CHAPTER - 1

INTRODUCTION

1.0 The equation of radiative transfer.

The basic equation of radiative transfer which governs the radiation field in a medium which absorbs, emits and scatters radiation is given by

$$-\frac{1}{k\rho}\frac{dI}{ds} = I - \mathfrak{S} \tag{1.1}$$

where I is the specific intensity, k is the absorption coefficient of the medium, ρ , the density of the medium, s, the thickness of the element of the mass considered and $\mathfrak{S}$ is called the source function which is the ratio of emission coefficient to absorption coefficient. The equation (1.1) is an integro - differential equation.

For problems in semi-infinite plane parallel medium with a constant net flux, the equation of transfer (1.1) becomes

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' \tag{1.2}$$

where

$\mu = \cos(\theta)$, θ being the angle made by the pencil of incident radiation with the outward drawn normal from an element of area $d\sigma$

$$\tau = \int_{z}^{\infty} k\rho dz = \text{Optical depth}$$

and $$p(\mu, \mu') = \frac{1}{2\pi}\int_{0}^{2\pi} p(\mu,\phi;\mu',\phi')d\phi' ,$$ ϕ being the azimuthal angle.

Here, $p(\mu\mu')$ is the phase function which governs the directional distribution of intensity. For Milne problem the equation of transfer is (Chandrasekhar (1960))

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')d\tau' \tag{1.3}$$

In equation (1.3) the intensity $I(\tau,\mu)$ is an unknown function of τ and μ. Approximate solutions of this equation can be obtained by means of Milne - Eddington approximation and by means of expansion of intensity $I(\tau,\mu)$ in a series of Legendre polynomials. On analyzing, these methods yield some difficulties which are due to the fact that there is trouble when the discontinuous function $I(\tau,\mu)$ is represented by a sum of simple functions. According to Kourganoff (1963) we are led to

Either (I) give up an analytic representation and restrict ourselves to a set of discrete ordinates;

Or (II) eliminate these discontinuous functions by transforming the integro-differential equation into an ordinary integral equation.

Or (III) introduce Fourier integrals (or Laplace transforms) which are well suited for use with discontinuous functions.

1.1 The Spherical Harmonic Method. (SHM)

The essential idea of the method which is due to Eddington is that we expand $I(\tau,\mu)$ in series of Legendre polynomials $P_j(\mu)$ and seek a solution of the equation of transfer. The Legendre polynomials $P_j(\mu)$ form a complete set of orthogonal functions in the interval (-1,1) which is just that through which μ varies. Accordingly we write,

$$I(\tau,\mu) = A_0(\tau)P_0(\mu) + A_1(\tau)P_1(\mu) + \ldots\ldots + A_m(\tau)P_m(\mu) \tag{1.1.1}$$

where the series is broken up after finite number of terms and the solution of the equation of the transfer is reduced to the determination of the functions $A_j(\tau)$. In grey case the mean intensity and the source functions are defined respectively,

$$\bar{I}(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)d\mu$$

$$\overline{\Im}(\tau) = 2\int_{-1}^{1} I(\tau,\mu)\mu d\mu = F$$

where F is the net integrated flux. In terms of $P_j(\mu)$, the above representation can be written as,

$$\bar{I}(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)P_0(\mu)d\mu \tag{1.1.2a}$$

$$\overline{\Im}(\tau) = 2\int_{-1}^{1} I(\tau,\mu)\mu P_1(\mu)d\mu = F \tag{1.1.2b}$$

Next, we use the orthogonality property of $P_j(\mu)$ which is given by

$$\int_{-1}^{1} P_m(\mu)P_n(\mu)d\mu = \begin{cases} 0, & if\ m \neq n \\ \dfrac{2}{2n+1}, & if\ m = n \end{cases} \tag{1.1.2c}$$

Therefore,

$$\bar{I}(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)P_0(\mu)d\mu$$

$$= \int_{-1}^{1} A_0(\tau)P_0^2(\mu)d\mu + \sum_{l=1}^{m}\int_{-1}^{1} A_1(\tau)P_l(\mu)P_0(\mu)d\mu = A_0(\mu) \tag{1.1.3}$$

and

$$\overline{\Im}(\tau) = 2\int_{-1}^{1} I(\tau,\mu)\mu P_1(\mu)d\mu =$$

$$= \int_{-1}^{1} A_0(\tau)P_0(\mu)P_1(\mu)d\mu + \int_{-1}^{1} A_1(\tau)P_1^2(\mu)d\mu + \sum_{l=2}^{m}\int_{-1}^{1} A_1(\tau)P_l(\mu)P_0(\mu)d\mu =$$

$$= \frac{2A_1(\tau)}{3} \tag{1.1.4}$$

We know that in Grey case the conservation of flux integral gives
$B = \bar{I}$, where B is Plank's function.

The equation of transfer for integrated radiation now takes the form

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - A_0(\tau) \tag{1.1.5}$$

We use recurrence formula for Legendre polynomials

$$(j+1)P_{j+1}(\mu) - (2j+1)\mu P_j(\mu) + jP_{j-1}(\mu) = 0 \tag{1.1.6}$$

in equation (1.1.5) we obtain

$$\sum_{l=0}^{m} A_l'(\tau) P_l(\mu) = \sum_{l=1}^{m} A_l(\tau) P_l(\mu) \tag{1.1.7}$$

If we compare the coefficients of $P_j(\mu)$ in (1.1.7) we find that

$$A_1'(\tau) = 0 \tag{1.1.8a}$$

$$A_0'(\tau) + \frac{2}{5}P_2'(\tau) = A_1'(\tau) \tag{1.1.8b}$$

$$A_2'(\tau) + \frac{4}{9}P_4'(\tau) = A_3'(\tau) \tag{1.1.8c}$$

So in general we have for j = 2,3, ...

$$\frac{j}{2j-1}\frac{dA_{j-1}(\tau)}{d\tau} + \frac{j+1}{2j+3}\frac{dA_{j+1}(\tau)}{d\tau} = A_j(\tau) \tag{1.1.8d}$$

For the simple case of j = 2 we obtain

$$A_1(\tau) = \frac{3}{4}F = C_1 \tag{1.1.9a}$$

$$A_0(\tau) = \frac{3}{4}F\tau - \frac{2}{5}A_2(\tau) + C_2 \tag{1.1.9b}$$

where C_1, C_2 are constants of integration. Extending for j = 2n in equation (1.1.7) we find that

$$\begin{aligned} DA_1 &= 0 \\ -A_2 + \frac{3}{7}DA_3 &= 0 \\ -A_3 + \frac{3}{5}A_2 + \frac{4}{9}DA_4 &= 0 \\ &\text{-----------------------} \\ &\text{-----------------------} \\ \frac{j}{2j-1}DA_{j-1} - A_j + \frac{j}{2j+3}DA_{j+1} &= 0 \end{aligned} \tag{1.1.10}$$

and

$$\frac{2n}{4n-1}DA_{2n-1} - A_{2n} = 0$$

From the first and last, together with those given by j = 4,6,...., 2n-2 $DA_3, DA_5, \ldots\ldots, DA_{2n-1}$ can be eliminated and another first integral is obtained. Therefore, we get a linear relation between $A_2, A_4, \ldots\ldots, A_{2n}$ with constants coefficients. The resolvent equation for any function A_m (m= 2,3,....2n) is of order 2n-2. So we get an equation of the form

$$F(D)A_m = 0 \tag{1.1.11}$$

where

$$F(D) = \begin{vmatrix} -1 & \frac{3}{7}D & 0 & \dots & 0 & 0 \\ \frac{3}{5}D & -1 & \frac{4}{9}D & \dots & 0 & 0 \\ 0 & \frac{4}{7}D & -1 & \dots & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & -1 & \frac{2n}{4n-1}D \\ 0 & 0 & 0 & .. & \frac{2n}{4n-1}D & -1 \end{vmatrix}$$

It contains only derivatives of even order and its characteristic equation $F(\lambda) = 0$ has all its roots real and of modulus greater than unity. Let the roots of $F(D) = 0$ be $\pm\alpha_j$, j = 2,3,....,n. where each $\alpha_j > 1$, then the solution will be given by,

$$A_j = \sum_{l=\pm 2}^{n} C_l e^{\mp \alpha_l \tau} \qquad (1.1.12)$$

where $C_{\pm j}$ (j = 2,3,...n) are 2n-2 constants of integration with C_0, C_1 are obtained from (1.1.9). The functions $A_2(\tau), A_3(\tau), \dots, A_{m-1}(\tau)$ will depend linearly on the same exponential and same constants of integration. We have to determine the constants $C_{\pm j}$ by means of the boundary conditions which are given below.

(ii) Net flux is constant, i.e., $\mathfrak{F}(\tau) = F =$ constant (1.1.13a)

(ii) The convergence of intensity as $\tau \to \infty$, i.e.,

$$I(\tau,\mu)e^{-\tau} \to 0 \quad as \quad \tau \to \infty \qquad (1.1.13b)$$

(iii) Absence of incident radiation from outside on the free surface , i.e.,

$$I(0,\mu) \equiv 0 \quad for\ \mu < 0 \tag{1.1.13c}$$

Now from (1.1.13a), the constancy of net flux gives $C_1 = \frac{3}{4}F$ and from (1.1.13b) we have

$$C_{-j} = 0, \text{ for } j = 2,3,.....n.$$

The remaining constants are to be determined from (1.1.13c) which implies that

$$I_{-}(0,\mu) \equiv 0 \text{ i.e.}$$

$$I_{-}(0,\mu) = \sum_{j=0}^{2n} A_j(0)P_j(\mu) \tag{1.1.14}$$

Equation (1.1.14) must be satisfied for all values of μ lying between - 1 and 0. Thus, we have a system of infinite number of linear homogeneous equations, and from this system of equation we have to determine a finite number (n) of unknowns $C_0, C_2, ..., C_n$. In equation (1.1.14) $A_j(0)$ are (2n+1) functions and each of these functions except A_1, is expressed as a linear combination of the above n constants. The same is true for $A_j(0)$. For, if we set $\tau = 0$ we have

$$A_1(0) = \frac{3}{4}F$$

$$A_0(0) = -\frac{2}{5}A_2(0) + C_2$$

$$A_2(0) = -\frac{20}{27}A_4(0) + C_3$$

etc.

Therefore we conclude that the system (1.1.14) is incompatible. We have no alternative but to choose arbitrarily n equations corresponding to n arbitrary values of μ (between - and 0) to determine the n constants of integration $C_0, C_2, ..., C_n$.

The equation (1.1.14) is satisfied for all values of μ in (0,1). This means that we are trying to determine n constants from an infinite set of linear equations. Hence, certain arbitrariness in the determination of the constants cannot be avoided. Use of various equivalent boundary conditions is an attempt to by pass it. For example Mark [1947] met it by choosing some strategic values of μ for which the condition (1.1.14) holds good.

