to be extremely powerful for tackling such type of boundary value problems. Ghosh (1972) applied the method to show the possibility of attenuation of microseismic waves due to the presence of an upward folding of the ocean bottom into the liquid. Following Knopoff and Gilbert, the moving crack is replaced by a set of virtual sources located at the fault surface HO. The displacement on the free surface is written as the sum of the contribution of these sources with the aid of suitable Green's function representation theorem.

Three particular cases of nonuniform motion of the crack are considered. Horizontal and vertical components of surface displacements due to Rayleigh waves produced by the propagating crack are determined and shown by means of graphs.

In the mathematical and physical structure of wave propagation phenomenon, the model assumed here is although over simplified, yet it brings forth some major features which are usually present in the ground motion.

2. Formulation of the Problem and Solution

The origin of the co-ordinate frame (x, y) is at the epicentre 0. It is assumed that a crack suddenly appearing at the focus H moves vertically upwards upto the free surface 0 with a nonuniform speed. The length of the crack measured from H at any time t is $h(t)$, which is assumed to be strictly monotonic increasing function of time t.

The Fourier transform $\bar{f}(x, y, \omega)$ of the function $f(x, y, t)$ is defined by

$$\bar{f}(x, y, \omega) = \int_{-\infty}^{\infty} f(x, y, t)e^{i\omega t}\, dt \qquad ...(1)$$

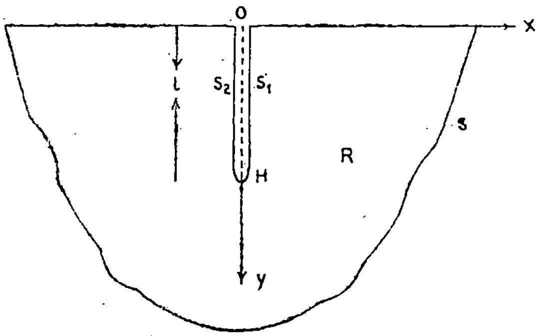


FIG. 1. Geometry of dip-slip fault.

Let $G_n^m(x, y \mid x_0, y_0)$, $(m, n = (x, y))$ be the component of Green's function $G^m(x, y \mid x_0, y_0)$ at the point (x, y) in the direction of n due to a point source of force in m-direction and situated at (x_0, y_0). If now $u(x, y)$ and $v(x, y)$ be the displacement components along x and y directions respectively and $P_{xx}^{(u,v)}$, $P_{xy}^{(u,v)}$ and $P_{yy}^{(u,v)}$ be the stress components then their Fourier transforms defined by (1) satisfy the following differential equations:

$$\frac{\partial \overline{P}_{xx}^{(u,v)}}{\partial x} + \frac{\partial \overline{P}_{xy}^{(u,v)}}{\partial y} + \rho\omega^2 \overline{u}(x, y) = 0 \qquad \ldots(2)$$

$$\frac{\partial \overline{P}_{xy}^{(u,v)}}{\partial x} + \frac{\partial \overline{P}_{yy}^{(uv)}}{\partial y} + \rho\omega^2 \overline{v}(x, y) = 0 \qquad \ldots(3)$$

$$\frac{\partial \overline{P}_{xx}[G^x(x,y \mid x_0, y_0)]}{\partial x} + \frac{\partial \overline{P}_{xy}[G^x(x,y \mid x_0, y_0)]}{\partial y} + \rho\omega^2 \overline{G}_x^x(x, y \mid x_0, y_0)$$
$$= -\delta(x - x_0)\,\delta(y - y_0) \qquad \ldots(4)$$

$$\frac{\partial \overline{P}_{xy}[G^x(x,y \mid x_0, y_0)]}{\partial x} + \frac{\partial \overline{P}_{yy}[G^x(x,y \mid x_0, y_0)]}{\partial y} + \rho\omega^2 \overline{G}_y^x(x, y \mid x_0, y_0) = 0 \qquad \ldots(5)$$

$$\frac{\partial \overline{P}_{xx}[G^y(x,y \mid x_0, y_0)]}{\partial x} + \frac{\partial \overline{P}_{xy}[G^y(x,y \mid x_0, y_0)]}{\partial y} + \rho\omega^2 \overline{G}_x^y(x, y \mid x_0, y_0) = 0 \qquad \ldots(6)$$

$$\frac{\partial \overline{P}_{xy}[G^y(x,y \mid x_0, y_0)]}{\partial x} + \frac{\partial \overline{P}_{yy}[G^y(x,y \mid x_0, y_0)]}{\partial y} + \rho\omega^2 \overline{G}_y^y(x'y \mid x_0, y_0)$$
$$= -\delta(x - x_0)\,\delta(y - y_0) \qquad \ldots(7)$$

where ρ is the density of the material and $\delta(\;)$ is Dirac's delta function.

