Chapter 1

Introduction

1.1 The equation of radiative transfer

The basic equation of radiative transfer which governs the radiation field of frequency ν in a medium which absorbs,emits and scatters radiation is given by

$$-\frac{1}{k_\nu \rho}\frac{dI_\nu}{ds} = I_\nu - \Im_\nu, \tag{1.1}$$

where I_ν is the specific intensity, k_ν is the absorption coefficient of the medium, ρ, the density of the medium, s, the thickness of the element of the mass considered and $\Im_\nu$ is called the source function which is the ratio of emission coefficient to absorption coefficient. The equation (1.1) is a differential equation.

1.1.1 Equation of transfer in different media and geometries:

(1) **Plane parallel medium:**
Here we assume a medium stratified in planes perpendicular to oz-axis and the uniformity of radiative properties in each plane layer is considered. We

define optical depth τ by

$$\tau_\nu = \int_s^\infty \kappa_\nu \rho ds, \tag{1.2}$$

where s is the height of the medium. Below there are some cases in these geometries.

(a) Local thermodynamic equilibrium with no scattering:

Here it is assumed that the circumstances are such that we can define at each point in the atmosphere a local temperature T such that the emission coefficient at that point is given in terms of the absorption coefficient by Kirchoff's law and the source function $\Im_\nu$ is given by

$$\Im_\nu = B_\nu(T), \tag{1.3}$$

where $B_\nu(T)$, the Planck function given by

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{exp(h\nu/kT) - 1}, \tag{1.4}$$

where h is the Planck constant, k the Boltzmann constant and T is the characteristic temperature. The equation of transfer in this case is

$$-\mu\frac{dI(\tau_\nu, \mu)}{d\tau_\nu} = I(\tau_\nu, \mu) - B_\nu(T), \tag{1.5}$$

where τ is the optical depth given in equation 1.2 and $\mu = \cos\theta$, θ being the angle the pencil of incident radiation makes with the outward drawn normal from an element of area $d\sigma$.

(b) Medium where the scattering is isotropic:

In a medium where the scattering is isotropic, the source function takes the form

$$\Im_\nu = \frac{1}{2}\int_{-1}^{+1} I(\tau, \mu')d\mu' \tag{1.6}$$

and the equation of transfer has the form

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{+1} I(\tau, \mu')d\mu'. \tag{1.7}$$

(c) In the case of anisotropically scattering medium, source function is

$$\Im_\nu = \frac{\varpi}{2}\int_{-1}^{+1} p(\mu,\mu')I(\tau, \mu')d\mu', \tag{1.8}$$

where $p(\mu,\mu')$ is the phase function which governs the directional distribution of intensity, ϖ is the albedo of single scattering which equals to 1 for single scattering (conservative case) and equals to zero for the case of true absorption. In nonconservative case $\varpi < 1$ and in the case of neutron transport $\varpi > 1$. If the scattering is isotropic $p(\mu,\mu') = 1$ and if the scattering is anisotropic then $p(\mu,\mu')$ has different well known form e.g.

$$\begin{aligned}
p(\mu,\mu') &= (1 + x\mu\mu'); && \textit{Planetary scattering,} \\
&= 1 + \frac{1}{2}P_2(\mu)P_2(\mu'); && \textit{Rayleigh scattering,} \\
&= 1 + \frac{\alpha}{2}P_2(\mu)P_2(\mu'); && \textit{Pomraning phase function, } \alpha = \frac{5}{5-3\lambda}, \\
&= 1 + b_0P_4(\mu); && \textit{Carlstedt \& Mullikin's phase function,} \\
&= 1 + 3gP_1(\mu)P_1(\mu') + 5g^2P_2(\mu)P_2(\mu') + 7g^3P_3(\mu)P_3(\mu'); \\
& && \textit{Henyey − Greenstein phase function,}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \omega_k P_k(\mu) P_k(\mu'); \qquad \textit{General phase function,}$$

(d) Coherent scattering:

If the frequency of absorption of radiation in an atom is the same with the frequency of emission, the scattering is said to be coherent, alternatively we can say, when the redistribution of energy following scattering is in same frequency we call the process to be one of coherent scattering. The equation of radiative transfer for coherent scattering [80] is in the form of

$$\mu \frac{dI(\tau, \mu)}{d\tau} = (1 + \eta) I(\tau, \mu) - (1 - \epsilon)\eta J_\nu - (1 + \epsilon\eta) B_\nu(T) \qquad (1.9)$$

where $\eta = \frac{l_\nu}{\kappa}$,

l_ν = line absorption coefficient and

κ = continuous absorption coefficient.

(e) Noncoherent scattering:

There are two sections in noncoherent scattering. Interlocking of lines and purely noncoherent scattering. When two, three, or more sub states of lower energy states have a common sub state in the upper energy state and an electron at the upper sub state coming from any of the lower sub states have equal probability to go down to any of the lower sub states and thus differing in the absorption and emission frequencies, the lines so formed are called interlocked to each other. The case may be totally reversed i.e., two, three, or more substates in the upper state may have a common substate in the lower state. The equation of transfer for r-th line of interlocked multiplet is [80]

$$\mu\frac{dI_r(\tau,\mu)}{d\tau} = (1+\eta_r)I_r(\tau,\mu) - (1+\epsilon\eta_r)B_\nu(T) - (1-\epsilon)\alpha_r\sum_{p=1}^{k}\frac{\eta_p}{2}\int_{-1}^{1} I_p(\tau,\mu')d\mu' \quad (1.10)$$

$r = 1, 2, \ldots k$

where

$$\alpha_r = \frac{\eta_r}{\eta_1+\eta_2+\ldots+\eta_k} \quad (1.11)$$

so that

$$\alpha_1+\alpha_2+\ldots+\alpha_k = 1 \quad (1.12)$$

and η_r , the ratio of line to the continuum absorption coefficient for the r-th line is independent of depth but is function of frequency. ϵ the coefficient of thermal emission, is independent of both frequency and depth. The equation of transfer for noncoherent scattering is of the form

$$\mu\frac{dI_\nu(\tau,\mu)}{d\tau} = (1+\epsilon)I_\nu(\tau,\mu) - (1-\epsilon\eta)B_\nu(T) - (1-\epsilon)\eta\{aJ_\nu + bJ_{\nu_0} + c\overline{J}\}. \quad (1.13)$$

(f) Grey medium:

A medium which possesses radiative properties independent of frequency is said to be gray medium. The equation of transfer for gray medium is

$$-\mu\frac{dI(\tau,\mu)}{d\tau} = (1-\varpi)\frac{n^2\sigma T^4}{\pi} + \frac{\varpi}{2}\int_{-1}^{+1} p(\mu,\mu')I(\tau,\mu')d\mu' \quad (1.14)$$

where n is the refractive index of the medium and σ is the Stefan constant.

In plane parallel system there are mainly two types of problem. One is semi-infinite atmosphere bounded at $\tau = 0$ and extended to infinity

($\tau \to \infty$) in the other direction and the other is finite atmosphere bounded by $\tau = 0$ and $\tau = \tau_1$.

(2) **Spherical geometry:**

The equation of transfer in spherical geometry is of the form:

$$\mu \frac{\partial I_\nu}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial I_\nu}{\partial \mu} = j_\nu - \kappa_\nu I_\nu. \tag{1.15}$$

In case of cylindrical geometry it takes the form

$$\sin\theta \cos\phi \frac{\partial I_\nu}{\partial r} - \frac{\sin\theta \sin\phi}{r} \frac{\partial I_\nu}{\partial \phi} = j_\nu - \kappa_\nu I_\nu. \tag{1.16}$$

The equation of transfer in three dimensional geometry are of the form

(a) Cartesian:

$$\mu \frac{\partial I_\nu}{\partial x} + \eta \frac{\partial I_\nu}{\partial y} + \xi \frac{\partial I_\nu}{\partial z} = j_\nu - \kappa_\nu I_\nu. \tag{1.17}$$

(b) General Spherical:

$$\mu \frac{\partial I_\nu}{\partial r} + \frac{\eta}{r} \frac{\partial I_\nu}{\partial \theta} + \frac{\xi}{r \sin\theta} \frac{\partial I_\nu}{\partial \theta} + \frac{1-\mu^2}{r} \frac{\partial I_\nu}{\partial \mu} + \frac{\xi \cot\theta}{r} \frac{\partial I_\nu}{\partial \phi} = j_\nu - \kappa_\nu I_\nu. \tag{1.18}$$

(c) General Cylindrical:

$$\mu \frac{\partial I_\nu}{\partial r} + \frac{\eta}{r} \frac{\partial I_\nu}{\partial \Theta} + \xi \frac{\partial I_\nu}{\partial z} - \frac{\eta}{r} \frac{\partial I_\nu}{\partial \phi} = j_\nu - \kappa_\nu I_\nu \tag{1.19}$$

where

$$I_r = I_r(r, \Theta, z, \theta, \phi),$$

$$\begin{aligned}
\xi &= \cos\theta, \\
\mu &= \sin\theta\cos\phi, \\
\eta &= \sin\theta\sin\phi, \\
\textit{and} \quad & \xi^2 + \mu^2 + \eta^2 = 1.
\end{aligned}$$

1.2 The Spherical Harmonic Methods. (SHM)

Originally Eddington[21] and Gratton[25] introduce the Single Interval Spherical Harmonic Method. Chandrasekhar[8]-[10] systematize the scheme and suggest a general procedure for solving integro- differential equation of Transfer by this method. The Spherical Harmonic Method was extensively used to solve varies problems of Radiative Transfer and Newton Transport. We shall reproduce here a short review of the development of the method in solving transfer problems.

