CHAPTER II

INDIFFERENCE DIGRAPHS – A GENERALIZATION OF INDIFFERENCE GRAPHS AND SEMIORDER *

2.1. Introduction

Indifference graphs were introduced and studied by Roberts [1969a]. An undirected graph is an *indifference graph* if there exists a real-valued function f on the vertices such that vertices u, v are adjacent if and only if $|f(u) - f(v)| \leq 1$. We may call f an *indifference representation* of G. Roberts characterized indifference graphs and proved that they are equivalent to *proper interval graphs* (intersection graphs of intervals in which no interval properly contains another) and to *unit interval graphs* (intersection graphs of unit-length intervals).

With this model in mind, we introduce an analogue of indifference graphs for directed graphs (henceforth "digraphs"). A digraph with edge set E is an *indifference digraph* if there exists an ordered pair of real-valued function f,g on the vertices satisfying $uv \in E$ if and only if $|f(u) - g(v)| \leq 1$.Here $f(v)$ and $g(v)$ are called the *source value* and *sink value* of the vertex v, and the pair f, g is called an *indifference representation*. Interchanging the source value and sink value of

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each vertex shows that the converse of an indifference digraph (obtained by changing the direction of each edge) is also an indifference digraph. A complete digraph (that is, a digraph in which every ordered pair of vertices forms an edge including the loops) is also an indifference digraph, as can be seen by assigning arbitrary source and sink values all within the interval [0,1].

Beineke and Zamfirescu [1982] and Sen *et al.* [1989a] introduced and studied (in different contexts) a natural analogue of intersection graph models for directed graphs. Let V be a family of ordered pairs of sets and assign a vertex v to each ordered pair. The first set assigned to v is called its *source set* S_v, and the second set is its *sink set* T_v. The *intersection digraph* of the family of pairs is the digraph $D = (V,E)$ such that $uv \in E$ if and only if $S_u \cap T_v \neq \phi$. An *interval digraph* is the intersection digraph of a family of ordered pairs of intervals on the real line; these were characterized by Şen *et al* [1989a,b]

By placing constraints on the source and sink intervals, we introduce two types of interval digraphs. *Unit interval digraphs* are interval digraphs with interval representations such that all the source intervals and sink intervals have unit length. *Proper interval digraphs* are interval digraphs with representations such that no source interval properly contains another source interval, and no sink interval properly contains another sink interval. Note that there is no restriction on the inclusion relationship between a source interval S_u and a sink interval T_v.

Given an indifference representation f,g, setting
$$S_u = [f(u)-\frac{1}{2}, f(u)+\frac{1}{2}] \text{ and } T_u = [g(u)-\frac{1}{2}, g(u)+\frac{1}{2}]$$
for all u provides an interval representation. Hence every indifference digraph is an interval digraph; indeed, it is a unit interval digraph, since these intervals have length 1. Conversely, setting $f(u)$, $g(u)$ to the midpoints of S_u, T_u in a unit interval representation yields an indifference representation, so indifference digraphs are precisely the unit interval digraphs. Section 2.2 contains an explicit interval digraph that is not an indifference digraph.

Making use of the characterizations of interval digraphs, we will give characterizations of the more restricted classes to prove that the classes of indifference digraphs, unit interval digraphs, and proper interval digraphs are all the same. The most important characterization, from which others are obtained, is a characterization of the adjacency matrices of these digraphs. We say that a 0,1-matrix has a *monotone consecutive arrangement* if there exist independent row and column permutations exhibiting the following structure: the 0's of the resulting matrix can be labeled R or C such that every position above *and* to the right of an R is an R, and every position below *and* to the left of a C is a C. The name arises from an equivalent restatement of the condi[illegible]tion described in Section 2.2.

Roberts [1969a] proved equivalence of the analogous classes of undirected graphs. Our results reduce to the earlier results on undirected graphs when we view undirected graphs as symmetric digraphs with loops (i.e., symmetric and reflexive binary relations). The adjacency matrix of the corresponding digraph is obtained by adding 1's on the diagonal; this is called the *augmented adjacency matrix* $A^*(G)$ for an undirected graph G. A symmetric digraph with loops has an indifference representation if and only if it has an indifference representation with $f = g$, because the symmetry implies that averaging f and g will not change the resulting edges. Conversely, every indifference representation with $f = g$ yields a symmetric digraph with loops. This establishes a bijection between indifference graphs and indifference digraphs representable using $f = g$.

Hence our characterization implies that G is an indifference graph if and only if $A^*(G)$ has a monotone conse[illegible]cutive arrangment. This reduces to Roberts characterization [1968] that G is an indifference graph if only if $A^*(G)$ has a column permutation so the 1's appear consecutively in each row (called the *consecutive ones property for rows*). A monotone consecutive arrangment exhibites the consecutive ones property for both rows and columns. Conversely, the symmetry of A^* allows us to apply any column permutation also to the rows to achieve the consecutive ones property for each simultaneously while leaving 1's on the diagonal; this is a monotone consecutive arrangement.

