C H A P T E R III

MOVING SOURCE PROBLEMS.

Problem 1. Displacement due to a uniformly moving line load over the plane boundary of an inhomogeneous elastic half-space.

Problem 2. Rayleigh waves due to a nonuniformly propagating dip-slip fault.

DISPLACEMENT DUE TO A UNIFORMLY MOVING LINE LOAD OVER THE PLANE BOUNDARY OF AN INHOMOGENEOUS ELASTIC HALF-SPACE.

INTRODUCTION: Since the publication of the classical paper by LAMB (1904) the problem of line and point sources in homogeneous media has attracted the attention of many investigators. But the corresponding problems for inhomogeneous media have not been discussed by many authors as yet. The problem of wave propagation in an inhomogeneous medium is important to geophysicists, because any realistic model of the Earth must take into account the continuous change in the elastic properties of the material in the vertical direction. Since the mathematical treatment of a complicated model is extremely difficult and since the approximation to such a problem does not lead to any worth while solution, so some simplifying assumptions are usually made. WILSON (1942) studied the propagation of surface waves in a semi-infinite medium, assuming the density to be constant and the coefficient of rigidity to be varying exponentially with depth. STONELEY (1934), however, considered the transmission of RAYLEIGH waves in a heterogeneous medium in which the rigidity varies linearly with depth. The field due to a point source in an inhomogeneous isotropic medium in which density is constant but the bulk modulus varies with depth according to the law $\lambda = \lambda_0(1+\epsilon z)^2$ has been considered by SINGH (1967).

In the present paper, considering an elastic medium in which the elastic parameters λ, μ and density ρ vary according to the law $\lambda = \mu = \mu_0 (1 + \epsilon z)^2$ and $\rho = \rho_0(1 + \epsilon z)^2$, the transient problem

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for a two-dimensional line load moving with uniform velocity v over the surface of the non-homogeneous semi-infinite medium is studied. The ground motion excited by the moving surface load occurs, for example, from nuclear blasts and from shock waves generated by supersonic aircrafts. These practical problems have been formulated mathematically by a two-dimensional normal line load which is suddenly created at t =o and moves subsequently with uniform velocity along the free surface. The method of solutions involves the use of the integral transform and CAGNIARD's (1962) method as modified by DE HOOP (1959). The application of CAGNIARD's method in the solution of transient problems in inhomogeneous media does not seem to have been discussed earlier.

This steadily moving line load problem, where t varies from $-\infty$ to ∞, has been solved by CHAKRAVARTY and DE (1971) following the method of COLE AND HUTH (1958). Of course, the transient solution for a point load moving over the surface of a homogeneous isotropic half-space had been thoroughly discussed by GAKENHEIMER and MIKLOWITZ (1969). An exact solution of the buried uniformly moving line-source problem has also been obtained by MITRA (1959).

FORMULATION OF THE PROBLEM: The inhomogeneous semi-infinite medium is supposed to occupy the region $z > o$ as shown in Fig.1. The x-axis is taken along the free surface, whereas the z-axis points vertically downwards into the medium. A concentrated line load, which is assumed to originate on the free surface at the origin at time $t = o$, moves with uniform velocity $v(v < \alpha, \beta)$ along the positive direction of the x-axis.

The equations of motion for a non-homogeneous medium in the absence of body forces are

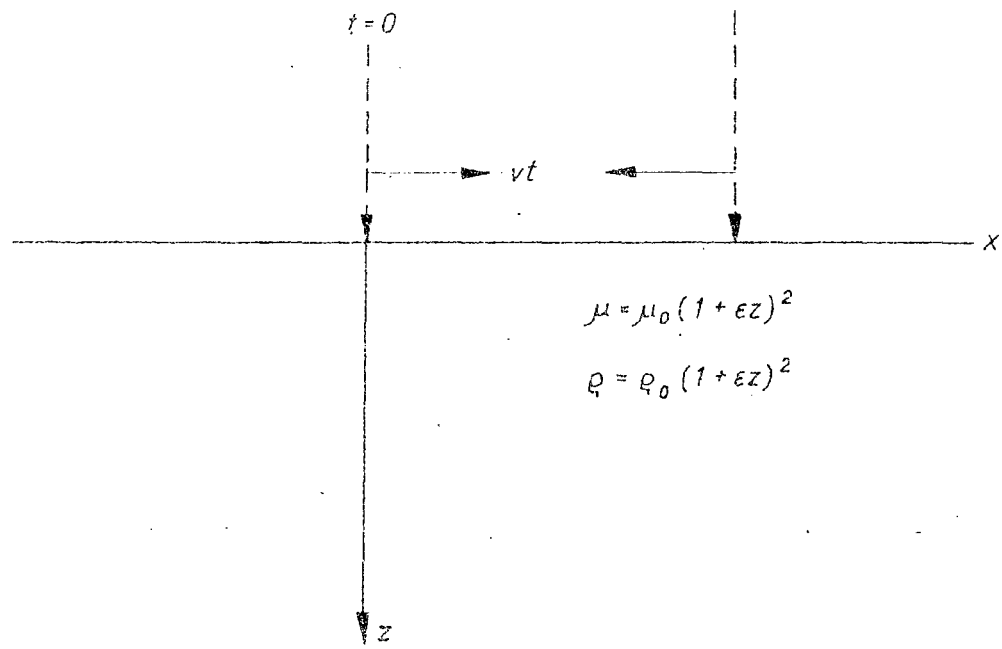
t = 0
vt
x
z
$\mu = \mu_0 (1 + \varepsilon z)^2$
$\varrho = \varrho_0 (1 + \varepsilon z)^2$

Fig.1

$$\frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]=\rho\frac{\partial^2 u}{\partial t^2}, \qquad (1)$$

$$\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}\right]=\rho\frac{\partial^2 w}{\partial t^2} \qquad (2)$$

u and w are the displacement components in the x- and z-directions, λ, μ are LAMÉ's constants and ρ is the density of the medium. It is assumed that

$$\lambda=\mu=\mu_0(1+\epsilon z)^2, \quad \rho=\rho_0(1+\epsilon z)^2, \qquad (3)$$

such that the velocity of propagation is independent of z. The equations (1) and (2) have to be solved subject to the boundary conditions

$$\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0 \qquad \text{at } z=0,$$

$$\tau_{zz}=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}=-P\delta(x-vt) \text{ at } z=0, t>0. \qquad (4)$$

$\delta(x-vt)$ is DIRAC's delta function.

FORMAL SOLUTION: In order to solve the equations (1) and (2), we make the substitution

$$U=u(1+\epsilon z) \text{ and } W=v(1+\epsilon z). \qquad (5)$$

This transforms the equations (1) and (2) into the forms

$$3\frac{\partial^2 U}{\partial x^2} + 2\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial z}\right) + \frac{\partial^2 U}{\partial z^2} = \frac{\rho_o}{\mu_o}\frac{\partial^2 U}{\partial t^2} \tag{6}$$

and

$$\frac{\partial^2 W}{\partial x^2} + 2\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial z}\right) + 3\frac{\partial^2 W}{\partial z^2} = \frac{\rho_o}{\mu_o}\frac{\partial^2 W}{\partial t^2}\,. \tag{7}$$

We introduce the FOURIER transform over x defined by

$$f_1(p,z,t) = \int_{-\infty}^{\infty} f(x,z,t)e^{ipx}\,dp$$

and then take the LAPLACE transform over t defined by

$$\bar{f}_1(p,z,s) = \int_0^{\infty} f_1(p,z,t)e^{-st}\,dt.$$

The equations (6) and (7) after these transformations take the forms

$$\left(\frac{d^2}{dz^2} - 3k_1^2\right)\bar{U}_1 = 2ip\frac{d\bar{W}_1}{dz} \tag{8}$$

and

$$\left(3\frac{d^2}{dz^2} - k_2^2\right)\bar{W}_1 = 2ip\frac{d\bar{U}_1}{dz}\,, \tag{9}$$

where

$$k_1^2 = p^2 + \frac{s^2}{\alpha^2}\,,\quad k_2^2 = p^2 + \frac{s^2}{\beta^2}\,,\quad \alpha^2 = 3\beta^2 = \frac{\lambda+2\mu}{\rho}$$

Using the conditions that the displacement components vanish as z approaches ∞, the solutions of (8) and (9) are

$$\bar{U}_1 = Ae^{-k_1 z} + Be^{-k_2 z} , \tag{10}$$

$$\bar{W}_1 = \frac{1}{ip}\left(Ak_1 e^{-k_1 z} + \frac{p^2 B}{k_2} e^{-k_2 z}\right). \tag{11}$$

Using (5), the above equations become

$$\bar{u}_1 = \frac{1}{1+\epsilon z}\left(Ae^{-k_1 z} + Be^{-k_2 z}\right) \tag{12}$$

and

$$\bar{w}_1 = \frac{1}{ip(1+\epsilon z)}\left(Ak_1 e^{-k_1 z} + \frac{p^2 B}{k_2} e^{-k_2 z}\right). \tag{13}$$

A and B have to be determined from the conditions

$$\frac{d\bar{u}_1}{dz} = ip\bar{w}_1 \text{ and } -ip/\mu_o\,\bar{u}_1 + 3/\mu_o \frac{d\bar{w}_1}{dz} = \frac{P}{ipv - s} \text{ on } z = o,$$

which are obtained by taking first the FOURIER and then the LAPLACE transform on both sides of equations (4). It is found that

$$A = \frac{ipP(\epsilon k_2 + k_2^2 + p^2)}{\mu_o(ipv-s)(k_1 k_2 - p^2)f(p)}, \quad B = -\frac{ipP(2k_1 k_2 + \epsilon k_2)}{\mu_o(ipv-s)(k_1 k_2 - p^2)f(p)},$$

where

$$f(p) = (p^2 - 3k_1 k_2) - 3\epsilon(k_1 + k_2) - 3\epsilon^2 .$$

Substituting the values of A and B in (12) and (13) and taking FOURIER inversion, we get

$$\bar{u} = \frac{ip}{2\pi/\mu_0(1+\epsilon z)} \int_{-\infty}^{\infty} \frac{p(\epsilon k_2 + k_2^2 + p^2)e^{-k_1 z} - pk_2(2k_1+\epsilon)e^{-k_2 z}}{(ipv - s)(k_1 k_2 - p^2)f(p)} e^{-ipx}\, dp \qquad (14)$$

and

$$\bar{w} = \frac{p}{2\pi/\mu_0(1+\epsilon z)} \int_{-\infty}^{\infty} \frac{k_1(\epsilon k_2 + k_2^2 + p^2)e^{-k_1 z} - p^2(2k_1+\epsilon)e^{-k_2 z}}{(ipv - s)(k_1 k_2 - p^2)f(p)} e^{-ipx}\, dp \qquad (15)$$