Kourganoff [1963] tried to reduce this arbitrariness by using the least square method but even then this arbitrariness cannot be removed completely. He imposed a minimum condition on $I(0\,\mu)$ and suggested that

$$\sigma = \int_0^1 [I(0,\mu)]^2 d\mu = \text{Minimum} \qquad (1.1.15)$$

This is equivalent to $I(0,\mu) \equiv 0$ for $-1 < \mu < 0$. From (1.1.14) and (1.1.15) we have

$$\sigma = \int_{-1}^{0} \Big[\sum_{j=0}^{2n} A_j(0) P_j(\mu) \Big]^2 d\mu = \text{Minimum} \qquad (1.1.16)$$

Differentiating σ partially w.r.t. $A_j(0)$ and using the orthogonal property of $P_j(\mu)$ we deduce that

$$\frac{2}{2i+1} A_i(0) = \sum_{j=0}^{2n} A_j(0) \int_0^1 P_j(\mu) P_i(\mu) d\mu\,, \quad i = 0,1,2,\ldots,2n \qquad (1.1.17)$$

Equation (1.1.17) states that there are now (2n+1) relations involving n unknowns. Therefore, we still have the arbitrariness and incompatibility. Thus, the arbitrariness in the determination of constants is minimized but not removed. Kourganoff [1963] traced this source of the defect to the fact that the function $I(\tau,\mu)$ which is discontinuous at the free surface at $\mu = 0$ was represented by a finite number of continuous terms. He suggested that the situation would improve if double interval representation of specific intensity is tried. His suggestion, was in fact, made by Yvon [footnote Kourganoff, p 301] and elaborately demonstrated by Mertens [1954]

1.2. A particular problem

We consider the equation of radiative transfer for a plane parallel, grey medium with azimuthal symmetry,

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - \frac{\omega}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' \tag{1.2.1}$$

where the phase function is assumed to be represented in the following form

$$p(\mu,\mu') = \sum_{n=0}^{\infty}(2n+1)f_n P_n(\mu)P_n(\mu'),\quad f_0 = 1 \tag{1.2.2}$$

We further assume that the intensity is represented in the following form,

$$I(\tau,\mu) = \sum_{m=0}^{\infty}\frac{2m+1}{4\pi}P_m(\mu)I_m(\tau) \tag{1.2.3}$$

Our problem is to determine $I_m(\tau)$ which are functions of single variable τ. Using (1.2.2) and (1.2.3) in (1.2.1) we obtain

$$\frac{1}{4\pi}\sum_{m=0}^{\infty}(2m+1)P_m(\mu)\left[\mu\frac{dI_m(\tau)}{d\tau} - I_m(\tau)\right]$$

$$= \frac{\omega}{2} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(2n+1)f_n P_n(\mu)\frac{2m+1}{4\pi}I_m(\tau)\int_{-1}^{1}P_n(\mu')P_m(\mu')d\mu' \tag{1.2.4}$$

If we apply the orthogonal property of the Legendre polynomial (Eq. 1.2c) and the recurrence formula (1.1.6) in the last equation we obtain

$$\sum_{m=0}^{\infty}\left[mP_m(\mu)\frac{dI_m(\tau)}{\tau} + (m+1)P_{m+1}(\mu)\frac{dI_m(\tau)}{d\tau}\right] -$$

$$- \sum_{m=0}^{\infty}(2m+1)P_m(\mu)I_m(\tau)\left[1+\omega f_m\right] = 0 \qquad (1.2.5)$$

On further reduction of (1.2.5) we find that

$$\sum_{m=0}^{\infty}\left[(m+1)\frac{dI_{m+1}(\tau)}{d\tau} + m\frac{dI_{m-1}(\tau)}{d\tau} - (2m+1)I_m(\tau)(1+\omega f_m)\right]P_m = 0 \qquad (1.2.6)$$

If (1.2.6) is valid for all μ, then the coefficients of $P_m(\mu)$ must vanish identically, i.e.

$$(m+1)\frac{dI_{m+1}(\tau)}{d\tau} + m\frac{dI_{m-1}(\tau)}{d\tau} - (2m+1)I_m(\tau)(1+\omega f_m) = 0 \qquad (1.2.7)$$

For isotropic scattering we have,

$$f_0 = 1, f_m = 0 \quad \text{for all m} > 1$$

Equations (1.2.7) are an infinite set of coupled ordinary differential equations for the function $I_m(\tau)$. In practice, however, only a finite number of equations m = N is considered and the term $I_{N+1}'(\tau)$ is neglected. Putting m = 0, 1,...., N we obtain the following system of equations

$$I_1' = I_0(1+\omega)\ ,\ f_0 = 1 \qquad (1.2.8i)$$

$$I_2' + I_0' - 3I_1(1 + \omega f_1) = 0 \tag{1.2.8ii}$$

$$3I_3' + 2I_1' - 5I_2(1 + \omega f_2) = 0 \tag{1.2.8iii}$$

....

....

$$NI_N' + (N-1)I_{N-2}' - (2N-1)I_{N-1}(1 + \omega f_{N-1}) = 0 \tag{1.2.8i,N}$$

$$NI_{N-1}' - (2N+1)I_N(1 + \omega f_N) = 0 \tag{1.2.8i,N+1}$$

The system (1.2.8) provides N + 1 simultaneous linear, ordinary differential equations for the (N + 1) unknown functions $I_0, I_1, ,......, I_N$ and is called the P_N approximation. The desired solution of the system of equations (1.2.8) can be written as a linear sum of the solution of the homogeneous part of these equations and a particular solution.

Let us assume a trial solution of the form

$$I_m(\tau) = g_m e^{k\tau}, \quad m = 0, 1, 2, ..., N \tag{1.2.9}$$

where g_m are arbitrary constants and k is the exponent which is to be determined. Using (1.2.9) in the equations (1.2.8) we obtain,

$$k\left[(m+1)g_{m+1} + mkg_{m-1}\right]e^{k\tau} - (2m+1)g_m(1 + \omega f_m)e^{k\tau} = 0$$

or simply we have,

$$k\left[(m+1)g_{m+1} + mkg_{m-1}\right] - (2m+1)(1 + \omega f_m)g_m = 0 \tag{1.2.10}$$

where $f_0 = 1$, $g_{N+1} = 0$.

We consider isotropic scattering. For this, we set

$$f_m = \delta_{0,m} = \begin{cases} 1, & if \ m = 0 \\ 0, & otherwise \end{cases} \qquad (1.2.11)$$

$$k\left[(m+1)g_{m+1} + mkg_{m-1}\right] - (2m+1)(1+\omega\delta_{0m})g_m = 0 \qquad (1.2.12)$$

where m = 0,1,2,.......,N

Equation (1.2.12) is a system of homogeneous algebraic equations. This system will have a non trivial solution if the determinant of the coefficient matrix vanishes, i.e. the determinant D(k), given below

$$\begin{vmatrix} -(1+\omega) & k & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ k & -3 & 2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 2k & -5 & 3k & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 3k & -7 & 4k & \dots & 0 & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 0 & \dots & k(n-2) & -(2N-3) & k(N-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & k(N-1) & -(2N-1) & kN \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & kN & -(2N+1) \end{vmatrix}$$

is zero or

$$D(k) = 0 \qquad (1.2.13)$$

For each value of ω we will get permissible values of k_i. Then for each value of k_i a set of $g_m(k_i)$, m = 0,1,2,...,N is determined from (1.2.12) and thus the solution to the homogeneous part is obtained as

$$I_m^H(\tau) = \sum_{i=0}^{N} L_i g_m(k_i) e^{k_i \tau}, m = 0,1,2...,N \qquad (1.2.14)$$

where L_i are (N+1) constants. They are determined from the boundary condition of the given problem. Once the functions $I_m(\tau)$ are known, the distribution of emergent intensity is calculated from (1.2.3). Davison (1957) discusses an alternative representation of the homogeneous solution of the equation (1.2.14) in terms of the auxiliary functions $H_m(\tau)$ in the form

$$I_m(\tau) = \sum_{i=0}^{N} \overline{L}_i H_m(k_i) e^{k_i \tau} \ , m = 0, 1, 2 \ldots, N \qquad (1.2.15)$$

where $\overline{L_i}$ are expansion coefficients, and the auxiliary function $H_n(k_i)$ is defined by

$$H_n(k_i) = (-1)^n \left\{ P_n\left(\frac{1}{k}\right) - \frac{\omega}{k}\left[Q_0\left(\frac{1}{k}\right) - Q_n\left(\frac{1}{k}\right)\right]\right\} \qquad (1.2.16)$$

Here P_n, Q_n are the Legendre polynomials and the Legendre functions of the second kind respectively.

1.2.1 P_1 approximation.

As a special case, we consider the take isotropic scattering. Here N = 1 and $f_m = \delta_{0,m}$. We obtain from (1.2.8)

$$I_1' - (1 + \omega) I_0 = 0 \qquad (1.2.1.1)$$

$$I_0' - 3 I_1 = 0 \qquad (1.2.1.2)$$

We wish to determine $I_0(\tau)$ and $I_1(\tau)$. But we have assumed

$$I(\tau, \mu) = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi} P_m(\mu) I_m(\tau) \qquad (1.2.1.2a)$$

Next using the orthogonal property of Legendre polynomials we obtain

$$\int_{-1}^{1} P_m(\mu) I(\tau,\mu) d\mu = \frac{I_m(\tau)}{2\pi} \tag{1.2.1.3}$$

This gives

$$I_m(\tau) = 2\pi \int_{-1}^{1} P_m(\mu) I(\tau,\mu) d\mu \tag{1.2.1.4}$$

Also, in case of first approximation m = 1, so that we have following two relations

$$I_0(\tau) = 2\pi \int_{-1}^{1} I(\tau,\mu) d\mu = G(\tau) = \text{Incident radiation} \tag{1.2.1.5a}$$

$$I_1(\tau) = 2\pi \int_{-1}^{1} \mu I(\tau,\mu) d\mu = F(\tau) = \text{Net flux} \tag{1.2.1.5b}$$

By replacing $I_0(\tau)$ and $I_1(\tau)$ by $G(\tau)$ and $F(\tau)$ respectively we obtain,

$$F'(\tau) - (1 + \omega) G(\tau) = 0 \tag{1.2.1.6a}$$

$$G'(\tau) - 3F(\tau) = 0 \tag{1.2.1.6b}$$

Equation (1.2.5) can be combined to yield a single differential for $G(\tau)$ or $F(\tau)$. Therefore, we have

$$G''(\tau) - 3(1 + \omega) G(\tau) = 0 \tag{1.2.1.7}$$

where the prime denotes differentiation w.r.t τ. Once $G(\tau)$ is determined from the solution of the equation (1.2.1.7) subject to a set of appropriate boundary conditions, the radiation intensity $I(\tau,\mu)$ is determined from the equation (1.2.1.2a) as

$$I(\tau,\mu) = \frac{1}{4\pi}\left[P_0(\mu)\Psi_0(\tau) + 3P_1(\mu)I_1(\tau)\right]$$

or

$$I(\tau,\mu) = \frac{1}{4\pi}\left[G(\tau) + 3\mu F(\tau)\right] \qquad (1.2.1.8)$$

1.2.2 P_2 - approximation.

Here N = 2 .Proceeding similarly we obtain the corresponding equations as

$$I_1' - I_0(1+\omega) = 0 \qquad (1.2.2.1a)$$

$$I_2' + I_0' - 3I_1(1+\omega f_1) = 0 \qquad (1.2.2.1b)$$

$$2I_1' - 5I_2(1+\omega f_2) = 0 \qquad (1.2.2.1c)$$

For isotropic scattering the above three equations reduce to

$$I_1' - I_0(1+\omega) = 0 \qquad (1.2.2.2a)$$

$$I_2' + I_0' - 3I_1 = 0 \qquad (1.2.2.2b)$$

$$2I_1' - 5I_2 = 0 \qquad (1.2.2.2c)$$

Like in the P_1 approximation these three equations can be treated in the same manner.

1.3 The necessity of modification.

From our discussion we have seen that in spherical harmonic method the specific intensity is expanded into a series of Legendre polynomials $P_l(\mu)$ which form a complete set of orthogonal polynomials within (-1,1). In general, for all practical purposes the series is truncated after N terms, where N is the required accuracy.However, the spherical harmomic method suffers from one serious defect; the difficulty of analytical representation of boundary conditions at the free surface,where the specific intensity is discontinuous. In plane parallel problems and in simple case cases of spherical models, the situation is met by using different types of equivalent boundary conditions but this method was still successful in yielding fairly accurate results. Kourganoff (1963) drew attention to some of the serious limitations of the single interval spherical harmonic method.

The origin of single interval spherical method is due to Eddington (1926), Gratton (1937). However, it was Chandrasekhar (1943,1945) who developed a systematic method and suggested a general procedure for solving integro-differential equation of transfer by this method to any order of approximations. This method was extensively used to solve various radiative transfer problems in plane parallel medium in stellar atmosphere and in neutron transport.

For more clarification, let us now restate the problem. The radiative transfer equation in simple model of semi-infinite, plane parallel, scattering medium is given by

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' \qquad (1.3.1)$$

where the symbols have their meanings described in Sec. (1.1). Further we have assumed that the specific intensity is represented as

$$I(\tau,\mu) = \sum_{l=0}^{L} I_l(\tau)P_l(\mu) \qquad (1.3.2)$$

Here the source function is given by

$$\Im(\tau) = \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' \qquad (1.3.3)$$

The boundary conditions under which the transfer equation to be solved is

(a) Absence of incident radiation from outside at the free surface defined by $\tau = 0$,

i.e. $I(0,-\mu) = 0$ for $0 < \mu \leq 1$ (1.3.4a)

(b) The convergence of intensity, i.e.

$$\Im(\tau)e^{-\tau} \rightarrow 0 \text{ as } \tau \rightarrow \infty \qquad (1.3.4b)$$

Substituting (1.3.2) in the transfer equation,using the recurrence formula for Legendre polynomial $P_l(\mu)$ and equating the coefficients of various Legendre polynomials, we get a set of linear differential equations. Chandrasekhar (1960) assumed a trial solution of the form

$$I_m(\tau) = g_m e^{-k\tau} \qquad (1.3.5)$$

where g_m , k are constants. Substituting (1.3.5) in the set of linear differential equations, we get a set of linear algebraic equations. As discussed earlier we get roots of k as $0, 0, k_i, (i = 2,3,\ldots,n)$. Then the solution of (1.3.5) will be as follows

$$I_l(\tau) = \sum_i c_i g_m(k_i) e^{-k_i\tau} \qquad (1.3.6)$$

where the constants are to be evaluated form the boundary conditions. While the exact boundary condition is realized at the lower boundary $\tau \rightarrow \infty$, it cannot be done at the at the free surface, i.e. at $\tau = 0$. Instead, equivalent boundary conditions are used. These are

(a) Chandrasekhar (1943, 1945) used the boundary condition

$$\frac{2}{2l+1} I_l(0) = \sum_{m=0}^{L} I_m(0) \int_{-1}^{1} P_m(\mu) P_l(\mu) d\mu \qquad (1.3.7a)$$

(b) Mark's (1947) boundary condition in connection with neutron transport problem in plane parallel medium in this case is

$$I(0,\mu_i) = 0 \qquad (1.3.8b)$$

where μ_i are some strategic values of μ within the range and were taken as the roots of the equation $P_{n+1}(\mu) = 0$,n being an odd integer.

