Chapter 1

Introduction

" There are more things in Heaven and Earth, Horatio, than are dreamt of in your philosophy. " —– William Shakespeare, Hamlet.

In 1915 Einstein proposed the general theory of relativity (GR) which revolutionized the theoretical understanding of the universe [2]. In 1917, Einstein applied his theory to obtain the model of the universe which initiated the study of relativistic cosmology [3]. As the astronomical observations during that time predicted a static universe, Einstein wanted to get a static model but failed to accommodate a static universe with his theory. In order to get a static universe, Einstein introduced a cosmological constant (Λ) which provides the repulsive force to obtain a stable static universe which is called Einstein's static universe (ESU). However, in 1922 Friedmann obtained relativistic solutions which describe an expanding universe without a cosmological constant [4]. But the cosmological solution was unattended and remained of academic interest for a long time. Later Lamaitre used the relativistic solution to formulate the Big Bang cosmological model [5]. However, in the year 1929 Hubble discovered that the spectral lines emitted from remote galaxies appear systematically shifted towards the red end of the spectrum indicating an expansion of the universe [6]. Hubble showed that galaxies recede in all directions and more distant ones recede more rapidly proportional to the distance between them. Hubble's discovery led Einstein to abandon the cosmological constant term (Λ), which he introduced to construct a static universe. Hubble's discovery indicates that the universe is expanding. If one extrapolates this cosmic expansion backward in time following the known laws of physics then the universe can be traced back to an extremely hot and dense state [7]. It led to a singularity, confirming initial explosion that occurred approximately 13.8 billion years ago called Big Bang. After the Big Bang, the universe began to expand and cooled sufficiently allowing nucleosynthesis, that led to the formation of stars and galaxies as we observe today. With the advent of modern science, cosmology is transforming from a speculative science to an experimental science. Modern cosmology relies on observational measurements to put constraints on the theoretical models leading to a better understanding of the universe.

1.1 The Big Bang cosmology

Penzias and Wilson [8], at the Bell Laboratories in New Jersey, while calibrating a sensitive microwave antenna designed for satellite communications found an unexpected “noise” with the antenna which is isotropic. Meanwhile, physicists at nearby Princeton University were busy calculating the expected characteristics of the radiation left over from the heat of the Big Bang [9]. It was expected that, there should be faint radiation permeating the entire universe. The “noise” detected at the Bell labs antenna was not an instrumental error instead it was the relic of the cosmic microwave background radiation (CMBR). The detection of CMBR provides the first strong evidence in favor of the Big Bang. In the early 1990s, the experiments conducted by Smoot and Mather with the NASA satellite Cosmic Background Explorer (COBE) Differential Microwave Radiometers (DMR) predicted that CMBR has a perfect thermal radiation spectrum with a peak at 2.73K [10, 11]. COBE measurements indicate that the temperature spectrum is not exactly uniform, instead there are temperature fluctuations embedded in it [12, 13]. Recently, NASA’s Wilkinson Microwave Anisotropy Probe (WMAP) provided a better picture of the small temperature variations by mapping the CMBR [14]. The tiny quantum fluctuations indicate that the density of the early universe indeed differed from place to place and they acted as the seeds for the structure formation in the universe. The Big Bang theory also predicts that the chemical composition of the universe indicates 25% Helium abundance.
Despite being fairly successful in predicting some of the observed features of the universe, Big Bang model suffers from some problems. The main drawbacks of the Big Bang are the singularity problem, the horizon problem, the small-scale inhomogeneity problem, and the flatness problem which finds no solution in the framework of the perfect fluid assumption [13, 15–19]. The present laws of physics cannot explain what happened during the Planck era (i.e., from 0 to 10^{-43}s). A flat universe which is pointed out by cosmological observations is permitted when the energy density becomes equal to the critical density.

1.2 Inflationary Universe

The concept of inflation in the early universe was introduced to get rid of the problems of the standard Big Bang model. It was Guth [20] and Sato [21, 22] who first proposed the temperature dependent phase transition independently. Although the inflationary universe solution by Starobinsky [23] was known, it is only after the seminal work of Guth [20] and Sato [21, 22] that the efficacy of inflation was understood. This is known as "old inflation". It corresponds to an exponential expansion realized with the first-order phase transition of the trapped universe which transits from a false vacuum phase to a true vacuum. A small causally coherent region thereafter grown rapidly to encompass the whole universe. The model suffers from a serious shortcoming where there is no graceful exit from the inflationary epoch [24]. Later Linde [25], and Albrecht and Steinhardt [26] in 1982 proposed a new version of the inflationary universe which is known as "new inflation". In this case, inflationary scenario is implemented due to the slow roll mechanism of the scalar field at the second-order phase transition. Although the model successfully resolved the issues of old inflation, soon it is found that the new inflation suffers from the issues of fine-tuning, spending enough time in the false vacuum leading to a sufficient amount of inflation. Subsequently, Linde in 1983 proposed an acceptable model of inflation as a variant of the slow-roll inflation called the "chaotic inflation" [27]. In the chaotic inflationary scenario, the universe emerged from a chaotic distribution of initial scalar field (ϕ) [28]. A scalar field $\phi > 3M_P$ to begin with and $V(\phi) \leq M_P^4$, where M_P is the Planck mass, lead to a universe which is physically acceptable. This upper bound of the scalar potential is required to avoid the quantum gravity regime and thus the chaotic inflation can originate close to the Planck time scale thereby solving the problem of initial conditions. In the literature a huge number of inflationary models are proposed in the last 40 years [16, 17, 29–35]. Inflation generates density perturbations which act as the seed for the large scale structures of the universe [36, 37]. Cosmic inflation is found successful in predicting a causal mechanism which generates a nearly scale-invariant spectra of cosmological perturbations. The tiny fluctuations predicted by cosmic inflation are detected recently by COBE and WMAP in the CMBR spectra [11, 38–40]. With the help of quantum field theory (QFT) and particle physics, modern cosmologists are trying to figure out how the minuscule quantum fluctuations evolved in the early universe, eventually becoming relevant is the cosmic scale and detected by the observational experiments namely, COBE, WMAP, PLANCK, etc. Modern precision cosmology is evolving with each passing day and several observational probes over the past decade namely, BOOMERANG, MAXIMA, PLANCK successfully confirmed many of the theoretical predictions. More recent experiments like James Webb Space Telescope (JWST), Primordial Inflation Polarization Explorer (PIPER), Galaxy Evolution Explorer (GALEX), two degree field (2dF), Sloan Digital Sky Survey (SDSS) among others have also confirmed the prediction of Big Bang cosmology with great accuracy. In recent times high precision observational cosmological missions opened up new possibilities to test the theoretical models of cosmology as well as

