

## CHAPTER 2

### SOME RESULTS ON THE GROWTH PROPERTIES OF DIFFERENTIAL MONOMIALS

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## 2.1 Introduction, Definitions and Notations.

We denote by  $\mathbb{C}$  the set of all finite complex numbers. Let  $f$  be a meromorphic function and  $g$  be an entire function defined on  $\mathbb{C}$ . In the sequel we use the following notation :

$\log^{[k]} x = \log(\log^{[k-1]} x)$  for  $k = 1, 2, 3, \dots$  and  $\log^{[0]} x = x$ .

We now recall the following definitions :

**Definition 2.1.1** *The order  $\rho_f$  and lower order  $\lambda_f$  of a meromorphic function  $f$  are defined as*

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} .$$

*If  $f$  is entire, one can easily verify that*

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r} .$$

**Definition 2.1.2** The hyper order  $\bar{\rho}_f$  and hyper lower order  $\bar{\lambda}_f$  of a meromorphic function  $f$  are defined as follows

$$\bar{\rho}_f = \limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f)}{\log r} .$$

If  $f$  is entire, then

$$\bar{\rho}_f = \limsup_{r \rightarrow \infty} \frac{\log^{[3]} M(r, f)}{\log r} \quad \text{and} \quad \bar{\lambda}_f = \liminf_{r \rightarrow \infty} \frac{\log^{[3]} M(r, f)}{\log r} .$$

**Definition 2.1.3** The type  $\sigma_f$  of a meromorphic function  $f$  is defined as

$$\sigma_f = \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}}, 0 < \rho_f < \infty .$$

When  $f$  is entire, then

$$\sigma_f = \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho_f}}, 0 < \rho_f < \infty .$$

**Definition 2.1.4** A function  $\lambda_f(r)$  is called a lower proximate order of a meromorphic function  $f$  of finite lower order  $\lambda_f$  if

- (i)  $\lambda_f(r)$  is non-negative and continuous for  $r \geq r_0$ , say
- (ii)  $\lambda_f(r)$  is differentiable for  $r \geq r_0$  except possibly at isolated points at which  $\lambda'_f(r+0)$  and  $\lambda'_f(r-0)$  exist,
- (iii)  $\lim_{r \rightarrow \infty} \lambda_f(r) = \lambda_f$ ,
- (iv)  $\lim_{r \rightarrow \infty} r \lambda'_f(r) \log r = 0$  and
- (v)  $\liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda_f(r)}} = 1$ .

**Definition 2.1.5** Let 'a' be a complex number, finite or infinite. The Nevanlinna deficiency and Valiron deficiency of 'a' with respect to a meromorphic function  $f$  are defined as

$$\delta(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)}$$

and

$$\Delta(a; f) = 1 - \liminf_{r \rightarrow \infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r \rightarrow \infty} \frac{m(r, a; f)}{T(r, f)} .$$



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**Definition 2.1.6** ([38]) For  $a \in \mathbb{C} \cup \{\infty\}$ , we denote by  $n(r, a; f \mid= 1)$ , the number of simple zeros of  $f - a$  in  $|z| \leq r$ .  $N(r, a; f \mid= 1)$  is defined in terms of  $n(r, a; f \mid= 1)$  in the usual way. We put  $\delta_1(a; f) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; f \mid= 1)}{T(r, f)}$ , the deficiency of 'a' corresponding to the simple a-points of  $f$  i.e., simple zeros of  $f - a$ .

Yang [37] proved that there exists at most a denumerable number of complex numbers  $a \in \mathbb{C} \cup \{\infty\}$  for which  $\delta_1(a; f) > 0$  and

$$\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4.$$

Let  $f$  be a transcendental meromorphic function defined in the open complex plane  $\mathbb{C}$ . Also let  $n_0, n_1, \dots, n_k$  be non-negative integers such that  $\sum_{i=0}^k n_i \geq 1$ . We call  $P[f] = bf^{n_0}(f^{(1)})^{n_1} \dots (f^{(k)})^{n_k}$  where  $T(r, b) = S(r, f)$  to

be a differential monomial generated by  $f$ . The numbers  $\gamma_P = \sum_{i=0}^k n_i$  and

$\Gamma_P = \sum_{i=0}^k (i+1)n_i$  are called the degree and the weight of  $P[f]$  respectively.