LAPLACE INVERSIONS: We assume

$$\bar{u} = \frac{p}{2\pi/\mu_0(1+\epsilon z)} (I_1 - I_2), \qquad (16)$$

where
$$I_1 = \int_{-\infty}^{\infty} \frac{ip(k_2 + k_2^2 + p^2)e^{-k_1 z - ipx}}{(ipv-s)(k_1 k_2 - p^2)f(p)} dp$$

and
$$I_2 = \int_{-\infty}^{\infty} \frac{ipk_2(2k_1+\epsilon)\, e^{-k_2 z - ipx}}{(ipv - s)(k_1 k_2 - p^2)f(p)} dp.$$

To find the inversions of I_1 and I_2, we adopt CAGNIARD's technique as modified by DEHOOP (1959). Accordingly, we put $p = -sh$ in I_1, which then reduces to the form

$$I_1 = \int_{-\infty}^{\infty} \frac{ih(\epsilon k_2' + sk_2'^2 + sh^2)e^{-s(k_1' z - ihx)}}{(ihv + 1)\phi(h)\,\psi(s,h)} dh, \qquad (17)$$

where

$$k_1'^2 = h^2 + \frac{1}{\alpha^2}, \quad k_2'^2 = h^2 + \frac{1}{\beta^2},$$

$$\phi(h) = (2k_1'k_2' - h^2)(h^2 - k_1'k_2') = -(h^4 - 4k_1'k_2' + 3k_1'k_2').$$

It has to be noted that $\phi(h)=0$ is the RAYLEIGH wave velocity equation corresponding to the homogeneous medium with $\lambda = \mu_o = \mu$ and

$$\psi(s,h)=s^2 + \frac{3s\epsilon(k_1' + k_2')}{3k_1'k_2' - h^2} + \frac{3\epsilon^2}{3k_1'k_2' - h^2} = (s-\epsilon m_1)(s-\epsilon m_2),$$

where $$m_{1,2} = \frac{-3(k_1' + k_2') \pm [9(k_1'-k_2')^2+12h^2]^{1/2}}{2(3k_1'k_2' - h^2)};$$

m_1 and m_2 are both negative. Breaking up $(\epsilon k_2' + sk_2'^2+sh^2)/\psi(s,h)$ into partial fractions, the equation (17) can be written as

$$I_1 = \int_{-\infty}^{\infty} \frac{ih(1-ihv)e^{-s(k_1'z-ihx)}}{(1 + h^2v^2)\phi(h)} \left(\frac{M}{s-\epsilon m_1} + \frac{N}{s-\epsilon m_2} \right) dh; \qquad (18)$$

similarly,

$$I_2 = \int_{-\infty}^{\infty} \frac{ih(1-ihv)e^{-s(k_2'z-ihx)}}{(1 + h^2v^2)\phi(h)} \left(\frac{S}{s-\epsilon m_1} + \frac{T}{s-\epsilon m_2} \right) dh. \qquad (19)$$

In (18) and (19),

$$M \equiv M(h) = \frac{k_2'+m_1(k_2'^2+h^2)}{m_1 - m_2}, \quad N \equiv N(h) = \frac{k_2'+m_2(k_2'^2+h^2)}{m_2 - m_1}$$

and

$$S \equiv S(h) = \frac{k_2'+2m_1k_1'k_2'}{m_1 - m_2}, \quad T \equiv T(h) = \frac{k_2'+2m_2k_1'k_2'}{m_2 - m_1}.$$

First let us consider the integral

$$\int_{-\infty}^{\infty} \frac{ih(1-ihv)e^{-s(k_1'z-ihx)}}{(1 + h^2v^2)\phi(h)} \frac{M}{s-\epsilon m_1} dh, \qquad (20)$$

which occurs in equation (13). In this integral the path of integration with respect to h, which is the real axis, is deformed in such a way that

$$z\left(h^2 + \frac{1}{\alpha^2}\right)^{1/2} - ihx = q,$$

where q is real and positive. The deformed path of integration is the branch Γ_1 (Fig.2) of a hyperbola, whose equation is

$$h = \frac{iqx \pm z\left(q^2 - \frac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2 + z^2}, \quad \frac{(x^2 + z^2)^{1/2}}{\alpha} < q < \infty.$$

In the course of deformation of the path of integration it is essential to know all the singularities of $M/[(1 + h^2v^2)\phi(h)]$ in the h-plane, which are the poles at $\pm (i/u)$, $\pm(i/v_R)$ and the branch points at $\pm(i/\alpha)$ and $\pm (i/\beta)$, where v_R is the RAYLEIGH WAVE velocity corresponding to the homogeneous medium when $\lambda = \mu = \mu_0$.
Since the hyperbolic path Γ_1 does not cross any of the singularities during its deformation, it is possible by virtue of CAUCHY's theorem and JORDAN'S lemma to replace the integration along the real h-axis by an integration along the hyperbolic path Γ_1. We write

$$h_+ = \frac{iqx+z\left(q^2 - \frac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2 + z^2}, \quad h_- = \frac{iqx-z\left(q^2 - \frac{x^2+z^2}{\alpha^2}\right)}{x^2 + z^2};$$

then

$$\frac{dh_\pm}{dq} = \frac{ix\left(q^2 - \frac{x^2+z^2}{\alpha^2}\right)^{1/2} \pm qz}{(x^2 + z^2)\left(q^2 - \frac{x^2 + z^2}{\alpha^2}\right)^{1/2}},$$ writing M_+ and M_- for $M(h_+)$ and $M(h_-)$

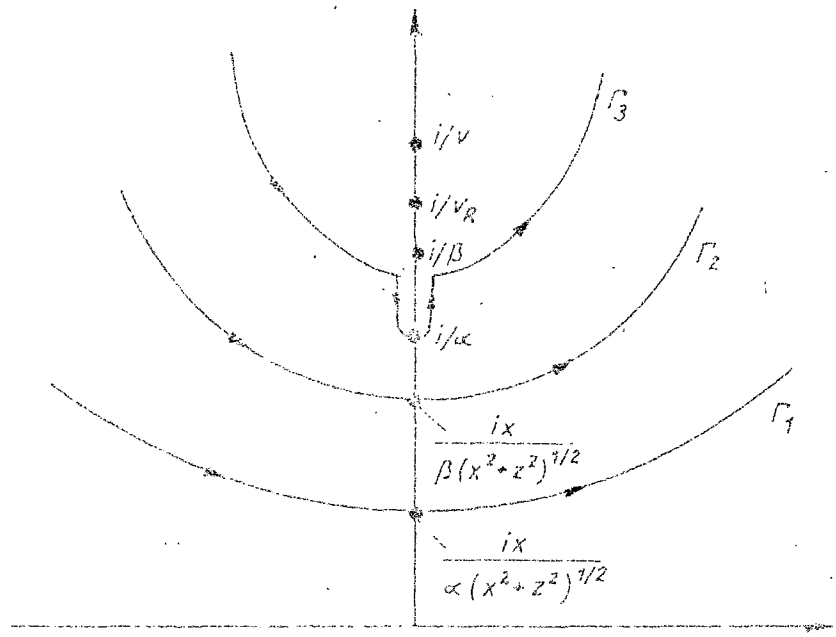
Γ_3
i/ν
i/ν_R
i/β
Γ_2
i/α
Γ_1
$\frac{ix}{\beta(x^2+z^2)^{1/2}}$
$\frac{ix}{\alpha(x^2+z^2)^{1/2}}$

Fig. 2

and using the facts that

$$h_- = -\bar{h}_+, \quad \frac{dh_-}{dq} = -\left(\frac{d\bar{h}_+}{dq}\right), \quad m_{1-} = \bar{m}_{1+}, \quad M_- = \bar{M}_+,$$

where $\bar{h}$ is the complex conjugate of h, the expression (20) takes the form

$$\int_{\frac{1}{a}\cdot(x^2+z^2)^{1/2}}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+ M_+}{(1+h_+^2 v^2)\phi(h_+)}\,\frac{e^{-sq}}{s-\epsilon m_{1+}}\,\frac{dh_+}{dq}\right] dq +$$

$$+\int_{\frac{1}{a}\cdot(x^2+z^2)^{1/2}}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2 M_+}{(1+h_+^2 v^2)\phi(h_+)}\,\frac{e^{-sq}}{s-\epsilon m_{1+}}\,\frac{dh_+}{dq}\right] dq.$$

Using the convolution theorem, the LAPLACE inversion of the above integral is

$$\int_0^t d\tau \int_{t_a}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+ M_+ e^{\epsilon(t-\tau)m_{1+}}}{(1+h_+^2 v^2)\phi(h_+)}\,\frac{dh_+}{dq}\right]\delta(\tau-q)\,dq +$$

$$+\int_0^t d\tau \int_{t_a}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2 M_+ e^{\epsilon(t-\tau)m_{1+}}}{(1+h_+^2 v^2)\phi(h_+)}\,\frac{dh_+}{dq}\right]\delta(\tau-q)\,dq,$$

where $t_a = (x^2 + z^2)^{1/2}/a$ is the arrival time of P-waves. By use of the properties of the δ-function, the above integrals can be written as

$$H(t-t_\alpha)\left[\int_{t_\alpha}^{t} -2\ \mathrm{Im}\left\{\frac{h_+ M_+ e^{\epsilon(t-\tau)m_{1+}}}{(1+h_+^2v^2)\phi(h_+)}\ \frac{dh_+}{d\tau}\right\}d\tau\ +\right.$$

$$\left.+\int_{t_\alpha}^{t} 2v\ \mathrm{Re}\left\{\frac{h_+^2 M_+ e^{\epsilon(t-\tau)m_{1+}}}{(1+h_+^2v^2)\phi(h_+)}\ \frac{dh_+}{d\tau}\right\}d\tau\right]. \qquad (21)$$

It should be noted that in the integrand of the above integral q has been replaced by every where.

In a similar manner the LAPLACE inversion of the other part of I_1 in (18) can be determined. It is found to be a similar expression as the expression in (21) except that M_+ and m_{1+} have to be replaced by N_+ and m_{2+} respectively. Thus the LAPLACE inversion of I_1 is

$$H(t-t_\alpha)\int_{t_\alpha}^{t} -2\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left\{M_+ e^{\epsilon(t-\tau)m_{1+}} + N_+ e^{\epsilon(t-\tau)m_{2+}}\ \frac{dh_+}{d\tau}\right\}\right]d\tau +$$

$$+ H(t-t_\alpha)\int_{t_\alpha}^{t} 2v\ \mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left\{M_+ e^{\epsilon(t-\tau)m_{1+}} + N_+ e^{\epsilon(t-\tau)m_{2+}}\ \frac{dh_+}{d\tau}\right\}\right]d\tau. \qquad (22)$$

Next we shall calulate the LAPLACE inversion of I_2 that occurs in (19). As before here also define

$$z\left(h^2 + \frac{1}{p^2}\right)^{1/2} - ihx = r\ ,$$

where r is real and positive.