Multiply eqn. (2) by $\bar{G}_x^x(x, y \mid x_0, y_0)$ and (4) by $\bar{u}(x, y)$ and subtract the latter from the former. Also multiply eqn. (3) by $\bar{G}_y^x(x, y \mid x_0, y_0)$ and (5) by $\bar{v}(x, y)$ and subtract the latter from the former. These two resulting equations are then added and integrated over the region R to yield the following equation:

$$\iint_R \Big\{ \frac{\partial}{\partial x}\Big[\bar{G}_x^x(x, y \mid x_0, y_0)\, \bar{P}_{xx}(u, v) + \bar{G}_y^x(x, y \mid x_0, y_0)\, \bar{P}_{xy}(u, v)$$

$$- \bar{u}\bar{P}_{xx}\Big[\mathbf{G}^x(x, y \mid x_0, y_0) \Big] - \bar{v}\, \bar{P}_{xy}\Big[\mathbf{G}_y^x(x, y \mid x_0, y_0) \Big]\Big]$$

$$+ \frac{\partial}{\partial y}\Big[\bar{G}_x^x(x, y \mid x_0, y_0) \bar{P}_{xy}(u,v) + \bar{G}_y^x(x, y \mid x_0, y_0)\, \bar{P}_{yy}(u, v)$$

$$- \bar{u}\bar{P}_{xy}\Big[\mathbf{G}^x(x, y \mid x_0, y_0) \Big] - \bar{v}\, \bar{P}_{yy}\Big[\mathbf{G}^x(x, y \mid x_0, y_0)\Big]\Big]\Big\}$$

$$\times\, dR = \bar{u}\,(x_0, y_0). \qquad \ldots(8)$$

For details of the analysis to obtain eqn. (8), we refer to the paper of Ghosh (1972).

Applying Green's theorem, the integral in (8) over the region R is converted to an integral over the curves S, S_1, S_2 (shown in Fig. 1) bounding the region R and we have

$$\int_{S+S_1+S_2} \Big\{ \bar{G}_x^x(x, y \mid x_0, y_0)\, P_{nx}(u, v) + \bar{G}_y^x(x, y \mid x_0, y_0)\, \bar{P}_{ny}(u, v)$$

$$- \bar{u}\bar{P}_{nx}[\mathbf{G}^x(x, y \mid x_0, y_0)] - \bar{v}\, \bar{P}_{ny}\Big[\mathbf{G}^x x, y \mid x_0, y_0) \Big]\Big\}\, ds = \bar{u}(x_0, y_0). \qquad \ldots(9)$$

Since the stresses due to (u, v) are zero on the free surfaces S, S_1, S_2 and the stresses due to Green's function are also zero on the free surface S, so we obtain from (9)

$$\int_0^l \{[\bar{u}]\, \bar{P}_{xx}[\mathbf{G}^x(0, y \mid x_0, y_0)] + [\bar{v}]\bar{P}_{xy}\, [\mathbf{G}^x(0, y \mid x_0, y_0)]\}\, dy = \bar{u}(x_0, y_0)$$

$$\ldots(10)$$

where $[u]$ and $[v]$ represent the jump discontinuity in displacement components across the crack HO and $HO = l$ is the length of the crack. Since we are considering a dip-slip fault, so there is no displacement discontinuity along x-direction across the fault surface. Consequently $[u] = 0$. Also, as we are interested in surface displacement only, so the equation (10) reduces to the form

$$\int_0^l [\bar{v}]\, \bar{P}_{xy}\, [\mathbf{G}^x(0, y \mid x_0, 0)]\, dy = \bar{u}(x_0, 0). \qquad \ldots(11)$$

