CHAPTER - II

A MODIFIED APPROACH TO THE LARGE AMPLITUDE FREE VIBRATION OF PARABOLIC PLATES

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PAPER - I

A MODIFIED APPROACH TO THE LARGE AMPLITUDE FREE VIBRATION OF PARABOLIC PLATES

ABSTRACT

A simplified method is developed following two different approaches, one proposed by Mazumder and other by Banerjee for the analysis of large amplitude transverse vibration of thin, isotropic, homogeneous elastic plates of arbitrary shapes. As an illustration of the method, an interesting example of the vibration of a parabolic plate is examined.

Numerical results are presented in tabular form and corresponding graphs are drawn and compared with available results.

Formulation of governing equations

Let us consider the large amplitude transverse free vibration of a thin homogeneous elastic parabolic plate of thickness h. With the xy-plane taken to be the middle plane of the plate and the z-axis directed perpendicular to the plane, the intersections between the deflected surface z = w (x, y) and the plane z = constant, yield contours which, after projection onto the z = 0 surface, are the level curves called 'lines of equal deflection'. Let us denote the family of such curves by the equation u(x, y) = constant. If the boundary C of the plate is subjected to any combination of clamping and simple support, then clearly it will belong to the family of lines of equal deflection and without loss of generality one may consider that u = 0 on the boundary.

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Let the transverse displacement of a point in its middle plane be denoted by w which is a function of the spatial coordinates (x, y) and the temporal variable t. When the plate vibrates in a normal mode, the deflected form maintained by the plate at any instant t may be described by the family of lines of equal deflection whose equation is u(x, y) = constant. Let us denote the family of curves u = constant, by C_u ,

Let us consider a portion Ω_u of the plate bounded by any closed contour C_u at any instant t. Using now D'Alembert's principle and summing up of the forces in the vertical direction, one obtains the following dynamical equation :

$$\int_{C_u} \left[Q_n - \frac{\partial M_{nt}}{\partial s} \right] ds + \iint_{\Omega_u} \left[ph \frac{\partial^2 w}{\partial t^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right] d\Omega = 0 \tag{90}$$

Here Q_n is the shearing force and the rate of change of the twisting moment is $\partial M_{nt}/\partial s$ along the contour, and the line integral represents the upward vertical contribution of the resultant tractions exerted upon this portion by the remainder, and N_x, N_y, N_{xy} represent the membrane forces acting on a small element $d\Omega$ lying within the contour C_u. The term represents the inertia force due to the vertical acceleration of the element $d\Omega$, ρh being the mass per unit area.

Considering the approach of Banerjee (1984) and substituting the well-known expressions for Q_n, M_{nt}, N_x, etc. into equation (90) as carried out by Mazumdar and Jones (1970), one obtains

$$\frac{\partial^3 w}{\partial u^3} \int_{C_u} R\, ds + \frac{\partial^2 w}{\partial u^2} \int_{C_u} F\, ds + \frac{\partial w}{\partial u} \int_{C_u} G\, ds = \iint_{\Omega_u} D[-\frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} + C_1 \alpha^2 .$$

$$\times \left(w_{xx} + w_{yy}\right) + \frac{6\lambda}{h^2} , \{ \nabla^2 w \left(w_x^2 + w_y^2\right) + 2 \left(w_x^2 w_{xx} + 2 w_x w_y w_{xy}\right)$$

$$+ w_y^2 w_{yy}) \}] \, dx\, dy = 0 \qquad .. (91)$$

where use has been made of the fact that w and its derivatives with respect to u are constant on the contour u = constant, and R, F, and G are given by the following relations

$$R=-Dt^{3/2}$$

$$F=-\left(D/t^{1/2}\right)\left[3u_{xx}\,u_x^2+3u_{yy}\,u_y^2+u_{xx}\,u_y^2+u_{yy}\,u_x^2+4u_{xy}\,u_x\,u_y\right]$$

$$G=-\left(D/t^{3/2}\right)[\,u_{xxx}\,u_x^3+u_{yyy}\,u_y^3+(2-\nu)(u_{xxx}\,u_x\,u_y^2+u_{yyy}\,u_y\,u_x^2$$

$$+\,u_{xyy}\,u_x^3+u_{xxy}\,u_y^3)+(2\nu-1)(u_{xyy}\,u_x\,u_y^2+u_{xxy}\,u_x^2\,u_y)$$

$$-2(1-\nu)\,u_{xy}\,(u_x\,u_y\,u_{xx}-u_y^2\,u_{xy}-u_x^2\,u_{xy}+u_x\,u_y\,u_{yy})$$

$$+(1-\nu)(u_{xx}-u_{yy})\,(u_{xx}\,u_y^2-u_{yy}\,u_x^2)]$$

$$+[2D(1-\nu)/t^{5/2}]\,[u_{xy}\,(u_x^2-u_y^2)-u_x u_y\,(u_{xx}-u_{yy})]^2 \qquad ..(92)$$

Here $t=u_x^2+u_y^2, D=Eh^3/12(1-\nu^2)$ the flexural rigidity, E is Young's modulus and $\alpha^2 C_1$ is to be determined from

$$\frac{\alpha^2 C_1 h^2}{12}=\frac{\partial u_1}{\partial x}+\frac{\nu\,\partial u}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad ..(93)$$

integrating over the whole area of the plate, α^2 being the normalized constant of integration, C_1 being a function of time, and u_1 , v are the components of displacement in the x and y directions, respectively. It is to be mentioned in this connection that use of Hamilton's principle and Euler's variational principle has been made in deriving the equations (91) and (93).

Large amplitude transverse vibration of parabolic plate

As an illustration of the method, let us consider the large amplitude free transverse vibration of a parabolic plate with clamped edges. The complete analysis of the problem of free vibration of any plate would require the determination of all the frequencies and the corresponding mode shapes. However, if attention is confined to symmetrical forms of vibration, then from symmetry considerations one may assume that the lines of equal deflection form a family of similar and similarly situated parabolas starting from the outer boundary as one of these lines. Therefore, the equation of the lines of equal deflection may conveniently be taken to be of the form

$$u(x,y)=y\left[\frac{a}{2}(2a-y)-x^2\right] \quad ..(94)$$

Calculation of the values of R, F, G and t now gives

$$t=4x^2y^2+a^4+a^2y^2+x^4-2a^3y+2ax^2y-2a^2x^2$$

$$R=-D(4x^2y^2+a^4+a^2y^2+x^4-2a^3y+2ax^2y-2a^2x^2)^{3/2}$$

$$F=Dt^{-1/2}[24x^2y^3+(3a+2y)(a^2-ay-x^2)^2-16x^2y(a^2-ay-x^2)]$$

$$G=+Dt^{-3/2}[(2-\nu)\,2(a^2-ay-x^2)+8(2\nu-1)x^2y^2(a^2-ay-x^2)-4(1-\nu)x\{4xy^2$$

$$\times(a^2-ay-x^2)+2x(a^2-ay-x^2)^2+8x^3y^2+2axy(a^2-ay-x^2)\}$$

$$-(1-\nu)(2y-a)\times\{2y(a^2-ay-x^2)-4ax^2y^2\}]$$

$$+2D(1-\nu)t^{-5/2}[2x\{4x^2y^2-(a^2-ay-x^2)^2+2xy(a^2-ay-x^2)(2y-a)]^2$$

..(95)

