Chapter 3

General Solution for a Class of Static Charged Spheres

3.1 Introduction

Exact solutions to the coupled Einstein-Maxwell system of equations are important for many physical applications, in particular in relativistic astrophysics. Such solutions may be used to model relativistic charged stars where the interior of the charged sphere matches with the Reissner-Nordstrom metric at the boundary. It is interesting to observe that, in the presence of charge, the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided. In this situation the gravitational attraction is counter balanced by the repulsive Coulombian force in addition to the pressure gradient. Consequently the Einstein-Maxwell system, for a charged star, has attracted considerable attention in various investigations. Earlier workers in this field include Bonnor [100], De and Raychaudhari [101] and Cooperstock and Cruz [102], to name a few. Exact solutions of the coupled Einstein-Maxwell equations on the background of spheroidal space-time were obtained by Tikekar [103], Patel *et al* [104] and Tikekar and Singh [105]. Dianyan [106] analyzed the Einstein-Maxwell system in a higher dimensional space-time. The charged sphere case has also been taken up by Tiwari and Ray [107], Guilfoyle [108] and Rao *et al* [109, 110]. Applications of the Einstein-Maxwell system in inhomogeneous cosmological models have been analyzed by Krasinskii [111]. Considering a particular charge distribution Felice *et al* [112] presented a model for a charged fluid sphere and discussed its stability. Stability of relativistic charged spheres has also been discussed by Anninos and Rothman [113]. In a recent article, Ivanov [71] has reviewed the known solutions for static charged fluid spheres in general relativity.

Although it is most unlikely that there exits a charged large astrophysical object, the possibility of a compact cold star acquiring a net charge by accretion from the surrounding medium has been studied recently by Treves and Turolla [69]. Similar observations have also been reported recently by Mak *et al* [70].

Our objective here is to study the gravitational behaviour of a general class of charged stars for a particular choice of the electric field intensity. In Section 3.2 we present the Einstein-Maxwell field equations for the static, spherically symmetric line element. The set of field equations is reduced to a single differential equation. The general solution of the condition for pressure isotropy is given in Section 3.3 in terms of Gegenbauer functions. We rewrite the general solution in terms of elementary functions. Also we relate our results to the solutions of other investigators. In Section 3.4 we provide a detailed analysis of the quantities of physical interest for a relativistic charged star. Finally in Section 3.5 we discuss the physical viability of our model and consider avenues for further work.

3.2 Einstein-Maxwell equations

We write the line element for static spherically symmetric charged spacetime in the form (in geometrised units with $8\pi G = c = 1$)

$$ds^2 = -e^{2\gamma(r)}dt^2 + e^{2\mu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.1}$$

in standard coordinates $x^i = (t, r, \theta, \phi)$. The quantities $\gamma(r)$ and $\mu(r)$ are the gravitational potentials.

The total energy-momentum tensor, for a charged sphere, has the standard form

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij} + 2\left[F_{ik}F_j^k - \frac{1}{4}F_{kl}F^{kl}g_{ij}\right] \tag{3.2}$$

where ρ represents the energy density, p is the isotropic pressure and u^i is the 4-velocity of the fluid. Because of spherical symmetry, the electric field E must be radial, the Maxwell stress tensor has only two non-zero component F_{tr} and F_{rt} and we define the intensity of the electric field as $E^2 = -F_{tr}F^{tr}$. F^{ij} satisfies the following relations:

$$\left(2F^{ij}\sqrt{-g}\right)_{,j} = \sqrt{-g}J^i = \sqrt{-g}\sigma u^i \tag{3.3}$$

$$F_{ij,k} = 0 \tag{3.4}$$

where, J^i and σ denote four-current and proper charge density of the fluid, respectively.

The Einstein-Maxwell equations, for the metric (3.1), then, reduce to

$$\rho + E^2 = \frac{(1 - e^{-2\mu})}{r^2} + \frac{2\mu' e^{-2\mu}}{r} \tag{3.5}$$

$$p - E^2 = \frac{2\gamma' e^{-2\mu}}{r} - \frac{(1 - e^{-2\mu})}{r^2} \tag{3.6}$$

$$p + E^2 = e^{-2\mu}\left(\gamma'' + \gamma'^2 - \gamma'\mu' + \frac{\gamma'}{r} - \frac{\mu'}{r}\right) \tag{3.7}$$

where primes denote differentiation with respect to r.