particle physics and quantum field theory.
The supernova cosmology project by Perlmutter *et.al.* [41], Schmidt *et.al.* [42] and Riess *et.al.* [43, 44] in the late 1990s predicted that our universe is not only expanding but accelerating. Consequently, our understanding of the late universe changed with this ground breaking discovery by Perlmutter, Schmidt and Riess who were awarded the Nobel prize in the year 2011. Cosmologists believe that the accelerated phase of the universe began after the matter-dominated phase. The standard model of cosmology based on GR alone cannot provide a satisfactory explanation for the late accelerating expansion. To accommodate the present acceleration a modification of the Einstein field equations by either modifying the matter sector or the geometrical sector of the equation is important. Ever since its first appearance, the cosmological constant (Λ) played a somewhat controversial role. Einstein first introduced the cosmological constant term to obtain a static universe, however, after Hubble's discovery, this term was removed. Cosmologists revived the idea of cosmological constant once again to set up the issues in the standard Big Bang model which can be solved using a non-zero cosmological constant. Later it was found that the Λ leads to a cosmological fine-tuning problem [45–48]. Consequently, a phase transition mechanism was introduced instead of a constant Λ [49, 50]. At present, one crucial question is still unsettled, what value of the cosmological constant will be a good fit with the observed late universe to accommodate the accelerated expansion. It is known at present that the Λ required in the early and late universe are quite different.

1.3 Gravitational Action

The gravitational action from which Einstein's field equation (EFE) can be derived is the Einstein-Hilbert action [51]:

$$S=\int\Big[\frac{c^4}{16\pi G}R+L_m\Big]\sqrt{-g}\,d^4x, \tag{1.1}$$

where, L_m is the matter Lagrangian, R is the Ricci scalar, G denotes the Newton's gravitational constant, c is the velocity of light. We have assumed $c=1$ in the rest of the thesis. From the principle of least action the variation of the action with respect to the inverse metric should be equal to zero ($\delta S=0$). Taking the variation of the above action we obtain,

$$\delta S=\int\Big[\frac{1}{16\pi G}\Big(\frac{\delta R}{\delta g^{\mu\nu}}+\frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\Big)+\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}\Big]\delta g^{\mu\nu}\sqrt{-g}d^4x. \tag{1.2}$$

The above equation is satisfied for any arbitrary $\delta g^{\mu\nu}$, which implies,

$$\frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -16\pi G\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}. \tag{1.3}$$

The equation of motion for the metric field $g^{\mu\nu}$ can be derived from eq. (1.3). Now, the energy-momentum tensor $T_{\mu\nu}$ is given by,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} = -2\frac{\delta L_m}{\delta g^{\mu\nu}} + g_{\mu\nu}L_m. \tag{1.4}$$

For, the left hand side of the above equation we have,

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}. \tag{1.5}$$

The variation of the metric determinant leads to,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{1.6}$$

So, one obtains,

$$\frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}g_{\mu\nu}. \tag{1.7}$$

Combining eqs. (1.4), (1.5) and (1.7), the Einstein's field equation yields,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R = 8\pi G\, T_{\mu\nu}. \tag{1.8}$$

Here, $R_{\mu\nu}$ is the Ricci tensor. The EFE in tensorial form is given by:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{1.9}$$

where, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ represents the Einstein tensor.
For an anisotropic fluid, the energy-momentum tensor becomes:

$$T_{\mu\nu} = (\rho + p_t)u_\mu u_\nu - p_t\, g_{\mu\nu} + (p_r - p_t)X_\mu X_\nu, \tag{1.10}$$

where, ρ, p_r and p_t represent the energy density, radial pressure, and the tangential pressure respectively. In the above, u_μ and X_μ denotes the four-velocity vector and the radial unit four velocity respectively, and they satisfy the relation $u^\mu u_\nu = 1$ and $X^\mu X_\nu = -1$. Now, the energy-momentum tensor for an isotropic perfect fluid can be expressed as,

$$T_{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + p\, g^{\mu\nu}, \tag{1.11}$$