Let $\overline{R},\overline{F}$, and $\overline{G}$ denote the mean values of R, F, and G on the contour u = constant. In this particular case, we shall take $\overline{R},\overline{F}$, and $\overline{G}$ as the arithmetic mean values of R, F, and G evaluated at the points of intersection of the lines u = constant, and x = 0, that is, at $A[0,\ a(1-(1-2u/a^3)^{1/2})]$ and $B[0,\ a(1+(1-2u/a^3)^{1/2})]$. We thus obtain

$$u=\frac{ay}{2}(2a-y) \quad ..(96)$$

that is

$$a-y=a(1-2u/a^3)^{1/2} \quad ..(97)$$

We have then

$$\overline{R}=\frac{R_B-R_A}{2}=-Da^6(1-2u/a^3)^{3/2} \quad ..(98)$$

$$\overline{F} = \frac{F_B - F_A}{2} = 5Da^3 (1 - 2u / a^3)^{1/2} \quad .. (99)$$

$$G = \frac{G_B - G_A}{2} = -4D(1 - \nu)(3/2 - 2u/a^3)/(1 - 2u/a^3)^{1/2} \quad .. (100)$$

Putting 1 - 2u / a^3 = v^2 and introducing equations (96, 97) and (98, 99, 100) into equation (91) and carrying out the necessary integration and simplification, we obtain

$$\frac{v^2 d^3 w}{dv^3} + \frac{2vd^2 w}{dv^2} + \left(\frac{1}{3}\alpha^2 C_1 vv^2 a^2 + \frac{8}{3}v^2 - \frac{2}{3}\right)\frac{dw}{dv} + 6\lambda v^2 \left(\frac{dw}{dv}\right)^2 - pha^4 v^2 \int_o^1 \frac{d^2 w}{dt^2} dv = 0$$

.. (101)

Putting $W = w/h$ and $\tau = t(pha^4 / D)^{1/2}$, we have from equation (101)

$$\frac{v^2 d^3 W}{dv^3} + \frac{2vd^2 W}{dv^2} + \left(\frac{1}{3}\alpha^2 C_1 v^2 a^2 + \frac{8}{3}v^3 - \frac{2}{3}\right)\frac{dW}{dv} + 6\lambda v^2 h^3 \left(\frac{dW}{dv}\right)^3 - v^2 \int_o^1 \frac{d^2 W}{d\tau^2} = 0$$

.. (102)

Let us now consider the boundary conditions at the edge of the plate. If the plate is assumed to be clamped, the corresponding boundary conditions are

$$w = 0 \Big|_{u=0} \quad and \quad \frac{dw}{du} = 0 \Big|_{u=0} \quad .. (103)$$

In this case the centre is a point on the y-axis which is obtained by considering the extreme value of the function u(x, y). For extremum, u_x = - 2xy = 0, that is, either x = 0, or, y = 0, but y ≠ 0, and u_y = a/2 (2a - 2y) - x^2 = 0. For x = 0, y = a and u = a^3 / 2. Therefore, the value of u (x, y) at the centre is found to be a^3 / 2. Then we have at the centre

$$\left(1 - 2u/a^3\right)^{1/2} \frac{dw}{du}\Big|_{u = a^3/2} = 0 \quad .. (104)$$

where

$$\frac{dw}{du}\Big|_{u = a^3/2}$$

is finite. Since $v^2 = 1 - 2u/a^3$, we have for u = 0, v = 1, and u = $a^3/2$, v = 0. The boundary conditions for w are

$$w = 0,\ v = 1 \quad \text{and} \quad \frac{dw}{dv} = 0,\ v = 0 \qquad ..(105)$$

Solution of the problem

Now any conventional method may be used to solve the differential equation (101). Let us solve this equation by the principle of Galerkin, assuming that

$$W = v^2 \left(1 - v^2\right)(a_o + a_1 v^2 + a_2 v^4 + ...) \qquad ..(106)$$

In fact, instead of taking the solution in the form of equation (106), we shall assume for W the first term approximation as

$$W = a_o W_o(\tau)\, v^2 (1 - v^2) \qquad ..(107)$$

It is to be noted that the first term approximation in the choice of this deflection W is available in the open literature and yields sufficiently accurate results. Substitution of the above expression (107) for W in the equation (102) yields an error function ε, which does not vanish, in general, since the expression for W is not an exact solution. Galerkin's principle requires that the error function ε must be orthogonal over the domain, that is

$$\int_o^1 \varepsilon\, v\, W\, dv = 0 \qquad ..(108)$$

Then from the equations (102), (107) and (108), we obtain

$$\frac{d^2 W_0(\tau)}{d\tau^2} + 239.95499\, W_0(\tau) + 1.03883\,\alpha^2 C_1 a^2\, W_0(\tau) + 29.223402\,\lambda a_0^2\, W_0^3(\tau) = 0$$

$$..(109)$$

The terms involving the in-plane displacements u_1 and v can be easily eliminated by considering suitable expressions for these displacements compatible with the boundary conditions and by subsequent integration, we obtain

$$\alpha^2 C_1 = (4a_0^2/3a^2)\, W_0^2(\tau) \qquad ..(110)$$

The constant λ is obtained by minimizing the total potential energy and in the case of the clamped edge boundary, it is found to be $\lambda = 2\mp^2$. Then equation (109) with the help of equation (110) becomes

$$\frac{d^2 W_0(\tau)}{d\tau^2} + \mu_1 W_0(\tau) + \mu_2 a_0^2 W_0^3(\tau) = 0 \qquad ..(111)$$

where $\mu_1 = 239.95499$ and $\mu_2 = 6.645389$.

If $\lambda = 0$, we obtain the result corresponding to that of Berger (4) and equation (109) reduces to

$$\frac{d^2 W_0(\tau)}{d\tau^2} + \mu_1 W_0(\tau) + \mu_{2B} a_0^2 W_0^3(\tau) = 0 \qquad ..(112)$$

where $\mu_{2B} = 1.385177$.

Again, when $\alpha^2 = 0$, we can derive the result for the *movable* edge boundary condition and equation (109) reduces to

$$\frac{d^2 W_0(\tau)}{d\tau^2} + \mu_1 W_0(\tau) + \mu_{2M} a_0^2 W_0^3(\tau) = 0 \qquad ..(113)$$

where $\mu_{2M} = 5.260212$.