The essential idea of the method is to expand specific intensity $I(\tau, \mu)$ in a series of Legendre polynomials $P_j(\mu)$, where $\mu = cos\theta$, θ being the angle between the intensity and outward drown normal to the plane surface at a depth τ in seeking a solution of equation of Radiative Transfer. The Legendre polynomials $P_j(\mu)$ form a complete set of orthogonal function in the interval (-1, 1), in which interval μ varies also. We write

$$I(\tau, \mu) = A_o(\tau)P_o(\mu) + A_1(\tau)P_1(\mu) + ... + A_m(\tau)P_m(\mu), \tag{1.20}$$

where the solution of the equation of the transfer is reduced to the determination of the functions $A_j(\tau)$.

In grey case the mean intensity and the source functions are defined respectively

$$\overline{I}(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau, \mu')d\mu' \tag{1.21}$$

and

$$\overline{\Im}(\tau) = 2\int_{-1}^{1} I(\tau, \mu')\mu'd\mu' = F, \tag{1.22}$$

where F is the net integrated flux. In terms of $P_j(\mu)$, the above representation can be written as

$$\overline{I}(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')P_0(\mu)d\mu' \tag{1.23}$$

and

$$\overline{\Im}(\tau) = 2\int_{-1}^{1} I(\tau,\mu')\mu' P_1(\mu)d\mu' = F. \tag{1.24}$$

Next, we use the orthogonality property of $P_j(\mu)$ which is given by

$$\int_{-1}^{1} P_m(\mu)P_n(\mu) = \begin{cases} 0, & if\ m \neq n \\ \frac{2}{2n+1}, & if\ m = n \end{cases} \tag{1.25}$$

Therefore, the mean intensity

$$\begin{aligned} \overline{I}(\tau) &= \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')P_o(\mu')d\mu' \\ &= \frac{1}{2}\left[A_0(\tau)\int_{-1}^{1} P_0^2(\mu')d\mu' + \sum_{l=1}^{m} A_1(\tau)\int_{-1}^{1} P_l(\mu')P_0(\mu')d\mu\right] \\ &= A_0(\mu) \end{aligned} \tag{1.26}$$

and source function

$$\begin{aligned} \overline{\Im}(\tau) &= 2\int_{-1}^{1} I(\tau,\mu')P_1(\mu')d\mu' \\ &= 2\Big[A_0(\tau)\int_{-1}^{1} P_0(\mu')P_1(\mu')d\mu' + A_1(\tau)\int_{-1}^{1} P_1^2(\mu')d\mu' \\ &\quad + \sum_{j=2}^{m}\left\{A_j(\tau)\int_{-1}^{1} P_j(\mu')P_1(\mu')d\mu'\right\}\Big] \\ &= \frac{4}{3}A_1(\tau). \end{aligned} \tag{1.27}$$

We know that in Grey case the conservation of flux integral gives $B = \overline{I}$, where B is Plank's function.

In grey case (taking $B = \overline{I}$) the equation of transfer for integrated radiation takes the from

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - A_0(\tau). \tag{1.28}$$

We use recurrence formula for Legendre polynomials

$$(j+1)P_{j+1}(\mu) - (2j+1)\mu P_j(\mu) + jP_{j-1}(\mu) = 0, \tag{1.29}$$

in equation (1.28), we obtain

$$\mu \sum_{j=0}^{m} A_j'(\tau)P_j(\mu) = \sum_{j=1}^{m} A_j(\tau)P_j(\mu). \tag{1.30}$$

i.e.,

$$\sum_{j=0}^{m} A_j'(\tau)\left[\frac{j}{2j+1}P_{j-1}(\mu) + \frac{j+1}{2j+1}P_{j+1}(\mu)\right] = \sum_{j=1}^{m} A_j(\tau)P_j(\mu). \tag{1.31}$$

If we compare the coefficients of $P_j(\mu)$ in (1.31), we can find

$$A_1'(\tau) = 0, \tag{1.32}$$

$$A_0'(\tau) + \frac{2}{5}A_2'(\tau) = A_1(\tau). \tag{1.33}$$

So in general, we have (for j = 2, 3, ...)

$$\frac{j}{2j-1}\frac{dA_{j-1}(\tau)}{d\tau} + \frac{j+1}{2j+3}\frac{dA_{j+1}(\tau)}{d\tau} = A_j(\tau). \tag{1.34}$$

This gives

$$A_1(\tau) = \frac{3}{4}F = C_1, \tag{1.35}$$

$$A_0(\tau) = \frac{3}{4}F\tau - \frac{2}{5}A_2(\tau) + C_0 \tag{1.36}$$

where C_1, C_0 are constants of integration.

Extending for m = 2n in equation (1.30) and ignoring the final equation
$\left[\frac{dA_{2n}}{d\tau} = 0\right]$ we find that $\left[writing\ ,\ \frac{d}{d\tau} \equiv D, A_j(\tau) \equiv A_j\right]$

$$\left.\begin{array}{ll} DA_1 & = 0,\ \ j = 1 \\ -A_2 + \frac{3}{7}DA_3 & = 0,\ \ j = 2 \\ \frac{j}{2j-1}DA_{j-1} - A_j + \frac{j+1}{2j+3}DA_{j+1} & = 0,\ \ j = 3, 4, \ldots 2n-1 \\ \frac{2n}{4n-1}DA_{2n-1} - A_{2n} & = 0.\ \ j = 2n \end{array}\right\} \qquad (1.37)$$

From the first and last, together with those given by $j = 4, 6, ..., 2n-2$, $DA_3, DA_5, ..., DA_{2n-1}$ can be eliminated and another first integral is obtained. Therefore, we get a liner relation between $A_2, A_4, ...A_{2n}$ with constants coefficients. The resolvent equation for any function A_m ($m = 2,\ 3, ..., 2n$) is of order (2n-2). So we get an equation of the from

$$F(D)A_m = 0, \qquad (1.38)$$

where

$$F(D) = \begin{vmatrix} -1 & \frac{3}{7}D & 0 & \ldots & 0 & 0 \\ \frac{3}{5}D & -1 & \frac{4}{9}D & \ldots & 0 & 0 \\ 0 & \frac{4}{7}D & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -1 & \frac{2n}{4n-1}D \\ 0 & 0 & 0 & \ldots & \frac{2n}{4n-1}D & -1 \end{vmatrix}. \qquad (1.39)$$

It contains only derivatives of even order and its characteristic equation $F(\lambda) = 0$ has all its roots real and of modulus greater than unity. Let the roots of $F(\lambda) = 0$ are $\pm\alpha_j$, (j = 2, 3, ..., n) where each $\alpha_j > 1$. Then the solution for one particular A_m (other than A_0 and A_1) will be given by

$$C_{-2}e^{\alpha_2\tau} + C_{-3}e^{\alpha_3\tau} + ...C_{-n}e^{\alpha_n\tau} + C_2e^{-\alpha_2\tau} + C_3e^{-\alpha_3\tau} + ... + C_ne^{-\alpha_n\tau}, \tag{1.40}$$

where $C_{\pm j}$ (j = 2, 3, ..., n) are (2n-2) constants of integration with C_0, C_1 are to be obtained from (1.35) and (1.36). The functions $A_2(\tau)$, $A_3(\tau)$, ...,$A_{m-1}(\tau)$ will depend linearly on the same exponentials and same constants of integration. We have to determine the constants $C_{\pm j}$ by means of the boundary conditions which are given below.

(i) Net flux is constant,i.e.,

$$S(\tau) = F = constant, \tag{1.41}$$

(ii) The convergence of intensity as $\tau \to \infty$, i.e.,

$$I(\tau, \mu)e^{-\tau} \to 0 \ as \ \tau \to \infty, \tag{1.42}$$

(iii) Absence of incident radiation from outside on the free surface,i.e.,

$$I(0, \mu) \equiv 0 \ for \ \mu < 0. \tag{1.43}$$

Now from (1.41) the constancy of net flux gives $C_1 = \frac{3}{4}F$ and from (1.42) we have

$$C_{-j} = 0, \ for \ j = 2, 3, ...n.$$

The remaining constant are to be determined from (1.43) which implies that

$I_{-}(0, \mu) \equiv \ 0 \ i.e.,$

$$I_{-}(0, \mu) = \sum_{j=o}^{2n} A_j(0)P_j(\mu) \equiv 0. \tag{1.44}$$

Equation (1.44) must be satisfied for all values of μ lying between -1 and 0. Thus, we have a system of infinite number of linear homogeneous equation, and from this system of equation we have to determine a finite number, n of unknowns $C_0, C_2, ..., C_n$.