Roberts introduced indifference graphs as a graph-theoretic concept related to semiorders, which are a special type of binary relation. In discussing a binary relation P, we use the notations xPy, $xy \in P$, and $x \longrightarrow y$ interchangeably, corresponding to several equivalent notions: 1) P as a binary relation, 2) P as the set of ordered pairs of a relation, and 3) P as the edges of a digraph. We use whichever notation is convenient. Luce [1956] and Scott and Suppes [1958] defined a *semiorder* to be an irreflexive binary relation (loopless digraph) satisfying 1) $ab, cd, \in P$ implies aPd or cPd, and 2) $ab, bc, \in P$ implies aPd or dPc, where in each case a, b, c, d are arbitrary (not necessarily distinct) elements (or vertices). The Scott-Suppes Theorem [1958] characterizes semiorders as those binary relations P for which there exists a real-valued function f such that xPy if and only if $f(x) > f(y)+\delta$ for some constant δ (which can be taken to be 1). This condition expresses P as a transitive orientation of the complement of an indifference graph; hence we call f a *coindifference representation*. When viewing an indifference graph as a symmetric digraph with loops, this result was rephrased by Roberts [1969a] by saying that a graph with edges E is an indifference graph if and only if there is a semiorder P such that $\bar{E} = P \cup P^{-1}$, where P^{-1} is the digraph obtained from P by reversing the directions on all the edges.

We introduce a generalization of semiorders that behaves in the analogous way for indifference digraphs. We obtain the Scott-Suppes Theorem and Roberts's rephrasing of it as special cases of the resulting theorem. We define a *generalized semiorder* to be a pair of disjoint binary relations (edge-disjoint digraphs) H_1, H_2 on the same set such that 1) aH_ib and cH_id imply aH_id or cH_ib ($i \in \{1, 2\}$), 2) aH_1b and cH_2b imply aH_1d or cH_2d, and 3) aH_1b and aH_2c imply dH_1b or dH_2c, where as above a, b, c, d are arbitrary (not necessarily distinct) elements.

Observe that if $H_2 = H_1^{-1}$, then conditions (2) and (3) coincide and disjointness forbids loops, so that H_1, H_2 are semiorders. We prove that a digraph with edge set E is an indifference digraph if and only if there is a generalized semiorder H_1, H_2 on the vertex set such that $\bar{E} = H_1 \cup H_2$. In particular, we prove the following generalization of the Scott-Suppes Theorem: (H_1, H_2) is a generalized semiorder on the elements A if and only if there are two functions $f, g: A \longrightarrow \mathbb{R}$ such that $x\ H_1 y \Leftrightarrow f(x) > g(y) + 1$ and $x\ H_2 y \Leftrightarrow g(y) > f(x) + 1$. The two functions f,g are called a *coindifference representation*.

To obtain the Scott-Suppes Theorem as a corollary, suppose that H is a binary relation and let $H_1 = H$, $H_2 = H^{-1}$. If H has a coindifference representation f (with threshold 1), then xH_1y if and only if $f(x) > f(y) + 1$, and similarly xH_2y if and only if $f(y) > f(x) + 1$. Hence (H_1, H_2) is a generalized semiorder, which implies that H is a semiorder as noted above. Conversely, suppose that H is a semiorder. This implies that (H_1, H_2) is a generalized semiorder, so we obtain a coindifference representation f,g satisfying $xH_1y \Leftrightarrow f(x) > g(y) + 1$ and $yH_2x \Leftrightarrow g(x) > f(y)+1$. Let $h(x) = [f(x) + g(x)]/2$. If xHy, then $h(x) > h(y) + 1$. If $h(x) > h(y) + 1$, then, xH_1y or yH_2x, i.e. xHy, and h is a coindifference representation of H.

We note that other generalizations of semiorders have been introduced, including the *bisemiorder* of Ducamp and Falmagne [1969] and the *double semiorder* of Cozzens and Roberts [1982] (see Sec 1.5.2). In both cases some analogoues of the results on undirected graphs were obtained.

2.2. Properties of adjacency matrices

We observed earlier that every indifference digraph is an interval digraph. We will need the characterizations of interval digraphs. First, a digraph is a *Ferrers digraph* or *Ferrers relation* if it satisfies condition (1) in the definition of semiorder: $ab, cd \in P$ implies aPd or cPb. This definition, introduced by Riguet [1952], is equivalent to forbidding the adjacency matrix to have a 2 by 2 submatrix that is a permutation matrix, or to requiring the successor sets (or predecessor sets) to be ordered by inclusion. The name arises from another characterization: the rows and columns of the adjacency matrix of a Ferrers digraph can be independently permuted so that the positions of the 1's form a Ferrers diagram.

Theorem 2.1. (Sen *et al* [1989a,b]. *For a digraph D, the following are equivalent:*

1) D is an interval digraph.

2) $\bar{D}$ is the union of two disjoint Ferrers digraphs.