The quantity

$$q(r) = \frac{1}{2}\int_0^r \sigma r^2 e^{\mu} dr = r^2 E \tag{3.8}$$

defines the total charge contained within the sphere of radius r.

Equations (3.6) and (3.7) may be combined to give

$$\gamma'' + \gamma'^2 - \gamma'\mu' - \frac{\gamma'}{r} - \frac{\mu'}{r} - \frac{(1 - e^{2\mu})}{r^2} - 2E^2e^{2\mu} = 0. \tag{3.9}$$

As discussed in chapter 2, in an attempt to reduce the complexity of the field equations Vaidya and Tikekar [63] gave an ansatz for one of the metric coefficients in equation (3.1), given by

$$e^{2\mu} = \frac{1 + \lambda r^2/R^2}{1 - r^2/R^2}, \tag{3.10}$$

where λ and R are the curvature parameters. This assumption produces relativistic stars with ultrahigh densities and pressures consistent with observations (Rhodes and Ruffini [114]). The assumption has the additional advantage of providing the solution of the field equations with a clear geometrical characterisation. The assumption has been used by Tikekar [103], Patel and Kopper [115] and Tikekar and Singh [105] to describe the behaviour of static charged objects for some restricted values of the parameter λ, e.g., $\lambda = 2$ & 7. Here we would like to generalize these results for an arbitrary λ; the requirements for a realistic solution, e.g., the need for causal signals, however, restrict the value of λ. This issue will be taken up in section 3.4. To solve equation (3.9), we also make the choice for one of the metric coefficients given by equation (3.10).

We now clarify a number of issues that pertain to the physical relevance of this class of static charged spheres. Our choice for the spatial geometry parametrized by λ generates models of static charged spheres which are consistent with densities of superdense stars. For a physically relevant solution, it is often required that an equation of state, relating energy density to the pressure should hold. We observe that the solutions presented here do not satisfy an equation of state, though they are relevant in the description of highly dense stars where we have little information about the matter content, atleast inside the core. However, we should point out that in the uncharged case, the solutions lead to an equation of state as demonstrated by Mukherjee *et al* [67]. In a more general treatment, we would also need to allow for the electric and magnetic properties of the barotropic matter content. However, in our present investigation, we consider only a special distribution of an electric field.

To carry out the solution, we introduce the transformation

$$\begin{aligned} \psi &= e^{\gamma} \\ x^2 &= 1 - \frac{r^2}{R^2}, \end{aligned}$$

so that (3.9) can be written as

$$\left(1 + \lambda - \lambda x^2\right) \psi_{xx} + \lambda x \psi_x + \lambda (\lambda + 1) \psi - \frac{2E^2 R^2 \left(1 + \lambda - \lambda x^2\right)^2}{1 - x^2} \psi = 0 \qquad (3.11)$$

where we have used (3.10). When $E = 0$, (3.11) becomes

$$\left(1 + \lambda - \lambda x^2\right) \psi_{xx} + \lambda x \psi_x + \lambda (\lambda + 1) \psi = 0 \qquad (3.12)$$

Equation (3.12) was comprehensively analysed by Maharaj and Leach [66] who found solutions in terms of elementary functions. The general solution, in terms of special functions, was presented by Mukherjee *et al* [67].

To solve the equation (3.11) we make the choice

$$E^2 = \frac{\alpha^2 \left(1 - x^2\right)}{R^2 \left(1 + \lambda - \lambda x^2\right)^2} \qquad (3.13)$$

where α is a constant. This choice for E generates a model for a charged star which is physically realistic as will be discussed in section 3.4. Also a specific upper bound on α and hence on E is found in this model.

