AN APPROACH TO SCALE FREE ANALYSIS AND DYNAMICS

Thesis resubmitted for the Degree of Doctor of Philosophy
in Science (Mathematics) under the
University of North Bengal

Submitted by

Santanu Raut

Under the supervision of

Dr. Dhurjati Prasad Datta

Department of Mathematics
University of North Bengal
Siliguri, Darjeeling · 734 013
West Bengal, India

2012

ACKNOWLEDGMENTS

It is indeed a matter of great joy for me to be able to express my heartiest regards and deepest sense of gratitude, appreciation and indebtness to my respectable teacher, Dr. Dhurjati Prasad Datta, Associate Professor, Department of Mathematics, University of North Bengal, Siliguri, West Bengal, India, for his wonderful supervision, excellent selection of the thesis problem, heartiest co-operation and valuable suggestions and sustained interest throughout the course of this dissertation and also for critically going through the manuscript.

I am grateful to all the honourable teachers, research scholars, students and staffs of my department for their kind support and inspirations through the entire course of my investigations. I wish to record also my greatest appreciations to all the teachers of my academic carrier in the different phases in my life. Special thanks are also due to my friend Ajoy Mukharjee for his counseling and assistance in the preparation of manuscript.

I take the pride to enunciate my all encompassing debt to my beloved mother and departed father and other close relatives who with their moral supports, endless blessings, and encouragements enabled me to complete this investigation and finally to prepare this thesis. I also wish to offer a very special thank to my wife for her constant inspiration and her resource of strength in moments of distress and hopelessness.

Date: 13.08.12

Santanu Raut

Santanu Raut

DEDICATION

To My parents

Abstract

In this thesis, a scale invariant analysis for a Cantor set like fractal subset C of the real line R is developed using the concepts of relative infinitesimals and the associated nonarchimedean absolute values. The meaning and salient properties of the scale invariant nonarchimedean valuation are discussed in detail through various examples. The valuation is shown to be related to an appropriate Cantor function, which is then realized as a locally constant function defined over the ultrametric Cantor space. The formalism of calculus and a valued measure are introduced on such an ultrametric space. The increments on such an ultrametric space are mediated by inversions. The valued measure is shown to equal the finite Hausdorff measure of the original Cantor set. The ordinary limit $x \to 0$, $x \in C$ is shown to be given by a limit of the form $x \log x^{-1} \to 0$, when $x \in R$. Next, we study an interesting new phenomenon called the growth of measure, exploiting the reparametrisation invariance of a locally constant function. The phenomenon is explained explicitly by showing how a measure zero Cantor set may become a positive measure set. The role and meaning of a higher order valuation is explained by constructing a class of Cantor sets having identical Hausdorff dimension and thickness. Next, we study the relevance of a novel class of nonsmooth solutions of the scale invariant ODE $t\frac{dx}{dt} = x$ in the context of ultrametric Cantor sets. Some applications of the new class of solutions in the longstanding problems of time asymmetry, $1/f$ noise, origin of q-deformed exponential in the chaos threshold of the logistic maps are also discussed.

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Chapter 1

INTRODUCTION

The world, we live in, is highly complex. A small seed, almost spherical in shape, and so may be considered to be geometrically simple, under right conditions, slowly grows, in successive steps, into a sampling and then gradually into a fully grown plant, with many flowers and new generations seeds. This well known mundane example can be considered to represent a model for the paradigm of complexity. The salient features of a complex system consist mainly of nonlinearity, scale invariance, self-similarity and so on. By nonlinearity we mean in this thesis that the governing dynamical principle inducing evolution of the system concerned may be described by one or more of nonlinear differential equation(s) and/or similar other equations and processes. Because of scale invariance the relevant dynamical variables can be represented by power laws of the form t^{α}, where t is a real variable and α is a constant. As a consequence, the growth of a complex system is expected to have influences from many different scales of a dynamical variable. Self-similarity finally means roughly "a part resembling exactly (or approximately) similar to the whole". More analytical definition is given latter. Fractals, a very active area of contemporary research in the field of nonlinear sciences, may be said to represent an example of a class of complex systems. Although there is still no generally acceptable definition, fractals are generally considered to be those subsets of the Euclidean space R^n which are highly irregular, nonsmooth and also enjoy some sort of scale invariance and self-similarity. Further, generation of such fractal subsets of R^n admit some non-linear process(es). Various natural objects and processes are known to reflect fractal like self-similarity and scale invariance. For example, large scale galaxy distribution, cloud boundaries, topographical surfaces of the planet earth, coastlines, turbulence in fluid, stock market fluctuations, structures of mammalian hearts and lungs and so on [1, 2]. The interest in the study of self-similarity and scale invariance of the global and local structures of the nature - ranging from the macroscopic cosmological scales down to the microscopic finer scales - is gaining momentum over the last few decades from extensive work of several mathematicians and physicists throughout the world [1, 2, 3, 4, 5].

Appearance of fractal like irregular (pathological) subsets in (real/complex) analysis dates back to the later half of nineteenth century when various examples of no-where non-differentiable continuous curves were studied. Weierstras's construction provided one such early example. Weierstras's curve $f(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^k t$, $1 < s < 2$, $\lambda > 1$ also enjoys self-similarity on all scales as represented by the scaling law $f(\lambda^{-1}t) \approx \lambda^{s-2} f(t)$, $\lambda >> 1$ [3]. As a result, a smaller portion of the said curve when suitably magnified will resemble the original curve. For about three decades after the construction of such functions, these were still considered to be rather pathological cases without any practical and/or analytical interest. In the recent years, the attitude has changed considerably. It has been realized that irregular sets provide a much better representation of many natural phenomena than the figures of classical (Euclidean/non-Euclidean) geometry. Perrin was the first physicist who pointed out their applications in the real physical world. His ground breaking work on the Brownian motion showed that trajectory of the diffusive Brownian particles are nowhere differentiable and have fractional dimension 3/2 [6]. In the fluid systems, passive scalars adverted by a turbulent fluid have isoscalar surfaces which are highly irregular. In dynamical systems attractors of some systems, for instance, the Lorenz attractor, are found to be continuous but nowhere differentiable [7]. Over the last few decades it has also become clear that the occurrence of chaos in a deterministic dynamical system such as logistic map for a suitable range of values of the control parameter requires formation of Cantor sets dynamically in a region of the so called strange (chaotic) attractor [6].

A Cantor set is a totally disconnected, compact and perfect subset of the real line. Cantor set is an example of a self-similar fractal set that arises, as indicated above, in various fields of applications. The chaotic attractors of a number of one dimensional maps; such as the logistic maps, tent map, turn out to be topologically equivalent to Cantor sets [8]. Cantor set also arises in electrical communications [1], in biological systems [2], and diffusion processes [9, 10]. Recently there have been a lot of interest in developing a framework of analysis on a Cantor like fractal sets [6, 11, 12, 13]. Because of the disconnected nature, methods of ordinary real analysis break down on a Cantor set. Various approaches based on the fractional derivatives [14, 15] and the measure theoretic harmonic analysis [16], functional analysis, probability theory [6] have already been considered at length in the literature. However, a simpler intuitively appealing approach is still considered to be welcome.

The present thesis is a part of an ongoing project that aims at developing a scale invariant analytical framework that would be suitable to construct a rigorous analysis on fractal subsets of R^n. In the present thesis, in particular, *we formulate a scale invariant analysis on Cantor like fractal subsets of* R. Since a Cantor set C is a totally disconnected, compact, perfect subset of R, the ordinary analysis of R can not be meaningfully extended over C, i.e., when a real variable x is assumed to live and undergo changes only over the points of C. More specifically, the concept of a derivative in the sense of rate of change of a dynamic quantity, namely, a function of time when time is supposed to vary over a Cantor set, (say)[1] can not be formulated consistently on such a set. The general trend in the literature is to bypass defining derivatives directly on such sets, by taking recourse to technically more involved approaches based on geometric measure theory [3], harmonic analysis [7], functional analysis on noncommutative [17] spaces, probability theory [6] and so on. The present scale invariant analysis utilizing the concepts of *relative and scale invariant infinitesimals* is not only simpler than the other contemporary approaches but also offers an elegant avenue extending the well known differential calculus of R over a Cantor set C in a conceptually

[1]The possibility of a time variation on a Cantor like fractal set is considered in Continuous Time Random Walk theories of statistical mechanics [9].

appealing manner. Recall that ordinary measure theoretic arguments can essentially establish an analytic statement on R upto a *Lebesgue measure zero set* only. Our analysis, on the otherhand, succeeds in deducing results which are valid *everywhere* in R. For instance, a Cantor function $\phi(x)$ can be defined classically as one which satisfies $\frac{d\phi}{dx} = 0$ *almost everywhere* in [0,1]. In the present scale invariant approach a Cantor function is shown to be locally constant *everywhere* in [0,1]. Further, the global variability of a Cantor function is shown to get exposed in a double logarithmic scale $\log \log x^{-1}$. We also define and study some evolutionary equation on such a Cantor set. The present approach rests on a novel extension of the usual ultrametric structure of a Cantor set into an inequivalent class of ultrametrics using a seemingly new concepts of relative infinitesimals that are shown to exist in the gaps of infinitesimally small neighbourhoods of 0, considered as an element of a Cantor set $C \subset [0, 1]$. In short, the present thesis represents a body of analytic results which are of interdisciplinary in nature involving various topics such as Cantor like fractal sets, nonstandard analysis, nonarchimedean spaces, real analysis, measure theory etc.

1.1 Main Results of the Thesis

In chapter 2, the salient features of several key notions such as fractals, ultrametric spaces, nonstandard analysis and Cantor sets, which will be useful in the subsequent development of the new analysis, are reviewed briefly.

In chapter 3, the basic concepts of relative infinitesimals and scale invariant infinitesimals are introduced (defined) and discussed in detail. Next, we introduce the novel definition of a scale invariant absolute value, which is shown to assign a nontrivial ultrametric valuation to such a scale invariant infinitesimal, thus raising the corresponding set of infinitesimals into an ultrametric space. We show that this nonarchimedean valuation, essentially, is defined by a suitable Cantor function associated with the original Cantor set. We then study some basic properties of topology and analytic structures on this ultrametric space of scale invariant infinitesimals. The definitions of limit, continuity and differentiability are formulated. The ultrametric and the corresponding induced topology are shown to represent respectively inequivalent classes in comparison to the natural ultrametric on a Cantor set. Next, we explain how this ultrametric structure is carried over to the entire Cantor set, thereby inducing an associated ultrametric structure in the said set. The chapter ends with a discussion of a *valued measure* that arise naturally in the above ultrametric space generalizing the standard metric Lebesgue measure. The valued measure turns out to give rise to directly the finite, nonzero Hausdorff s-measure of the underlying Cantor set when s denotes the Hausdorff dimension of the set.

In chapter 4, several explicit examples, namely, the middle third Cantor set, middle α- Cantor set and (p,q) Cantor set are reexamined in the light of present scale invariant analytic framework. To clarify the basic analytic ingredients, namely, the relative infinitesimals and associated absolute values, we present here an *independent* set of arguments detailing the *origin, actual role, and significance* of the above concepts of relative infinitesimals and associated valuations in the context of a family of homogeneous Cantor sets [24, 25, 26]. It is shown that the singleton set of the zero of the real line is replaced by a nontrivial zero measure set of relative infinitesimals, which are supposed to live in an inverted Cantor set, defined as the collection of the closure of the gaps of the original Cantor set in the neighbourhood of 0. Further more, it is verified explicitly, in each of the above distinct cases, that the valued infinitesimals induce a finer structure in the neighbourhood of each Cantor point, leading to a multiplicative structure defined on a Cantor set. The nonarchimedean valuation realized here as an appropriate Cantor function is next interpreted as a locally constant function satisfying the equation $\frac{dv}{dx} = 0$. Although locally constant in the neighbourhood of a point, such a function , nevertheless, can enjoy global variability. The chapter ends with a discussion of the global variability of the locally constant function in the usual topology [25].

In chapter 5, another *independent* analysis is presented on the derivation of a smooth multiplicative representation of an element of a Cantor set. This is expected to offer new *insights* into the *mechanism of smoothening* of a Cantor function at the points of Cantor set. The analysis is based on the standard classical analysis arguments exposing the nondifferentiability of a Cantor function $\phi(x)$ at $x \in C$. Our scale invariant analysis leading to the above results are presented again in the context of the classical middle third Cantor set, as well as in the (p, q) type Cantor set [24, 25].

In chapter 6, some new results leading to the differential jump measure on a Cantor Set are presented. It exposes the *precise* nature of variability of a nontrivial valuation. The ordinary limit $x \to 0$ on the real line R is shown to extend over to a sublinear limit $x \log x^{-1} \to 0$, when x is assumed to vary over a Cantor set. Further, the incremental measure of smooth self similar jump processes is determined. It corresponds to the multiplicative increment which is realized as a smooth measure and may be considered to contribute an independent component in the ordinary measure of R [28].

An interesting new phenomenon, called the *growth of measure* is studied in chapter 7 [26]. Using the reparametrisation invariance of the valuation it is shown how the scale factors of a Lebesgue measure zero Cantor set might get *deformed* leading to a *deformed* Cantor set with a positive measure. The definition of a new *valuated exponent* is introduced which is shown to yield the fatness exponent in the case of a positive measure (fat) Cantor set. Here, we also study a class of Cantor set having identical Hausdorff dimensions and thickness. However, the higher order valuated exponent, introduced here, may be exploited to distinguish such sets.

In chapter 8, a class of an exact, higher order derivative discontinuous (nonsmooth) solutions to the simplest scale invariant ordinary differential equation $t\frac{d\tau}{dt} = \tau$ is derived using a novel iteration procedure revealing the possible presence of a nontrivial selfsimilar multiplicative structure in such a solution [27]. The new class of solutions are shown to break the reflection symmetry of original differential equation. The existence of such non-trivial solutions, which can be put in a rigorous setting in the context of a nonstandard model of real analysis, can be interpreted in an extended framework of calculus accommodating (random) inversions as a valid mode of changes over and above the usual mode of linear increments in the real analysis.

In chapter 9, a few interesting applications of the above class of nonsmooth solutions are presented in the context of some selected topics of nonlinear dynamical systems [27]. First, we discuss how the class of nonsmooth solutions might lead to a new paradigm in realizing and reinterpreting randomness that appears so abundantly in nonlinear deterministic models. Next, we argue that the reflection asymmetry of the class of nonsmooth solutions may be reinterpreted as a novel framework to understand the origin of *time asymmetry* in any evolutionary process [47, 49]. The origin and genesis of universally present *flicker* $(1/f)$ noise in diverse natural, biological , financial processes is still considered to be a riddle by many authors [53, 57]. In Sec.9.4, we discuss the relevance of nonsmooth solutions to the flicker noise problem. Because of the presence of multiscale stochastic behaviours, the nonsmooth solutions naturally become relevant in understanding flicker noise. In the final two subsections, we show how a derivation of the q-exponential power law dynamics of the sensitivity to initial conditions of a logistic map in the edge of chaos can be formulated in the present framework [53, 54]. We further show how a hyperbolic type distribution arise naturally at the asymptotic late time $(t \to \infty)$ limit even from a normally distributed variate.

In chapter 10, we show that the above scale free differential equation which is actually not defined on a Cantor set, even in the usual (ultrametric) sense, is raised to an equation which is well defined on a Cantor

Set C [25]. The derivation becomes possible as every point of a Cantor set C is replaced by the closure of collection of gaps of another Cantor Set $\widetilde{C}$, called an inverted Cantor set, where the relative infinitesimals are supposed to live in. We also rederive local constancy of a Cantor function and the valuation is realized now as the so called nonsmooth solutions of the said scale invariant equation.

In the concluding chapter 11, we summarize our main results and also indicate briefly how the present formalism may be extended further.

Chapter 2

BACKGROUND INFORMATION

2.1 Fractals

2.1.1 Introduction

The word 'fractal' is coined by Mandelbort from a Latin word *fractus* that denotes "a stone's shape after it was hit hard". It describes objects that are too irregular to fit into a traditional smooth (Euclidean) geometric setting. A fractal is usually considered to be an object of inquiry having finer structures. An important characteristic of fractal objects is the occurrence of some sort of a self-similarity. Generally, it can be expressed as a union of subsets, each of which is a reduced copy of the full set. Therefore the structure of the original set can be realized at smaller scales. Usually, the topological dimension of an object (i.e. a set)(in any dynamical problem) is a non-negative integer. But the dimension of a fractal set is a fractional (more correctly, real) number strictly exceeding the topological dimension. One of the major motivation in the study of fractals is the possibility of describing complex natural phenomena by only a finite set of parameters. But the methods of ordinary calculus is inapplicable in fractals as they are generally not smooth and are made of many fragmented geometric shapes. Therefore a proper development and application of an appropriate analytical framework for fractal sets is of considerable interest in the contemporary literature [3].

The main characteristic features of a fractal set $F \subset R^n$ can be stated as

1. It has fine structures on arbitrarily small scales.

2. F is too irregular to be described in the traditional geometrical setting.

3. It has fractional (fractal) dimension that exceeds its topological dimension.

4. The fractal set F has some sort of exact or approximate self-similarity.

5. The set F may be generated recursively following a simple (finite) set of rule.

Definition 1. *(self-similarity) The mapping $S_1, \ldots S_k : R^n \to R^n$ are called similarity transformations when $|S_i(x) - S_i(y)| = c_i\,|x - y|$ $(x, y \in R^n)$ and $0 < c_i < 1$ (c_i is called the scaling ratio of S_i). Each S_i transforms subsets of R^n into geometrically similar sets. A set that is invariant under such a collection of similarities is called a self-similar set.*

Middle third Cantor set, Sierpinski gasket and von Koch curve are all examples of self-similar sets.

2.1.2 Fractional Dimension

Conventionally, the dimension of a object is usually a non-negative integer that specify the number of coordinates that are necessary to describe the object (i.e. the elements of the set concerned) precisely. But fractal sets can not generally be described by simply by a finite set of coordinates. Thus, we have to look for a different definition of dimension that does not depend on the coordinates. Here, we shall consider two important dimensions namely Hausdorff dimension and Box dimension. We first briefly discuss some desirable properties which a definition of dimension is expected to satisfy [3]. Let $E, F \subset R^n$, then

1. Monotonicity: If $E \subset F$ then $\dim E \leq \dim F$.

2. Stability: $\dim(E \cup F) = \max(\dim E, \dim F)$.

3. Countable stability: $\dim\ (U_{i=1}^{\infty} F_i) = \sup\limits_{1 \leq i \leq \infty} \dim F_i$.

From (2) and (3) it is ensured that if we combine a set with other set having lower dimensions, the dimension will be same.

4. Geometric invariance: $\dim f(F) = \dim F$ provided f is a translation, rotation, similarity or affinity transformation.

5. Lipschitz invariance: Let $f : F \to R^n$ is a bi-lipschitz transformation, i,e. if $\exists\ c_1$ and c_2 s.t. $c_1 \mid x - y \mid \leq \mid f(x) - f(y) \mid \leq c_2 \mid x - y \mid$, $x, y \in F$, and $0 < c_1 \leq c_2 < \infty$, then $\dim f(F) = \dim F$.

6. Countable sets: If F is finite or countable then $\dim F = 0$.

7. Open sets: If F is a open subset of R^n then $\dim F = n$.

8. Smooth manifold: If F is a n dimensional smooth manifold then $\dim F = n$.

Definitions of dimension generally satisfy monotonicity and stability, but some definitions fail to exhibit countable stability and may even ascribe a set of positive dimension to a countable set.

2.1.3 Hausdorff measure

Let U be a nonempty subset of n dimensional Euclidean space R^n. Diameter of U is defined as $\mid U \mid = \sup\{\mid x-y \mid : x, y \in U\}$. If $\{U_i\}$ be countable (or finite) collection of sets of diameter at most δ that cover F, i.e. $F \subset \cup_{i=1}^{\infty} U_i$, with $0 < \mid U_i \mid \leq \delta$, for each i, then we say that $\{U_i\}$is a $\delta-$cover of F. Suppose that F is a subset of $R^n$and we define

$$H_\delta^s(F) = \inf\left\{\sum_{i=1}^{\infty} \mid U_i \mid^s : \{U_i\} \text{ is a } \delta - \text{cover of } F\right\} \tag{2.1}$$

where s is a non-negative number and infimum is taken with all possible $\delta-$covers.

As δ decreases, the class of permissible covers of F is reduced. Therefore, the infimum increases and so approaches a limit as $\delta \to 0$. We write

$$H^s(F) = \lim_{\delta \to o} H_\delta^s(F). \tag{2.2}$$

This limit exits for any subset F of R^n, though the limiting value can be 0 or ∞. We call $H^s(F)$, the $s-$dimensional Hausdorff measure of F. If $\{F_i\}$ is any countable collection of disjoint Borel sets, then

$$H^s\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} H^s(F_i). \tag{2.3}$$

Hausdorff measures generalize the familiar ideas of length, area, volume, etc. It may be shown that, for a subset of R^n, n-dimensional Hausdorff measure, to within a constant multiple, equals n-dimensional Lebesgue measure, i.e. the usual n-dimensional volume. More precisely if F is a borel subset of R^n, then

$$H^n(F) = c_n vol^n(F) \tag{2.4}$$

where the constant $c_n = 2^n \left(\frac{n}{2}\right)!/\pi^{\frac{n}{2}}$ is the reciprocal of the volume of an n-dimensional ball of diameter 1. Similarly, for 'nice' lower dimension subsets of R^n, we have $H^0(F)$ is the number of points of F; $H^1(F)$ gives the length of a smooth curve F; $H^2(F) = \left(\frac{4}{\pi}\right) \times area(F)$ if F is a smooth surface; $H^3(F) = \left(\frac{6}{\pi}\right) \times vol(F)$; and $H^m(F) = c_m vol^m(F)$ if F is a smooth m-dimensional sub-manifold of R^n.

The scaling properties of length, area and volume are well known. On magnification by a factor λ, the length of a curve is multiplied by λ, the area of a plane region is multiplied by λ^2, and the volume of a 3-dimensional object is multiplied by λ^3. As might be anticipated, s-dimensional Hausdorff measure scales with a factor λ^s. Such scaling properties are fundamental to the theory of fractals.

Lemma 1. *[3]* Scaling Property: *If $F \subset R^n$, and $\lambda > 0$, then $H^s(\lambda F) = \lambda^s H^s(F)$, where $\lambda F = \{\lambda x : x \in F\}$, i.e. the set F scaled by a factor λ.*

2.1.4 Hausdorff dimension

As $H^s_\delta(F)$ is non-increasing with s so that $H^s(F)$ is also non-increasing with s. Now, if $t > s$, and $\{U_i\}$ is $\delta-$cover of F, we have

$$\sum \mid U_i \mid^t \leq \delta^{t-s} \sum \mid U_i \mid^s \tag{2.5}$$

and so taking infimum for each fixed s,

$$H^t_\delta(F) \leq \delta^{t-s} H^s_\delta(F). \tag{2.6}$$

Letting $\delta \to 0$, we see that if $H^s(F) < \infty$, then $H^t(F) = 0$ for $t > s$. Thus it shows that there is a critical value of s at which $H^s(F)$ jumps from ∞ to 0. This critical value is called the Hausdorff dimension of F,

$$H^s(F) = \begin{cases} \infty & if\ s < \dim_H F \\ 0 & if\ s > \dim_H F. \end{cases} \tag{2.7}$$

The critical value $s_0 : 0 < s_0 < 1$ at which $H^s(F)$ jumps from ∞ to 0 is the Hausdorff dimension of F. It can be shown that for a totally disconnected uncountable set F, $0 < H^s(F) < \infty \Longleftrightarrow 0 < s_0 < 1$.

2.1.5 Box Dimension

Box dimension is one of the most widely used dimension. It is easy to use both analytically and numerically. Let F be any nonempty bounded subset of R^n and let $N_\delta(F)$ be the smallest number of sets of diameter *at most* δ which can cover F. Then the lower and upper box dimensions of F are respectively defined as

$$\underline{\dim}_b(F) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

$$\overline{\dim}_b(F) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Box dimension of F is defined as $\dim_b(F) = \underline{\dim}_b(F) = \overline{\dim}_b(F)$, whenever two limits are equal.

Box dimension is very simple but it gives in some cases inadmissible results. For example, Box dimension of countable sets may have dimension one. Consider the set of rational numbers of [0,1]. Let us cover this set by a partition with interval δ. Then $N(\delta) = \frac{1}{\delta}$. And hence the Box dimension of the set is 1. But the set of rational number is a countable set. The union of countable zero dimensional singleton sets has dimension zero. Inspite of this paradoxical result this definition is widely used mainly for its simplicity and geometrical appeal.

In case of Box dimension we essentially cover the set with boxes of fixed sizes where as in the Hausdorff dimension we allow all the sizes smaller than δ. This is the crucial difference between two definitions. One can give simple arguments to show that the Hausdorff dimension of the set of rational number is indeed zero. Set of rational number is countable and hence we can label each rational number by a positive integer K. Now cover the K th rational number by an interval of length $\frac{\delta}{2^k}$. Then the sum becomes $\sum \frac{\delta^s}{2^{ks}}$. It is also bounded by $K\delta^s$. Now as $\delta \to 0$, for $s > 0$, the limit becomes zero. Hence Hausdorff dimension of the set becomes zero.

2.1.6 Examples of Fractals

We give here a few simple recursive constructions leading to a few interesting examples of fractal sets which arise significantly in various applications in recent literature [3]. The middle third Cantor set is one of the most well known and easily constructed fractals. It is constructed from a unit interval by a sequence of deletion operations of removing middle third portion of certain relative length at each scale. Stated in another way, the classical middle third Cantor set, for example, consists of all the points between 0 and 1 that can be represented using only 0's and 2's in ternary representation. It is self-similar, because at every scale, it is equal to two copies of itself, if each copy is shrunk by a factor of $\frac{1}{3}$ and translated. Another familiar fractal, though not considered in this work, is Von Koch curve. To construct Von Koch curve one have to begin again with the unit interval and remove then middle third of it by replacing the other two sides of the equilateral triangle based on the removed segments. Repeating the process the sequence of polygonal curves approaches a limiting curve, called the Von koch curve. Similarly, Sierpinski gasket is obtained by repeatedly removing of an equilateral triangle by the three trainless of half the hight. The highly intricate structure of the Julia set is constructed from the quadratic function $f(z) = z^2 + c$, for a suitable constant c. Although, the set is not strictly self-similar like as Cantor set or Von koch curve, it is quasi self-similar. These are a few examples of sets that are commonly known as fractals.