Considering the equations (2), (3), (6) and (7), following the same procedure, we get

$$\int_0^l [v]\, \bar{P}_{xy}\, [G^y(0, y \mid x_0, 0)]\, dy = \bar{v}\,(x_0, 0). \qquad \ldots(12)$$

The Fourier transforms of the Green's functions are

$$\bar{G}^x_x(x, y \mid x, 0) = \frac{1}{\mu\ \pi} \int_{-\infty}^{\infty} \frac{\nu_2\, e^{-i\xi(x-x_0)}}{R(\xi)} \left[2\xi^2 e^{-\nu_1 y} - (2\xi^2 - k_2^2)\, e^{-\nu_2 y}\right] d\xi$$

$$\bar{G}^x_y(x, y \mid x_0, 0) = \frac{-i}{2\pi\mu} \int_{-\infty}^{\infty} \frac{\xi\, e^{-i\xi(x-x_0)}}{R(\xi)} \left[2\nu_1\nu_2 e^{-\nu_1 y} - (2\xi^2 - k_2^2) e^{-\nu_2 y}\right] d\xi$$

$$\bar{G}^y_x(x, y \mid x_0, 0) = \frac{-i}{2\pi\mu} \int_{-\infty}^{\infty} \frac{\xi\, e^{-i\xi(x-x_0)}}{R(\xi)} \left[(2\xi^2 - k_2^2)\, e^{-\nu_1 y} - 2\nu_1\nu_2\, e^{-\nu_2 y}\right] d\xi$$

$$\bar{G}^y_y(x, y \mid x_0, 0) = \frac{-1}{2\pi\mu} \int_{-\infty}^{\infty} \frac{\nu_1\, e^{-i\xi(x-x_0)}}{R(\xi)} \left[(2\xi^2 - k_2^2)\, e^{-\nu_1 y} - 2\xi^2\, e^{-\nu_2 y}\right] d\xi. \qquad \ldots(13)$$

Here $k_1^2 = \omega^2/\alpha^2$, $k_2^2 = \omega^2/\beta^2$, $\nu_1 = \sqrt{\xi^2 - k_1^2}$, $\nu_2 = \sqrt{\xi^2 - k_2^2}$ and $R(\xi) = 4\xi^2\nu_1\nu_2 - (\xi^2 + \nu_2^2)^2$; α, β are respectively P-wave and S-wave velocities. The values of $\nu_1(\xi)$ and $\nu_2(\xi)$ are to be so chosen that with such values the expression for the displacement decay exponentially as $y \to \infty$ for real $\nu_1(\xi)$ and $\nu_2(\xi)$.

The Fourier transform of the stress $P_{xy}[G^x(0, y \mid x_0, 0)]$ on the line of faulting HO is given by

$$\bar{P}_{xy}[G^x(0, y \mid x_0, 0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi x_0}}{R(\xi)} \left[(2\xi^2 - k_2^2)\, e^{-\nu_2 y} - 4\xi^2\nu_1\nu_2\, e^{-\nu_1 y}\right] d\xi. \qquad \ldots(14)$$

Since we want to determine the surface displacement due to Rayleigh waves, we need to determine the value of the integral in (14) for Rayleigh pole contribution only, for which we refer to Mal and Knopoff (1968). The Rayleigh pole contribution to the integral (14) is evaluated by following the method prescribed by Lapwood (1949) and $\bar{P}_{xy}[G^x(0, y \mid x_0, 0)]$ is found to be

$$
\left.
\begin{aligned}
& i\omega A_1 \left\{\exp\left(-\frac{\omega}{\alpha c_R}\sqrt{\alpha^2-c_R^2}\,y\right)-\exp\left(-\frac{\omega}{\beta c_R}\sqrt{\beta^2-c_R^2}\,y\right)\right\} \\
& \qquad \times \exp\left(i\frac{\omega x_0}{c_R}\right) \text{ for } \omega>0 \\
\text{and} \\
& i\omega A_1 \left\{\exp\left(\frac{\omega}{\alpha c_R}\sqrt{\alpha^2-c_R^2}\,y\right)-\exp\left(\frac{\omega}{\beta c_R}\sqrt{\beta^2-c_R^2}\,y\right)\right\} \\
& \qquad \times \exp\left(i\frac{\omega x_0}{c_R}\right), \text{ for } \omega<0
\end{aligned}
\right\} \quad \ldots(15)
$$