(c) Marshak's (1948) equivalent boundary condition was

$$\int_{-1}^{0} I(0,\mu) P_{2l-1}(\mu) d\mu = 0 \,, l = 1, 2, ..., n \qquad (1.3.8c)$$

Mertens (1954) represented $I(\tau,\mu)$ as $I_+(\tau,\mu)$ and $I_-(\tau,\mu)$ at the two separate ranges (0,1) and (-1,0) for μ. He wrote

$$I_+(\tau,\mu) = \sum_{l=0}^{L} (2l+1) I_l^+(\tau) \mu P_l(2\mu - 1) \,, 0 \le \mu \le 1 \qquad (1.3.9a)$$

$$I_-(\tau,\mu) = \sum_{l=0}^{L} (2l+1) I_l^-(\tau) \mu P_l(2\mu + 1) \,, -1 \le \mu \le 0 \qquad (1.3.9b)$$

The boundary conditions were taken as

$$I_l^-(0) = 0 \,, l = 0, 1, 2, ..., n \qquad (1.3.10a)$$

$$I_l^+(\tau) e^{-\tau} \to 0 \text{ as } \tau \to \infty \qquad (1.3.10b)$$

$$I_l^-(\tau)e^{-\tau} \to 0 \quad \text{as } \tau \to \infty \tag{1.3.10c}$$

With this representation, the main steps of the single interval spherical harmonic method are gone thorough necessary adaptation. The end results were encouraging. As far as the transfer problems in plane parallel medium is concerned, this method was free from two main defects of the single interval spherical harmonic method, namely,

(i) The exact boundary condition could be used here.

(ii) There was no preference for odd order approximations over the even order ones contrary to the suggestions of Davison and Sykes (1955)

However, some new difficulties arose; these are listed below.

(i) It was found to adversely effect the critical size calculation of neutron transport.

(ii) The extrapolated end points calculated for neutron transport in slab medium were found to be unreliable.

(iii) It was found that the method was adoptable to the solution of transfer problems in spherical geometry. The method did not give correct flux integral in the spherical case.

Double interval spherical harmonics method has been used with some modifications by Sykes(1951), Gross and Zeiring (1955), Max Krook (1955, 1999) and others. They however, are essentially equivalent to Merten's method and share it's limitations.

On a close approximation of Merten's (1954) representation of specific intensity, it was found that in an attempt to provide for discontinuity at the free surface, the discontinuity of representation at μ=0 had been carried to the interior. Wilson and Sen (1963) introduced at this stage a modified double interval spherical harmonic method which retained the advantages of the double interval representation of Mertens and at the same time removed its defects. The aim was to select a suitable spherical harmonic method which would be equally effective for tackling transfer problems in plane parallel and spherical medium.

1.4 Modifications of SHM.

In this section we will demonstrate various modifications of the SHM that have been taken place over the years. The passage from single interval SHM to double interval SHM has been elaborately described by Wilson and Sen (1990) and they gave an exhaustive account for the modification. Kourganoff (1963) made a critical analysis of the single interval SHM used in the case of Milne problem and raised objections against it. For example i) exact boundary conditions cannot be prescribed and ii) certain arbitrariness persists in the determination of the constants. He suggested double interval SHM. Here, we list the modifications of double interval SHM that have been used by various workers in the field of radiative transfer. Most of the forms are covered by Wilson and Sen (1990) and we give a few additional ones.

I) Merten's (1954) Form

$$I_+(\tau,\mu) = \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)P_l(2\mu-1)\,, \quad 0<\mu\leq 1 \tag{1.4.1a}$$

$$I_-(\tau,\mu) = \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)P_l(2\mu+1)\,, \quad -1\leq\mu<0 \tag{1.4.1b}$$

where P_l are Legendre polynomials.

II) Wilson and Sen's (1965) Forms

A. *Plane parallel medium*

$$I_+(\tau,\mu) = A(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1), \quad 0\leq\mu\leq 1 \tag{1.4.2a}$$

and

$$I_-(\tau,\mu) = A(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), \quad -1\leq\mu\leq 0 \tag{1.4.2b}$$

where A(τ) is an arbitrary function of τ only, τ being the optical depth, μ, the cosine of the angle made by the pencil radiation with the outward drawn normal.

B. *Spherical geometry.*

$$I_+(\tau,\mu) = A(r) + \sum_{l=0}^{l_0}(2l+1)I_l^+(r)\mu P_l(2\mu-1), \quad 0 \le \mu \le 1 \qquad (1.4.3a)$$

and

$$I_-(\tau,\mu) = A(r) + \sum_{l=0}^{l_0}(2l+1)I_l^-(r)\mu P_l(2\mu+1), \quad -1 \le \mu \le 0 \qquad (1.4.3b)$$

where $A(r)$ is a function r only, r being the distance measured outward from the center of the sphere, μ, the cosine of the angle measured from the positive direction of the radius vector.

III) Wan, Wilson and Sen's (1977) Form

$$i_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)i_l^+(\tau)\mu P_l(2\mu-1), \quad 0 \le \mu \le 1 \qquad (1.4.4a)$$

and

$$i_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)i_l^-(\tau)\mu P_l(2\mu+1), \quad -1 \le \mu \le 0 \qquad (1.4.4b)$$

IV) Karanjai and Talukdar's (1992) Form

$$I_+(\tau,\mu) = I(0,0)\left[A\tau + \phi(\mu) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1)\right], \quad 0 \le \mu \le 1 \qquad (1.4.5a)$$

and

$$I_{-}(\tau,\mu) = I(0,0)\left[A\tau + \phi(\mu) + \sum_{l=0}^{l_0}(2l+1)I_l^{-}(\tau)\mu P_l(2\mu+1)\right], \quad -1 \le \mu \le 0 \qquad (1.4.5b)$$

where I(0,0) is the specific intensity at the surface in the direction normal to the surface and is a constant and A is an arbritary constant. This form was also used by Bishnu.

V) Raychaudhuri and Karanjai's (1993) Form

$$I^{+}(\tau,\mu) = I(0,0)\left[\phi(\tau) + \psi(\mu) + \sum_{l=0}^{l_0}(2l+1)I_l^{+}(\tau)\mu P_l(2\mu-1)\right], \quad 0 \le \mu \le 1 \qquad (1.4.6a)$$

and

$$I^{-}(\tau,\mu) = I(0,0)\left[\phi(\tau) + \psi(\mu) + \sum_{l=0}^{l_0}(2l+1)I_l^{-}(\tau)\mu P_l(2\mu+1)\right], \quad -1 \le \mu \le 0 \qquad (1.4.6b)$$

1.5 Application of SHM in solving Radiative Transfer (RT) Problems.

Wilson and Sen (1963) introduced a modification of the double interval SHM for solving the equation of radiative transfer. Their work was mainly an attempt to eliminate the objections raised against the method of Yvon and retaining Yvon's advantages. Wilson and Sen's (1963) work ensured the continuity of the intensity in a direction perpendicular to the outward drawn normal for all values of optical thickness. The basic equation of transfer for plane parallel medium was taken as

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')d\mu' \tag{1.5.1}$$

where $I(\tau,\mu)$ is the specific intensity of radiation at an optical depth τ and in a direction θ with the outward drawn normal $\mu = \cos(\theta)$. The optical thickness is given by

$$\tau = \int_{z}^{\infty} \kappa\rho dz$$

where κ is the scattering coefficient and ρ is the density of the medium. They considered the boundary conditions

i) Absence of incident radiation from outside at the free surface, i.e.,

$$I(0,\mu) \equiv 0 \text{ for } -1 \leq \mu \leq 0 \tag{1.5.2a}$$

ii) The convergence of intensity, i.e.,

$$I(\tau,\mu)e^{-\tau} \rightarrow 0 \text{ as } \tau \rightarrow \infty \tag{1.5.2b}$$

They represented the intensity $I(\tau,\mu)$ by two different expansions in the intervals (0,1) and (-1,0) and these are respectively as

$$I_+(\tau,\mu) = A\tau + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu - 1), \qquad 0 \le \mu \le 1 \qquad (1.5.3a)$$

and

$$I_-(\tau,\mu) = A\tau + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu + 1), \qquad -1 \le \mu \le 0 \qquad (1.5.3b)$$

They evaluated q(τ) and I(0,μ)/F for both first approximation (i.e., P_1 - approximations) and second approximations (i.e., P_2 - approximations) and compared the results with Mertens (1954) and Chandrasekhar (1960). They have shown that the second approximation is distinctly superior to first approximation.

Wilson and Sen (1964a) extended their earlier work [Wilson and Sen (1963)] to solve the equation of radiative transfer in plane geometry in case of anisotropic scattering. They used the general phase function and used it to discuss the case of Rayleigh phase function. The basic equation considered by them is once again

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' \qquad (1.5.4)$$

where

$$p(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} p(\mu,\phi;\mu'\phi')d\mu',$$

p(μ,φ;μ′,φ′) is the phase function giving the measure of the probability of a ray in the direction (μ′,φ′) being scattered into the direction (μ,φ). The other symbols have their meanings described in the previous page. In this case they also took the same boundary conditions given by Eq. (1.5.2a) and (1.5.2b) and considered the same forms of intensity given by Eqs. (1.5.3a) and (1.5.3b) respectively. However, they assumed the form of the phase function to be

$$p(\mu,\mu') = \sum_{k=0}^{\infty} w_k P_k(\mu) P_k(\mu') \qquad (1.5.5)$$

where w_k 'S are simply constants and P_k's are Legendre polynomials. They considered the Rayleigh phase function for an example and used it to obtain results of I(0,μ)/F for the P_1 - approximations and compared the results with those of Chandrasekhar (1944).

Wilson and Sen (1964b) extended their modified SHM [Wilson and Sen (1963 and 1964a)] to solve the equation of transfer in spherical geometry. By this method, they solved the classical problem of diffusion of radiation through a homogeneous sphere, the radius being either finite or infinite. They considered the equation of transfer appropriate to the problem of diffusion of radiation on the homogeneous sphere as,

$$\mu \frac{\partial I(r,\mu)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I(r,\mu)}{\partial \mu} + I(r,\mu) = \frac{1}{2} \int_{-1}^{1} I(r,\mu) d\mu \qquad (1.5.6)$$

where r is the distance measured outward from the center of the sphere and μ is the cosine of the angle measured from the positive direction of the radius vector, I(r,μ) is the specific intensity of radiation at a distance r in the direction of $\theta = cos^{-1}(\mu)$. They considered, [Vide Wilson and Sen (1963 and 1964a)] the two different expansions of intensity and these are

$$I_+(r,\mu) = A(r) + \sum_{l=0}^{l_0} (2l+1) \mu I_l^+(r) P_l(2\mu - 1), \quad 0 \leq \mu \leq 1 \qquad (1.5.7a)$$

$$I_-(r,\mu) = A(r) + \sum_{l=0}^{l_0} (2l+1) \mu I_l^-(r) P_l(2\mu + 1), \quad -1 \leq \mu \leq 0 \qquad (1.5.7b)$$

where $A(r)$ is a function of r only. The equation (1.5.6) was solved with the following boundary conditions

$$\left.\begin{aligned} A(R) &= 0 \\ I_0^-(R) &= 0 \\ I_1^-(R) &= 0 \end{aligned}\right\} \tag{1.5.8}$$

They obtained results for the first approximation and calculated the mean intensity $J(r)$ and compared the results with those of Chandrasekhar (1944).