In the chapter we prove some new results depending on the comparative growth properties of composite entire or meromorphic functions and differential monomials generated by one of the factors which improve some earlier theorems.

## 2.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 2.2.1** ([1]) *If  $f$  is meromorphic and  $g$  is entire then for all sufficiently large values of  $r$ ,*

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f) .$$

**Lemma 2.2.2** ([3]) *Let  $f$  be meromorphic and  $g$  be entire and suppose that  $0 < \mu < \rho_g \leq \infty$ . Then for a sequence of values of  $r$  tending to infinity,*

$$T(r, f \circ g) \geq T(\exp(r^\mu), f) .$$

**Lemma 2.2.3 ([23])** *Let  $f$  be a transcendental meromorphic function of finite order or of non-zero lower order and  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = 4$ , then*

$$\lim_{r \rightarrow \infty} \frac{T(r, P[f])}{T(r, f)} = \Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f),$$

where  $\Theta(\infty; f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, f)}{T(r, f)}$ .

**Lemma 2.2.4** *If  $f$  be a transcendental meromorphic function of finite order or of non-zero lower order and  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = 4$ , then the order and*

*lower order of  $P[f]$  are same as those of  $f$ . Also the type of  $P[f]$  is  $\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}$  times that of  $f$  when  $f$  is of finite positive order i.e.,  $\rho_{P[f]} = \rho_f$ ,  $\lambda_{P[f]} = \lambda_f$  and  $\sigma_{P[f]} = \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}\sigma_f$ .*

**Proof.** By Lemma 2.2.3,  $\lim_{r \rightarrow \infty} \frac{\log T(r, P[f])}{\log T(r, f)}$  exists and is equal to 1. So

$$\begin{aligned} \rho_{P[f]} &= \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, P[f])}{\log T(r, f)} \\ &= \rho_f \cdot 1 = \rho_f. \end{aligned}$$

In a similar manner,

$$\begin{aligned} \lambda_{P[f]} &= \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, P[f])}{\log T(r, f)} \\ &= \lambda_f \cdot 1 = \lambda_f. \end{aligned}$$

Again

$$\begin{aligned} \sigma_{P[f]} &= \limsup_{r \rightarrow \infty} \frac{T(r, P[f])}{r^{\rho_{P[f]}}} \\ &= \lim_{r \rightarrow \infty} \frac{T(r, P[f])}{T(r, f)} \cdot \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho_f}} \\ &= \{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; f)\}\sigma_f. \end{aligned}$$

This proves the lemma. ■

**Lemma 2.2.5** *Let  $f$  be a transcendental meromorphic function of finite order or of non-zero lower order and  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = 4$ . Then the hyper order (hyper lower order) of  $P[f]$  and  $f$  are equal.*

**Proof.** By Lemma 2.2.3,  $\lim_{r \rightarrow \infty} \frac{\log^{[2]} T(r, P[f])}{\log^{[2]} T(r, f)}$  exists and is equal to 1. So

$$\begin{aligned} \bar{\rho}_{P[f]} &= \limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f)}{\log r} \cdot \lim_{r \rightarrow \infty} \frac{\log^{[2]} T(r, P[f])}{\log^{[2]} T(r, f)} \\ &= \bar{\rho}_f \cdot 1 = \bar{\rho}_f. \end{aligned}$$

In a similar way,

$$\begin{aligned} \bar{\lambda}_{P[f]} &= \liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f)}{\log r} \cdot \lim_{r \rightarrow \infty} \frac{\log^{[2]} T(r, P[f])}{\log^{[2]} T(r, f)} \\ &= \bar{\lambda}_f \cdot 1 = \bar{\lambda}_f. \end{aligned}$$