3) The rows and columns of the adjacency matrix of D can be permuted independently such that each 0 can be labeled with R or C in such a way that every position to the right of an R is an R and every position below a C is a C. ∎

A matrix satisfying condition (3) above has the *partitionable zeros property*; note that it is a weaker form of the monotone consecutive arrangement condition described earlier. This characterization enables us to show that the indifference digraphs are properly contained in the interval digraphs.

Example 2.1 *An interval digraph that is not an indifference digraph.* The adjacency matrix below satisfies the characterizations above, so the corresponding digraph is an interval digraph.

1	1	0	0
1	1	1	1
0	1	1	0
0	0	0	0

Let v_1, v_2, v_3, v_4 denote the vertices in order, and suppose there is an indifference representation with $f(v_i) = f_i$ and $g(v_i) = g_i$. All sink values must lie in the interval $[f_2 - 1, f_2 + 1]$. Since edges are determined by absolute value of $f_i - g_j$, we may assume $g_4 \geq f_2$. Now the 0's in rows 1 and 3 force $f_1, f_3 < f_2$ and then g_3, $g_1 > f_2$. However, a 2 by 2 permutation submatrix has no indifference representation in which both source values are less than both sink values; the larger source value must be more than 1 away from one of the sink values, which then cannot be within distance 1 of the smaller source value. ∎

A more straightforward characterization of interval digraphs (Sen *et al* [1989a]) can be restricted in a natural way to characterize proper interval digraphs. A *generalized complete bipartite subdigraph* (GBS) of a digraph is a subdigraph consisting of vertex sets X,Y and edges from all of X to all of Y. The word ``generalized'' indicates the fact that X,Y need not be disjoint; any submatrix of 1's in the adjacency matrix yields a GBS. If $\mathbf{B} = \{(X_k, Y_k)\}$ is a collection of GBS's we can form the incidence matrix between the vertices V and the source sets $\{X_k\}$, called the *V,X-matrix*, and similary the *V,Y martix* between the vertices and sink sets. Since the rows in the two matrices can be viewed as source sets and sink sets, with the GBS's corresponding to elements that can be placed in order along a line, we obtain:

Theorem 2.2 (Sen *et al* [1989a]). *A digraph is an interval digraph if and only if its edges can be covered by a collection of GBS's* $\mathbb{B}$ *that can be indexed so that the 1's in each row of the V,X-matrix and the V,Y-matrix appear consecutively.*

To obtain proper sets of intervals, we want the resulting matrices to have the *proper consecutive ones property* (for rows) introduced by Fishburn [1985]. This is defined to be the existence of a column ordering so that 1's in each row appear consecutively and do not properly contain the 1's in any other row. Such an ordering ``exhibits'' the property. By transforming a proper interval representation into GBS's corresponding to the endpoints of all intervals, and conversely by transforming membership in an appropriate sequence of GBS's into intervals, we obtain

Theorem 2.3 *A digraph is a proper interval digraph if and only if its edges can be covered by a collection of GBS's* $\mathbb{B}$ *that can be indexed so that the 1's in the rows of the V,X-matrix and in the rows of the V,Y-matrix exhibit the proper consecutive ones property.* ■

Note that, given a matrix with the proper consecutive ones property for rows, we can permute the rows to exhibit a monotone consecutive arrangement. The converse does not hold, as a monotone consecutive arrangement allows proper inclusion of 1's in rows. We will see in Theorem 2.5 that the condition of Theorem 2.3 for proper interval digraphs is equivalent to requiring the weaker monotone consecutive condition for the incidence matrices of some family of *maximal* GBS's that cover D. We will need a rephrasing of the monotone consecutive condition that focuses explicity on how the 1's in rows must behave. The following characterization is the source of the name "monotone consecutive arrangement". Note that zero rows or columns can be placed at the bottom or right without affecting the existence of a monotone consecutive arrangement.

Lemma 2.1 *A 0,1-matrix with* n *non-zero rows has a monotone consecutive arrangement if and only if it has independent row and column permutations such that the 1's appear consecutively in* *each row and the value $\{a_i\}$ and $\{b_i\}$ denoting the initial column and final column of the interval of 1's in row i satisfy* $a_1 \le \ldots \le a_n$ *and* $b_1 \le \ldots \le b_n$.

Proof. The labeling of the 0's in a monotone consecutive arrangement guarantees $a_{i-1} \le a_i$ and $b_{i+1} \ge b_i$. Conversely, if M is permuted so sequences a and b are monotone, let a 0 in position (i,j) have label C if $j < a_i$, or label R if $J > b_i$. By construction, the labeling condition is satisfied in rows, and monotonicity of the sequences guarantees that it is satisfied in columns. ∎

Finally, it is worthwhile to note that although a monotone consecutive arrangement implies the consecutive ones property for both rows and columns, the converse does not hold.