The choice of E, lets us to rewrite equation (3.11) as

$$\left(1 + \lambda - \lambda x^2\right) \psi_{xx} + \lambda x \psi_x + \left[\lambda (\lambda + 1) - 2\alpha^2\right] \psi = 0 \qquad (3.14)$$

We integrate (3.14) in the next section. The choice (3.13) for the electric field intensity was made so as to retain the characterisation of the solution in terms of the parameter λ. Patel and Koppar [115] chose a similar form for E corresponding to the parameter value $\lambda = 2$.

3.3 General Solution

To obtain the general solution we let

$$z = \left(\frac{\lambda}{\lambda+1}\right)^{1/2} x$$
$$\beta^2 = \lambda + 2 - \frac{2\alpha^2}{\lambda}$$

We can then write (3.14) as the third order equation

$$(1-z^2)\psi_{zzz} - z\psi_{zz} + \beta^2\psi_z = 0 \tag{3.15}$$

If we treat ψ_z as the dependent variable then the general solution of (3.15) is given by

$$\psi_z = A_1 T_\beta^{-1/2}(z) + A_2(1-z^2)^{1/2} T_{\beta-1}^{1/2}(z) \tag{3.16}$$

where $T_\beta^{-1/2}$ and $T_{\beta-1}^{1/2}$ are Gegenbauer functions ([99], p. 547), and A_1 and A_2 are constants. When β is zero or a positive integer then the functions reduce to polynomials. On using the properties of the Gegenbauer functions, we can eliminate the derivative in (3.16) and obtain the representation

$$\psi(z) = CT_{\beta+1}^{-3/2}(z) + D(1-z^2)^{3/2} T_{\beta-2}^{3/2}(z), \tag{3.17}$$

where C and D are constants. We can also represent the solution in the form

$$\psi(\zeta) = A\left[\frac{\cos[(\beta+1)\zeta+\omega]}{\beta+1} - \frac{\cos[(\beta-1)\zeta+\omega]}{\beta-1}\right], \tag{3.18}$$

if we let $\zeta = \cos^{-1} z$ for real z, with $0 < z \leq \beta$. In (3.18), A and ω are constants.

When $\beta^2 = \lambda + 2$ (i.e. $\alpha = 0$) we regain the general solution of Mukherjee *et al* [67] for uncharged spheres. In the case $\beta^2 = 4 - \alpha^2$ (i.e. $\alpha \neq 0$) we regain the particular solution of Patel *et al* [104] for charged spheres with the geometry corresponding to the parameter $\lambda = 2$.

3.4 Physical analysis

The form of the solution enables us to study the physical features in a qualitative fashion. The energy density, pressure and charge density are given, respectively, by

$$\rho = \frac{1}{R^2(1-z^2)}\left[1+\frac{2}{(\lambda+1)(1-z^2)} - \frac{\alpha^2[(\lambda+1)(1-z^2)-1]}{\lambda(1+\lambda)^2(1-z^2)}\right] \tag{3.19}$$

$$p = -\frac{1}{R^2(1-z^2)}\left[1+\frac{2z\psi_z}{(\lambda+1)\psi} - \frac{\alpha^2[(\lambda+1)(1-z^2)-1]}{\lambda(1+\lambda)^2(1-z^2)}\right] \tag{3.20}$$

$$\sigma = \frac{2\alpha z(1-z^2)^{-5/2}}{\sqrt{\lambda}R^2(1+\lambda)^2}\left[2+(\lambda+1)(1-z^2)\right]. \tag{3.21}$$

When $\alpha = 0$ we regain from (3.19) and (3.20) the expressions of Mukherjee *et al* [67] as a special case. A detailed physical study in the particular case $\lambda = 2$ by Patel and Koppar [115] shows that $\rho \geq 0$ and $p \geq 0$ so that the weak energy conditions are satisfied. Our general solution permits a similar analysis to be performed for all real values of λ. The quantities ρ, p, σ have simple forms; they are well-behaved and bounded in the interior of the charged sphere. It is clear that these quantities are finite and regular at the centre. We have shown in Table 3.1, some values of ρ and p for $\lambda = 1$ and $\alpha = 0$ & 0.4. For a high value of $\lambda \geq 50$, the effect of the charge distribution ($\alpha < 0.43$) is negligible.