Thus the lemma is established. ■

**Lemma 2.2.6** *For a meromorphic function  $f$  of finite lower order, lower proximate order exists.*

*The lemma can be proved in the line of Theorem 1[19] and so the proof is omitted.*

**Lemma 2.2.7** *Let  $f$  be a meromorphic function of finite lower order  $\lambda_f$ . Then for  $\delta (> 0)$  the function  $r^{\lambda_f + \delta - \lambda_f(r)}$  is ultimately an increasing function of  $r$ .*

**Proof.** Since

$$\frac{d}{dr} r^{\lambda_f + \delta - \lambda_f(r)} = \{\lambda_f + \delta - \lambda_f(r) - r\lambda'_f(r) \log r\} r^{\lambda_f + \delta - \lambda_f(r) - 1} > 0$$

for all sufficiently large values of  $r$ , the lemma follows. ■

**Lemma 2.2.8** ([18]) *Let  $f$  be an entire function of finite lower order. If there exist entire functions  $a_i$  ( $i = 1, 2, \dots, n$ ;  $n \leq \infty$ ) satisfying  $T(r, a_i) = o\{T(r, f)\}$  and  $\sum_{i=1}^n \delta(a_i; f) = 1$ , then  $\lim_{r \rightarrow \infty} \frac{T(r, f)}{\log M(r, f)} = \frac{1}{\pi}$ .*

## 2.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 2.3.1** *Let  $f$  be a meromorphic function and  $g$  be a transcendental entire function satisfying*

- (i)  $\lambda_f, \lambda_g$  are both finite and
- (ii)  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ . Then

$$\liminf_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, P[g])} \leq \frac{3 \cdot \rho_f \cdot 2^{\lambda_g}}{\Gamma_P - (\Gamma_P - \gamma_P) \Theta(\infty; g)}.$$

**Proof.** If  $\rho_f = \infty$ , the result is obvious: So we suppose that  $\rho_f < \infty$ . Since  $T(r, g) \leq \log^+ M(r, g)$ , in view of Lemma 2.2.1, we get for all sufficiently large values of  $r$ ,

$$T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f).$$

$$\text{i.e., } \log T(r, f \circ g) \leq \log \{1 + o(1)\} + \log T(M(r, g), f)$$

$$\text{i.e., } \log T(r, f \circ g) \leq o(1) + (\rho_f + \varepsilon) \log M(r, g)$$

$$\text{i.e., } \liminf_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, g)} \leq (\rho_f + \varepsilon) \liminf_{r \rightarrow \infty} \frac{\log M(r, g)}{T(r, g)}.$$

Since  $\varepsilon (> 0)$  is arbitrary, it follows that

$$\liminf_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, g)} \leq \rho_f \cdot \liminf_{r \rightarrow \infty} \frac{\log M(r, g)}{T(r, g)}. \quad (2.3.1)$$

As  $\liminf_{r \rightarrow \infty} \frac{T(r, g)}{r^{\lambda_g(r)}} = 1$ , so for given  $\varepsilon (0 < \varepsilon < 1)$  we get for a sequence of values of  $r$  tending to infinity,

$$T(r, g) \leq (1 + \varepsilon) r^{\lambda_g(r)} \quad (2.3.2)$$

and for all sufficiently large values of  $r$ ,

$$T(r, g) \geq (1 - \varepsilon) r^{\lambda_g(r)}. \quad (2.3.3)$$

Since  $\log M(r, g) \leq 3T(2r, g)$  we have by (2.3.2), for a sequence of values of  $r$  tending to infinity,

$$\log M(r, g) \leq 3T(2r, g) \leq 3(1 + \varepsilon)(2r)^{\lambda_g(2r)}. \quad (2.3.4)$$