Therefore, in general, we conclude that the system (1.44) is incompatible. We have no alternative but to choose arbitrarily n equations corresponding to n arbitrary values of μ (between -1 and 0) to determine the n constants of integration $C_0, C_2, ..., C_n$.

To overcome the difficulty arises in the solution, use of various equivalent boundary conditions to the boundary condition stated above, has been used. For example, Mark[1947] met it by choosing some strategic values of μ for which the condition (1.44) holds good. Davison and Sykes [17] suggested that in the above approximation odd order in the Legendre polynomials yield better results than even order Legendre polynomials (due to the insufficiency of the number of roots in the even order approximation), i.e., the specific intensity $I(\tau, \mu)$ will be of the form

$$I(\tau, \mu) = \sum_{l=0}^{N}(2l+1)I_l(\tau)P_l(\mu). \quad (1.45)$$

Viskanta [70] in a separate study has concluded that in lower order approximations, P_3 approximation is desirable.

Kourganoff [46] tried to reduce this arbitrariness by using the least square method but even then this arbitrariness cannot be removed completely. He imposed a minimum condition on $I(0, \mu)$and suggested that

$$\sigma = \int_{-1}^{0}[I(0, \mu)]^2 d\mu = Minimum. \quad (1.46)$$

This is equivalent to $I(0, \mu) \equiv 0$ for $-1 < \mu < 0$. From(1.43) and (1.44) we have

$$\sigma = \int_{-1}^{0}\left[\sum_{j=0}^{2n} A_j(0)P_j(\mu)\right]^2 d\mu = Minimum. \quad (1.47)$$

Differentiating σ partially w.r.t. $A_j(0)$ and using the orthogonal property of $P_j(\mu)$, we deduce that

$$\frac{2}{2i+1}A_i(0) = \sum_{j=0}^{2n} A_j(0)\int_0^1 P_i(\mu)P_j(\mu)d\mu, \;\; (i = 0, 1, 2, ..., 2n). \quad (1.48)$$

Equation (1.48) states that there are now (2n+1) relations involving n unknowns $C_0, C_2, ..., C_n$. Therefore, we still have the arbitrariness and incompatibility. Thus, the arbitrariness in the determination of constants is minimized but yet not removed.

Kourganoff [46] traced this source of the defect to the fact that in equation (1.44) the function $I(0,\mu)$ which is represented by a finite sums of Legendre polynomials, is discontinuous at the free surface $\mu = 0$.

1.2.1 The necessity of modification

From our discussion we have seen that in spherical harmonic method the specific intensity is expanded into a series of Legendre polynomials $P_l(\mu)$ which form a complete set of orthogonal polynomials within (-1, 1). In general, for all practical purpose the series is truncated after N terms, where N is the required accuracy. However, the spherical harmonic method suffers from one serious defect, the difficulty of analytical representation of boundary conditions at the free surface, where the specific intensity is discontinuous. In plane parallel problems and in simple case of spherical models, the situation is met by using different types of equivalent boundary conditions but this method was still successful in yielding fairly accurate results. Kourganoff (1963) drew attention to some of the serious limitations of the single interval spherical harmonic method.

Let us now take a simple model of semi-infinite, plane parallel, scattering medium for axially symmetric radiative transfer problem in diffuse radiation equation in simple model of , scattering medium is given by

$$\mu\frac{dI(\tau,\mu)}{dI} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu', \quad (1.49)$$

where the symbols have their meanings described in section (1.1). Further we have assumed that the specific intensity is represented as

$$I(\tau,\mu)=\sum_{l=0}^{L} I_l(\tau)P_l(\mu). \tag{1.50}$$

Here the source function is given by

$$s(\tau)=\frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu'. \tag{1.51}$$

The boundary conditions under which the transfer equation to be solved is

(a) The atmosphere is bounded by the plane $\tau=0$ and that it receives no radiation form the exterior, i. e.,

$$I(0,-\mu)=0 \ for \ 0<\mu\leq 1. \tag{1.52}$$

(b) The convergence of intensity,i.e.

$$S(\tau)e^{-\tau}\rightarrow 0 \ as \ \tau\rightarrow\infty. \tag{1.53}$$

Substituting (1.50) in the transfer equation, using the recurrence formula for Legendre polynomial $P_i(\mu)$ and equation the coefficients of various Legendre polynomials, we get a set of linear differential equations. Chandrasekhar (1960) assumed a trial solution of the form

$$I_m(\tau)=g_m e^{-k\tau}, \tag{1.54}$$

where g_m, k are constants. Substituting (1.54) in the set of linear differential equations,we get a set of linear algebraic equations. As discussed earlier we get the roots 0, 0, k_i, (i =2,3,......,n). Then the solution of (1.54) will be as follows

$$I_l(\tau)=\sum_i c_i g_m(k_i)e^{-k_i\tau}, \tag{1.55}$$

where the constants are to be evaluated from the boundary conditions. While the exact boundary condition is realized at the lower boundary $\tau\rightarrow\infty$, it cannot be done at the free surface, i.e. at $\tau=0$. Instead, equivalent boundary conditions are used. These are

(a) Chandrasekhar ([8] & [10]) used the boundary condition

$$\frac{2}{2l+1} I_l(0) = \sum_{m=0}^{L} I_m(0) \int_{-1}^{1} P_m(\mu) P_l(\mu) d\mu. \tag{1.56}$$

(b) Mark's [48] boundary condition in connection with neutron transport problem in plane parallel medium in this case is

$$I(0, \mu_i) = 0, \tag{1.57}$$

where μ_i are some strategic values of μ within the range and were taken as the roots of the equation $P_{n+1}(\mu) = 0$, n being an odd integer.

(c) Marshak's [49] equivalent boundary condition was

$$\int_{-1}^{0} I(0, \mu) P_{2l-1}(\mu) d\mu = 0, \;\; l = 1, 2, ...n. \tag{1.58}$$

Kourganoff [46] (1963) made a critical analysis of the single interval SHM used in the case of Milne problem and raised objections against it. His conclusions are (i) exact boundary conditions cannot be prescribed and (ii) certain arbitrariness persists in the determination of the constants.

1.2.2 Double Interval Spherical Harmonic Method

Owing to difficulty arises to the solution of radiative transfer Kourganoff suggested that the situation would improve if double interval representation of specific intensity is tried. His suggestion, was in fact, made by Yvon [footnote Kourganoff, p.101] and elaborately demonstrated by Mertens[51].

Mertens (1954) divided the total range of μ which was (-1, 1) into two separate ranges (-1, 0) and (0, 1) and represented $I(\tau, \mu)$ as $I_+(\tau, \mu)$ and $I_-(\tau, \mu)$ at the two separate ranges (0,1) and $(-1, 0)$ for μ. He suggested

$$I_+(\tau, \mu) = \sum_{l=0}^{L} (2l+1) I_l^+(\tau) \mu P_l(2\mu - 1), 0 \leq \mu \leq 1 \tag{1.59}$$

$$I_-(\tau,\mu) = \sum_{l=0}^{L}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), -1 \le \mu \le 0. \quad (1.60)$$

The boundary conditions were taken as

$$I_l^-(0) = 0, \quad l = 0, 1, 2, ...n \quad (1.61)$$

$$I_l^+(\tau)e^{-\tau} \to 0 \ as \ \tau \to \infty, \quad (1.62)$$

$$I_l^-(\tau)e^{-\tau} \to 0 \ as \ \tau \to \infty. \quad (1.63)$$

With this representation, the main steps of the single interval spherical harmonic method are gone thorough necessary adaptation. The end results were encouraging. As far as the transfer problems in plane parallel medium is concerned, this method was free from two main defects of the single interval spherical harmonic method, namely,

(i) This exact boundary condition could be used here.

(ii) There was no preference for odd order approximations over the even order ones contrary to the suggestions of Davison and Sykes (1955)

However, some new difficulties arose; these are listed below.

(i) It was found to adversely effect the critical size calculation of neutron transport.

(ii) The extrapolated end points calculated for neutron transport in slab medium were found to be unreliable.

(iii) It was found that the method was adoptable to the solution of transfer problems in spherical geometry. The method did not give correct flux integral in the spherical case.

1.2.3 Modifications of Double Interval SHM.

In this section we will demonstrate various modifications of the DISHM that have been taken place over the years. Double interval spherical harmonics method has been used with some modifications by Sykes[64], Gross and Zeiring [26], Max Krook [50] and others. They however, are essentially equivalent to Merten's method and share it's limitations.

On a close approximation of Merten's [1.60] (1954) representation of specific intensity, it was found that in an attempt to discontinuity at the free surface, the discontinuity of representation at $\mu = 0$ had been carried to the interior. Wilson and Sen[74] introduced at this stage a modified double interval spherical harmonic method which retained the advantages of the double interval representation of Mertens and at the same time removed its defects. The aim was to select a suitable spherical harmonic method which would be equally effective for tackling transfer problems in plane parallel and spherical medium.

The passage from single interval SHM to double interval SHM has been elaborately described by Wilson and Sen (1990) and they gave an exhaustive account for the modification. Most of the forms are covered by Wilson and Sen(1990) and we give a few additional ones. Here, we list the modifications of double interval SHM that have been used by various workers in the field of radiative transfer.