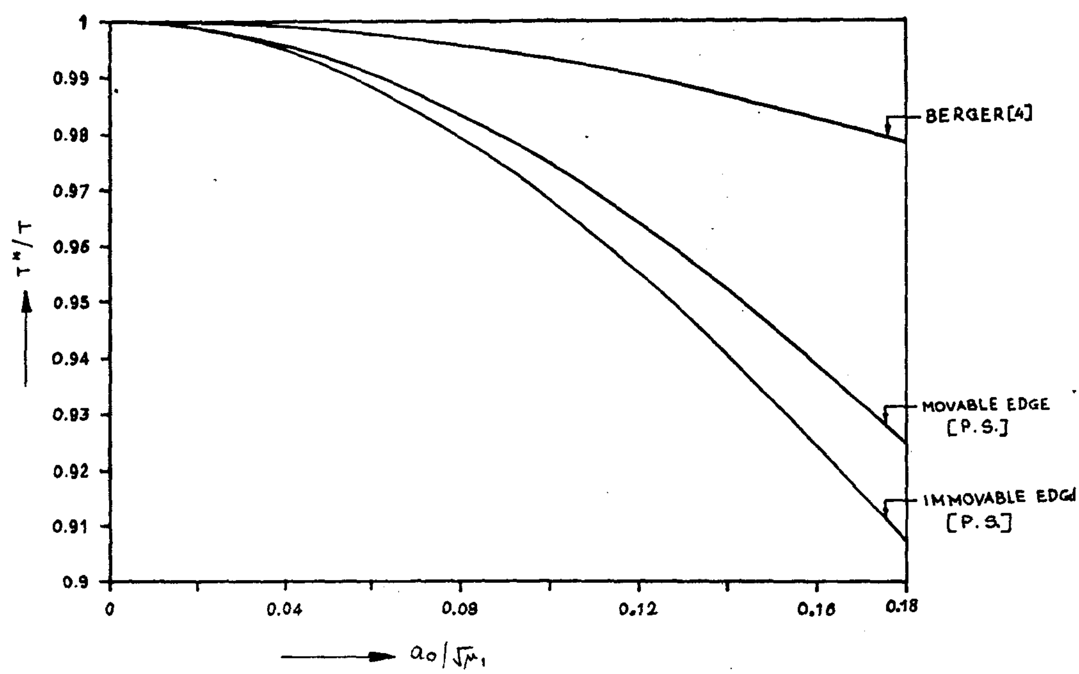
Numerical results

Table 1 presents the numerical results for a comparative study of the nonlinear to linear time period ratios versus the ratio of the non-dimensional amplitude to linear frequency for a clamped parabolic plate with immovable as well as with movable edges, as obtained in the present study and by Berger's approach [4], with $\nu = 0.3$, and $\lambda = 2\nu^2$. The graphs corresponding to the results given in Table 1 are shown in Figure 1.

	T^* / T for immovable edge		T^* / T for movable edge	
$a_o / \sqrt{\mu_1}$	Present study	Berger's Method	Present study	Berger's Method
0.00	1.00	1.00	1.00	Absurd
0.02	0.998673	0.999723	0.998949	Absurd
0.04	0.994725	0.998898	0.995818	Absurd
0.06	0.988248	0.997516	0.990664	Absurd
0.08	0.979390	0.995596	0.983580	Absurd
0.10	0.968342	0.993277	0.974693	Absurd
0.12	0.955334	0.990173	0.964150	Absurd
0.14	0.940617	0.986695	0.952121	Absurd
0.16	0.924452	0.982727	0.938766	Absurd
0.18	0.907102	0.978288	0.924330	Absurd

Observations and conclusions

It is observed that the results of the present study are sufficiently accurate for both movable and immovable edge conditions. For parabolic plates, the maximum numerical difference in the different values of T^* / T given by Berger's approach and the present study is only 0.0712, although the difference in percentage is significant. This is due to the fact that Berger's approach is purely an approximate one based on the neglect of e_2, the so-called second strain invariant in the potential energy expression, whereas in the present study no term in the potential energy expression is neglected. Thus, as is expected, the results of the present study are more accurate. As the results for the present problem are not available through any other method, it has not been possible to compare the results of the present analysis with them. Furthermore, it may be reiterated that the results for different plates can be obtained from a single differential equation, for different choices of u for different contours; as a result, the method described here seems more advantageous than those previously reported in the published literature. Finally, it may also be pointed out that the calculations of R, F, G, for parabolic plates are a little more involved than those of the corresponding results for other plates.



[P.S.]—Present Study

Figure 1.

PAPER - II

LARGE AMPLITUDE FREE VIBRATION OF SANDWICH PARABOLIC PLATES

ABSTRACT :

The field of sandwich construction is gaining importance in recent year as a result of improvements in manufacturing techniques. As new manufacturing methods are now being developed which make the use of sandwiches economically feasible.

The present method is a modification of two approaches, one by Mazumdar and another by Banerjee for the analysis of large amplitude transverse vibration of thin isotropic, homogeneous sandwich elastic plates of arbitrary shapes. As an illustration of the method, an interesting example of the vibration of a sandwich parabolic plate with an isotropic core is examined. Graphs corresponding to numerical results are drawn and compared to available results.

FORMULATION OF GOVERNING EQUATIONS

Let us consider the large amplitude transverse free vibration of a thin homogeneous elastic sandwich parabolic plate with an isotropic core of thickness h as well as isotropic upper and lower faces of identical thickness t_1 , while the faces respond to the bending and membrane actions of the plate, the core is assumed to transfer only shear deformations. Moreover, compared to the core thickness h, the face thickness t_1 is supposed to be thin enough ($t_1 << h$) to ignore a variation of stress in the thickness direction of the faces. The xy-plane is taken to be the middle plane of the sandwich plate

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and the z-axis directed perpendicular to the plane. The intersections between the deflected surface z = w (x, y) and the plane z = constant, yield contours which after projection on the z = 0 surface, are the level curves called 'lines of equal deflection'. Let us denote the family of such curves by the equation u (x, y) = constant. If the boundary C of the sandwich plate is subjected to any combination of clamping and simple support, then clearly it will belong to the family of lines of equal deflection and without loss of generality one may consider that u = 0 on the boundary.

Let the transverse displacement of a point in its middle plane be denoted by w which is a function of the spatial co-ordinates (x, y) and the temporal variable t. When the sandwich plate vibrates in a normal mode, the deflected form maintained by the sandwich plate at any instant t may be described by the family of lines of equal deflection whose equation is u (x, y) = constant. Let us denote the family of curves u = constant by Cu .

Let us consider a portion Ω_u of the sandwich plate bounded by a closed contour C_u at any instant t. Using D'Alembert's principle and summing up of forces in the vertical direction, one obtains the following dynamical equation :

$$\int_{C_u}\left[Q_n-\frac{\partial M_{nt}}{\partial S}\right]ds+\int\int_{\Omega_u}\left[\rho h\frac{\partial^2 w}{\partial t^2}+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+N_y\frac{\partial^2 w}{\partial y^2}\right]d\Omega=0 \qquad ..(114)$$

Here Q_n is the shearing force and the rate of change of the twisting moment $\frac{\partial M_{nt}}{\partial s}$ is along the contour, and the line integral represents the upward vertical contribution of the resultant tractions exerted upon this portion by the remainder. N_x , N_y , N_{xy} represent the membrane forces acting on a small element $d\Omega$ lying within the contour C_u . The first term within the double integral represents the inertia force due to the vertical acceleration of the element $d\Omega$, ρh being the mass per unit area.

By virtue of Hooke's law for isotropic elastic materials, the strain energy per unit area of both the faces can be represented as

$$\overline{V}_o^f = \frac{Et_1}{1-\nu^2}[\left(\varepsilon_x^m\right)^2 + \frac{1}{4}\left(\frac{\partial r}{\partial x}\right)^2 + \left(\varepsilon_y^m\right)^2$$

$$+ 2\nu\left(\varepsilon_x^m \varepsilon_y^m + \frac{1}{4}\frac{\partial r}{\partial x}\frac{\partial s}{\partial y}\right) + \frac{(1-\nu)}{2}\{\left(\gamma_{xy}^m\right)^2$$

$$+\frac{1}{4}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial s}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial r}{\partial y}\frac{\partial s}{\partial x}\}] \quad ..(115)$$

where

$$\varepsilon_x^m = \frac{1}{2}\left(\varepsilon_x^u + \varepsilon_x^l\right),$$

$$\varepsilon_y^m = \frac{1}{2}\left(\varepsilon_y^u + \varepsilon_y^l\right),$$

$$\gamma_{xy}^m = \frac{1}{2}\left(\gamma_{xy}^u + \gamma_{xy}^l\right) \quad ..(116)$$

are averaged values of both face strain components.