Combining (2.3.3) and (2.3.4) we obtain for a sequence of values of  $r$  tending to infinity,

$$\frac{\log M(r, g)}{T(r, g)} \leq \frac{3(1 + \varepsilon)(2r)^{\lambda_g(2r)}}{(1 - \varepsilon)r^{\lambda_g(r)}}.$$

Now for any  $\delta > 0$ , for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} \frac{\log M(r, g)}{T(r, g)} &\leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot \frac{(2r)^{\lambda_g + \delta}}{(2r)^{\lambda_g + \delta - \lambda_g(2r)}} \cdot \frac{1}{r^{\lambda_g(r)}} \\ \text{i.e., } \frac{\log M(r, g)}{T(r, g)} &\leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \cdot 2^{\lambda_g + \delta} \end{aligned} \quad (2.3.5)$$

because  $r^{\lambda_g + \delta - \lambda_g(2r)}$  is ultimately an increasing function of  $r$  by Lemma 2.2.7. Since  $\varepsilon (> 0)$  and  $\delta (> 0)$  are arbitrary, it follows from (2.3.5) that

$$\liminf_{r \rightarrow \infty} \frac{\log M(r, g)}{T(r, g)} \leq 3 \cdot 2^{\lambda_g}. \quad (2.3.6)$$

Thus from (2.3.1) and (2.3.6) we obtain that

$$\liminf_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, g)} \leq 3 \cdot \rho_f \cdot 2^{\lambda_g}. \quad (2.3.7)$$

Now in view of Lemma 2.2.3 and (2.3.7) we get that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, P[g])} &= \liminf_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, g)} \cdot \lim_{r \rightarrow \infty} \frac{T(r, g)}{T(r, P[g])} \\ &\leq \frac{3 \cdot \rho_f \cdot 2^{\lambda_g}}{\{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)\}}. \end{aligned}$$

This proves the theorem. ■

**Theorem 2.3.2** *Let  $f$  be meromorphic and  $g$  be transcendental entire such that  $\rho_f < \infty$ ,  $\lambda_g < \infty$  and  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; g) = 4$ . Then*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P[g])} \leq 1.$$

**Proof.** Since  $T(r, g) \leq \log^+ M(r, g)$ , in view of Lemma 2.2.1, we get for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T(r, f \circ g) &\leq \log T(M(r, g), f) + \log \{1 + o(1)\} \\ \text{i.e., } \log T(r, f \circ g) &\leq (\rho_f + \varepsilon) \log M(r, g) + o(1) \\ \text{i.e., } \log^{[2]} T(r, f \circ g) &\leq \log^{[2]} M(r, g) + O(1). \end{aligned} \quad (2.3.8)$$

It is well known that for any entire function  $g$ ,  $\log M(r, g) \leq 3T(2r, g)$  {cf.[16]}. Then for  $0 < \varepsilon < 1$  and  $\delta (> 0)$ , for a sequence of values of  $r$  tending to infinity it follows from (2.3.5) that

$$\log^{[2]} M(r, g) \leq \log T(r, g) + O(1). \quad (2.3.9)$$

Now combining (2.3.8) and (2.3.9) we obtain for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} \log^{[2]} T(r, f \circ g) &\leq \log T(r, g) + O(1) \\ \text{i.e., } \liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, g)} &\leq 1. \end{aligned} \quad (2.3.10)$$

As by Lemma 2.2.3,  $\lim_{r \rightarrow \infty} \frac{\log T(r, g)}{\log T(r, P[g])}$  exists and is equal to 1, then from (2.3.10) we get that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, P[g])} &= \liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log T(r, g)} \cdot \lim_{r \rightarrow \infty} \frac{\log T(r, g)}{\log T(r, P[g])} \\ &\leq 1.1 = 1. \end{aligned}$$

Thus the theorem is established. ■

**Theorem 2.3.3** *Let  $f$  and  $g$  be two transcendental entire functions such that  $\rho_g < \lambda_f \leq \rho_f < \infty$  and  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = \sum_{a \in \text{CU}\{\infty\}} \delta_1(a; g) = 4$ . Also there exist entire functions  $a_i$  ( $i = 1, 2, \dots, n$ ;  $n \leq \infty$ ) with*