(I) Merten's [51] From:

$$I_+(\tau, \mu) = \sum_{l=0}^{l_0} (2l+1) I_l^+(\tau) P_l(2\mu - 1), \quad 0 \leq \mu \leq 1 \tag{1.64}$$

$$I_-(\tau, \mu) = \sum_{l=0}^{l_0} (2l+1) I_l^-(\tau) P_l(2\mu + 1), \quad -1 \leq \mu \leq 0 \tag{1.65}$$

whereP_l are Legendre polynomials.

(II) Wilson and Sen's forms:

A. Plane parallel medium [74]

$$I_+(\tau,\mu) = A(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1), \quad 0 \le \mu \le 1 \tag{1.66}$$

and

$$I_-(\tau,\mu) = A(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), \quad -1 \le \mu \le 0. \tag{1.67}$$

where $A(\tau) = A\tau$ for slab media, τ being the optical depth, μ, the cosine of the angle made by the pencil radiation with the outward drawn normal. The only draw back of this method is that the unknown function $A(\tau)$ which appears in the representation had to be given a suitable form depending on the problem on hand.

B. Spherical geometry.

$$I_+(\tau,\mu) = A(r) + \sum_{l=0}^{l_0}(2l+1)I_l^+(r)\mu P_l(2\mu-1), \quad 0 \le \mu \le 1 \tag{1.68}$$

and

$$I_-(\tau,\mu) = A(r) + \sum_{l=0}^{l_0}(2l+1)I_l^-(r)\mu P_l(2\mu+1), \quad -1 \le \mu \le 0, \tag{1.69}$$

where A(r) is a function r only, r being the distance measured outward form the center of the sphere, μ, the cosine of the angle measured form the positive direction of the radius vector.

(III) Wan, Wilson and Sen's [72] Form:

$$i_+(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)i_l^+(\tau)\mu P_l(2\mu-1), \quad 0 \le \mu \le 1 \tag{1.70}$$

and

$$i_-(\tau,\mu) = \phi(\tau) + \sum_{l=0}^{l_0}(2l+1)i_l^-(\tau)\mu P_l(2\mu+1), \quad -1 \le \mu \le 0 \tag{1.71}$$

(IV) Karanjai and Talukdar's [39] Form:

$$I_+(\tau,\mu) = I(0,0)\left[A\tau + \phi(\mu) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1)\right], \quad 0 \le \mu \le 1 \tag{1.72}$$

and

$$I_{-}(\tau,\mu) = I(0,0)\left[A\tau + \phi(\mu) + \sum_{l=0}^{l_0}(2l+1)I_l^{-}(\tau)\mu P_l(2\mu+1)\right], -1 \le \mu \le 0 \tag{1.73}$$

where I(0,0) is the specific intensity at the surface in the direction normal to the surface and is a constant and A is an arbitrary constant. This form was also used by Bishnu [2].

(V) Raychaudhuri and Karanjai's [57] From:

$$I^{+}(\tau,\mu) = I(0,0)\left[\phi(\tau) + \psi(\mu) + \sum_{l=0}^{l_0}(2l+1)I_l^{+}(\tau)\mu P_l(2\mu-1)\right], 0 \le \mu \le 1 \tag{1.74}$$

and

$$I^{-}(\tau,\mu) = I(0,0)\left[\phi(\tau) + \psi(\mu) + \sum_{l=0}^{l_0}(2l+1)I_l^{-}(\tau)\mu P_l(2\mu+1)\right], -1 \le \mu \le 0. \tag{1.75}$$

1.2.4 Work done so far by the use of SHM in solving Radiative Transfer Problems

The Spherical Harmonic Method has been extensively used by Eddington [21], Gratton [25], Chandrasekhar [8], Sen [60], Marshak [49] and others to solve various radiative transfer problems in plane as well as spherical atmospheres. Marshak and Davison [17] used the same method for solving the problems of neutron transport. Subsequently Davison has also extended the method to the case of spherical and cylindrical geometries. Henceforward Wilson and Sen [74] called this method as single-interval spherical harmonic method.

Kourganoff [46] drew attention to certain arbitrariness that arises in the solution and suggested that the method, suitably modified, could prove useful in solving the integro-differential equation of transfer. He mention about a possible modification that could be made i. e. the replacement of the single interval intensity representation by a double interval one as suggested by Yvon. Sykes and Maxkrook have also utilized the concept of double interval representation of intensity in different context of the method of Gaussian quadrature and Moments respectively. But the representation of intensities was discontinuous in a direction perpendicular to the outward drown normal, not only at free surface but at all depths.

A modification of the double interval Spherical Harmonic Method of Yvon for solving the equation of radiative transfer in the plane geometry was suggested by Wilson and Sen [74]. Wilson and Sen's (1963) work removes the discontinuity in the representation of intensity in the direction perpendicular to the outward drown normal for all values of optical thickness. The equation of transfer for plane parallel medium was taken as

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = (\tau,\mu) - \frac{1}{2}\int_{-1}^{1} I(\tau,\mu')d\mu' \tag{1.76}$$

where $I(\tau,\mu)$ is the specific intensity of radiation at an optical depth τ and in a direction θ with the outward drawn normal $\mu = cos(\theta)$. The optical thickness is given by

$$\tau = \int_{z}^{\infty} k\rho dz$$

where k is the scattering coefficient and ρ is the density of the medium. They considered the boundary conditions

(i) Absence of incident radiation from outside at the free surface, i.e.,

$$I(0,\mu) = 0 \quad for \quad -1 \le \mu \le 0 \tag{1.77}$$

(ii) The convergence of intensity, i.e.,

$$I(\tau,\mu)e^{-\tau} \rightarrow 0 \quad as \quad \tau \rightarrow \infty \tag{1.78}$$

They represented the intensity $I(\tau,\mu)$ by two different expansions in the intervals (0,1) and (-1,0) and these are respectively as

$$I_+(\tau,\mu) = A(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1), \quad 0 \le \mu \le 1 \tag{1.79}$$

and

$$I_-(\tau,\mu) = A(\tau) + \sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), \quad -1 \leq \mu \leq 0 \qquad (1.80)$$

they evaluated $q(\tau)$ and $I(0,\mu)/F$ for both first approximation (i.e., P_1 approximations) and second approximations (i.e., P_2 approximations) and compared the result with Mertens (1954) and Chandrasekhar (1960). They have shown that the second approximation is distinctly superior to first approximation.

In 1964, Wilson and Sen [75] extended their earlier work to the case of anisotropic scattering to solve the equation of radiative transfer. They took the R.T. equation in plane geometry as

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - \frac{1}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' \qquad (1.81)$$

where, $p(\mu,\Phi;\mu',\Phi')$ is the phase function expresses the probability that radiation from the primed direction will be scattered into the unprimed direction. They consider the general phase function given by

$$p(\mu,\mu') = \sum_{k=0}^{\infty}\omega_k P_k(\mu)P_k(\mu') \qquad (1.82)$$

where ω_k's are constants and P_k's are Legendre Polynomials. The meaning of the other symbols described previously. In this case they also took the same boundary conditions given by Equations (1.77) and (1.78) and took the same form of intensity given by equations (1.79) and (1.80) respectively.

They demonstrate DISHM with particular case to the Rayleigh phase function which can be obtain by putting

$$\omega_0 = 1, \quad \omega_1 = 0, \quad \omega_2 = \frac{1}{2}, \quad \omega_n = 0 \ \ for \ n > 2, \qquad (1.83)$$

in equation (1.82). They evaluated $q(\tau)$ and $I(0,\mu)/F$ for the p_1 approximations and compared the result with those of Chandrasekhar (1944).

they showed that the first approximation of the DISHM is directly superior to the second approximation of Chandrasekhar[9].

Wilson and Sen [76] extended their modified double interval spherical harmonic method, in which the continuity in the representation of intensity in the two half intervals has been proposed, directly to solve the classical problem of diffusion of radiation through a homogeneous sphere, the radius being either finite or infinite. They considered the equation of transfer appropriate to the problem of radiation on the homogeneous sphere as,

$$\mu\frac{\partial I(r,\mu)}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial I(r,\mu)}{\partial \mu}+I(r,\mu)=\frac{1}{2}\int_{-1}^{1}I(r,\mu')d\mu' \qquad (1.84)$$

where r is the distance measured outward from the center of the sphere and μ is the cosine of the angle measured from the positive direction of the radius vector, I(r,μ) is the specific intensity of radiation at a distance r in the direction of $\theta=cos^{-1}(\mu)$. They considered, ([74]and [75]) the two different expansions of intensity and these are

$$I_{+}(r,\mu)=A(r)+\sum_{l=0}^{l_0}(2l+1)\mu I_1^{+}(r)P_l(2\mu-1), \ \ 0\leq\mu\leq 1 \qquad (1.85)$$

$$I_{-}(r,\mu)=A(r)+\sum_{l=0}^{l_0}(2l+1)\mu I_1^{-}(r)P_l(2\mu+1), \ \ -1\leq\mu\leq 0 \qquad (1.86)$$

where A(r) is a function of r only. The equation (1.84) was solved with the following boundary conditions

$$\left.\begin{aligned} A(R)&=0\\ I_0^{-}(R)&=0\\ I_1^{-}(R)&=0.\end{aligned}\right\} \qquad (1.87)$$

The results obtained in this case of the infinite sphere in the first approximation were found to compare favorably with those of Chandrasekhar [10] obtained by the method of Gaussian quadrature.