So,

$$h_{\pm} = \frac{ixx \pm z\left(r^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2} \quad . \tag{23}$$

Case 1: A path along which r is real and non-negative is the hyperbolic path Γ_2 (Fig.2) represented parmetrically by the above equation with $r > (x^2+z^2)^{1/2}/\beta$, provided the path where it cuts the imaginary axis, viz. $h = ix/\beta(x^2+z^2)^{1/2}$, lies below the branch point i/α, which occurs when $x < \beta z/(\alpha^2-\beta^2)^{1/2}$. In this case the path Γ_2 (Fig.2) does not cross any of the singularities during the deformation. Following the same procedure as that done in case of I_1, the LAPLACE inversion of I_2 is found to be

$$H(t-t_\beta)\int_{t_\beta}^{t} -2\,\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left\{S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right\}\frac{dh_+}{d\tau}\right]d\tau +$$

$$+ H(t-t_\beta)\int_{t_\beta}^{t} 2v\,\mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left\{S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right\}\frac{dh_+}{d\tau}\right]d\tau, \tag{24}$$

where $t_\beta = (x^2+z^2)^{1/2}/\beta$ is the arrival time of S-waves, and h_+ occurring in the above expression is obtained by replacing r by in the expression for h_+ as given in (23).

Case 2: If $x > \beta z/(\alpha^2-\beta^2)^{1/2}$, the point $ix/\beta(x^2+z^2)^{1/2}$ lies above the branch point i/α. Therefore, the path of integration in the h-plane has to be deformed to the path Γ_3 (Fig.2) round the branch point i/α as shown in Fig.2.

We consider the integral

$$\int_{-\infty}^{\infty} \frac{ih(1-ihv)S\, e^{-s(k_2'z-ihx)}}{(1+h^2v^2)\phi(h)(s-\epsilon m_1)}\, dh \qquad (25)$$

occurring in I_2 of equation (19). Here too we put

$$z(h^2+\frac{1}{\beta^2})^{1/2} - ihx = r.$$

On the two finite straight line portions of the path Γ_3, h is given by

$$h_{\pm} = \pm\eta + \frac{i\left\{rx - z\left(\frac{x^2+z^2}{\beta^2}\right)^{1/2}\right\}}{x^2+z^2},$$

where finally η should be made to tend to zero, and on the remaining portions of the path of Γ_3,

$$h_{\pm} = \frac{irx \pm z\left(r^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}.$$

On the straight line portions of the path Γ_3, r varies from $r = t_{\alpha\beta}$ to $r = t_\beta$, where $t_{\alpha\beta} = x/\alpha + z(1/\beta^2 - 1/\alpha^2)^{1/2}$ is the arrival time of PS-waves. The expression in (25) can then be written in the form

$$\int_{t_{\alpha\beta}}^{t_\beta} \left[\frac{ih_+(1-ih_+v)S_+}{(1+h_+^2v^2)\phi(h_+)(s-\epsilon m_{1+})}\frac{dh_+}{dr} - \frac{ih_-(1-ih_-v)}{(1+h_-^2v^2)\phi(h_-)(s-\epsilon m_{1-})}\frac{dh_-}{dr}\right]e^{-sr}\,dr +$$

$$+\int_{t_\beta}^{\infty}\left[\frac{ih_+(1-ih_+v)S_+}{(1+h_+^2v^2)\phi(h_+(s-\epsilon m_{1+})}\frac{dh_+}{dr} - \frac{ih_-(1-ih_-v)}{(1+h_-^2v^2)\phi(h_-)(s-\epsilon m_{1-})}\frac{dh_-}{dr}\right]e^{-sr}\,dr. \qquad (26)$$

Noting that $h_- = -\bar{h}_+$, $dh_-/dr = -(d\bar{h}_+/dr)$ and $S_- = \bar{S}_+$ on the path Γ_3, the expression (26) takes the form

$$\int_{t_{\alpha\beta}}^{t_\beta} -2\ \mathrm{Im}\left[\frac{h_+S_+}{(1+h_+^2v^2)\phi(h_+)}\ \frac{e^{-sr}}{s-\epsilon m_{1+}}\ \frac{dh_+}{dr}\right]dr + \int_{t_{\alpha\beta}}^{t_\beta} 2v\,\mathrm{Re}\left[\frac{h_+^2S_+}{(1+h_+^2v^2)\phi(h_+)}\ \times\right.$$

$$\left.\times\ \frac{e^{-sr}}{s-\epsilon m_{1+}}\ \frac{dh_+}{dr}\right]dr + \int_{t_\beta}^{\infty} -2\ \mathrm{Im}\left[\frac{h_+S_+}{(1+h_+^2v^2)\phi(h_+)}\ \frac{e^{-sr}}{s-\epsilon m_{1+}}\ \frac{dh_+}{dr}\right]dr +$$

$$+ \int_{t_\beta}^{\infty} 2v\ \mathrm{Re}\left[\frac{h_+^2S_+}{(1+h_+^2v^2)\phi(h_+)}\ \frac{e^{-sr}}{s-\epsilon m_{1+}}\ \frac{dh_+}{dr}\right]dr. \qquad (27)$$

To transform the other integral of I_2 occurring in (19), a similar procedure is applied, and finally I_2 in (19) takes the following form:

$$I_2 = \int_{t_{\alpha\beta}}^{t_\beta} -2\ \mathrm{Im}\left[\frac{h_+e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]dr +$$

$$+ \int_{t_\beta}^{\infty} -2\ \mathrm{Im}\left[\frac{h_+e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]dr +$$

$$+ \int_{t_{\alpha\beta}}^{t_\beta} 2v\ \mathrm{Re}\left[\frac{h_+^2e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]dr +$$

$$+ \int_{t_\beta}^{\infty} 2v\ \mathrm{Re}\left[\frac{h_+^2e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]dr. \qquad (28)$$

It must be remembered that the value of h_+ when r lies in $[t_{\alpha\beta}, t_\beta]$ has to be taken as

$$h_+ = \frac{i\left[rx - z\left(\frac{x^2+z^2}{\beta^2} - r^2\right)^{1/2}\right]}{x^2 + z^2},$$

and for r lying in $[t_\beta, \infty)$,

$$h_+ = \frac{irx + z\left(r^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2 + z^2}.$$

Next the LAPLACE inversion of I_2 in (28) has to be calculated. By applying the convolution theorem, the LAPLACE inversion of the first integral of I_2 in (28) is found to be

$$\int_0^t d\tau \int_{t_{\alpha\beta}}^{t_\beta} -2\ \mathrm{Im}\left[\frac{h_+ e^{-s\tau}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]\delta(\tau - r)dr =$$

$$= \int_0^t d\tau \int_{t_{\alpha\beta}}^{\infty} -2\ \mathrm{Im}\left[\frac{h_+ e^{-s\tau}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]\delta(\tau - r)dr -$$

$$- \int_0^t d\tau \int_{t_\beta}^{\infty} -2\ \mathrm{Im}\left[\frac{h_+ e^{-s\tau}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]\delta(\tau - r)dr,$$

and it takes the following form when the δ-function property is used:

$$H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -2\ \mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right]d\tau -$$

$$-H(t-t_{\beta})\int_{t_\beta}^{t} -2\ \mathrm{Im}\left[\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right]d\tau. \qquad (29)$$

It can be shown that the last term of (29) is cancelled with the LAPLACE inversion of the second integral in (28).
Similarly, the LAPLACE inversion of the other integrals of (29) can be determined, and finally , after simplification, we get the LAPLACE inversion of I_2 as

$$H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -2\,\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_+}{d\tau}\right]d\tau -$$

$$-H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} 2v\,\mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_+}{d\tau}\right]d\tau . \qquad (30)$$

Combining the results of the inverse LAPLACE transforms of I_1 and I_2 from (22) and (24) it follows that

$$u(x,z,t) = \frac{P}{\pi/\mu_o(1+\epsilon z)}\,X$$

$$X\left[H(t-t_\alpha)\int_{t_\alpha}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}(M_+e^{\epsilon(t-\tau)m_{1+}}+N_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_+}{d\tau}\right\}d\tau +\right.$$

$$+H(t-t_\alpha)\int_{t_\alpha}^{t} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}(M_+e^{\epsilon(t-\tau)m_{1+}}+N_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_+}{d\tau}\right\}d\tau -$$

$$-H(t-t_\beta)\int_{t_\beta}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_+}{d\tau}\right\}d\tau -$$

$$\left.-H(t-t_\beta)\int_{t_\beta}^{t} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_+}{d\tau}\right\}d\tau\right] \qquad (31)$$

for $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, and when $x > \beta z/(\alpha^2 - \beta^2)^{1/2}$, from (22) and (30) it follows that

$$u(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)} \; X$$

$$X\Bigg[H(t-t_\alpha)\int_{t_\alpha}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-\tau)m_{1+}} + N_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+H(t-t_\alpha)\int_{t_\alpha}^{t} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-\tau)m_{1+}} + N_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau -$$

$$-H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}} + T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau -$$

$$-H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}} + T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\Bigg] . \tag{32}$$

Carrying on a similar procedure as done for the evaluation of the displacement along the x-direction, the expression for the displacement along the z-direction can also be determined from (13) and is found to be equal to

$$w(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)} \; X$$

$$X\Bigg[H(t-t_\alpha)\int_{t_\alpha}^{t} \mathrm{Re}\left\{\frac{k'_{1+}}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-\tau)m_{1+}} + N_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+ H(t-t_\alpha)\int_{t}^{t} v\,\mathrm{Im}\left\{\frac{k'_{1+}h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M e^{\epsilon(t-\tau)m_{1+}} + N e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+ H(t-t_\beta)\int_{t_\beta}^{t} \mathrm{Re}\left\{\frac{h_+^2}{k'_{2+}(1+h_+^2v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+ H(t-t_\beta)\int_{t_\beta}^{t} v\ \mathrm{Im}\left\{\frac{h_+^3}{k'_{2+}(1+h_+^2v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\Bigg] \qquad (33)$$

for $x < \beta z/(\alpha^2-\beta^2)^{1/2}$, and if $x > \beta z/(\alpha^2-\beta^2)^{1/2}$:

$$w(x,z,t) = \frac{p}{\pi/\mu_0(1+\epsilon z)} \; X$$

$$X\Bigg[H(t-t_\alpha)\int_{t_\alpha}^{t} \mathrm{Re}\left\{\frac{k'_{1+}}{(1+h_+^2v^2)\phi(h_+)}\left(M_+ e^{\epsilon(t-\tau)m_{1+}} + N_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+H(t-t_\alpha)\int_{t_\alpha}^{t} v\ \mathrm{Im}\left\{\frac{k'_{1+}h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M_+ e^{\epsilon(t-\tau)m_{1+}} + N_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} \mathrm{Re}\left\{\frac{h_+^2}{k'_{2+}(1+h_+^2v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} v\ \mathrm{Im}\left\{\frac{h_+^3}{k'_{2+}(1+h_+^2v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\Bigg] \qquad (34)$$

where $k'_{1+} = \left(h_+^2 + \frac{1}{\alpha^2}\right)^{1/2}$ and $k'_{2+} = \left(h_+^2 + \frac{1}{\beta^2}\right)^{1/2}$.