where, u^{μ} denotes the four velocity of the observer comoving with the fluid and ρ and p denotes the energy density and the pressure of the fluid respectively. Modern cosmology is based on a fundamental principle known as the cosmological principle which states that the spatial distribution of matter in our universe is homogeneous and isotropic when viewed on a large enough scale $(l > 100Mpc)$. The homogeneous and isotropic spacetime is best described by the Robertson-Walker metric which is given by,

$$ds^2 = dt^2 - a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\Big], \tag{1.12}$$

where, $k = 0, \pm 1$ denotes the spatially flat, closed and open geometry respectively for a fixed time. The distance scale between galaxies are measured using $a(t)$, which is known as the scale factor of the universe. Using the above metric in Einstein's field equation (1.9) one obtains the dynamical equations which are given by,

$$\Big(\frac{\dot{a}}{a}\Big)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.13}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G\, p. \tag{1.14}$$

where, (1.13) and (1.14) are the time-time and space-space components respectively. The conservation equation can be written as,

$$\dot{\rho} + 3\Big(\frac{\dot{a}}{a}\Big)(\rho + p) = 0. \tag{1.15}$$

One obtains the Raychaudhuri equation in this case which is given by,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.16}$$

If $\ddot{a} > 0$ then it corresponds to accelerating universe and if $\ddot{a} < 0$ then it corresponds to decelerating universe. For ordinary matters namely perfect fluid, radiation, dust, etc. the pressure of the fluid is positive along with its density.

1.4 Modified Einstein Field Equations

1.4.1 Modification of the Matter sector

The time-time component of EFE given by eq. (1.13) can be expressed in terms of Hubble parameter as,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{1.17}$$

where, $H = \frac{\dot{a}}{a}$ and $c = 1$. A critical density is defined as $\rho_c = \frac{3H^2}{8\pi G}$, for which the universe becomes flat. The energy density of the cosmic fluid can be normalized with respect to the critical density ρ_c by constructing a density parameter $\Omega = \frac{\rho}{\rho_c}$. The difference in the density parameter from 1 determines the shape of the universe. If $\Omega > 1$, then it corresponds to a closed universe, and if $\Omega < 1$ it denotes an open universe. If the density parameter is unity then the universe is flat. The cosmic fluid can be a combination of a number of different matter fields and the total energy density of the universe is $\rho = \sum_i \rho_i$, where i is the number of different fields. The corresponding density parameter can be expressed as $\Omega = \sum_i \Omega_i$, where $\Omega_i = \frac{\rho_i}{\rho_c}$. One can recast eq.(1.17) using the expression of density parameter as,

$$\Omega_{total} - \frac{k}{H^2 a^2} = 1. \tag{1.18}$$

Cosmological observations namely the CMBR indicates that the universe is spatially flat. This corresponds to $\Omega_{total} = 1$. Current estimations predicted that the contribution in Ω_{total} is due to the normal baryonic matter as well as gravitating non-baryonic dark matter which is almost $\Omega_M = 0.3 \pm 0.1$. The rest of the contribution is coming from dark energy, $\Omega_{DE} \sim 0.7$. The observational evidence of such an energy first came from the supernovae observations. Similar to DM, DE only interacts *via* gravity making its detection extremely difficult. It is homogeneous having extremely low density (almost $10^{-27} kg/m^3$) and has a strong negative pressure [52, 48]. The Raychaudhuri equation in cosmology demands an accelerated expansion of the universe in the presence of cosmic fluid with negative pressure. The DE serves as a promising candidate which provide an explanation for the present accelerated expansion. The DE violates the strong energy condition (SEC) and the cosmological constant (Λ) is considered to be a candidate for DE. However, it suffers from two cosmological constant problems, the fine tuning and the coincidence problem [45]. To overcome these issues a number of different DE models have been proposed in the recent times namely, quintessence [53–59], phantom [60], K-essence [61–65], tachyon [66–71], Chaplygin gas (generalized as well as modified) [72–77] Chameleon [78–80] etc. to name a few. In most of these models, the DE density is considered to be dynamic in nature instead of static, that varies with time slowly attaining the ΛCDM (Λ-cold dark matter) model at the present epoch.

I. Holographic Dark Energy models

"The Holographic Principle" was first proposed by 't Hooft [81] in the framework of string theory with a property of quantum gravity phenomena which assumes that the three-dimensional universe can be considered as an image, and the information can be stored on a two-dimensional projection similar to a hologram (*i.e.* the description of a volume of space can be thought of as encoded on a lower dimensional boundary to the region). The holographic principle was later given a precise interpretation in string theory by Susskind [82]. According to Susskind, "The three-dimensional world of ordinary experience-the universe filled with galaxies, stars, planets, houses, boulders, and people—is a hologram, an image of reality cited on a distant two-dimensional ($2d$) surface". All the relevant information about the three-dimensional ($3d$) universe can be found within the $2d$. In the recent past, the idea of the holographic principle was applied in cosmology to explain the cosmic acceleration [83–88]. The holographic principle further inspired from the development of black hole thermodynamics. The maximal entropy of any region is proportional to the square of the radius but not the cube. This is motivated from the theorem of black hole thermodynamics known as the Bekenstein-Hawking theorem [89–94]. According to Bekenstein, black holes are maximal entropy objects and the entropy of the black hole is directly proportional to the area of its event horizon. So the entropy of the matter inside a black hole will be less than this and one concludes, $S_{matter} \leq S_{BH} = \frac{A}{4}$, where A is the area of the event horizon. In 1997, Maldacena proposed that AdS/CFT correspondence is connected to the holographic principle which has a sound theoretical footing [95]. A successful explanation of the accelerating expansion of the universe was put forward by Li in 2004 [96], making use of the holographic principle to construct a viable DE model. This class of models are referred to as "Holographic Dark Energy" (HDE) models. In this case the DE density, ρ_{DE}, depends only on two quantities on the boundary of the universe, the reduced Planck mass, $M_P \equiv \frac{1}{8\pi G}$ and the cosmological length scale L which is chosen to be the future event horizon of the universe. The HDE density can be expressed using dimensional analysis [97], as