2.1.7 Occurrence of Fractals

There are abundance of natural objects and processes that have a fractal like structure. Fractal structures provide different ways of modeling biological systems [4]. The usage of fractals are common, for instance, in root system analysis, in the study of variation of shapes of dental crown pattern, and also in the analysis of cancer cells images. Human lungs, breathing patterns of mammals, branching of trees and so on, also have self-similar fractal like spatial and/or temporal structures. Fractal models are also used to understand the shape of neurons, growth of bacterial colonies, forest tree distributions, population distribution in metros and large cities. Geometry of fractals also appear in cloud boundaries, topographical surfaces, coastlines, turbulence in fluids, and in daily fluctuations in stock markets and other related fields of applied sciences [1]. However, most of these natural and biological/ financial objects are not actual fractals. Their fractal features disappear if they are viewed at sufficiently small scales. Only over a certain level of scales they appear and/or behave like fractals.

2.2 Ultrametric Space

An ultrametric space is a special kind of metric space in which the triangle inequality is replaced by a stronger inequality $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ [20, 21, 22, 23]. The associated metric is also called non-Archimedean metric. To a beginner, properties of an ultrametric space may seem rather unusual but they, nevertheless, appear naturally in many applications. Many self-similar sets such as Cantor sets are bilipschitz equivalent to ultrametric spaces. Thus, ultrametric space is very relevant in the study of Cantor sets as well as other self similar sets.

Definition 2. *An ultrametric (or nonarchimedean metric) on a set X is a mapping $d : X \times X \to R$ with the following properties.*

(i) For $a, b \in X$, $d(a, b) \geqslant 0$ and $d(a, b) = 0$ if and only if $a = b$.

(ii) For $a, b \in X$, $d(a, b) = d(b, a)$.

(iii) For $a, b, c \in X$, $d(a, c) \leq \max\{d(a, b), d(b, c)\}$ (strong triangle inequality).

Note that if $d(a, b) \neq d(b, c)$, then $d(a, b) = \max\{d(a, b), d(b, c)\}$.

Definition 3. *Let K be a field. A nonarchimedean absolute value (norm) on K is a mapping $|.| : K \to R$ such that for any $a, b \in K$, we have*

(i) $|a| \geq 0$.

(ii) $|a| = 0$ *if and only if* $a = 0$.

(iii) $|ab| = |a| \, |b|$.

(iv) $|a + b| \leq \max(|a|, |b|)$.

It follows from the definition 3(iv) that $|n.1| \leq 1$, for any $n \in Z$. Also, if $|a| \neq |b|$ for some $a, b \in K$, then the triangle inequality becomes an equality $|a + b| = \max(|a|, |b|)$.

The set K becomes a ultrametric space via the metric induced by the non-archimedean norm $d(a, b) = |a - b|$. The subsets $\overline{D}(a, r) = \{b \in K : |b - a| \leq r\}$ for any $a \in K$ and any real number $r > 0$ are called closed balls or simply balls in K, Likewise the open balls $D(a, r) = \{b \in K : |b - a| < r\}$ form the neighbourhood of a in the metric space K.

A Cauchy sequence in X is a sequence $\{x_n\}_{n=1}^{\infty}$ such that for any $\epsilon > 0$, there exists $N \in \mathbf{N}$, ($\mathbf{N}$ being the set of Natural numbers), such that for all $m, n \geq N$ $(m, n \in \mathbf{N})$, $d(x_m, x_n) < \epsilon$ Note that by the strong triangle inequality, this is equivalent to $d(x_{n+1}, x_n) < \epsilon$ for all $n \in \mathbf{N}$. Further, a sequence $\{x_n\}$ converges to a non-zero limit x iff the sequence is eventually constant in the ultrametric, i.e. $|x_n| = |x|$ for $n > N$. As a consequence, nonarchimedean analysis turns out to be, in many situations, simpler than the traditional analysis on an archimedean field. X is complete if every Cauchy sequence converges to a limit (necessarily unique because of (i)).

Definition 4. *The field K is called nonarchimedean if it is equipped with* *a nonarchimedean absolute value such that corresponding metric space K is complete (that is, every Cauchy sequence in K converges).*

In the following, a few salient features of ultrametric topology is enlisted. The strong triangle inequality leads to several geometrical as well as topologically interesting consequences. For example,

Proposition 1. *Let K be a complete non- archimedean field. Then*

i) Given $a,\ b \in K$ and $s \geq r > 0$ such that $a \in D(b,s)$, we have $D(a,r) \subset D(b,s)$, and $D(a,s) = D(b,s)$.

ii) Given $a,\ b \in K$ and $s \geq r > 0$ such that $a \in \overline{D}(b,s)$, we have $\overline{D}(a,r) \subset \overline{D}(b,s)$, and $\overline{D}(a,s) = \overline{D}(b,s)$.

iii) If $D_1, D_2 \subseteq C_k$ are two balls such that $D_1 \cap D_2 \neq \phi$ then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.

iv) All balls in K are both open and closed topologically. Such balls are called clopen balls.

v) K is totally disconnected as topological space; that is, only nonempty connected subsets are singletons.

Proof. i) Given $x \in D(a,r)$, because $|x-a| < r \leq s$ and $|a-b| < s$, we have $|x-b| \leq \max\{|x-a|, |a-b|\} < s$, and hence $x \in D(b,s)$. The reverse inclusion in the case $r = s$ is similar.

ii) Given $x \in \overline{D}(a,r)$, because $|x-a| < r \leq s$ and $|a-b| < s$, we have $|x-b| \leq \max\{|x-a|, |a-b|\} < s$, and hence $x \in \overline{D}(b,s)$. The reverse inclusion in the case $r = s$ is similar.

iii) Pick $c \in D_1 \cap D_2$. From (i) and (ii) we can say that c is a centre of each ball. That is each D_i can be written as either $D(c, r_i)$ or

$\overline{D}(c, r_i)$. After possibly exchanging D_1 and D_2, either $r_1 > r_2$ or else $r_1 = r_2$ with either D_1 closed or D_2 open (or both) . Then $D_1 \supseteq D_2$.

iv) To show that an open ball $D(a, r)$ is topologically closed, pick any $x \in C_k \backslash D(a, r)$. If the two balls $D(x, r)$ and $D(a, r)$ intersects, then one contains the other by part (iii) and hence they coincide by part (i). That contradicts our assumption that $x \notin D(a, r)$, and therefore $D(a, r) \subseteq C_k \backslash D(a, r)$, as desired.

To show that a closed ball $\overline{D}(a, r)$ is open, pick any $x \in \overline{D}(a, r)$. Since the balls $\overline{D}(x, r)$ and $\overline{D}(a, r)$ intersects at x, one contains the other by part (iii), and hence coincide by part (ii). Thus, $D(x, a) \subseteq \overline{D}(x, r) = \overline{D}(a, r)$, as desired.

v) Suppose $X \subseteq K$ is set containing two distinct points a, b. Let $r = |a - b| > 0$. Then by part (iv), $X \cap D(a, r) \ni a$ and $X \setminus D(a, r) \ni b$ are both nonempty open subsets of X, and hence X is disconnected. Thus, the only connected subsets of K are the empty set and singletons.

■

Thus, it is clear that a non-archimedean ball does not have well defined centres; indeed every point of the ball can be called its centre. Further, the clopen balls form the basis for the induced topology on a ultrametric space.

Example 1. *Let n be a natural number. Then it has a unique representation as a product of powers of distinct primes. Let p be a prime from this product of primes. Denote by $ord_p n$, exponent of p in this representation and put $|n|_p = p^{-ord_p n}$, called the p-adic norm on the set of natural numbers . Indeed, (i) and (ii) of Definition obviously hold for the defined* *norm. Moreover, the stronger inequality (iii) also holds. This definition of norm can easily be extended over the set of rational Q. Let* $Q \ni r = \frac{n}{m}$, $(n,\ m) = 1,\ \ m \neq 0$. *Then* $|r|_p = p^{-ord_p r}$, $ord_p r = ord_p n - ord_p m$. *The completion of the p-adic norm over the field of rational Q leads to the local field* Q_p *for each prime p.*

The set of real numbers R with usual metric is Archimedean. That is, it satisfies the Archimedean axioms that can be geometrically stated as follows. Let us consider a segment of real line of length s and another smaller segment of length l. Then there exists a natural number n such that $n.l > s$. That is to say, if we append a short segment of line to itself sufficient number of times we get a longer segment. Let us now give a similar geometric argument to clarify the significance of the corresponding non-archimedean property: In the ring of p-adic integers Z_p, defined by $|x|_p \leq 1,\ x \in Z_p$, appending a segment to itself one could make the resulting segment shorter than the original one. Let $p = 2$ and let L be some 'segment of length' $\frac{1}{2}$, say, $L = 2$. Then doubling the segment we, obtain a 'segment' $2{\cdot}L = 4$, and for which we have $|4|_2 = \frac{1}{2^2}$. Thus the 'doubled segment' becomes twice as short as the original in a 2-adic space, in contradistinction with our usual Archimedean commonsense. The origin of this peculiarity is again hidden in the strong triangle inequality that plays a crucial role in a non-Archimedean space.

We say K is discretely valued if its value group $|K| = \{|x| : x \in K\}$ is a discrete subgroup of R; that means it must be isomorphic to Z. Some examples of complete, discretely valued, ultrametric fields are

(a) the field of formal power series $K\,(G)$ over the field K.

(b) the local fields Q_p of p-adic numbers.

(c) any finite extension of either of these.

On the other hand, infinitesimals in nonstandard analysis are considered as examples of non-discretely valued ultrametric field. Here, any infinitesimal number can be represented as a certain equivalence classes of sequence of real numbers.

Before closing this subsection, let us prove an important result revealing the generic structure of a discretely valued field K for which the multiplicative value group $|K^{\times}|$ is a discrete subset of $R_+^{\times}$.

Proposition 2. *[18] The subgroup $|K^{\times}| \subseteq R_+^{\times}$ either is dense or is discrete; in the latter case there is a real number $0 < r < 1$ such that $|K^{\times}| = r^Z$.*

Proof. Let us assume that the multiplicative group $|K^{\times}|$ is not dense in $R_+^{\times}$. Then the additive group $\log|K^{\times}|$ is not dense in R. Set $\rho = \sup(\log|K^{\times}| \cap (-\infty, 0))$. We claim that ρ actually is the maximum of this set. Otherwise there is a sequence $\rho_1 < \rho_2 < \cdots$ in $\log|K^{\times}|$ which converges to ρ. But then $(\rho_i - \rho_{i+1})$ is a sequence in $(\log|K^{\times}| \cap (-\infty, 0))$ converging to zero which implies that $\rho = 0$. In that case we find for any $\epsilon > 0$ a $\sigma \in \log|K^{\times}|$ such that $-\epsilon < \sigma < 0$. Consider now an arbitrary $\tau \in R$ and choose an integer $m \in Z$ such that $m\sigma \leq \tau < (m-1)\,\sigma$. It follows that $0 \leq \tau - m\sigma < -\sigma < \epsilon$ and hence that $\log|K^{\times}|$ is dense in R which is a contradiction. This establishes the existence of this maximum and consequently also the existence of $r = \max(|K^{\times}| \cap (0,1))$. Given any $s \in |K^{\times}|$ there is an $m \in Z$ such that $r^{m+1} < s \leq r^m$. We then have $r < s/r^m \leq 1$ which by the maximality of r, implies that $s = r^m$. This shows that $|K^{\times}| = r^{Z}$. ■

2.3 Nonstandard Analysis

Robinson showed that proper extension $^*R$ of the field of real numbers R could be constructed, which contains infinitely small and infinitely large numbers [35]. The theory, first evolved by using free ultrafilters and equivalence classes of sequences of reals, was later formalized by Nelson as an axiomatic extension of Zermelo set theory. We do not intend to give here a detailed account of the field which is now developed as a major branch of Mathematical analysis. We shall just recall the results which we think to be relevant for our subsequent presentation [36].

Let us briefly recall the ultrapower construction of Robinsion. Though less direct than the axiomatic approach, it allows one to get a more intuitive contact with the origin of the new structure. Indeed the new infinite and infinitesimal numbers are formulated as equivalence classes of sequences of real numbers, in a way quite similar to the construction of R from rationals.

Let N be the set of natural numbers. A free ultrafilter u on N is defined as follows:

u is a non empty set of subsets of N [$p(N) \supset u \supset \phi$], such that:

(1) $\phi \in u$.

(2) $A \in u$ and $B \in u \Longrightarrow A \cap B \in u$.

(3) $A \in u$ and $B \in p(N)$ and $B \supset A \Longrightarrow B \in u$.

(4) $B \in p(N) \Longrightarrow$ either $B \in u$ or $\{j \in N : j \notin B\} \in u$, but not both.

(5) $B \in p(N)$ and B is finite $\Longrightarrow B \notin u$.

Then the set $^*R$ is defined as the set of the equivalence classes of all sequences of real numbers modulo the equivalence relation:

$a \equiv b$, provided $\{j : a_j = b_j\} \in u,$, a and b being the two sequences $\{a_j\}$ and $\{b_j\}$.

Similarly, a given relation is said to hold between elements of $^*R$ if it holds termwise for a set of indices which belongs to the ultrafilter. For example:

$a < b \Longleftrightarrow \{j : a_j < b_j\} \in u.$

R is isomorphic to a subset of $^*R$, since one can identify any real $r \in R$ with the class of sequences $\{r, r, \ldots\}$. It is the axiom of maximality (4) which ensures $^*R$ to be an order field. In particular, thanks to this axiom, a sequences which takes its value in a finite set of numbers is equivalent to one of these numbers, depending on the particular ultrafilter u. This allows one to solve the problem of zero divisors: indeed the fact that $(0, 1, 0, 1, \ldots).(1, 0, 1, 0, \ldots) = (0, 0, 0, \ldots)$ does not imply that there are zero divisors, since axiom (4) ensures that one of the sequences is equal to 0 and other to 1.

That $^*R$ contains new elements with respect to R become evident when one considers the sequence $\{\omega_j = j\} = \{1, 2, 3, \ldots n, \ldots\}$. The equivalence class of this sequence, ω, is larger than any $r \in R, \{j : \omega_j > r\} \in u$, so that what ever $r \in R$, $\omega > r$. It is straightforward that the inverse of ω is infinitesimal.

Hence the set $^*R$ of hyper-real numbers, as it is also called sometime in the literature, is a totally ordered and non-Archimedean field, of which the set R of standard numbers is a subset. $^*R$ contains infinite elements, i.e. numbers A such that $\forall n \in N$, $|A| > n$ (where N refers to the integers). It also contains infinitesimal elements, i.e. numbers ϵ such that $\forall n \neq 0 \in N$, $|\epsilon| < \frac{1}{n}$. A finite element C is also defined formally as: $\exists\, n \in N$, $|C| < n$. Now all hyper-integers $^*N$ (of which N and the set of infinite hyper-integers $^*N_\infty$ are subsets), hyper-rationales $^*Q$, positive or negative numbers, odd or even hyperintegers, etc. may be defined systematically.

An important result is that any finite number a can be spilt up in a unique way as the sum of a standard real number $r \in R$ and an infinitesimal number $\epsilon \in J : a = r + \epsilon$. In other words the set of finite hyper-reals consists of a set of new real numbers (a) clustered infinitesimally closely around each ordinary real r. The set of these additional numbers $\{a\}$ is called monad of r. More generally, one may demonstrate that any hyper-real number A may be decomposed in a unique way as $A = N + r + \epsilon$, where $N \in {^*N}$, $r \in R \cap [0, 1)$ and $\epsilon \in J$.

The real r is said to be the 'standard part' of finite hyper-real a, this function being denoted by $r = st(a)$. This new operation, " take the standard part of " play a crucial role in the theory, since it allows one to solve the contradictions which prevented previous attempts, such as Leibniz's, to be developed rigorously. Indeed, apart from the strict equality "=", one introduces an equivalence relation, "$\approx$", meaning "infinitely close to" defined by $a \approx b \Longleftrightarrow st(a - b) = 0$. Hence the two numbers of the same monad are infinitely close to one another, but not strictly equal. Similarly the derivative of a function will be written in the form $\frac{df}{dx} = st\left\{\left[f\left(x + \epsilon\right) - f\left(x\right)\right]/\epsilon\right\}$, provided the expression is finite and independent of ϵ.

2.4 *Cantor set*

2.4.1 Introduction

A Cantor set is a compact, perfect, totally disconnected, metrisable topological space. In this thesis we consider a Cantor set C that is realized as a (proper) subset of the real line. It is of measure zero if the Lebesgue measure of the set is zero. Otherwise this has a positive measure. Such a set is also said to be a fat Cantor set. Topological dimension of a Cantor set is also zero. Although, both the linear Lebesgue measure and the usual sense of dimension are trivial, a Cantor has the cardinality of the continuum c. To reveal the intricate geometric structure of such a set, nonlinear Hausdorff measure and Hausdorff dimension are generally considered to be most useful. A set C is said to be an s- set if the corresponding Hausdorff measure has a finite non-zero value viz; $0 < H^s(C) < \infty$ [3]. The real number s then denotes the Hausdorff dimension of the Cantor set.

As already noted in introduction, a Cantor set is an example of a self-similar fractal set that arises in various fields of applications. The chaotic attractors of a number of one dimensional maps; such as the logistic maps, turn out to be topologically equivalent to Cantor sets. Cantor set also arises in electrical communications [1], in biological systems [2], and diffusion processes [10]. Recently there have been a lot of interest in developing a framework of analysis on a Cantor like fractal sets [12, 13]. Because of the disconnected nature, methods of ordinary real analysis break down on a Cantor set. Various approaches based on the fractional derivatives [14, 15] and the measure theoretic harmonic analysis [16] have already been considered at length in the literature. Parvate and Gangal [19], for instance, considered the so called staircase functions, having a Cantor function like properties, in their formulation of the analysis. Their approach is based mainly on developing a formalism for replacing the linear Lebesgue measure (variable) viz., $x \in C \subset [0,1]$ by a nonlinear Hausdorff measure theoretic variable, viz., the integral staircase function $S_C^s(x) \approx x^s$ when $x(\approx 0) \in C$ and s is the Hausdorff dimension of C. However, a simpler intuitively appealing approach is still considered to be welcome.

2.4.2 Basic Definitions

A Cantor set C is defined as a countable intersection of finite unions of closed (and bounded) subsets of R. For definiteness, let $C \subset I = [0,1]$. Then, by definition, $C = \bigcap_1^{\infty} F_n = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{p^n} F_{nm}$ where $F_{nm} \subset I$ are closed with $F_{00} = I$. Equivalently, C is also defined as $C = I - \bigcup_{i=1}^{\infty} O_i$ where O_i are open intervals which are deleted recursively from I. Consequently, a Cantor set is often defined as the limit set of an iterated function system (IFS) $f = \{f_i \mid f_i : I \to I,\ i = 1, 2...., p\}$ so that $C = f\ (C)$. For definiteness, we consider binary Cantor sets in which each application of the IFS removes an open interval from a closed subinterval splitting it into two disjoint closed subintervals of the form

$$F = F_0 \cup O \cup F_1. \tag{2.8}$$

The deleted interval O is called the gap and the two closed components are the bridges. As an example let us consider a middle α Cantor set C_α

which arises as the limit set under the IFS

$$f_i(x) = \beta x + i(1-\beta), \quad i = 0, 1 \tag{2.9}$$

where the scale factor β is defined by $\alpha + 2\beta = 1$. Each iteration of the IFS removes an open interval (i.e. a gap) of length proportional to α from a closed subinterval of I, leaving out two bridges of size proportional to β each. The IFS (2.8) satisfies the *open set condition* (OSC) if there exists a non-empty bounded open set S such that $\bigcup_i f_i(S) \subseteq S$. It follows accordingly that $\beta \in (0, \frac{1}{2})$. Since the total length of the disjoint open intervals viz., $\sum_{i=1}^{\infty} \mid O_i \mid = \sum_{i=1}^{\infty} \alpha(2\beta)^{i-1} = 1$, the middle α Cantor set is of measure zero with the Hausdorff dimension $s = \frac{\log 2}{\log \frac{1}{\beta}}$. For latter reference, let us recall that a point $x \in C_\alpha$ has the unique infinite word representation

$$x = (1-\beta)\sum_{0}^{\infty} x_i \beta^i = x_0 x_1 x_2 \cdots, \; x_i \in \{0, 1\}.$$

More generally, when q open intervals each of size α are deleted leaving out p equal closed intervals of size β so that $q\alpha + p\beta = 1$, then the OSC gives $\beta \in (0, \frac{1}{p})$. The length of the deleted open intervals add up to 1 viz., $\Sigma(q\alpha)(p\beta)^{n-1} = 1$. The corresponding measure zero set $C_{\alpha,p}$ has the Hausdorff dimension $\frac{\log p}{\log \frac{1}{\beta}}$.

Returning to the discussion of the binary Cantor set we recall that the set C_α is also a homogeneous and uniform Cantor set. It is homogeneous since the scale factors in each component of the IFS are same. The set is uniform because each deleted open interval also is of constant proportion α of the length of the previous (defining) closed interval.

A positive 1-set $\tilde{C}$, on the other hand, is obtained if the deletion process removes open intervals of variable sizes.

Example 2. *Let at each step we remove α_n portion of the length of each component of the previous closed set F_{n-1} so that $F_{n-1} = F_{n0} \cup O_n \cup F_{n1}$ and $\mid O_n \mid = \alpha_n \mid F_{n-1} \mid, \mid F_{n0} \mid = \mid F_{n1} \mid = \frac{1}{2}(1-\alpha_n) \mid F_{n-1} \mid$. By induction, each of 2^n components of F_n has length $\mid F_{ni} \mid = \frac{1}{2^n}\prod_{0}^{n}(1-\alpha_j)$, $i = 1, 2, \cdots 2^n$. Consequently, $m\ (\tilde{C}) = \lim_{n\to\infty} \mid F_{n-1} \mid = \prod_{0}^{\infty}(1-\alpha_i) > 0$ when $\Sigma\ \alpha_n < \infty$.*

Example 3. *Suppose at the nth step an open interval of length $\frac{\delta}{3^n}$, $(0 < \delta < 1)$ is removed from each of the 2^n components of F_n. The length of each component of F_n is $\frac{1}{2^n}\ (1 - \frac{\delta}{3} - \cdots \frac{2^{n-1}\delta}{3^n})$. The sum of the lengths of all the open intervals removed is $\Sigma\ \frac{2^n\delta}{3^{n+1}} = \delta$, so that $m\ (\tilde{C}) = 1 - \delta$.*

In both the above examples, $0 < H^s(\tilde{C}) = m(\tilde{C}) = l < \infty$ when $s = 1$. Even as the Cantor set $\tilde{C}$ is nowhere dense in I, it has a non-zero measure because of the fact that the defining iteration process now removes the open interval at a slower rate in comparison to the middle α set C_α. In fact, the relative difference of the lengths of the deleted open middle α and α_n intervals $O_{\alpha n}$ and $\tilde{O}_{\alpha n}$ respectively is given by

$$\begin{aligned} \Big| \mid O_{\alpha n} \mid - \mid \tilde{O}_{\alpha n} \mid \Big| &= \left| \alpha\beta^n - \frac{1}{2^n}\alpha_n \prod_{1}^{n}(1-\alpha_i) \right| \\ &= \left| 1 - \frac{\alpha_n}{\alpha}\prod_{1}^{n}\left(\frac{1-\alpha_i}{1-\alpha}\right) \right| |O_{\alpha n}| \\ &\geq |1-\gamma||O_{\alpha n}| \end{aligned}$$

where $\frac{\alpha_n}{\alpha}\prod_1^n(\frac{1-\alpha_i}{1-\alpha}) \to \gamma$ for $n \to \infty$. Note that the above lower bound exists, otherwise $\tilde{C}$ would have been a set of measure zero. For the modified middle $\frac{1}{3}$rd set (Example 2) one has the exact equality

$$\Big| \mid O_{\frac{1}{3}n} \mid - \mid \tilde{O}_\delta \mid \Big| = (1-\delta) \mid O_{\frac{1}{3}n} \mid .$$

The emergence of the positive measure of $\tilde{C}$ can, therefore, be explained in a dynamical sense provided the said set $\tilde{C}$ is seen as being evolved from a given zero measure set because of a possible *principle* allowing for a deformation of scales in the deletion process. As is evident the relative scaling of the lengths of infinitesimal elements of the deleted open intervals $\tilde{O}_{\alpha n}$ over the corresponding α interval $O_{\alpha n}$, indeed captures the origin of the positive measure even in the totally disconnected perfect set $\tilde{C}$. In the following we offer a new ultrametric explanation of the growth of the positive measure of $\tilde{C}$ over C_α [chapter 7]. The class of ultrametrics that we consider is not only *scale invariant* (in the sense of a power law), but also *reparametrisation invariant* (that is, the invariance under a reparatrisation invariance of the form $X(t) \to \tilde{X}(t) = X(f(t))$ where the otherwise arbitrary function f satisfies the conditions $f' > 0$, along with the boundary condition $f(0) = 0,\ f(1) = 1$). As will be explained later, the scale variation can, therefore, be interpreted as a reflection of the underlying reparametrisation invariance of the valuation. Incidentally, we note that for a measure 1 set the functions defining an IFS may not have a closed form [45].

2.4.3 Thickness

The measure of thickness of a Cantor set has various applications in number theory and dynamical systems [39, 40]. To recall the definition, let $F_i = I - \bigcup_{l=0}^{i} O_l$ form a defining sequence of the Cantor set C. The 2^i components of F_i are the closed intervals F_{ij}, $j = 1, 2, \cdots 2^i$, which are the bridges. The deleted intervals O_i are the gaps of C. Let O_{i_k} denote the open deleted subinterval of a bridge F_{ij} of F_i which divides F_{ij} into two smaller bridges F_{ij}^L and F_{ij}^R of F_i. Let

$$\tau(F_i) = \inf_j \left\{ \frac{| F_{ij}^L |}{| O_{i_k} |}, \frac{| F_{ij}^R |}{| O_{i_k} |} \right\}.$$

The thickness $\tau(C)$ is defined by $\tau(C) = \sup_i \tau\ (F_i)$, where sup is evaluated over the defining sequences of C. For a set A containing an interval, $\tau(A) = \infty$, by definition.