- (i)  $T(r, a_i) = o\{T(r, g)\}$  as  $r \rightarrow \infty$  for  $i = 1, 2, \dots, n$  and
- (ii)  $\sum_{i=1}^n \delta(a_i; g) = 1$ . Then

$$\lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ g)\}^2}{T(r, P[f]) T(r, P[g])} = 0.$$

**Proof.** In view of the inequality  $T(r, g) \leq \log^+ M(r, g)$  and Lemma 2.2.1 we obtain for all sufficiently large values of  $r$ ,

$$\begin{aligned} T(r, f \circ g) &\leq \{1 + o(1)\} T(M(r, g), f) \\ \text{i.e., } \log T(r, f \circ g) &\leq \log \{1 + o(1)\} + \log T(M(r, g), f) \\ \text{i.e., } \log T(r, f \circ g) &\leq o(1) + (\rho_f + \varepsilon) \log M(r, g) \\ \text{i.e., } \log T(r, f \circ g) &\leq o(1) + (\rho_f + \varepsilon) r^{\rho_g + \varepsilon}. \end{aligned} \quad (2.3.11)$$

Again in view of Lemma 2.2.4, we get for all sufficiently large values of  $r$

$$\begin{aligned} \log T(r, P[f]) &> (\lambda_{P[f]} - \varepsilon) \log r \\ \text{i.e., } \log T(r, P[f]) &> (\lambda_f - \varepsilon) \log r \\ \text{i.e., } T(r, P[f]) &> r^{\lambda_f - \varepsilon}. \end{aligned} \quad (2.3.12)$$

Now combining (2.3.11) and (2.3.12) it follows for all sufficiently large values of  $r$

$$\frac{\log T(r, f \circ g)}{T(r, P[f])} \leq \frac{o(1) + (\rho_f + \varepsilon) r^{\rho_g + \varepsilon}}{r^{\lambda_f - \varepsilon}}. \quad (2.3.13)$$

Since  $\rho_g < \lambda_f$ , we can choose  $\varepsilon (> 0)$  in such a way that

$$\rho_g + \varepsilon < \lambda_f - \varepsilon. \quad (2.3.14)$$

So in view of (2.3.13) and (2.3.14) it follows that

$$\lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, P[f])} = 0. \quad (2.3.15)$$

Again from Lemma 2.2.3 and Lemma 2.2.8 we get for all sufficiently large values of  $r$ ,

$$\begin{aligned} \frac{\log T(r, f \circ g)}{T(r, P[g])} &\leq \frac{o(1) + (\rho_f + \varepsilon) \log M(r, g)}{T(r, P[g])} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, P[g])} &\leq (\rho_f + \varepsilon) \limsup_{r \rightarrow \infty} \frac{\log M(r, g)}{T(r, P[g])} \\ \text{i.e., } \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, P[g])} &\leq (\rho_f + \varepsilon) \limsup_{r \rightarrow \infty} \frac{\log M(r, g)}{T(r, g)} \cdot \lim_{r \rightarrow \infty} \frac{T(r, g)}{T(r, P[g])} \\ &= (\rho_f + \varepsilon) \cdot \frac{\pi}{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)}. \end{aligned}$$

Since  $\varepsilon (> 0)$  is arbitrary it follows from above that

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, P[g])} \leq \rho_f \cdot \pi \cdot \frac{1}{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)}. \quad (2.3.16)$$

Combining (2.3.15) and (2.3.16) we obtain that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\{\log T(r, f \circ g)\}^2}{T(r, P[f])T(r, P[g])} &= \lim_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, P[f])} \cdot \limsup_{r \rightarrow \infty} \frac{\log T(r, f \circ g)}{T(r, P[g])} \\ &\leq 0 \cdot \frac{\pi \rho_f}{\Gamma_P - (\Gamma_P - \gamma_P)\Theta(\infty; g)} = 0 \end{aligned}$$

$$\text{i.e., } \lim_{r \rightarrow \infty} \frac{\{\log T(r, f \circ g)\}^2}{T(r, P[f])T(r, P[g])} = 0.$$