The radius is fixed by putting $p = 0$ at the boundary $r = b$, which gives the condition

$$\frac{\psi_z(z_b)}{\psi(z_b)} = \frac{\lambda+1}{2z_b}\left[\frac{\alpha^2 b^2}{R^2(1+\lambda)(1-z_b^2)} - 1\right], \tag{3.22}$$

where

$$z_b = \left(\frac{\lambda}{\lambda+1}\right)^{1/2}\left(1-\frac{b^2}{R^2}\right)^{1/2},$$

is the value of z at the boundary.

The model has five parameters, A, ω, R ,λ, and α. Two of these may, in principle, be determined by matching the interior solution at $r = b$, to the Reissner-Nordstrom

$\alpha = 0$		$\alpha = 0.4$	
ρ	p	ρ	p
.4567	0	.3800	0
.4842	.0128	.3939	.0037
.5026	.0214	.4089	.0098
.5225	.0307	.4250	.0163
.5441	.0407	.4426	.0233
.5676	.0516	.4617	.0309
.5933	.0635	.4826	.0391
.6188	.0753	.5034	.0472

Table 3.1: Energy-density and Pressure in GeVfm^{-3} for $\lambda = 1$ and $\alpha = 0$ & 0.4, for a typical compact star of mass $M = 0.88\ M_{\odot}$.

metric, given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.23)$$

where M and q denote the total mass and charge, respectively, as measured by an observer at infinity. The two parameters A and ω in (3.18) can now be determined conveniently by noting that the equation (3.22) is independent of A, and the relation

$$\psi(z_b) = \left(1 - \frac{b^2}{R^2}\right)^{1/2} \left(1 + \frac{\lambda b^2}{R^2}\right)^{-1/2},$$

is valid. In the uncharged case, given a value of λ (which may be looked upon as specifying an equation of state) one may determine the radius of the star. The mass is already determined by the matching conditions. In the charged case, one needs also to specify either α or q. Thus we have an exactly solvable model.

To see the effect of the charge distribution, we calculate the radius of a typical highly compact star of mass $M = 0.88\ M_{\odot}$. For $\lambda = 1$, if we let the radius in the uncharged case to be 7.7 *km*, in the presence of charge with $\alpha = 0.4$, the radius increases to 8.18 *km*. However if we assume a high value of λ, say $\lambda = 100$, there is practically no effect of charge on the radius. It is, however, useful to study the constraints that are imposed on the parameters from physical considerations.

Since the pressure $p \geq 0$ inside the sphere, we require that

$$\frac{\psi_z}{\psi} \leq \frac{\alpha^2 r^2}{R^2(\lambda+1)2z(1-z^2)} - \frac{\lambda+1}{2z}. \tag{3.24}$$

Again, from (3.19) and (3.20) we obtain the expression

$$\frac{dp}{d\rho} = \frac{z(1-z^2)^2(\psi_z/\psi)^2 - (1-z^2)(\psi_z/\psi) + \alpha^2\left[\frac{2z}{\lambda+1} + \frac{z^3}{\lambda} - \frac{3z}{\lambda}\right]}{z(1-z^2)(\lambda+1) + 4z - \alpha^2\left[\frac{2z}{\lambda+1} - \frac{z}{\lambda} - \frac{z^3}{\lambda}\right]}$$

For causality not to be violated the speed of sound must be less than the speed of light so that $\frac{dp}{d\rho} < 1$. This condition and (3.24) give

$$\frac{1}{(1-z^2)}\left[\frac{1}{2z} - D\right] \leq \frac{\psi_z}{\psi} \leq \frac{\alpha^2 r^2}{R^2(\lambda+1)2z(1-z^2)} - \frac{\lambda+1}{2z} \tag{3.25}$$

where

$$D = \left[4 + \frac{1}{4z^2} + (\lambda+1)(1-z^2) + \frac{4\alpha^2}{\lambda(\lambda+1)}\right]^{1/2}.$$

The constraint (3.25) follows from the joint requirements of positive pressure and causal signals.