Example 2.1 *(continued).* The adjacency matrix has the consecutive ones property for both rows and columns, but it does not yield an indifference digraph. By the characterization we claim, it thus does not have a monotone consecutive arrangement. For clarity and motivation, we give a short direct proof of that. In a monotone consecutive arrangement, the 0's of a 2 by 2 permutation submatrix cannot be both C or both R. By symmetry of v_1 and v_3 in this digraph, we may assume that the 0 in position (1,3) is labeled R and the 0 in position (3,1) is labeled C.

Since row 2 has all 1's, it thus must be under row 1 and over row 3 in any monotone consecutive arrangement. This implies that the 0's of column 4 cannot be both *C* or both *R*. With one of each in these rows, column 4 must occur after column 2 and before column 2 in any monotone consecutive arrangement, which is impossible. ∎

2.3 Equivalence of digraph classes

In this section, we prove characterizations of the digraph classes we have been considering. First, we characterize the adjacency matrices and show that the classes are equivalent.

Theorem 2.4 *If D is a digraph, the following conditions are equivalent.*

A) D is an indifference digraph.

B) D is a unit interval digraph.

C) D is a proper interval digraph.

D) The adjacency matrix of D has a monotone consecutive arrangement.

Proof. We noted A ⇔ B ⇒ C in the introduction. For C ⇒ D, suppose that *D* is a proper interval digraph. By Theorem 2.3,we have a collection **B** = $\{(X_k, Y_k)\}$ of GBS's that cover *D*, such that for the *V*,*X*-and *V*,*Y*-matrices, the 1's in each row appear consecutively and do not properly contain the 1's of any other row. We may assume that each row has a 1, else the corresponding row or column of the adjacency matrix can be placed at the end, which will not affect the existence of a monotone arrangement.

Let $a(v)$, $b(v)$ denote the first and last columns contaning 1 in the row of the V,X-matrix corresponding to v; similarly define $c(v)$, $d(v)$ from the V,Y-matrix. Index the vertices $u_1,\ldots,u_n$ so that $a(u_1) \leq \ldots \leq a(u_n)$; the proper consecutive ones property implies also that $b(u_1) \leq \ldots \leq b(u_n)$. Similarly, index them as $v_1,\ldots,v_n$ so that $c(v_1) \leq \ldots \leq c(v_n)$ and $d(v_1) \leq \ldots \leq d(v_n)$. Let $a_i = a(u_i)$, $b_i = b(u_i)$, $c_j = c(v_j)$, $d_j = d(v_j)$. We may assume that all source sets and sink sets are nonempty, which means that for any $(X_k, Y_k) \in \mathbf{B}$ there exist i,j such that $k \in [a_i, b_i]$ and $k \in [c_j, d_j]$.

We claim that the row ordering $u_1,\ldots,u_n$ and column ordering $v_1,\ldots,v_n$ of the adjacency matrix exhibit a monotone consecutive arrangement. The edges with tail u_i are covered by the GBS's with source sets $X_{a_i},\ldots,X_{b_i}$; hence the successors (out-neighbors) of u_i are $Y_{a_i} \cup \ldots \cup Y_{b_i}$.
Let $\alpha_i = \min \{j : a_i \in [c_j, d_j]\}$ and $\beta_i = \max \{j : b_i \in [c_j, d_j]\}$. The proper consecu‡tive ones property of the V,Y-matrix (and lack of 0 rows except possibly at the end) implies that the ones in row i of the adjacency matrix are $\{j : \alpha_i \leq j \leq \beta_i\}$. It also implies $\alpha_i \leq \alpha_{i+1}$ because $a_i \leq a_{i+1}$ and $\beta_i \leq \beta_{i+1}$ because $b_i \leq b_{i+1}$. By Lemma 2.1, we have a monotone consecutive arrangement.

To prove D $\rightarrow$ A, we take a monotone consecutive arrangement of an m by n 0,1-matrix M with rows $u_1,\ldots,u_m$ and columns $v_1,\ldots,v_n$ and construct an indifference representation, writing $f_i = f(u_i)$ and $g_j = g(v_j)$. To avoid technical degeneracies, we use induction on $m + n$. In particular, if the matrix has a $\frac{A|O}{O|B}$ decomposition, then we can take representations of the two smaller matrices by induction and shift one by a large constant to obtain a representation of M (similarly for zero rows or columns). Hence we may assume that every row and column has a 1 (including position (1,1)) and that each consecutive pair of rows have 1's in some common column.

From the monotone consecutive arrangement, we have nondecreasing sequences $\langle a_i\rangle$ and $\langle b_i\rangle$ such that the sucessor set of u_i is $Q_i = \{V_{a_i},\ldots,V_{b_i}\}$; the existence of a common 1 is equivalent to $a_i \le b_{i-1}$. We distinguish the *maximal* successor sets, those not contained in anv other. We present an iterative algorithm such that, at the end of stage i, we have specified $g_1 < \ldots < g_{b_i}$ and $f_1 \le \ldots \le f_i$ to form an indifference representation of the first i rows and g_{b_i} columns. Furthermore, the values are chosen so that $g_{b_i} - 1 = f_i = g_{a_i} + 1$ if Q_i is maximal and $g_{b_i} - 1 < f_i < g_{a_i} + 1$ if Q_i is not maximal. As an initialization, we can specify any desired value; we choose $g_1 = 1$, and by convention we take $b_0 = 1$ and $g_0 = -1$ to treat statge 1 inductively.