For the middle α- Cantor set C_α, it follows that

$$\frac{| F_{ij}^L |}{| O_{i_k} |} = \frac{| F_{ij}^R |}{| O_{i_k} |} = \frac{\beta \mid F_{ij} \mid}{\alpha \mid F_{ij} \mid} = \frac{\beta}{\alpha}$$

so that $\tau(C) = \frac{\beta}{\alpha}$.

For a positive measure set $\tilde{C}$, on the other hand, one has $\frac{|F_{ij}^L|}{|O_{i_k}|} = \frac{|F_{ij}^R|}{|O_{i_k}|} = \frac{1-\alpha_n}{2\alpha_n}$, leading to $\tau(\tilde{C}) = \infty$, as expected.

2.4.4 Cantor Function

A Cantor function [7] is a nonconstant and non-decreasing continuous function $\phi : [0, 1] \to [0, 1]$ such that $\phi'(x) = 0$ a.e. with points of nondifferentiability x lying, for instance, in the Cantor set C_α .

To construct ϕ explicitly, let $\phi(0) = 0$, $\phi(1) = 1$. Assign $\phi(x)$ a constant value $\phi(x) = i2^{-n}$, $i = 1, 2, ..., 2^n - 1$ on each of the deleted open intervals (including the end points of the deleted interval) of C_α. Next, let $x \in C_\alpha$. Then, at the nth iteration, x belongs to the interior of exactly one of the 2^n remaining closed intervals each of length β^n. Let $[a_n, b_n]$ be one such intervals. Then $b_n - a_n = \beta^n$. Moreover, $\phi(b_n) - \phi(a_n) = 2^{-n}$. At the next iteration, assuming $x \in [a_{n+1}, b_{n+1}]$, ($a_n = a_{n+1}$), say, we have $\phi(a_n) \leq \phi(a_{n+1}) < \phi(b_{n+1}) \leq \phi(b_n)$. Define $\phi(x) = \lim_{n\to\infty} \phi(a_n) = \lim_{n\to\infty} \phi(b_n)$. Then $\phi : [0, 1] \to [0, 1]$ is a continuous, non-decreasing function. Also $\phi'(x) = 0$ for $x \in I \backslash C_\alpha$ when it is not differentiable at any $x \in C_\alpha$. (c.f., [24, 25]).

2.4.5 Ultrametric

The topology of a Cantor set is equivalent to an ultrametric topology. As already noted, a point $x \in C_\alpha$ has the unique infinite word representation

$$x = (1 - \beta)\sum_0^\infty x_i\beta^i = x_0x_1x_2\cdots, \quad x_i \in \{0, 1\}.$$

Let $L(x, y) = n$ such that $x_0 = y_0, \cdots, x_{n-1} = y_{n-1}$, $x_n \neq y_n$, $x, y \in C_\alpha$. The ultrametric $\tilde{d}_u$ is defined by $\tilde{d}_u(x, y) = p^{-L(x,y)}$ for any $p > 1$. This ultrametric is equivalent to the usual metric

$$C_1\tilde{d}_u(x, y) \leq d(x, y) \leq C_2\tilde{d}_u(x, y)$$

for two positive constants C_1 and C_2, where $d(x, y)$ is the usual metric.

The Cantor set thus consists of towers of closed balls (intervals) with countable intersection property. Further the fundamental neighbourhood system of any point consists of clopen balls.

Chapter 3

ULTRAMETRIC CANTOR SET

3.1 Introduction

As already mentioned, a main objective of the present work is to develop a non-archimedean framework [20] of a scale invariant analysis which will be naturally relevant on a Cantor set [24, 25, 26]. Since a Cantor set is compact, perfect, totally disconnected set, the conventional framework of real analysis is known to break down. For definiteness, we consider the Cantor set C to be a subset of the unit interval $I = [0, 1]$. We introduce a non-archimedean absolute value on C exploiting a concept of relative infinitesimals which correspond to the arbitrarily small elements $\tilde{x}$ of $I \backslash C$ satisfying $0 < \tilde{x} < \epsilon < x$, $\epsilon \to 0^+$ (together with an inversion rule) relative to the scale ϵ for a given $x \in C$ close to 0. As a consequence, increments among infinitesimals as well as between an infinitesimal and a (real) point of the Cantor set are accomplished by *inversions*, rather than by *translations* that is generally considered in standard real analysis. One of the main results in this chapter is the relationship of the nontrivial valuation with a Cantor function. Indeed, it is proved that the valuation is indeed given by an appropriate Cantor function. In ref.[24], we presented the details of the construction in the light of the middle third Cantor set. In ref.[25], the analysis is extended to a more general class of homogenous

Cantor sets.

We first introduce the definitions of relative infinitesimals, scale invariant infinitesimals and the associated class of inequivalent ultrametrics [24, 25, 26] and discuss the salient properties, mainly, in the context of a class of homogeneous Cantor sets. Next, we discuss and expose the relation of valuation with a Cantor function that arise naturally in the present context. At the next step, we present the arguments extending the ultrametric structure on the scale invariant infinitesimals over the whole Cantor set C. In the next subsection, we present the results on *valued measure* that can be defined on C. We show the valued measure of C gives rise to directly the finite Hausdorff measure of C. The nontrivial convergence of sequences of the form ϵ^{n-nl}, $0 < \epsilon,\ l < 1$ are treated subsequently. The usual limit 0 is replaced by the constant l in the present ultrametrics. This establishes the *metric as well as the topological inequivalence of these scale invariant ultrametrics.* The final subsection contains a discussion of differentiability that could be defined rigorously in the present ultrametric framework.

3.2 Non-archimedean Analysis: Ultrametrics

3.2.1 Absolute Value

Definition 5. *[26] Given an arbitrarily small* $x \in C - \{0\}$ *(in the sense that* $x \to 0^+$ *on* $C - \{0\}$*),* $\exists\ \epsilon \in I$ *and* $\epsilon < x$ *and an open interval* $\tilde{I} \subset (0, \epsilon)$ *such that* $\tilde{I} \cap C = \Phi$*, the null set. This follows from the total disconnectedness of* C*. An element* $\tilde{x}$ *in* $\tilde{I}$ *satisfying* $0 < \tilde{x} < \epsilon < x$ *and* *the* inversion *rule*

$$\frac{\tilde{x}}{\epsilon} = \lambda(\epsilon)\frac{\epsilon}{x} \tag{3.1}$$

for a real constant $\lambda = \lambda(\epsilon)$ $(0 \ll \lambda(\epsilon) \leq 1)$ *is called a* relative infinitesimal *relative to the* scale ϵ. *The infinitesimal gap* $O_{\text{inf}} \subset \tilde{I}$ *is, by definition, the set of these relative infinitesimals satisfying the inversion rule (3.1), as* $\epsilon \to 0^+$, *in an asymptotic sense.*

Definition 6. *The non-empty set* $\tilde{O}_{\text{inf}} = \lim_{\epsilon \to 0^+} \{ \frac{\tilde{x}}{\epsilon} \}$, *is called the set of (positive) scale free (invariant) infinitesimals.*

Remark 1. Any finite (nonzero) $x \in C$ may be said to carry the trivial scale of 1 (unit, say). Nontrivial scales ϵ are said to emerge when $x \to 0^+$. Idea is that as x gets smaller and smaller in the sense that x becomes smaller than any preassigend positive number δ, and so becomes indistinguishable from 0 from the point of view of the unit scale 1, the Definition 5 now gives us a mechanism of zooming out the classical $\delta-$neighbourhood of 0 and then identifying it as an ultrametric neighbourhood (see below). The consideration of asymptotic limit as $\epsilon \to 0^+$ for a fixed but, nevertheless, arbitrarily small $x > 0$ allows one to erase traces of possible spurious (trivial) scales and also to probe and analyze nontrivial effects, if any, over and above the standard classical analytic results (cf, Remark 4(1)). Note that the framework of the classical analysis (calculus) does not naturally accommodate a scale. Nontrivial scales, however, arise in the context of a Cantor set. The new class of nontrivial scales now provides a new tool to explore the rich analytic and geometric structures of such a Cantor set.

Definition 7. *To each $x \in C$, $\exists$ an arbitrarily small $\epsilon > 0$ and a (relative) infinitesimal neighbourhood $\mathbf{I}_\epsilon(x) = (x - \epsilon, x + \epsilon) \subset I$, $x \neq 0, 1$ such that $C \cap \mathbf{I}_\epsilon(x) = \{x\}$. Points in $\mathbf{I}_\epsilon(x)$ are called the relative infinitesimal neighbours in I of $x \in C$.*

Remark 2. For each choice of x and ϵ, we have a unique $\tilde{x}$ for a given $\lambda \in (0, 1)$. Consequently, by varying λ in an open subinterval of (0,1), we get an open interval of relative infinitesimals in the interval $(0, \epsilon)$, all of which are related to x by the inversion formula. In the limit $\epsilon \to 0^+$, $O_{\text{inf}} = \Phi$, in the usual topology. However, the corresponding set of *scale invariant infinitesimals* $\tilde{O}_{\text{inf}} = \lim_{\epsilon \to 0^+} \{\tilde{X} \mid \tilde{X} = \frac{\tilde{x}}{\epsilon} \approx \mu\epsilon^\alpha \pm o(\epsilon^\beta)\}$ where μ is a constant and $1 > \beta > \alpha \geq 0$, may be a non-null subset of (0,1) (for instance, when $\alpha = 0$, in particular) (for an explanation of the asymptotic expansion of $\tilde{X}$ see Remarks 4.1 and 4.3). Notice that constants α, β and μ may, however, depend on λ. Notice also that the infinitesimal gap $O_{\text{inf}} = O(x, \epsilon, \lambda)$ depends on ϵ, but apparently also on the arbitrarily small element x of the Cantor set along with the parameter λ appearing in the inversion law. But x and ϵ are very closely related, so that O_{inf} essentially depends only on ϵ and $\lambda(\epsilon)$.

For a point x from a Cantor set C, it is natural to assume that the scale ϵ is determined by the privileged scale of the Cantor set. Two relative infinitesimals $\tilde{x}$ and $\tilde{y}$ must necessarily satisfy the condition $0 < \tilde{x} \leq \tilde{y} < \tilde{x} + \tilde{y} < \epsilon$. As indicated already, the inversion rule maps an open interval of (relative) infinitesimals of size determined by the parameter λ to an arbitrarily small element x of C.

Lemma 2. $\mathbf{I}_\epsilon(x) = x + \mathbf{I}_0$, $\mathbf{I}_0 = \mathbf{I}_0^+ \cup \mathbf{I}_0^-$, $\mathbf{I}_0^- = \{ -\tilde{x} \mid \tilde{x} \in \mathbf{I}_o^+ \}$ *and* $\mathbf{I}_0^+ \risingdotseq O_{\text{inf}}$. *Further* $\exists$ *a bijection between* $\mathbf{I}_0^+$ *and* $(0,1)$ *for a given* ϵ.

Proof. Let $y \in \mathbf{I}_\epsilon(x)$. Then $y = x \pm \tilde{x}$, $0 < \tilde{x} < \epsilon < z$, so that $\tilde{x} = \lambda \frac{\epsilon^2}{z}$ for a fixed z and a variable λ. Thus $y \in x + \mathbf{I}_0$. The other inclusion also follows similarly. Finally, the bijection is given by the mapping $\tilde{x} \to \frac{\tilde{x}}{\epsilon}$. ■

Definition 8. *[26] Given* $\tilde{x} \in \mathbf{I}_o$, *we define a scale free absolute value of* $\tilde{x}$ *by* $v : \mathbf{I}_0 \to [0,1]$ *where*

$$v(\tilde{x}) = \lim_{\epsilon \to 0} \log_{\epsilon^{-1}} \frac{\epsilon}{|\tilde{x}|} \tag{3.2}$$

and $v(0) = 0$.

Lemma 3. v *is a non-archimedean semi-norm over* $\mathbf{I}_0$.

Notation 1. *By semi-norm we mean (i)* $v(\tilde{x}) > 0$, $\tilde{x} \neq 0$. *(ii)* $v(-\tilde{x}) = v(\tilde{x})$. *(iii)* $v(\tilde{x} + \tilde{y}) \leq \max\{ v(\tilde{x}), v(\tilde{y}) \}$. *Property (iii) is called the strong (ultrametric) triangle inequality [20]. Note that this definition of seminorm on a set differs from the seminorm on a vector space. However, this suffices our purpose here.*

Proof. The case (i) and (ii) follow from the definition. To prove (iii) let $0 < \tilde{x} \leq \tilde{y} < \tilde{x} + \tilde{y} < \epsilon$. Then $v(\tilde{y}) \leq v(\tilde{x})$ and hence $v(\tilde{x} + \tilde{y}) = \lim_{\epsilon \to 0} \log_{\epsilon^{-1}} \frac{\epsilon}{\tilde{x}+\tilde{y}} \leq \lim_{\epsilon \to 0} \log_{\epsilon^{-1}} \frac{\epsilon}{\tilde{x}} = v(\tilde{x}) = \max\{ v(\tilde{x}), v(\tilde{y}) \}$. Moreover, $v(\tilde{x} - \tilde{y}) = v(\tilde{x} + (-\tilde{y})) \leq \max\{ v(\tilde{x}), v(\tilde{y}) \}$. ■

Remark 3. As remarked already, the set of infinitesimals $O_{\text{inf}} = \Phi$ when $\epsilon \to 0$. However, the corresponding asymptotic expression for the scale free (invariant) infinitesimals is nontrivial, in the sense that the associated valuations (3.2) can be shown to exist as finite real numbers. This also gives an explicit construction of infinitesimals and the associated absolute value.

Choosing $\epsilon = \beta^r$, the Cantor set scale factor, the scale free infinitesimal gaps can be identified as $\tilde{O}^m_{inf} = (0, \beta^m)$ when $\epsilon \to 0$ is realized as $n \to \infty$, $r = n + m$, $m = 1, 2, \ldots$. Assign nonzero constant valuation $v(\tilde{x}_m) = \alpha_m$ $\forall\ \tilde{X}_m = \tilde{x}_m/\epsilon \in \tilde{O}^m_{\rm inf}$. The set of all possible scale free infinitesimals $\cup \tilde{O}^m_{inf} \subset (0, 1)$ is now realized as nested clopen circles $S_m : \{\tilde{x}_m : \mathrm{v}(\tilde{\mathrm{x}}_{\mathrm{m}}) = \alpha_m\}$. The ordinary 0 of C is replaced by this set of scale free infinitesimals $0 \to \ \mathbf{0} = O_{\rm inf}/ \sim = \{0, \cup S_m\}$, $\mathbf{0}$ being the equivalence class under the equivalence relation $\sim$, where $x \sim y$ means $v(x) = \mathrm{v(y)}$. The existence of $\tilde{x}$ could also be conceived dynamically as a computational model [24, 25, 29], in which a number, for instance, 0 is identified as an interval $[-\epsilon, \epsilon]$ at an accuracy level determined by $\epsilon = \beta^n$.

Remark 4. 1. The concept of infinitesimals and the associated absolute value considered here become significant only in a limiting problem (or process), which is reflected in the explicit presence of "$\lim\limits_{\epsilon \to 0}$" in the relevant definitions. Recall that for a continuous real valued function $f(x)$, the statement $\lim\limits_{x \to 0} f(x) = f(0)$, means that $x \to 0$ essentially is $x = 0$. This may be considered to be a *passive* evaluation (interpretation) of limit. The present approach is *dynamic*, in the sense that it offers not only a more refined evaluation of the limit, but also provides a clue how one may induce new (nonlinear) structures (ingredients) in the limiting (asymptotic) process. The inversion rule (3.1) is one such nonlinear structure which may act nontrivially as one investigates more carefully the *motion* of a real variable x (and hence of the associated scale $\epsilon < x$) as it goes to 0 more and more accurately. Notice that at any "instant", elements defined by inequalities $0 < \tilde{x} < \epsilon < x$ in the limiting process, are well defined; relative infinitesimals are *meaningful* only in that *dynamic* sense (classically, these are all zero, as x itself *is* zero). Scale invariant infinitesimals $\tilde{X}$, however, may or may not be zero classically. $\tilde{X} = \mu$ $(\neq 0)$, a constant, for instance, is nonzero even when x and ϵ go to zero. On the other hand, $\tilde{X} = \epsilon^{\alpha}$, $0 < \alpha < 1$, of course, vanish classically, but as shown below, are nontrivial in the present formalism. As a consequence, relative (and scale invariant) infinitesimals may be said to *exist* even as real numbers in this dynamic sense. The accompanying metric $|.|_u$, however, is an ultrametric. Notice that the 'limit' above refers to the standard limit on R with usual metric.

2. However, a genuine (nontrivial) scale free infinitesimal $\tilde{X}$ can not be a constant. Let $\tilde{x}_0 = \mu\epsilon$, $0 < \mu < 1$, μ being a constant. Then $v(x_0)$ $= \lim_{\epsilon \to 0} \log_\epsilon \mu = 0$, so that $\tilde{x}_0$ is essentially the trivial infinitesimal 0 (more precisely, such a relative infinitesimal belongs to the equivalence class of 0).

3. The scale free infinitesimals of the form $\tilde{X}_m \approx \epsilon^{\alpha_m}$ go to 0 at a slower rate compared to the linear motion of the scale ϵ. The associated nontrivial absolute value $v(\tilde{x}_m) = \alpha_m$ essentially quantifies this decelerated motion.

Definition 9. *The set* $B_r(a) = \{ x \mid v(x-a) < r \}$ *is called an open ball in* $\mathbf{I}_0$. *The set* $\bar{B}_r(a) = \{ x \mid v(x-a) \leq r \}$ *is a closed ball in* $\mathbf{I}_0$.

Lemma 4. (*i*) *Every open ball is closed and vice-versa (clopen ball)* (*ii*) *every point $b \in B_r(a)$ is a centre of $B_r(a)$. (iii) Any two balls in $\mathbf{I}_0$ are either disjoint or one is contained in another. (iv) $\mathbf{I}_0$ is the union of at most a countable family of clopen balls.*

Proof follows directly from the ultrametric inequality and the fact that $\mathbf{I}_0$ is an open set. It also follows that in the topology determined by the semi-norm, O_{inf} is a totally disconnected set. We next show that a closed ball in O_{inf} is compact.

Lemma 5. *[24] A closed (clopen) ball in O_{inf} is both complete and compact.*

Proof. The proof follows from the following observation. Given $\epsilon > 0$, consider a closed interval $[a, b] \subset O_{\text{inf}}$ (in the usual topology) such that $0 < a < b < \epsilon$. The valuation v realizes this closed interval as an ultrametric (sub)space U of O_{inf} which is an union of atmost a countable family of disjoint clopen balls (Lemma 4). Completeness now follows from the standard ultrametric properties: a sequence $\{x_n\} \subset U$ is Cauchy $\Leftrightarrow$ $v(x_n - x_m) \to 0 \Leftrightarrow v(x_{n+1} - x_n) \to 0 \Rightarrow \exists\, N > 0$ such that $v(x_{n+1}) = v(x_n)$ for $n \geq N$. Noting that for a nonzero infinitesimal x_n, the associated valuation is nonzero, it follows that $x_n \to x_N \in U$ in the ultrametric in the sense that $v(x_n) = v(x_N)$ as $n \to \infty$. Compactness is a consequence of the fact that any sequence in U has a convergent subsequence. Indeed, a sequence $\{x_n\} \subset U$ can not diverge (and can atmost be oscillating) in the given ultrametric since $0 \leq v(x_n) \leq 1$. ■

As a result, O_{inf} is the union of countable family of disjoint closed (clopen) balls, in each of which $v(\tilde{x})$ can assume a constant value. With this assumption, $v : O_{\text{inf}} \to [0,1]$ is discretely valued. Next, to restore the product rule viz : $v(\tilde{x}\tilde{y}) = v(\tilde{x}).v(\tilde{y})$, we note that given $\tilde{x}$ and ϵ, $0 < \tilde{x} < \epsilon$, there exist $0 < \sigma(\epsilon) < 1$ and $a : O_{\text{inf}} \to R$ such that

$$\frac{\tilde{x}}{\epsilon} = \epsilon^{\sigma^{a(\tilde{x})}}.\epsilon^{t(\tilde{x},\epsilon)} \tag{3.3}$$

so that $v(\tilde{x}) = \sigma^{a(\tilde{x})}$ for an indeterminate vanishingly small $t : O_{\text{inf}} \to R$ i.e. $t(\tilde{x},\epsilon) \to 0$ as $\epsilon \to 0^+$. For the given Cantor set C there is a unique (natural) choice of σ dictated by the scale factors of C viz : $\sigma = p^{-n} = r^{-ns}$, $s = \frac{\log p}{\log r}$, for some natural number n.

The mapping $a(\tilde{x})$ is a valuation and satisfies (i) $a(\tilde{x}\tilde{y}) = a(\tilde{x}) + a(\tilde{y})$, (ii) $a(\tilde{x}+\tilde{y}) \geq \min\{\ a(\tilde{x}), a(\tilde{y})\ \}$. Now discreteness of $v(\tilde{x})$ implies range $\{\ a(\tilde{x})\ \} = \{\ a_n \mid n \in Z^+\ \}$. Again for a given scale ϵ, O_{inf} is identified with a copy of $(0,1)$ (by Lemma 1) which is clopen in the semi-norm. Thus O_{inf} is covered by a finite number of disjoint clopen balls $B(\tilde{x}_n)$ (say), $\tilde{x}_n \in O_{\text{inf}}$. Because of finiteness, values of $a(\tilde{x})$ on each of the balls can be ordered $0 = a_0 < a_1 < \cdots < a_n = s_0$ (say). Let $v_0 = v(B(\tilde{x}_n)) = \sigma^{s_0}$. Then we can write $v_i = v(B(\tilde{x}_i)) = \alpha_i v_0 = \alpha_i\ \sigma^{s_0}$ for an ascending sequence $\alpha_i > 0$, $i = 0, 1, \ldots, n$. We also note that $a_0 = 0$ corresponding to the unit $\tilde{x}_u$ so that $v\,(\tilde{x}_u) = 1$.

From equation (3.3) we have $\frac{\tilde{x}_u}{\epsilon} = \epsilon^{1+t(\tilde{x},\epsilon)}$ and so it follows that $\tilde{x} \in O_{\text{inf}}$ will admit a factorization

$$\frac{\tilde{x}}{\epsilon} = \frac{\tilde{x}_i}{\epsilon}.\frac{\tilde{x}_u}{\epsilon^2} \tag{3.4}$$

since $\tilde{x} \in B(\tilde{x}_i)$ for some i.

Thus

$$\tilde{x} = \tilde{x}_i\ (1 + \tilde{x}_\epsilon) \tag{3.5}$$

where $\tilde{x}_u = \epsilon^2(1 + \tilde{x}_\epsilon)$, $\tilde{x}_\epsilon \in O_{\text{inf}}$, so that $v(\tilde{x}) = v(\tilde{x}_i)$, as $v(\tilde{x}_\epsilon) < 1$.

We thus have,

Lemma 6. *[25] v is a discretely valued non-archimedean absolute value on O_{inf} Any infinitesimal $\tilde{x} \in O_{\text{inf}}$ have the decomposition given by equation* (3.5) *so that v has the canonical form*

$$v(\tilde{x}) = \alpha_i \ \sigma^{s_0}, \ \tilde{x} \in B(x_i). \tag{3.6}$$

Definition 10. *The infinitesimals given by equation (3.5) and having absolute value* (3.6) *are called valued infinitesimals.*

Theorem 1. *[26] The non-archimedean infinitesimal absolute value (3.2) is given by a Cantor function associated with the Cantor set containing the relative infinitesimals. Conversely, given a Cantor function, there exists a class of infinitesimals, determined by the Cantor function, that live in an extended ultrametric neighbourhood of 0, denoted* **0**.

Proof. The (infinitesimal) absolute value (valuation)* $v(\tilde{x})$, as a mapping from O_{inf} to $I \subset R$, is continuous. The equation (3.6), however, defines $v(\cdot)$ only for points in the clopen balls $B(a_i)$, $i = 1, 2, \cdots$. The definition can be extended continuously over the entire set O_{inf} for points outside the clopen balls. Indeed, let for a given primary scale ϵ, σ_i be the secondary scale. Let also that $y \in O_{\text{inf}} / \cup B(a_i)$. Then there exist $y_i \in B(a_i)$, $y_{i+1} \in B(a_{i+1})$ such that $y_i < y < y_{i+1}$, and $v(y_{i+1}) - v(y_i) = (\alpha_{i+1} - \alpha_i)\sigma_i$ (to be precise, the selection of the sequence y_i actually requires one to invoke the axiom of choice). Clearly, the sequence

*In algebraic number theory valuation means usually the exponent $a(\tilde{x})$ in (3.3). We however often use the word valuation to denote non-archimedean absolute value as well.

$v(y_{i+1})$ is increasing and $v(y_i)$ is decreasing. Thus, $v(y) := \lim v(y_i)$, as $i \to \infty$. We have thus proved that the scale invariant valuation $v(\tilde{x})$ is indeed given by a Cantor function. Conversely, given a Cantor function $\phi(x)$, $x \in [0,1]$, one can define a set of infinitesimals by the asymptotic formula $\tilde{x} \approx \epsilon\epsilon^{\phi(\tilde{x}/\epsilon)}$ as $\epsilon \to 0$ that is assumed to live in a nontrivial neighbourhood of 0. ■

With this class of valuations, the seminorm now extends to a non-archimedean absolute value, satisfying also the product rule (iv) $v(\tilde{x}\ \tilde{y}) = v(\tilde{x}).v(\ \tilde{y})$.

We now make use these valued infinitesimals to define a non-trivial absolute value on C in the following steps [24, 25].

(i) Given $x \in C$ define a set of multiplicative neighbours of x which are induced by the valued infinitesimals $\tilde{x}_i \in \mathbf{I}_0^+$ by

$$X_{\pm}^i = x.\ x^{\mp v(\tilde{x}_i)} \tag{3.7}$$

where $v(\tilde{x}_i) = \alpha_i \sigma^{s_0}$ and $\alpha_i = \alpha_i(x)$ may now depend on x. We note that the non-archimedean topology induced by v makes the infinitesimal neighbourhood of 0^+ in I totally disconnected. Equation (3.7) thus introduces a finer infinitesimal subdivisions in the neighbourhood of $x \in C$.