ε_x^u, ε_x^l etc. are strains in the upper and lower faces respectively and

$$r = u^u - u^l, \quad S = v^u - v^l \quad ..(117)$$

where

u^u, u^l etc. are displacements in the upper and lower faces respectively.

Introducing two invariants of the averaged strains

$$I_1^m = \varepsilon_x^m + \varepsilon_y^m,$$

$$I_2^m = \varepsilon_x^m \varepsilon_y^m - \frac{1}{4}\left(\gamma_{xy}^m\right)^2 \quad ..(118)$$

into the equation (115), the equation (115) in the following form:

$$\overline{V}_0^f = \frac{Et_1}{1-\nu^2}[(I_1^m)^2 - 2(1-\nu)I_2^m + \frac{1}{4}\{(\frac{\partial r}{\partial s})^2$$

$$+\left(\frac{\partial s}{\partial y}\right)^2+2\nu\frac{\partial r}{\partial x}\frac{\partial s}{\partial y}\}+\frac{1-\nu}{8}\left(\frac{\partial r}{\partial y}+\frac{\partial s}{\partial x}\right)^2] \qquad ..(119)$$

Since the shear strains of the core can be expressed as

$$\gamma_{xz}=\frac{1}{h}(u^1-u^u)+\frac{\partial w}{\partial x},$$

$$\gamma_{yz}=\frac{1}{h}(v^1-v^u)+\frac{\partial w}{\partial y}, \qquad ..(120)$$

the strain energy per unit area of the isotropic core due to shear can be obtained

$$\overline{V}_0^c=\frac{1}{2}hG^{/}[\left(\frac{r}{h}\right)^2+\left(\frac{s}{h}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2$$

$$+\left(\frac{\partial w}{\partial y}\right)^2-\frac{2}{h}\left(r\frac{\partial w}{\partial x}+s\frac{\partial w}{\partial y}\right)$$

$$..(121)$$

In consequence, the total strain energy per unit area of the sandwich plate is

$$\overline{V}_0=\overline{V}_0^f+\overline{V}_0^c \qquad ..(122)$$

The strain energy expression (122) per unit area is then modified following a method similar to that utilized by Dutta and Banerjee (1991), and the expression (122) can finally be written as

$$V=\frac{Et_1}{1-\nu^2}[\left(I'^m_1\right)^2+\frac{\lambda}{4}\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}^2$$

$$+\frac{1}{4}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2+2\nu\frac{\partial r}{\partial x}\frac{\partial s}{\partial y}\right\}$$

$$+\frac{1-\nu}{8}\left(\frac{\partial r}{\partial y}+\frac{\partial s}{\partial x}\right)^2]+\frac{1}{2}hG'.$$

$$[\left(\frac{r}{h}\right)^2 + \left(\frac{s}{h}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2$$

$$+\left(\frac{\partial w}{\partial y}\right)^2 - \frac{2}{h}\left(r\frac{\partial w}{\partial x} + s\frac{\partial w}{\partial y}\right)] \qquad ..(123)$$

where

$$I_1^{\prime m} = \frac{\partial}{\partial x}\left(u^u + u^l\right) + \nu\frac{\partial}{\partial y}\left(v^u + v^l\right)$$

$$+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\nu}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad ..(124)$$

where λ is a constant to be determined from the principle of minimum potential energy. as given in the analysis of Pal and Bera (1995).

Applying Euler's variational principle so as to minimize the total potential energy of the present elastic system of the sandwich plate, we arrive at the following differential equations :

$$I_1^{\prime m} = \frac{\partial P}{\partial x} + \nu\frac{\partial Q}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \nu\left(\frac{\partial w}{\partial y}\right)^2\right\} = constant = \alpha^2 C_1,\ say \qquad ..(125)$$

where

$P = u^u + u^l$, $Q = v^u + v^l$, and

$$\frac{Et_1}{2(1-\nu^2)}\left[\frac{\partial^2 r}{\partial x^2} + \nu\frac{\partial^2 s}{\partial x\,\partial y}\right] + \frac{Et_1}{4(1+\nu)}\left[\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 s}{\partial x\,\partial y}\right] - G'\left[\frac{r}{h} - \frac{\partial w}{\partial x}\right] = 0 \qquad ..(126)$$

$$\frac{Et_1}{2(1-\nu^2)}\left[\frac{\partial^2 s}{\partial y^2} + \nu\frac{\partial^2 r}{\partial x\,\partial y}\right] + \frac{Et_1}{4(1+\nu)}\left[\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 r}{\partial x\,\partial y}\right] - G'\left[\frac{s}{h} - \frac{\partial w}{\partial y}\right] = 0 \qquad ..(127)$$

$$\frac{Et_1}{1-\nu^2} I_1^{\prime m}\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right] + hG'\,\Delta^2 w - G'\left(\frac{\partial r}{\partial x} + \frac{\partial s}{\partial y}\right)$$

$$\frac{Et_1}{1-\nu^2}[\{(\frac{\partial w}{\partial x})^2+(\frac{\partial w}{\partial y})^2\} \quad \Delta^2 w + 2(\frac{\partial w}{\partial x})^2 \frac{\partial^2 w}{\partial x^2}$$

$$+ 2(\frac{\partial w}{\partial y})^2 \frac{\partial^2 w}{\partial y^2} + 4 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}] = 0 \qquad ..(128)$$

where

$$\Delta^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Rewriting the equations (126) and (127), we obtain

$$[\frac{Et_1}{2(1-\nu^2)} \Delta^2 - \frac{G'}{h}](\frac{\partial r}{\partial x} + \frac{\partial s}{\partial y}) + G' \Delta^2 w = 0 \qquad ..(129)$$

Combining the equations (128) and (129) and following the procedure of Dutta and Banerjee (1991), we get

$$[\frac{Et_1}{2(1-\nu^2)} \Delta^2 - \frac{G'}{h}][\frac{2Et_1}{G'(1+\nu^2)} I_1''' \{\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\} + h\Delta^2 w.$$

$$\frac{Et_1 \lambda}{G'(1-\nu^2)}\{[(\frac{\partial w}{\partial x})^2 + (\frac{\partial w}{\partial y})^2]\Delta^2 w + 2(\frac{\partial w}{\partial x})^2 \frac{\partial^2 w}{\partial x^2}$$

$$+ 2(\frac{\partial w}{\partial x})^2 \frac{\partial^2 w}{\partial y^2} + 4 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \}] + G' \Delta^2 w$$

$$+ N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \qquad ..(130)$$

Following the approach of Pal and Bera (1995) and substituting the well-known expressions for Q_n, M_{nt}, N_x, etc. and introducing the equation (130) into the equation (114), we obtain the following equation :

$$\frac{\partial^3 w}{\partial u^3} \int_{C_u} R\,ds + \frac{\partial^2 w}{\partial u^2} \int_{C_u} F\,ds + \frac{\partial w}{\partial u} \int_{C_u} ds \iint_{\Omega_u} D(-\rho h / D \frac{\partial^2 w}{\partial t^2})$$