Wilson and Sen (1965b) extended their modified form of SHM to solve the transfer problem in an isotropically scattering, spherically symmetric,finite stellar atmosphere with $k\rho \propto r^{-2}$. They considered the following transfer equation

$$\mu \frac{\partial I(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I(r,\mu)}{\partial \mu} + k\rho I(r,\mu) = \frac{1}{2}k\rho \int_{-1}^{1} I(r,\mu)d\mu \tag{1.5.9}$$

where r is the distance measured outward from the center of the sphere, and μ is the cosine of the angle measured from the positive direction of the radius vector, $I(r,\mu)$ is the specific intensity of radiation at a distance r in the $cos^{-1}(\mu)$ direction, ρ is the density of the material and k,mass absorption coefficient. They used the following boundary conditions

I) $I(R,\mu) = 0$ for $-1 \le \mu \le 0$, R is the radius (1.5.10a)

II) The convergence of intensity as $r \to 0$ (1.5.10b)

They considered the same two forms of intensity [Wilson and Sen (1964b)] given by the equations (1.5.7a) and (1.5.7b). They also pointed out that the function A(r) which appeared in Eqs. (1.5.7a) and (1.5.7b) depend on the nature of the physical problem and solved the problem for the first approximation to evaluate J(x),the mean intensity at x = 2 where $x = \frac{k_0 C}{r}$ for two different boundaries given by $R = 2k_0 C$ and $R = k_0 C$. Their results were compared with those of Chandrasekhar (1960).

Wilson and Sen (1965c) once again extended their modifications of the SHM to solve the problem of radiative transfer in spherically symmetric, finite planetary nebular shell with $k\rho \propto r^{-2}$. The equation of transfer considered by them was

$$\mu \frac{\partial I}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I}{\partial \mu} = -k\rho \left[I - \frac{1}{2} \int_{-1}^{1} I d\mu - \frac{S}{4} \frac{r_1^2}{r^2} e^{-(\tau_1 - \tau)} \right] \qquad (1.5.11)$$

where τ_1 is the radial optical thickness of the nebular shell and πS is the net flux of the radiant energy incident on each square centimeter of the inner surface (radius $= r_1$) of the nebula. Following Wilson and Sen (1964b) they considered the same two forms of the intensity [cf. Eqs. (1.5.7a) & (1.5.7b)]. They assumed the boundary conditions

a) There is no incident radiation on the outer boundary defined by r = R, i.e.,

$$I(R,\mu) = 0 \quad \text{for} \quad -1 \leq \mu \leq 0 \qquad (1.5.12a)$$

b) The diffuse flux across the inner surface ($r = r_1$) vanishes, i.e.,

$$F_{r=r_1} = 0 \qquad (1.5.12b)$$

They evaluated J(τ) at τ = and R = C using the first approximation. Here C is connected to kρ by the relation $k\rho = \frac{C}{r^n}$, n > 1 . The results were compared with Sen (1949)

Bishnu(1968) gave an alternative modification of the double interval SHM.He assumed plane parallel scattering atmosphere with spherical symmetry and the appropriate equation of transfer was taken to be

$$\mu \frac{dI(t,\mu)}{dt} = I(t,\mu) - \frac{1}{2}\int_{-1}^{1} I(t,\mu')d\mu' \tag{1.5.13}$$

where the symbols have their usual meanings [Vide. Wilson and Sen (1963)]. The forms of intensity taken by him are as follows

$$I_+(t,\mu) = I(0,0)\left[At + \phi(\mu) + \sum_{l=0}^{l=l_0}(2l + 1)I_l^+(t)\mu P_l(2\mu - 1)\right], \quad 0 \le \mu \le 1, \tag{1.5.14a}$$

$$I_-(t,\mu) = I(0,0)\left[At + \phi(\mu) + \sum_{l=0}^{l=l_0}(2l + 1)I_l^-(t)\mu P_l(2\mu + 1)\right], \quad -1 \le \mu \le 0, \tag{1.5.14b}$$

He used the boundary conditions given by Wilson and Sen (1963) and evaluated the H - functions and made comparisons with those of Chandrasekhar (1960).

Canosa and Penafiel (1973) proposed a direct method for the numerical solution of the spherical harmonics approximations to the equation of radiative transfer in plane parallel atmospheres. The spherical harmonics equations are a two-point boundary value problem for a system of ordinary differential equations of first order. These are then reduced to an algebraic problem by finite difference method.Since the matrices of the problem are non-convergent, the round off error grows exponentially. Canosa and Penafiel(1973) avoided this difficulty by applying a parallel shooting method. The method was applied to homogeneous atmospheres and Rayleigh and Mie phase functions were used. The basic equation they took is

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\tau,\mu,\phi;\mu',\phi') I(\tau,\mu',\phi') d\mu' d\phi' + S(\tau,\mu,\phi) \qquad (1.5.15)$$

where τ is the optical depth measured from top of the atmosphere, μ is the cosine of the zenith angle measured with respect to the positive τ axis,ϕ is the azimuthal angle, p is the general phase function, S is the source of the incident radiation and I is the intensity of radiation.They dealt with only the “average intensity”form of the equation of transfer and this is given by

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I = \frac{1}{2} \sum_{l=0}^{L} \frac{2l+1}{2} a_l(\tau) P_l(\tau) \int_{-1}^{1} P_l(\mu') I(\tau,\mu',\phi') d\mu' d\phi' +$$

$$+ \frac{1}{4} F e^{-\frac{\tau}{\mu_0}} \sum_{l=0}^{l_0} \frac{2l+1}{2} a_l(\tau) P_l(\mu_0) P_l(\mu) \qquad (1.5.16)$$

where ω_l are connected to a_l by the following relation

$$\omega_l = \frac{2l+1}{2} a_l(\tau) \qquad (1.5.17)$$

The form of the intensity taken by them was

$$I(\tau,\mu) = \sum_{l=0}^{L} \frac{2l+1}{2} f_l(\tau) P_l(\mu) \qquad (1.5.18)$$

and assumed normalized phase function

$$P(cos\theta) = \sum_{l=0}^{L} \omega_l P_l(cos\theta) \qquad (1.5.19)$$

Test computations were performed on Rayleigh and Mie phase functions.

Devaux et al (1973) discussed a critical study of four methods of solution of the equation of transfer (principle of invariance,doubling,spherical harmonics,successive orders of scaling) and compared both the accuracy of the results and required computation time. The SHM seems to have significant advantages over the others.

Wan,Wilson and Sen (1977) used the modified SHM in solving the radiative transfer problem in an isothermal slab with Rayleigh scattering. They considered the model consisting of an isothermal plane parallel slab of optical thickness τ_0 confined between gray and diffuse walls that absorb and anisotropically scatter radiant energy. The equation of transfer for such a model was [Dayan and Tien (1976)]

$$\mu \frac{\partial i}{\partial \tau} + i = (1 - \omega_0) i_b + \frac{\omega_0}{2} \int_{-1}^{1} p(\mu,\mu') i(\mu') d\mu' = S(\tau,\mu) \qquad (1.5.20)$$

where I is the intensity, i_b, the black body intensity, ω_0,the albedo for single scattering, p, the phase function, S, the source function, μ, the cosine of the angle measured from the positive direction of optical depth τ. The Rayleigh phase function was considered and this is given by

$$p(\mu,\mu') = \frac{3}{8}\left[\left(3 - \mu^2\right) + \left(3\mu^2 - 1\right)\mu'^2\right] \qquad (1.5.21)$$

They took the forms of the intensity given by

$$i_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0} (2l+1) i_l^+(\tau) \mu P_l(2\mu - 1), \qquad 0 \le \mu \le 1 \qquad (1.5.22a)$$

and

$$i_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0} (2l+1) i_l^-(\tau) \mu P_l(2\mu + 1), \qquad -1 \le \mu \le 0 \qquad (1.5.22b)$$

The boundary conditions used are

$$\left.\begin{aligned} i(0,\mu) &= B_1 \quad for \quad 0 \le \mu \le 0 \\ i(\tau_0,\mu) &= B_2 \quad for \quad -1 \le \mu \le 0 \end{aligned}\right\}$$

In their problem they took both B_1 , B_2 to be zero and applied the first approximation to find the zeroth,first and second moments of intensity and compared results with that of Dayan and Tien (1976)

Peraiah (1979) discussed a numerical method for obtaining solution of radiative transfer equation in spherically symmetric media spherical harmonic approximation. Peraiah (1979) approximated the angle derivative by an orthonormal polynomial and this is represented by a matrix called curvature matrix, for a given beam of rays. He considered the radiative transfer equation in spherical symmetry as

$$\frac{\mu}{r^2}\frac{\partial}{\partial r}\left[r^2 I(r,\mu)\right] + \frac{1}{r}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)I(r,\mu)\right] + k(r)I(r,\mu) =$$

$$= k(r)\left[\left[1-w(r)\right]B(r) + \frac{1}{2}w(r)\int_{-1}^{1} P(r,\mu,\mu')I(r,\mu')d\mu'\right]$$

where k(r) is the absorption coefficient, k(r) $\ge$ 0 and w(r) is the albedo for single scattering. $0 \le w(r) \le 1$. I(r,μ) is the monochromatic specific intensity of the ray making an angle $cos^{-1}(\mu)$ with the radius vector at the radial point r. B(r) is the Plank's function at r and P(r,μ,μ′) is the phase function and it is assumed to be isotropic. The angles are discretized such that $0 < \mu_1 < \mu_2 < .... < \mu_m \le 1$. They expanded the specific intensity as

$$I(\mu) = \sum_{m=0}^{M} \alpha_m P_m(\mu)$$

and calculated the emergent intensities in two cases.

Karp,Greenstadt and Fillmore (1980) discussed the SHM for solving the equation of radiative transfer for a plane parallel planetary atmosphere. They assumed that all the inhomogeneities were confined to the vertical direction and each layer of the atmosphere was taken to be homogeneous but with arbitrary optical thickness. They considered the equation of transfer with monochomatic radiation as

$$\mu \frac{dI(\tau;\mu,\phi)}{d\tau} = I(\tau;\mu,\phi) - J(\tau;\mu,\phi) \tag{1.5.23}$$

where $I(\tau;\mu,\phi)$ is the specific intensity, μ is the cosine of the zenith angle, ϕ is the azimuth angle measured from Sun's meridian and $J(\tau;\mu,\phi)$ is the source function defined by

$$J(\tau;\mu,\phi) = \frac{1}{4}P(\tau;\mu,\phi;-\mu_0,\phi_0)F_0 e^{-\frac{\tau}{\mu_0}} + \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} P(\tau;\mu,\phi;\mu',\phi')I(\tau,\mu',\phi')d\mu' d\phi' \tag{1.5.24}$$

where the external illumination πF_0 was assumed to be unidirectional. The boundary conditions were

$$I(0;\mu<0,\phi) = I(\tau_b;\mu>0,\phi) = 0$$

Their technique for solving the above problem was mainly based on SHM and they developed an algorithm for the method and computed the angle dependent intensity at all points in the atmosphere.

Karp and Petrack(1983) compared SHM and DOM (Discrete ordinate method) for azimuth dependent intensity calculations. They showed that for higher terms of the Fourier expansions of the intensity the results were exact at the zeros of the Legendre polynomials and considered the equation of transfer for plane parallel,scattering and absorbing atmosphere

$$\mu \frac{dI(\mu,\phi,\tau)}{d\tau} = I(\mu,\phi,\tau) + \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} I(\mu',\phi',\tau)p(\tau,\mu,\mu';\phi,\phi')d\mu' d\phi' \qquad (1.5.25)$$

where $P(\tau;\mu,\mu',\phi,\phi')$ is the scattering phase function and other symbols have their usual meanings. They expanded (1.5.25) in a Fourier series in ϕ and obtained the following equations

$$\mu \frac{dI^m(\mu,\tau)}{d\tau} = I^m(\mu,\tau) - \int_{-1}^{1} I(\mu',\tau)P^m(\tau;\mu,\mu')d\mu' \qquad (1.5.26)$$

They considered the following form of phase function

$$P^m(\tau;\mu,\mu') = \sum_{l=m}^{L+m} \beta_l(\tau)Y_l^m(\mu)Y_l^m(\mu')$$

where $Y_l^m(\mu)$ are normalized spherical harmonics (or the associated Legendre polynomials) and $\beta_l(\tau)$ are expansion coefficients of the scattering phase function. The form of the intensity considered by them is

$$I^m(\tau,\mu) = \sum_{l=m}^{L+m} \frac{2l+1}{2} f_l^m(\tau)Y_l^m(\mu) \qquad (1.5.27)$$

where $f_l^m(\tau)$ are the moments of the intensity. They showed that the azimuth-dependent intensities computed from spherical harmonics method was exact at the zeros of the Legendre polynomials. This was used to set up the connection between SHM and DOM.