To begin stage i, we set the sink values through g_{b_i}. If $b_i = b_{i-1}$, then these are already known. If $b_i > b_{i-1}$, then we need to set the sink values above b_{i-1}. Since $a_i \leq b_{i-1}$, the value g_{a_i} has been set. Choose $g_{b_{i-1}} + 1, \ldots\ldots, g_{b_i}$ as an arbitrary increasing sequence in the open interval $(f_{i-1} + 1,\ g_{a_i} + 2)$, expect set $g_{b_i} = g_{a_i} + 2$ if Q_i is maximal. This interval is nonempty because $f_{i-1} \leq a_{i-1} + 1 \leq a_i + 1$, and the first inequality is strict if Q_{i-1} is not maximal, while the second is strict if Q_{i-1} is maximal and $b_i > b_{i-1}$. The resulting inequality $g_{b_{i-1}} + 1 > f_{i-1} + 1$ guarantees that the representation is correct on the new columns and old rows. Note also that $f_{i-1} + 1 \geq g_{b_{i-1}}$, so the sink values remain increasing.

To complete stage i, we choose f_i to add row i to the representation and preserve the stated restrictions. We must have $g_{b_i} - 1 \leq f_i \leq g_{a_i} + 1$ (with strict inequalities if Q_i is not maximal and equalities if Q_i is maximal), and $f_i > g_{a_{i-1}} + 1$. Let $f_i = (\lambda + \mu)/2$, where $\lambda = \max\ \{g_{a_{i-1}} + 1,\ g_{b_i} - 1\}$ and $\mu = g_{a_i} + 1$. By the construction of g_{b_i}, we have $\lambda \leq \mu$, with equality if and only if Q_i is maximal. This guarantees that f_i satisfies all the restrictions, and the monotonicity of $\{a_i\}$ and $\{b_i\}$ guarantees $f_i \geq f_{i-1}$. ∎

Example 2.2 To illustrate the above construction procedure consider the rearranged adjacency matrix below :

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
u_1	1	0	0	0	0	0	0	0	0	0	0	0
u_2	1	1	1	1	0	0	0	0	0	0	0	0
u_3	0	0	1	1	1	1	0	0	0	0	0	0
u_4	0	0	0	1	1	1	0	0	0	0	0	0
u_5	0	0	0	0	1	1	1	0	0	0	0	0
u_6	0	0	0	0	1	1	1	1	1	0	0	0
u_7	0	0	0	0	1	1	1	1	1	1	0	0
u_8	0	0	0	0	0	0	1	1	1	1	0	0
u_9	0	0	0	0	0	0	0	1	1	1	0	0
u_{10}	0	0	0	0	0	0	0	0	1	1	1	0
u_{11}	0	0	0	0	0	0	0	0	0	1	1	1
u_{12}	0	0	0	0	0	0	0	0	0	0	1	1

Maximal successor sets are S_2, S_3, S_7, S_{10}, and S_{11}, where

$S_2 = \{ v_1, \ldots, v_4 \}$ $S_3 = \{ v_3, \ldots, v_6 \}$ $S_7 = \{ v_5, \ldots, v_{10} \}$

$S_{10} = \{ v_9, \ldots, v_{11} \}$ $S_{11} = \{ v_{10}, \ldots, v_{12} \}$

Step 1. Consider the first maximal set S_2. Put arbitrarily $g_1 = 1$. Then $g_4 = g_1 + 2 = 3$

Now to find f_1 we write $\lambda = g_1 - 1 = 0$

and $\mu = g_1 + 1 = 2$

So $f_1 = \frac{\lambda + \mu}{2} = 1$

Step 2. We have to choose $g_2 > f_1 + 1 = 2$

Choose arbitrarily $g_2 = 2.6$ and $g_3 = 2.8$

Since S_2 is maximal $f_2 = \frac{g_1 + g_4}{2} = 2$

Step 3. Since S_3 is maximal, $g_6 = g_3 + 2 = 4.8$

Choose g_5 arbitrarily satisfying $g_5 > f_2 + 1$

and $g_4 < g_5 < g_6$. Let $g_5 = 4$

$$\text{Then } f_3 = \frac{g_3 + g_6}{2}$$
$$= \frac{2.8 + 4.8}{2} = 3.8$$

Step 4. Here the sink values of all the successors sets are already known.