(ii) We define the new absolute value of $x \in C$ by

$$\| x \| = \inf \log_{x^{-1}} \frac{X_+^i}{x} = \inf \log_{x^{-1}} \frac{x}{X_-^i} \tag{3.8}$$

so that $\| x \| = \sigma^s$ where $\sigma^s = \inf \alpha_i \sigma^{s_0}$ and the infimum is over all i. It thus follows that

Corollary 1. *$\|\ .\ \| : C \to R_+$ is a non-archimedean absolute value on a Cantor set C.*

We now define [24, 25] the valued measure $\mu_v : C \to R_+$ by

(a) $\mu_v\,(\Phi) = 0$, Φ the null set.

(b) $\mu_v[(0, x)] = ||x||$, when $x \in C$.

(c) For any $E \subset C$, on the other hand, we have $\mu_v(E) = \lim_{\delta \to 0} \inf \sum_i \{d_{\text{na}}\,(I_i)\}$, where $I_i \in \tilde{I}_\delta$ and the infimum is over all countable δ- covers $\tilde{I}_\delta$ of E by clopen balls. Moreover, $d_{\text{na}}(I_i)$=the non-archimedean diameter of $I_i = \sup\{||x - y|| : x, y \in I_i\}$. Denoting the diameter in the usual (Euclidean) sense by $d(I_i)$, one notes that $d_{\text{na}}(I_i) \leq \{d(I_i)\}^s$, since $x, y \in C$ and $|x - y| = d$, imply $||x - y|| = \epsilon^s \leq d^s$, as the scale ϵ satisfies, by definition, $\epsilon \leq d \leq \delta$.

Thus μ_v is a metric (Lebesgue outer) measure on C realized as a non-archimedean ultrametric space. Now to compare this with the Hausdorff s measure, we first note that $\mu_v[E] \leq \mu_s[E]$ since $d_{\text{na}}(I_i) \leq \{d(I_i)\}^s$ for a given cover of (Euclidean) size ϵ. Next, for a cover of clopen balls of sizes ϵ_i, we have $\sum_i \{d_{\text{na}}(I_i)\} = \sum_i \epsilon_i^s$. For the Hausdorff measure, on the other hand, covers by any arbitrary sets are considered. Using the monotonicity of measures it follows that

$$\inf \sum_i \{(d(I_i))^s\} \leq \inf \sum_i \{d_{\text{na}}(I_i)\} \tag{3.9}$$

so that letting $\epsilon \to 0$ we have $\mu_v[E] \geq \mu_s[E]$. Hence

$$\mu_v[E] = \mu_s[E] \tag{3.10}$$

for any subset E of C. Finally, for s = dimension of C, $\mu_s[C]$ is finite and hence the valued measure of C is also finite. Notice that the valued measure selects *naturally* the finite Hausdorff measure of the Cantor set, when s is it's Hausdorff dimension.

3.2.2 Topological Inequivalence

The metric properties of the present ultrametric are indeed distinct from the natural ultrametric (c.f., [24, 25]), since the Lebesgue measure of C in the natural ultrametric is zero, but in the present case, the corresponding valued measure equals the Hausdorff measure. More importantly, topologies induced by the two ultrametrics are also different, as it is shown in the following example [26].

Example 4. *The sequence $\epsilon_n = \epsilon^{n-nl},\ 0 < \epsilon < 1,\ 0 < l < 1$ converges to 0 in the usual metric (ultrametric), but converges to l in the present ultrametric. For a sufficiently large n, choose ϵ^n as non-trivial scale factor and then relative infinitesimals are $\tilde{\epsilon}_n = \lambda^{-1}\epsilon^{n+nl},\ 0 \ll \lambda < 1$. Then, letting the secondary scale $\epsilon \to 0$, we have $v(\tilde{\epsilon}_n) = \lim \log_{\epsilon^{-n}}(\epsilon^n/\tilde{\epsilon}_n) = l$ and hence $||\epsilon_n|| = l$, by equation (3.8), for a sufficiently large n. Thus, $\{\epsilon_n\} \to l$ in the ultrametric $||.||$.*

Letting $\epsilon = \tilde{\epsilon}^m$, the sequence ϵ^{n-nl} is replaced by $\tilde{\epsilon}^{N-Nl},\ N = nm$, so that the limit $\epsilon \to 0$ of the secondary scale is well defined, since it is realized as $m \to \infty$.

Note, however, that the sequence $\{\epsilon^n\}$ converges to 0, even in $||.||$. For a sufficiently large but fixed n, we choose ϵ^{n+1} as the scale factor, so that $\tilde{\epsilon}_n = \lambda^{-1}\epsilon^{n+2}$, are relative infinitesimals and $v(\tilde{\epsilon}_n) = \frac{1}{n+1}$. More generally, *for scales* ϵ^{n+r}, r *being a nonnegative real, we have* $v(\tilde{\epsilon}_n) = \frac{r}{n+r}$. *Thus,* $||\epsilon^n|| = \inf_r \frac{r}{n+r} = 0$.

Incidentally, by letting $\bar{\epsilon} = \epsilon^{1-l}$, *one may like to conclude that* $||\bar{\epsilon}^n|| = 0$, *which would contradict our original claim that* $||\epsilon^{n-nl}|| = l$. *But this demonstrates the basic fact that* $||\epsilon^n|| = 0$ *since the sequence* ϵ^n *does not have any* natural (nontrivial) scale *other than* ϵ^n *itself. The sequence of the form* ϵ^{n-nl} *has, however, access to the natural scale* ϵ^n *and hence affords to have a nontrivial limiting ultrametric value* l *when the ultrametric absolute value is evaluated using the natural scale. Clearly, the limiting value depends on the choice of a nontrivial scale. For a given choice of scale the limit of course is unique.*

This example also gives an alternative proof that the metric $||.||$ is really an ultrametric. This follows because of the eventual constancy of a converging sequence (to a non zero limit) under an ultrametric (c.f., page 22).

3.2.3 Differentiability

To discuss the formalism of the Calculus on C we change the notations of section 3.2 a little. Let X denote a valued infinitesimal while an arbitrarily small real $x \in I$ denote the scale ϵ. The set of infinitesimals is covered by n clopen balls B_n in each of which v is constant. Let

$$\tilde{v}_n(x) = v(\ X_n(x)\ ) = \log_{x^{-1}} \frac{x}{X_n} = \alpha_n\ x^{s_0} \tag{3.11}$$

so that $X_n = x.\ x^{\tilde{v}_n(x)} \in B_n$. For each x, $\tilde{v}_n$ is constant on B_n.

Definition 11. *A function $f : C \to R$ is said to have the limit $l \in R$ as x approaches $x_0 \in C$ on C if given $\epsilon > 0, \exists\ \delta > 0$ s.t. $|f(x) - l| < \epsilon$ $\forall\ 0 < \|x - x_0\| < \delta$.*

Definition 12. *[25] A function $f : C \to I$ is said to be differentiable at $x_0 \in C$ if given $\epsilon > 0$, $\exists$ a finite l and $\delta > 0$ such that*

$$\left| \frac{| f(x) - f(x_0) |}{\| x - x_0 \|} - l \right| < \epsilon \tag{3.12}$$

when $0 <\| x - x_0 \|< \delta$ and we write $f'(x_0) = l$.

Now $\| x - x_0 \| = \inf \tilde{v}_n(x - x_0) = \log_{x_0^{-1}} \frac{x_0}{X}$, where the valued infinitesimal X ($\propto (x - x_0)) \in \tilde{B}$, an open sub-interval of $[0, 1]$ in the usual topology and $\tilde{B}$ is the ball which corresponds to the infimum of $\tilde{v}_n$. Further $f(x) - f(x_0) = (\log x_0)^{-1} \tilde{f}(X)$, since $x = x_0.x_0^{\pm v(x)}$, and $\tilde{f}$ is a differentiable function on $\tilde{B}$ in the usual sense. Thus equation (3.12), viz., the equality $f'(x_0) = l$, extends over $\tilde{B}$ as a scale free differential equation

$$\frac{d\tilde{f}}{d \log X} = l. \tag{3.13}$$

Definition 13. *[25] Let $f : C \to C$ be a mapping on a Cantor set C to itself. Then f is differentiable at $x_0 \in C$ if $\exists\ l$ such that given $\epsilon > 0, \exists$ $\delta > 0$ so that*

$$\left| \frac{\| f(x) - f(x_0) \|}{\| x - x_0 \|} - l \right| < \epsilon \tag{3.14}$$

when $0 <\| x - x_0 \|< \delta$.

As before we write $f'(x_0) = l$ (with an abuse of notation). It follows that the above equality now extends over to a scale free equation of the form

$$\frac{d \log \tilde{f}(X)}{d \log X} = l \tag{3.15}$$

where notations are analogous to the above.

Remark 5. The discrete point like structures of C are replaced by infinitesimal open intervals over which the ordinary continuum calculus is carried over on logarithmic variables via the scale invariant non-archimedean metric. We consider some applications in chapter 4 (Sec.4.4) and chapter 10.

Chapter 4

ULTRAMETRIC CANTOR SETS: EXAMPLES

4.1 Introduction

In this chapter, we present two examples, one on the classical triadic Cantor set and another on a general class of homogeneous (p, q) Cantor sets and explain various properties of valued infinitesimals and related concepts. To justify the analytic framework of Chapter 3, we present here *independent arguments* showing the actual process how a nontrivial valuation could arise on a Cantor set. The valuation in each of these two models is shown to be related to an appropriate Cantor function $\phi(x)$. The Cantor function $\phi(x)$, in the non-archimedean framework, is also shown to be extended to a locally constant function for *any* $x \in I$. Further, we verify the multiplicative representation equation (4.3) that exists because of the nontrivial infinitesimals and the scale invariant ultrametric for every element of the Cantor set, explicitly, in either of the classical middle third set, middle α set and the (p, q) Cantor set separately. Finally, the variability of the locally constant $\phi(x)$ is reinterpreted in the usual topology as an effect of relative infinitesimals which become dominant by inversion at an appropriate log scale.

4.2 Middle third Cantor set and Cantor function

4.2.1 Valuation

Let us now investigate in detail the well known triadic Cantor set C in the light of the analytic framework developed in Chapter 3 [24]. Indeed, we are going to show in detail how the concept of relative infinitesimals and associated valuations may actually arise in the context of the classical Cantor set. The relation of the valuation with the corresponding Cantor function will also be explained.

Suppose we begin with the set $C_0 = [0, 1]$. In relation to the *scale* 1, C_0 is essentially considered to be a doublet $\{0, 1\}$, in the sense that real numbers $0 < x < 1$ are *undetectable* in the assigned scale, and hence all such numbers might be identified with 0. We denote this 0 as $0_0 = [0, 1)$, the set of *infinitesimals*. However, the possible existence of *infinitesimals* are ignored at this scale and so 0 is considered simply as a singleton $\{0\}$ only. At the next level, we choose a smaller scale $\epsilon = 1/3$ (say), so that only the elements in $[0, 1/3) \subset C_0$ are now identified with 0, so that $0_1 = [0, 1/3)$, which is actually $0_1 = 0_0$ in the unit of $1/3$. Relative to this nontrivial scale $1/3$, we now assign the ultrametric valuation v to 0_1. In principle, all possible ultrametric valuations are admissible here. One has to make *a priori choice* to *select* the most appropriate valuation in a given application. In the context of the triadic Cantor set, there happens to be a unique choice relating it to the Cantor's function, as explained below.

Recall that the valuation induces a nontrivial topology in 0_1. Accordingly, the set is covered by n number of disjoint clopen intervals of *valued* infinitesimals. At the level 1, $n = 1$, which is actually the clopen interval I_{11} of length $1/3$ and displaced appropriately to the middle of the 1/3rd Cantor set, viz, $I_{11} = [1/3, 2/3]$ (in the ordinary representation this is the deleted open interval, including the two end points of neighbouring closed intervals). The value assigned to these valued infinitesimals is the constant $v(I_{11}) = 1/2$, where, of course, $v(0) = 0$. In principle, again, v could assume any constant value. Our choice is guided by the triadic Cantor function. Thus the valued set of infinitesimals, at the scale $1/3$, turns out to be $0_1 = \{0, 1/2\}$.

How does this valued set of infinitesimals enlight the ordinary construction of the Cantor set? Let $C_1 = F_{11} \cup F_{12}$ where $F_{11} = [0, 1/3]$ and $F_{12} = [2/3, 1]$. The value awarded to the deleted middle open interval is now inherited by these two closed (clopen) intervals, and so $||F_{11}|| = 1/3^s$ and $||F_{12}|| = 1/3^s$, recalling that $2 = 3^s$, s being the Hausdorff dimension $s = \log 2/\log 3$.

At the next level, when the scale is $\epsilon = 1/3^2$, the above interpretation can be easily extended. The zero set is now made of 3 clopen sets $0_2 = I_{20} \cup I_{21} \cup I_{22}$ where $I_{20} = [1/9, 2/9]$, $I_{21} = [3/9, 6/9]$ and $I_{22} = [7/9, 8/9]$. The value assigned to each of these sets are respectively, $v(I_{20}) = 1/4$, $v(I_{21}) = 2/4$ and $v(I_{22}) = 3/4$, so that the valued infinitesimals are given by $0_2 = \{0, 1/4, 2/4, 3/4\}$. Notice that the new members of the valued infinitesimals are derived as the mean value of two consecutive values from those (including 1 as well) at the previous level. These valued infinitesimals now, in turn, assign equal value to the 4 closed intervals in the ordinary level 2 Cantor set $C_2 = F_{20} \cup F_{21} \cup F_{22} \cup F_{23}$ where $F_{20} = [0, 1/9]$ and etc, viz. $||F_{2i}|| = 1/2^2 = 1/3^{2s}$, $i = 0, 1, 2, 3$. Notice that, in the sense of Sec. 2, the valued infinitesimals 0_2 induces a *fine structure* in the neighbourhood of F_{2i} : for a $x \in F_{2i}$, we now have valued neighbours $X^{\pm} = xx^{\pm k 3^{-2s}}$, $k = 1, 2, 3$. Clearly, $||F_{2i}|| = ||x|| = 1/3^{2s}$, the infimum of all possible valued members, so misses the above fine structures. It also follows that the limit set of this triadic construction reproduces the Cantor function (c.f., Example 2) as the the valuation $v : [0, 1] \to [0, 1]$, defined originally on the *inverted* Cantor set $\mathbf{0} = \bigcap_n \bigcup_k I_{nk}$, and then extended on [0,1] by continuity.

Remark 5: The continuity in the present ultrametric topology is defined in the usual manner (*c.f.*, Definition 11). Further, v on $\mathbf{0}$ is an example of *locally constant* function relative to the $||.||$-topology and will be shown (in the next section) to satisfy the differential equation

$$x\frac{dv(x)}{dx} = 0. \tag{4.1}$$

We may interpret this as follows: Considered as a function on $\mathbf{0}$ (or C), v is constant in clopen sets I_{nk}(or $\mathrm{F_{nk}}$) for fixed values of both n and k, but experiences variability as either of these vary. This variability is not only continuous, but continuously first order differentiable as well. In contrast, v on $\{I_{nk}$ or $(F_{nk})\}$ is a discontinuous function in the usual topology.

4.2.2 Multiplicative structure

Let C be the standard middle $\frac{1}{3}$rd Cantor set. As will become clear our discussion will apply generically to any measure zero Cantor set (*c.f.*, *Sec.*4.2.3). The Cantor set C offers us with a privileged set of scales $\epsilon_n = 3^{-n}$. For a sufficiently large n viz : as $n \to \infty$, suppose an infinitesimal gap of the form $(0, \epsilon_{n+m})$ is decomposed into a finite number M of open subintervals $\tilde{I}_i$ of relative infinitesimals with constant valuations defined by

$$v(\tilde{x}_i) = i\ 2^{-n}\ ,\ i = 1, 2, \cdots, M,\ \tilde{x}_i \in I_i. \tag{4.2}$$

The valuation assigned by (4.2) is the triadic Cantor function $\phi : I \to I$ so that $M = 2^m - 1$ corresponding to the scale $\epsilon_m = 3^{-m}$. We call infinitesimals $\tilde{x}$ leaving in the island $\tilde{I}_i \subset (0, \epsilon_{n+m})$ of *valued infinitesimals* having the valuation (4.2) induced by the Cantor function. Any element x of the original Cantor set C is now endowed with a set of valued neighbours

$$X^i_{\pm} = x.x^{\mp v(\tilde{x}_i)}. \tag{4.3}$$

Finally, the element x is assigned the ultrametric valuation

$$\| x \| = \inf_i \log_{x^{-1}} \frac{X^i_+}{x} = \inf_i \log_{x^{-1}} \frac{x}{X^i_-} \tag{4.4}$$

so that $\| x \| = 2^{-n} = 3^{-ns}$ where $s = \frac{\log 2}{\log 3}$, the Hausdorff dimension of the triadic Cantor set C and $n \to \infty$. As it turns out, this valuation exactly reproduces the nontrivial measure of [17] derived in the context of noncommutative geometry (c.f., definition of valued measure in Sec.3.2.).

Now, to make contact with the absolute value (3.2) and the inversion rule (3.1), let for a sufficiently large but fixed n, $\tilde{x}_i \in \tilde{I}_i$ has the form (we *set for definiteness $m = 0$)*

$$\tilde{x}_i = 3^{-n} \cdot 3^{-n \cdot i2^{-r}} \times a_i \tag{4.5}$$

where $ni = 2^r \cdot k_i$, k_i being, in general, a sufficiently large real number and $a_i = \Sigma a_{ij} 3^{-j} \in O_i$, a gap of size $3^{-r}$of the Cantor set C and $a_{ij} \in \{0,1,2\}$. Then $0 < \tilde{x}_i < 3^{-n}$ and $v(\tilde{x}_i) = i.2^{-r}$. One also verifies that for an $a_i = 1 + a_{0i}$, $v(a_i) = \lim_{n\to\infty} \log_{3^n}(a_i/3^{-n}) = 1$. By the inversion rule (3.1) the elements $\tilde{x}_i$ of the ball $\tilde{I}_i$ now connect to an $x_i \in C$ given by

$$x_i = 3^{-n} \cdot 3^{-n(-\cdot i2^{-r})} \times b_i, \; b_i = \Sigma \; b_{ij} \; 3^{-j}, \; b_{ij} \in \{0,2\} \tag{4.6}$$

where $\lambda = a_i \times b_i \in (0,1)$. (Infinitesimal) Scales $\epsilon_n = 3^{-n}$, are the *primary* scales when the scales 3^{-k_i}(or equivalently 2^{-r}) are the *secondary* scales.

Finally, to verify the new multiplicative representation one notes that there exists $y_i \in C$ in a neighbourhood of x_i so that

$$y_i = 3^{-n} \; c_i, \quad c_i = \Sigma \; c_{ij} \; 3^{-j}, \; c_{ij} \in \{0,2\}$$

and

$$x_i = y_i . y_i^{-i.2^{-r}} \tag{4.7}$$

so as to satisfy the identity

$$c_i^{n-k_i} = b_i^n . \tag{4.8}$$

To verify that (4.8) is not empty we note that for the end points $\frac{1}{3}$ and $\frac{2}{3}$, both belonging to C, (4.8) means $\left(\frac{2}{3}\right)^n = \left(\frac{1}{3}\right)^{n-k_1}$ yielding $k_1 = ns$, $s = \frac{\log 2}{\log 3}$. For this value of k_1, (4.8) now tells that $c_i^{1-s} = b_i$ so that $c_i = (\frac{1}{3})^r$ and $b_i = \left(\frac{2}{3}\right)^r$ for a suitable r. Similar estimates for k_i are available for other (consecutive) end points of (higher order) gaps. It thus follows that the representation (4.3) is realized at the level of the finite Hausdorff measure of the set, when the value of the constant k is real (rather than a natural number).

4.2.3 Multiplicative Structure: Middle α set

Next, we consider C_α Cantor set. Here in each iteration we removes an open interval of length proportional to α from a closed interval $I = [0, 1]$, leaving out two open intervals of size β each. Therefore this Cantor set offers us with a privileged set of scales $\epsilon_n = r^{-n}$. For a sufficiently large n viz : as $n \to \infty$, suppose an infinitesimal gap of the form $(0, \epsilon_{n+m})$ is decomposed into a finite number M of open subintervals $\tilde{I}_i$ of relative infinitesimals with constant valuations defined by

$$v(\tilde{x}_i) = i\ 2^{-n}\ ,\ i = 1, 2, \cdots, M,\ \ \tilde{x}_i \in I_i. \tag{4.9}$$

The valuation assigned by equation (4.9) is the Cantor function $\phi : I \to I$ so that $M = 2^m - 1$ corresponding to the scale $\epsilon_m = r^{-m}$. We call infinitesimals $\tilde{x}$ leaving in the island $\tilde{I}_i \subset (0, \epsilon_{n+m})$ of *valued infinitesimals* having the valuation equation (4.9) induced by the Cantor function. Any element x of the original Cantor set is now endowed with a set of valued neighbours

$$X^i_{\pm} = x \cdot x^{\mp v(\tilde{x}_i)}. \tag{4.10}$$

Finally, the element x is assigned the ultrametric valuation

$$\| x \| = \inf_i \log_{x^{-1}} \frac{X^i_+}{x} = \inf_i \log_{x^{-1}} \frac{x}{X^i_-} \tag{4.11}$$

so that $\| x \| = 2^{-n} = \beta^{ns}$ where $s = \frac{\log 2}{\log \frac{1}{\beta}}$, the Hausdorff dimension of the C_α Cantor set and $n \to \infty$.

Now, to make contact with the absolute value (3.2) and the inversion rule (3.1), let for a sufficiently large but fixed n, $\tilde{x}_i \in \tilde{I}_i$ has the form (we set for definiteness $m = 0$)

$$\tilde{x}_i = \beta^n \cdot \beta^{n \cdot i 2^{-r}} \times a_i \tag{4.12}$$

where $ni = 2^r \cdot k_i$, k_i being, in general, a sufficiently large real number and $a_i = \Sigma a_{ij} (1-\beta) \beta^j \in O_i$, a gap of size $\beta^r$of the Cantor set and $a_{ij} \in \{0,1,2\}$. Then $0 < \tilde{x}_i < \beta^n$ and $v(\tilde{x}_i) = i.2^{-r}$. One also verifies that for an $a_i = 1 + a_{0i}$, $v(a_i) = \lim_{n\to\infty} \log_{\beta^{-n}}(a_i/\beta^n) = 1$. By the inversion rule (3.1) the elements $\tilde{x}_i$ of the ball $\tilde{I}_i$ now connect to an $x_i \in C$ given by

$$x_i = \beta^n \cdot \beta^{n(-\cdot ip^{-r})} \times b_i, \; b_i = \Sigma \, b_{ij} \, (1-\beta)\beta^j, \; b_{ij} \in \{0, 2, \} \tag{4.13}$$

where $\lambda = a_i \times b_i \in (0, 1)$. (Infinitesimal) Scales $\epsilon_n = r^{-n}$, are the *primary* scales when the scales β^{k_i}(or equivalently 2^{-r}) are the *secondary* scales.

Finally, to verify the new multiplicative representation one notes that there exists $y_i \in C$ in a neighbourhood of x_i so that

$$y_i = \beta^n\, c_i, \quad c_i = \Sigma\, c_{ij}\, (1-\beta)\, \beta^j, \ \ c_{ij} \in \{0, 2\}$$

and

$$x_i = y_i . y_i^{-i.2^{-r}} \tag{4.14}$$

so as to satisfy the identity

$$c_i^{n-k_i} = b_i^n. \tag{4.15}$$

Accordingly, it follows that a gap O in I/C (which is a connected interval in the usual topology) containing a point x of the Cantor set C is indeed realized as an "infinitesimal" Cantor set in the valuation defined by the Cantor function associated with the original Cantor set C itself. One thus concludes that

Proposition 3. *Any element x of an ultrametric Cantor set C is endowed with a class of valued neighbours having the multiplicative representation of the form (4.3) (or (4.10)) and the non-archimedean absolute value $||x|| = \inf_i \, v(\tilde{x}_i)$.*

4.3 (p,q) Cantor set and Cantor function

We first show that the value $v(x)$ awarded to the valued infinitesimals $X \in B_i$, $i = 1, 2, \ldots, n$ is given by the Cantor function $\phi : I \to I$ with points of discontinuity in $\phi'(x)$, in the usual sense, are in C [26]. In the new formalism this discontinuity is removed in a scale invariant way using logarithmic differentiability over (valued) infinitesimal open line segments replacing each $x \in C$. Our definition of $v(x)$ is guided by the given Cantor set C so as to retrieve the finite Hausdorff measure uniquely via the construction of the valued measure.

Let us denote the valued scale free infinitesimals by $[0,1)$, denoted here by $\tilde{C}$. The interval $[0,1)$ here is a copy of the scale free infinitesimals $\mathbf{I}^+$ for an arbitrary small ϵ_0 (say). The valued infinitesimals in $[0,1)$ then introduce a new set of scales of the form r^{-n} (in the unit of ϵ_0) so that the scales introduced in definition 1 are now parameterized as $\epsilon = \epsilon_0\, r^{-n}$. The choice of the 'secondary' scales r^{-n} are motivated by the finite level Cantor set C. At the ordinary level i.e. at the scale 1 (corresponding to $n = 0$), there is no valued infinitesimal (at the level of ordinary real calculus) except the trivial 0. So relative to the finite scale (given by $\delta = \frac{\epsilon}{\epsilon_0} = 1$) $[0,1)$ reduces to the singleton $\{0\}$. At the next level, we choose the smaller scale $\delta = \frac{1}{r}$. Consequently, elements in $[0,\frac{1}{r})$ are undetectable and identified with 0, again in the usual sense. Presently we have, however, the following.

We assume that the void (emptiness) of 0 reflects in an inverted manner the structure of the Cantor set C that is available at the finite scale. That is to say, at the first iteration of C from I, q open intervals are removed leaving out p closed intervals F_{1n}, $n = 1,2,\ldots,p$. At the scale $\frac{1}{r}$ in the void of $\tilde{C}$, on the other hand, there now emerges (by "inversion") q open islands (intervals) $\mathbf{I}_{1i}$, $i = 1,2,\ldots,q$. By definition, $\mathbf{I}_{1i}$ contains, for each i, the so called valued infinitesimals X_i which are assigned the values $v(X_i) = \phi(X_i) = \frac{i}{p}$, $i = 1,2,\ldots,q, X_i \in I_{1i}$.