where

$$
A_1=\frac{1}{4c_R}
$$

$$
\times\frac{\sqrt{\left(\alpha^2-c_R^2\right)\left(\beta^2-c_R^2\right)}}{\frac{2\alpha}{\beta}\left(2\beta^2-c_R^2\right)-\beta'\left(\frac{\alpha^2-c_R^2}{\beta^2-c_R^2}\right)^{1/2}-\alpha^2\left(\frac{\beta^2-c_R^2}{\alpha^2\cdot c_R^2}\right)^{1/2}-2\sqrt{\left(\alpha^2-c_R^2\right)\left(\beta^2-c_R^2\right)}} \quad \ldots(16)
$$

and c_R is the Rayleigh wave velocity.

Similarly, the contribution from the Rayleigh pole to $\bar{P}_{xy}[G^y(0, y \mid x_0, 0)]$ is fonnd to be

$$
\left.
\begin{aligned}
& B_1\omega \left\{\exp\left(-\frac{\omega}{\alpha c_R}\sqrt{\alpha^2-c_R^2}\,y\right)-\exp\left(-\frac{\omega}{\beta c_R}\sqrt{\beta^2-c_R^2}\,y\right)\right\} \\
& \qquad \times \exp\left(i\frac{\omega x_0}{c_R}\right), \text{ for } \omega>0 \\
\text{and} \\
& -B_1\omega \left\{\exp\left(\frac{\omega}{\alpha c_R}\sqrt{\alpha^2-c_R^2}\,y\right)-\exp\left(\frac{\omega}{\beta c_R}\sqrt{\beta^2-c_R^2}\,y\right)\right\} \\
& \qquad \times \exp\left(i\frac{\omega x_0}{c_R}\right), \text{ for } \omega<0
\end{aligned}
\right\} \quad \ldots(17)
$$

where

$$
B_1=\frac{1}{2\beta c_R}
$$

$$
\times\frac{\left(2\beta^2-c_R^2\right)\sqrt{\alpha^2-c_R^2}}{\frac{2\alpha}{\beta}\left(2\beta^2-c_R^2\right)-\beta^2\left(\frac{\alpha^2-c_R^2}{\beta^2-c_R^2}\right)^{1/2}-\alpha^2\left(\frac{\beta^2-c_R^2}{\alpha^2-c_R^2}\right)^{1/2}-2\sqrt{\left(\alpha^2-c_R^2\right)\left(\beta^2-c_R^2\right)}}. \quad \ldots(18)
$$

The discontinuity in displacement along the line of faulting at any time t and at a depth y below the free surface is assumed to be

$$[v] = DH[h(t) - (l-y)]\,[H(y) - H(y-l)]$$
$$= DH[t - r(y)]\,[H(y) - H(y-l)] \qquad \ldots(19)$$

$H(\ )$ is Heaviside step function and $r(y) = h^{-1}(l-y)$, which is the inverse function of h and it exists as $h(t)$ is strictly monotonic increasing function. Fourier transform of eqn. (19) is given by

$$\overline{[v]} = D[H(y) - H(y-l)] \int_{-\infty}^{\infty} H[t - r(y)] e^{i\omega t}\, dt$$

$$= D[H(y) - H(y-l)] \int_{r(y)}^{\infty} e^{i\omega t}\, dt = D[H(y) - H(y-l)]$$
$$\times \left(\pi\delta(\omega) + \frac{i}{\omega}\right) e^{i\omega r(y)}. \qquad \ldots(20)$$

Putting the value of $\overline{[v]}$ from (20) in (11) and then taking Fourier inversion of (11) and changing the order of integration one obtains

$$u(x_0, 0) = \frac{D}{2\pi} \int_0^l dy \int_{-\infty}^{\infty} e^{i\omega(r(y)-t)} \left(\pi\delta(\omega) + \frac{i}{\omega}\right) \bar{P}_{xy}\, [\mathbf{G}^x(0, y \mid x_0, 0)]\, d\omega.$$