Wells and Sidorowich (1985) used the SHM to solve the radiative transfer problems with extreme forward scattering and demonstrated the method by using slab geometry. They developed the computational techniques with some test results.

Aronson (1986) compared P_N approximations with double P_N approximations for highly anisotropic scattering. It is known that the double P_N transport calculations generally give better results than the corresponding (2N-1) orders transport calculations with the full range spherical harmonics. They examined the results of both methods by considering different terms of the phase function for both haze model and the cloud model and the following form of phase function was taken.

$$f(\theta) = \omega \sum b_l P_l(\cos\theta), \quad b_0 = 1$$

ω is the single scattering albedo and $\cos^{-1}(\mu_0)$ is the angle made by the pencil of radiation with the normal. For haze model they took 82 terms in the phase function and for the cloud model 300 terms of the phase function were taken. They concluded that D_N approximations (double P_N) were almost always better than the corresponding P_N calculations.

Garcia and Siewert (1986) developed a generalized spherical harmonics solution for all components (m ≥ 0) in a Fourier representation of the Stokes vector basic to the scattering of polarized light. Following Benassi, Garcia and Siewert (1985), they denoted I(τ,μ,ϕ) as the density vector with the four Stokes parameters the components and considered the following equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau,\mu,\phi) + I(\tau,\mu,\phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} P(\mu,\mu',\phi-\phi') I(\tau,\mu,\mu') d\mu' d\phi' \qquad (1.5.28)$$

where P(μ,μ',ϕ-ϕ') is the phase matrix. They solved the equation (1.5.28) subject to the following boundary conditions

$$I(0,\mu,\phi) = \pi \delta(\mu - \mu_0) \delta(\phi - \phi_0) F \qquad (1.5.29a)$$

$$I(\tau_0, -\mu,\phi) = \frac{\lambda_0}{\pi} L \int_0^{2\pi} \int_0^{1} I(\tau_0,\mu',\phi') \mu' d\mu' d\phi' \qquad (1.5.29b)$$

where λ_0 is the coefficient for lambert reflection, L = diag{1,0,0,0} and F is the flux vector (with the components F_I, F_Q, F_U, F_V assumed to be given). They used the generalized SHM to solve the complete and general polarization problem for a plane parallel layer.

Kamiuto (1986) applied Chebyshev collocation method for solving the radiative transfer equation. Kamiuto expanded the radiant intensity in a series of Legendre polynomials, and using the orthogonal properties of Legendre polynomials, he obtained the spherical harmonics equations. These equations are a system of ordinary differential equations of first order but such a system cannot be solved by means of a simple finite difference scheme because of the ocurance of the numerical instability which is due to the accumulation of round off errors. In order to get rid of this difficulty Kamiuto proposed the collocation method technique.

He considered the equation of transfer in plane parallel homogeneous dispersive medium as

$$\mu \frac{dI(\tau,\mu,\phi)}{d\tau} + I(\tau,\mu,\phi) = S(\tau,\mu,\phi) + \frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(cos\theta) I(\tau,\mu',\phi')\, d\mu' \, d\phi' \qquad (1.5.30)$$

where $P(\cos\theta)$ is the phase function and $S(\tau,\mu,\phi)$ is the source of incident radiation;the other symbols have their usual meanings. The intensity form was taken to be

$$I(\tau,\mu) = \sum_{l=0}^{L} \frac{2l+1}{2} P_l(\mu) f_l(\tau) \qquad (1.5.31)$$

He used Marshak's equivalent boundary conditions which are stated below

$$\int_0^1 I(-1,\mu)\mu^{2i-1} d\mu = 0 \text{ and } \int_{-1}^{0} I(1,\mu)\mu^{2i-1} d\mu = 0 \qquad (1.5.32)$$

The method was applied to Rayleigh and moderately Henyey-Greenstein phase functions. He assumed ω to be unity and varied τ from 0.5 to 16.

Takeuchi (1988) applied a new formulation of the SHM to solve certain problems in atmospheric sciences. He considered the equation of transfer for plane parallel atmospheric models with all inhomogeneities confined in the vertical direction and with axially symmetric phase functions. Takeuchi (1988) showed that the spherical harmonics approximation reduces the m-th Fourier component of the equation of radiative transfer to a system of infinite homogeneous ordinary differential equations. He then expanded each term of the equation by the normalized associated Legendre polynomials $Q_l^m(\mu)$ which are related to the associated Legendre polynomials $P_l^m(\mu)$ by

$$Q_l^m(\mu) = [(2l+1)(l-m)!]^{\frac{1}{2}} / [2(l+m)!]^{\frac{1}{2}} P_l^m(\tau) \qquad (1.5.33)$$

The scattering phase function was taken to be

$$\phi(\tau;\mu_\alpha) = \omega_0(\tau)\sum_{l=0}^{\infty}\omega_l(\tau)P_l(\mu_\alpha) \qquad (1.5.34)$$

Takeuchi (1988) showed that the adaptation of the normalized associated Legendre polynomials $S_l^m(\mu)$ made SHM very easy to handle. The eigenvalues and the eigenvectors inherent in each scattering layer are obtained by using single-value decomposition.

Tine,Aiello,Belleni and Pestellini (1992) applied the SHM to find the solution of transfer equation in media with spherical symmetry but with radially varying parameters and anisotropic scattering.They considered the following form of radiative transfer equation

$$\mu \frac{\partial I(r,\mu)}{\partial r} + \frac{1-r^2}{r}\frac{\partial I(r,\mu)}{\partial \mu} = -\alpha(r) I(r,\mu) + \frac{\alpha^{sca}(r)}{2}\int_{-1}^{1} I(r,\mu')P(\mu,\mu')d\mu' +$$

$$+ \epsilon^*(r), \qquad (1.5.35a)$$

with the boundary condition $I(r,\mu) = I^{est}(\mu), \quad \mu \le 0$ (1.5.35b)

In equation (1.5.33a), α(r) and $\alpha^{sca}(r)$ are respectively the extinction and scattering coefficients for unit length and $P(\mu,\mu')$ is the phase function. In Eq. (1.5.33b), $I^{est}(\mu)$ is the externally incident radiation, if any. They considered the following form of the phase function

$$P(\mu,\mu') = \sum_{l=0}^{N} \sigma_l P_l(\mu) P_l(\mu')$$

and the form of the intensity is $I(r,\mu) = \sum_{l=0}^{L} (2l+1) F_l(r) P_l(\mu)$

They applied the technique to Henyey-Greenstein phase function and for this they took ω = g = .5 and τ = .1.

Biswas and Karanjai (1992) used a modified double interval SHM to solve the equation of radiative transfer with Rayleigh phase function with thin atmosphere. The following equation of transfer was considered

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{1}{2}\int_{-1}^{1} p(\mu,\mu') I(\tau,\mu') d\mu' \qquad (1.5.36)$$

The boundary conditions for their problem are

I) isotropic radiation field at $\tau = 0$ i.e.

$$I(0,\mu) = I_0 \quad \text{say} \quad 0 \le \mu \le 1$$

II) no incoming radiation at $\tau = \tau_0$, i.e.

$$I(\tau_0,\mu) = 0\,, \qquad -1 \le \mu \le 0$$

They considered the following forms of intensity

$$I_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1), \quad 0 \le \mu \le 1 \tag{1.5.37a}$$

and

$$I_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), \quad -1 \le \mu \le 0 \tag{1.5.37b}$$

where $\phi(\tau)$ is a function of τ only and the nature of this depends on the extent of the medium and the boundary conditions.

Talukdar and Karanjai (1992) applied a modified SHM to solve the equation of transfer with the general phase function. They considered the equation of transfer in plane parallel atmosphere with axial symmetry and given by

$$\mu\frac{\partial I(\tau,\mu)}{\partial \tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\mu(\tau,\mu')d\mu' \tag{1.5.38}$$

where the symbols have their usual meanings and phase function was taken to be

$$p(\mu,\mu') = \sum_{k=0}^{\infty} w_k P_k(\mu) P_k(\mu') \qquad (1.5.39)$$

The form of intensity taken by them was

$$I_+(\tau,\mu) = I(0,0)\left[A\tau + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1)\right], \quad 0 \le \mu \le 1 \qquad (1.5.40a)$$

$$I_-(\tau,\mu) = I(0,0)\left[A\tau + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1)\right], \quad -1 \le \mu \le 0 \qquad (1.5.40b)$$

Test computations were performed on Rayleigh and Henyey-Greenstein phase functions.

Tezcan and Yildtz (1993) expressed the P_L solutions for extremely anisotropic scattering in terms of solutions of isotropic scattering. To see the use of transformed solutions; applications for variation of the critical thickness with backward scattering are given.Numerical results are compared with the zeroth order results of elementary solutions and with the F_N method.

Siewert (1993a) used the SHM to develop solutions to a class of multi group on non-gray radiation transport problems. The multi group problem considered allows an anisotropic scattering law and transfer from any group to any group. In addition to spherical harmonics solutions for the case of a homogeneous radiative transfer equation, a particular solution for the P_N method is derived for the case of multi group radiative transfer in a homogeneous plane parallel medium that contains group sources that vary with position and direction. Computational aspects of the developed solutions are discussed and numerical results for a test case are reported.

1.6 Application of SHM to Heat Transfer (HT) Problems.

Le Sage (1965) considered the problem of the transport of thermal radiation through an absorbing medium between two parallel walls held at fixed different temperatures and applied the double SHM to this problem. The walls are assumed to radiate isotropically and to have an absorptivity of unity (i.e., black body). Le Sage considered the following integro-differential equation

$$\mu \frac{dI(\xi,\mu)}{d\xi} + I(\tau,\mu) = \frac{1}{2}\int_{-1}^{1} I(\xi,\mu')d\mu' \tag{1.6.1}$$

where ξ is the optical depth. By applying Yvon's (1957) method, the integro-differential equation is reduced to a set of ordinary differential equations. He used the following boundary conditions

$$\frac{\sigma T_0^4}{\pi} = I_0^+(0), \quad I_n^+(0) = 0 \tag{1.6.2a}$$

$$\frac{\sigma T_L^4}{\pi} = I_0^-(\xi_L), \quad I_n^-(\xi_L) = 0 \tag{1.6.2b}$$

where T = local temperature, σ = Stefan-Boltzmann constant. He obtained normalized flux in both double P_1 and double P_5 approximations.

Ou and Liou (1982) applied SHM to the basic three dimensional radiative transfer equation in terms of the three most commonly used coordinate systems (Cartesian, cylindrical and spherical). The finite expansion for both intensity and the phase function are inserted into the basic equation and a set of coupled partial differential equations are obtained by means of orthogonal property of spherical harmonics. Ou and Liou (1957) then formulated the first order approximations together with all possible boundary conditions assuming that the model medium considered is subject to internal emission only. The number of partial differential equations in this case is four, and a modified

Helmholtz equation is subsequently obtained. Examples of the computation based on the first order approximation for the three coordinate systems in one-dimensional space are then presented to demonstrate the practicality of the method.

Menguc and Viskanta (1983) examined critically the accuracy of the three methods, namely, the two flux, the spherical harmonic and the discrete ordinate methods for radiative transfer problems. They assumed the medium to be planar, participating and highly forward scattering. Comparisons were made for the results of the three methods with the F_N method. The equation of transfer for a plane layer of participating (absorbing, emitting and scattering) medium having azimuthal symmetry was taken as

$$\frac{\mu}{\epsilon_0}\frac{\partial \psi}{\partial \eta} + \psi(\eta,\mu) = (1-\omega)\beta(\eta) + \frac{\omega}{2}\int_{-1}^{1} p(\mu_0)\psi(\eta,\mu')d\mu' \qquad (1.6.3)$$

where ψ is normalized intensity with respect to I_0, the intensity of isotropic radiation field), η is the normalized distance, μ is the direction cosine, τ_0 is the optical depth of the layer, ω is the albedo for single scattering and β is the dimensionless emission (B/I_0). The boundary conditions were taken to be

$$\psi(\eta,\mu) = 1 \quad \text{at} \quad \eta = 0 \quad (\mu > 0) \qquad (1.6.4a)$$

$$\psi(\eta,-\mu) = 1 \quad \text{at} \quad \eta = 1 \quad (\mu > 0) \qquad (1.6.4b)$$

They considered the general phase function which is given below

$$p(\mu_0) = 1 + \sum_{m=1}^{\infty} a_m P_m(\mu_0) \qquad (1.6.5)$$

where a_m are expansion coefficients and P_m are the Legendre polynomials of the first kind. The scattering angle μ_0 which is the angle between the incoming ray and the scattered ray is connected to μ by the following relation.

$$\mu_0 = \mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}cos(\phi-\phi') \tag{1.6.6}$$

In order to apply the SHM they represented the radiation intensity by the following expansion

$$\psi(\eta,\mu) = \sum_{n=0}^{\infty}\frac{2n+1}{4\pi}P_n(\mu)\psi_n(\eta) \tag{1.6.7}$$

They evaluated radiative heat fluxes for each of the three methods and compared the results with the F_N method. In case of SHM they computed upto P_9 approximations i.e., nine terms of the phase function.