From (3.25) we generate a lower bound for the spheroidal parameter

$$\lambda > \frac{-7 + \left[49 - 17(16\alpha^2 - 3)\right]^{1/2}}{17}$$

When $\alpha = 0$, we have $\lambda > \frac{3}{17}$, which is the lower bound on the spheroidal parameter for uncharged stars. Otherwise for real λ we require that $\alpha < 0.607$, although for a real and positive λ, $\alpha < 0.43$. This condition on α places an upper bound on the value of the electric field intensity via (3.13). Specifically the bound on E in this model is given by

$$E^2 < \frac{0.185\left[\lambda - (\lambda+1)z^2\right]}{R^2\left[\lambda + \lambda^2 - \lambda(\lambda+1)z^2\right]^2}$$

where z lies in the interval (z_0, z_b). Finally we observe that an upper bound on possible values of b/R may be obtained from (3.24). This upper limit on the boundary of the star is given by

$$\frac{b^2}{R^2} \leq \frac{-6\alpha^2 - 12\lambda + 2\alpha^2\lambda - 12\lambda^2 + K}{\alpha^4 - 2\alpha^2\lambda + 5\lambda^2 - 2\alpha^2\lambda^2 + 6\lambda^3 + \lambda^4}$$

where

$$\begin{aligned} K^2 &= (6\alpha^2 + 12\lambda - 2\alpha^2\lambda + 12\lambda^2)^2 \\ &\quad -(3 - 16\alpha^2 - 14\lambda - 17\lambda^2)(\alpha^4 - 2\alpha^2\lambda + 5\lambda^2 - 2\alpha^2\lambda^2 + 6\lambda^3 + \lambda^4) \end{aligned}$$

When $\alpha = 0$ we regain the upper bound on b/R for an uncharged star. We have shown in Table 3.2 the values of b/R for the cases $\alpha = 0$ and $\alpha = 0.4$. In Fig.3.1 we have shown the effects of both λ and α on the upper bound of the possible values of b/R of a charged star. In Fig.3.2 we have plotted the upper bound on $\frac{b}{R}$ against $Log_{10}\lambda$ for two cases: (a) $\alpha = 0$ and (b) $\alpha = 0.4$. We observe that the effect of charge, through the presence of the parameter α, is to raise the upper bound, and this allows for a wider range in the causal behaviour.

The condition $\frac{dp}{d\rho} < 1$ leads to

$$z(1 - z^2)^2(\frac{\psi_z}{\psi})^2 - (1 - z^2)(\frac{\psi_z}{\psi}) - z(1 - z^2)(1 + \lambda) - 4z - \frac{4\alpha^2 z}{\lambda(1 + \lambda)} < 0. \qquad (3.26)$$

When applied to the centre of the star ($z = z_0$), one gets another interesting constraint on the parameters. Table 3.3 gives the minimum allowed values of $\frac{\psi_z(z_0)}{\psi(z_0)}$ for different λ and two different values of α. Thus, $\frac{\psi_z(z_0)}{\psi(z_0)}$ can attain a lower value for a charged star, indicating a wider range for the parameter ω. For example, for $\lambda = 1$, (a) $\omega > 1.7024$, for $\alpha = 0$ and (b) $\omega > 1.6726$ for $\alpha = 0.4$. The effect decreases with increasing λ. Imposition of other constraints however will further restrict the values of ω.

λ	$\frac{b^2}{R^2}$		
	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$
.5	.4365	.4600	.5255
1	.5166	.5251	.5502
2	.5000	.5030	.5119
5	.3809	.3815	.3835
10	.2631	.2632	.2636
20	.1611	.1612	.1612
100	.0390	.0390	.0390

Table 3.2: Upper bound on $\frac{b^2}{R^2}$ for different values of α and λ

λ	$\left.\frac{\psi_z(z_0)}{\psi(z_0)}\right\|_{Min.}$		
	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$
.5	-2.2978	-2.3639	-2.5555
1	-3.2762	-3.3102	-3.41072
2	-5.1181	-5.1353	-5.1867
5	-10.5267	-10.5337	-10.5545
10	-19.4957	-19.4991	-19.5096
20	-37.4150	-37.4167	-37.4220
100	-180.7230	-180.7240	-180.7250