We note that at the scale $\delta = \frac{1}{r}$, there are p voids in $\tilde{C}$. At the next level of the scale $\frac{1}{r^2}$, there emerges again in each void q islands of open intervals, so that there are now pq number of total islands $\mathbf{I}_{2i}$, $i = 1, 2, \ldots, pq$. The value assigned to each of these valued islands of infinitesimals are $v(X_j) = \phi(X_j) = \frac{j}{p^2}$, $j = 1, 2, \ldots, pq$, where $X_j \in \mathbf{I}_{2j}$. Continuing this iteration, at the n th level, the (secondary) scale is $\delta = \frac{1}{r^n}$ and the number of open intervals $\mathbf{I}_{nj}$ of infinitesimals are now $q(1 + p + p^2 + \cdots + p^n) = N$ (say) with corresponding values

$$v(X_j) = \phi(X_j) = \frac{j}{p^n},\ j = 1, 2, \ldots, N \tag{4.16}$$

where $X_j \in \mathbf{I}_{nj}$. Thus v and hence the Cantor function ϕ is defined on the "inverted Cantor set" $\tilde{C} = \bigcap_n \bigcup_j \mathbf{I}_{nj}$ and is extended to $\phi : I \to I$ by continuity following equations like

$$\begin{aligned} \phi(\beta_n) - \phi(\alpha_n) &= \frac{1}{p^n}, \\ \beta_n - \alpha_n &= \frac{1}{r^n} \text{ where } x \in [\alpha_n, \beta_n] \subset I. \end{aligned}$$

We note that the absolute value $\|.\|$ awarded to each block of the Cantor intervals F_{nk} are

$$\| F_{nk} \| = r^{-ns} \tag{4.17}$$

for each $k = 1, 2, \ldots, p^n$ where $C = \bigcap_n \bigcup_k F_{nk}$ and so $s = \frac{\log p}{\log r}$, since the valued set of infinitesimals induces fine structures to an element in F_{nk} viz. for an $y \in F_{nk}$, we now have the infinitesimal neighbours $Y_\pm^j = y.\ y^{\mp jp^{-n}}, j = 1, 2, \ldots, N$.

Clearly, the absolute value in equation (4.17) corresponds to the minimum of $v(x)$ at the nth iteration. Thus the valuations defined as the associated Cantor function leads to a valued measure on C that equals the corresponding Hausdorff measure with $s = \frac{\log p}{\log r}$.

Let us now recall that the solutions of $\phi'(x) = 0$ in a non-archimedean space are locally constant functions [20]. To show that Cantor function $\phi : I \to I$ is a locally constant function, let us recall that the Cantor set C is constructed recursively as $C = \bigcap_n \bigcup_k F_{nk}$. The set I, on the other hand, is written as $I = \bigcap_n [(\bigcup_{k=1}^{p^n} \tilde{F}_{nk}) \cup (\bigcup_{j=1}^{N} \mathbf{I}_{nj})]$, the open interval $\tilde{F}_{nk}$ being F_{nk} with end points removed (recall that $\mathbf{I}_{nj}$ are closed in the ultrametric topology). By definition $v(\mathbf{I}_{nj}) = a_{nj}$ a constant for each n and j. We set $v(\tilde{F}_{nk}) = 0$ as $n \to \infty$. This equality is to be understood in the following sense. At an infinitesimal scale $\epsilon_0 \to 0^+$ the zero value of $\tilde{F}_{nk}$ becomes finitely valued recursively for each n since a Cantor point $x \in C$ is replaced by a copy of the (inverted) Cantor set $\tilde{C}$ with finite number of closed intervals like $\mathbf{I}_{nj}$. The derivatives of ϕ vanishes not only for each n and j but *even as* $n \to \infty$ (and $\epsilon \to 0$, for each arbitrarily small but fixed ϵ_0). Thus, the equality $\phi'(x) = 0$ on I/C, in the ordinary sense, gets extended to every $x \in C$ when the Cantor set is reinterpreted as a nonarchimedean space. The removal of the usual derivative discontinuities is also explained dynamically as due to the fact that the approach to an actual Cantor set point x is accomplished in the nonarchimedean setting by inversion. That is to say, as a variable $X \in I$ approaches $x \in C$, the usual linear shift in I is replaced by infinitesimal hoppings between two nieghbouring elements of the form $X_+/x \propto x/X_-$.

4.3.1 Multiplicative structure in (p, q) Cantor set

We now show that any element of the (p, q) Cantor set also has the multiplicative representation. We divide the interval $I = [0, 1]$ into r

number of closed subintervals each of length $\frac{1}{r}$ and delete q number of open subintervals from them so that $p+q=r$. Therefore this Cantor set offers us with a privileged set of scales $\epsilon_n = r^{-n}$. For a sufficiently large n viz : as $n \to \infty$, suppose an infinitesimal gap of the form $(0, \epsilon_{n+m})$ is decomposed into a finite number M of open subintervals $\tilde{I}_i$ of relative infinitesimals with constant valuations defined by

$$v(\tilde{x}_i) = i\, p^{-n} \ , \ i = 1, 2, \cdots, M, \ \tilde{x}_i \in I_i. \tag{4.18}$$

The valuation assigned by (4.18) is the triadic Cantor function $\phi : I \to I$ so that $M = q(1 + p + p^2 + \cdots + p^m)$ corresponding to the scale $\epsilon_m = r^{-m}$. We call infinitesimals $\tilde{x}$ leaving in the island $\tilde{I}_i \subset (0, \epsilon_{n+m})$ *valued infinitesimals* having the valuation (4.18) induced by the Cantor function. Any element x of the original Cantor set is now endowed with a set of valued neighbours

$$X^i_{\pm} = x \cdot x^{\mp v(\tilde{x}_i)}. \tag{4.19}$$

Finally, the element x is assigned the ultrametric valuation

$$\| x \| \ = \ \inf_i \log_{x^{-1}} \frac{X^i_+}{x} \ = \ \inf_i \log_{x^{-1}} \frac{x}{X^i_-} \tag{4.20}$$

so that $\| \ x \ \| = p^{-n} = r^{-ns}$ where $s = \frac{\log p}{\log r}$, the Hausdorff dimension of the (p, q) Cantor set and $n \to \infty$. As it turns out, this valuation exactly reproduces the nontrivial measure of [17] derived in the context of noncommutative geometry (c.f., definition of valued measure in Sec.3.2.)

Now, to make contact with the absolute value (3.2) and the inversion rule (3.1), let for a sufficiently large but fixed n, $\tilde{x}_i \in \tilde{I}_i$ has the form (we set for definiteness $m = 0$)

$$\tilde{x}_i = r^{-n} \cdot r^{-n \cdot ip^{-t}} \times a_i \tag{4.21}$$

where $ni = 2^t \cdot k_i$, k_i being, in general, a sufficiently large real number and $a_i = \Sigma a_{ij} r^{-j} \in O_i$, a gap of size $r^{-t}$of the Cantor set and $a_{ij} \in \{0, 1, 2, \ldots, N\}$ where $N = r - 1$. Then $0 < \tilde{x}_i < r^{-n}$ and $v(\tilde{x}_i) = i.p^{-t}$. One also verifies that for an $a_i = 1 + a_{0i}$, $v(a_i) = \lim_{n \to \infty} \log_{r^n}(a_i / r^{-n}) = 1$. By the inversion rule (3.1) the elements $\tilde{x}_i$ of the ball $\tilde{I}_i$ now connect to an $x_i \in C$ given by

$$x_i = r^{-n} \cdot r^{-n(-\cdot ip^{-t})} \times b_i, \; b_i = \Sigma \, b_{ij} \, r^{-j}, \; b_{ij} \in \{0, 2, 4 \ldots N\} \tag{4.22}$$

where we consider the special case that the closed interval $I = [0, 1]$ is divided into r number of closed intervals with r odd and $\frac{r-1}{2}$ number of open intervals in the even places are deleted, and $\lambda = a_i \times b_i \in (0, 1)$. Infinitesimal scales $\epsilon_n = r^{-n}$, are the *primary* scales when the scales r^{-k_i}(or equivalently p^{-t}) are the *secondary* scales.

Finally, to verify the new multiplicative representation one notes that there exists $y_i \in C$ in a neighbourhood of x_i so that

$$y_i = r^{-n} \, c_i, \quad c_i = \Sigma \, c_{ij} \, r^{-j}, \quad c_{ij} \in \{0, 2, \ldots N\}$$

and

$$x_i = y_i . y_i^{-i.p^{-t}} \tag{4.23}$$

so as to satisfy the identity

$$c_i^{n-k_i} = b_i^n. \tag{4.24}$$

To verify that (4.24) is not empty we note that for the end points $\frac{1}{r}$ and $\frac{p}{r}$, both belonging to C, (4.24) means $\left(\frac{p}{r}\right)^n = \left(\frac{1}{r}\right)^{n-k_1}$ yielding $k_1 = ns$, $s = \frac{\log p}{\log r}$. For this value of k_1, (4.24) now tells that $c_i^{1-s} = b_i$ so that $c_i = (\frac{1}{r})^t$ and $b_i = \left(\frac{p}{r}\right)^t$ for a suitable t. Similar estimates for k_i are available for other (consecutive) end points of (higher order) gaps. It thus follows that the representation (4.19) is realized at the level of the finite Hausdorff measure of the set, when the value of the constant k is real (rather than a natural number).

4.4 Locally Constant Cantor Function and Usual topology

The variability of the locally constant function $\phi : I \to I$ may, even, be captured in the usual topology as follows [25].
Indeed, we show that

$$\frac{d\phi}{dx} = 0 \tag{4.25}$$

for finite values of $x \in I$ is transformed into

$$\frac{d\phi}{dv(\tilde{x})} = -O(1)\phi \tag{4.26}$$

for an infinitesimal $\tilde{x}$ satisfying $\frac{x}{\epsilon} = \lambda \ \frac{\epsilon}{\tilde{x}} = \epsilon^{-v(\tilde{x})}$, $0 < \tilde{x} < \epsilon \leq x$, $x \to 0^+$, $x \in I$, $\lambda > 0$, when one interprets 0 in relation to the scale ϵ as $O(\ \delta = \frac{\epsilon^2}{x} \log \epsilon^{-1}\ )$. However, this follows once one notes that equation (4.25) means, in the ordinary sense, $d\phi = 0 = O(\delta)$, $dx \neq 0$, for a finite $x \in I$. But, as $x \to \epsilon$, that is, as $dx \to 0 = O(\delta)$, the ordinary variable x is replaced by the ultrametric extension $x = \epsilon.\epsilon^{-v(\tilde{x})}$ so that $d\log x = dv(\tilde{x})\log\epsilon^{-1} = O(\delta)$. On the other hand, the constant function ϕ (equation (4.25)), now, in the presence of smaller scale infinitesimals, has the form $\phi = \phi_0\epsilon^{k_0 v(\tilde{x})}$ for a real constant k_0. Equation (4.26) thus follows. The variability of $\phi(x)$ in the usual topology is thus explained as an effect of the relative infinitesimals which are *insignificant* relative to the finite scale of $x \in C$, but attain a dominant status in the appropriate logarithmic variable $v(\tilde{x}) = \log_{\epsilon^{-1}} \frac{\epsilon}{\tilde{x}}$. It is also of interest to compare the present case with computation. In the ordinary framework, the scale ϵ stands for the level of accuracy in a computational problem. The infinitesimals in $(0, \epsilon)$ are "valueless" in the sense that these have no effect on the actual computation. The open interval $(0, \epsilon)$ is thus effectively identified with $\{0\}$. In the present framework, the zero element 0 is, however, identified with a smaller interval of the form $(0, \delta)$ where $\delta = \eta\epsilon\log\epsilon^{-1}$ and $0 < \eta \lesssim 1$. The valued infinitesimals in the interval (δ, ϵ) are already shown to have significant influence on the structure of the Cantor set. The variability of $\phi(x)$ as given by equation (4.26) is revealed, on the other hand, in relation to an infinitesimal variable lying in $(0, \delta)$.

Chapter 5

CANTOR FUNCTION: FROM NONDIFFERENTIABILITY TO DIFFERENTIABILITY

5.1 Introduction

In the previous chapter, we explicitly verified the multiplicative structure (4.3) in the context of middle third Cantor set and similar other more general class of homogeneous sets. Clearly such a representation is valid for any general Cantor set. We also presented independent analysis explaining how the nontrivial valuation is related to an associated Cantor function. We studied in detail both the middle third and (p,q) Cantor set and also discussed the variability of the valuation vis a vis Cantor function both in ultrametric and usual topology. In this Chapter, we present *another new independent* derivation of the multiplicative structure explaining explicitly the smoothening of Cantor function at the points of a Cantor set. We begin by first recalling the usual proof of the nondifferentiability of a Cantor function $\phi(x)$ at $x \in C$. The analysis here clearly brings out the precise points where the present scale invariant approach supersedes the classical analytic results. This also offers another justification in favour of the existence of infinitesimals introduced in Chapter 3.

5.2 Middle third Cantor set

In this example, we present an explicit construction of multiplicative neighbours of $x \in C$ using the Cantor function $f_C : I \to I$. In the following we denote this function instead by $\tilde{X}(x)$. To recall again the definition of the Cantor function, consider the $\frac{1}{3}$-rd Cantor set: $r = 3,\ p = 2$. Let $x = \Sigma a_i 3^{-i}$ be the ternary representation of $x \in C$ where a_i may be either 0 or 2. We set $x = \frac{2}{3}\psi(\tilde{X})$ where $\psi(\tilde{X}) = \Sigma \frac{b_i}{3^{i-1}}$ and $\tilde{X} = \Sigma b_i 2^{-i} \in I \backslash C,\ b_i \in \{0, 1\}$.

Then $\tilde{X} = \tilde{X}\ (x)$ defined as the inverse of the above functional equation is the Cantor function $\tilde{X} : [0,1] \to [0,1]$. By continuity, this extends over C as well.

Let us recall that at the k -th step of the iterative construction of the Cantor set, the initial closed interval I fragments into 2^k smaller closed intervals $I_j^k = [x_{2j-1}, x_{2j}],\ j = 1, 2....,$ each of length 3^{-k}.

Then $x_{2j} - x_{2j-1} = 3^{-k}$. Definition of the Cantor function also gives that

$$\tilde{X}(x_{2j}) - \tilde{X}(x_{2j-1}) = 2^{-k}. \tag{5.1}$$

Let $x \in C$. Then $x \in I_j^k$ for some j. It thus follows

$$\tilde{X}(x_{2j}) - \tilde{X}(x_{2j-1}) = \frac{3^k}{2^k}(x_{2j} - x_{2j-1}). \tag{5.2}$$

This equality is at the heart of the standard proof of the nondifferentiability of the Cantor function [42]. Now, to see how such a nondifferentiability is removed in the present framework, let $\tilde{X}(x_{2j}) = X_+,\ \tilde{X}(x_{2j-1}) = X_-,\ x_{2j} = x_+,\ x_{2j-1} = x_-$. Suppose also that

$$3^k(x_+ - x) \to k\log\sigma_+, \quad 3^k(x - x_-) \to k\log\sigma_- \tag{5.3}$$

and

$$2^k(\tilde{X}_+ - \tilde{X}) \to k\log X'_+, \; 2^k(\tilde{X} - \tilde{X}_-) \to k\log X'_- \tag{5.4}$$

for infinitely large $k \to \infty$. The limiting value of equation (5.2) thus becomes

$$\log X'_+ + \log X'_- = \log\sigma_+ + \log\sigma_-. \tag{5.5}$$

Now, using the inequality $\frac{\alpha+\gamma}{\beta+\delta} \leq \max(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}), \alpha, \gamma \geq 0, \beta, \delta > 0$, equation (5.5) yields

$$\max\left(\frac{\log X'_+}{\log\sigma_+}, \frac{\log X'_-}{\log\sigma_-}\right) \geq 1. \tag{5.6}$$

But equation (5.5) shows that $\sigma_+ = \sigma_-^{-1} = \sigma$ (say) and $X_+ = X_-^{-1}$, as $k \to \infty$, so that equation (5.6) reduces to

$$X'_+ = \sigma^{1+j}, \; X'_- = \sigma^{-(1+j)}, \; j \geq 0. \tag{5.7}$$

Setting $\sigma^{-1}X'_+ = \frac{X_+}{x}$ and $\sigma X'_- = \frac{X_-}{x}$, we finally get the multiplicative neighbours of $x \in C$ as

$$X_\pm = x\sigma^{\pm j}. \tag{5.8}$$

Notice that $\sigma \approx 1$. In the notation of Section 2, $\sigma = x^{\tau^s}$, τ being a valued infinitesimal. The inequality equation (5.6) is reminiscent of the strong triangle inequality for the non-archimedean valuation. We also remark that the clue to the substitutions of the form equation (5.3) and equation (5.4) arise from our basic definitions of relative infinitesimals and the associated scale invariant norms of Sec.3.2.

5.3 (p,q) *Cantor set*

Here, we extend the above analysis to the more general class of Cantor set and the associated Cantor function. Indeed, we again verify the emergence of equation (5.8) from the classical Cantor function equations (5.1) and (5.2) viz. :

$$\phi(\beta_k)-\phi(\alpha_k)=\frac{1}{p^k} \text{ and } \beta_k-\alpha_k=\frac{1}{r^k}. \tag{5.9}$$

We have

$$\phi(\beta_k)-\phi(\alpha_k)=\left(\frac{r}{p}\right)^k(\beta_k-\alpha_k). \tag{5.10}$$

Let $\phi(\beta_k)=\tilde{\phi}_+$, $\phi(\alpha_k)=\tilde{\phi}_-$, $\beta_k=x_+$, $\alpha_k=x_-$. Suppose also that $r^k(x_+-x)\to k\log\sigma_+$, $r^k(x-x_-)\to k\log\sigma_-$, $p^k(\tilde{\phi}-\tilde{\phi}_-)\to k\log\phi'_-$ and $p^k(\tilde{\phi}_+-\tilde{\phi})\to k\log\phi'_+$ as $k\to\infty$.

Equation (5.10) becomes

$$\log\phi'_+ + \log\phi'_- = \log\sigma_+ + \log\sigma_- \tag{5.11}$$

which leads to

$$\frac{\log\phi'_+}{\log\sigma_+}=\frac{\log\phi'_-}{\log\sigma_-}=\frac{\log\phi'_+ + \log\phi'_-}{\log\sigma_+ + \log\sigma_-}=1. \tag{5.12}$$

Equation (5.12) is essentially the left and right branches of equation (4.25) at $x\in C$, in the appropriate logarithmic variables, where the multiplicative neighbours of x, in the present derivation, is given by the limiting form of the Cantor function defined by

$$\phi'_+=\sigma^{1+i},\ \phi'_-=\sigma^{-(1+i)},\ i\geq 0 \tag{5.13}$$

which follows from the inequality $\frac{\alpha+\gamma}{\beta+\delta} \leq \max(\frac{\alpha}{\beta}, \frac{\gamma}{\delta})$, $\alpha, \gamma \geq 0$, $\beta, \delta > 0$ and equation (5.11) so that

$$\left(\frac{\log \phi'_+}{\log \sigma_+}, \frac{\log \phi'_-}{\log \sigma_-}\right) \geq 1. \tag{5.14}$$

Setting $\sigma^{-1}\phi'_+ = \{\frac{X_+}{x}\}^i$, $\sigma\phi'_- = \{\frac{X_-}{x}\}^i$ and $\sigma = x^{-v(\tilde{x})}$ the multiplicative neighbours of x are obtained as

$$X_\pm = x.\ x^{\mp v(\tilde{x})}. \tag{5.15}$$

The Cantor function $\phi(\tilde{x})$ over the infinitesimals $\tilde{x}$ is thus given by

$$\phi(\tilde{x}) = \log_{x^{-1}} \frac{X(\tilde{x})}{x} = v(\tilde{x}) \tag{5.16}$$

thereby retrieving the variability of ϕ relative to v trivially viz : $d\phi = dv$.

We note that this again explains explicitly the removal of derivative discontinuities as encoded in equation (5.10) in the present formalism. The divergence of either the left or right derivative at an $x \in C$, that arises due to the divergence of $(r/p)^k$, $k \to \infty$, is smoothed out in the logarithmic variables that replace the ordinary limiting variables as in equations (5.11) and (5.12), which, in fact, correspond to equation (4.26). We conclude that the multiplicative non-archimedean structure given by equation (5.15) induces a smoothening effect on the discontinuity of $\phi'(x)$ in the usual topology.

Chapter 6

DIFFERENTIAL MEASURE

6.1 *Introduction*

In this Chapter, we present a few more new results on a Cantor set exposing the precise nature of variability of a nontrivial valuation and hence of a Cantor function [28]. Next, we analyze how an ordinary limiting variable $R \ni x \to 0$ is extended to a sublinear variation $x \log x^{-1} \to 0$ when $x \in C \subset R$. Finally, we derive the differential measure on a Cantor set C.

6.2 *Cantor Set: A Few More New Results*

It is shown in Chapter 3 that the ultrametric so defined by equation (3.2) is both metrically and topologically inequivalent to the usual ultrametric that a Cantor set carries naturally [26]. For a point x_0 in a Cantor set $C \subset I = [a, b]$, the representation (3.7) now gives rise to a scale invariant ultrametric extension $\tilde{X}_{\pm} = X_{\pm}/x_0 = x_0^{\mp v(\tilde{x})}$ where the transition between two infinitesimally close scale invariant neighbours is supposed to be mediated by inversions of the form $\tilde{X} \to \tilde{X}^{-h}$ for a real h, which determines the jump size. Notice that $\tilde{X}$ (and equivalently, $v(\tilde{x})$) is a locally constant Cantor like function and solves $\frac{d\tilde{X}}{dx} = 0$ everywhere in I. The ordinary discontinuity of a Cantor function at an $x_0 \in C$ is removed, since in the present ultrametric extension, the point x_0 in C is replaced by an *inverted Cantor set* which is the closure of *gaps* of an infinitesimal Cantor set C_i that is assumed to be the residence Cantor set for the relevant infinitesimals $\tilde{x}$ living in the extended neighbourhood $\mathbf{0}$ of 0. The gaps of C_i constitute a disjoint family of connected clopen intervals (represented in a scale invariant manner) over each of which scale invariant equations of the form (8.1) are well defined [26]. Consequently, the valuation $v(\tilde{x})$, redefined slightly in the modified form

$$(x/x_0)^{\tilde{v}(x)} = x_0^{v(\tilde{x})} \tag{6.1}$$

(that is, $\tilde{v}(x)/v(\tilde{x}) = \log x_0/\log(x/x_0)$, exposing the relative variation of $\tilde{v}$ over v), x assuming values from the gaps in the neighbourhood of x_0, is realized as a smooth function defined recursively in a scale invariant way by the equation

$$\frac{d\tilde{v}(x)}{d\xi} = -\tilde{v}(x) \tag{6.2}$$

where $\xi = \log\log(x/x_0)$, $x \in C_i$. Recall that $\tilde{x}$ resides in the gaps of non-trivial neighbourhood of 0 instead. As a consequence, $\tilde{v}$ may be written as $\tilde{v}(x) = (\log(x/x_0)^k)^{-1}$, where k may be allowed to assume values from a set of scale factors related to that of the Cantor set. This form is clearly consistent with (6.1). Assuming x is drawn from a specific gap of a given size, the same, written more effectively as $\tilde{v}(x)/v(\tilde{x}) = (\log_{x_0}(x/x_0))^{-1}$, yields, in the limit of vanishingly small gaps (i.e., as $x \to x_0$ and vice versa), the limiting value $\tilde{v}_0(x)/v_0(\tilde{x}) = 1/s$, since $\lim \log_{x_0}(x/x_0) = s$ *equals the finite Hausdorff dimension of the Cantor set* C_i. Let us first note that if one replaces the Cantor set by a segment of a line of the form $(0,\delta)$, then $x/x_0 = 1$, in that limit ($\delta \to 0$) gives $s = 0$, which is consistent with the fact that the line segment reduces to a point, viz. 0. In the general case, $x/x_0 \propto N$, the number of clopen balls that covers the fattened gap of the form $(x, x_0) \subset (0,\delta)$ (size of balls are determined by the gap). Letting $x_0 \to 0$ (as the relevant scale factors β^n), the above limit therefore mimics the box dimension, which also equals the Hausdorff dimension of the Cantor set concerned. The topological inequivalence of the present ultrametric arises from the possible dichotomy in the choice of C_i.

6.3 Limit on a Cantor set

Let us now show that when the ordinary 0 of R is replaced by an infinitesimal Cantor set C_i, the ordinary limit $\epsilon \to 0$ in R is altered. This follows because of scale invariant dynamic infinitesimals with valuations given by $\log \tilde{x}/\epsilon \approx v(\tilde{x}) \log \epsilon^{-1} \approx \epsilon \log \epsilon^{-1}$, when the relative infinitesimals $\tilde{x}$ is considered to lie on a fattened (connected) gap, so that the ultrametric valuation may be assumed to coincide with the usual (Euclidean) value viz. $v(\tilde{x}) \approx \epsilon$. However, assuming $\tilde{\epsilon}$ ($= \beta^n$, $n \to \infty$) to be an infinitesimal scale of the Cantor set concerned, we also have $\log \tilde{x}/\tilde{\epsilon} \approx \tilde{\epsilon}^s \log \tilde{\epsilon}^{-1}$, since the valuation is identified with the associated Cantor function $\phi(\tilde{x}) \approx \tilde{\epsilon}^s \approx \epsilon$, s being, as usual, the corresponding Hausdorff dimension. Reverting back to the ordinary scale ϵ (and keeping in view the associated scale invariance), this scaling can be identified with $\epsilon^{\tilde{s}} \approx O(1)\epsilon \log \epsilon^{-1}$, for an $\tilde{s}$ given by $\tilde{s} \approx 1 - \frac{\log\log \epsilon^{-1}}{\log \epsilon^{-1}}$. *As a consequence,* *in the presence of an ultrametric space, in the neighbourhood of 0 the ordinary limit* $\epsilon \to 0$ *is replaced by the sublinear limit*

$$\epsilon^{\tilde{s}} = \epsilon \log \epsilon^{-1} \to 0, \tag{6.3}$$

$0 < \tilde{s} < 1$, *as* $\epsilon \to 0$. This is one of the main results of this thesis. A real variable t in R approaching to (or flowing out from) 0 will experience this scale invariant sublinear behaviour in an incredibly small neighbourhood of **0** in **R** and should have a deep significance in number theory and other areas [29, 32]. If the variable t changes only over a Cantor set C and approaches 0 through points of C by hoppings (inversion induced jumps) then the above sublinear asymptotic behaviour is obviously remain valid. As a consequence the limit $\epsilon \to 0$ on C is interpreted as the sublinear limit $\epsilon \log \epsilon^{-1} \to 0^+$ on R instead.