Benassi, Cotta and Siewert (1983) applied SHM to solve a certain problem in heat transfer. The appropriate equation of transfer is

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} + I(\tau,\mu) = \frac{\omega}{2}\sum_{l=0}^{L}\frac{2l+1}{2}\beta_l P_l(\mu)\int_{-1}^{1}P_l(\mu')I(\tau,\mu')d\mu' + (1-\omega)\frac{\sigma}{\pi}T^4(\tau) \tag{1.6.8}$$

with the boundary conditions

$$I(0,\mu) = \epsilon_1\frac{\sigma}{\pi}T_1^4 + \rho_1^s I(0,-\mu) + 2\rho_1^d\int_0^1 I(0,-\mu')\mu' d\mu', \ \mu>0 \tag{1.6.9a}$$

and

$$I(\tau_0,-\mu) = \epsilon_2\frac{\sigma}{\pi}T_2^4 + \rho_2^s I(\tau_0,\mu) + 2\rho_2^d\int_0^1 I(\tau_0,\mu')\mu' d\mu', \ \mu>0 \tag{1.6.9b}$$

where T_1 and T_2 refer to the two boundary temperatures, ρ_α^s and ρ_α^d, $\alpha = 1,2$ are the coefficients for specular and diffuse reflection and ϵ_1 , ϵ_2 are the emissivities. They used SHM to compute the partial heat fluxes in an anisotropically scattering plane parallel medium. Their intensity form is

$$I(\tau,\mu) = \sum_{l=0}^{N} \frac{2L+1}{2} P_l(\mu) \sum_{j=1}^{J} \left[A_j e^{-\tau/\xi_j} + (-l)^l B_j e^{-(\tau_0 - \tau)/\xi_j} \right] g_l(\xi_j) + I_p(\tau,\mu) \qquad (1.6.10)$$

where $I_p(\tau,\mu)$ denotes a particular solution to Eq. (1.6.8) corresponding to the inhomogeneous source term,

$$S(\tau) = (1 - \omega) \frac{\sigma}{\pi} T^4(\tau) \qquad (1.6.11)$$

The polynomials $g_l(\xi)$ are those of Chandrasekhar (1960)

Based on the work of Menguc and Viskanta (1983), Wan, Wilson and Sen (1986) used a modified double interval SHM for solving the radiative transfer equation in plane parallel finite medium scattering anisotropically. Following Menguc and Viskanta (1983), Wan, Wilson and Sen (1986) compared the results of their method with the F_N method which is considered to be standard. The transfer equation for their model is

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{\omega}{2} \int_{-1}^{1} p(\mu,\mu') I(\tau,\mu') d\mu' \qquad (1.6.12)$$

where the symbols have their usual meanings. The phase function is linearly anisotropic

$$p(\mu,\mu') = 1 + a\mu\mu' \qquad (1.6.13)$$

The boundary conditions for this problem are

I) isotropic radiation field I_0 at $\tau = 0$, i.e.,

$$I(0,\mu) = I_0 = 1 \text{ say } \quad 0 \le \mu \le 1 \tag{1.6.14a}$$

II) no incoming radiation at $\tau = \tau_0$, i.e.,

$$I(\tau_0,\mu) = 0 , \qquad -1 \le \mu \le 0 \tag{1.6.14b}$$

They considered the following forms of intensity

$$I_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1), \quad 0 \le \mu \le 1 \tag{1.6.15a}$$

and

$$I_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), \quad -1 \le \mu \le 0 \tag{1.6.15b}$$

where $\phi(\tau)$ is a function of τ only and they suggested that the form of $\phi(\tau)$ has to be chosen suitably based on the boundary conditions and the extent of the medium. They evaluated the heat fluxes at $\tau = 0$ and $\tau = \tau_0$ and compared their results with the F_N method and with Menguc and Viskanta (1983).

Using P_N approximations, Swathi, Tong and Cunnington (1987) obtained results on the behavior of the hemispherical reflectance when a two layer porous composite wall is irradiated by plane incident rays. A number of workers made important contributions to the subject of radiative transfer in composite or inhomogeneous materials. Based on this, the earlier studies can be classified into two categories, 1) those using isotropic scattering and 2) those using anisotropic scattering. Swathi, Tong and Cunnington (1987) used anisotropic scattering for the composite materials.However, they assumed that the anisotropy to be variable. This was done by using linear anisotropic phase function with a variable coefficient. The SHM was used to obtain the hemispherical reflectance.

The equation of transfer considered by them is

$$\mu \frac{\partial I_i}{\partial \tau} + I_i(\tau,\mu) = \frac{\omega_i}{2} \int_{-1}^{1} \phi_i(\mu,\mu') I_i(\tau,\mu') d\mu' \tag{1.6.16}$$

where I is the intensity normalized with respect to the incident intensity, ω, the single scattering albedo, τ, the optical depth, μ, the cosine of the angle between the direction of propagation, $\phi(\mu,\mu')$, the scattering phase function and subscript I denotes either layer 1 or layer 2 (i =1,2). They employed the following phase function

$$\phi_i(\mu,\mu') = 1 + 3 f_i \mu \mu' \tag{1.6.17}$$

where f_i is a coefficient bounded between 1/3 and - 1/3 and the boundary conditions are

$$I_1(0,\mu) = \begin{cases} 1, & \text{diffuse incidence} \\ \delta(\mu - 1), & \text{for } \mu > 0 \text{ (normal incidence)} \end{cases} \tag{1.6.18}$$

and the interface conditions are

$$I_1(\tau_1, -\mu) = I_2(\tau_1, -\mu), \quad \text{for } \mu > 0 \tag{1.6.19a}$$

$$I_1(\tau_1,\mu) = I_2(\tau_1,\mu), \quad \text{for } \mu > 0 \tag{1.6.19b}$$

The intensity form is

$$I_i(\tau,\mu) = \sum_{m=0}^{N} \frac{2m+1}{4\pi} P_m(\mu) \psi_{i,m}(\tau) \tag{1.6.20}$$

Here N is the order of the approximation. They have shown that P_N approximations converge to F_N approximations as N is increased and pointed out that the accuracy of the P_N method increases with increasing albedo.

Menguc and Iyer (1988) discussed various features in solving accurately the general solution of the radiative transfer equation in different geometries. They developed an approximate hybrid model to solve the radiative transfer equation in inhomogeneous,absorbing,emitting and anisotropically scattering, one dimensional plane parallel and two dimensional cylindrical media bounded by emitting and reflecting walls. The original idea for such an hybrid model was first given by Yvon (1957) and by Schiff and Ziering (1958,1960). The formulation of Menguc and Iyer (1988) differs from Schiff and Ziering because Schiff and Zeiring employed an eigen value method for solving the differential equation which is difficult to use and does not account for the anisotropic scattering. On the other hand, Menzuc and Iyer's formulation accounted for anisotropic scattering. First, they applied double SHM which is applied to a physical system in one dimensional plane parallel, absorbing and emitting and anisotropic scattering medium with azimuthal symmetry and the equation of transfer is

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = S(\tau,\mu) \tag{1.6.21}$$

Here, τ is the optical thickness and the source function $S(\tau,\mu)$ is defined by

$$S(\tau,\mu) = (1-\omega)I_b[T(\tau)] + \frac{\omega}{4\pi}\int_{\phi'=0}^{2\pi}\int_{\mu'=-1}^{1}\Phi(\tau,\mu')I(\tau,\mu,\mu')d\mu' d\phi' \tag{1.6.22}$$

They applied Marshak's boundary conditions and evaluated heat fluxes for different ω (albedo) and τ.

Next they applied octuple spherical harmonics approximation for the radiative transfer equation in asymmetrical cylindrical enclosures. The analysis in this case was almost the same as in the previous case and in this case also they evaluated the heat fluxes. However, they pointed out that the derivations of the equation governing the model were very tedious and this is the main drawback of their model. Also, the computer code they used is inefficient to employ for their calculations but they suggested that if the custom-made numerical code is used then both the speed and the accuracy of the solution would increase.

Li and Tong (1990) analyzed the radiative heat transfer in emitting,absorbing and scattering spherical gray isothermal media. They assumed the phase function to be linearly anisotropic and the medium was confined in the space between two concentric spheres which diffusely emit and specularly diffusely reflect radiation. They obtained the approximate solution of the equation of radiative transfer by using SHM. The governing equation for their model is

$$\mu \frac{\partial i(\tau,\mu)}{\partial \tau} + \frac{1-\mu^2}{r}\frac{\partial i(\tau,\mu)}{\partial \mu} + i(\tau,\mu) = \frac{(1-\omega)\sigma T^4}{\pi} + \frac{\omega}{2}\int_{-1}^{1} p(\mu,\mu')i(\tau,\mu')d\mu \quad (1.6.23)$$

where $I(\tau,\mu)$ is the radiant intensity, T is the temperature of the medium the medium is assumed to be gray and is characterized by absorption coefficient σ_a and scattering coefficient σ_b. They used linearly anisotropic phase function and the form of the intensity is

$$i(\tau,\mu) = \sum_{m=0}^{N} \frac{2m+1}{4\pi} P_m(\mu)\psi_m(\tau) \quad (1.6.24)$$

where $\psi_m(\tau)$ are functions of τ. However, they considered the approximation to be finite, .i.e., they used P_N th approximation where N is odd. The boundary conditions for inner and outer sphere are taken as

$$i(\tau_a,\mu) = \epsilon_a \frac{\sigma}{\pi} T_a^4 + \rho_a^s i(\tau_a,-\mu) + 2\rho_a^d \int_0^1 i(\tau_a,-\mu')\mu' d\mu' \,,\; \mu > 0 \quad (1.6.25a)$$

and

$$i(\tau_b,\mu) = \epsilon_b \frac{\sigma}{\pi} T_b^4 + \rho_b^s i(\tau_a,-\mu) + 2\rho_b^d \int_0^1 i(\tau_b,-\mu')\mu' d\mu' , \; \mu > 0 \qquad (1.6.25b)$$

They obtained P_1 approximations analytically and higher approximations P_3, P_5, P_7, P_9 and P_{11} are obtained numerically.

The P_N method has been used by Siewert and Thomas (1990,1991) to obtain a particular solution for solving radiative heat transfer problems in 1) plane geometry and 2) geometry with spherical symmetry. Siewert and Thomas (1990) considered the equation of transfer in plane geometry as

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{\omega}{2} \sum_{l=0}^{L} \frac{2l+1}{2} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau,\mu') d\mu' + (1-\omega) \frac{\sigma}{\pi} T^4(\tau) \qquad (1.6.26)$$

for $\tau \in (0,\tau_0)$, $\mu \in [-1,1]$ and the boundary conditions are

$$I(0,\mu) = \epsilon_1 \frac{\sigma}{\pi} T_1^4 + \rho_1^s I(0,-\mu) + 2\rho_1^d \int_0^1 I(0,-\mu')\mu' d\mu' , \; \mu > 0 \qquad (1.6.27a)$$

and

$$I(\tau_0,-\mu) = \epsilon_2 \frac{\sigma}{\pi} T_2^4 + \rho_2^s I(\tau_0,\mu) + 2\rho_2^d \int_0^1 I(\tau_0,\mu')\mu' d\mu' , \; \mu > 0 \qquad (1.6.27b)$$

where T_1 and T_2 refer to two boundary temperatures, ρ_α^s and ρ_α^d , $\alpha = 1$ and 2 are the coefficients for specular and diffuse reflection and ϵ_1, ϵ_1 are the emissivities.

In a geometry with spherical symmetry Siewert and Thomas (1991) considered the equation

$$\mu \frac{\partial I(r,\mu)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I(r,\mu)}{\partial \mu} + I(r,\mu) = \frac{\omega}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(r,\mu) d\mu' + S(r) \qquad (1.6.28)$$

for $r \in (R_1, R_2)$, $\mu \in (-1,1)$ and the boundary conditions are

$$I(R_1,\mu) = \epsilon_1 \frac{\sigma_n}{\pi} T_1^4 + \rho_1^s I(R_1,-\mu) + 2\rho_1^d \int_0^1 I(R_1,-\mu')\mu' d\mu' \qquad (1.6.29a)$$

and

$$I(R_2,-\mu) = \epsilon_2 \frac{\sigma_n}{\pi} T_2^4 + \rho_2^s I(R_2,\mu) + 2\rho_2^d \int_0^1 I(R_2,\mu')\mu' d\mu' \qquad (1.6.29b)$$

for $\mu \in [0,1]$.