$$+\left[\frac{E t_1}{2(1-\nu^2)}\Delta^2 - \frac{G'}{h}\right]\left[\frac{E t_1}{G'(1-\nu^2)}\alpha^2 C_1 \left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right.$$

$$+ h\Delta^2 w + \frac{E t_1 \lambda}{G'(1-\nu^2)}\left\{\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\Delta^2 w\right.$$

$$\left.\left. + 2\left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 w}{\partial x^2} + 2\left(\frac{\partial w}{\partial y}\right)^2 \frac{\partial^2 w}{\partial y^2} + 4\frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}\right] \quad ..(131)$$

where use has been made of the fact that w and its derivatives with respect to u are constant and R , F , G are given by the following relations : $R = -D\beta^{3/2}$

$$F = -D\beta^{\frac{1}{2}}\left[3u_{xx}u_x^2 + 3u_{yy}u_y^2 + u_{xx}u_y^2 + u_{yy}u_x^2 + 4u_{xy}u_x u_y\right],$$

$$G = -D\beta^{\frac{3}{2}}\left[u_{xxx}u_x^3 + u_{yyy}u_y^3 + (2-\nu)\left(u_{xxx}u_x u_y^2 + u_{yyy}u_y u_x^2 + u_{xyy}u_x^3 + u_{xxy}u_y^3\right) + (2\nu-1)\right.$$

$$.\ u_{xyy}u_x u_y^2 + u_{xxy}u_x^2 u_y) - 2(1-\nu)u_{xy}$$

$$.(u_x u_y u_{xx} - u_y^2 u_{xy} - u_x^2 u_{xy} + u_x u_y u_{yy})$$

$$\left. + (1-\nu)(u_{xx} - u_{yy})(u_{xx}u_y^2 - u_{yy}u_x^2)\right]$$

$$+\left[2D(1-\nu)\beta^{\frac{5}{2}}\right]\left[u_{xy}(u_x^2 - u_y^2) - u_x u_y(u_{xx} - u_{yy})\right]^2$$

$$..(132)$$

Here $\beta = u_x^2 + u_y^2$, $D = \dfrac{E t_1 h^2}{2(1-\nu^2)}$, is the flexural rigidity, E is the Young's modulus and $\alpha^2 C_1$ is to be determined from

$$I_1^{/m} = \alpha^2 C_1 = \frac{\partial P}{\partial x} + \nu\frac{\partial Q}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \nu\left(\frac{\partial w}{\partial y}\right)^2\right] \quad ..(133)$$

Large Amplitude Transverse Vibration of a Sandwich Parabolic Plate :

As an illustration of the method, let us consider the large amplitude free transverse vibration of a sandwich parabolic plate with clamped edges. The complete analysis of the problem of free vibration of any plate would require the determination of all the frequencies and the corresponding mode shapes. However, if attention is confined to symmetrical forms of vibration, then from symmetry considerations one may assume that the lines of equal deflection form a family of similar and similarly situated parabolas starting from the outer boundary as one of the lines. Let the contour of the plate be bounded by the parabola

$$x^2 = (a/2)\,(2a - y) \qquad ..(134)$$

and the line

$$y = 0 \qquad ..(135)$$

as shown in the Fig. 1 (P-56)

In agreement with the method outlined above , we will select for the equation of the lines of equal deflection as

$$u(x,y) = y\,[\frac{a}{2}\,(2a - y) - x^2] \qquad ..(136)$$

Evidently this expression vanishes along the boundary of the plate.

Calculation of the values of R, F, G and β now gives

$$\beta = 4x^2y^2 + a^4 + a^2y^2 + x^4 - 2a^3y + 2ax^2y - 2a^2x^2$$

$$R = D(4x^2y^2 + a^4 + a^2y^2 + x^4 - 2a^3y + 2ax^2y - 2a^2x^2)^{\frac{3}{2}}$$

$$F = D\beta^{\frac{1}{2}}\ [\ 24x^2y^3 + (3a+2y)(a^2 - ay - x^2)^2 - 16x^2y\ (a^2 - ay - x^2)]$$

$$G = D\beta^{\frac{3}{2}}\ [2(2-\nu)(a^2 - ay - x^2) + 8(2\nu - 1)x^2y^2$$

$$(a^2 - ay - x^2) - 4(1-\nu)x\{4xy^2(a^2 - ay - x^2)$$

$$+ 2x(a^2 - ay - x^2)^2 + 8x^3y^2 + 2axy(a^2 - ay - x^2)\}$$

$$-(1-\nu)(2y-a)\{2y(a^2 - ay - x^2) - 4ax^2y^2\}]$$

$$+ 2D(1-\nu)\beta^{\frac{5}{2}}\ [\ 2x\{4x^2y^2 - (a^2 - ay - x^2)^2$$

$$+ 2xy(a^2 - ay - x^2)(2y - a)\ ]^2$$

.. (137)

Let $\overline{R}$, $\overline{F}$, and $\overline{G}$ denote the mean values of R, F, and G respectively on the contour u = constant. In this particular case, we shall take $\overline{R}$, $\overline{F}$, and $\overline{G}$ as the arithmetic mean values of R, F, G evaluated at the point of intersection of the lines u = constant and x = 0, at

$$A[0, a(1-(1-\frac{2u}{a^3}))^{\frac{1}{2}}\ ] \text{ and } B[0, a(1+(1-\frac{2u}{a^3}))^{\frac{1}{2}}\ ].$$

We have then

$$\overline{R} = \frac{R_B - R_A}{2} = -Da^6(1 - \frac{2u}{a^3})^{3/2}$$

$$\overline{F} = \frac{F_B - F_A}{2} = 5Da^3(1 - \frac{2u}{a^3})^{1/2}$$

$$\overline{G} = \frac{G_B - G_A}{2} = -4D(1-\nu)\left(\frac{3}{2} - \frac{2u}{a^3}\right)\left(1 - \frac{2u}{a^3}\right)^{1/2} \quad ..(138)$$

Setting $1 - \frac{2u}{a^3} = v^2$ and introducing (138) into the equation (131) and carrying out the necessary integration, we obtain

$$\frac{d^2 w}{dv^3} + \frac{2}{v}\frac{d^2 w}{dv^2} + \left\{ -\frac{2}{v^2} + 4(1-\nu)\left(1+\frac{1}{2v^2}\right)\right\}\frac{dw}{dv} - \frac{4B\alpha^2 C_1 a^2 v}{h}$$

$$\frac{dw}{dv} - \frac{2B\lambda}{h}\left(\frac{dw}{dv}\right)^3 - \frac{Bh}{a^2}\left[2\frac{d^2 w}{dv^3} + \left(\frac{4a^3 - a^2 + 3a^2 v - av^2}{v^2} - \frac{6}{v}\right)\right.$$

$$\frac{d^2 w}{dv^2} + \left(\frac{6}{v^2} - \frac{4a^3 - a^2 + 3a^2 v - av^2}{v^3}\right)\frac{dw}{dv} - \frac{2B^2}{G'}\left[2\alpha^2 C_1 \left\{ \frac{2v}{a^2}\frac{d^3 w}{dv^3}\right.\right.$$

$$+\left\{ \frac{v}{v}\left(3 - \frac{6}{a^2}\right) + \frac{4a^4 - a^3 - a^2 v^2}{a^3 v^2}\right\}\frac{d^2 w}{dv^2} + \left\{ \frac{v}{v^2}\left(\frac{6}{a^2} - 3\right)\right.$$

$$-\frac{4a^4 - a^3 - a^2 v^2}{a^3 v^3}\left.\right\}\frac{dw}{dv} + \lambda\left\{ \left(\frac{dw}{dv}\right)^3 \left(\frac{2(2-a)}{a^3 v^2} - \frac{2(2a^2+1)}{a^4 v^2}\right)\right.$$

$$\left.\left.+\left(\frac{dw}{dv}\right)^2\left(\frac{1}{v} + \frac{1}{va^3}\right) - \frac{1}{a^3}\frac{dw}{dv}\frac{d^2 w}{dv^2} + \frac{2(2a^2+1)}{a^4 v^2}\left(\frac{dw}{dv}\right)^2 \frac{d^2 w}{dv^2}\right\}\right]$$

$$-\frac{a^4 \rho h}{D}\int_0^1 \frac{d^2 w}{dt^2}\, dv = 0 \quad ..(139)$$

where $B = \frac{E t_1}{2(1-\nu^2)}$.