This proves the theorem. ■

**Theorem 2.3.4** *If  $f$  and  $g$  be two entire functions with  $f$  to be transcendental satisfying the following conditions:*

(i)  $\lambda_f > 0$ , (ii)  $\bar{\rho}_f < \infty$ , (iii)  $0 < \lambda_g \leq \rho_g$  and (iv)  $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ , then

$$\limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \geq \max \left\{ \frac{\lambda_g}{\bar{\lambda}_f}, \frac{\rho_g}{\bar{\rho}_f} \right\}.$$

**Proof.** We know that for  $r > 0$  (cf. [27]) and for all sufficiently large values of  $r$ ,

$$T(r, f \circ g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) + o(1), f \right\}. \quad (2.3.17)$$

Since  $\lambda_f$  and  $\lambda_g$  are the lower orders of  $f$  and  $g$  respectively then for given  $\varepsilon (> 0)$  and for all large values of  $r$  we obtain that

$\log M(r, f) > r^{\lambda_f - \varepsilon}$  and  $\log M(r, g) > r^{\lambda_g - \varepsilon}$  where  $0 < \varepsilon < \min \{\lambda_f, \lambda_g\}$ .

So from (2.3.17) we have for all sufficiently large values of  $r$ ,

$$T(r, f \circ g) \geq \frac{1}{3} \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) + o(1) \right\}^{\lambda_f - \varepsilon}$$

$$\text{i.e., } T(r, f \circ g) \geq \frac{1}{3} \left\{ \frac{1}{9} M \left( \frac{r}{4}, g \right) \right\}^{\lambda_f - \varepsilon}$$

$$\text{i.e., } \log T(r, f \circ g) \geq O(1) + (\lambda_f - \varepsilon) \log M \left( \frac{r}{4}, g \right)$$

$$\text{i.e., } \log T(r, f \circ g) \geq O(1) + (\lambda_f - \varepsilon) \left( \frac{r}{4} \right)^{\lambda_g - \varepsilon}$$

$$\text{i.e., } \log^{[2]} T(r, f \circ g) \geq O(1) + (\lambda_g - \varepsilon) \log r. \quad (2.3.18)$$

Again in view of Lemma 2.2.5, we get for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} \log^{[2]} T(r, P[f]) &\leq (\bar{\lambda}_{P[f]} + \varepsilon) \log r \\ \text{i.e., } \log^{[2]} T(r, P[f]) &\leq (\bar{\lambda}_f + \varepsilon) \log r. \end{aligned} \quad (2.3.19)$$

Combining (2.3.18) and (2.3.19), it follows for a sequence of values of  $r$  tending to infinity,

$$\frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \geq \frac{O(1) + (\lambda_g - \varepsilon) \log r}{(\bar{\lambda}_f + \varepsilon) \log r}.$$

Since  $\varepsilon (> 0)$  is arbitrary, we obtain that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \geq \frac{\lambda_g}{\bar{\lambda}_f}. \quad (2.3.20)$$

Again from (2.3.17) we get for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} \log T(r, f \circ g) &\geq O(1) + (\lambda_f - \varepsilon) \left(\frac{r}{4}\right)^{\rho_g - \varepsilon} \\ \text{i.e., } \log^{[2]} T(r, f \circ g) &\geq O(1) + (\rho_g - \varepsilon) \log r. \end{aligned} \quad (2.3.21)$$

Also in view of Lemma 2.2.5, for all sufficiently large values of  $r$ , we have

$$\log^{[2]} T(r, P[f]) \leq (\bar{\rho}_{P[f]} + \varepsilon) \log r = (\bar{\rho}_f + \varepsilon) \log r. \quad (2.3.22)$$