$$\rho_{DE} = c_1 M_P^4 + c_2 M_P^2 L^{-2} + c_3 L^{-4} + ... \tag{1.19}$$

where the c's are constants. The first term in the series is discarded because of the fine-tuning problem (the famous discrepancy between the theoretical and observed value which is of the order of 10^{120}) [98]. Later Cohen, Kaplan and Nelson [99] pointed out that the c_1 term is also not compatible with the holographic principle. So, the first term in the above expansion is discarded and the expansion starts immediately from the second term. The higher order terms in ρ_{DE} are negligible compared to the second term, so the HDE density can be expressed as,

$$\rho_{DE} = 3C^2 M_P^2 L^{-2}, \tag{1.20}$$

where, C is a constant. The interesting point to note here is that the above expression of DE density is obtained by applying the holographic principle and dimensional analysis rather than using a Lagrangian approach. Due to their unique origin, HDE models differ considerably from the other DE models.

In HDE models, the correct choice for the characteristic length scale L is important. The most common choice for the characteristic length scale is the Hubble radius $L = \frac{1}{H}$ [100, 101]. However, Hsu [102] pointed out that such a choice leads to a wrong EoS for the DE. Fischler and Susskind [103], however, chose the particle horizon as the IR cut-off, the choice also fails to accommodate the late time acceleration of the universe. Li [96], pointed out that it is better to represent L by the future event horizon, which is found consistent with the cosmological observations. The issue with the original HDE model is that if one considers the Hubble radius as the IR cutoff in conjunction with the Bekenstein entropy then the dark matter and the HDE are functions of the scale factor only. To avoid this issue several new cut-offs were introduced in the literature [104]. Another interesting approach is to use generalised entropies to construct new HDE models. In the literature generalised entropies like the Sharma-Mittal [105], Tsallis and Rényi [106–111] entropies are also employed to construct HDE models. These holographic dark energy models can be used with both interacting and non-interacting fluids.

II. Chaplygin Gas

The general non-linear equation of state (nEoS) for a cosmic fluid is,

$$p = f(\rho), \tag{1.21}$$

where, $f(\rho)$ is in generally a polynomial function of the energy density ρ. In 1904, Chaplygin first introduced an equation of state in aerodynamics to obtain negative pressure region required for the uplift of an aeroplane. Recently the scope of use of the Chaplygin gas (CG) is extended in cosmology for the understanding of late universe because a fluid with negative pressure is required to describe dark energy. The CG nEoS is given by [112],

$$p = -\frac{A}{\rho}, \tag{1.22}$$

where, A is a positive constant. CG was considered for an alternative to quintessence. But CG is ruled out in cosmology, as cosmological models are not consistent with the observational data of supernovae Ia (SNIa), Baryon Acoustic Oscillation (BAO), Cosmic Microwave Background (CMB) and so on. Subsequently, the EoS for CG is generalized to incorporate the observational universe. The modified EoS for the generalized Chaplygin gas (GCG) is given by [72, 73, 75],

$$p = -\frac{A}{\rho^{\alpha}}, \tag{1.23}$$

with, $0 \leq \alpha \leq 1$. However, $\alpha = 1$ reduces to simple Chaplygin gas. Now the EoS for GCG contains two free parameters, A and α. It is known that GCG permits a satisfactory explanation of the background dynamics and other features of a homogeneous isotropic universe. The most interesting property of GCG is that at high energy GCG behaves almost like a pressureless dust whereas at low energy it behaves like dark energy which provides a negative pressure. The feature of the GCG which corresponds to almost dust ($p = 0$) at high density does not agree completely with our universe. Later it is known that the GCG suffers from serious problems at the perturbation level [113]. The matter power spectrum of GCG exhibits strong oscillations or instabilities, unless GCG model reduces to ΛCDM, since the oscillations for the baryon component in the GCG leads to undesirable features in CMB spectrum. Thus a further modification to the GCG is considered by adding a positive linear term in density to the EoS, popularly known as modified Chaplygin gas (MCG) [74]. The equation of state for the MCG is given by:

$$p = A\rho - \frac{B}{\rho^{\alpha}}, \tag{1.24}$$

where, A, B, α are positive constants with $0 \leq \alpha \leq 1$. The above equation reduces to GCG model for $B = 0$. For negative values of α the above EoS represents cosmic fluids with dissipative effects. Also, a special case exists for $\alpha = -\frac{1}{2}$, when the EoS corresponds to an ever-existing Emergent Universe [114, 1]. It may be pointed out here that MCG is a single fluid model which unifies DM and DE. The EoS for MCG can be employed to model dark stars which are remnants of continued gravitational collapse [115]. The idea is that dark energy provides sufficient repulsion to halt further collapse leading to a stable bounded configuration free of horizons and singularities. The MCG corresponds to a barotropic fluid for the early universe when the size of the universe was small (with $A = \frac{1}{3}$ corresponding to radiation and $A = 0$ corresponding to matter). Thus MCG at one extreme behaves as an ordinary fluid and on the other hand, it behaves as a cosmological constant. In between these two limiting cases, it behaves as a mixture of both. It is important to note that MCG is a good candidate for the cosmological model building which is worthy of investigation in cosmology.