Substituting the value of $\bar{P}_{xy}$ from (15) in the above equation we get

$$u(x_0, 0) = -\frac{A_1 D}{\pi} \int_0^l dy \int_0^{\infty} \left\{ \left[\exp\left(-\frac{\omega}{\alpha c_R}\sqrt{\alpha^2 - c_R^2}\, y\right) - \exp\left(-\frac{\omega}{\beta c_R}\sqrt{\beta^2 - c_R^2}\, y\right)\right] \cos\left(r(y) - t + \frac{x_0}{c_R}\right)\omega \right\} d\omega$$

or

$$\frac{u(x_0, 0)}{D} = -\frac{A_1 C_R}{\pi} \int_0^l \left\{ \frac{\alpha\sqrt{\alpha^2 - c_R^2}\, y}{(\alpha^2 - c_R^2)\, y^2 + \alpha^2 (c_R r(y) - tc_R + x_0)^2} - \frac{\beta\sqrt{\beta^2 - c_R^2}\, y}{(\beta^2 - c_R^2)\, y^2 + \beta^2 (c_R r(y) - tc_R + x_0)^2} \right\} dy. \qquad \ldots(21)$$

Similarly, we obtain

$$\frac{v(x_0, 0)}{D} = -\frac{B_1 C_R}{\pi} \int_0^l \left\{ \frac{\alpha^2 (c_R r(y) - tc_R + x_0)}{(\alpha^2 - c_R^2) y^2 + \alpha^2 (c_R r(y) - tc_R + x_0)^2} \right.$$

(*equation continued on p.* 1001)

$$- \frac{\beta^2(c_R\, r(y) - tc_R + x_0)}{(\beta^2 - c_R^2)y^2 + \beta^2(c_R\, r(y) - tc_R + x_0)^2} \Big\} dy. \qquad ...(22)$$

3. Different Cases of Nonuniform Crack Speed

In this section we determine the Rayleigh wave displacement on the free surface for different nonuniform motions of the vertical crack.

Case 1—Here it is assumed that $h(t) = ct$, where c is the constant velocity of propagation of the fault and we have

$$t = \frac{l - y}{c} = r(y)\,. \qquad ...(23)$$

Substituting the value of $r(y)$ from (23) in (21) and (22) and integrating the resulting equation one gets

$$\frac{u(x_0, 0)}{D} = - \frac{AG}{4\pi p} \frac{c^2}{C_R^2} \left[\frac{A}{B} \left\{ \frac{1}{2} \ln \frac{A^2 + X^2}{T^2} + \frac{c_R}{c_A} \tan^{-1} \frac{A}{X} \right\} - \frac{G}{H} \left\{ \frac{1}{2} \ln \frac{G^2 + X^2}{T^2} + \frac{c_R}{c_G} \tan^{-1} \frac{G}{X} \right\} \right] \qquad ...(24)$$

$$\frac{v(x_0, 0)}{D} = - \frac{A(1 + G^2)}{2\pi P} \frac{c}{c_R} \left[\frac{1}{B} \left\{ \frac{c_A}{c_R} \tan^{-1} \frac{A}{X} - \frac{1}{2} \ln \frac{A^2 + X^2}{T^2} \right\} - \frac{1}{H} \left\{ \frac{c_G}{c_R} \tan^{-1} \frac{G}{X} - \frac{1}{2} \ln \frac{G^2 + X^2}{T^2} \right\} \right] \qquad ...(25)$$

where

$$A = \sqrt{1 - \frac{c_R^2}{\beta^2} \frac{\beta^2}{\alpha^2}},\ B = 1 + \frac{c^2}{c_R^2}\left(1 - \frac{c_R^2}{\beta^2} \cdot \frac{\beta^2}{\alpha^2}\right),\ G = \sqrt{1 - \frac{c_R^2}{\beta^2}}$$

$$H = 1 + \frac{c^2}{c_R^2}\left(1 - \frac{c_R^2}{\beta^2}\right),\ X = \frac{x_0}{l} - \frac{tc_R}{l},\ T = \frac{c_R}{c} - \frac{tc_R}{l} + \frac{x_0}{l}$$

and

$$P = 2(1 + G^2) - \frac{A}{G} - \frac{G}{A} - 2AG.$$

$T = 0$ implies $t = \dfrac{x_0}{c_R} + \dfrac{l}{c}$ which is the time taken to reach the point $(x_0, 0)$ by Rayleigh wave, which is emitted from the epicentre 0 when the crack reaches the free surface and $X = 0$ implies $t = x_0/c_R$ which is the time taken to reach the point $(x_0, 0)$ by the Rayleigh wave generated at H as soon as the crack appears at H.