Now from (2.3.21) and (2.3.22) it follows for a sequence of values of  $r$  tending to infinity,

$$\frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \geq \frac{O(1) + (\rho_g - \varepsilon) \log r}{(\bar{\rho}_f + \varepsilon) \log r}.$$

As  $\varepsilon (0 < \varepsilon < \rho_g)$  is arbitrary, we obtain from above that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \geq \frac{\rho_g}{\bar{\rho}_f}. \quad (2.3.23)$$

Therefore from (2.3.20) and (2.3.23) we get that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \geq \max \left\{ \frac{\lambda_g}{\bar{\lambda}_f}, \frac{\rho_g}{\bar{\rho}_f} \right\}.$$

Thus the theorem is established. ■

**Theorem 2.3.5** *Let  $f$  be transcendental meromorphic and  $g$  be entire such that (i)  $0 < \bar{\lambda}_f < \bar{\rho}_f$ , (ii)  $\rho_g < \infty$ , (iii)  $\rho_f < \infty$  and (iv)  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = 4$ .*

*Then*

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \leq \min \left\{ \frac{\lambda_g}{\bar{\lambda}_f}, \frac{\rho_g}{\bar{\rho}_f} \right\}.$$

**Proof.** In view of Lemma 2.2.1 and the inequality  $T(r, g) \leq \log^+ M(r, g)$ , we obtain for all sufficiently large values of  $r$ ,

$$\log T(r, f \circ g) \leq o(1) + (\rho_f + \varepsilon) \log M(r, g). \quad (2.3.24)$$

Also for a sequence of values of  $r$  tending to infinity,

$$\log M(r, g) \leq r^{\lambda_g + \varepsilon}. \quad (2.3.25)$$

Combining (2.3.24) and (2.3.25) it follows for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} \log T(r, f \circ g) &\leq o(1) + (\rho_f + \varepsilon) r^{\lambda_g + \varepsilon} \\ \text{i.e., } \log T(r, f \circ g) &\leq r^{\lambda_g + \varepsilon} \{(\rho_f + \varepsilon) + o(1)\} \\ \text{i.e., } \log^{[2]} T(r, f \circ g) &\leq O(1) + (\lambda_g + \varepsilon) \log r. \end{aligned} \quad (2.3.26)$$

Again in view of Lemma 2.2.5, we have for all sufficiently large values of  $r$ ,

$$\log^{[2]} T(r, P[f]) > (\bar{\lambda}_{P[f]} - \varepsilon) \log r = (\bar{\lambda}_f - \varepsilon) \log r. \quad (2.3.27)$$

Now from (2.3.26) and (2.3.27) we get for a sequence of values of  $r$  tending to infinity,

$$\frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \leq \frac{O(1) + (\lambda_g + \varepsilon) \log r}{(\bar{\lambda}_f - \varepsilon) \log r}.$$

As  $\varepsilon (> 0)$  is arbitrary, it follows that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \leq \frac{\lambda_g}{\bar{\lambda}_f}. \quad (2.3.28)$$

In view of Lemma 2.2.1 we get for all sufficiently large values of  $r$ ,

$$\log^{[2]} T(r, f \circ g) \leq O(1) + (\rho_g + \varepsilon) \log r. \quad (2.3.29)$$

Also for a sequence of values of  $r$  tending to infinity,

$$\log^{[2]} T(r, P[f]) > (\bar{\rho}_{P[f]} - \varepsilon) \log r = (\bar{\rho}_f - \varepsilon) \log r. \quad (2.3.30)$$

Combining (2.3.29) and (2.3.30) we have for a sequence of values of  $r$  tending to infinity,

$$\frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \leq \frac{O(1) + (\rho_g + \varepsilon) \log r}{(\bar{\rho}_f - \varepsilon) \log r}.$$

Since  $\varepsilon (> 0)$  is arbitrary, it follows from above that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \leq \frac{\rho_g}{\bar{\rho}_f}. \quad (2.3.31)$$

Now from (2.3.28) and (2.3.31) we get that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} \leq \min \left\{ \frac{\lambda_g}{\bar{\lambda}_f}, \frac{\rho_g}{\bar{\rho}_f} \right\}.$$

This proves the theorem. ■

The following theorem is a natural consequence of Theorem 2.3.4 and Theorem 2.3.5.