Setting $W = \frac{w}{h}$ and $\tau = \frac{1}{\sqrt{\frac{\rho h a^4}{D}}}$, we can transform the equation (139) in the following form :

$$v^2 \frac{d^2 W}{dv^3} + 2v\frac{2}{v}\frac{d^2 W}{dv^2} + v^2\left\{\frac{4(1-\nu)v^2 - 2\vartheta}{v^2} - \frac{4B\alpha^2 C_1 a^2 v}{h}\right\}\frac{dW}{dv}$$

$$- 2 Bh\lambda \left(\frac{dW}{dv}\right)^3 v^2 - \frac{Bhv^2}{a^2} \left[2\frac{d^3 W}{dv^3} + \frac{4a^3 - a^2 + 3v(a^2-2) - av^2}{v^2} \right.$$

$$\left(\frac{d^2 W}{dv^2} - \frac{1}{v}\frac{dW}{dv}\right) - \frac{2B^2 v^2}{G'} \left[2\alpha^2 C_1 \left(\frac{2v}{a^2}\frac{d^3 W}{dv^3} + \left\{\frac{3v}{v}\left(1-\frac{2}{a^2}\right)\right.\right.\right.$$

$$+ \frac{4a^2 - a - v^2}{av^2}\Big\}\left(\frac{d^2 W}{dv^2} - \frac{1}{v}\frac{dW}{dv}\right)\Big) + \lambda \left\{ \frac{2h^2}{a^3 v^2} \left(\frac{dW}{dv}\right)^3 \left(2 - 3a - \frac{1}{a}\right)\right.$$

$$\left. + \frac{h}{v}\left(1+\frac{1}{a^3}\right)\left(\frac{dW}{dv}\right)^2 - \frac{h}{a^3}\frac{d^2 W}{dv^2}\frac{dW}{dv} + \frac{2h^2(2a^2+1)}{a^4 v}\left(\frac{dW}{dv}\right)^2 \frac{d^2 W}{dv^2} \right\} \Big]$$

$$- v^2 \int_0^1 \frac{d^2 W}{d\tau^2} dv = 0 \qquad .. (140)$$

SOLUTION OF THE PROBLEM :

Any conventional method may be used to solve the differential equation (140). Let us solve this equation by the principle of Galerkin, assuming W in the following form :

$$W(v,\tau) = W_0(\tau) v^2 (1-v^2)(a_0 + a_1 v^2 + a_2 v^2 + \ldots\ldots\ldots.) \qquad .. (141)$$

In fact, taking the solution in the form of equation (141), we shall assume for W the first term approximation as

$$W(v, \tau) = a_0 W_0(\tau) v^2 (1 - v^2) \qquad .. (142)$$

It is to be noted that the first term approximation in the choice of this deflection W is available in the open literature and yields sufficiently accurate results. Substitution of the above expression (142) for W in the equation (140) yields an error function $\in$, which does not vanish, in general, since the expression for W is not an exact solution. Galerkin's principle requires that the error function $\in$ must be orthogonal over the domain, that is ,

$$\int_0^1 \in v W \, dv = 0 \qquad ..(143)$$

Then from the equation (140), (142) and (143), we obtain

$$\frac{d^2 W_o(\tau)}{d\tau^2} - \{ \frac{12.2857 a^2 B v}{h} - \frac{B^2}{G'} (\frac{142.8571}{a} - 953.1428 a - 306.8571) \}$$

$$\alpha^2 C_1 W_o(\tau) - \{16.8571 v - 243.7857 - (\frac{35.7143}{a} - 76.7143$$

$$-238.2857 a) Bh \} W_o(\tau) + (9.8571 + \frac{2293.2857}{a^3}) \frac{B^2 h \lambda}{G'} a_o W_o^2(\tau)$$

$$- \{ 9.7143 B h \lambda - \frac{B^2 h^2 \lambda}{G' a^3} (16868.571 - 23261.714 a$$

$$- \frac{7413.7142}{a}) \} a_o^2 W_o^3(\tau) = 0 \qquad ..(144)$$

The terms involving the in-plane displacements P and Q can be easily eliminated by considering suitable expressions for these displacements compatible with the boundary conditions and by subsequent integration, we obtain

$$\alpha^2 C_1 = \frac{2 a_o^2 W_o^2(\tau) h v}{3 a^2 t_1} \qquad ..(145)$$

The equation (144) with the help of the equation (145) then becomes

$$\frac{d^2 W_o(\tau)}{d\tau^2} + \mu_1 W_o(\tau) + \mu_2 (\frac{a_o}{h}) [W_o(\tau)]^2 + \mu_3 (\frac{a_o}{h})^2 [W_o(\tau)]^3 = 0 \qquad ..(146)$$

where

$$\mu_1 = [16.8571 v - 243.7857 - (\frac{35.7143}{a} - 76.7143 - 238.2857 a) Bh]$$

$$\mu_2 = \frac{B^2 h^2 \lambda}{G'} (9.8571 + \frac{2293.2857}{a^3})$$

$$\mu_3 = -h^2 [B (8.1905 \frac{v^2}{t_1} - 9.7143 \lambda h) - \frac{B^2 h}{G' a^2} \{ \frac{v}{t_1} (\frac{95.2381}{a}$$

$$-635.4285a - 204.5714) + \frac{h\lambda}{a}(16868.571$$

$$- 23261.714a - \frac{7413.7142}{a}))\}] \qquad ..(147)$$

Making $G' \to \propto$, we can easily identify the equation (140) with that already obtained by Pal and Bera (1995).