1.4.2 Modification of the Gravitational sector

It is known that inflation is one of the essential ingredients in modern cosmology. Inflation can be obtained by adding an R^2 term in the Einstein-Hilbert action as shown by Starobinsky [23]. It is also now felt that a modification of the Einstein-Hilbert gravitational action with terms polynomial in R in the present scale are important to look into the problems of the late universe. The additional terms will address the DE sector of the matter present in the universe. Efforts were made following the conventional approach to quantize the gravitational field as was done in the case of electromagnetism [116]. However, as general relativity is difficult to renormalize [117, 118], the quantization technique fails to work. Later, a renormalizable theory of gravitation was proposed at the cost of modification of the general relativity theory [119–121]. Modern theories of renormalization deals with the concept of effective field theory. It is known that quantization of gravitational field is quite a challenge because it produces a deviation from GR [122–124]. Two popular approaches are String theory and Loop quantum gravity (for a detailed review on these topics see Refs. [125–131]). There are other attempts to unify GR and quantum theories which essentially modify the gravitational field equations taking into account quantum corrections. However, such corrections in the theory are effective at small scales or at high enough energies. Recently, there has been a surge of research activity focusing on the modification of general relativity on both the small and large scales of the observed universe because of the presence of experimental cosmological missions in space. Eddington was one of the first few people who initiated the search for an alternative theory of gravity keeping the conceptual basis of Einstein's theory intact [132]. Brans and Dicke developed an alternative theory of gravity which is popularly known as the Brans-Dicke theory [133–135]. However, the theory may not lead to GR. A number of scalar-tensor theories have been developed since then. The Dvali-Gabadadze-Porrati (DGP) gravity [136], brane world gravity [137], TeVeS (Tensor-Vector-Scalar) gravity [138], Einstein-Aether theory [139], etc. come out for a modification of GR. The possibility of a modified theory of gravity that matches the observational data and which can emerge as a classical limit of quantum gravity seems plausible. The modified gravity must follow two main assumptions: (i) the models must fit observations which also include the solar system tests, (ii) they must validate the results coming from high energy physics and quantum gravity

in the near future.
It is known that the cosmological models involving particle physics often need more than five dimensions for their consistent formulation which opened up new avenues in higher dimensions. Riemann proposed the concept of non-Euclidean geometry which led to the formalisms describing the geometry in any arbitrary dimensional manifolds. Motivated by this concept Clifford, Helmholtz and Hinton speculated a higher dimensional spacetime in the past. However, the first important application of higher dimensional theories came up from the independent work of Kaluza and Klein who tried to unify gravity with electromagnetism by introducing an extra dimension [140, 141]. The theory is popularly known as the Kaluza-Klein theory (KK henceforth). The first cosmological model in 5 dimensions was constructed by Chodos [142]. But KK theory did not work well as a number of issues came up for its acceptability in that form. Long after that, the advent of string theory led to the construction of higher dimensional cosmology. It is known that string theory is applicable to bosons in general and is consistent for spacetimes with 26 dimensions. If fermions are included, however, the number of spacetime dimensions, however, reduces to 10, which is known as the superstring theory (SST) [143]. Although a consistent theory of quantum gravity is yet to come, these theories are considered as promising candidates for quantum gravity. Thus higher dimensional cosmology requires special attentions. Cosmological models will be studied in the thesis taking different theories of modified gravity in higher dimensions.

I. $f(R)$-Gravity:

The gravitational action as given in eq. (1.1) is modified as,

$$S=\frac{1}{16\pi G}\int f(R)\sqrt{-g}\,d^4x+S_m(g_{\mu\nu},\phi). \tag{1.25}$$

where, S_m is the matter action with the matter fields denoted by ϕ. Here $f(R)$ in generally includes higher order curvature invariants ($R_{\mu\nu}R^{\mu\nu}$) and are often considered to be a polynomial in R. The major advantage of $f(R)$ gravitational theory is that it is simple and can be generalized to exhibit some of the basic characteristics of higher order gravitational theories. The higher-order theories of gravity described by $f(R)$ avoids the fatal Ostrogradski instability [144] which is a kinetic instability involving ghosts. In the case of $f(R)$ gravity, modifications to GR appear naturally in the low energy limit of the effective actions of superstring theories. The $f(R)$ gravity is conformally related to GR with a self-interacting scalar field [145, 146], and both early inflation and the late acceleration of the universe can be accommodated. In 1980 Starobinsky obtained an inflationary solution in a higher derivative gravity [23] long before the advent of inflation was actually realized. The inflationary scenario was obtained by Starobinsky by adding a R^2 term in the gravitational action, which arises from vacuum polarization. It is important to mention here that EFEs can be obtained from the Einstein-Hilbert action using two different methods, the standard metric variation and the Palatini formalism [147]. In the Palatini formalism, the metric and the connection are assumed to be independent variables and the action is varied with respect to both of them with the assumption that the matter Lagrangian does not depend on the connection. For actions with Lagrangian linear in R, both the variational principles yield same field equations for relativistic models. However, for a general form of the action, this is not true [148]. In the literature, late time accelerating attractor solutions are obtained for a theory in polynomials of R^2, $R^{\mu\nu}R_{\mu\nu}$ and $R^{\mu\nu\gamma\sigma}R_{\mu\nu\gamma\sigma}$ in both the metric and the Palatini formalisms [149–154]. The cosmological models can be tested using observational data from Supernova (SN), Cosmic Microwave Background (CMB) shift parameter, Baryon Acoustic Oscillations (BAO) and Big Bang Nucleosynthesis (BBN) [155–162]. In the literature, several functional forms of $f(R)$ have been proposed to obtain cosmological models. To test the viability of the models, solar system tests can be performed [163–168]. The results appearing from these tests are important to keep or rule out a $f(R)$-theory. A cosmological model which fails to pass solar system tests also accommodates an accelerating universe [169] thus local considerations are therefore bypassed and an independent test can be devised by studying the cosmological behaviour. The observational constraints on $f(R)$ gravity is relevant because the baryons do not see the modifications to general relativity and can evade the solar system tests in a somewhat artificial way [170–176]