It should be remembered that in the first two integrals of the equations (31), (32), (33) and (34)

$$h_+ = \frac{i\rho x + z\left(\tau^2 - \frac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2+z^2}, \quad t_\alpha \leq \tau \leq t,$$

and in the last two integrals of those equations

$$h_+ = \frac{i\tau x + z\left(\tau^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}, \quad t_\beta \leq \tau \leq t,$$

where as in the last two integrals of (32) and (34)

$$h_+ = \frac{i\left[\tau x - z\left(\frac{x^2+z^2}{\beta^2} - \tau^2\right)^{1/2}\right]}{x^2+z^2}, \quad t_{\alpha\beta} \leq \tau \leq t_\beta.$$

WAVE FRONT EXPANSION: The wave forms of the solutions given in (31) to (34) are evaluated by approximate estimation of the above integrals in the neighbourhood of the time of the first arrival of the different waves. To facilitate this evaluation we put $\tau = A + a$, where A is the lower limit of the integrals in question and a varies from 0 to t-A. Then when $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, from (31) we get

$$u(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)} X$$

$$X\left[H(t-t_\alpha)\int_o^{t-t_\alpha} - \mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-t_\alpha-a)}m_{1+} + N_+e^{\epsilon(t-t_\alpha-a)}m_{2+}\right)\frac{dh_+}{da}\right\}da + H(t-t_\alpha)\int_o^{t-t_\alpha} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-t_\alpha a)}m_{1+} + N_+e^{\epsilon(t-t_\alpha-a)}\frac{dh_+}{da}\right\}da + H(t-t_\beta)\int_o^{t-t_\beta}\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-t_\beta-a)}m_{1+} + T_+e^{\epsilon(t-t_\beta-a)}m_{2+}\right)\frac{dh_+}{da}\right\}da + H(t-t_\beta)\int_o^{t-t_\beta} -v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-t_\beta-a)}m_{1+} + T_+e^{\epsilon(t-t_\beta-a)}m_{2+}\right)\frac{dh_+}{da}\right\}da\right]. \tag{35}$$

For $x > \beta z/(\alpha^2-\beta^2)^{1/2}$,

$$u(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)} X$$

$$X\left[H(t-t_\alpha)\int_o^{t-t_\alpha} - \mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-t_\alpha-a)}m_{1+} + N_+e^{\epsilon(t-t_\alpha-a)}m_{2+}\right)\frac{dh_+}{da}\right\}da + H(t-t_\alpha)\int_o^{t-t_\alpha} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-t_\alpha-a)}m_{1+} + N_+e^{\epsilon(t-t_\alpha-a)}m_{2+}\right)\frac{dh_+}{da}\right\}da +$$

$$+H(t-t_{\alpha\beta})\int_{o}^{t-t_{\alpha\beta}}\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-t_{\alpha\beta}-a)}m_{1+}+T_+e^{\epsilon(t-t_{\alpha\beta}-a)}m_{2+}\right)\frac{dh_+}{da}\right\}da +$$

$$+H(t-t_{\alpha\beta})\int_{o}^{t-t_{\alpha\beta}}-v\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-t_{\alpha\beta}-a)}m_{1+}+T_+e^{\epsilon(t-t_{\alpha\beta}-a)}m_{2+}\right)\frac{dh_+}{da}\right\}da\Bigg]. \tag{36}$$

A similar type of expressions for $w(x,z,t)$ can be written by substituting $\tau = A + a$ in the equations (33) and (34). For approximate evaluation of the integrals (35) and (36) just after the arrival of the corresponding wave fronts it has to be noted that

$$e^{\epsilon(t-A-a)}m_{1+} \to 1, \quad e^{\epsilon(t-A-a)}m_{2+} \to 1 \text{ and } a \to 0 \text{ as } t \to A,$$

where A is the arrival time of a typical wave front. So, when $A = t_\alpha$, using the facts that

$$h_+ \to \frac{ix}{\alpha(x^2+z^2)^{1/2}}, \quad h_+^2 \to \frac{-x^2}{\alpha^2(x^2+z^2)}$$

and

$$\frac{(M_+ + N_+)}{(1+h_+^2v^2)\phi(h_+)} \to$$

$$\to \frac{(x^2+z^2)^2\,\alpha^4\left\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\right\}}{\left\{x^2(\alpha^2-v^2)+\alpha^2z^2\right\}\left[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4\beta x^2z\left\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\right\}^{1/2}\right]}$$

as $a \to 0$ and that

$$\frac{dh_+}{da} = \frac{\left(h_+^2 + \frac{1}{a^2}\right)^{1/2}}{\left\{2\frac{(x^2+z^2)^{1/2}}{a} + a\right\}^{1/2}} \frac{1}{a^{1/2}} .$$

the first two integrals of the equations (35) and (36) just after the arrival of P-waves can approximately be evaluated to the form

$$u(x,z,t) = -\frac{\sqrt{2}\pi H(t-t_\alpha)xz\alpha^{3/2}(x^2+z^2)^{1/4}\left\{vx+\alpha(x^2+z^2)^{1/2}\right\}\left\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\right\}(t-t_\alpha)^{1/2}}{\pi\mu_0(1+\epsilon z)\, x^2(\alpha^2-v^2)+\alpha^2z^2\left[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4x^2z\beta\ X\right.}$$

$$\left. X\left\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\right\}^{1/2}\right].$$

Similarly, the approximate value of w just after the arrival of P-waves is given by

$$w(x,z,t) = \frac{\sqrt{2}\pi H(t-t_\alpha)z^2\alpha^{3/2}(x^2+z^2)^{1/4}\left\{vx+\alpha(x^2+z^2)^{1/2}\right\}\left\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\right\}(t-t_\alpha)^{1/2}}{\pi\mu_0(1+\epsilon z)\left\{x^2(\alpha^2-v^2)+\alpha^2z^2\right\}\left[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4x^2z\beta\ X\right.}$$

$$\left. X\left\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\right\}^{1/2}\right].$$

The same method is applied for approximate evaluation of u and w just after the arrival of S-waves. It should be remembered in this case that

$$\Lambda = t_\beta, \qquad h_+ = \frac{i\tau x+z\left(\tau^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}, \quad t_\beta < \tau < t.$$

The effects of u_1 and w_1 on the displacement components u and v due to S-waves just after their arrival are found to be

$$u_1(x,z,t)=$$

$$=\frac{2\sqrt{2PH(t-t_\beta)}xz^2\alpha\rho^{3/2}(x^2+z^2)^{1/4}\left\{vx+\beta(x^2+z^2)^{1/2}\right\}\left\{\beta^2z^2-x^2(\alpha^2-\beta^2)\right\}^{1/2}(t-t_\beta)^{1/2}}{\pi\mu_0(1+\epsilon z)\left\{x^2(\beta^2-v^2)+\beta^2z^2\right\}\left[\alpha^2x^2(3z^2-x^2)-3\beta^2z^2(x^2+z^2)-4x^2z\alpha\ X\right.}$$

$$\left.X\left\{\beta^2z^2-x^2(\alpha^2-\beta^2)\right\}^{1/2}\right],$$

$$w_1(x,z,t)=$$

$$=-\frac{2\sqrt{2PH(t-t_\beta)}x^2z\alpha\rho^{3/2}(x^2+z^2)^{1/4}\left\{vx+\beta(x^2+z^2)^{1/2}\right\}\left\{\beta^2z^2-x^2(\alpha^2-\beta^2)\right\}^{1/2}(t-t_\beta)^{1/2}}{\pi\mu_0(1+\epsilon z)\left\{x^2(\beta^2-v^2)+\beta^2z^2\right\}\left[\alpha^2x^2(3z^2-x^2)-3\beta^2z^2(x^2+z^2)-4x^2z\alpha\ X\right.}$$

$$\left.X\left\{\beta^2z^2-x^2(\alpha^2-\beta^2)\right\}^{1/2}\right]$$

for $0 < x < \beta z/(\alpha^2-\beta^2)^{1/2}$, and in order to obtain the displacement due to S-wave arrival in the region $x > \beta z/(\alpha^2-\beta^2)^{1/2}$, we write last two integrals of (32) in the form

$$\int_{t_{\alpha\beta}}^{t_\beta}\left[-\mathrm{Im}\left\{\frac{h_+}{(1+v^2h_+^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh}{d\tau}\right\}+\right.$$

$$\left.+v\mathrm{Re}\left\{\frac{h_+^2}{(1+v^2h_+^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}\right]d\tau+$$

$$+\int_{t_\beta}^{t}\left[-\mathrm{Im}\left\{\frac{h_+}{(1+v^2h_+^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}+\right.$$

$$\left.+v\mathrm{Re}\left\{\frac{h_+^2}{(1+v^2h_+^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}\right]d\tau. \quad (37)$$

This is to be remembered that for $t_\beta < \tau < t$

$$h_+ = \frac{1}{x^2 + z^2}\left[i\tau x + z\left(\tau^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}\right]$$

The displacement due to the first arrival of S-wave is obtained from the last integral of (37) as $t \to t_\beta + 0$ and this is found to be

$$u_1(x,z,t) = \frac{-8\sqrt{2}\ PH(t-t_\beta)\ \alpha^2\beta^{3/2}x^3z^3(x^2+z^2)^{1/4}}{\pi\mu_0(1+\epsilon z)\left\{x^2(\beta^2-v^2)+\beta^2z^2\right\}} \times$$

$$\times \frac{\left\{x^2(\alpha^2-\beta^2)-\beta^2z^2\right\}\left\{\beta(x^2+z^2)^{1/2}+vx\right\}(t-t_\beta)^{1/2}}{\left\{x^2\alpha^2(3z^2-x^2)-3z^2\beta^2(x^2+z^2)\right\}^2+16\alpha^2x^4z^2\left\{x^2(\alpha^2-\beta^2)-\beta^2z^2\right\}}$$

Adopting the similar procedure for the last two integrals of (34) we have, for $t \to t_\beta + 0$ in the region $x > \beta z/(\alpha^2-\beta^2)^{1/2}$

$$w_1(x,z,t) = \frac{8\sqrt{2}\ PH(t-t_\beta)\alpha^2\beta^{3/2}x^4z^2(x^2+z^2)^{1/4}}{\pi\mu_0(1+\epsilon z)\left\{x^2(\beta^2-v^2)+\beta^2z^2\right\}} \times$$

$$\times \frac{\left\{x^2(\alpha^2-\beta^2)-\beta^2z^2\right\}\left\{\beta(x^2+z^2)^{1/2}+vx\right\}(t-t_\beta)^{1/2}}{\left\{x^2\alpha^2(3z^2-x^2)-3z^2\beta^2(x^2+z^2)\right\}^2+16\alpha^2x^4z^2\left\{x^2(\alpha^2-\beta^2)-\beta^2z^2\right\}}$$

In the region $x > \beta z/(\alpha^2-\beta^2)^{1/2}$, PS-waves exist and arrive earlier than S-waves.