They took

$$S(r) = (1-\omega)\frac{\sigma n^2}{\pi} T^4(r) \qquad (1.6.30)$$

where the inhomogeneous source term, β_l define the scattering law, $r \in (R_1, R_2)$, is the optical variable, μ is the direction cosine measured from the r - axis, ω is the albedo for single scattering. T_1 and T_2 refer to two boundary temperatures, ϵ_1, ϵ_1 are the emissivities and ρ_α^s and ρ_α^d, $\alpha = 1$ and 2 are the coefficients of specular and diffuse reflection. Also n is the index of refraction and σ is the Stefan-Boltzmann constant. The temperature distribution is considered to be specified and that $T(R_1) = T_1$ and $T(R_2) = T_2$.

Following Davison (1957) and Aronson (1984a,b) the form of the intensity considered is

$$I(r,\mu) = \sum_{l=0}^{N} \frac{2L+1}{2} P_l(\mu) \sum_{j=1}^{J} \left[A_j K_l(r/\xi_j) + (-l)^l B_j i_l(r/\xi_j) \right] g_l(\xi_j) + I_p(r,\mu) \qquad (1.6.31)$$

where $I_p(r,\mu)$ is the particular solution to the Eq. (1.6.28) corresponding to the inhomogeneous source term S(r). $g_l(\xi_j)$ are Chandrasekhar polynomials. The P_N eigenvalues are given by ξ_j, j=1,2,....J=(N+1)/2. The modified spherical Bessel functions of the first kind and the third kind are denoted by

$$i_l(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} I_{l+1/2}(z), \quad k_l(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} K_{l+1/2}(z).$$

The arbitrary constants $\{A_j\}, \{B_j\}$ are determined by using the boundary conditions.

Siewert (1993) discussed a post processing technique which is used with the SHM to develop accurate results for the calculation of radiative intensity in a homogeneous plane parallel medium. He assumed that the medium contains a source that varies with position and considered anisotropic scattering. The equation of transfer considered is given below.

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{\omega}{2}\sum_{l=0}^{L}\frac{2l+1}{2}\beta_l P_l(\mu)\int_{-1}^{1} P_l(\mu')I(\tau,\mu')d\mu' + S(\tau) \tag{1.6.32}$$

for $\tau \in (0,\tau_0)$ and $\mu \in [-1,1]$. The boundary conditions are

$$I(0,\mu) = F_1(\mu) + \rho_1^s I(0,-\mu) + 2\rho_1^d \int_0^1 I(0,-\mu)\mu d\mu \tag{1.6.33a}$$

and

$$I(\tau_0,-\mu) = F_2(\mu) + \rho_2^s I(\tau_0,\mu) + 2\rho_2^d \int_0^1 I(\tau_0,\mu)\mu d\mu \tag{1.6.33b}$$

for $\mu \in$ (-1,0]. Here the symbols β_l, ρ_α^s, ρ_α^d, $I(\tau,\mu)$, τ, μ, $S(\tau)$ have been described in Siewert and Thomas (1990,1991). They assumed that the functions $F_1(\mu)$, $F_2(\mu)$ and the inhomogeneous source term are assumed to be given. He has taken, as in Siewert and Thomas (1991) the following solution to Eq. (1.6.32)

$$I(\tau,\mu) = \sum_{l=0}^{N} \frac{2L+1}{2} P_l(\mu) \sum_{j=1}^{J} \left[A_j e^{-\tau/\xi_j} + (-l)^l B_j e^{-(\tau_0 - \tau)/\xi_j} \right] g_l(\xi_j) + I_p(\tau,\mu) \qquad (1.6.34)$$

where the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are determined from the boundary conditions.

Karanjai and Biswas (1992b) applied a modified SHM to solve the radiative transfer equation. They considered a plane parallel finite medium with isotropic radiation field. The equation of transfer is

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{1}{2} \int_{-1}^{1} p(\mu,\mu') I(\tau,\mu') d\mu' \qquad (1.6.35)$$

and the phase function is

$$p(\mu,\mu') = 1 + \omega_1 P_1(\mu) P_1(\mu') + \omega_2 P_2(\mu) P_2(\mu')$$

They have taken the following forms of intensity

$$I_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0} (2l+1) I_l^+(\tau) \mu P_l(2\mu - 1), \quad 0 \le \mu \le 1 \qquad (1.6.36a)$$

and

$$I_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0} (2l+1) I_l^-(\tau) \mu P_l(2\mu + 1), \quad -1 \le \mu \le 0 \qquad (1.6.36b)$$

They evaluated various constants by applying these forms and used the phase functions like Rayleigh and isotropic.

1.7 Application of SHM to Neutron Transport and Other Problem.

The SHM for neutron transport problems was first applied by Wick (1943) and Marshak (1947) and developed in detail by Mark (1944,1945). Davison and Sykes (1958) discussed the application of SHM in both plane and spherical geometry. In case plane geometry, Davison and Sykes (1958) assumed that the neutron flux is a function of Cartesian coordinate (x say) only and consequently the angular distribution depends only on x. They considered the constant cross-section isotropically scattering Boltzmann integro-differential equation

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \frac{\psi(x,\mu)}{l} = \frac{c}{2l} \int_{-1}^{1} \psi(x,\mu') d\mu' \qquad (1.7.1)$$

where $\psi(x,\mu)$ is the angular distribution of the neutrons and l is the mean free path. They expanded $\psi(x,\mu)$ into spherical harmonics in μ so that

$$\psi(x,\mu) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) P_n(\mu) \psi_n(x) \qquad (1.7.2)$$

where $P_n(\mu)$ are Legendre polynomials. Further, they assumed that

$$\psi_n(x) = 2\pi \int_{-1}^{1} \psi(x,\mu) P_n(\mu) d\mu \qquad (1.7.3)$$

Multiplying (1.7.1) by $(2n+1)P_n(\mu)$ and integrating over all μ, using the recurrence formula for Legendre polynomials one obtains

$$(n+1)\psi'_{n-1}(x) + n\psi'_{n-1}(x) + (2n+1)\frac{1 - c\delta_{0,n}}{l}\psi'_n(x) = 0 \qquad (1.7.4)$$

where dashes denote differentiation with respect to x. The quantities $\psi_n(x)$ are called spherical harmonic moments of the angular distribution. It is easily seen that first two moments $\psi_0(x)$ and $\psi_1(x)$ are identical with the flux $\rho(x)$ and the current density *j(x)* respectively. Equations (1.7.4) are an infinite system of differential equations with an infinite number of unknowns. In practice, however, only a finite number N,say is taken. Davison and Sykes (1958) assumed a trial solution of the form

$$\psi_n(x) = g_n e^{\nu x/l}, \quad n=0,1,2,......,N \tag{1.7.5}$$

g_n are some constants and obtained the solution

$$\psi_n(x) = \sum_j \hat{A}_j g_n(\nu_j) e^{\nu_j x/l}, \quad n=0,1,2,.....,N \tag{1.7.6}$$

where $\hat{A}_j$ are arbitrary constants which are determined by using appropriate boundary conditions. Davison and Sykes (1958) introduced a set of auxiliary functions $G_n(\nu)$ defined by the following equations

$$G_n(\nu) = (-1)^n \left\{ P_n\left(\frac{1}{\nu}\right) - \frac{c}{\nu}\left[Q_0\left(\frac{1}{\nu}\right) P_n\left(\frac{1}{\nu}\right) - Q_n\left(\frac{1}{\nu}\right)\right]\right\} \tag{1.7.7}$$

where P_n are Legendre polynomials and Q_n are Legendre functions of the second kind. With the help of (1.7.7) Davison and Sykes rewrote (1.7.6) as

$$\psi_n(x) = \sum_j \hat{A}_j G_n(\nu_j) e^{\nu_j x/l}, \quad n=0,1,2,.....,N \tag{1.7.8}$$

The boundary conditions. 1. *Conditions at an interface between media.*

For a stationary problem in plane geometry with constant cross-section this is given by

$\psi(x,\mu)$ is a continuous function of x for any μ except (possibly) $\mu = 0$ (1.7.9a) or equivalently,

$$\psi_n(x) \text{ is a continuous for } n = 0,1,\ldots\ldots. N \qquad (1.7.9b)$$

Davison and Sykes also stated that odd order approximations are superior to even order approximations.

2. *Conditions at a free surface.*

In this case Davison and Sykes considered the free surface at $x = 0$ and the medium occupies the space $x > 0$. The exact boundary conditions are

$$\psi(0,\mu) = 0 \quad \text{for} \qquad \mu > 0 \qquad (1.7.10)$$

This,however,constitutes an infinite number of conditions which cannot all be exactly satisfied in an approximation of finite order. In P_n approximations, only (N+1)/2 conditions are satisfied. Davison and Sykes proposed that alternative Mark's (1945) or Marshak's boundary conditions should be used. Mark's boundary conditions for this problem are

$$\psi(0,\mu_j) = \ , \ j = 1,2,\ldots\ldots.., \frac{N+1}{2}. \quad \mu_j > 0 \qquad (1.7.11a)$$

or equivalently,

$$P_{n+1}(\mu_j) = 0, \quad \mu_j > 0 \qquad (1.7.11b)$$

and Marshak's boundary conditions are

$$\int_0^1 \psi(0,\mu)\mu^{2j-1} d\mu = 0, \quad j = 1,2,\ldots.., \frac{N+1}{2} \qquad (1.7.12)$$

Davison and Sykes stated that in a low order approximation, Marshak's boundary conditions give better results, but the accuracy would increase faster in higher approximations when Mark's conditions are used.

3. *Conditions at a surface exposed to the neutrons*

The medium is taken as $x > 0$ with the surface $x = 0$ exposed, the exact bounadry conditions are

$$\psi(0,\mu) = F(\mu), \text{ for } \mu > 0 \tag{1.7.13}$$

where $F(\mu)$ is some known function. For this problem, Davison and Sykes suggested that a generalized Mark's or Marshak's boundary conditions should be used.

In case of spherical geometry, the corresponding Boltzmann equation for neutron transport is [Davison and Sykes (1958)]

$$\mu\frac{\partial\psi(r,\mu)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial\psi(r,\mu)}{\partial\mu} + \frac{\psi(r,\mu)}{l} = \frac{c}{4\pi}\int\int\psi(r,\mu')d\Omega' \tag{1.7.14}$$

where Davison and Sykes assumed spherically symmetrical system, in which neutron flux depends only on radial component.

Writing $$\psi(r,\mu) = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)P_n(\mu)\psi_n(\mu) \tag{1.7.15}$$

and proceeding in the same manner as in the plane case the following system of equations is obtained

$$(n+1)\left[\frac{d}{dr} + \frac{n+2}{r}\right]\psi_{n+1}(r) + n\left[\frac{d}{dr} - \frac{n-1}{r}\right]\psi_{n-1}(r) + (2n+1)\frac{1-c\delta_{0,n}}{l}\psi_n(r) \tag{1.7.16}$$

In the P_n approximations, this infinite system of equations is converted into a finite one by neglecting $\psi_{N+1}(r)$ and retaining only the first N+1 equations of (1.7.16). It is easily verified that the resulting system of equations is satisfied if

$$\psi_n(r) = G_n(\nu_j)\frac{1}{\nu_j}K_{n+\frac{1}{2}}(-\nu_j r/l) \tag{1.7.17}$$

where $K_s(z)$ is the modified Bessel function of the second kind, $G_n(\nu)$ are defined by (1.7.7) The general solution of (1.7.16) in the P_n approximations can be written as

$$\psi_n(r) = \sum_j A_j \nu_j G_n(\nu_j)\left(\frac{2l}{\pi \nu_j r}\right)^{\frac{1}{2}} K_{n+\frac{1}{2}}(-\nu_j r/l) \tag{1.7.18}$$

Putting (1.7.18) in (1.7.15) the angular distribution in the P_n approximations can easily be obtained.

The functions $G_n(\nu)$ figure in the solutions for both the plane and the spherical case; the reasons for this have been given by Davison and Sykes (1958)

The boundary conditions.

1. *Conditions at the origin*

Apart from the boundary conditions, Davison and Sykes also prescribed a condition at the origin

$$\psi_n(0) \text{ is finite } \quad (n=0,1,\dots,N) \tag{1.7.19}$$

2. *Conditions at infinity.*

In this case the boundary conditions will be the same as in the plane case except that a supply of neutrons from infinity is now understood in the sense of total supply rather than supply per unit area.