Wilson and Sen [77] extended their work to solve the equation of radiative transfer for finite, isotropically scattering, spherically symmetric, stellar atmosphere with k $\rho \propto r^{-2}$. The equation of transfer appropriate to the problem had been taken as

$$\mu\frac{\partial I(r,\mu)}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial I(r,\mu)}{\partial \mu}+k\rho I(r,\mu)=\frac{1}{2}k\rho\int_{-1}^{1}I(r,\mu)d\mu' \qquad (1.88)$$

where r is the distance measured outward from the center of the sphere, and μ is the cosine of the angle measured from the positive direction of the radius vector, I(r,μ) is the specific intensity of radiation at a distance r in the cos^{-1} (μ) direction, ρ is the density of the material and k, mass scattering coefficient. They consider the following boundary conditions

$$(i)\ I(R,\mu)=0\ for\ \ -1\le\mu\le 0, \qquad (1.89)$$

where R is the radius of the sphere, which indicates the absence of incident radiation at the outer boundary of the sphere.

The convergence of intensity as

$$(ii)\ r\ \rightarrow 0. \qquad (1.90)$$

They considered the same two forms of intensity given by the equations (1.85) and (1.86). They also pointed out that the function A(r) which appeared in equations (1.85) and (1.86) depend on the nature of the physical problem and solved the problem for the first approximation to evaluate J(x), the mean intensity at x =2 where x = $\frac{k_0C}{r}$ for two different boundaries given by $R=2k_0C\ and\ R=k_0C$. Their results were compared with those of Chandrasekhar [12](1960)

Wilson and Sen [78] once again extended their modified Spherical Harmonic Method to solve the problem of radiative transfer in spherically symmetric, finite planetary nebular shell with k $\rho \propto r^{-2}$. They considered the equation of transfer

$$\mu\frac{\partial I}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu}=-k\rho\left[I-\frac{1}{2}\int_{-1}^{1}Id\mu-\frac{S}{4}\frac{r_1^2}{r^2}e^{-(\tau_1-\tau)}\right], \qquad (1.91)$$

where τ_1 is the radial optical thickness of the nebular shell and πS is the net flux of the radiant energy incident on each square centimeter of the inner surface (radius $= r_1$)of the nebula. They considered the same two forms of the intensity [cf. equations (1.85) and (1.86)]. They assumed the Milne boundary conditions

i) There is no incident radiation on the outer boundary defined by r = R, i.e.,

$$I(R,\mu)=0 \; for \; -1\leq\mu\leq 0 \qquad (1.92)$$

ii) The diffuse flux across the inner surface ($r = r_1$) vanishes, i.e.,

$$F_{r=r_1}=0 \qquad (1.93)$$

they took k$\rho = \frac{C}{r^n}, n > 1$ and evaluated J(τ) at $\tau_1 = 2$ and R = C using the first approximation. The results were compared with those of Sen [60] obtained by the single interval spherical method.

Bishnu [2] solved the equation of radiative transfer for plane parallel scattering atmosphere with spherical symmetry. He assumed the appropriate equation of transfer

$$\mu\frac{dI(t,\mu)}{dt}=I(t,\mu)-\frac{1}{2}\int_{-1}^{1}I(t,\mu')d\mu', \qquad (1.94)$$

where the symbols have their usual meanings [vide.Wilson and Sen(1963)]. He gave an alternative modification of the double interval SHM. The forms of intensity taken by him are as follows

$$I_+(t,\mu)=I(0,0)\left[At+\phi(\mu)+\sum_{l=0}^{l=1_0}(2l+1)I_1^+(t)\mu P_1(2\mu-1)\right], 0\leq\mu\leq 1, \qquad (1.95)$$

$$I_{-}(t,\mu) = I(0,0)\left[At + \phi(\mu) + \sum_{l=0}^{l=l_0}(2l+1)I_1^{-}(t)\mu P_1(2\mu+1)\right], -1 \leq \mu \leq 0. \tag{1.96}$$

where A is a constant and $\phi(\mu)$ defined by

$$\begin{aligned} \phi(\mu) &= 1, \quad in \quad 0 \leq \mu \leq 1, \\ \phi(\mu) &= 0, \quad in \quad -1 \leq \mu \leq 0. \end{aligned} \tag{1.97}$$

He used the boundary conditions given by Wilson and Sen [74] and evaluated the H-functions and made comparisons with those of Chandrasekhar [12].

Canosa and Penafiel [6] proposed a direct method for the numerical solution of the spherical harmonics equations are a two-point boundary value problem for a system of ordinary differential equations of first order . These are then reduced to an algebraic problem by finite difference method. Canosa and Penafiel [6] avoided this difficulty by applying a parallel shooting method. The method was applied to homogeneous atmospheres and Rayleigh and Mie phase functions were used. The basic equation was taken as

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} + I = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} p(\tau,\mu,\Phi;\mu',\Phi')I(\tau,\mu',\Phi')d\mu' d\Phi' + S(\tau,\mu,\Phi) \tag{1.98}$$

where τ is the optical depth measured from top of the atmosphere, μ is the cosine of the zenith angle measured with respect to the positive τ axis, Φ is the azimuthal angle, p is the general phase function, S is the source of the incident radiation and I is the intensity of radiation. They dealt with only the 'average intensity' from of the equation of transfer and this is given by

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} + I = \frac{1}{2}\sum_{l=0}^{L}\frac{2l+1}{2}a_l(\tau)P_l(\tau)\int_{-1}^{1} P_l(\mu')I(\tau,\mu',\Phi')d\mu' d\Phi'$$

$$+\frac{1}{4}Fe^{-\frac{\tau}{\mu_0}}\sum_{l=0}^{l_0}\frac{2l+1}{2}a_l(\tau)P_l(\mu_0)P_l(\mu) \quad (1.99)$$

where ω_l are connected to a_l by the following relation

$$\omega_l = \frac{2l+1}{2}a_l(\tau). \quad (1.100)$$

The form of the intensity taken by them was

$$I(\tau,\mu) = \sum_{l=0}^{L}\frac{2l+1}{2}f_l(\tau)P_l(\mu), \quad (1.101)$$

and assumed normalized phase function

$$P(cos\theta) = \sum_{l=0}^{L}\omega_l P_l(cos\theta). \quad (1.102)$$

Test computations were performed on Rayleigh and Mie phase functions.

Devaux, Fouguart, Herman and Lenoble [20] discussed a critical study of four methods of solution of the equation of transfer (principle of invariance, doubling, spherical harmonics, successive orders of scattering). They compared both the accuracy of the results and required computation time.The Spherical Harmonic Method seems to have significant advantages.

Wan, Wilson and Sen [71] used the modified SHM in solving the radiative transfer problem in an isothermal slab with Rayleigh scattering. They considered the model consisting of an isothermal plane parallel slab of optical thickness τ_0 confined between gray and diffuse walls that absorb and anisotropically scatter radiant energy. The equation of transfer for such a model was

$$\mu\frac{\partial I}{\partial \tau} + I = (1-\omega_0)I_b + \left(\frac{\omega_0}{2}\right)\oint_{-1}^{1} I(\tau,\mu')p(\mu,\mu')d\mu' - S(\tau,\mu) \quad (1.103)$$

where I is the intensity; I_b, the black body intensity; ω_0, the albedo for single scattering; p, the phase function; S, the source function; μ, the cosine of the angle measured from the positive direction of optical depth τ. The Rayleigh phase function was considered and this is given by

$$p(\mu,\mu') = \frac{3}{8}\left[(3-\mu^2) + (3\mu^2-1)\mu'^2\right]. \tag{1.104}$$

The forms of the intensity taken by them was

$$I_+(\tau,\mu) = \phi(\tau) + \sum_l (2l+I) I_l^+(\tau)\mu P_l(2\mu - I), \;\; 0 \le \mu \le 1 \tag{1.105}$$

and

$$I_-(\tau,\mu) = \phi(\tau) + \sum_l (2l+I) I_l^-(\tau)\mu P_l(2\mu + I), \;\; -1 \le \mu \le 0. \tag{1.106}$$

where $\phi(\tau)$ is a function of τ only.