To justify the above claim, let us suppose that the original Cantor set C and the infinitesimal Cantor set C_i have Hausdorff dimensions s and s' respectively. Any point $\mathbf{x}$ of the fattened set $\mathbf{C} = C + C_i$ is given as $\mathbf{x} = x + \tilde{x}$, $x \in C$, $\tilde{x} \in C_i$. It is well known that $\mathbf{C} = I$, for *almost every* s', for a given s [38]. Accordingly, it follows that given a Lebesgue measure zero Cantor set C, the above smooth differentiable structure is a.s (*almost surely*) realized on $\mathbf{C}$, which is nothing but I, though in an appropriate (scale free) logarithmic variable.

We note that similar behaviour is also reported recently in the context of diffusion in an ultrametric Cantor set in a noncommutative space [17]. Finally, the ultrametric induced by the valuation $v(\tilde{x})$ coincides with the natural ultrametric only when the scaling properties of C_i coincide with that of C. In this chapter we adhere to the latter possibility.

To understand more clearly the above smooth scale invariant structure let us consider the classical middle third Cantor set $C_{1/3}$ with scale factors $\tilde{\epsilon} = 3^{-n}$. A point $x_0 = 3^{-n}\sum a_i 3^{-i}$, $a_i \in \{0, 2\}$ of $C_{1/3}$ is raised to the scale free $\mathbf{x}$ which is a variable living in a family of fattened gaps, attached and structured hierarchically at the point x_0 (or equivalently, by scale invariance, at 1), over each of which scale free equations of the form equation (6.2) are valid. The infinitesimals are elements of the gaps "closest" to 0, viz. the open intervals $3^{-n-m}(1, 2)$, in the limit $n \to \infty$, for a fixed $0 < m < n$, which are assigned nontrivial values akin to the Cantor function $v(\tilde{x}) = i3^{-ms}$, $i = 1, 2, \ldots 2^m - 1$ [31]. Over each of the finite size gaps, on the otherhand, the valuation $v(x)$ is awarded as $v(x) = 3^{-sn}$. Both these valuations are not only continuous but also smooth since the corresponding Cantor function is realized as a smooth function via the logarithmic ansatz for a substitution of the form $3^n\Delta x_n = 3^n(x - x_n) \to n \log \frac{\mathbf{x}}{x_n}$, as $n \to \infty$, thereby removing the derivative discontinuity at the points of scale changes (c.f., Sec.5.2), so that $\frac{dv(x)}{dx} = 0$, every where on the Cantor set concerned (c.f., chapter 3). Notice that gaps scale as $\epsilon = 2^{-n}$ (recall the binary representation for points on a connected segment of the real line) when a closed interval containing points like $x_0 \in C_{1/3}$ scales as 3^{-n}, so that the Hausdorff dimension is $s = \log_3 2$. By equation (6.1), the variability of the valuation $\tilde{v}(x)$ in the limit of vanishing gap sizes is obtained as $\tilde{v}_0(x) \propto 3^{-sn}s^{-1}, n \to \infty$.

6.4 Differential increments

Next, we determine the incremental *measure*, denoted $d_j\tilde{X}$, of smooth self similar jump processes of (gap) "size" (in the sense of a weight) ϵ (2^{-n}, for $C_{1/3}$, say) in the neighbourhood of the scale invariant 1. To this end, let us first recall that pure translations follow a linear law: $y = Tx = x + h$. The *instantaneous pure jumps* (of unit length close to the scale invariant 1), on the other hand, follow a *hyperbolic law*: $\tilde{X} \to Y = \tilde{X}^{-1} \Rightarrow \log Y + \log X = 0$, which tells, in turn, that the corresponding translational increment, even in the log scale, is indeed zero. This actually is the case for the valuation defined in terms of the locally constant Cantor function (c.f., Chapter 3). The (manifestly scale invariant) multiplicative valuation $\tilde{v}(x) = \log_{x^{-1}}(X/x)$ is, however, gives the correct linear measure for a single jump relative to the point x (for the above hyperbolic type jump, $v(x) = 1$ relative to x itself as the scale). The corresponding *multiplicative increment* is denoted as $\delta_j\tilde{X} = (x/x_0)^{\tilde{v}(x)}$. More importantly, this valuation is realized as a smooth measure and may be considered to contribute an independent component in the ordinary measure of R. Further, the total self similar jump mediated increments over a spectrum of gaps of various sizes of the forms $\epsilon_n = 2^{-n}\epsilon_0$ in the neighbourhood of a (middle third) Cantor point x_0 (say) is now obtained as

$$\Delta_j\tilde{X} = (x/x_0)^{s^{-1}2^{-m}\sum_n 2^{-n}} \tag{6.4}$$

which in the limit $x \to x_0$, that is, $m \to \infty$ yields the *jump differential* $d_j\tilde{X} = (d\tilde{x})^{s^{-1}}$, where $\tilde{x} = \lim(x/x_0)^{2^{-m}}$, is a deformed variable close to 1. Such a variable ($\neq 1$ exactly) exists because of a nontrivial g.l.b. of gap sizes (another manifestation of the sublinear asymptotic). Incidentally, we note that the essential singularity in $s = 0$ tells that in the absence of inversion mediated jumps, the whole structure of gaps collapses to a point (singleton set, devoid of any nontrivial infinitesimals). The divergence in the jump measure then reflects the ordinary nondifferentiable structure of the Cantor set. On the otherhand, on any connected segment of R, $s = 1$, and the jump measure reduces to the ordinary linear measure dx. To summarize, *the significantly new insight that emerges from the above analysis is that an infinitesimal scale invariant increment on an ultrametric space must have the form* $\tilde{X} = 1+\epsilon^{1/s}$, $\epsilon \to 0^+$ *on a connected segment close to 0.* Recalling $\tilde{\epsilon} = \epsilon^{1/s}$, the above infinitesimal jump increment $\tilde{X} = 1 \pm \tilde{\epsilon}$ reduces to the usual increment on a Cantor set in the usual metric, but at the cost of the smooth structure.

Chapter 7

GROWTH OF MEASURE: APPLICATIONS

7.1 Introduction

A point of the original Cantor set C is identified with the closure of the set of gaps of $\tilde{C}$. The increments on such an ultrametric space is accomplished by inversion rule. An interesting phenomenon, called *growth of measure*, is studied on such an ultrametric space [26]. Using the reparametrisation invariance of the valuation it is shown how the scale factors of a Lebesgue measure zero Cantor set might get *deformed* leading to a *deformed* Cantor set with a positive measure. The definition of a new *valuated exponent* is introduced which is shown to yield the fatness exponent in the case of a positive measure (fat) Cantor set.

7.2 Reparametrisation Invariance and Measure

We studied the valued ultrametric structure of a measure zero Cantor set. Here we study a few more general properties of the valued ultrametricity. We note, at first, that the valuation $v(\tilde{x})$ is LC Cantor function corresponding to a homogeneous Cantor set C. As a consequence, v satisfies the equation

$$\frac{d}{dx}v(\tilde{x}(x)) = 0. \tag{7.1}$$

and hence, v is, not only a LCF, but more importantly is a *reparametrisation* invariant object (c.f., page 33) As a result, v does not require to be an explicit function of the original variable x but may be a function instead of *any* monotonic, continuously first differentiable function of x. By the same token, v does not depend explicitly on the scale ϵ inherited from the original (mother) Cantor set, as we did in the examples of chapter 4. In the following example, we show that relative infinitesimals may instead live in a positive measure Cantor set. Notice that in the general representation of the valued ultrametric in equation (3.2), the parameter may be a constant independent of an explicit ϵ.

Example 5. *Suppose that equations (4.5) and (4.6) are replaced by*

$$\tilde{x} = \beta^n \left(\beta^{n^\delta}\right)^{n\beta_n(1+\gamma_m)} \times a \tag{7.2}$$

where $a = (1-\beta)\left(1 + \overset{\infty}{\underset{1}{\Sigma}} a_i \beta^i\right)$, $a_i \in \{0,1\}$, $\beta = \frac{1}{2}(1-\alpha)$, *and* β_n *and* γ_n *are two non-increasing sequence of positive numbers such that* $\beta_n \to 0$ *as* $n \to \infty$ *and* m *may be independent of* n *or may vary with* n *more slowly, and* $\delta > 1$ *is a constant.*

Although $\beta_n \to 0$ *as* $n \to \infty$ *, the valuation* $v(\tilde{x})$ *could be non-trivial, since*

$$\begin{aligned} v(\tilde{x}) &= \lim_{n\to\infty} \log_{\beta^{-n}} \frac{\beta^n}{\tilde{x}} \\ &= \lim_{n\to\infty} \left[n^\delta \beta_n (1+\gamma_m) + \log_{\beta^{-n}} a \right] \\ &= l + \tilde{\gamma}_{m_n}(\delta) \end{aligned} \tag{7.3}$$

when we assume $n^\delta \beta_n \to l$ *as* $n \to \infty$ *and* $n^\delta \beta_n \gamma_m \to \tilde{\gamma}_{m_n}(\delta)$ *is a subdominant slowly varying non-increasing sequence, for a real* $m_n > 0$. *The representation (7.2) tells that a scale free infinitesimal* $\frac{\tilde{x}}{\beta^n}$ *may live in a Cantor set* $\tilde{C}_p$, *so that* $m(\tilde{C}_p) = l$. *Let the original Cantor set be a middle* α *set* C_α *with the uniform scale factor* $\beta = \frac{1}{2}(1-\alpha)$. *For the positive measure set* $\tilde{C}_p$ *the scale factor at the nth iteration is* $\tilde{\beta}_n = 2^{-n}\sum_{i=1}^{n}(1-\alpha_i)$ *and* $l = m\left(\tilde{C}_p\right) = \prod_{i=1}^{\infty}(1-\alpha_i) = \lim_{n\to\infty} 2^n \tilde{\beta}_n$. *Let us choose* $\delta > 1$ *such that* $\tilde{\beta}_n = \beta^{n^\delta}$. *Then* $n^\delta \beta_n \to l$ *tells that* $\beta_n \approx \frac{l \log \beta}{\log \tilde{\beta}_n}$ *as* $n \to \infty$. *Thus the dominant term* l *of the valuation* $v(\tilde{x})$ *is a constant while the subdominant asymptotic* $\tilde{\gamma}_m(\delta)$ *could be a genuine LCF (i.e. a Cantor function for a sub dominant Cantor like set* C_s *(say)), precise determination of which depends on the explicit model of the Cantor set* $\tilde{C}_p$. *It follows, therefore, from equations (4.3) and (4.4) the ultrametric valuation of* $x \in C_\alpha$ *now has the form*

$$\| x \| = l + \tilde{\gamma}_{m_n}(\delta). \tag{7.4}$$

For larger and larger values of n $(\to \infty)$, *we can disregard the subdominant term (since* $\tilde{\gamma}_{m_n} \to 0$ *as* $m_n \to \infty$*) so that*

$$\| x \| = l \quad \forall\, x \in C_\alpha,\ x \neq 0. \tag{7.5}$$

Clearly the trivial ultrametric (7.5) reveals that the mother set C_α *must get deformed to a positive measure set* C_p *so that* $\mu_v(C_p) = m(C_p) = l$, *when the reparametrisation invariance of LC correction factors is invoked. Indeed, we have* $\| x - y \| = l$ *for any two* $x, y \in C_p$. *Thus, any single clopen* *ball* $B(x_0)$, $x_0 \in C_p$ *(say) covers the compact* C_p *and hence* $\mu_v(C_p) = d_u(B(x_0)) = l$.

To summarize, we have shown that any element $x \in C_\alpha$ when deformed by the non-trivial, reparametrisation invariant valuation of relative infinitesimals, is identified with an element of a 1-set C_p. Because of this invariance, the relative infinitesimals may be assumed to live in a positive measure set $\tilde{C}_p$, which, in turn, determines the measure (size) of the deformed set C_p. Since each element $x \in C \subset [0,1]$ is written as the arithmetic sum of two elements $x_0 \in C_\alpha$ and $x_1 \in C_{\alpha'}$ ($C_{\alpha'}$ being the Cantor set of infinitesimal neighbours of x_0), it follows from a theorem of Solomyak [46] that for $\beta = \frac{1}{2}(1-\alpha) \in \left(0, \frac{1}{2}\right)$, there exists $C_{\alpha'}$ for a.e. $\beta' = \frac{1}{2}(1-\alpha') \in \left(0, \frac{1}{2}\right)$ so that $C_\alpha + C_{\alpha'}$, has positive measure and $\frac{1}{\log \frac{1}{\beta}} + \frac{1}{\log \frac{1}{\beta'}} > \frac{1}{\log 2}$. This, therefore, constitutes an alternative proof for the said assertion. Indeed, in the above construction, the set of infinitesimals $C_{\alpha'}$ itself is a 1-set $\tilde{C}_p$.

It follows, accordingly, that a slower rate of removal of middle open sets compared to a measure zero Cantor set hides a positive measure in an infinitesimal scaling factor which is exposed under the present scale invariant valuation. The uniform rate of deletion in the case of a measure zero set is violated because of the underlying reparametrisation invariance. Further, in a dynamical process leading to a Cantor set, a positive measure Cantor set C_p is favoured a.s (almost surely) compared to a measure zero set C_α since relative infinitesimal neighbours a.s. lie in a Cantor set $C_{\alpha'}$ satisfying the above constraints.

The generic result that follows from this example is stated thus

Theorem 2. *Because of the reparametrisation invariance of the infinitesimal valuation, a measure zero Cantor set C_α is a.s. deformed to a positive measure Cantor set C_p, the measure of which is determined by the Cantor set $\tilde{C}_p$ in which the relative infinitesimals are supposed to live.*

Next to expose the significance of the sub-dominant term, let us first define a renormalized valuation $v_R(\tilde{x})$:

$$v_R(\tilde{x}) = \log_{\beta^n} \log_{\beta^n} \left[\frac{\tilde{x}}{(\beta^n)^{1+v_0(\tilde{x})}} \right], \quad n \to \infty \tag{7.6}$$

where $v_0(\tilde{x}) = l < 1$ is the dominant valuation of the infinitesimal $\tilde{x}$. The LCF $\tilde{\gamma}_{m_n}(\delta)$ is now given by (c.f.,(chapter 6))

$$\tilde{\gamma}_{m_n}(\delta) = \alpha_i \beta^{m_n \rho(\delta)} \tag{7.7}$$

where the $\delta-$dependent constant ρ is called a *renormalised valuated exponent* and the non-zero constant α_i assumes values from a finite set for a secondary scale β^{m_n}. As will become clear the valuated exponent ρ is useful to distinguish two sets with identical Hausdorff dimensions.

7.2.1 Applications

1. Middle third Cantor sets:

As an application of the renormalised valuated exponent, let us first consider a class of s- sets where $s = \log_3 2$, constructed as a slight variation of the process of Example 2, Sec.2.4. Let $I = [0, 1]$. Also let

$0 << \delta_n = 3^{-(n+1)\alpha_n} \lesssim 1$, $n, = 1, 2, \cdots$ (so that $\delta_n^{-1} \gtrsim 1$), be a non-increasing sequence. For definiteness, one may choose $\alpha_n = q^{-n}$, for a sufficiently large positive integer n and $q > 1$. In that case α_n may be considered to belong to the range set of an appropriate Cantor function. Delete the middle open interval of length 1/3. Next, delete a length $3^{(-2(1+\alpha_1))}$ from each of the two closed subintervals. Then, delete the length $3^{(-3(1-\alpha_2))}$ from each of 2^2 closed subintervals. Call these two operations together O_1. O_n consists of two steps: deletion of 2^{n+1} open intervals of length $3^{-(n+1)(1+\alpha_n)}$, which is succeeded by the next deletion of lengths $3^{-(n+2)(1-\alpha_{n+1})}$ from 2^{n+2} remaining closed subintervals. Notice that we are considering a set of fluctuating scale factors, i.e., in the $(n+1)$th step open intervals of slightly smaller sizes compared to the middle third set are removed. In the next step, however, open intervals of slightly bigger sizes are removed. As a consequence, we get a family of limit sets which are indistinguishable and equivalent to the middle third Cantor set at the level of the Hausdorff dimension, but nevertheless, distinguishable at the level of renormalised valuated exponents. Indeed, the total length of deleted open intervals viz., $\frac{1}{3}+\frac{2}{3^2}\delta_1+\frac{2^2}{3^3}\delta_2^{-1}+\cdots = 1+\sum u_n$ equals 1, when the series of real numbers $\sum u_n$ vanishes. The sequence u_n is determined by the sequence α_n, i.e., $\alpha_n = \log_{3^{(n+1)}}(1 - \frac{3^{(n+1)}}{2^n}\tilde{u}_n)^{-1}$ and $\alpha_{n+1} = \log_{3^{(n+2)}}(1 + \frac{3^{(n+2)}}{2^{n+1}}\tilde{u}_{n+1})$ so that $u_n = -\tilde{u}_n$, $u_{n+1} = \tilde{u}_{n+1}$. Clearly, such a series exists. Hence, all such sets are of measure zero.

Now, to determine the Hausdorff dimension, we first note that the scaling of closed intervals (bridges) follows the recurrence $2l_n = l_{n-1} - \delta_n^{\pm 1}3^{-(n+1)}$, where + sign goes with an odd n and the - sign with n even and l_n denotes the length of each closed interval at level n. Accordingly, $l_n = \frac{1}{3.2^n}[1 - \frac{\delta_1}{3} - \frac{2\delta^{-1}}{3^2} - \cdots \frac{2^{n-1}\delta_n^{\pm 1}}{3^n}] \approx \frac{\delta_{n+1}^{\pm 1}}{3^{n+2}}$, for a sufficiently large n. As a consequence, the scale factors behave as either $\beta_{n+1} = 3^{-(n+1)(1+\alpha_n)}$ or $\beta_{n+2} = 3^{-(n+2)(1-\alpha_{n+1})}$ respectively, and hence, the lower and upper box dimensions and the Hausdorff dimension are all equal and equal to $\lim_{n\to\infty} \frac{\log 2}{\log 3^{(1\pm\alpha_n)}} = \log_3 2$.

One may also estimate the thickness of these sets easily. Because of the above scaling, the limiting length of the closed intervals (bridges) coincides with that of the corresponding gap (viz., $\delta_{n+1}^{\pm 1} 3^{-(n+2)}$) at the nth level. It follows therefore that the ratio of sizes of bridges and gaps (c.f., Sec.2.1) has the limiting value 1. Hence, thickness of all these sets coincides with that of the classical middle third Cantor set as well.

However, a higher order (renormalised) valuated exponent can indeed reveal the local dissimilarities of such an $s-$set. Extending the representations (4.6) and (7.2) a little further to suit the present problem, we would now have for an element x of the $s-$set,

$$x_{i\pm} = 3^{-n} \cdot 3^{-n(-i2^{-m_n(1\pm\alpha_{m_n})})} \times b, \ ||b|| = 1 \tag{7.8}$$

where i assumes values from a finite set and $m_n \to \infty$ at a slower rate as $n \to \infty$, so that a renormalised valuation is defined as

$$v_R(x) = \inf_i \log_{2^{-m_n}} \log_{3^{-n}}(x_{i+}/x_0) = \alpha_{m_n}, \ x_0 = 3^{-n} \cdot 3^{-n(-i2^{-m_n})}. \tag{7.9}$$

It now follows from the definition of α_{m_n}, that one can find a sufficiently large natural number $q >> 1$ such that $\alpha_{m_n} = q^{-m_n}$. Consequently, we obtain $v_R(x) = \alpha_{m_n} = 3^{-\rho \tilde{m}_n}$, where $\rho = \log_{3^r} q > 0$ is the *valuated exponent*, for suitable positive integers r and $\tilde{m}_n$.

Now, to justify the existence of such a q, let us first assume $\tilde{u}_{2m} = u^1_m$ and $\tilde{u}_{2m+1} = u^2_m$ such that $\sum u^i_m = l$. Then $\sum_2^\infty u_n = \sum \tilde{u}_{2m+1} - \sum \tilde{u}_{2m} = l - l = 0$. Consequently, $\alpha_{2m} = \log_{3^{(2m+1)}}(1 - \frac{3^{(2m+1)}}{2^{2m}} u^1_m)^{-1}$ and $\alpha_{2m+1} = \log_{3^{(2m+2)}}(1 + \frac{3^{(2m+2)}}{2^{2m+1}} u^2_m)$.

Let $\eta_{2m} = \frac{3^{(2m+1)}}{2^{2m}} u^1_m$ and $\tilde{\eta}_{2m+1} = \frac{3^{(2m+2)}}{2^{2m+1}} u^2_m$. Then the functions $(1 - \eta_{2m})^{-1}$ and $1 + \tilde{\eta}_{2m+1}$ are identified as LCF of the form (7.1), in the neighbourhood of 1. Using scale invariance, we can then choose for x in equation (23) as $x = 3^{-n}(1 - \eta_n)$ (or $x = 3^{-n}(1 + \tilde{\eta}_n)$), and the scale factor $\epsilon = 3^{-n}$. Thus, there exists a Cantor function $\tau(\tilde{x})$, $\tilde{x} = x/\epsilon$ such that $\log_{\epsilon^{-1}} \tilde{x}^{-1} = \tau(\tilde{x})$ (or $\log_{\epsilon^{-1}} \tilde{x} = \tau(\tilde{x})$). As a result, there exists positive integers q and m_n so that the sequence $\{q^{-m_n}\} \subset \text{Range}(\tau(\tilde{x}))$. More generally, because of the local constancy, the limiting form α_n could be $\alpha_n = \tilde{l} + q^{-m_n}$, where $\tilde{l}$ is a non-negative constant, $0 \leq \tilde{l} < 1$.

We remark that the exponent ρ may be considered to be the inverse of the Hausdorff dimension of a *residual* Cantor set that would remain attached with infinitesimal scales in a neighbourhood of a point (of the original Cantor set). For the classical middle third Cantor set $\alpha_n = 0 \; \forall n$ and so $\rho = \infty$, which is consistent with the fact that the residual set is null. Since, sets with infinite Hausdorff dimension $s = \infty$ are excluded, by definition, ρ indeed is positive $\rho > 0$.

2. 1-sets: Irregular 1-sets [3] are positive measure Cantor sets and are generally classified on the basis of fatness and/or uncertainty exponents.

The LC renormalised valuation (7.6) and (7.7) now tells that $v_R(\tilde{x})$ is a Cantor function corresponding to a subdominant residual Cantor set C_s, and so has the form $v_R(\tilde{x}) = \alpha_i \beta^{m_n \rho}$. As for the s-sets, the valuated exponent $\rho > 0$ equals the inverse of the Hausdorff dimension of the residual set C_s. For $\rho = \infty$, the double exponential factor in (7.2) drops out (i.e., reduces to the trivial factor β^n), and hence the 1-set is a regular set [3] having connected components (actually corresponds to a nonfractal set). Consequently, $0 < \rho \leq \infty$.

Now, to compare with the fatness exponent [37, 43], we first recall the relationship between the uncertainty exponent α, $0 < \alpha \leq 1$ [44] and the fatness exponent $\tilde{\beta}$, $0 < \tilde{\beta} \leq \infty$. It is shown [37] that $\tilde{\beta} = \alpha$ in [0,1], so that there is essentially the fatness exponent that has to be considered. We claim that $\rho = \tilde{\beta}$. The parameter $\tilde{\beta}$ is defined as

$$\tilde{\beta} = \lim_{\epsilon \to 0} \frac{\log[\mu(\epsilon) - \mu(0)]}{\log \epsilon} \tag{7.10}$$

where $\mu(\epsilon)$ is a LC measure which tells the scaling of smaller gap sizes when the smaller gaps are coarse grained by fattening by the amount ϵ and $\mu(0)$ equals the positive (Lebesgue) measure of the set. In our multiplicative representation (c.f.,(7.2) and (7.7)), the fattening size is $\epsilon = \beta^n$ and

$$x = \beta^n \left(\beta^n\right)^{-(l+k\beta^{n\rho})} \times b \tag{7.11}$$

where k is a constant independent of β, so that the exponent ρ is defined by (7.10) when we identify $\mu(\beta^n) = \log_{\beta^n}(x/\beta^n)$. Notice that $\mu(0) =$

$\lim_{n\to\infty} \log_{\beta^n}(x/\beta^n) = l$. Notice also that the measure μ here is nothing but the valuation of relative infinitesimals at the fattened scale ϵ, which equals the full measure of the Cantor set $\tilde{C}_p$ at the scale ϵ (c.f., Example 4) where the infinitesimals live. Because of the reparametrisation invariance, we may suppose that $\tilde{C}_p$ is determined by the original 1-set and vice versa. At the scale ϵ, the gaps of $\tilde{C}_p$ are fattened by the amount ϵ, and in the presence of a positive measure, the said valuation is determined by the sum of the fattened gap sizes. For a zero measure set, this valuation, on the other hand, is determined instead by the finite Hausdorff measure, upto a finer (double logarithmic) scale correction that arises from the possible presence of local fine structures (c.f., above application). This observation proves the claim.

Chapter 8

SCALE FREE ORDINARY DIFFERENTIAL EQUATION: NOVEL SOLUTIONS

8.1 Introduction

Here, we argue that the finitely differentiable scale free solutions to the simplest scale free initial value problem (IVP) [30, 31, 32, 33]

$$t\frac{\mathrm{d}\tau}{\mathrm{d}t} = \tau, \ \ \tau(1) = 1 \tag{8.1}$$

should be able to offer an ideal framework for many complex phenomena. We present a novel dynamical treatment of linear ODEs when the time (i.e. the independent real) variable t is assumed to have a random element [31]. We show how a judicious use of the golden mean partition of unity, $\nu^2 + \nu = 1$, $\nu = (\sqrt{5} - 1)/2$, not only allows time to undergo random changes (flips) by inversions, $t_- \to t_-^{-1} = t_+$, $t_\pm = 1 \pm \eta$, in the vicinity of an instant $t = 1$ (say), but also unveils the possible existence of a class of random, second derivative discontinuous, scale free solutions to equation (8.1). One of the major aim is to extend a framework of calculus accommodating inversions as a valid mode of changes (increments) besides ordinary translations. The freedom of random inversions provides a dynamic, evolutionary character to the second derivative discontinuous solutions, with a privileged sense of time's arrow. This solution, though approximate ($\sim$O(η^2) =0), in the ordinary real number system R, is, however, exact in an nonstandard real number set **R**. Finally, we show that the 'approximate' solution is in fact generic, in the sense that the more accurate, in fact the exact solution, derived by generating successive self- similar corrections to an initially approximate solution, fails to yield the exact solution even in the limit of infinite number of iterations.