II. $f(R,T)$-Gravity:

The $f(R)$ gravity can be modified further by introducing an explicit coupling of the Ricci scalar and the matter Lagrangian density L_m [177–185]. The coupling results in a non-geodesic motion of massive particles, also an extra force arises which is orthogonal to the four-velocity. The gravitational field equations for $f(R,L_m)$ gravity can be used to build cosmological models which behave as a model of interacting DE [186]. Recently, Harko *et.al.* [187] proposed a new class of modified gravity based on the geometry matter coupling described by a Lagrangian with an arbitrary function $f(R,T)$, where T is the trace of the energy-momentum tensor. The dependence on T may arise from either exotic imperfect fluid or quantum effects (conformal anomaly). For different choices of the matter Lagrangian L_m different sets of field equations are obtained. It was shown that for a particular choice of $f(T)$ the FRW cosmological models can be reconstructed [187].

The gravitational action of eq. (1.33) can be modified in case of $f(R,T)$ gravity as,

$$S = \frac{1}{16\pi G}\int f(R,T)\sqrt{-g}d^4x + \int L_m\sqrt{-g}d^4x. \tag{1.26}$$

The trace of the energy-momentum tensor is given by, $T = g^{\mu\nu}T_{\mu\nu}$. The field equation in $f(R,T)$ gravity is,

$$f_R(R,T)\,R_{\mu\nu} - \frac{1}{2}f(R,T)\,g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\,f_R(R,T) = 8\pi G\,T_{\mu\nu} - .$$

$$f_T(R,T)\,T_{\mu\nu} - f_T(R,T)\,\Theta_{\mu\nu}, \tag{1.27}$$

where, $f_R = \frac{\partial}{\partial R}$ and $f_T = \frac{\partial}{\partial T}$ and $\Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}$. One important point to note is that in the $f(R,T)$ gravity, the covariant divergence of the stress-energy tensor is non-zero. Thus the field equations are expressed in terms of an effective energy-momentum tensor $T^{eff}_{\mu\nu}$, which, however, satisfies the energy conservation equation.

III. $f(Q)$-Gravity:

An equivalent representation of GR is recently constructed considering a flat spacetime described by a metric having asymmetric connections. This is known as the teleparallel description of gravity and in this case gravity is entirely described by torsion, which is called Teleparallel equivalent of GR (TEGR) [188]. Thus the action $f(\mathscr{T})$ where $\mathscr{T}$ is the torsion scalar [189–192], arose from torsion, is interesting for the study of late time cosmology. Another alternative approach can be considered where one can obtain GR in a flat spacetime without torsion. In this approach, gravity is attributed to non-metricity. The same physical theory can be reproduced by the Einstein-Hilbert action $\int\sqrt{-g}Rd^4x$, the TEGR $\int\sqrt{-g}\mathscr{T}\,d^4x$ and the action of symmetric teleparallel equivalent of GR (STEGR) whose action is given by $\int\sqrt{-g}Q\,d^4x$ [193, 194]. Here, Q represents the non-metricity scalar.
In the literature, modified theories of gravity namely $f(R)$ [195, 149–154] and $f(\mathscr{T})$ are considered for academic model building. Recently, $f(Q)$ theory is used in the literature [196–202] for the description of the universe. It is important to note that at the cosmological background level the $f(Q)$ models are indistinguishable from that of the $f(\mathscr{T})$ models. However, major difference between these two theories arise while considering cosmological perturbations [203].

The gravitational action is,

$$S = \int \sqrt{-g}\Big[-\frac{1}{16\pi G}f(Q) + L_m\Big]d^4x. \tag{1.28}$$

GR can be reproduced upto a boundary term if one considers $f = \frac{Q}{8\pi G}$ which represents the STEGR. The field equation is,

$$\frac{2}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}\, f_Q\, P^\alpha_{\mu\nu}) + \frac{1}{2}g_{\mu\nu}\, f + f_Q(P_{\mu\alpha\beta}Q_\nu^{\alpha\beta} - 2Q_{\alpha\beta\mu}P^{\alpha\beta}_\nu) = T_{\mu\nu}, \tag{1.29}$$