We approximately calculate the last two integrals of (36), which will give the effect of u_2 on the displacement components u due to PS-waves just after their arrival. In this case

$$A = t_{\alpha\beta},\ h_+ = \frac{i\left[\tau x - z\left(\frac{x^2+z^2}{\beta^2} - \tau^2\right)\right]}{x^2+z^2},\ t_{\alpha\beta} \leq \tau \leq t_\beta .$$

Then

$$h_+ \to i\left[\frac{1}{\alpha} + \frac{\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2}}{x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - \frac{z}{\alpha}}\, a\right],$$

$$\frac{S+T}{(1+h_+^2 v^2)\phi(h_+)} \to \frac{i\,2\sqrt{2}\,\alpha^6\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{3/4} a^{1/2}}{(\alpha^2 - v^2)\left\{\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z\right\}^{1/2}}$$

and

$$\frac{dh_+}{da} \to \frac{i\alpha\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2}}{\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z},$$

where in these expressions terms containing higher order of a are neglected because $a \to 0$ as $t \to t_{\alpha\beta}$, and we get

$$u_2(x,z,t) = \frac{4\sqrt{2}\rho H(t-t_{\alpha\beta})\alpha^5\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{5/4}}{3\pi\mu\sigma(1+z)(\alpha-v)\left[\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z\right]^{3/2}} (t-t_{\alpha\beta})^{3/2} .$$

Similarly, the effect on the displacement component w just after the arrival of PS-waves is given by

$$w_2(x,z,t) = -\frac{4\sqrt{2}\,PH(t-t_{\alpha\beta})\alpha^4\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)^{3/4}}{3\pi/\mu_0(1+z\epsilon)(\alpha-v)\left\{\alpha x\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)^{1/2}-z\right\}^{3/2}}(t-t_{\alpha\beta})^{3/2}$$

We now find out the effects of u_3 and w_3 on the displacement components u and w in the neighbourhood of the point C(Fig.3), where S- and PS-waves arrive at the same time. In this case $t_\beta = t_{\alpha\beta}$ and

$$u_3(x,z,t) = -\frac{4\,PH\,(t-t_{\alpha\beta})\,2^{3/4}\,\alpha^{1/4}\,\beta^{3/2}\,z^{5/2}}{3\pi/\mu_0(1+\epsilon z)\,x^{13/4}\,(\alpha-v)}(t-t_{\alpha\beta})^{3/4},$$

$$w_3(x,z,t) = \frac{4\,PH\,(t-t_{\alpha\beta})\,2^{3/4}\,\alpha^{1/4}\,\beta^{3/2}\,z^{3/2}}{3\pi/\mu_0(1+\epsilon z)\,x^{9/4}\,(\alpha-v)}(t-t_{\alpha\beta})^{3/4}.$$

CONCLUDING REMARKS: It is found from the integrals (31) to (34) that the effect of inhomogeneity enters into the expressions for u and v through the factors $e^{\epsilon(t-T)\,m_1}$ and $e^{\epsilon(t-T)\,m_2}$ in the corresponding integrands. So, if these two factors are absent, and that is so if $\epsilon = 0$, a parallel case for a homogeneous medium is obtained.

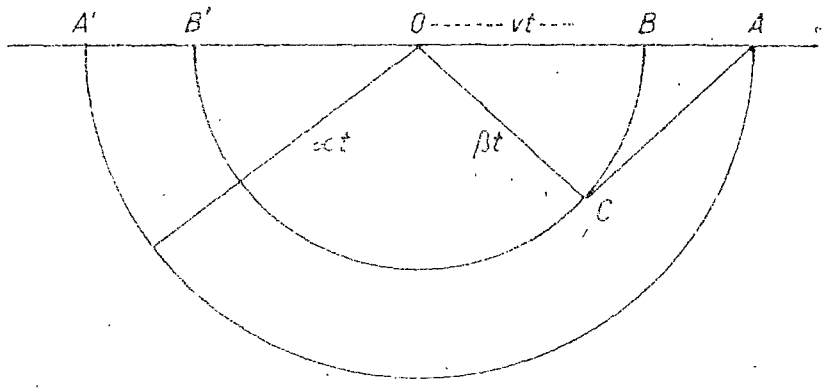

Fig.3

Also, it is interesting to note that in the neighbourhood of points just after the arrival of the different wave fronts the displacement components are independent of ϵ ,i.e., at any point, the effect of the first arrival of wave fronts on the displacement components is the same for homogeneous as well as for inhomogeneous media. But as time goes on, ϵ occurring in the exponential terms of the integrals (31) to (34) for u and w will have its effect, and consequently, the amplitude of the wave fronts will decay exponentially with time due to inhomogeneity of the medium.

RAYLEIGH WAVES DUE TO NONUNIFORMLY PROPAGATING DIP-SLIP FAULT

INTRODUCTION: The study of dynamic crack propagation is very important in geophysics and in earthquake engineering science. In geophysics it is desirable to formulate the earthquake source in terms of physical parameters and to study the long period waves over a large distance and for a long time. Also in structural engineering it is essential to know the nature of surface waves covering a large distance. At aparticular place the ground motion produced by the earthquake is a very complicated function of the nature of propagation of the crack and the geological properties of the place as well. Most of the known solutions of the moving crack are restricted by the assumption of constant velocity of propagation, which is not in general expected. Mal (1972) discussed Rayleigh wave propagation by a finite fault moving with constant velocity. He represented the shear failure by a jump in the tangential components of displacement across the fault surface. Achenbach and Abo-Zeno (1972) analysed the wave motions generated by a vertical strike slip fault on which motion is opposed by a frictional shear stress and which is assumed to increase linearly with depth. Freund (1973) discussed wave motions as expected in case of a nonuniformly expanding line load. Fossum and Freund (1975) considered a model in which a plane strain shear crack moves from rest at a nonuniform rate under the action of general loading. First motion response of an elastic half space due to a

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nonuniformly moving dislocation by Cagniard De-Hoop technique is determined by Roy (1973). In a recent paper Markenscoff and Clifton (1981) analyzed the motion of an edge dislocation starting from rest and moving thereafter nonuniformly on its slip plane by means of Laplace transform, where the inversion of the transform is accomplished by Cagniard De-Hoop method.

In the present paper an idealised earthquake model is considered. A fault break along a horizontal line at a finite depth below the free surface is assumed to appear suddenly and to move vertically upward with nonuniform motion upto the free surface. A discontinuity in components of displacement across the fault break is prescribed. The displacement components on the free surface due to Rayleigh waves are determined for nonuniform motion of the crack.

To find the solution of the problem the technique developed by Knopoff and Gilbert with appropriate modification is used. The technique is found to be extrememely powerful for tackling such type of boundary value problems. Ghosh (1972) applied the method to show the possibility of attenuation of microseismic waves due to the presence of an upward folding of the ocean bottom into the liquid. Following Knopoff and Gilbert, the moving crack is replaced by a set of virtual sources located at the fault surface HO. The displacement on the free surface is written as the sum of the contribution of these sources with the aid of suitable Green's function representation theorem.

Three particular cases of nonuniform motion of the crack are considered. Horizontal and vertical components of surface displacements due to Rayleigh waves produced by the propagating crack are determined and shown by means of graphs.

In the mathematical and physical structure of wave propagation phenomenon, the model assumed here is although over simplified, yet it brings forth some major features which are usually present in the ground motion.

FORMULATION OF THE PROBLEM AND SOLUTION: The origin of the coordinate frame (x,y) is at the epicentre o. It is assumed that a crack suddenly appearing at the focus H moves vertically upwards upto the free surface o with a nonuniform speed. The length of the crack measured from H at any time t is h(t), which is assumed to be strictly monotonic increasing function of time t.

The Fourier transform $\bar{f}(x,y,w)$ of the function f(x,y,t) is defined by

$$\bar{f}(x,y,w) = \int_{-\infty}^{\infty} f(x,y,t)\, e^{iwt}\, dt \qquad (1)$$

Let $G_n^m(x,y|x_o,y_o)$, (m,n = (x,y)) be the component of Green's function $\bar{G}^m(x,y|x_o,y_o)$ at the point (x,y) in the direction of n due to a point source of force in m-direction and situated at (x_o,y_o). If now u(x,y) and v(x,y) be the displacement components along x and y directions respectively and $P_{xx}^{(u,v)}$, $P_{xy}^{(u,v)}$ and $P_{yy}^{(u,v)}$ be the stress components then their Fourier transforms defined by (1) satisfy the following differential equations.

$$\frac{\partial \bar{P}_{xx}^{(u,v)}}{\partial x} + \frac{\partial \bar{P}_{xy}^{(u,v)}}{\partial y} + \rho w^2 \bar{u}(x,y) = 0 \qquad (2)$$

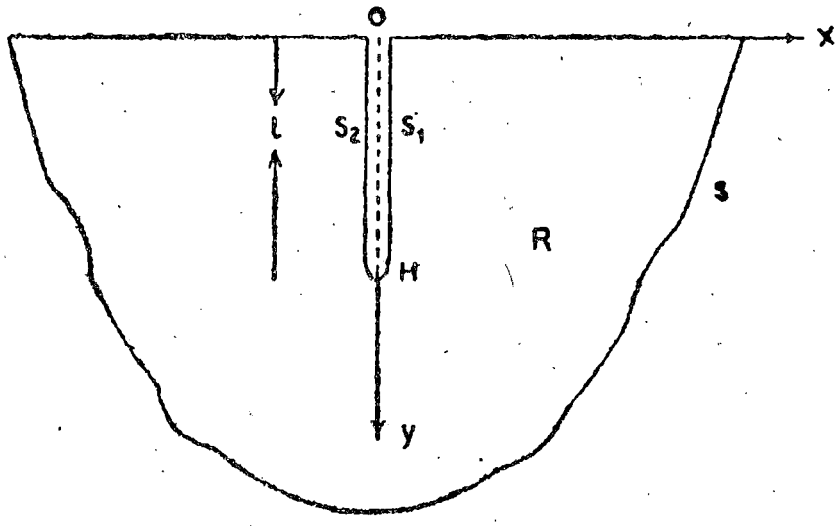


FIG. 1. Geometry of dip-slip fault.