Case 2—In this case it is assumed that the crack starts to move vertically upward with a finite velocity a and has a retardation b. Here at a time t after the formation of the crack

$$h(t)\ (\leqslant l) = at - \frac{1}{2} bt^2$$

so that

$$r(y) = \frac{2}{a}\,\frac{1(1-z)}{1+[1-(2bl/a^2)(1-z)]^{1/2}}. \qquad \ldots(26)$$

Substituting the value of $r(y)$ in (21) and (22) we obtain

$$\frac{u(x_0, 0)}{D} = -\frac{AG}{\pi P}\int_0^1 \left[\frac{Az}{Az^2 + \left\{\frac{c_R}{a}\,\frac{2(1-z)}{1+[1-2F(1-z)]^{1/2}} + X\right\}^2} - \frac{Gz}{G^2z^2 + \left\{\frac{c_R}{a}\,\frac{2(1-z)}{1+[1-2F(1-z)]^{1/2}} + X\right\}^2}\right] dz \qquad \ldots(27)$$

$$\frac{v(x_0, 0)}{D} = -\frac{A}{\pi P}\,\frac{c_R}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\left(1-\frac{c_R^2}{2\beta^2}\right)$$

$$\times \int_0^1 \left(2\,\frac{c_R}{a}\,\frac{1-z}{1+(1-2F(1-z))^{1/2}} + X\right) z^2\,dz \Big/$$

$$\left\{A^2z^2 + \left(2\,\frac{c_R}{a}\,\frac{1-z}{1+[1-2F(1-z)]^{1/2}} + X\right)^2\right\}$$

$$\times \left\{G^2z^2 + \left(2\,\frac{c_R}{a}\,\frac{1-z}{1+[1-2F(1-z)]^{1/2}} + X\right)^2\right\} \qquad \ldots(28)$$

where $F = bl/a^2$ and the other constants have the same values mentioned earlier. It may be noted that the integrands in equations (27) and (28) have a singularity at $z = 0$ provided

$$2\frac{c_R}{a}\,\frac{1}{1+\sqrt{1-2F}} + X = 0, \text{ which implies that}$$

$$t = \frac{x_0}{c_R} + \frac{2l}{a(1+\sqrt{1-(2bl/a^2)})}.$$

This is the time to reach the point $(x_0, 0/$ by the Rayleigh wave emitted from the epicentre 0 just after the arrival of the crack at this point.

Case 3— Finally let the crack at a depth 1 below the free surface, start to move vertically upward with infinitely large velocity which gradually decays with time. Accordingly $h(t)$ is taken in the form

$h(t) = D_1\sqrt{t}$ where D_1 is a constant.

or

$$D_1\sqrt{t} = 1 - y.$$

Therefore

$$r(y) = [(l-y)/D_1] \qquad \ldots(29)$$

As before substituting the value of $r(y)$ in (21) and (22) and making a change of variable of the integration, we have

$$\frac{u(x_0, 0)}{D} = -\frac{K^2AG}{4\pi P}\int_0^1\Bigg[\frac{A}{z^4-4z^3+(K^2A^2+2M+4)^2 z^2 - 4Mz+M^2} - \frac{G}{z^4-4z^3+(K^2G^2+2M+4)^2 z^2 - 4Mz+M^2}\Bigg] zdz \quad \ldots(30)$$