Then $\lambda = \max (g_3 + 1,\ g_6 - 1) = \max(3.8, 3.8) = 3.8$,

$\mu = g_4 + 1 = 4$

So $f_4 = \frac{\lambda + \mu}{2} = 3.9$

Step 5. Here $g_7 > f_4 + 1 = 4.9$ Let $g_7 = 5$

Then $\lambda = \max (g_4 + 1,\ g_7 - 1) = 4$

$\mu = f_5 = \frac{\lambda + \mu}{2} = 4.5$

Step 6. Here $g_8 > f_5 + 1 = 5.5$ Let $g_8 = 5.6$

Then $g_9 = 5.8$, $g_{10} = g_5 + 2 = 6$ since s_7 is maximal

Then $\lambda = \max (g_4 + 1,\ g_9 - 1) = \max (4, 4.8)$

$\mu = g_5 + 1 = 5$

and $f_6 = \frac{\lambda + \mu}{2} = 4.9$

Step 7. Since s_7 is maximal $f_7 = \frac{g_5 + g_{10}}{2} = 5$

Step 8. All the sink values are known

$\lambda = \max(g_6 + 1, g_{10} - 1) = \max(5.8, 5) = 5.8$

$\mu = g_7 + 1 = 6$

So $f_8 = 5.9$

Step 9. $\lambda = \max(g_7 + 1, g_{10} - 1) = \max(6,5) = 6$

$\mu = g_8 + 1 = 6.6$

$f_9 = \frac{\lambda + \mu}{2} = 6.3$

Step 10. To find g_{11} we note that s_{10} is maximal

Then $g_{11} = g_9 + 2 = 7.8$

$f_{10} = 6.8$

Step 11. s_{11} is maximal. So $g_{12} = g_{10} + 2 = 8$

$f_{11} = 7$

Step 12. Here $\lambda = \max(g_{10} + 1, g_{12} - 1) = \max(7,7) = 7$

$\mu = g_{11} + 1 = 8.8$

So $f_{12} = \frac{\lambda + \mu}{2} = 7.9$.

Arranging in increasing order, the values of $f(v_i)$ and $g(v_j)$ can be tabulated as below.

i	1	2	3	4	5	6	7	8	9	10	11	12
$f(u_i)$	1	2	3.8	3.9	4.5	4.9	5	5.9	6.3	6.8	7	7.9
$g(v_i)$	1	2.6	2.8	3	4	4.8	5	5.6	5.8	6	7.8	8

Using the adjacency matrix characterization, we can now give an alternate characterization of proper interval digraphs as a variation on Theorem 2.3. We can restrict our attention to maximal GBS's if we allow the flexibility of a monotone consecutive arrangement of the incidence matrices instead of requiring the proper consecutive ones property.

Theorem 2.5. *A digraph D is a proper interval digraph (or indifference digraph) if and only of it has a covering by a family of maximal GBS's* $B = \{(X_k, Y_k)\}$ *numbered so that the V,X-matrix and V,Y-matrix exhibit monotone consecutive arrangements.*

Proof. For necessity, we may assume we have a monotone consecutive arrangement of the adjacency matrix, with $[a_i, b_i]$ being the interval of columns with 1's in row i. Again we implicitly use induction on $m + n$ to avoid degeneracies and assume that there is a 1 in every row and column and a common column in any two consecutive rows. There is a natural set of maximal GBS's associated with such a configuration. As suggested by viewing the matrix in the example, these are the maximal rectangular blocks of ones determined by corners of the region of 1's in the matrix. The upper corners are the positions of the form (i, b_i) with $b_i > b_{i-1}$; the lower corners have the form (j, a_j) with $a_j < a_{j+1}$. If $i \le j$ and $a_j \le b_i$, then these form a maximal GBS $B(i,j)$.

To select an appropriate family $\{B_k\}$, begin with $B_1 = B(1,r)$, where $r = \max\ \{i : a_i = 1\}$. Having determined $B_{k-1} = B(c_{k-1}, d_{k-1})$, let α be the next row below c_{k-1} having an upper corner (if any), and let β be the next row below d_{k-1} having a lower corner (if any). Since we have avoided degeneracy, we must have $\alpha \le d_{k-1}$ or $a_\beta \le b_{c_{k-1}}$ (unless B_{k-1} completes the covering), which implies that $B(\alpha, d_{k-1})$ or $B(c_{k-1}, \beta)$ can be chosen as $B_k = B(c_k, d_k)$. Since we shift by one corner at a time, the resulting sequence covers all the 1's.

In this sequence, the vertices X_k are consecutive set $u_{c_k}, \ldots, u_{d_k}$. Since each step we take increases c_k or d_k while leaving the other fixed, $\{c_k\}$ and $\{d_k\}$ are monotone, and the transpose of the V,X-matrix has a monotone consecutive arrangement. Since the original definition is invariant under transpose, the V,X-matrix also has a monotone consecutive arrangement. Applying the same argument to the transpose of the original adjacency matrix shows that the V,Y-matrix of **B** also has a monotone consecutive arrangement.