The new solution breaks the reflection symmetry ($t \to -t$) of the ODE. We also show here that besides these finitely differentiable (C^{2^n-1}) time asymmetric solutions as well as the infinitely differentiable, time (reflection) symmetric standard solution of equation (8.1), possesses another new class of fluctuating solutions which are both infinitely differentiable and time symmetric. Because of these nontrivial classes of finitely and infinitely differentiable fluctuating solutions, a real variable t can undergo changes not only by linear translations, but by inversions ($t \to 1/t$), in the neighbourhood of each real t. We next discuss how this could define a nonstandard extension of the real number system. This also clarifies the origin of an intrinsic randomness at as fundamental a level as the real number system. Consequently, every real number is identified with an equivalence class of a continuum of new, infinitesimally separated elements, which are in a state of "random fluctuations" (c.f., Sec.7.2).

8.2 Mathematical results

Because of the novelty of the result, it is instructive to give a fairly complete derivation of such solutions [30]. To this end, let us first construct the solution in the neighbourhood of $t = 1$. We need to introduce following notations.

Let $t_{n\pm} = 1 \pm \eta_n$, $t_0 \equiv t$, $0 < \eta_n << 1$, $\alpha_n = 1 + \epsilon_n$, $n = 0, 1, 2, \ldots$,

and $\epsilon_0 = 0$, $0 < \epsilon_n < 1 (n \neq 0)$, such that $\epsilon_n \to 0$, as $n \to \infty$ (we retain the symbol α_0 for the sake of symmetry). Next, we write $t'_{n\pm} = 1 \pm \alpha_n \eta'_n$, so that $\alpha_n t_{n-} = t'_{n-}$. Consequently, $\eta'_n = \eta_n - \frac{\epsilon_n}{\alpha_n}$. Here, α_n (and ϵ_n) are scaling parameters. A useful choice, however, is $\epsilon_n = \epsilon^{2^n}$, $\epsilon = \epsilon_1$ (the reason will become clear later). As will become evident, $\eta_{n+1} = \alpha_n^2 \eta_n'^2$.

To construct a nontrivial solution (with the initial condition $\tau(1) = 1$), we begin with an initial approximate solution in the small scale variable η_0. To this end, let

$$\tau(t) = \begin{cases} \tau_- \; if \; t \lessapprox 1 \\ \tau_+ \; if \; t \gtrapprox 1 \end{cases}, \quad \tau_-(t_-) = (1/t'_+) f_{1-}(\eta_0), \; \tau_+(t_+) = t_+ \qquad (8.2)$$

be an exact solution of equation (8.1). This is obviously true for the right hand component τ_+. To verify the same for the nontrivial component τ_-, we differentiate it with respect to t_-, and use the scale invariance of equation (8.1). Utilizing $\alpha_0 t_- = t'_-$, one obtains

$$t'_- \frac{d\tau_-}{dt'_-} = \tau_- \left(\frac{t'_-}{t'_+} - t'_- \frac{f'_{1-}}{f_{1-}} \right) \qquad (8.3)$$

where $f'_{1-} = \frac{df_{1-}}{d\bar{\eta}_0}$, $\bar{\eta}_0 = \alpha_0 \eta'_0$. Consequently, equation (8.3) would be an exact solution if and only if f_{1-} solves exactly the self-similar equation

$$t_{1-} \frac{df_{1-}}{dt_{1-}} = f_{1-} \qquad (8.4)$$

in the smaller logarithmic variable $\ln t_{1-}^{-1}$, where $t_{1-} = 1 - \alpha_0^2 \eta_0'^2 \equiv 1 - \eta_1$. The self-similar replica equation (8.4) follows from the equality

$$\frac{t'_-}{t'_+} - t'_- \frac{f'_{1-}}{f_{1-}} = 1 \tag{8.5}$$

so that τ_- is an exact solution of equation (8.1). The exact (nontrivial part of the) solution could thus be written recursively as

$$\tau_- = C\frac{1}{t_+}\frac{1}{t'_{1+}}\cdots\frac{1}{t'_{(n-1)+}}f_{n-}(\eta'_n) \tag{8.6}$$

where f_n satisfies the nth generation self-similar equation

$$t_{n-}\frac{\mathrm{d}f_{n-}}{\mathrm{d}t_{n-}} = f_{n-} \tag{8.7}$$

and $t_{n-} = 1 - \alpha^2_{n-1}\eta'^2_{n-1} \equiv 1 - \eta_n$. We also note that $t'_+ = t_+$, since $\alpha_0 = 1$.

Plugging in the initial condition $\tau_\pm = 1$ at $t_\pm = 1$ (viz., $\eta_0 = 0$), one obtains finally the desired solution as

$$\tau_- = C\frac{1}{t_+}\frac{1}{t'_{1+}}\frac{1}{t'_{2+}}\ldots, \quad \tau_+ = t_+ \tag{8.8}$$

where $C = t'_{1+}(0)t'_{2+}(0)\ldots$. Notice that $C \neq 1$, since $\eta'_1 = -\epsilon_1/\alpha_1$, $\eta'_2 = \epsilon_1^2 - \epsilon_2/\alpha_2$, etc, when $\eta_0 = 0$.

A remark is in order here [27, 30].

The solution (8.8) follows from equation (8.6) only if the sequence $\{f_{n-}\}$ is convergent. In fact, we prove that $f_\infty = \lim_{n\to\infty} f_{n-} = 1$. Let $\tau_n = \frac{1}{t_+}\frac{1}{t'_{1+}}\cdots\frac{1}{t'_{n+}}$. Then for η_0 sufficiently small and $\epsilon_n \to 0$ for $n \to \infty$, the sequence $\{\tau_n\}$ is convergent (to a nonzero value), since $t'_{n+} \to 1$ as $n \to \infty$. Accordingly, for $\epsilon > 0$, $\exists N_1$ such that $|\tau_m - \tau_n| < \epsilon$ for $m, n > N_1$ $(m > n)$. As a result, $0 < k_1 < \tau_n < k_2$, $k_1, k_2 \sim \mathrm{O}(1)$, for

$n > N_2$ for a sufficiently large N_2. Again, f_{n-}, being defined by equation (8.7), is uniformly bounded in a neighbourhood of $t = 1$, so that $|f_{n-}| < k$ for $n > N_2$. The desired convergence now follows from the Cauchy convergence criterion, since $|f_{n-} - f_{m-}| = |\tau_n^{-1}||f_{m-}||\tau_m - \tau_n| < k_1^{-1}k\epsilon$ $\forall m, n > N$, $N = \max(N_1, N_2)$. Finally, equation (8.7), in the asymptotic limit $n \to \infty$, yields $f_\infty = \frac{\mathrm{d}f_{n-}}{\mathrm{d}t_{n-}}|_\infty = \frac{\mathrm{d}\tau}{\mathrm{d}t}|_{t=1} = \tau(1) = 1$.

Now to test the continuity of the derivatives of the solution (8.8) at $t_\pm = 1$, i.e., at $\eta_0 = 0$, we note that η'_n is a polynomial in η_0, of degree 2^n, being defined recursively by $\eta'_n = \eta_n - \frac{\epsilon_n}{\alpha_n}$, $\eta_n = \alpha_{n-1}^2 \eta'^2_{n-1}$. As a result $\frac{\mathrm{d}\eta'_n}{\mathrm{d}\eta_0} = 0$, but $\frac{\mathrm{d}^2\eta'_n}{\mathrm{d}\eta_0^2} \neq 0$, at $\eta_0 = 0$. One thus obtains

$$\frac{\mathrm{d}\tau_-}{\mathrm{d}\eta_0} = -\tau_-\{\frac{1}{1+\eta_0} + (\frac{\alpha_1}{1+\alpha_1\eta'_1})\frac{\mathrm{d}\eta'_1}{\mathrm{d}\eta_0} + (\frac{\alpha_2}{1+\alpha_2\eta'_2})\frac{\mathrm{d}\eta'_2}{\mathrm{d}\eta_0} + \ldots\} \qquad (8.9)$$

so that $\frac{\mathrm{d}\tau_-}{\mathrm{d}t_-} = 1 = \frac{\mathrm{d}\tau_+}{\mathrm{d}t_+}$ at $\eta_0 = 0$ which means that the first derivative of the solution is indeed continuous for all η_0. However, as is verified easily from equation (8.9), the second derivative of τ_- at $\eta_0 = 0$ is not zero, as one expects on the basis of the standard solution $\tau_s = t$. Indeed, one can verify that $\frac{\mathrm{d}^2\tau_-}{\mathrm{d}\eta_0^2} = 2(1 - \frac{1+\epsilon_1}{1-\epsilon_1} - \ldots) \neq 0$ at $\eta_0 = 0$, unless $\epsilon_n = 0$, for all n. In this special case, i.e., when $\epsilon_n = 0$, $\forall n$, our solution (8.8) reduces to the standard solution, since $\tau_- = \frac{1}{1+\eta_0}\frac{1}{1+\eta_0^2}\frac{1}{1+\eta_0^4}\ldots = 1 - \eta_0 = t_-$.

It thus follows that the solution (8.8), with nonzero scaling parameters, is indeed nontrivial, because of this second derivative discontinuity at $\eta_0 = 0$, that is at $t = 1$. In fact, the scaling invariance of equation (8.1) tells also that, $t = 1$ could be realized as $t \to t/t_0 = 1$, so that the nontrivial solution (8.8) actually holds in the neighbourhood of every real number t_0, the 2nd derivative being discontinuous at $t = t_0$. Let us note here that $\tau_- = \tau_{s-}(1 + O(\eta_0^2))$, besides the arbitrariness of the scaling parameters ϵ_n. Combining the standard and the new solutions together, one can finally write down a more general one parameter class of solutions

$$\tau_g(t) = t(1 + \phi(t)), \ \phi(t) = \epsilon t^{-1} \tau(t). \tag{8.10}$$

Note that

$$t \frac{d\phi}{dt} = 0 \tag{8.11}$$

because τ is an exact solution of equation (8.1). The 2nd derivative discontinuity of τ, however, tells that ϕ can not be considered simply as an ordinary constant.

8.3 Salient features

The salient features of this solution are the following.

1. The solution has discontinuous second derivative at $t = 1$. The said discontinuity is an effect of an infinity of nonzero rescaling parameters ϵ_n. For a finite set of ϵ_n (or in the special case when $\epsilon_n = 0, \forall n$), one gets back the standard solution. Moreover, the scale invariance is realized only in a one sided manner. The scaling $\alpha_n t_- = t'_-$ does not mean $\alpha_n t_+ = t'_+$.

2. It also follows that the solution (8.8) is indeed an exact solution of equation (8.1) when the ordinary real variable $t = 1 - \eta_0$ (near $t = 1$) is replaced by the fat real variable $\mathbf{t}_-^{-1} = \Pi_0^\infty t'_{n+}$. The fat variable $\mathbf{t}$ leaves in $\mathbf{R}$, a nonstandard extension of the ordinary real number set R, inhabiting

infinitesimal scales (variables) $\ln t'_n \approx \alpha_n \eta'_n$. All these variables can be treated as independent because of the arbitrary scaling parameters ϵ_n.

3. Finally, it is easy to verify that equation (8.1) possesses yet another (nontrivial) class of infinitely differentiable solutions of the form

$$\tau'(t) = \begin{cases} \tau'_- \ if \ t \lessapprox 1 \\ \tau'_+ \ if \ t \gtrapprox 1 \ , \end{cases} \qquad (8.12)$$

$\tau'_-(t_-) = (1/t_+)f(\eta_0)$, $\tau_+(t_+)' = (1/t_-)(f(\eta_0), f(\eta_0) = \frac{1}{t'_{1+}}\frac{1}{t'_{2+}}\cdots$

which is, however, distinct from the standard solution. Note that the infinite differentiability is restored because of identical self similar corrections in $\tau'_\pm$. However, as it should be evident from the above derivations, the iteration schemes for both τ'_- and τ'_+ could be run independently with different sets of scaling factors ϵ_n and ϵ'_n respectively, leading again to second derivative discontinuity. Besides these second derivative discontinuous solutions, equation (8.1), also accommodates a larger class of C^{2^n-1} solutions. Consequently, the simple ODE (8.1) accommodates indeed an astonishingly rich set of solutions belonging to different differentiability classes.

We have presented new families of higher derivative discontinuous solutions of the ODE (8.1), which apparently do not respect the Picard's theorem. The origin of this violation could be traced to the fact that a variable in **R** may undergo changes (increments) via the extended SL(2, R)-like group actions. These solutions break explicitly the parity symmetry of the underlying ODE (for details, see Sec.9.3).

Chapter 9

NOVEL SOLUTIONS: APPLICATIONS

9.1 Introduction

The origin of the arrow of time is still considered to be a puzzling problem in theoretical physics [47]. Another difficult problem is the $1/f$ noise [48], a footprint of complexity. The ubiquity of $1/f$-like noise in diverse natural and biological processes seems, in particular, to signal to certain key, but still not clearly understood, dynamical principles that might be at work universally at the heart of any given dynamical process. Recently, there seems to have been an emerging urge in literature [4] for a new principle for understanding complex, intrinsically irreversible, processes in nature.

In this chapter, we explain how both the above problems might acquire a natural resolution in the class of such novel solutions. Both the problems were originally discussed in [27, 31, 32]. We also reinterpret, in the present framework, some new results on unimodal logistic map [51, 52] at the chaos threshold. At the end, we show how a hyperbolic type distribution arises naturally at the asymptotic late time ($t \to \infty$) limit even from a normally distributed variate.

9.2 Randomness

Is randomness a fundamental principle (law) controlling our life and all natural processes? Or, is it simply a projection of our limitations in comprehending such complex phenomena? Is there any well defined boundary separating simple and complex? These and similar related questions on the actual status of randomness are being vigorously investigated in the literature [4, 49]. The new class of discontinuous solutions sheds altogether a new light on the ontological status of randomness. (It is believed that randomness in quantum mechanics arises at a fundamental level. However, the Schrödinger equation, the governing equation of any quantal state, is purely deterministic and time symmetric. The random behaviour is ascribed only through an extraneous hypothesis of a 'collapsed state' at the level of measurement (see,[4]))

Let us note that equation (8.1) is the simplest ODE, t being an ordinary real variable. In the framework of the conventional analysis, one can not, in any way, expect, at such an elementary level, a random behaviour in its, so called unique (Picard's), solution. (The chaos and unpredictability could arise only in the presence of explicit nonlinearity in higher order ODEs.) However, the nontrivial scaling, along with the initial ansatz (8.2) and (8.8), reveals not only the self similarity of C^{2^n-1} solutions over scales η^{2^n}, but also exposes a subtle role of *decision making* and randomness in generating nontrivial late t behaviour of the solution.

A basic assumption in the framework of the standard calculus is that a real variable t changes by linear translation only. Further, t assumes (attains) every real number exactly. However, in every computational problem within a well specified error bar, a real number is determined only up to a finite degree of accuracy ϵ_0, say. Suppose, for example, in a computation, a real variable t is determined upto an accuracy of ± 0.01, so that t here effectively stands for the closure of set $t_\epsilon \equiv \{t \pm \epsilon,\ \epsilon < \epsilon_0 = 0.01\}$ with cardinality c of the continuum. We call ϵ an 'infinitesimally small' real number (variable). Now, any laboratory computational problem (experiment) is run only over a finite time span, and the influence of such infinitesimally small ϵ's, being insignificantly small, could in fact be disregarded. Consequently, the variable t could be written near 1, for instance, as $t_1 = 1 + \eta$, where η is an ordinary real variable close to 0, having 'exact' values as long as one disregards infinitesimal numbers ($< \epsilon_0$) due to *practical limitations.* At the level of mathematical analysis, such practical limitations being indicative only of natural (physical/biological) imperfections that should not jeopardy the existence of an abstract theory of sets, real number system, calculus and so on, shaping the logical framework for an exact and deterministic understanding of natural processes. That such an attempt would remain as an unfulfilled dream is not surprising in view of the new class of discontinuous solutions. As discussed in [30](see also [31, 32, 33]), the C^{2^n-1} solution shows that the real number set R should actually be identified with a nonstandard real number set $\mathbf{R}$ [35] so that every real number t *is a fat* real (hyperreal) number $\mathbf{t}$, which means that $t \equiv \mathbf{t}$. Accordingly, a real number t could not be represented simply by a structureless point, but in fact is embedded in a sea of irreducible fluctuations of infinitesimally small numbers $\mathbf{t} = tt_f$,

$t_f = 1 + \phi$ denoting *universal* random corrections from infinitesimals ϕ (c.f., equation(8.9)). The origin of randomness is obviously tied to the *freedom* of injecting an infinite sequence of arbitrary scaling parameters ϵ_n into the C^{2^n-1} solutions because of scale invariance of equation (8.1), introducing small scale uncertainty (indeterminacy) in the original variable. (Note that scaling at each stage introduces a degree of uncertainty in the original variable viz., $\delta\eta_1 = |\eta_1' - \eta_1| = \epsilon_1/\alpha_1$ and so on.) It is also clear that the set of infinitesimals $\mathbf{0} = \{\pm\phi\}$ has a Cantor set like structure viz., discrete, dust like points separated by voids of all possible sizes and having the cardinality 2^c [31, 32]. Accordingly, an infinitesimal variable could change (within an infinitesimal neighbourhood of a point, say, 1) only by discrete *jumps* (inversions) of the form $t_- = t_+^{-\alpha}$, $t_\pm = 1 \pm \phi$, to cross the gulf of emptiness, length of jumps being arbitrary because of an arbitrary α. Note that, the value of a small real number η_0 is uncertain not only upto $O(\eta_0^2)$, but also because of the arbitrary parameter ϵ. Note also that the solution (8.8) proves explicitly that inversion is also a valid mode of change for a real variable, at least in an *infinitesimal* neighbourhood of an ordinary real number. We note further that two solutions τ_g and τ_s are indistinguishable for $t \sim O(1)$ and $\eta_0^2 << 1$. However, for a sufficiently large $t \sim O(\epsilon^{-1})(\equiv O(\eta_0^{-2}))$, the behaviours of two solutions would clearly be different. Finally, the order of discontinuity could be "controlled" by an application of an *intelligent decision* invoking a nonzero $\epsilon_n \neq 0$, $\epsilon_m = 0$, $m < n$ only at the nth level of iteration. This *freedom of decision making* could either be utilized at a pretty early stage of iterations, for instance, $n = 1$, say, making the system corresponding to the ODE (8.1) *fully intelligent*, or be *postponed* indefinitely ($n = \infty$) reproducing the standard Picard's solution for a material (non-intelligent) system [33]. We note that this randomness and potential intelligence being intrinsic properties of the solutions of equation (8.1) *are indelibly rooted to the real number system and hence could not simply be interpreted as due to some coarse graining effect* analogous to thermodynamics and statistical mechanics [4].

The new mathematical results explored here (and in[30, 31, 32, 33]) would likely to have some profound implications. With intelligence (and decision making) emerging as a fundamentally new ingredient (degree of freedom) from the mathematical analysis, the traditional framework of a physical theory viz., space, time, matter, energy (or in a relativistic theory, spacetime and energy) might, in future, be extended and replaced by a *truly dynamical* framework consisting of *intelligence, space, time, matter, energy* as envisioned in Ref.[4]. To explore the dynamic properties of the new solutions further, we now examine the origin of time's arrow in the following.

9.3 Reflection symmetry breaking and time

It is well known that *time* is *directed*, that is to say, we all have a sense of a forward moving time. The problem of time asymmetry [47] points to a fundamental dichotomy between the (Newtonian) 'time' in physics and mathematics and that of our (objective) experiences. The Newtonian time is non-directed. There is no way to distinguish between a space like variable x with a time variable t. Further, most of the fundamental equations of physics viz, the Newton's classical equation of motion, the General Relativistic field equations and the quantum mechanical Schrodinger equation, are time reversal symmetric. However, the existence of C^{2^n-1}solutions of equation (8.1) presents us with a new scenario! One is now obliged to re-examine the conventionally accepted standard notions under this new light. As mentioned already, we show here that time *does* indeed have an arrow, which is inherited, not only by all (physical /biological /social) dynamical systems, but it is also indelibly inscribed even to a real number. The concept of time thus turns out to be more fundamental compared to space and may even be considered at par with the real number system (and hence to the existence of intelligence as a fundamental entity)!

To see how a time sense is attached to C^{2^n} solutions, we recall first that the variable t is, in general, non-dynamical, and need not denote the (forward) flow of time. In fact, it simply behaves as a labeling parameter. Further, the inversion $t_- \leftrightarrow t_-^{-1} = t_+$ may at most be considered as a reversible random fluctuation between $t_\pm$ (for a given $\eta > 0$) with equal probability 1/2. In the usual treatment of ordinary calculus and classical dynamics, t is a non-random ordinary variable, and the above inversion reduces to the symmetry of equation (8.1) under reversal of sign (parity) $t \rightarrow -t$. The infinitely differentiable standard solution $\tau_s(t)$, written (in the notation of equation (8.2)) as $\tau_{s-} = t_-$, $\tau_{s+} = t_+$, $(\tau_s(1) = 1)$ is obviously symmetric under this reversible inversion $(\eta \rightarrow -\eta)$. The nontrivial solution $\tau(t)$ in equation (8.8), however, constitutes an explicit example where *this parity invariance is dynamically broken*, viz.; when the inversion is realized in an irreversible (one-sided, directed) sense.

To state the above more precisely, let $P : Pt_\pm = t_\mp$ denote the reflection transformation near $t = 1$ ($P\eta = -\eta$ near $\eta = 0$). Clearly, equation (8.1) is parity symmetric. So is the standard solution $\tau_{s\pm} = t_\pm$ (since $P\tau_s = \tau_s$). However, the solution (8.8) breaks this discrete symmetry spontaneously: $\tau_-^P = P\tau_+ = t_-$, $\tau_+^P = P\tau_- = C\frac{1}{t_-}\frac{1}{t'_{1+}}\frac{1}{t'_{2+}}\ldots$, which is of course a solution of equation (8.1), but clearly differs from the original solution, $\tau_\pm^P \neq \tau_\pm$.

To see more clearly how this one-sided inversion is realized, let $t \to 1^-$ from the initial point $t \approx 0$. Then at *a* point in the infinitesimal neighbourhood of $t_- \lessapprox 1$, the solution τ_- carries (transfers) t_- instantaneously to t_+ by an inversion $\tau_- \approx 1/t_+$ (we disregard here the $O(\eta_0^2)$ and lower order self similar fluctuations) and subsequently the solution follows the (standard) path $\tau_+ = t_+$ in the small scale variable $\eta \gtrapprox 0$, as it is now free to follow the standard path till it grows to $O(\lessapprox 1)$, when second order transition to the next smaller scale variable by inversion becomes permissible: $\eta = 1/\eta_+, \eta_+ = 1 + \bar{\eta}$, $\bar{\eta} \gtrapprox 0$, and so on. Clearly, the generic pattern of (irreversible) (time asymmetric) evolution in $\tau(t)$ over smaller and smaller scales resembles more and more closely the infinite continued fraction of the golden mean: $\tau_-(t) = t$, $0 < t \uparrow\lessapprox 1, \to \tau_- = 1/(1+\eta)$, $0 < \eta \uparrow\lessapprox 1, \to \tau_- = 1/(1 + 1/(1+\bar{\eta}))$, $\bar{\eta} \gtrapprox 0$, and so, $\tau_-(t) \to \nu$, $\nu = (\sqrt{5}-1)/2$, the golden mean, as $t \to \infty$. Here, $t \uparrow$ means that t is an increasing variable. We note that the (macroscopic) variable t is *reversible* as long as $t \sim 1$. Subsequently, this parity symmetry is broken by a random inversion, leading the evolution irreversibly to

9.4 Logistic Maps

Let us now point out an interesting application in noninveribile one dimension maps in nonlinear dynamics. There have been some new, previously unexposed, asymptotic scaling properties of the iterates of unimodal logistic maps at the edge of chaos [51, 52]. For definiteness, we consider here the dynamics of the sensitivity ξ_t to the initial conditions for large iteration time t, at the chaos threshold $\mu = \mu_\infty = 1.40115...$ of the map $f_\mu(x) = 1 - \mu|x|^2$, $-1 \leq x \leq 1$ (for notations see [51]). For a sufficiently large t, the ordinary exponential behaviour of sensitivity gives away to a power law behaviour, and is shown to have the $q-$exponential form given by

$$\xi_t = \exp_q(\lambda_q t) \equiv [1 + (1-q)\lambda_q t)]^{1/(1-q)}. \tag{9.1}$$

The standard exponential dependence $\xi_t = \exp(\lambda_1 t)$ is retrieved at the limit $q \to 1$. The system is said to be strongly insensitive (sensitive) to initial conditions if $\lambda_1 < 0$ ($\lambda_1 > 0$). The behaviour, however, gets altered at the edge of chaos. Using the Feigenbaum's RG doubling transformation $\hat{R}f(x) = \alpha f(f(x/\alpha))$ n times to the fixed point map $g(x)$ viz., $g(x) = \hat{R}^n g(x) \equiv \alpha^n g^{2^n}(x/\alpha^n)$, the values of the $q-$Lyapunov coefficient λ_q and q are determined to be $\lambda_q = \ln\alpha/\ln 2\,(>0)$, $q = 1 - \ln 2/\ln\alpha\,(<1)$, α being one of the universal Feigenbaum constants, $\alpha = 2.50290....$ Consequently, the critical dynamics corresponds to *weak chaos.* Notice that the dynamics at the chaos threshold, being the most prominent and readily accessible to numerical experiments, among the critical points of a quadratic map, reveals a *universal* concerted behaviour, described by the fixed point solution of the RG doubling transformation. One, however, needs to explain the origin of weak chaos, i.e., the system being weakly *sensitive*, rather than being only weakly *insensitive*, to initial conditions, at the chaos threshold, when the system approaches the threshold from the left through period doubling route (for $\mu < \mu_\infty$).