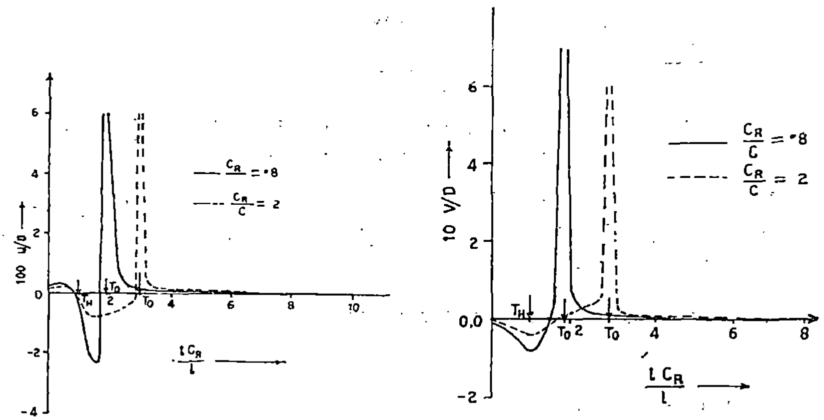
$$\frac{v(x_0, 0)}{D} = \frac{K^3Ac_R^2}{\pi P\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\left(1-\frac{c_R^2}{2\beta^2}\right) \times \int_0^1 \frac{z^2(z^2-2z+M)dz}{\{K^2A^2z^2+(z^2-2z+M)^2\}\{K^2G^2z^2+(z^2-2z+M)^2\}} \quad \ldots(31)$$

where $K^2 = D_1^2/lc_R$, $M = 1-(tc_R/l)(D_1^2/lc_R)+(x_0/l)(D_1^2/lc_R)$ and the other constants have same values as mentioned before. Again, the integrands in eqns. (30) and (31) are singular at $z=0$ if $M=0$. This corresponds to $t=(x_0/c_R)+(l^2/D_1^2)$ which is the time of arrival at $(x_0, 0)$ of the Rayleigh wave which is generated at 0 just after the arrival of the crack on the free surface.

4. Numerical Results and Conclusion

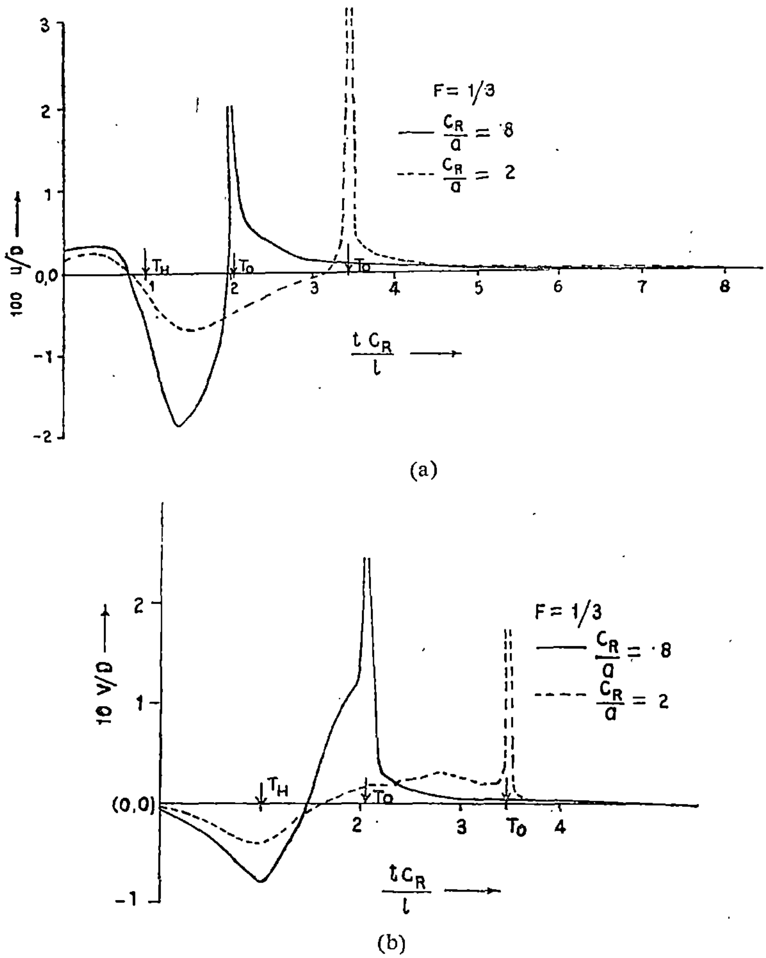
When the earth material is under tension or compression in a direction parallel to the free surface, shear failure occurs on a fault plane. In general, this failure moves with nonuniform speed. Numerical computations are carried out for poisson solid ($\alpha/\beta = \sqrt{3}$) and for $c_R/\beta = 0.9194$. The quantities $A, B, G, H, X, T, P, K, M, F$ defined in section 3 are all dimensionless.