**Theorem 2.3.6** *Let  $f$  be a transcendental entire function and  $g$  be an entire function such that (i)  $0 < \bar{\lambda}_f \leq \bar{\rho}_f < \infty$ , (ii)  $0 < \lambda_f \leq \rho_f < \infty$ , (iii)  $0 < \lambda_g \leq \rho_g < \infty$  and (iv)  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = 4$ . Then*

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])} &\leq \min \left\{ \frac{\lambda_g}{\bar{\lambda}_f}, \frac{\rho_g}{\bar{\rho}_f} \right\} \\ &\leq \max \left\{ \frac{\lambda_g}{\bar{\lambda}_f}, \frac{\rho_g}{\bar{\rho}_f} \right\} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r, P[f])}. \end{aligned}$$

**Theorem 2.3.7** *Let  $f$  be transcendental meromorphic and  $g$  be entire such that  $0 < \lambda_f \leq \rho_f < \infty$  and  $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = 4$ . Then*

$$\limsup_{r \rightarrow \infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f \circ g)}{\log^{[2]} T(\exp(r^\mu), P[f])} = \infty \quad \text{where } 0 < \mu < \rho_g.$$

**Proof.** Let  $0 < \mu < \mu' < \rho_g$ . Then in view of Lemma 2.2.2 we get for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} \log T(r, f \circ g) &\geq \log T\left(\exp\left(r^{\mu'}\right), f\right) \\ \text{i.e., } \log T(r, f \circ g) &\geq (\lambda_f - \varepsilon) \log\left(\exp\left(r^{\mu'}\right)\right) \\ \text{i.e., } \log T(r, f \circ g) &\geq (\lambda_f - \varepsilon) r^{\mu'} \\ \text{i.e., } \log^{[2]} T(r, f \circ g) &\geq O(1) + \mu' \log r. \end{aligned}$$

So for a sequence of values of  $r$  tending to infinity,

$$\begin{aligned} \log^{[2]} T\left(\exp\left(r^{\rho_g}\right), f \circ g\right) &\geq O(1) + \mu' \log\left(\exp\left(r^{\rho_g}\right)\right) \\ \text{i.e., } \log^{[2]} T\left(\exp\left(r^{\rho_g}\right), f \circ g\right) &\geq O(1) + \mu' r^{\rho_g}. \end{aligned} \quad (2.3.32)$$

Again in view of Lemma 2.2.4, we have for all sufficiently large values of  $r$ ,

$$\begin{aligned} \log T\left(\exp\left(r^{\mu}\right), P[f]\right) &\leq (\rho_{P[f]} + \varepsilon) \log\left(\exp\left(r^{\mu}\right)\right) \\ \text{i.e., } \log T\left(\exp\left(r^{\mu}\right), P[f]\right) &\leq (\rho_f + \varepsilon) r^{\mu} \\ \text{i.e., } \log^{[2]} T\left(\exp\left(r^{\mu}\right), P[f]\right) &\leq O(1) + \mu \log r. \end{aligned} \quad (2.3.33)$$

Now combining (2.3.32) and (2.3.33) we obtain for a sequence of values of  $r$  tending to infinity,

$$\frac{\log^{[2]} T\left(\exp\left(r^{\rho_g}\right), f \circ g\right)}{\log^{[2]} T\left(\exp\left(r^{\mu}\right), P[f]\right)} \geq \frac{O(1) + \mu' r^{\rho_g}}{\mu \log r},$$

from which the theorem follows. ■

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