Table 3.3: Minimum values of $\psi_z(z_0)/\psi(z_0)$ permitted by causal behaviour

3.5 Discussions

We have found a non-trivial family of a new class of solutions to the Einstein-Maxwell equations for a particular choice of the electric field intensity E (with the associated parameter α) and the general spheroidal parameter λ. This form for E had previously been shown to lead to physically acceptable solutions for $\lambda = 2$ [104]. The generalisation to all allowed values of λ may be useful as uncharged compact stars can be described in this model with large values of λ. Some of these stars could become charged due to accretion [69], [70], in particular charges may accumulate in a shell on the boundary. Description of these stars, in terms of a core-envelope model, may be done by making use of the solution given here. We have also performed a qualitative analysis on the physical properties of the charged sphere. Where relevant we have regained the uncharged limit of quantities of interest. We note the very interesting interplay between λ and the charge parameter α; the presence of charge allows for a wider range for physical parameters. In particular we have shown that $\alpha \neq 0$ permits causal signals over a wider range of values of $\frac{b}{R}$ than is the case for uncharged stars. The upper bound on $\frac{b}{R}$ of the charged star is constrained by both λ and α, as shown in Fig3.1.

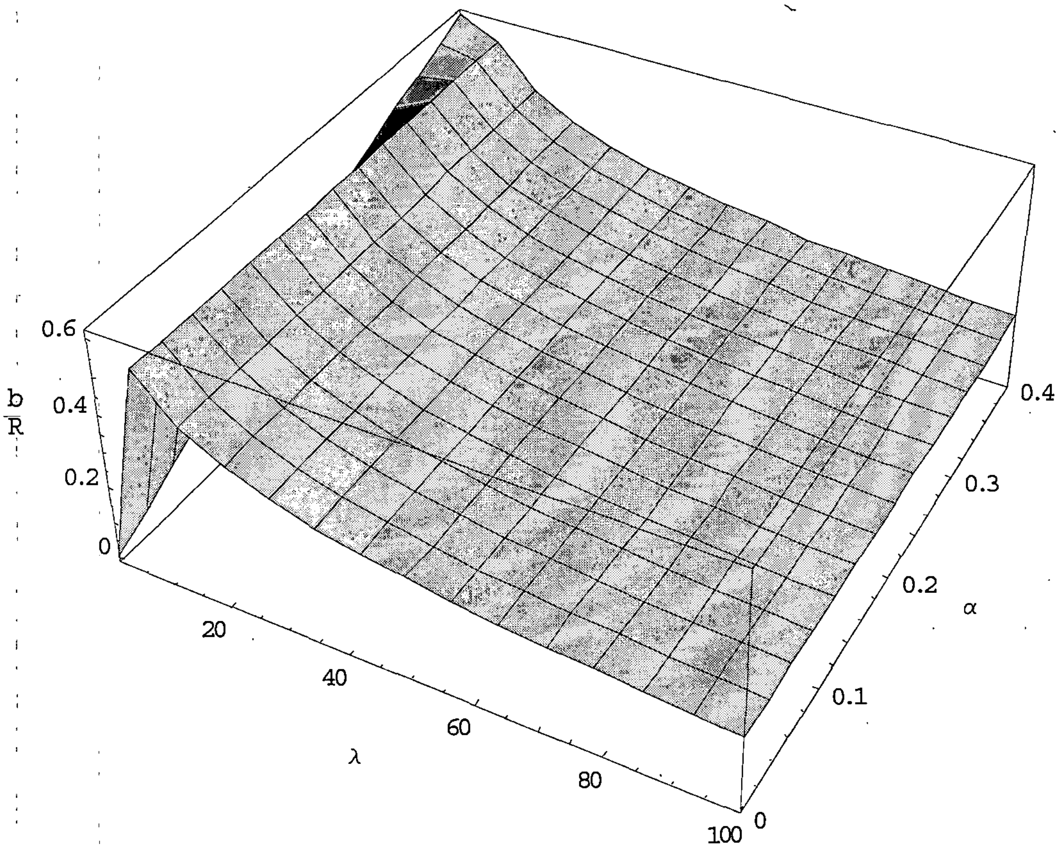

Figure 3.1: Upper bound on the value of $\frac{b}{R}$ plotted against λ & α.