They took the function $\phi(\tau)$ as

$$\phi(\tau) = I_0[1 + A\ exp(-\tau) + B\ exp(\tau)]. \tag{1.107}$$

The boundary conditions used are

$$I(0,\mu) = B_1 \quad for \quad 0 \le \mu \le 1 \tag{1.108}$$

$$I(\tau_0,\mu) = B_2 \;\; for \;\; -1 \le \mu \le 0. \tag{1.109}$$

In their problem they took both B_1, B_2 to be zero and applied the first approximation to find the zeroth, first and second moments of intensity and compared results with that of Dayan and Tien (1976)

In another paper Wan, Wilson and Sen [72] used the same form of intensity function (1.105) and (1.106) to solve the integro-differential equation of transfer

$$\mu\frac{\partial I}{\partial \tau} + I = \frac{1}{2}\omega\int_{-1}^{1} p(\mu,\mu') I(\tau,\mu') d\mu'. \tag{1.110}$$

But they restricted $\phi(\tau)$ in different ways

$$\phi(\tau) = A\ exp(-\tau) + B\ exp(\tau) \quad for \quad \tau_0\ small, \tag{1.111}$$

$$\phi(\tau) = A\ exp(-\tau) + (A + B\tau) \quad for \quad \tau_0\ large. \tag{1.112}$$

Wan, Wilson and Sen [72] have shown that the double interval spherical harmonic method yields results comparable to the exact values even in the first order of approximation for the radiative transfer problem in a planner medium irradiated by an isotropic radiation field.
They [72] shown that the Double Interval Spherical Harmonic Method admirably suit to solve the Menguc and Viskanta's model. The results computed are found to be in good agreement with those obtained by other methods.

Peraiah [54] discussed a numerical method for obtaining solution of equation of radiative transfer in spherically symmetric media by spherical harmonic approximation. Peraiah [54] approximated the angle derivative by an orthonormal polynomial and this is represented by a matrix called curvature matrix, for a given beam of rays.The radiative transfer equation in spherical symmetry was considered as

$$\frac{\mu}{r^2}\frac{\partial}{\partial r}[r^2 I(r,\mu)] + \frac{1}{r}\frac{\partial}{\partial \mu}[(1-\mu^2)I(r,\mu)] + k(r)I(r,\mu)$$
$$= k(r)\left[[1-w(r)]B(r) + \frac{1}{2}w(r)\int_{-1}^{1} p(r,\mu,\mu')I(r,\mu')d\mu'\right], \tag{1.113}$$

where $k(r)$ is the absorption coefficient, $k(r) \geq 0$ and $w(r)$ is the albedo for scattering. $0 \leq \ w(r) \leq \ 1$. $I(r,\mu)$ is the monochromatic specific intensity of the ray making an angle $cos^{-1}(\mu)$ with the radius vector at the radial point r. $B(r)$ is the Plank's function at r and $p(r,\mu,\mu')$ is the phase function and it is assumed to be isotropic.

The angles are discretized such that $0 < \mu_1 < \mu_2 < .... < \mu_m \leq 1.$

They expanded the specific intensity as

$$I(\mu) = \sum_{m=o}^{M} \alpha_m P_m(\mu) \tag{1.114}$$

and calculated the emergent intensities in two cases.

Karp, Greenstadt and Fillmore [43] presented a method for solving the equation of radiative transfer in a vertically inhomogeneous planetary atmosphere. This method, based on the spherical harmonics expansion, can be used to compute models with an arbitrarily large optical thickness and any scattering phase function. They considered the equation of transfer with monochromatic radiation as

$$\mu \frac{dI(\tau;\mu,\Phi)}{d\tau} = I(\tau;\mu,\Phi) - J(\tau;\mu,\Phi) \tag{1.115}$$

where $I(\tau;\mu,\Phi)$ is the specific intensity, μ is the cosine of the zenith angle, Φ is the azimuth angle measured from Sun's meridian and $J(\tau;\mu,\Phi)$ is the source function defined by

$$\begin{aligned} & J(\tau;\mu,\Phi) \\ = \; & \frac{1}{4} P(\tau;\mu,\Phi;-\mu_0,\Phi_0) F_0 e^{-\frac{\tau}{\mu_0}} \\ + \; & \frac{1}{4\pi} \int_{-1}^{1} \int_{-1}^{2\pi} P(\tau;\mu,\Phi;\mu',\Phi') I(\tau;\mu',\Phi') d\mu' d\Phi', \end{aligned} \tag{1.116}$$

where the external illumination πF_0 was assumed to be unidirectional. The boundary conditions were

$$I(0;\mu < 0,\Phi) = I(\tau_b;\mu > 0,\Phi) = 0. \tag{1.117}$$

Their technique for solving the above problem was mainly based on Spherical Harmonic Method. To illustrate the range of validity of this method, they compute the plane albedo from model atmospheres containing clouds with optical thicknesses ranging from 0 to 10^6.

Karp and Petrack [44] find the relationship between the discrete ordinates and spherical harmonics methods.It is known that the azimuth-averaged component of the intensity computed from the spherical harmonics method for solving the equation of radiative transfer is 'exact' at the Gauss quadrature points. .They shown that a similar relation holds for higher terms in the Fourier expansion of the intensity but that the results are 'exact' at the zeros of the associated Legendre polynomials. they considered the equation of transfer for plane parallel, scattering and absorbing atmosphere

$$\mu\frac{dI(\mu,\Phi,\tau)}{d\tau} = I(\mu,\Phi,\tau) + \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} I(\mu',\Phi',\tau)P(\tau,\mu,\mu';\Phi,\Phi')d\mu' d\Phi' \tag{1.118}$$

where $P(\tau,\mu,\mu';\Phi,\Phi')$ is the scattering phase function. They expanded 1.118 in a Fourier series in Φ and obtained the following equation

$$\mu\frac{dI^m(\mu,\tau)}{d\tau} = I^m(\mu,\tau) - \int_{-1}^{1} I(\mu',\tau)P^m(\tau;\mu,\mu')d\mu'. \tag{1.119}$$

They considered the following form of phase function

$$P^m(\tau;\mu,\mu') = \sum_{l=m}^{L+m} \beta_l(\tau)Y_l^m(\mu)Y_l^m(\mu') \tag{1.120}$$

where $Y_l^m(\mu)$ are normalized spherical harmonics (or the associated Legendre polynomials) and $\beta_l(\tau)$ are expansion coefficients of the scattering phase function. The form of intensity considered by them is

$$I^m(\tau,\mu) = \sum_{l=m}^{L+m} \frac{2l+1}{2} f_l^m(\tau)Y_l^m(\mu), \tag{1.121}$$

where $f_l^m(\tau)$ are the moments of the intensity. A discrete ordinates quadrature scheme, based on the zeros of the associated Legendre polynomials, was shown to maintain the correspondence of the methods for these problems, as well as providing a better set of points than other methods in use.

Wan, Wilson and Sen [73] shown that the DISHM can be easily extended to radiative transfer problem in a plane medium with reflective boundaries. They took the radiative transfer equation for this model

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau}+I(\tau,\mu)=\frac{\omega}{2}\int_{-1}^{+1}p(\mu,\mu')I(\tau,\mu')d\mu', \qquad (1.122)$$

where I is the intensity; ω is the albedo of single scattering; τ the optical depth; $p(\mu,\mu')$ the phase function, μ being the angle measured from the direction of increasing τ.

They consider the case where the boundaries have emissivities and both specular and diffuse reflectivities, such as

$$I(0,\mu)=\epsilon_1+\rho_1^s I(0,-\mu)+2\rho_1^d\int_0^1 I(0,-\mu')\mu' d\mu', \quad \mu>0 \qquad (1.123)$$

and

$$I(\tau_0,-\mu)=\epsilon_2+\rho_2^s I(\tau_0,\mu)+2\rho_2^d\int_0^1 I(\tau_0,\mu')\mu' d\mu', \quad \mu>0, \qquad (1.124)$$

where ϵ is the boundary emissivity, ρ^s, ρ^d being the specular and diffuse reflectivities of the boundaries.

Intensities $I_+(\tau,\mu)$ and $I_-(\tau,\mu)$ are represented for μ in the interval (0,1) and (-1,0) respectively as

$$I_+(\tau,\mu)=\phi(\tau)+\sum_{l=0}^{l_0}(2l+1)I_l^+(\tau)\mu P_l(2\mu-1), \quad 0\le\mu\le 1 \qquad (1.125)$$

and

$$I_-(\tau,\mu)=\phi(\tau)+\sum_{l=0}^{l_0}(2l+1)I_l^-(\tau)\mu P_l(2\mu+1), \quad -1\le\mu\le 0 \qquad (1.126)$$

with

$$\phi(\tau) = A\ exp(-\tau) + B\ exp(\tau).$$

They took planetary phase function of the form

$$p(\mu, \mu') = 1 + a\ \mu\ \mu'.$$

In three different table for ω = 0.2, 0.8 and 0.95, they showed that the values of flux obtained by the present method are in fair agreement with the exact values.

Generalized spherical harmonic solution for radiative transfer models which includes polarization effects is done by R. D. M. Garcia and C. E. Siewert [24]. This generalized spherical harmonic solution is developed for all components ($m \geq 0$) in a Fourier representation of the Stokes vector basic to the scattering of polarized light and computational aspects of the solution are discussed in detail. The established solution is used in regard to two test problems to obtain numerical results, accurate, in general, to five significant figures, for the four Stokes parameters. They denoted I (τ, μ, Φ) as density vector with the four stokes parameters the components and considered the following equation of transfer

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu,\Phi) + I(\tau,\mu,\Phi) = \frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\mu',\Phi-\Phi')I(\tau,\mu,\mu')d\mu' d\Phi' \tag{1.127}$$

where $P(\mu, \mu', \Phi - \Phi')$ is the phase matrix. They solved the equation 1.127 subject to the following boundary conditions

$$I(0,\mu,\Phi) = \pi\delta(\mu-\mu_0)\delta(\Phi-\Phi_0)F \tag{1.128}$$

$$I(\tau_0,-\mu,\Phi) = \frac{\lambda_0}{\pi}L\int_0^{2\pi}\int_0^{i} I(\tau_0,\mu',\Phi')\mu' d\mu' d\Phi' \tag{1.129}$$

where λ_0 is the coefficient for lambert reflection, $L = diag\{1, 0, 0, 0\}$ and F is the flux vector.