Figures 2, 3 and 4 show the variation of components of displacement with



(a) (b)

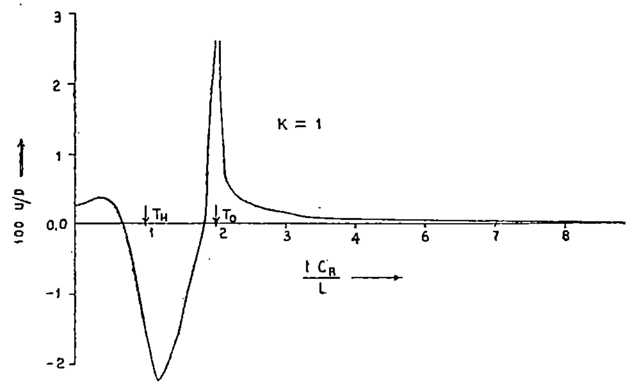
Figs. 2(a, b). Horizontal and vertical components of displacement versus time. Case of crack propagation with uniform velocity.



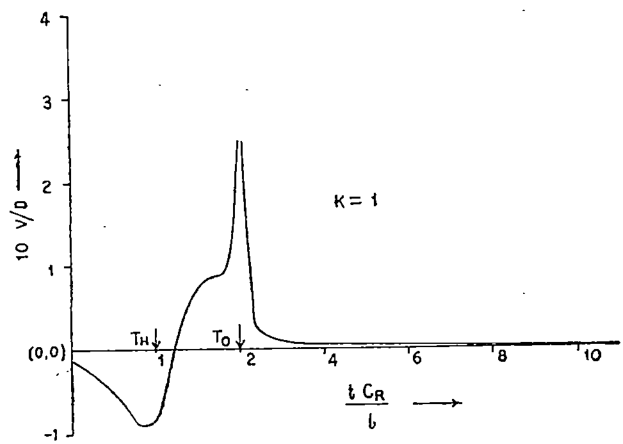
FIGS. 3(a, b). Horizontal and Vertical components of displacement versus time. Case of crack propagation with retardation and finite starting velocity.

time. The dimensionless displacement components are plotted against dimensionless time T_H and T_0 indicated in the figures correspond to the arrival times at $(x_0, 0)$ of the Rayleigh waves from the focus H and the fault break at 0. Figures 2(a, b) correspond to the case 1 of section 3 where the constant velocity of propagation of the crack ($c_R/c = 0.8$ and 2) is assumed. Figures 3(a, b) correspond to the case where the crack starts with a finite velocity a and has a retardation b. Here also two cases $c_R/a = 0.8$ and 2 with the assumption that $F = 1/3$, are considered. Figures 4(a, b) depicts the case 3 where the initial velocity of crack propagation is assumed to be infinitely large and K is taken to be equal to 1.

From eqns. (21) and (22) it may be noted that $u(x_0, 0)/D$ and $v(x_0, 0)/D$ are



(a)



(b)

FIGS. 4(a, b). Horizontal and Vertical components of displacement versus time. Case of crack propagation with retardation and infinite starting velocity.

functions $(x_0/l) - (c_R t/l)$. Therefore in all computational works, without any loss of generality x_0/l has been taken to be equal to 1, because any change in value of x_0/l will merely cause a shifting of the graphs along the direction of $c_R t/l$.

We find that in each case the strongest ground motion occurs at $T^0 = c_R t/l$ which correspond to the arrival time of Rayleigh waves from the surface break at 0. Also it is found that though the nature of the graphs in three different cases differ between T_H and T_0 but their natures are almost the same after the arrival of Rayleigh waves from the surface break. This may be explained from the fact that the main contribution to the ground motion due to Rayleigh wave is from a small portion of the fault near the surface after the arrival of Rayleigh wave from the surface break. So the contribution from the details of crack initiation becomes insignificant after T_0. It may be mentioned in this connection that though contribution of Rayleigh wave to the ground motion is significant at large distances from the epicentre, the effect of body waves near the epicentral region cannot be ignored. This effect may be incorporated if we consider in addition to the contribution from Rayleigh pole, the contribution from the branch line integrals arising from the evaluation of stresses due to Green's function.

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