For sufficiency, let **B** be a collection of GBS's covering D that is indexed so the V,X-matrix and V,Y-matrix exhibit monotone consecutive arrangements. By Thoerem 2.3, it suffices to show that if there is any proper inclusion between the 1's of a consecutive pair of rows in the V,X -matrix or V,Y-matrix, then we can add one GBS to the sequence to reduce the number of proper inclusions.

By symmetry, we may assume there is such an inclusion in the V,X -matrix, with $a_i < a_{i+1}$ but $b_i = b_{i+1} = k$. Define a new GBS (X', Y') by $X' = X_k - \{u_j : j \leq i\}$ and $Y' = Y_k$, and insert this into the sequence immediately following (X_k, Y_k). Now $b_i < b_{i+1}$, and the other such inequalities remain unchanged. ∎

2.4 W-Consecutive Arrangements

Tucker [1972] gave a complete characterization of consecutive 1's property of a (0,1) matrix in terms of some forbidden configurations of the matrix. [illegible] In his paper Tucker associated an $m \times n$ (0,1) matrix M with an unoriented bipartite graph $G = (V_1, V_2, A)$ where V_1 and V_2 are two disjoint sets of vertices and A is the set of undirected edges joining a vertex of V_1 with a vertex of V_2 such that for $x_i \in V_1$ and $y_j \in V_2$, $x_i A y_j \Leftrightarrow$ entry (i,j) is 1. He has shown that the consecutive 1's property in the columns of M has an equivalent formulation in the bipartite graph G viz. that the vertices of V_1 can be ordered so that for each x in V_2, $N(x) = \{y, xAy\}$ is a consecutive set of vertices in V_1. Tucker called such an ordering a V_1-consecutive arrangement of G. In order to characterize a matrix having an MCA we extend the idea to define a *W-consecutive arrangement* of its associated bipartite graph andshow that they are equivalent.

Below we associate with a given digraph D, an undirected bipartite graph G, analogous to that in Tucker [1972]. Which we shall call the *bipartite graph associated with D*, or simply the *associated graph of D.*

Let $G = (V_1, V_2, A)$ be a connected bipartite graph defined on a vertex set $W = V_1 \cup V_2$. The graph G will be said to have a *W- consecutive arrangement* if there exists an order R in W such that for any three elements $x, y, z \in W$, if x preceeds y and y preceeds z (i.e., $x \leq_R y \leq_R z$), then y is adjacent to any path connecting x and z.

We will now prove the following theorem.

Theorem 2.6. *A digraph D is an indifference digraph if and only if the corresponding associated bipartite graph G has a W-consecutive arrangement.*

Proof. (*Necessary*). Let D be an indifference digraph. Then by theorem 2.4 , its adjacency matrix has a monotone consecutive one's arrangement. Let us denote the rearranged adjacency matrix by $A'(D)$.

Consider the lower stair in the monotone consecutive arrangement so that the positions just above or just to the right of this stair are all 1's. These positions define a natural ordering of W. We call this ordering R and shall prove that the order R is the required one for the W-consecutive arrangement. It will be observed that this order is not unique and that there are other orders serving our purpose.

Let x, y, z be any three vertices belonging to $W = U \cup V$ such that $x \underset{R}{<} y \underset{R}{<} z$. When x, y, z all belong exclusively to either U or V the proof is same as that of Tucker because the matrix has both U and V-consecutive arrangement in R. Now suppose that the vertices x, y, z are distributed among both U and V. As a particular case let $y, z \in V$ and $x \in V$. (other cases may be similarly shown). Let $P = (r_0, q_1, r_1 \ldots . q_n)$ be a path from $r_0 = x$ to $q_n = z$. Let K be the smallest i, $0 < 1 < n$ such that $y \underset{R}{<} q_1$ (and $q_k \underset{R}{<} y$). Thus q_{k-1}, $q_k \in N(r_{k-1})$ and the consecutivity of $N(r_{k-1})$ implies $y \in N(r_{k-1})$. So y is adjacent to P, the path joining x and z.

(Sufficient.) Suppose that the associated bipartite graph G has a W-consecutive arrangement. The W-consecutive arrangement gives us the natural orders of the sets U and V separately. Let $x_1 \ldots . x_n$, and $y_1 \ldots . y_n$, be the orders of U and V separately. Then if we rearrange the adjacency matrix of D with $x_1 \ldots . x_n$ as rows and $y_1 \ldots . y_n$, as columns, the given order of W immediately provide us with a stair partition of the rearranged matrix.

Let now the entry (i,j) in the above matrix be 1 so that $x_i A y_j$. Suppose that the entry (i,j) occurs in the lower sector of the partition so that $y_j <_{\mathbf{R}} x_i$. Let ρ be the greatest integer such that $y_\rho <_{\mathbf{R}} x_i$ and accordingly $y_j <_{\mathbf{R}} y_{j+1} <_{\mathbf{R}} \ldots . <_{\mathbf{R}} y_\rho <_{\mathbf{R}} x_i$ (and $x_i <_{\mathbf{R}} y_{\rho+1}$). Note that the possible x_j's in the above chain between y_j and x_i (when $j \neq \rho$) are not shown here.