For $\lambda = 0$, we obtain the result corresponding to that of Berger (1955) and the equation (146) reduces to

$$\frac{d^2 W_o(\tau)}{d\tau^2} + \mu_{1B} W_o(\tau) + \mu_{3B}\left(\frac{a_o}{h}\right)^2 [W_o(\tau)]^3 = 0 \qquad ..(148)$$

where

$$\mu_{1B} = \mu_1$$

$$\mu_{2B} = 0,$$

$$\mu_{3B} = -h^2 \left[\frac{8.1905\nu^2 B}{t_1} - \frac{B^2 h\nu}{G' a^2 t_1}\left(\frac{95.2381}{a}\right.\right.$$

$$\left.\left.-635.4285a - 204.5714\right)\right] \qquad ..(149)$$

When $\alpha^2 = 0$, we can derive the equation for the **movable edge** boundary conditions and in this case the equation (144) reduces to

$$\frac{d^2 W_0(\tau)}{d\tau^2} + \mu_{1M} W_0(\tau) + \mu_{2M}[W_0(\tau)]^2 + \mu_{3M}[W_0(\tau)]^3 = 0 \qquad ..(150)$$

where

$$\mu_{1M} = \mu_1,$$

$$\mu_{2M} = \mu_2,$$

$$\mu_{3M} = -h^2 \{ 9.7143 Bh\lambda + \frac{B^2 h^2 \lambda}{a^3 G'}(16868.571$$

$$- 23261.714\, a - \frac{7413.7142}{a}) \}$$

If the initial conditions are $W_o = 1$ and $\frac{d W_o}{d \tau} = 0$ at $\tau = 0$, then the solution of the equation (146) can be written as

$$\frac{\omega_1^*}{\omega_1} = [\, 1 + (\frac{a_o}{h})^2 \{ \frac{3}{4} \frac{\mu_3}{\mu_1} - \frac{5}{6} (\frac{\mu_2}{\mu_1})^2 \} \,]^{\frac{1}{2}} \tag{151}$$

where ω_1 and ω_1^* are the linear and nonlinear frequencies respectively.

Numerical Results and Discussion :

For numerical calculation, we use the following values for the geometry of the plate and material constants assumed by Alwan (1964) :

$$a = 1m, \; E = 7347.201 \times 10^6 \; kg/m^2$$

$$h = 1.7 \times 10^{-2}\, m, \; G' = 4218.4884 \times 10^3 \; kg/m^2$$

$$t_1 = 0.6 \times 10^{-3}\, m, \; \nu = 0.3\, , \; \lambda = 3\, \nu^2 \, . \qquad ..(152)$$

Fig. 2 shows the graphs corresponding to the numerical results for a comparative study of the non-linear to linear frequency ratios versus the ratio of the non-dimensional amplitude for a clamped sandwich parabolic plate with immovable edges, as obtained in the present study [PS] and by Berger's method (1955). Fig. 3 shows the graph corresponding to **movable edges** obtained in the present analysis and no such graph is available through Berger's method, the reason for which has already been pointed out. This is the most important achievement of the present analysis.

It is observed that the results of the present study are sufficiently accurate for both movable as well as immovable edge conditions. For sandwich parabolic plates, the maximum numerical difference in the different values of ω_1^* / ω_1 given by Berger's method and the present study is only 0.083, although the percentage in difference is significant. This is due to the fact that Berger's method is purely an approximate one based on the neglect of e_2, the so-called second strain invariant in the potential energy expression, whereas in the present study no term in the potential energy expression is neglected rather modified through a modified technique which helps not only for the improvement of the results in the case of immovable edges but also in the determination of the results for the **movable edge** boundary. Thus the results of the present analysis is more accountable. As the results for the movable boundary in the present problem are not available through any other method, it has not been possible to compare the results of the present analysis with them. Furthermore, it may be reiterated that the results for the sandwich plates of other shapes can be obtained from a single differential equation, for different choices of u for different contours, as a result, the method described here seems to be more advantageous than those of the previous authors. Finally, it may also be pointed out that the calculations of R, F, G for sandwich parabolic plates are a little more involved than those of the corresponding results for other plates.