3. *Conditions at an interface.*

The conditions at the interface between two media in direct contact are purely local conditions and will not depend on the geometry. They are therefore, the same as in the plane case.

This also applies to Marshak's condition at the free surface, but the situation with respect to Mark's condition is more complicated.

4. *Conditions at the surface of a gap.*

In the case of plane symmetry, gaps have no effects, but in the spherical case,they must be taken into account. Davison and Sykes considered them in two ways. First, the gap is treated as a medium where the mean free path is finite, and then the interface conditions are applied. Second,the surfaces of the gap may be considered directly, and then the exposed surface conditions are applied.

Kofink (1958) made thorough analysis of the SHM in connection neutron transport problem. Kofink [part I,1958] discussed the relation between P_n method (i.e. the SHM) and Gauss quadrature method for the solution of Milne problem. It is often pointed out that both SHM and Gauss quadrature method dealing with monoenergetic transport equation are closely related, but they are not identical; for the later as applied to the Milne problem is a non-analytic approximation, whereas all functions used in the SHM are continuous. Kofink (1958) defined a general solution to Milne problem and gave comparisons with Gauss quadrature solution. He considered the appropriate Boltzmann transport equation of the form

$$\mu \frac{\partial f(\zeta,\mu)}{\partial \zeta} + f(\zeta,\mu) = \frac{1}{2}(1 - \nu_0)\int_{-1}^{1} f(\zeta,\mu')d\mu' + \frac{3}{2}(1 - \nu_1)\mu \int_{-1}^{1} f(\zeta,\mu')\mu' d\mu' \tag{1.7.20}$$

where $\mu = \cos\vartheta$, ϑ is the angle between the positive z-axis and the direction of the directed flux $f(\zeta,\mu)$.The other notations are described below

$$c = 1 - \nu_0 = \frac{\sum_s + \nu\sum}{\sum}, \quad \nu_0 = \frac{\sum_a - \nu\sum_f}{\sum}$$

with $\sum = \sum_s + \sum a$, $\quad \sum_a = \sum_c + \sum_f$

where $\sum_s, \sum_a, \sum_f, \sum_c$ are respectively macroscopic cross-sections for scattering,absorption,fission and capture. Further

$\zeta = \sum z = \frac{z}{l_0}$ = distance in units of the mean free path l_0 and $\sum = \frac{1}{l_0}$ =total cross-section..

In order to apply SHM, he assumed

$$f(\zeta,\mu) = \sum_{l=0}^{\infty} f_{l0}(\zeta) P_l(\mu) \tag{1.7.21}$$

Inserting Eq. (1.7.21) in (1.7.20) an infinite systems of differential equations is obtained. He assumed a trial solution of the form

$$f_{l0}(\zeta) = (2l+1) g_l(\lambda) \exp[-\zeta/\lambda]$$

As usual, the summation appearing (1.7.21),for all practical purposes must be a terminating one, i.e. a finite number N say.Kofink reported that the number of P_L approximations depends upon the parameter c.He showed that this number L_1 say increases very quickly for small,for example when $c = 1/3$ P_{11} approximations are required and for c = 1/4 one needs P_{29} approximations.

For an illustration, Kofink has chosen Milne problem with anisotropic scattering, the scattering law is given by

$$W(cos\Theta) = \frac{1}{4\pi}\left[1 + 3s_1 cos\Theta\right]$$

and approximate partial solution by SHM (or the P_L) is given by

$$f^L(\zeta,\mu;\lambda_k) = e^{-\zeta/\lambda_k} \sum_{l=0}^{L} (2l+1) g_l(\lambda_k) P_l(\mu) \tag{1.7.22}$$

where λ_k are L+1 eigenvalues. Kofink assumed that the left half space $\zeta \leq 0$ is the medium and the right half space $\zeta \geq 0$ is vacuum.

Zeiring and Schiff (1958) applied the method of half range polynomials to neutron transport theory. Their problem consists of monoenergetic neutrons in an elastic, isotropically scattering medium. They extended Yvon's method to the one group neutron transport equation applied the same to the problem of semi-infinite and finite slab and derived the general solution for the isotropic scattering in the N-th approximation by Yvon's method. They considered the transport equation for one group theory with isotropic scattering in plane geometry as

$$\mu \frac{d\phi(\mu,x)}{dx} + \frac{\phi(\mu,x)}{l} = \frac{C}{2l}\int_{-1}^{1}\phi(\mu',x)d\mu' + S(\mu,x) \qquad (1.7.23)$$

The neutron flux $\phi(x)$ is expanded as

$$\phi^{\pm}(\mu,x) = \sum_{n}(2n+1)B_n^{\pm}(x)P_n^{\pm}(\mu)$$

where $B_n^{\pm}(x)$ are expansion coefficients and

$$P_n^{+}(\mu) = P_n(2\mu - 1),\ 0 \leq \mu \leq 1,\ P_n^{-}(\mu) = P_n(2\mu + 1),\ -1 \leq \mu \leq 0$$

Zeiring and Schiff (1958) obtained results for $P_0^{\pm}$ and $P_1^{\pm}$ approximations in case of semi-infinite slab and $P_0^{\pm}, P_1^{\pm}, P_2^{\pm}$ approximations for finite slab. In each case, they have used the appropriate boundary conditions. The results have been compared with the exact solution of the classical Milne problem for an isotropic scattering medium.

Wilson and Sen in their earlier papers [Wilson and Sen (1963, 1964a,1964b)] suggested a modification of the double interval SHM and applied that to solve certain transfer problems in plane and spherical geometries. Wilson and Sen (1965a) applied the same modification of the SHM to solve the problem of neutron transport with a finite spherical core. They considered an infinite source free noncapturing medium surrounding a sphere of radius a which absorbs all neutrons falling on it, they assumed a current density in the direction - r (r being measured from the center of the black core) in the medium which scatters neutrons isotropically. The appropriate transfer equation considered by them is

$$\mu \frac{\partial \Psi(r,\mu)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial \Psi(r,\mu)}{\partial \mu} + \Psi(r,\mu) = \frac{1}{2} \int_{-1}^{1} \Psi(r,\mu) d\mu \qquad (1.7.24)$$

where r is the distance measured outward from the center of the spherical core and μ is the cosine of the angle measured from the positive direction of the radius vector. $\Psi(r,\mu)$ is the angular distribution of the neutrons at a distance r in the direction $cos^{-1}(\mu)$. The neutron density is defined by

$$J(r) = \int_{-1}^{1} \Psi(r,\mu) d\mu$$

Following Wilson and Sen (1963) they represented $\Psi(r,\mu)$ by two different expansions,viz,

$$\Psi_{+}(r,\mu) = A(r) + \sum_{l=0}^{l_0} (2l+1)\mu \Psi_l^{+}(r) P_l(2\mu - 1)\,, \quad 0 \le \mu \le 1 \qquad (1.7.25a)$$

and

$$\Psi_{-}(r,\mu) = A(r) + \sum_{l=0}^{l_0} (2l+1)\mu \Psi_l^{-}(r) P_l(2\mu + 1)\,, \quad -1 \le \mu \le 0 \qquad (1.7.25b)$$

where $A(r)$ is a function of r only. The equation (1.7.23) was solved with the following boundary conditions

$$\Psi(a,\mu) = 0, \quad \text{for} \quad 0 \le \mu \le 1 \tag{1.7.26}$$

where a is the radius of the finite spherical core. They evaluated $J(r)$ for the P_1 approximation and compared the results with those of Marshak (1947).

Kobayashi (1985) deduced a solution of the neutron transport equation in multi dimension of the SHM by applying the finite Fourier transformation. They derived the spherical harmonics equations by making use of a quadrature formula related to the associated Legendre functions. He considered the neutron equation as

$$\Omega\nabla f(r,\Omega) + f(r,\Omega) = \frac{c}{4\pi}\int_{4\pi} f(r,\Omega')d\mu' + \frac{1}{4\pi}S(r) \tag{1.7.27}$$

where Ω is the unit vector in the direction of the neutron and c is the number of secondary collisions. The function $f(r,\Omega)$ and $S(r)$ are the angular flux and source respectively. He assumed that the angular distribution of the scattering and the external source are isotropic. He also showed that the characteristic roots of the spherical harmonics equations for the higher components are the roots of the associated Legendre functions.

Matasuek and Milosevic (1986) presented a generalization of a procedure to solve multi group spherical harmonics equation for two dimensional system in r-z geometry. They derived the expressions for the axial and the radial dependence of the group values of neutron flux moments. They applied a combined analytical-numerical technique in r-z geometry. They considered a system of J (j=1,2,...J) transport equations and by the application of SHM the systems of J equations were reduced to an infinite set of partial differential equations. They applied numerical techniques to solve these partial differential equations.

Kobayashi,Oigawa and Yamagata (1986) applied the SHM for the solution of multigroup neutron transport equation in x-y geometry. They derived the spherical harmonics equations which are second order differential equations. They considered the multigroup transport equation in the form [Vide Kaplan and Davis (1967)] and deduced necessary spherical harmonics equations. From these spherical harmonics equations a set of finite difference equations is derived which can be solved iteratively from the lower order of angular moment to the highest order moment. They evaluated total flux for P_1, P_3, P_5,P_7 approximations and compared their results with that Fletcher (1981) and Gelbard and Crawford (1972). They discussed the rate of convergence of this method for the P_3 approximation and finally suggested that the code that was used for this method could be improved upon and pointed out two main drawbacks of it.

Kamiuto and Seki (1987) appliedP_1 approximation to an inverse scattering problem. They reported that if only a limited number of radiative properties,such as albedo and asymmetry factor are known then the inverse scattering problem can be solved by P_1 or higher approximation. They fundamental equation considered by them is

$$\mu \frac{dI(\eta,\mu)}{d\eta} + \tau_0 I(\eta,\mu) = \frac{\tau_0 \omega}{2}\int_{-1}^{1} P(\mu,\mu')I(\eta,\mu')d\mu' \qquad (1.7.28)$$

where τ_0 is the optical thickness of the medium,ω the albedo and $P(\mu,\mu')$ the azimuthally averaged phase function given by the following Henyey-Greenstein phase function.

$$P(\mu,\mu') = \sum_{n=0}^{\infty}(2n+1)\hat{g}^{n}P_n(\mu)P_n(\mu') \qquad (1.7.29)$$

where $\hat{g}$ is the asymmetry factor. The boundary conditions are

$$\left.\begin{aligned} I(0,\mu>0) &= \frac{1}{\pi} \\ I(1,\mu<0) &= 0 \end{aligned}\right\} \qquad (1.7.30)$$

In order to determine the spherical harmonics equations, Kamiuto and Seki (1987) expanded the intensity by the following form

$$I(\eta,\mu) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n(\mu) P_n(\mu') \qquad (1.7.31)$$

Substituting (1.7.30) in (1.7.27) and using the orthogonal properties of Legendre polynomials, the spherical harmonics equations (of the lowest order) are obtained.Next, using Marshak's (1947) boundary conditions for these spherical harmonics equations, Kamiuto and Seki obtained, at P_1 approximation, the hemispherical transmittance which is given by

$$T(\tau_0) = 4\sqrt{A^*} \Big/ \left[(7 - 4\omega - 3\omega\hat{g}) \sinh \tau_0 \sqrt{A^*} + 4\sqrt{A^*} \cosh \tau_0 \sqrt{A^*} \right] \; for \;\; \omega \neq 1 \qquad (1.7.32a)$$

$$= 1 \Big/ \left[3(1 - \hat{g})\tau_0 / 4 + 1 \right], \;\; for \;\; \omega = 1 \qquad (1.7.32b)$$

where $A^* = 3(1 - \omega)(1 - \omega\hat{g})$

The right-hand sides of the equations (1.7.31) have two parameters $\hat{g}$ and ω. Kamiuto and Seki (1987) determined these parameters by applying three methods i) Asymptotic expansion method , ii) unconstrained least square method, iii) constrained least square method. They varied $\hat{g}$ for - .85 ,0 and .85 and ω for .1 ,.3, .5, .7, .9, 1. Once these parameters are evaluated, the hemispherical transmittance can be determined and the equation of transfer (1.7.28) can be solved by Barkstrom's

(1976) method.

Siewert (1993c) applied SHM to develop a solution to an inverse source problem in radiative transfer. It is assumed that, with the exception of the inhomogeneous source term, all aspects of radiation-transport problems are known, and Siewert (193) determined the inhomogeneous source term from specified angular distribution of radiation existing the two surfaces of a homogeneous plane-parallel medium. Ansiotropic radiative transfer model and general reflecting boundary conditions are considered.