The established solution was used in regard to two lest problems to obtain numerical results, accurate, in general, to five significant figures, for the four stokes parameters.

Takeuchi [66] described a method for numerical solution with spherical harmonics of the equation of transfer for plane-parallel atmospheric models with arbitrary inhomogeneities in the vertical direction and with axially-symmetric phase functions. Takeuchi (1988) showed that the spherical harmonics approximation reduces the m-th Fourier component of the equation of radiative transfer to a system of infinite homogeneous ordinary differential equations He expanded each term of the equation by the normalized associated Legendre polynomials $Q_l^m(\mu)$ which are related to the associated Legendre polynomials $P_l^m(\mu)$ by

$$Q_l^m(\mu) = \frac{[(2l+1)(l-m)!]^{\frac{1}{2}}}{[2(l+m)!]^{\frac{1}{2}}} P_1^m(\tau). \tag{1.130}$$

The scattering phase function was taken to be

$$\Phi(\tau;\mu_\alpha) = \omega_0(\tau)\sum_{l=o}^{\infty}\omega_l(\tau)P_l(\mu_\alpha). \tag{1.131}$$

Takeuchi [66] showed that the adaptation of the normalized associated Legendre polynomials $S_l^m(\mu)$ made SHM very easy to handle. The eigenvectors inherent in each scattering layer are obtained by using-value decomposition.

His method had the following features: (1) high efficiency in finding eigenvalues and eigenvectors associated with characteristics of the scattering layer; (2) stable expressions for the condition linking the radiance at the top to the bottom of the layer; (3) economy in calculations for a multi-layer problem. A smoothing method to obtain reasonable intensity patterns was also discussed.

Tine, Aiello, Belleni and Pestellini [67] extend the spherical harmonics method (SHM) to solve the radiative transfer equation for the case of radially-varying extinction and scattering coefficients. The formalism was first introduced and discussed. They considered the following form of radiative transfer equation

$$\mu\frac{\partial I(r,\mu)}{\partial r}+\frac{l-r^2}{r}\frac{\partial I(r,\mu)}{\partial \mu}=-\alpha(r)I(r,\mu)+\frac{\alpha^{sca}(r)}{2}\int_{-1}^{1}I(r,\mu')p(\mu,\mu')d\mu'+\epsilon(r), \tag{1.132}$$

with the boundary condition

$$I(r,\mu)=I^{est}(\mu),\ \ \mu\leq 0. \tag{1.133}$$

In equation (1.132), $\alpha(r)$ and $\alpha^{sca}(r)$ are respectively the scattering coefficients for unit length and $P(\mu,\mu')$ is the phase function. In equation (1.132), $I^{est}(\mu)$ is the externally incident radiation, if any. They considered the following from of the phase function

$$p(\mu,\mu')=\sum_{l=0}^{N}\sigma_l P_l(\mu)P_l(\mu') \tag{1.134}$$

and the form of the intensity is

$$I(r,\mu)=\sum_{l=0}^{L}(2l+l)F_l(r)P_l(\mu).$$

They applied the technique to Henyey-Greenstein phase function and some numerical results of astrophysical importance were then presented.

Biswas and Karanjai [36] used modified double interval SHM to solve the equation of radiative transfer with anisotropic scattering in a thin atmosphere. They considered the following equation of transfer

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau}+I(\tau,\mu)=\frac{1}{2}\int_{-1}^{1}P(\mu,\mu')I(\tau,\mu')d\mu' \tag{1.135}$$

and the boundary conditions for their problem were

I) an isotropic radiation field I_0 at $\tau=0$, i.e.

$$I(0,\mu)=I_0\ =1(say)\ \ 0\leq\mu\leq 1$$

II) no incoming radiation at $\tau=\tau_0$, i.e.

$$I(\tau_0,\mu)=0,\ \ -1\leq\mu\leq 0.$$

They considered two different forms of intensities in two different direction followed by Wilson and Sen [74]. They had taken Rayleigh phase function and find the corresponding results and in another paper [37] they took the phase function

$$p(\mu, \mu') = 1 + \omega_1 P_1(\mu)P_1(\mu') + \omega_2 P_2(\mu)P_2(\mu') \tag{1.136}$$

and showed the similarity of results.

Talukdar and Karanjai [39] applied a modified SHM to solve the equation of transfer with the general phase function. They considered the equation of transfer in plane parallel atmosphere with axial symmetry and given by

$$\mu\frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - \frac{1}{2}\int_{-1}^{1} P(\mu, \mu')I(\tau, \mu')d\mu' \tag{1.137}$$

where the symbols have their usual meanings and phase function was taken to be

$$p(\mu, \mu') = \sum_{k=0} w_k P_k(\mu)P_k(\mu'). \tag{1.138}$$

The form of intensity considered by them was

$$I_+(\tau, \mu) = I(0, 0)\left[A(\tau) + \sum_{l=0}^{l_0}(2l + l)I_l^+(\tau)\mu P_l(2\mu - 1)\right], \quad 0 \le \mu \le 1 \tag{1.139}$$

$$I_-(\tau, \mu) = I(0, 0)\left[A(\tau) + \sum_{l=0}^{l_0}(2l + l)I_l^-(\tau)\mu P_l(2\mu + 1)\right], \quad -1 \le \mu \le 0 \tag{1.140}$$

They solved the equation (1.137) using general phase function (1.138) and derived the solution in the case of different phase functions, viz. Rayleigh, Planetary, and Henyey-Greenstein.

Raychaudhuri and Karanjai [59] used a new modification of the form of intensity

$$I^{+}(\tau,\mu)=I(0,0)\left[\phi(\tau)+\psi(\mu)+\sum_{l=0}^{l_0}(2l+1)I_l^{+}(\tau)\mu P_l(2\mu-1)\right],\ \ 0\leq\mu\leq 1$$

$$I^{-}(\tau,\mu)=I(0,0)\left[\phi(\tau)+\psi(\mu)+\sum_{l=0}^{l_0}(2l+1)I_l^{-}(\tau)\mu P_l(2\mu+1)\right],\ \ -1\leq\mu\leq 0.$$

where $I(0.0)$ is some constant, $\phi(\tau)$ is a function of τ only and $\psi(\mu)$ is given by

$$\psi(\mu)=\begin{cases}1, & if\quad 0\leq\mu\leq 1,\\ 1, & if\quad -1<\mu\leq 0,\end{cases}\tag{1.141}$$

in the double interval spherical harmonic method to solve the integro-differential equation of radiative transfer. The method was used to solve problems in case of plane geometry and is extended to the problem in plane geometry in case finite atmosphere with Rayleigh phase function where the form of the function $\phi(\tau)$ has been taken to be $\phi(\tau)=Ae^{\tau}+Be^{-\tau}$, where A and B are constants and are determined.

Szu-Cheng S. Ou and Kuo-Nan Liou [65] make a generalization of the spherical harmonic method to the radiative transfer in multi-dimensional space. The basic radiative transfer equation in three-dimensional space is expressed in terms of three commonly used coordinate systems, namely, Cartesian, cylindrical and spherical coordinates. The concept of a transformation processes between the Cartesian system and two other systems. The spherical harmonic method is then applied to decompose the radiative transfer equation into a set of coupled partial differential equation for all three systems in terms of partial differential operators. They obtain Helmholtz equation by truncating the number of partial differential equations into four along with farther mathematical analysis. For each coordinate system, analytical solutions have been done in terms of infinite series whenever the equation is solvable by the technique of separation of variables with proper boundary conditions. Numerical computations are carried out for one dimensional radiative transfer to illustrate the applicability of the technique developed in this study.

The spherical harmonic method has been used to obtain solution of radiative transfer equation by M.A. Atalay [1] for a slab with reflective boundaries. An absorbing, emitting, non-isothermal, gray medium is considered with lineally anisotropic scattering. Under the condition of the thermal equilibrium, the slab boundaries are subjected to specular and diffuse reflection. The analytical form of solutions is obtained for both conservative and non-conservative cases. The accuracy of the method was verified by the normal-mode expansion technique. The predictions of heat flux were found to in good agreement with the benchmark data.

The equation of radiative transfer in the Melin Eddington model for interlocked doublets has been solved by Mukherjee and Karanjai [52]. They took the equation of transfer for doublets

$$\begin{aligned} \mu\frac{dI_r(\tau,\mu)}{d\tau} &= (1+\eta_r)I_r(\tau,\mu) - (1+\epsilon\eta_r)(a+b\tau) \\ &\quad -(1-\epsilon)\alpha_r\frac{1}{2}\eta_p\left[\int_{-1}^{1} I_p(\tau,\mu')d\mu'\right], \qquad (1.142) \\ r &= 1,2. \end{aligned}$$

They showed the corresponding results for 1st and 2nd approximation.