(where we use gravitational unit $8\pi G = 1$), $f_Q = \frac{\partial f}{\partial Q}$, $P^\alpha_{\mu\nu} = -\frac{1}{2}L^\alpha_{\mu\nu} + \frac{1}{4}(Q^\alpha - \tilde{Q}^\alpha)g_{\mu\nu} - \frac{1}{4}\delta^\alpha_\mu Q_\nu$ is the non-metricity conjugate, $Q_{\alpha\beta\mu}$ is the non-metricity tensor. The non-metricity scalar in this case is defined as,

$$Q = -Q_{\alpha\mu\nu}P^{\alpha\mu\nu}. \tag{1.30}$$

Raising one index the above field equation can be recast in a more compact form as,

$$\frac{2}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}\, f_Q\, P_\nu^{\alpha\mu}) + \frac{1}{2}\delta^\mu_\nu\, f + f_Q\, P^{\mu\alpha\beta}Q_{\nu\alpha\beta} = T^\mu_\nu. \tag{1.31}$$

It is known that at the small-scale quasi-static limit the cosmological predictions drawn from $f(Q)$ and $f(\mathscr{T})$ gravity models are found to coincide. However, at large scales $f(Q)$ models generically propagate with two scalar degrees of freedom which are in general absent in $f(\mathscr{T})$ gravity [203].

IV. Gauss-Bonnet Gravity

Lancoz first proposed a generalized theory of gravity considering higher order terms in Ricci scalar, Ricci tensor and Riemann [204]. The theory was later generalized by Lovelock [205]. Subsequently, it is shown that the Einstein-Gauss-Bonnet theory is a special case [206]. The advantage of the theory is that it requires an extra field apart from those already present in GR and the field equations of the theory can be written without involving derivatives higher than the second order of the metric thus preventing the Ostrogradsky instability [207]. The Gauss-Bonnet (GB) term is given by, $\mathscr{G} \equiv R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$. Zweibach long ago showed that the string corrections due to the Einstein action up to first order in the slope parameter and fourth power of momenta is proportional to the GB terms [208]. Subsequent analysis revealed that the field redefinition theorem of 't Hooft and Veltman can be applied in the GB terms [209]. Applying field redefinition theorem on the Einstein shell ($R_{\mu\nu} = 0$) for an action having curvature squared terms of the form $R + \alpha R^2_{\mu\nu} + \beta R^2$ transforms into R

(neglecting the higher order curvature invariants) by the redefinition,

$$g'_{\mu\nu} = g_{\mu\nu} + \alpha R_{\mu\nu} + g_{\mu\nu}\frac{\alpha+2\beta}{2-D}R \tag{1.32}$$

with D being the number of dimensions. Deser and coworkers later showed that the actions $R+\alpha(GB)$ and $R+\alpha R^2_{\mu\nu\gamma\delta}$ on the linearized Einstein shell are invariant [210, 211] which can be generalized to the other higher order ghost terms. The leading order of the α expansion of heterotic superstring theory gives rise to GB terms [212–214]. Nojiri *et.al.* [215] found that Ghosts can be eliminated by implementing a theoretical framework in the case of Gauss-Bonnet gravity. In four dimensions, however, the GB terms do not contribute to the dynamics of evolution. That is why, a dilaton coupling with the GB terms is essential to study the dynamics of the evolution significantly. The contribution arising from Einstein-Gauss-Bonnet (EGB) gravity in four dimensions was considered trivial for a long time. However, recently Glavan and Lin [216] showed that by re-scaling the coupling constant of the theory allowed a noticeable effect of EGB gravity even in four dimensions. The action with GB term is a modification of the Einstein-Hilbert action.

1.5 Aim of the Work

The objective of the proposed research work is to study theoretical and observational aspects of cosmological models in the framework of general relativity and modified theories of gravity. In the above models normal and exotic matters will be also taken up. The source of present accelerating phase of the universe is not yet understood. Determination of dynamical equation of state for dark energy is an important issue in theoretical physics. The relativistic cosmological models will be used to study the presence of dark energy in the light of the recent observational data and the model parameters will be estimated. The following problems are identified:

1. To study the Emergent Universe model which is singularity free in the presence of non-linear equation of states. The observational data will be used to constrain the equation of state parameters.

2. To study dark matter and dark energy in the universe in the theoretical framework of modified theories of gravity. Dark energy is an important ingredient to understand the present observed universe, basically the accelerating phase at the present epoch.

3. To study cosmological issues in higher dimensions.

4. To study compact objects in modified gravity.

5. To study the primordial black holes (PBH) in the early universe in the framework of different theories of gravity.

1.6 Methodology:

1.6.1 Field equations in $f(R)$-gravity

In $f(R)$-gravity theories, the gravitational action is,

$$S = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\, d^4x + S_m(g_{\mu\nu},\phi). \tag{1.33}$$

The field equations are obtained by varying the action with respect to $g_{\mu\nu}$ as,

$$f'(R)\, R_{\mu\nu} - \frac{1}{2} f(R)\, g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box]\, f'(R) = 8\pi G\, T_{\mu\nu}, \tag{1.34}$$

where $f'(R) = \frac{\partial f(R)}{\partial R}$, ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric and $\Box = \nabla^\mu \nabla_\mu$. The surface terms are ignored in the above equation which can be done by adding a total divergence to the action. Eq. (1.34) can be expressed in the following form,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R = \frac{8\pi G\, T_{\mu\nu}}{f'(R)} + g_{\mu\nu}\frac{[f(R) - R\, f'(R)]}{2f'(R)} + \frac{[\nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)]}{f'(R)}. \tag{1.35}$$

Defining the extra terms other than $T_{\mu\nu}$ on the right hand side of eq. (1.35) as the effective stress-energy tensor we get,

$$G_{\mu\nu} = \frac{8\pi G}{f'(R)}(T_{\mu\nu} + T^{eff}_{\mu\nu}). \tag{1.36}$$

The effective stress-energy tensor contains linear second derivative terms. The effective energy density derived, in this case is not positive definite and the energy conditions are violated similar to the case of scalar-tensor gravity.