To reinterpret above observations in the present context, we note first of all that the fixed point equation $\tilde{g}(x) = g(x)$ where $\tilde{g}(x) = \alpha g(g(x/\alpha))$, is a solution of equation (8.1), viz., $d\tilde{g}/dg = \tilde{g}/g$. Consequently, one expects that the critical dynamics of unimodal quadratic maps at the onset of chaos would be linked directly to our nontrivial solutions to equation (8.1). As an explicit example, let us investigate the origin of the q-exponential type power law dynamics from the ordinary exponential one

$$\frac{d\xi}{dt} = \lambda_1 \xi. \tag{9.2}$$

As the critical point is approached from left (say), the Lyapunov exponent $\lambda_1 \to 0^-$, and hence gets replaced, in our extended framework, by a 'dynamic' infinitesimal (ordinary zero is an equivalence class of infinitesimals) of the form $\lambda_1 \to -\epsilon\lambda_p\phi(t_1)$, where $\epsilon\,(> 0)$ is an infinitesimal ($\epsilon \neq 0$, ϵ^2 =O(0)) scaling parameter, $\phi = t_1\tau(t_1^{-1})$ ($t_1 = \epsilon t$), and $\lambda_p > 0$, $p = p(\epsilon)$ are two generalized constants (c.f., equation (8.11)), both of O(1). However, when the variation in the intrinsic time like variable ϕ becomes relevant at the scale $t \sim 1/\epsilon$, that of λ_p would be relevant only at a longer scale viz., $t \sim 1/\epsilon^2$ or more and could be considered as an ordinary constant. The reason for introducing the (third) generalized constant (again with a slower variation than ϕ) p will become clear below. At a critical point, Equation (9.2) now reduces to the RG -like equation

$$t_{1+}\frac{\mathrm{d}\xi}{\mathrm{d}t_{1+}} = \lambda_p \xi. \tag{9.3}$$

To explain the derivation of the above equation let us proceed in steps.

(i) The rescaled variable t_1, defined by $t_1 = \epsilon t$, is O(1) when $t \sim$ O$(1/\epsilon)$. Consequently, the critical dynamics would be revealed only at *a sufficiently long time scale* $t_{1-} = 1 - \eta \to 1^-$ i.e., $t \to (1/\epsilon)^-$. That means *the dynamics at the chaos threshold needs to be probed in an extended framework, viz., in the sense of a limit as the control parameter* μ *approaches* μ_∞ *from left (say) through period-doubling cascade* instead of simply replacing μ by μ_∞ in the map and then iterating. In any computational problem, this extended framework is automatically realized, because of the inherent finite bit (decimal) representation of a real number, such as μ_∞, exact value of which could only be approached recursively by increasing its accuracy. The infinitesimal ϵ (along with an O(1) variability as represented by $\lambda_p \phi(t_1)$) then simply corresponds to the infinite trailing bits in the finitely represented real number e.g., μ_∞.

(ii) The critical point equation (9.3) follows when one makes use of the relation $d \ln t_{1-} = -d \ln t_{1+}$, which is valid for infinitesimal η with O(η^2)=0 [33]. Clearly, the solution to the above equation is $\xi = [1 + p\lambda_p \eta]^{1/p}$ which corresponds exactly to the q exponential (9.1) provided we choose $q = 1 - p$, $p\lambda_p = 1$.

(iii) The q exponential solution is, however, valid not only in the 'infinitesimal' neighbourhood of $t_1 = 1$, but for arbitrarily large t_1, because of the scale invariance of equation (9.3). Indeed, writing $2^n t_1 = 1 + \tilde{t}$, the q exponential sensitivity takes the form $\xi = [1 + (1-q)\lambda_q \tilde{t}]^{1/(1-q)}$, $\tilde{t}$ being large.

(iv) The scale factors 2^n correspond to the times to determine the trajectory positions x_{2^n} of the logistic map with an initial position x_{in-} [51]. It follows therefore that the ratio of the sensitivities at times 2^n and 2^{n+1} viz., $\frac{\xi_{2^{n+1}}}{\xi_{2^n}} = (\frac{2^n t_1}{2^{n+1} t_1})^{1/p} = \alpha$ when p remains constant at $1/p = \ln \alpha / \ln 2$ upto a time $T \approx 2^N$, $N \approx |\ln \xi_{in} / \ln \alpha|$, ξ_{in} being the initial value of sensitivity, corresponding to the initial iterate $n = 0$.

(v) Indeed, to obtain the later estimate, we note that $\xi_{in}^p = t_1$, which follows from equation (9.3). For a non-zero (sufficiently large) n, it now follows that $\xi_n \equiv (2^n \xi_{in}^p)^{1/p} = \alpha^n$ which translates to $p = \frac{\ln 2}{\ln \alpha}(1 + \frac{\ln \xi_{in}}{n \ln \alpha})$. Consequently, a possible variation in p would be revealed only when $n \approx |\ln \xi_{in} / \ln \alpha|$, as claimed. Note that q exponential form in the neighbourhood of $t_1 = 1$ belongs to the class of solutions (8.12) provided the generalized constant p is given by $1/p = \ln f / \ln t_1 - 1$, $(dp / \ln t_1) = 0$.

The change in sensitivity from strongly insensitive case ($\lambda_1 < 0$) to weakly sensitive ($\lambda_q > 0$, $q < 1$) power law behaviour is thus explained as an effect of nontrivial infinitesimals and associated inversion $t_{1-} t_{1+} \approx 1$ in the infinitesimal neighbourhood of $t_1 = 1$. The origin of Feigenbaum's constant α (notice our use of $\xi_n = \alpha^n$ in the above derivation) in the present formalism along with other relevant issues will be considered elsewhere. Another interesting problem is to identify the golden mean number ν in the critical dynamics.

9.5 $1/f$ spectrum

The relevance of higher derivative discontinuous solutions to the origin in $1/f$ noise problem have been discussed in detail in[[31, 32]]. We note here that in the extended framework of a dynamical theory, accommodating these solutions, any physical, variable t, say time, is replaced by $t^{1+\sigma}$, $\sigma = \ln(1+\phi)/\ln t$ (c.f., Sec.3) where σ typically is small $O(\epsilon)$ for any $\epsilon > 0$. The nonzero exponent σ introduces a small stochastic fluctuations over the ordinary (time) variable t. Clearly, these small scale stochastic fluctuations, existing purely in the real number system, would remain insignificant for any terrestrial (laboratory) *inanimate* system which persists over a moderate time scale, such as the motion of a (classical) particle under gravity. However, even for simple electrical circuits where the voltage fluctuation spectrum $S_V(f)$ is known to vary proportionally with the thermal fluctuation spectrum $S_T(f)$, the origin of $1/f$ noise as observed in [53] could be naturally ascribed to the C^{2^n-1} solutions of

$$c\frac{\mathrm{d}T}{\mathrm{d}t} = -g(T - T_0). \tag{9.4}$$

This equation describes the macroscopic (equilibrium) variations of the temperature (T) of a resistive system with heat capacity c, coupled through a thermal conductance g to a heat source at temperature T_0 [53]. According to the conventional knowledge one does not expect $1/f$ spectrum from such a simple, purely deterministic, linear uniscale system. One needs, in fact, to consider extraneous nonlinear effects from environment to explain the origin of the generic $1/f$ fluctuations.

However, according to the present analysis, even this simple system would behave stochastically because of small scale, intrinsic fluctuations in the time variable t. These scale free fluctuations could influence the late time behaviour of the system provided *the system is 'allowed' to survive* over a period $t >> 1/\epsilon$, $\epsilon = g/c$. Notice that ordinarily a system following purely equation (9.4) is assumed to relax to the equilibrium temperature T_0 after $t \approx 1/\epsilon$. The late time variability that is observed in any resistive system is then ascribed normally to the complex nature of the resistive medium and/or (nonlinear) interactions with environment [53], asking for an explicit modeling. The generic observation of $1/f$ fluctuations in metal films and semiconductors still eludes a universal explanation for its microscopic origin in the framework of conventional dynamical theories [56].

In view of C^{2^n-1} solutions, we now have an extended framework to re-examine the above problem. The solution of equation (9.4) now have the form $T(t) - T_0 = t^{\sigma}e^{-\epsilon t}$, σ being a small fluctuating variable. As noted already, this random exponent would lead to small scale stochastic modulations over the (mean) macroscopic decay mode, as observed in physical systems. These small scale (power law) fluctuations would persists even far beyond the ordinary relaxation time. Accordingly, a time series of temperature fluctuations ($T_f(t) = (T(t) - T_0)e^{\tau}$) recorded over a period of a few decades (1 to 10^4, say), in the unit of the dimensionless time $\tau = \epsilon t$, would reveal a scale free $1/f$ type variability. For, the two point autocorrelation function of this intrinsic fluctuations has the form $C(t) =< T_f(t)T_f(0) >= c < T_f(t) >= c < t^{\sigma} >\approx ct^{<\sigma>}$, $T_f(0) = c$, where $< \sigma >$ is the expectation value of the random exponent and $c \sim$O(1) is the initial (background) noise in the system. The associated probability distribution would have a generic late t behaviour, resembling infinitely divisible Levi type distributions (see below). The corresponding power spectrum of this stochastic scale free fluctuation is given by $S(f) \sim 1/f^{1-<\sigma>}$. We note that an arbitrarily small nonzero σ is sufficient to generate a $1/f$-like fluctuation. In other words, intrinsically random, infinitesimal scales associated with the time variable t could act as a perennial source of small scale fluctuations leading to the universal low frequency $1/f$ spectrum. However, an accurate determination of the intensity of the fluctuations (viz, the constant of proportionality in the observed spectrum $S_V(f) \propto V^2/f^a$, $a \approx 1$) [53, 56] may require further work. The relevance of number theory to $1/f$ noise problem is also pointed out by Planat [57]. El Naschie suggested that the exponent β of the $1/f^\beta$ noise to semiconductors would be related to the golden mean in the framework of the E-infinity theory [58].

9.6 Hyperbolic distribution

In Ref.[31] we show that the scale free infinitesimal fluctuations follow a nongaussian, Bramewell-Holdsworth-Pinton (BHP) [59] distribution. Here we show how a hyperbolic, power law tail gets superposed generically in *any* distribution when the concerned random variate is assumed to leave in **R**. To see this it suffices to consider only a normally distributed variate, because by the central limit theorem a normal probability density acts as the attractor for any probability density with finite moments.

Let t be a zero mean normal variate, with unit standard deviation. The corresponding fat variate could be written as $\mathbf{t}^2 = t^2 + \phi$, $\phi = \epsilon(t)\ln t^2$, being a random infinitesimal satisfying equation (8.11) in the logarithmic variable $\ln t$ (c.f., Appendix). Consequently, the normal density function $\propto e^{-t^2/2}$ gets a generic power law tail $t^{-\epsilon}e^{-t^2/2}$. This generic power law tail in the present extended formalism should become important in the future studies on the statistics of rare events. We close with the remark that occasional detections of exceptional events in an experiment of, for instance, a normal variate over a prolonged period could be explained by this slowly varying tail. Note that the power law variability would become visible only in the asymptotic limits ($t \to \pm\infty$) because of an infinitesimal ϵ, so that ϕ remains vanishingly small in any laboratory experiments over a finite time scale.

9.7 Appendix

A continuously differentiable function $f(t)$ of a real variable could be defined as an integral of the ODE $\frac{dx}{dt} = f'(t)$. For the gaussian $e^{-t^2/2}$, the relevant equation is $\frac{dx}{dt} = -te^{-t^2/2}$, and hence the corresponding hyperreal (fat) extension is given by $\mathbf{t}^2 = t^2 + \epsilon(t)\ln t^2$. The extension of the linear variable t is given by $\mathbf{t} = t + \epsilon(t)\ln t$. The extended exponential $e^{\mathbf{t}} = t^{\epsilon}e^{t}$ would therefore have a slowly fluctuating power law tail. Note that the variable t gets the infinitesimal correction term in the logarithmic variable $\ln t$, when the infinitesimal ϵ satisfies the equation

$$\sigma\frac{d\epsilon}{d\sigma} = -\epsilon,\ \sigma = \ln t$$

Chapter 10

DIFFERENTIAL EQUATION AND CANTOR SET

10.1 Introduction

A multiplicative representation (8.9) for the solution of Cauchy problem (8.1) is discussed in detail in the previous chapter. Because of it's multiplicative structure there is a direct relevance of the said problem in the context of a Cantor set (or, equivalently to an ultrametric space). Here we assume that the problem IVP (8.1) is defined on a Cantor set $\mathbf{C}$ which is realized as an inequivalent ultrametric space. As a consequence, each point of a Cantor set is replaced by the infinitesimal copy of an inverted Cantor set $\mathbf{C}_i$. Thus, the said scale invariant equation of chapter 9 actually is well defined on a closed set of the form $\widetilde{\mathbf{C}} = \mathbf{C} + \mathbf{C}_i$ that replaces a given Cantor set and so the above multiplicative model (8.8) is given a novel interpretation. As already established (c.f., Sec. 7.2), $\widetilde{C}$ is almost surely I or a positive measure set C_p. The results of this chapter are presented in Ref.[26].

10.2 Mathematical results

We study the relationship of the scale free DE of the form

$$x\frac{dX}{dx} = X \tag{10.1}$$

with a Cantor set C. As a preparation, let us recall how the simplest Cauchy problem

$$\frac{dX}{dx} = 1, \ X\ (1) = 1 \tag{10.2}$$

is solved on the interval $I = [0, 1]$. One considers a partition $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$. The desired result $X\ (x) = x$ is obtained as a limit of a sum: $\lim_{\substack{\Delta x_j \to 0 \\ n \to \infty}} \sum_{j=1}^{i} \Delta\, x_j$ where $x_{i-1} < x < x_i$. The scale free Cauchy problem

$$x\frac{dX}{dx} = X, \ X\ (1) = 1 \tag{10.3}$$

is also solved exactly in an analogous fashion.

First, we note that the neighbourhood of a point x_0 is mapped to that of $x = 1$ by a rescaling $x \to \frac{x}{x_0}$. So we concentrate only in the neighbour of $x = 1$. Let $x_\pm = 1 \pm \eta$, $X_\pm = X\ (x_\pm)$. Then the DE in (10.3) splits into two branches

$$x_\pm \frac{dX_\pm}{dx_\pm} = X_\pm. \tag{10.4}$$

The equation is already solved in chapter 8 and the solution is expressed in a non-trivial form

$$X_- = \prod_{i=0}^{\infty} \frac{1}{1 + \eta^{2^i}}. \tag{10.5}$$

The right hand branch, however, has the form

$$X_+ = \frac{1}{1-\eta} \prod_{i=1}^{\infty} \frac{1}{1+\eta^{2^i}}. \tag{10.6}$$

The infinite product representation of X_-, for instance, is interpreted as follows. The first iterated value exceeds the exact value by an amount $\frac{\eta^2}{1\pm\eta}$ which is canceled progressively in a self -similar manner over smaller and smaller inverted scales log $\left(1-\eta^{2^i}\right)^{-1}$, $i = 1, 2, 3, \cdots$.

We note that the higher order correction factors $X_c = \prod_{i=1}^{\infty} \frac{1}{1+\eta^{2^i}}$ may therefore be re-interpreted as a deletion process: viz; a portion of a line segment is deleted progressively and self -similarly, analogous to the formation of a Cantor set. Alternately, a product of the form $(1-\eta)\ (1+\eta) = 1-\eta^2$ could be considered to represent a deletion: a length of size η in the neighbourhood of 1^-, is deleted progressively as $(1-\eta)(1+\eta)(1+\eta^2)\cdots\left(1+\eta^{2^{n-1}}\right) = \left(1-\eta^{2^n}\right)$, $n \to \infty$.

A second possibility is to interpret the multiplicative iteration process defined above as a dynamical process in which the dynamic (independent) variable undergoes increments not by the usual linear translations but by inversions (hoppings) over smaller and smaller sizes. This then provides one with a mechanism of deletion process stated above. Actually, these logarithmic scales inhabits concomitant smaller scales of the from η^i. To justify this in a greater detail, let us assume that the scale free problem (10.3) is now defined on a closed subset $\tilde{\mathbf{C}} \subseteq I$, called an *inverted Cantor set*, where $\tilde{\mathbf{C}} = \bigcup_i \tilde{I}_i$ is a countable union of disjoint closed intervals

$\tilde{I}_i$ of varying sizes. $\tilde{I}_i$ in fact, is the closure of a corresponding gap O_i (inclusive of the end points) of the original Cantor set C. Suppose, a dynamic variable (say, a particle) in motion on this set $\tilde{\mathrm{C}}$, hops between the end points of each of such disjoint closed intervals $\tilde{I}_i$, following the scale free DE (10.3). Let $|\tilde{I}_0| = \eta$ be the maximum hopping size. Then the smaller hopping sizes are η proportion of the remaining sizes of the set $\tilde{\mathrm{C}}$ viz., $|\tilde{I}_i| = \eta\ (1-\eta)^{i-1} \equiv \eta_i$ (say), when we assume $|\tilde{\mathrm{C}}| = 1$. Because of the rescaling symmetry (scale invariance) of the DE (10.3) each of the component intervals $\tilde{I}_i$ could be imagined to have been symmetrically placed at 1 with end points, say, at $x_{0\pm} = \frac{1}{2}\ (1 \pm \eta)$. Now, the particle at left end points x_{0-} of $\tilde{I}_0$ hops to the right end point x_{0+} following the rule

$$x_{0-} \to x_{0-}^{-1} = x_{0+}\, X_1 \tag{10.7}$$

so that we have, using equation (10.6)

$$(1-\eta)^{-1} = (1+\eta)\ X_c,\ X_c = \prod_{i=1}^{\infty}\left(1+\eta^{2^i}\right) \tag{10.8}$$

because of the scale invariance. Equation (10.8) tells that hopping motion of the type considered above, of any given size η is accomplished by an infinite cascade of self similar smaller scale inverted motions of sizes η^{2^i}, $i = 1, 2, \cdots$. The total length covered by all these self-similar jumps , viz., 1, is reached multiplicatively i.e. as $1 = \lim_{n\to\infty}\left(1-\eta^{2^n}\right)$, reminiscent of an ultrametric limiting process. Notice that in the ordinary sense, the total jump size is determined additively as an infinite series viz., $\sum_1^{\infty} \eta\ (1-\eta)^{i-1} = \Sigma\eta_i\ =\ 1$.

Accordingly, each $x \in C$ is replaced by infinitesimal copy of an inverted Cantor set $\tilde{\mathbf{C}}$. Because of scale invariance, the DE (10.3) at an $x \in C$, which is actually not defined in the usual (even in the natural ultrametric) sense, is now raised to an equation which is well defined on a closed set of the form $\tilde{\mathbf{C}}$.

We note that the solution (10.5) and (10.6) is the standard solution derived in an unconventional way and interpreted non-trivially. On a Cantor set, however, the equation (10.3) can accommodate a host of new solutions in consonance with the multiplicative model interpretation. The origin of these new solutions could be explained in the context of locally constant functions (LCF). To justify, in a most natural way, the existence of locally constant functions, let us write a solution of equation (10.3) in the form

$$X = x.x^{\phi(x^{-1})}. \tag{10.9}$$

The function $\phi(x)$ here represents a LCF and is defined by the scale free equation on logarithmic variables, viz:

$$\log x^{-1} \frac{d\phi}{d \log x^{-1}} = \phi. \tag{10.10}$$

Clearly $\phi(x)$ corresponds to our non-trivial valuation (3.2) denoted $v(\tilde{x}) = \mid \tilde{x} \mid_u$. To verify $v(\tilde{x})$, indeed is a LCF, we note that

$$\frac{d}{dx} v(\tilde{x}) = \lim_{\epsilon \to 0} \frac{d}{dx} \left(\frac{\log x}{\log \epsilon} + 1 \right) = 0. \tag{10.11}$$

Equation (10.10), on the other hand, reveals the variability of a LCF over smaller logarithmic scales. Of course, the valuation also passes this test

$$\log v(\tilde{x}) = \log\log \tfrac{x}{\epsilon} + \log\log_\epsilon \lambda - \log\log \epsilon^{-1}$$

leading to equation (10.10) in the inverted rescaled real variable $\frac{x}{\epsilon}$ (in the log log scale). We have already seen that $v(\tilde{x})$ relates to an appropriate Cantor function. Consequently, a Cantor function $\phi(x)$ is shown to be a LCF with variability over log log scales. Equation (10.9) constitutes an ultrametric extension not only of a Cantor set, but of any connected interval of R.

We summerise the above findings as a theorem thus extending theorem 1 in a significant manner.

Theorem 3. *An element of an ultrametric Cantor set C is replaced by the set of gaps of the Cantor set C_i where relative infinitesimals are supposed to live in. Increments on such an extended Cantor set C is accomplished by following an inversion rule of the form (3.1). A scale free differential equation of the form equation (10.3) is well defined on such an ultrametric space and accommodates Cantor functions as locally constant functions. The associated infinitesimal valuation $v(\tilde{x})$ is a locally constant function with variability over double logarithmic scales.*

Chapter 11

CONCLUDING REMARKS

A scale invariant analytical framework on a Cantor set like fractal subsets of R is presented. Since a Cantor set C is a totally disconnected, compact, perfect subset of R, the ordinary analysis of R can not be meaningfully extended over C, i.e., when a real variable x is assumed to live and undergo changes only over the points of C. The usual ultrametric structure of a Cantor set is extended to an inequivalent class of ultrametrics exploiting the concept of relative infinitesimals those are supposed to exist on the gaps of another Cantor set in the neighbourhood of 0. A variable living on a Cantor set are shown to undergo changes by smooth, inversion induced jumps. As a consequence, the derivative discontinuity of a Cantor function at a point of the Cantor set are removed, making the global variability over double logarithmic scales of such a function everywhere smooth. Since the ultrametric valuations of scale invariant infinitesimals turn out to be a Cantor function, the corresponding scale invariant analysis formulated over a Cantor set also happens to ascribe a smooth differentiable structure over the Cantor set. This is the main advantage of the present analysis over the other competing approaches in the literature [6, 11, 10, 12].

In this thesis we report on only a few specific applications of this formalism: namely, (i) the precise form of differential jump measure on a Cantor set is derived, (ii) the interesting possibility of growth of measure is explained exploiting the underlying reparametrisation invariance of the nontrivial valuation, (iii) the variability in a family of Cantor sets with identical Hausdorff dimension and thickness are shown to be exposed by an application of the higher order contribution in the scale invariant valuation, and finally (iv) the simplest scale invariant ODE $t\frac{dx}{dt} = x$ is shown to admit a novel class of solutions which can be modeled to exist on a Cantor set. Besides these, we also have studied a few applications of the novel solutions and related ideas in some wellknown longstanding problems such as (i) the issue of randomness and time asymmetry (ii) chaos threshold in a logistic map and (iii) the origin of the ubiquitous $1/f$ noise (signal).

The present study seems to open up a large number of interesting new areas of investigations in various interdisciplinary branches, such as analysis, geometry, number theory, nonlinear dynamics, to name a few. A few extensions of the present scale invariant formulation of analysis on a Lebesgue measure zero Cantor set are already studied recently, in the context of the ordinary Calculus on R, leading to an interesting new proof of the Prime Number Theorem [29]. The scale invariant extension of R leads to a new deformed space $\mathcal{R}$ of real numbers. An analysis of the diffusion equation on this deformed space is recently completed with a novel interpretation of the emergence of anomalous mean square fluctuations when a diffusive system is allowed to execute motion over infinitely long time scales [28]. Applications of similar ideas in various other well known differential equations as well as in dynamical system theories remain to be undertaken in future.

This thesis is prepared on the basis of the following publications:

[1] D P Datta and S Raut, The arrow of time, complexity, and the scale free analysis, *Chaos, Solitons & Fractals*, 28, 581-589, (2006).

[2] S Raut and D P Datta, Analysis on a fractal set, *Fractals*, 17(1), 45-52, (2009).

[3] S Raut and D P Datta, Nonarchimedean scale invariance and Cantor sets, *Fractals*, 18, 111-118, (2010).

[4] D P Datta, S Raut, A Raychaudhuri, Ultrametric Cantor sets and Growth of Measure, *p-Adic Numbers, Ultrametric Analysis and Applications*, Vol.3, No.1, pp.7–22, (2011).

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ADDENDUM-ERRATUM WITH CORRECTIONS

I take this opportunity to sincerely apologise for my inadvertent laps in submitting the original version of the thesis with several typographical and linguistic errors as one of the lesser edited format of the manuscript which remained in the computer memory was erroneously picked up for taking printout just before the submission of the thesis.

I would also like to take this opportunity to thank the honourable Examiner for a very meticulous reading and for some illuminating comments and remarks allowing I suppose a much improved clarity in the presentation.

Detailed corrections entered

A. Typographical/Language Corrections

p.14 Both are corrected in appropriate places.

p.16 (2.6) is corrected. Repetition around (2.7) is removed.

p.17 Indicated correction included.

Pp. 19,20 Triangle inequality corrected.

p.20, l.6 corrected.

p.21, Proposition 1 All indicated corrections are included.

P.22 Cauchy convergence statement is a typo: correct version should be $|x_n| = |x|$.

p.22, l.4,5 & p.23 the format is corrected by reordering.

p.28 l.2-5 Repetition is removed.

p.43 l.7 (below Lemma 5) this is yet another misprint being copied and pasted from older files. Corrected as $v(x_{n+1}) = v(x_n)$ for $n > N$. Proof is rewritten.

p.43, l.4 above (3.3) corrected as "can assume".

p.44 statement of Theorem 1 is corrected as indicated.

p.48 Definition 12, quantifiers corrected.

p.49 Theorem 2 reframed as Remark 5 in p.50

p.57 below (4.9) symbol N removed in p. 59, similar corrections are entered in p.64.

p.58 Proposition 3 is corrected in p. 60

B. Technical Remarks and Clarifications

p.24 definition of multiplicative group is included and Proposition 2 rewritten and corrected as indicated.

p.37. To clarify Def.5 a new Remark 1 is written with further motivation and explanation. Perhaps this will suffice.

p.38 Def.7 is tried to put in a more understandable manner.

p.39 The examiners remark is respected in Notation 1.

p.45. The choice of sequence y_i perhaps needs axiom of choice. Also (3.8) is corrected and a footnote remark is made on valuation.

p.47 a new paragraph is added at the end of p.48 to explain nontrivial limit.

p.99 Corrected as "Further, most of the fundamental equations —" with some additional remarks citing some basic equations of motion, in p.101.