1.3 Coherent Scattering

When the frequency of absorption of radiation in an atom is the same with the frequency of emission, the scattering is said to be coherent otherwise it will be called non-coherent.
We consider the scattering as coherent or non-coherent according to our theoretical point of view. When an atom absorbs energy of certain frequency, the probability that the energy will be re-emitted in the same frequency will be maximum if the atom,

(I) is at rest,

(II) is in the lowest quantum state,

(III) is in a weak radiation field .

Other than of any of the above three conditions the scattering will be non-coherent.

1.3.1 Development of the equation

Let us take a stellar atmosphere consisting of plane parallel layers bounded at x = 0 and extending to infinity in the negative direction of the coefficient of absorption k_ν is a function of the frequency of the radiation. Let $I\nu(\theta)$drdw be the flow a radiation in frequencies from ν to $\nu + d\nu$ within the solid angle $d\omega$ in a direction making an angle θ with the outward normal and let $4\pi j_\nu d\nu$ be the rate of emission within ν to $\nu + d\nu$ by unit mass of the atmosphere assuming the emission to be isotropic, then the equation is

$$cos\frac{dI_\nu(\theta)}{\rho dx} = -k_\nu I_\nu(\theta) + j_\nu. \qquad (1.143)$$

The absorption coefficient K_ν may be divided into two parts (i) continuous absorption - here variation with frequency within the absorption line may be neglected. (ii) line absorption - a rapidly changing part.
Accordingly we write

$$k_\nu = k + l_\nu.$$

The line absorption coefficient l_ν is the absorption coefficient for the atoms present per unit mass of the atmosphere.

Now we can write the equation (1.143) as

$$cos\frac{dI_\nu(\theta)}{\rho dx} = -(k + l_\nu)I_\nu(\theta) + j_\nu. \qquad (1.144)$$

The emission in the frequency ν per unit mass of the atmosphere is consist of two parts (i) re-emission from the selectively absorbing atoms, and (ii) re-emission from the atmospheric part which give rise to the continuous absorption. The emission (i) is the coherent scattering in the frequency ν due to spontaneous emission of quanta by atoms excited by radiation. If we assume that the direction of emission from an excited atom is independent of the direction of the absorbed quantum, the re-emission by an assembly of atoms may be taken as isotropic. Thus, if all the energy absorbed by the atoms is emitted as isotropic radiation, the emission (i) will be the absorption averaged over all directions of incidence of the radiation. this will be given by

$$\frac{1}{4\pi}\int l_\nu I_\nu(\theta)d\omega = l_\nu J_\nu. \tag{1.145}$$

We consider that the emission (ii) takes place as in local thermodynamic equilibrium, and is given by k $B(\nu, T)$. Hence

$$j_\nu = l_\nu J_\nu + kB(\nu, T). \tag{1.146}$$

With the emission coefficient given by (1.146), equation (1.144)describes the coherent formation of an absorption line when the atoms are subject only to radiative transitions between two energy levels.

Consider now the effect of collisions on the emission of radiation by excited atoms. If an atom experiences a collision during the natural lifetime in the excited state, the energy of excitation may be transformed into kinetic energy, and the atom and colliding particle rebound from their super-elastic collision with increased velocity, sharing the energy of excitation between them . If the fraction ϵ of atoms excited by radiation of frequency ν is prevented from contributing to the coherent re-emission by super-elastic collisions, the line emission given by (1.145) will be reduced to $(1-\epsilon)l_\nu J_\nu$. Now in thermodynamic equilibrium the conversion of radiation into kinetic energy by super-elastic collisions is exactly balanced by the reverse process of conversion of kinetic energy into radiation by way of inelastic collisions Thus the emission in the frequency ν by atoms excited by inelastic collisions may be found from the fact that it balances the amount $\epsilon l_\nu J_\nu$ lost to the radiation field by super-elastic collisions when J_ν has its equilibrium value $B(\nu, T)$. The total emission is therefore

$$j_\nu = (1-\epsilon)l_\nu J_\nu + \epsilon l_\nu B(\nu, T) + kB(\nu, T), \tag{1.147}$$

and equation (1.143) may then be written in the form

$$cos\theta\frac{dI_\nu(\theta)}{\rho dx} = -(k+l_\nu)I_\nu(\theta) + (1-\epsilon)l_\nu J_\nu + (k+\epsilon l_\nu)B(\nu,T), \quad (1.148)$$

Now, if we set $\eta = \frac{l_\nu}{k}$ the equation (1.148) becomes

$$\mu\frac{dI(\tau,\mu)}{d\tau} = (1+\eta)I(\tau,\mu) - (1-\epsilon)\eta J_\nu - (1+\epsilon\eta)B(\nu,T), \quad (1.149)$$

a result by Woolley and Stibbs [80] and due to Eddington [22].

1.3.2 Work done so far in solving Coherent Scattering Problems

Busbridge [3] solved the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a linear function of optical depth, viz.

$$B_\nu(T) = b_0 + b_1\tau \quad (1.150)$$

by a modified principle of inverience method. The exact solution to the equation of coherent line formation in an η-constant atmosphere in which

$$B_\nu = a_\nu + b_\nu\tau_\nu \quad (1.151)$$

has been carried out by Chandrasekhar [11]. Chandrasekhar [12] solved the same problem by the method of discrete ordinates. The same problem has also been solved by Eddington's [22] method (when η_ν, the ratio of line to the continuous absorption coefficient is constant) and by Stromgren's [63] method (when η_ν has small but arbitrary variation with depth).

Dasgupta [16] applied the method of Laplace transform and Wiener-Hopf technique to find an exact solution of the transfer equation for coherent scattering in stellar atmosphere with Planck's function as a sum of elementary functions

$$B_\nu(T) = b_0 + b_1\tau + \sum_{r=2}^{n} b_r E_r(\tau),$$

by use of a new representation of the H-function obtained by Dasgupta [15].

Karanjai and Deb [38] obtained an approximate solution of the equation of transfer for coherent scattering in stellar atmospheres with plank's function as a nonlinear function of optical depth [Exponential atmosphere] given by [Woolley and Stibbs]

$$cos\theta\frac{dI_\nu(\theta)}{\rho dx} = -(k+I_\nu)I_\nu(\theta) + (1-\epsilon)I_\nu J_\nu + (k+\epsilon I_\nu)B_\nu(T) \quad (1.152)$$

By the method of Eddington. They considered

$$B_\nu(T) = b_0 + b_1 e^{-\beta\tau}, \quad (1.153)$$

where β, b_0, b_1 are positive constants.

Similar problem has been solved by Deb and Karanjai [19], by the method of Busbridge.

Oliveira, Cardona, Vilhena and Barros [53] described a semi-analytical numerical method for coherent isotropic scattering time dependent radiative transfer problems in slab geometry. The numerical method which they used was based on a combination of two classes of numerical methods, (i) the spherical methods and (ii) the Laplace transform method applied to the radiative transfer equation in the discrete ordinate formulation. At the end of the paper they showed some numerical experiments for a typical model problem.

Efimov, Kryzhevoi, Wandenfels and Wehrse [23] in their paper, presented a method for the fast and accurate solution of the radiative transfer equation in plane-parallel media with coherent isotropic scattering. Their approach was analytical used the formation of meromorphic functions in order to obtain the angle and depth variation of the radiation field. They applied the method to the particular case of a finite slab whose midplane is the symmetry plane.

Uitenbroek [68] investigate in their paper the influence of partial frequency redistribution (PRD) on radiative cooling due to the Ca II K line in a hydrostatic model of the quiet Sun, and in a series of 20 snapshots from a chromospheric radiation hydrodynamics simulation. They found that the approximation of complete of the net radiative rates because of its neglect of coherent scattering.

Keller and Tomas [45] have been presented in their paper the approximate solutions obtained for a radiative transfer problem that represents a highly idealized description of the multiple scattering of solar resonance radiation in the nearby inter-stellar medium. The problem of a point source in a centre of a spherically symmetric cavity embedded in an infinite uniform medium is solved for a range of cavity radii. They were calculated first and second order scattering contributions and the Eddington approximation is used to estimate the higher order components of the radiation field. It was shown that for coherent scattering at a very large cavity radii, the back scattered intensity from the cavity approaches three times the value deducted from the optically thin solution. It is concluded by the authors that an accurate analysis of sky background will require including not only the frequency redistribution, but also the correct spatial distribution of density.

A new formalism has been developed by Shia, Run-Lie [62] for partially coherent wave scattering in a random medium. In this formalism the coherent wave is the solution of a phenomenological wave equation and a simple integral equation is satisfied by the mutual coherence function of the wave field. A simple problem is solved to find the mutual coherence function produced by a laser beam in the atmosphere. The problem of multiple scattering of non-polarized light in a planetary body of arbitrary shape illustrated by a parallel beam is formulated using the integral equation approach. The analysis makes down a direct relation between the microscopic symmetry of the phase function for each scattering event and for the entire planet body and the intimate connection between these symmetry relationship and variational principle.