1.6.2 Field equations in $f(R,\mathscr{G})$-gravity

The $f(R)$-gravity models can be further generalized by adding the Gauss-Bonnet term in the gravitational action, where,

$$\mathscr{G} \equiv R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}. \tag{1.37}$$

The gravitational action for the $f(R,\mathscr{G})$-gravity is,

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,f(R,\mathscr{G})+S_m(g^{\mu\nu},\phi). \tag{1.38}$$

The field equation is given by,

$$G_{\mu\nu}=16\pi G\,T_{\mu\nu}+\zeta_{\mu\nu}, \tag{1.39}$$

where, $\zeta_{\mu\nu}=\nabla_\mu\nabla_\nu f(R)-g_{\mu\nu}\Box f_R+2R\nabla_\mu\nabla_\nu f_{\mathscr{G}}-2g_{\mu\nu}R\Box f_{\mathscr{G}}-4R^\lambda_\mu\nabla_\lambda\nabla_\nu f_{\mathscr{G}}-4R^\lambda_\nu\nabla_\lambda\nabla_\mu f_{\mathscr{G}}+$ $4R_{\mu\nu}\Box f_{\mathscr{G}}+4g_{\mu\nu}R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_{\mathscr{G}}+4R_{\mu\alpha\beta\nu}\nabla^\alpha\nabla^\beta f_{\mathscr{G}}-\frac{1}{2}g_{\mu\nu}X+(1-f_R)G_{\mu\nu}$,
$f_R=\frac{\partial f(R,\mathscr{G})}{\partial R}$, $f_{\mathscr{G}}=\frac{\partial f(R,\mathscr{G})}{\partial \mathscr{G}}$ and $X=f_RR+f_{\mathscr{G}}\mathscr{G}-f(R,\mathscr{G})$. The above field equation is highly non-linear and can be solved for specific functional forms of $f(R,\mathscr{G})$.

1.6.3 System of autonomous differential equations

An autonomous system is a set of ordinary differential equations (ODEs) that do not have an explicit dependence on the independent variable [217–219]. In the autonomous system if the independent variable is time then they are known as time-invariant systems.
We begin with an outline of a pair of autonomous linear homogeneous differential equations which are,

$$\frac{dx}{dt}=ax+by, \tag{1.40}$$

$$\frac{dy}{dt}=cx+dy. \tag{1.41}$$

The above system of equations can be expressed in matrix form as,

$$X'=AX, \tag{1.42}$$

where, $X=\begin{pmatrix}x\\y\end{pmatrix}$ and $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, thereby it is expressed as,

$$AX=0, \tag{1.43}$$

which can be solved for te equilibrium points. If the coefficient matrix A is non-singular ($i.e.$ $|A|\neq 0$) then the above equation has a unique solution at $X=0$. For a singular coefficient matrix the above system has an infinite number of equilibrium points or critical points.
The types of the critical points are obtained by determining the eigenvalues λ_1 and λ_2 of the matrix A. For a non-singular A there are four types of critical points in general.
• **Node:** If λ_1 and λ_2 are real and of the same sign so that $\lambda_1.\lambda_2>0$ then the equilibrium

point is a node.
• **Saddle:** If λ_1 and λ_2 are real and non-zero having opposite signs (*i.e.* $\lambda_1.\lambda_2 < 0$), then the equilibrium point is a saddle.
• **Focus:** If λ_1 and λ_2 are complex quantities with equal and non-zero real parts *i.e.* $Re\lambda_1 = \Re\lambda_2 \neq 0$ then the equilibrium point is a focus.
• **Center:** If λ_1 and λ_2 are purely imaginary *i.e.* $Re\lambda_1 = \Re\lambda_2 = 0$ then the equilibrium point is a center.
The stability of the equilibrium points can be done by the general theorem of stability. For real negative eigenvalues or real negative parts of the complex eigenvalues the equilibrium point is asymptotically stable. The example of such points are stable nodes. However, for positive real part of at least one eigenvalue the equilibrium point is unstable. The point may be a saddle in that case. For second order systems the nature of stable points can be obtained by drawing phase space trajectories. This method will be applied in the analysis of ODEs obtained in cosmology.

1.6.4 The Energy Conditions

In GR we use the following energy conditions (EC) to determine the composition of matter. The four energy conditions (EC) are [51, 220],

• Null Energy Condition (NEC): $T_{\mu\nu}K^{\mu}K^{\nu} \geq 0$, where K^{μ} is a null vector. For a perfect fluid the NEC becomes $\rho + p \geq 0$;

• Weak Energy Condition (WEC): $T_{\mu\nu}X^{\mu}X^{\nu} \geq 0$, where X^{μ} is a timelike vector field. For a perfect fluid the WEC becomes $\rho \geq 0$; $\rho + p > 0$;

• Strong Energy Condition (SEC): $(T_{\mu\nu} - \frac{1}{2}Tg)\mu\nu)X^{\mu}X^{\nu} \geq 0$. For a perfect fluid the WEC becomes$\rho + 3p \geq 0$;

• Dominant Energy Condition: $\rho \geq |p|$.

For anisotropic fluids, the energy conditions can be expressed in a more general form. The validity of the strong energy condition automatically implies a decelerating universe.