From the definition of W-consecutive arrangement all the vertices lying within y_j and x_i are adjacent to any path joining x_i and y_j. Then those vertices must be adjacent to either x_i or y_j. Since G is bipartite the intermediate vertices can not be adjacent to y_j and so all those vertices must be adjacent to x_i. In otherwords

$$x_i y_j \in A \text{ for } k = j + 1,\ j + 2, \ldots . \rho$$

Thus if (i,j) is in the bottom sector then all the $(i,\ j + 1)$, $(i,\ j + 2) \ldots .$upto the stair will be 1.

Now considering x_i's within the chain between y_j and x_i, we can similarly show that all the $(i-1,j)$, $(i-2,j)$....upto the stair will be 1.

Similar arguments hold good for the case when (i,j) belong to the upper sector. Hence the digraph is an indifference digraph. ∎

2.5. Generalized semiorders and indifference digraphs

Using Theorem 2.4, it is easy to prove the generalization of Roberts restatement of the Scott-Suppes Theorem, as described in the introduction. The generalization of the Scott-Suppes Theorem itself follows as a corollary. Complementation is taken with respect to the complete relation $V \times V$, including loops.

Theorem 2.7. *A digraph $D = (V,E)$ is an indifference graph if and only if $\bar{E} = H_1 \cup H_2$, where (H_1,H_2) is a generalized semiorder on V.*

Proof. Necessity follows easily from Theorem 2.4 by considering a monotone consecutive arrangement of the adjacency matrix. Let $[a_i, b_i]$ be the interval of 1's in row i. Let H_1 correspond to the 0's labeled R (those after b_i) and H_2 to the 0's labeled C (those before a_i). Hence $\bar{E} = H_1 \cup H_2$, and H_1,

H_2 are disjoint. Monotonicity implies that H_1 and H_2 are Ferrers digraphs, which is equivalent to the generalized semiorder condition (1) that uH_iv and xH_iy imply uH_iy or xH_iv ($i \in \{1,2\}$).

For the other conditions, let u_i, u_k be the (not necessarily distinct) source vertices, and let v_j, v_l be the (not necessarily distinct) sink vertices, indexed as in the monotone consecutive arrangement. To prove (2) $u_iH_1v_j$ and $u_kH_2v_j$ imply $u_iH_1v_l$ or $u_kH_2v_l$, note that the hypothesis implies $j > b_i$ and $j < a_k$. If $l \geq j$, then $l > b_i$ and $u_iH_1v_l$, while if $l \leq j$, then $l < a_k$ and $u_kH_2v_l$. When we appeal to the fact that the transpose of a monotone consecutive arrangement is also a monotone consecutive arrangement, the proof is the same for (3) $u_iH_1v_j$ and $u_iH_2v_l$ imply $u_kH_1v_j$ or $u_kH_2v_l$.

For sufficiency, suppose $\bar{E} = H_1 \cup H_2$, where (H_1, H_2) is a generalized semiorder on V. As noted in the introduction, we can view H_1, H_2 as relations or as digraphs on V; let $d_i^+(u)$ denote the set of successors (out-degree) of u in H_i; similarly define d_i^- for in-degree. We claim it is not possible to have both $d_1^+(u) > d_1^+(u')$ and $d_2^+(u) > d_2^+(u')$. If these hold, then there are vertices v,v' such that $uv \in H_1$ but $u'v \notin H_1$ and $uv' \in H_2$ but $u'v' \notin H_2$, but this is precisely the configuration forbidden by condition (3). Similarly, the combination $d_1^-(v) > d_1^-(v')$ and $d_2^-(v) > d_2^-(v')$ is forbidden by condition (2). This implies that we can order the elements as $u_1,\ldots\ldots,u_n$ and as $v_1,\ldots\ldots,v_n$ so that for $1 \le i < n$ we have $d_1^+(u_i) \le d_1^+(u_{i+1})$, $d_2^+(u_i) \ge d_2^+(u_{i+1})$, $d_1^-(v_i) \ge d_1^-(v_{i+1})$, and $d_2^-(v_i) \le d_2^-(v_{i+1})$. By condition (1), H_1 and H_2 are Ferrers digraphs, and this ordering of the degrees simultaneously places the relations of H_1 in the upper right and H_2 in the lower left so that any position above or to the right of a position in H_1 is also in H_1, and any position above or to the right of a position in H_2 is also in H_2. In other words, this ordering of the rows and columns is a monotone consecutive arrangement. ∎

The generalization of the Scott-Suppes Theorem is just a rephrasing of this result.

Corollary 2.1. A pair (H_1,H_2) of relations on the same set is a generalized semiorder if and only if it has a coindifference representation f, g; that is, real-valued functions f, g exist on the elements so that xH_1y if and only if $f(x) > g(y) + 1$, and xH_2y if and only if $g(y) > f(x) + 1$. ∎