$$\frac{\partial \bar{P}_{xy}(u,v)}{\partial x} + \frac{\partial \bar{P}_{yy}(u,v)}{\partial y} + \rho w^2 \bar{v}(x,y) = 0 \qquad (3)$$

$$\frac{\partial \bar{P}_{xx}}{\partial x}\left[\bar{G}^x(x,y|x_o,y_o)\right] + \frac{\partial \bar{P}_{xy}}{\partial y}\left[\bar{G}^x(x,y|x_o,y_o)\right] + \rho w^2 \bar{G}^x_x(x,y|x_o,y_o) =$$

$$= -\delta(x-x_o)\,\delta(y-y_o) \qquad (4)$$

$$\frac{\partial \bar{P}_{xy}}{\partial x}\left[\bar{G}^x(x,y|x_o,y_o)\right] + \frac{\partial \bar{P}_{yy}}{\partial y}\left[\bar{G}^x(x,y|x_o,y_o)\right] + \rho w^2 \bar{G}^x_y(x,y|x_o,y_o)=0 \qquad (5)$$

$$\frac{\partial \bar{P}_{xx}}{\partial x}\left[\bar{G}^y(x,y|x_o,y_o)\right] + \frac{\partial \bar{P}_{xy}}{\partial y}\left[\bar{G}^y(x,y|x_o,y_o)\right] + 2\,\bar{G}^y_x(x,y|x_o,y_o)=0 \qquad (6)$$

$$\frac{\partial \bar{P}_{xy}}{\partial x}\left[\bar{G}^y(x,y|x_o,y_o)\right] + \frac{\partial \bar{P}_{yy}}{\partial y}\left[\bar{G}^y(x,y|x_o,y_o)\right] + \rho w^2 \bar{G}^y_y(x,y|x_o,y_o) =$$

$$= -\delta(x-x_o)\,\delta(y-y_o) \qquad (7)$$

where ρ is the density of the material and $\delta(\ )$ is Dirac's delta function.

Multiply equation (2) by $\bar{G}^{x}_{x}(x,y|x_o,y_o)$ and (4) by $\bar{u}(x,y)$ and subtract the latter from the former. Also multiply equation (3) by $\bar{G}^{x}_{y}(x,y|x_o,y_o)$ and (5) by $\bar{v}(x,y)$ and subtract the latter from the former. These two resulting equations are then added and integrated over the region R to yield the following equation.

$$\iint_R \Big\{ \frac{\partial}{\partial x}\Big[\bar{G}^{x}_{x}(x,y|x_o,y_o)\,\bar{P}_{xx}(u,v) + \bar{G}^{x}_{y}(x,y|x_o,y_o)\,\bar{P}_{xy}(u,v) - \bar{u}\,\bar{P}_{xx}[\bar{G}^{x}(x,y|x_o,y_o)] - \bar{v}\,\bar{P}_{xy}[\bar{G}^{x}_{y}(x,y|x_o,y_o)]\Big] + \frac{\partial}{\partial y}\Big[\bar{G}^{x}_{x}(x,y|x_o,y_o)\,\bar{P}_{xy}(u,v) + \bar{G}^{x}_{y}(x,y|x_o,y_o)\,\bar{P}_{yy}(u,v) - \bar{u}\,\bar{P}_{xy}[\bar{G}^{x}(x,y|x_o,y_o)] - \bar{v}\,\bar{P}_{yy}[\bar{G}^{x}(x,y|x_o,y_o)]\Big] \Big\} dR = \bar{u}(x_o,y_o) \tag{8}$$

For details of the analysis to obtain the equation (8), we refer to the paper of Ghosh (1972).

Applying Green's theorem, the integral in (8) over the region R is converted to an integral over the curves S, S_1, S_2 (shown in Fig.1.) bounding the region R and We have

$$\int_{S+S_1+S_2} \bar{G}^{x}_{x}(x,y|x_o,y_o)\,\bar{P}_{nx}(u,v) + \bar{G}^{x}_{y}(x,y|x_o,y_o)\,\bar{P}_{ny}(u,v) - \bar{u}\,\bar{P}_{nx}[\bar{G}^{x}(x,y|x_o,y_o)] - \bar{v}\,\bar{P}_{ny}[\bar{G}^{x}(x,y|x_o,y_o)]\,ds = \bar{u}(x_o,y_o), \tag{9}$$

n is the direction of the outward normal at ds.

Since the stresses due to (u,v) are zero on the free surfaces S, S_1, S_2 and the stresses due to Green's function are also zero on the free surface S, so we obtain from (9)

$$\int_0^1 \left\{ [\bar{u}]\, \bar{P}_{xx}\, [\vec{G}^x(o,y|x_o,y_o)] + [\bar{v}]\, \bar{P}_{xy}\, [\vec{G}^x(o,y|x_o,y_o)] \right\} dy =$$

$$= \bar{u}(x_o,y_o) \qquad (10)$$

Where $[u]$ and $[v]$ represent the jump discontinuity in displacement components across the crack HO and HO = ℓ is the length of the crack. Since we are considering a dip-slip fault, so there is no displacement discontinuity along x-direction across the fault surface. Consequently $[u] = o$. Also, as we are interested in surface displacement only, so the equation (10) reduces to the form

$$\int_0^1 [\bar{v}]\, \bar{P}_{xy}\, [\vec{G}^x(o,y|x_o,o)]\, dy = \bar{u}\,(x_o,\, o) \qquad (11)$$

Considering the equations (2), (3), (6) and (7) and following the same procedure, we get

$$\int_0^1 [\bar{v}]\, \bar{P}_{xy} [\vec{G}^y(o,y|x_o,o)]\, dy = \bar{v}\,(x_o,o) \qquad (12)$$

The Fourier transforms of the Green's functions are

$$\bar{G}^x_x(x,y|x_o,o) = \frac{1}{2\pi/\mu}\int_{-\infty}^{\infty} \frac{\nu_2\, e^{-i(x-x_o)}}{R(\xi)} \left[2\xi^2 e^{-\nu_1 y} - (2\xi^2 - k_2^2) e^{-\nu_2 y}\right] d\xi,$$

$$\bar{G}^x_y(x,y|x_o,o) = \frac{-i}{2\pi/\mu}\int_{-\infty}^{\infty} \frac{\xi\, e^{-i\xi(x-x_o)}}{R(\xi)} \left[2\nu_1\ \nu_2\, e^{-\nu_1 y} (2\xi^2 - k_2^2) e^{-\nu_2 y}\right] d\xi, \quad (13)$$

$$\bar{G}^y_x(x,y|x_o,o) = \frac{-i}{2\pi/\mu}\int_{-\infty}^{\infty} \frac{\xi\, e^{-i\xi(x-x_o)}}{R(\xi)} \left[(2\xi^2 - k_2^2) e^{-\nu_1 y} - 2\nu_1\ \nu_2\, e^{-\nu_2 y}\right] d\xi,$$

$$\bar{G}^y_y(x,y|x_o,o) = \frac{-1}{2\pi/\mu}\int_{-\infty}^{\infty} \frac{\nu_1\, e^{-i\xi(x-x_o)}}{R(\xi)} \left[(2\xi^2 - k_2^2) e^{-\nu_1 y} - 2\xi^2 e^{-\nu_2 y}\right] d\xi.$$

Here $k_1^2 = w^2/\alpha^2$, $k_2^2 = w^2/\beta^2$, $\nu_1 = \sqrt{\xi^2 - k_1^2}$, $\nu_2 = \sqrt{\xi^2 - k_2^2}$ and

$R(\xi) = 4\xi^2\ \nu_1\ \nu_2 - (\xi^2 + \nu_2^2)^2$; α, β are respectively P-wave and S-wave velocities. The values of $\nu_1(\xi)$ and $\nu_2(\xi)$ are to be so chosen that with such values the expression for the displacement decay exponentially as $y \to \infty$ for real $\nu_1(\xi)$ and $\nu_2(\xi)$.

The Fourier transform of the stress $P_{xy}\left[\bar{G}^x(o,y|x_o,o)\right]$ on the line of faulting HO is given by

$$\bar{P}_{xy}\left[\bar{G}^x(o,y|x_o,o)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi x_o}}{R(\xi)} \left[(2\xi^2 - k_2^2) e^{-\nu_2 y} - 4\xi^2 \nu_1\ \nu_2\, e^{-\nu_1 y}\right] d\xi. \quad (14)$$

Since we want to determine the surface displacement due to Rayleigh waves, we need to determine the value of the integral in (14) for Rayleigh pole contribution only, for which we refer to Mal and Knopoff (1968). The Rayleigh pole contribution to the integral (14) is evaluated by following the method prescribed by Lapwood (1949) and $\bar{P}_{xy}[\vec{G}^x(o,y|x_o,o)]$ is found to be

$$iwA_1\left\{\exp\left(-\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2}y\right)-\exp\left(-\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2}y\right)\right\}\exp\left(i\frac{wx_o}{c_R}\right),\ \text{for } w>o \tag{15}$$

$$\text{and } iwA_1\left\{\exp\left(\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2}y\right)-\exp\left(\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2}y\right)\right\}\exp\left(i\frac{wx_o}{c_R}\right),\ \text{for } w<o$$

$$\text{where } A_1=\frac{1}{4c_R}\,\frac{\sqrt{(\alpha^2-c_R^2)(\beta^2-c_R^2)}}{2\frac{\alpha}{\beta}(2\beta^2-c_R^2)-\beta^2\left(\frac{\alpha^2-c_R^2}{\beta^2-c_R^2}\right)^{1/2}-\alpha^2\left(\frac{\beta^2-c_R^2}{\alpha^2-c_R^2}\right)^{1/2}-2\sqrt{(\alpha^2-c_R^2)(\beta^2-c_R^2)}} \tag{16}$$

and c_R is the Rayleigh wave velocity.