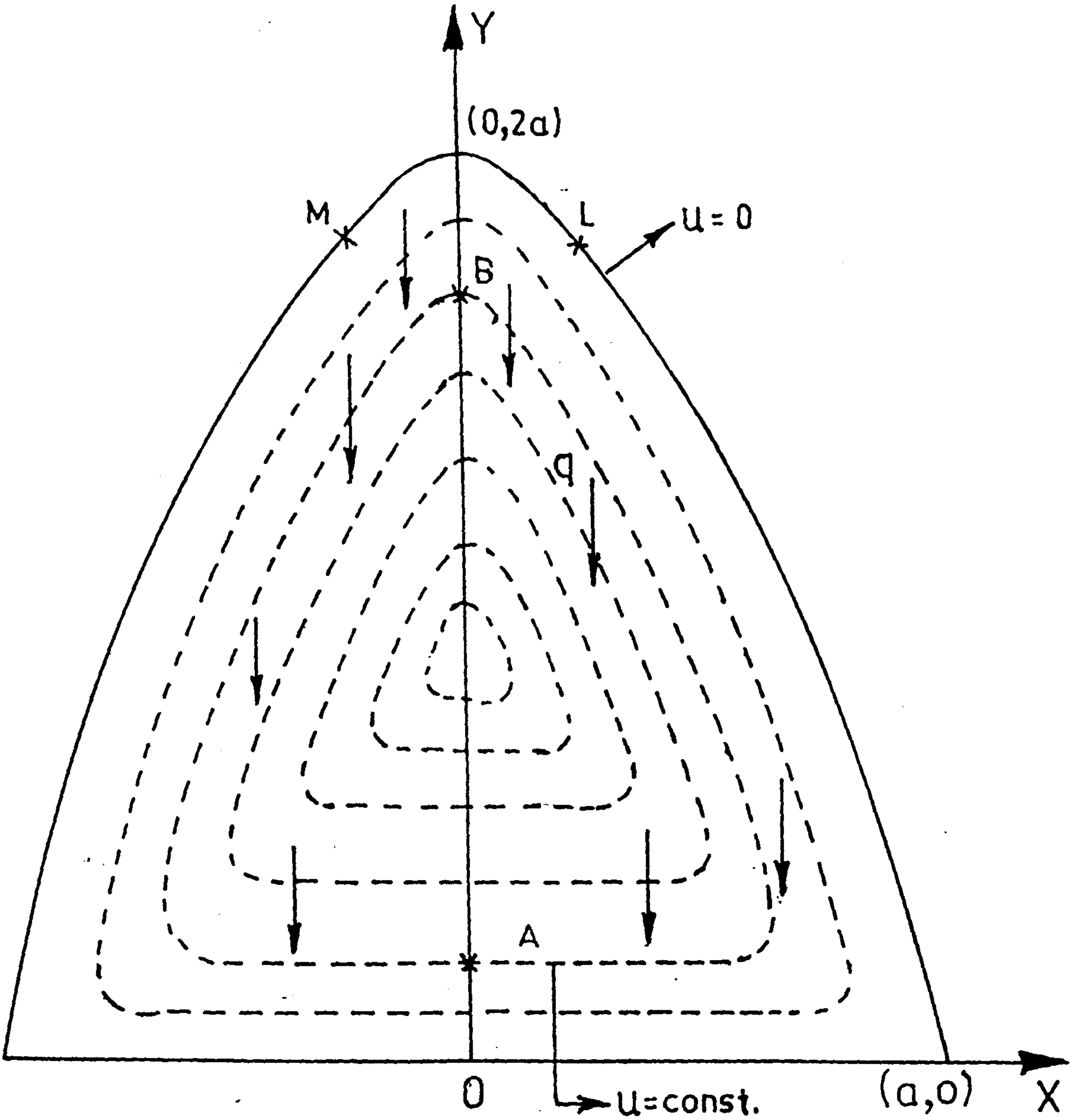

Fig.1: Parabolic Plate

Nonlinear to Linear Frequency Ratios vs. Nondimensional Amplitudes

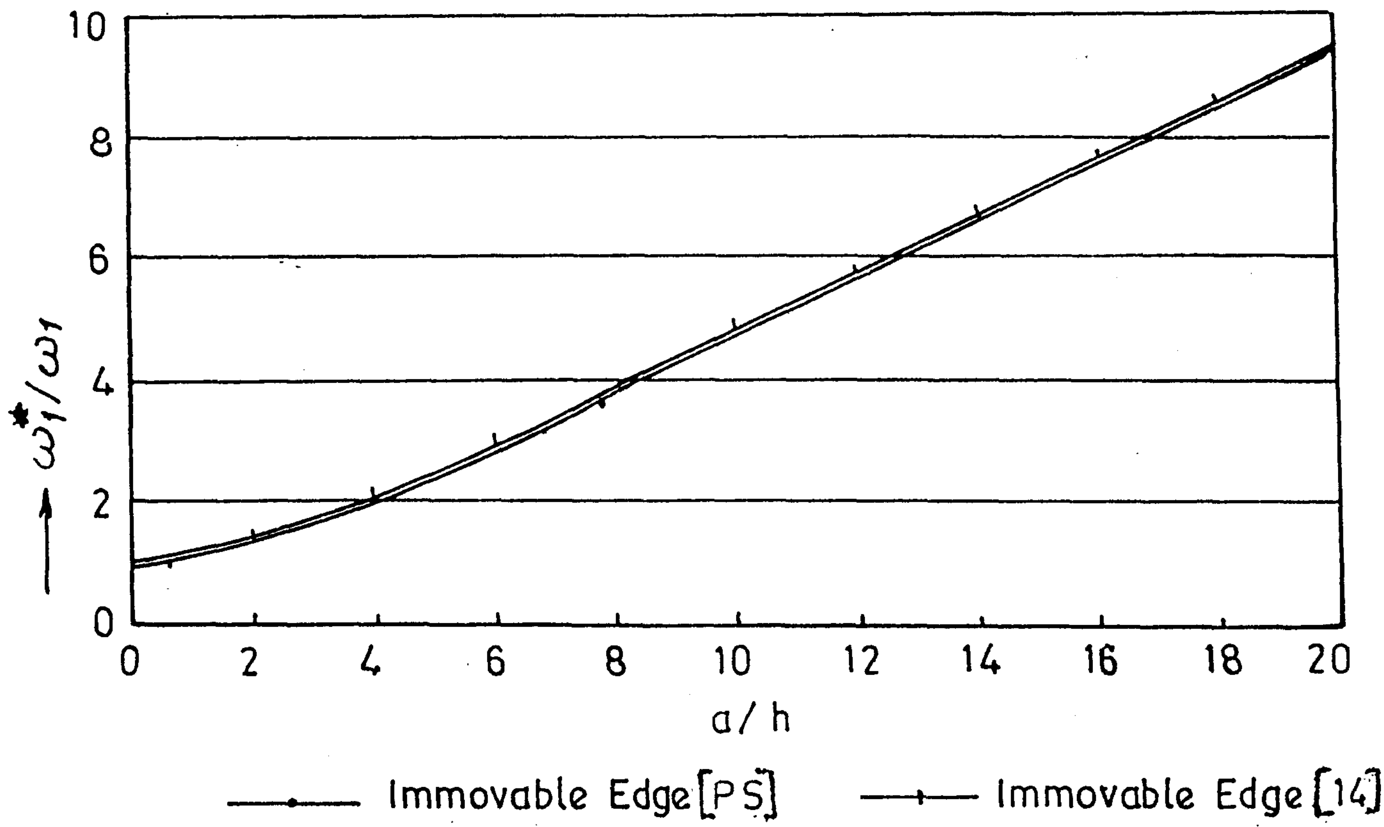

Fig.2: Vibration of Parabolic Plates

Nonlinear to Linear Frequency Ratios vs. Nondimensional Amplitudes

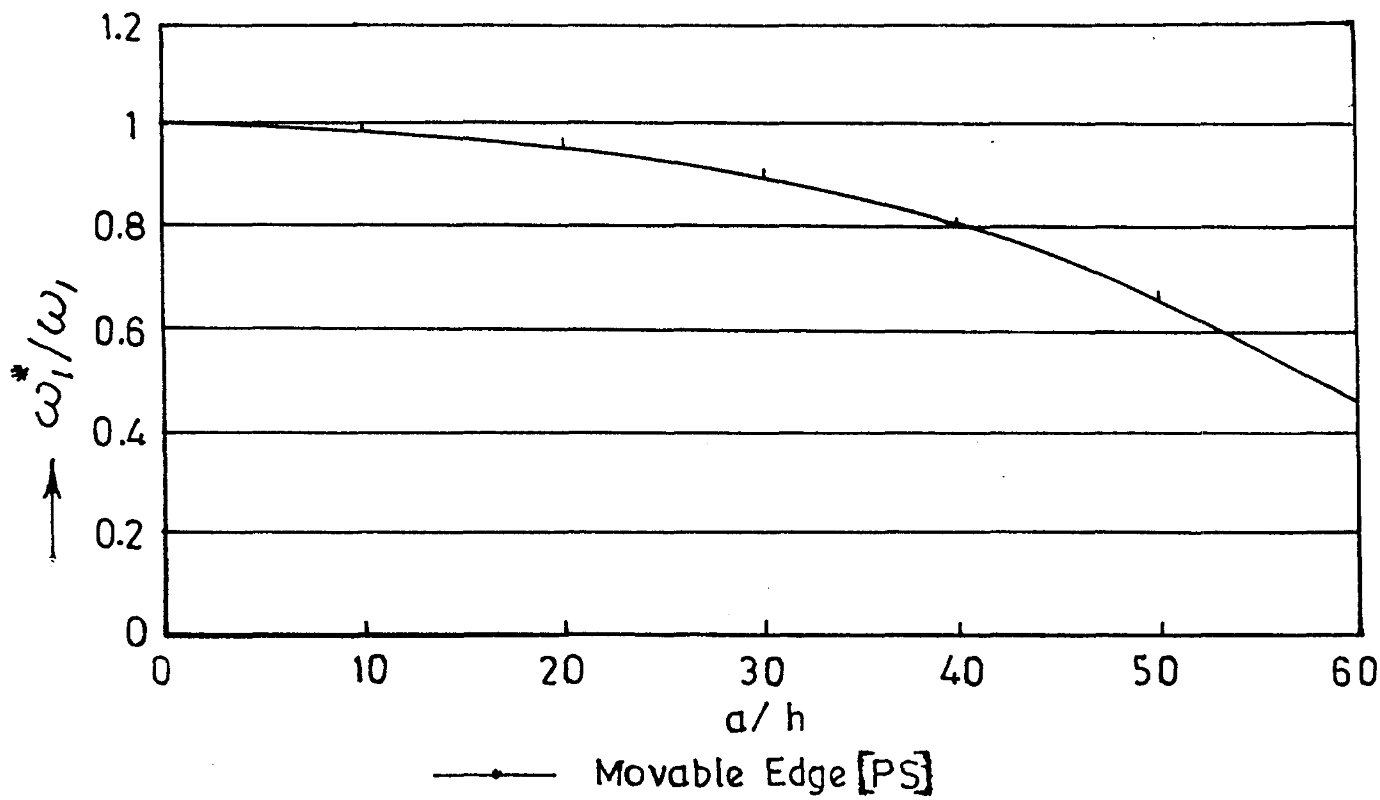

Fig. 3: Vibration of Parabolic Plates