Chapter 5

CANTOR FUNCTION: FROM NONDIFFERENTIABILITY TO DIFFERENTIABILITY

5.1 Introduction

In the previous chapter, we explicitly verified the multiplicative structure (4.3) in the context of middle third Cantor set and similar other more general class of homogeneous sets. Clearly such a representation is valid for any general Cantor set. We also presented independent analysis explaining how the nontrivial valuation is related to an associated Cantor function. We studied in detail both the middle third and (p,q) Cantor set and also discussed the variability of the valuation vis a vis Cantor function both in ultrametric and usual topology. In this Chapter, we present *another new independent* derivation of the multiplicative structure explaining explicitly the smoothening of Cantor function at the points of a Cantor set. We begin by first recalling the usual proof of the nondifferentiability of a Cantor function $\phi(x)$ at $x \in C$. The analysis here clearly brings out the precise points where the present scale invariant approach supersedes the classical analytic results. This also offers another justification in favour of the existence of infinitesimals introduced in Chapter 3.

5.2 *Middle third Cantor set*

In this example, we present an explicit construction of multiplicative neighbours of $x \in C$ using the Cantor function $f_C : I \to I$. In the following we denote this function instead by $\tilde{X}(x)$. To recall again the definition of the Cantor function, consider the $\frac{1}{3}$-rd Cantor set: $r = 3,\ p = 2$. Let $x = \Sigma a_i 3^{-i}$ be the ternary representation of $x \in C$ where a_i may be either 0 or 2. We set $x = \frac{2}{3}\psi(\tilde{X})$ where $\psi(\tilde{X}) = \Sigma \frac{b_i}{3^{i-1}}$ and $\tilde{X} = \Sigma b_i 2^{-i} \in I \backslash C,\ b_i \in \{0, 1\}$.

Then $\tilde{X} = \tilde{X}\ (x)$ defined as the inverse of the above functional equation is the Cantor function $\tilde{X} : [0, 1] \to [0, 1]$. By continuity, this extends over C as well.

Let us recall that at the k -th step of the iterative construction of the Cantor set, the initial closed interval I fragments into 2^k smaller closed intervals $I_j^k = [x_{2j-1}, x_{2j}],\ j = 1, 2....,$ each of length 3^{-k}.

Then $x_{2j} - x_{2j-1} = 3^{-k}$. Definition of the Cantor function also gives that

$$\tilde{X}(x_{2j}) - \tilde{X}(x_{2j-1}) = 2^{-k}. \tag{5.1}$$

Let $x \in C$. Then $x \in I_j^k$ for some j. It thus follows

$$\tilde{X}(x_{2j}) - \tilde{X}(x_{2j-1}) = \frac{3^k}{2^k}(x_{2j} - x_{2j-1}). \tag{5.2}$$

This equality is at the heart of the standard proof of the nondifferentiability of the Cantor function [42]. Now, to see how such a nondifferentiability is removed in the present framework, let $\tilde{X}(x_{2j}) = X_+,\ \tilde{X}(x_{2j-1}) = X_-,\ x_{2j} = x_+,\ x_{2j-1} = x_-$. Suppose also that

$$3^k(x_+ - x) \to k\log\sigma_+, \quad 3^k(x - x_-) \to k\log\sigma_- \tag{5.3}$$

and

$$2^k(\tilde{X}_+ - \tilde{X}) \to k\log X_+^{'}, \; 2^k(\tilde{X} - \tilde{X}_-) \to k\log X_-^{\prime} \tag{5.4}$$

for infinitely large $k \to \infty$. The limiting value of equation (5.2) thus becomes

$$\log X_+^{\prime} + \log X_-^{\prime} = \log\sigma_+ + \log\sigma_-. \tag{5.5}$$

Now, using the inequality $\frac{\alpha+\gamma}{\beta+\delta} \leq \max(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}), \alpha, \gamma \geq 0, \beta, \delta > 0$, equation (5.5) yields

$$\max\left(\frac{\log X_+^{'}}{\log\sigma_+}, \frac{\log X_-^{'}}{\log\sigma_-}\right) \geq 1. \tag{5.6}$$