Similarly, the contribution from the Rayleigh pole to $\bar{P}_{xy}[\vec{G}^y(o,y|x_o,o)]$ is found to be

$$B_1w\left\{\exp\left(-\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2}y\right)-\exp\left(-\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2}y\right)\right\}\exp\left(i\frac{wx_o}{c_R}\right),\ \text{for } w>o \tag{17}$$

$$\text{and } -B_1w\left\{\exp\left(\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2}y\right)-\exp\left(\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2}y\right)\right\}\exp\left(i\frac{wx_o}{c_R}\right),\ \text{for } w<o$$

where

$$B_1 = \frac{1}{2\beta c_R} \frac{(2\beta^2 - c_R^2)\sqrt{\alpha^2 - c_R^2}}{\frac{2\alpha}{\beta}(2\beta^2 - c_R^2) - \beta^2\left(\frac{\alpha^2 - c_R^2}{\beta^2 - c_R^2}\right)^{1/2} - \alpha^2\left(\frac{\beta^2 - c_R^2}{\alpha^2 - c_R^2}\right)^{1/2} - 2\sqrt{(\alpha^2 - c_R^2)(\beta^2 - c_R^2)}} \tag{18}$$

The discontinuity in displacement along the line of faulting at any time t and at a depth y below the free surface is assumed to be

$$[\, v \,] = DH[\, h(t) - (1 - y) \,]\ [\, H(y) - H\,(y - 1) \,]$$

$$= DH[\, t - r(y) \,]\ [\, H(y) - H\,(y - 1) \,] \tag{19}$$

H( ) is Heaviside step function and $r(y) = h^{-1}(1-y)$, which is the inverse function of h and it exists as h(t) is strictly monotonic increasing function. Fourier transform of the equation (19) is given by

$$[\, \bar{v} \,] = D[\, H(y) - H(y - 1) \,] \int_{-\infty}^{\infty} H[\, t - r(y) \,]\ e^{iwt}\, dt$$

$$= D[\, H(y) - H(y - 1) \,] \int_{\bar{r}(y)}^{\infty} e^{iwt}\, dt = D[\, H(y) - H$$

$$(y - 1) \,]\left(\pi\delta(w) + \frac{i}{w}\right) e^{iwr(y)} \tag{20}$$

Putting the value of $[\, \bar{v} \,]$ from (20) in (11) and then taking Fourier inversion of (11) and changing the order of integration one obtains

$$u(x_o,o)= \frac{D}{2\pi} \int_0^1 dy \int_{-\infty}^{\infty} e^{iw(r(y)-t)} \left(\pi\,\delta(w)+\frac{1}{w}\right) \bar{P}_{xy} \left[\, \vec{G}^x(o,y|x_o,o)\, \right] dw.$$

Substituting the value of $\bar{P}_{xy}$ from (15) in the above equation we get

$$u(x_o,o)= -\frac{A_1 D}{\pi} \int_0^1 dy \int_0^{\infty} \left\{ \left[\exp\left(-\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2}\, y\right) - \right.\right.$$

$$\left.\left. - \exp\left(-\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2}\, y\right)\right] \cos\left(r(y)-t+\frac{x_o}{c_R}\right) w \right\} dw$$

or

$$\frac{u(x_o,o)}{D} = -\frac{A_1 c_R}{\pi} \int_0^1 \left\{ \frac{\alpha\sqrt{\alpha^2-c_R^2}\, y}{(\alpha^2-c_R^2)\, y^2 + \alpha^2\,(c_R\, r(y) - t\, c_R + x_o)^2} - \right.$$

$$\left. - \frac{\beta\sqrt{\beta^2-c_R^2}\, y}{(\beta^2-c_R^2)\, y^2 + \beta^2\,(c_R\, r(y) - t\, c_R + x_o)^2} \right\} dy \qquad (21)$$

Similarly, we obtain

$$\frac{v(x_o,o)}{D} = -\frac{B_1 c_R}{\pi} \int_0^1 \left\{ \frac{\alpha^2 (c_R\, r(y) - t\, c_R + x_o)}{(\alpha^2-c_R^2)\, y^2 + \alpha^2 (c_R\, r(y) - t c_R + x_o)^2} - \right.$$

$$\left. - \frac{\beta^2 (c_R\, r(y) - t\, c_R + x_o)}{(\beta^2-c_R^2)\, y^2 + \beta^2 (c_R\, r(y) - t\, c_R + x_o)^2} \right\} dy \qquad (22)$$

DIFFERENT CASES OF NONUNIFORM CRACK SPEED: In this section we determine the Rayleigh wave displacement on the free surface

for different nonuniform motions of the vertical crack.

Case 1. Here it is assumed that $h(t) = ct$, where c is the constant velocity of propagation of the fault and we have

$$t = \frac{1-y}{c} = r(y) \ . \tag{23}$$

Substituting the value of $r(y)$ from (23) in (21) and (22) and integrating the resulting equation one gets

$$\frac{u(x_0,0)}{D} = -\frac{AG}{4\pi P}\frac{c^2}{c_R^2}\left[\frac{A}{B}\left\{\frac{1}{2}\ln\frac{A^2+X^2}{T^2} + \frac{c_R}{cA}\tan^{-1}\frac{A}{X}\right\} - \frac{G}{H}\left\{\frac{1}{2}\ln\frac{G^2+X^2}{T^2} + \frac{c_R}{cG}\tan^{-1}\frac{G}{X}\right\}\right] \tag{24}$$

$$\frac{v(x_0,0)}{D} = -\frac{A(1+G^2)}{2\pi P}\frac{c}{c_R}\left[\frac{1}{B}\left\{\frac{cA}{c_R}\tan^{-1}\frac{A}{X} - \frac{1}{2}\ln\frac{A^2+X^2}{T^2}\right\} - \frac{1}{H}\left\{\frac{cG}{c_R}\tan^{-1}\frac{G}{X} - \frac{1}{2}\ln\frac{G^2+X^2}{T^2}\right\}\right] \tag{25}$$

where $A = \sqrt{1 - \frac{c_R^2}{\beta^2}\frac{\beta^2}{\alpha^2}}$, $B = 1 + \frac{c^2}{c_R^2}\left(1 - \frac{c_R^2}{\beta^2}\cdot\frac{\beta^2}{\alpha^2}\right)$, $G = \sqrt{1 - \frac{c_R^2}{\beta^2}}$,

$H = 1 + \frac{c^2}{c_R^2}\left(1 - \frac{c_R^2}{\beta^2}\right)$, $X = \frac{x_0}{l} - \frac{t\,c_R}{l}$, $T = \frac{c_R}{c} - \frac{t\,c_R}{l} + \frac{x_0}{l}$ and

$P = 2\left(1 + G^2\right) - \frac{A}{G} - \frac{G}{A} - 2AG.$

$T = 0$ implies $t = \frac{x_o}{c_R} + \frac{1}{c}$ which is the time taken to reach the point (x_o, o) by Rayleigh wave, which is emitted from the epicentre O when the crack reaches the free surface and $X = 0$ implies $t = x_o / c_R$ which is the time taken to reach the point (x_o, o) by the Rayleigh wave generated at H as soon as the crack appears at H.

Case 2. In this case it is assumed that the crack starts to move vertically upward with a finite velocity a and has a retardation b. Here at a time t after the formation of the crack

$$h(t) \; (\leq l) = at - \frac{1}{2} bt^2, \text{ so that}$$

$$r(y) = \frac{2}{a} \frac{l(1 - z)}{1 + \sqrt{1 - \frac{2bl}{a^2}(1-z)}} \qquad (26)$$

Substituting the value of r(y) in (21) and (22) we obtain

$$\frac{u(x_o, o)}{D} = -\frac{AG}{\pi P} \int_o^1 \left[\frac{Az}{Az^2 + \left\{ \frac{c_R}{a} \frac{2(1-z)}{1+\sqrt{1-2F(1-z)}} + X \right\}^2} - \right.$$

$$\left. - \frac{Gz}{G^2 z^2 + \left\{ \frac{c_R}{a} \frac{2(1-z)}{1+\sqrt{1-2F(1-z)}} + X \right\}^2} \right] dz , \qquad (27)$$

$$\frac{v(x_o, o)}{D} = -\frac{A}{\pi P} \frac{c_R}{\beta^2} \left(1 - \frac{\beta^2}{\alpha^2}\right) \left(1 - \frac{c_R^2}{2\beta^2}\right) \times$$

$$\times \int_0^1 \frac{(2\frac{c_R}{a}\frac{1-z}{1+\sqrt{(1-2F(1-z))}} + X)z^2\,dz}{\left\{A^2z^2+(2\frac{c_R}{a}\frac{1-z}{1+\sqrt{1-2F(1-z)}}+X)^2\right\}\left\{G^2z^2+(2\frac{c_R}{a}\frac{1-z}{1+\sqrt{1-2F(1-z)}}+X)^2\right\}} \qquad (28)$$

where $F = \frac{bl}{a^2}$ and the other constants have the same values mentioned earlier. It may be noted that the integrands in equation (27) and (28) have a singularity at $z = o$ provided

$$2\frac{c_R}{a}\frac{1}{1+\sqrt{1-2F}} + X = 0 \text{ , which implies that}$$

$$t = \frac{x_o}{c_R} + \frac{2l}{a(1+\sqrt{1-\frac{2bl}{a^2}})} \text{ .}$$

This is the time to reach the point (x_o, o) by the Rayleigh wave emitted from the epicentre O just after the arrival of the crack at this point.

Case 3. Finally let the crack at a depth l below the free surface, start to move vertically upward with infinitely large velocity which gradually decays with time. Accordingly h(t) is taken in the form

$$h(t) = D_1 \sqrt{t} \text{ where } D_1 \text{ is a constant.}$$

$$\text{or} \quad D_1\sqrt{t} = l - y$$

$$\text{Therefore} \quad r(y) = [\,(l - y) / D_1\,]^2 \qquad (29)$$

As before substituting the value of r(y) in (21) and (22) and making a change of variable of the integration , we have

$$\frac{u(x_o,o)}{D} = -\frac{K^2 A G}{4\pi P}\int_o^1 \left[\frac{A}{z^4-4z^3+(K^2A^2+2M+4)^2\, z^2-4Mz+M^2} - \right.$$

$$\left. - \frac{G}{z^4-4z^3+(K^2G^2+2M+4)^2\, z^2-4Mz+M^2}\right] z\,dz, \qquad (30)$$

$$\frac{v(x_o,o)}{D} = \frac{K^3 A c_R^2}{\pi P \beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\left(1-\frac{c_R^2}{2\beta^2}\right) \times$$

$$\times \int_o^1 \frac{z^2(z^2-2z+M)\,dz}{\left\{K^2A^2z^2+(z^2-2z+M)^2\right\}\left\{K^2G^2z^2+(z^2-2z+M)^2\right\}} \qquad (31)$$

where $K^2 = D_1^2 / lc_R$, $M = 1-(tc_R/l)(D_1^2/lc_R)+(x_o/l)(D_1^2/lc_R)$ and the other constants have same values as mentioned before. Again, the integrands in equation (30) and (31) are singular at $z = o$ if $M = 0$. This corresponds to $t = (x_o/c_R)+(l^2/D_1^2)$ which is the time of arrival at (x_o,o) of the Rayleigh wave which is generated at o just after the arrival of the crack on the free surface.