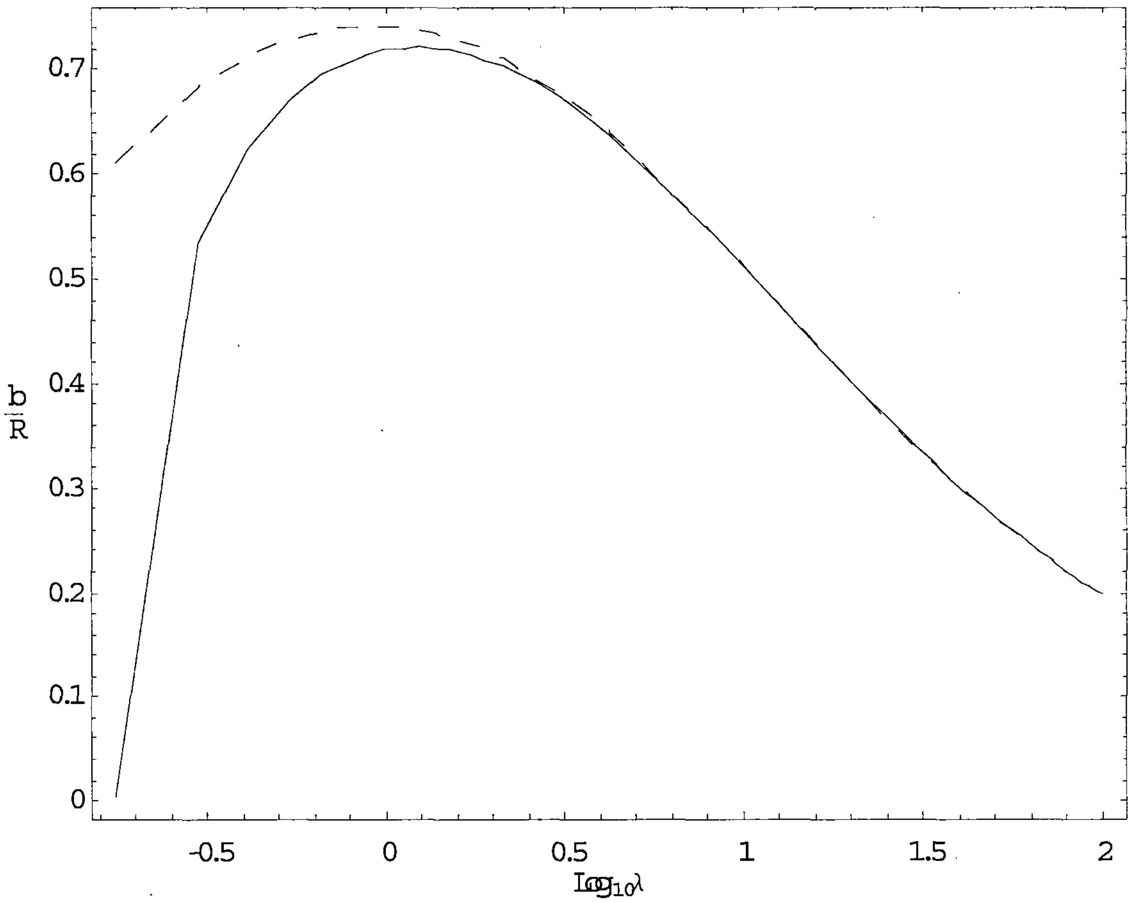

Figure 3.2: Upper bound on the value of $\frac{b}{R}$ plotted against $Log_{10}\lambda$ for $\alpha = 0$ (solid line) and $\alpha = 0.4$ (dashed line).