But equation (5.5) shows that $\sigma_+ = \sigma_-^{-1} = \sigma$ (say) and $X_+ = X_-^{-1}$, as $k \to \infty$, so that equation (5.6) reduces to

$$X_+^{'} = \sigma^{1+j}, \; X_-^{'} = \sigma^{-(1+j)}, \; j \geq 0. \tag{5.7}$$

Setting $\sigma^{-1}X_+^{'} = \frac{X_+}{x}$ and $\sigma X_-^{'} = \frac{X_-}{x}$, we finally get the multiplicative neighbours of $x \in C$ as

$$X_\pm = x\sigma^{\pm j}. \tag{5.8}$$

Notice that $\sigma \approx 1$. In the notation of Section 2, $\sigma = x^{\tau^s}$, τ being a valued infinitesimal. The inequality equation (5.6) is reminiscent of the strong triangle inequality for the non-archimedean valuation. We also remark that the clue to the substitutions of the form equation (5.3) and equation (5.4) arise from our basic definitions of relative infinitesimals and the associated scale invariant norms of Sec.3.2.

5.3 (p,q) *Cantor set*

Here, we extend the above analysis to the more general class of Cantor set and the associated Cantor function. Indeed, we again verify the emergence of equation (5.8) from the classical Cantor function equations (5.1) and (5.2) viz. :

$$\phi(\beta_k)-\phi(\alpha_k)=\frac{1}{p^k} \text{ and } \beta_k-\alpha_k=\frac{1}{r^k}. \tag{5.9}$$

We have

$$\phi(\beta_k)-\phi(\alpha_k)=\left(\frac{r}{p}\right)^k(\ \beta_k-\alpha_k). \tag{5.10}$$

Let $\phi(\beta_k)=\tilde{\phi}_+$, $\phi(\alpha_k)=\tilde{\phi}_-$, $\beta_k=x_+$, $\alpha_k=x_-$. Suppose also that $r^k(x_+-x)\to k\log\sigma_+$, $r^k(x-x_-)\to k\log\sigma_-$, $p^k(\tilde{\phi}-\tilde{\phi}_-)\to k\log\phi'_-$ and $p^k(\tilde{\phi}_+-\tilde{\phi})\to k\log\phi'_+$ as $k\to\infty$.

Equation (5.10) becomes

$$\log\phi'_+ + \log\phi'_- = \log\sigma_+ + \log\sigma_- \tag{5.11}$$

which leads to

$$\frac{\log\phi'_+}{\log\sigma_+}=\frac{\log\phi'_-}{\log\sigma_-}=\frac{\log\phi'_+ + \log\phi'_-}{\log\sigma_+ + \log\sigma_-}=1. \tag{5.12}$$

Equation (5.12) is essentially the left and right branches of equation (4.25) at $x\in C$, in the appropriate logarithmic variables, where the multiplicative neighbours of x, in the present derivation, is given by the limiting form of the Cantor function defined by

$$\phi'_+=\sigma^{1+i},\ \phi'_-=\sigma^{-(1+i)},\ i\geq 0 \tag{5.13}$$

which follows from the inequality $\frac{\alpha+\gamma}{\beta+\delta} \leq \max(\frac{\alpha}{\beta}, \frac{\gamma}{\delta})$, $\alpha, \gamma \geq 0$, $\beta, \delta > 0$ and equation (5.11) so that

$$\left(\frac{\log \phi'_+}{\log \sigma_+}, \frac{\log \phi'_-}{\log \sigma_-}\right) \geq 1. \tag{5.14}$$

Setting $\sigma^{-1}\phi'_+ = \{\frac{X_+}{x}\}^i$, $\sigma\phi'_- = \{\frac{X_-}{x}\}^i$ and $\sigma = x^{-v(\tilde{x})}$ the multiplicative neighbours of x are obtained as

$$X_{\pm} = x.\ x^{\mp v(\tilde{x})}. \tag{5.15}$$

The Cantor function $\phi(\tilde{x})$ over the infinitesimals $\tilde{x}$ is thus given by

$$\phi(\tilde{x}) = \log_{x^{-1}} \frac{X(\tilde{x})}{x} = v(\tilde{x}) \tag{5.16}$$

thereby retrieving the variability of ϕ relative to v trivially viz : $d\phi = dv$.

We note that this again explains explicitly the removal of derivative discontinuities as encoded in equation (5.10) in the present formalism. The divergence of either the left or right derivative at an $x \in C$, that arises due to the divergence of $(r/p)^k$, $k \to \infty$, is smoothed out in the logarithmic variables that replace the ordinary limiting variables as in equations (5.11) and (5.12), which, in fact, correspond to equation (4.26). We conclude that the multiplicative non-archimedean structure given by equation (5.15) induces a smoothening effect on the discontinuity of $\phi'(x)$ in the usual topology.