NUMERICAL RESUMTS AND CONCLUSION: When the earth material is under tension or compression in a direction parallel to the free surface,

shear failure occurs on a fault plane. In general, this failure moves with nonuniform speed. Numerical computations are carried out for poisson solid ($\alpha/\beta = \sqrt{3}$) and for $c_R/\beta = .9194$. The quantities A,B,G,H,X,T,F,K,M,F defined in section 3 are all dimensionless.

Figures 2,3 and 4 show the variation of componants of displacement with time. The dimensionless displacement components are plotted against dimensionless time. T_H and T_o indicated in the figures correspond to the arrival times at (x_o,o) of the Rayleigh waves from the focus H and the fault break at o.

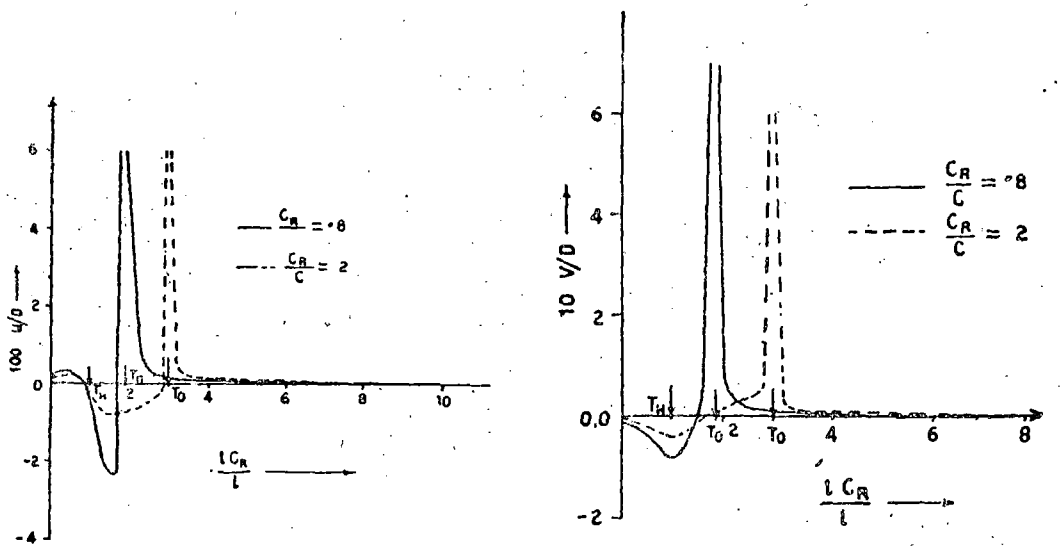
Figs.2(a-b) correspond to the case 1 of section 3 where the constant velocity of propagation of the crack ($c_R/c = .8$ and 2) is assumed.

Figs.3(a-b) correspond to the case where the crack starts with a finite velocity aand has a retardation b. Here also two cases $c_R/a = .8$ and 2 with the assumption that $F = 1/3$, are considered.

Figs.4(a-b) depicts the case 3 where the initial velocity of crack propagation is assumed to be infinitely large and K is taken to be equal to 1.

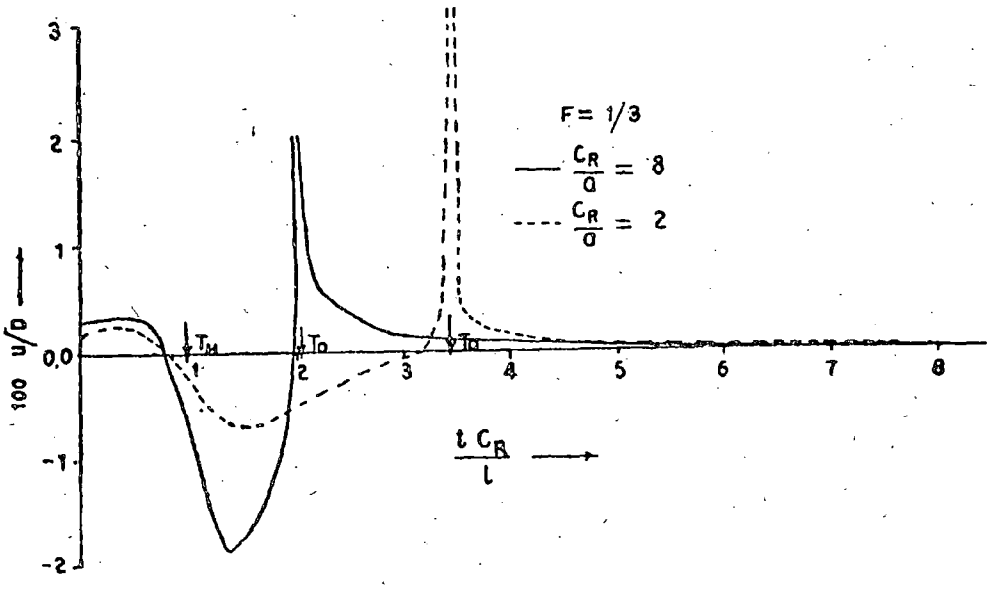
From equations (21) and (22) it may be noted that $u(x_o,o)/D$ and $v(x_o,o)/D$ are functions $(x_o/l)-(c_Rt/l)$. Therefore in all computational works, without any loss of generality x_o/l has been taken to be equal to 1, because any change in value of x_o/l will merely cause a shifting of the graphs along the direction of c_Rt/l.

We find that in each case the strongest ground motion occure at $T_o = c_Rt/l$ which correspond to the arrival time of Rayleigh waves from the surface break at O. Also it is found that though the nature of the graphs in three different cases differ between T_H and T_o but their natures are almost the same after the arrival of

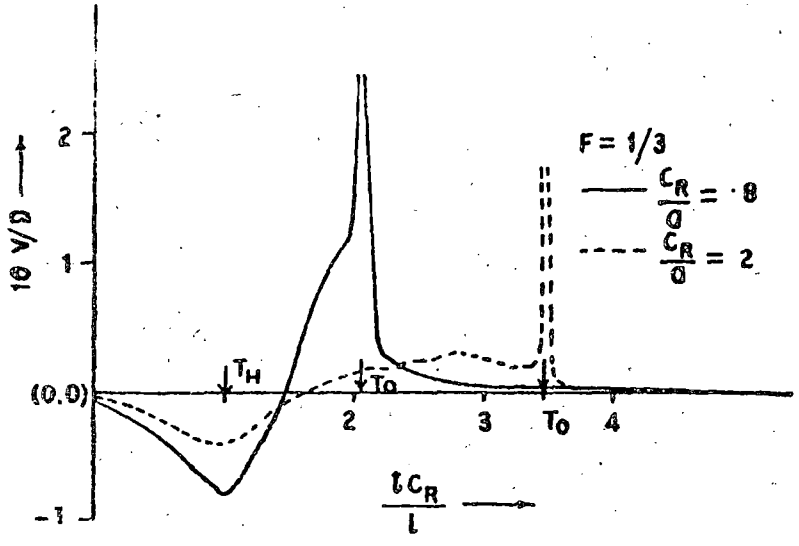


(a) (b)

FIGS. 2(a, b). Horizontal and vertical components of displacement versus time. Case of crack propagation with uniform velocity.

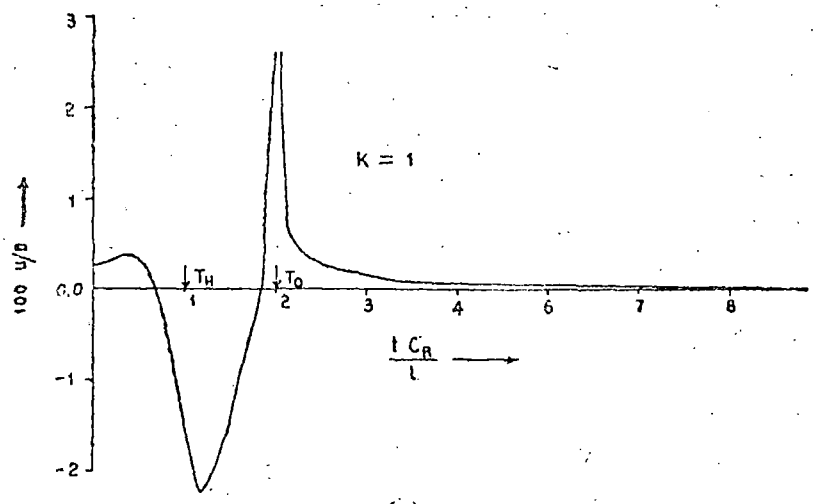


(a)

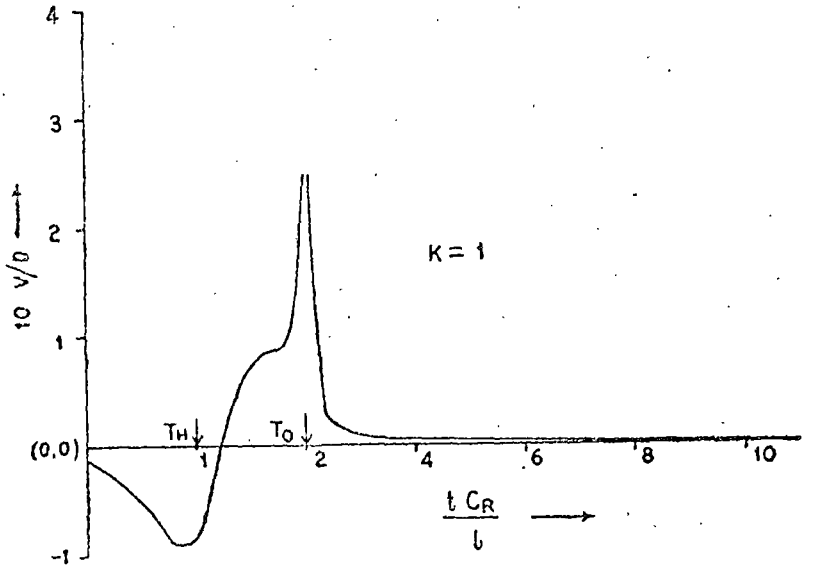


(b)

FIGS. 3(a, b). Horizontal and Vertical components of displacement versus time. Case of crack propagation with retardation and finite starting velocity.



(a)



(b)

FIGS. 4(a, b). Horizontal and Vertical components of displacement versus time. Case of crack propagation with retardation and infinite starting velocity.

Rayleigh waves from the surface break. This may be explained from the fact that the main contribution to the ground motion due to Rayleigh wave is from a small portion of the fault near the surface after the arrival of Rayleigh wave from the surface break. So the contribution from the details of crack initiation becomes insignificant after T_o.

It may be mentioned in this connection that though contribution of Rayleigh wave to the ground motion is significant at large distances from the epicentre, the effect of body waves near the epicentral region can not be ignored. This effect may be incorporated if we consider in addition to the contribution from Rayleigh pole, the contribution from the branch line integrals arising from the evaluation of stresses due to Green's function.