ON SOME PROBLEMS OF WAVE PROPAGATION IN SEMI-INFINITE ELASTIC MEDIA

By

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THE THESIS SUBMITTED TO
UNIVERSITY OF NORTH BENGAL
FOR DOCTOR OF PHILOSOPHY (SCIENCE)
AUGUST, 1984

ACKNOWLEDGEMENTS

The author wishes to express his immense gratitude to his supervisor Professor Mohan Lal Ghosh, D.Sc.(Cal) of North Bengal University. Without his sustained encouragement and inspiring guidance this thesis would not have been completed. He is also thankful to Professor Dr. M. Mitra of Barasat Government College whose illuminating discussion on one of the topics presented here helped the author a lot.

The author expresses his thanks to all his departmental colleagues and friends for their inspiration and encouragement. Finally, he acknowledges with thanks the services of Mr. Chittaranjan Nayek of the North Bengal University Research Service Centre for computational work, of the artist Mr. Bhupesh Kumar Bagchi, and of Junior Assistant cum typist Mr. Mohan Kumar Modak of the Department of Mathematics, North Bengal University. Finally, the author expresses his sincere thanks to his parents, wife and son for their unwavering patience and cooperation during the process of work.

Some financial assistance from the North Bengal University is acknowledged with thanks.

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August, 1984.

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INTRODUCTION

The theory of propagation of waves in elastic solids was developed in the last century. The names of the academicians that may be associated for pioneer theoretical work on this line are, STOKES, POISSON, RAYLEIGH, KELVIN and others, who extended the theory of elasticity to the problem of vibrating bodies and propagation of waves in elastic material. During the first quarter of this century the subject lost much of its glamour and interest. This is perhaps partly because of the attractions of the new fields opened up by the discoveries in atomic physics and partly because of a gap between the advancement of theoretical and experimental work, as there were no practical methods available in laboratory for observing the passage of stress waves in elastic materials. But there has been a remarkable revival of interest in the subject as far back as thirties. Extensive study in seismic wave propagation, earthquake engineering and research on geophysical phenomena attract a number of theoretical and practical workers. Since then the interest in the subject has been gathering momentum. With the advent of sophisticated instruement, electronic techniques and high speed computers, the subject has become a very important field of research. A large number of original papers on both experimental and theoretical aspects of the subject have been appearing with various information.

Most of the experimental works carried out on the wave propagation are concerned with studying propagation in specimens of comparatively simple geometrical shape, the results of this experiment could be compared directly with exact or approximate theoretical predictions. The agreement, with experimental results and theoretical predictions, inspires confidence in taking up complicated problems and makes possible theoretical predictions and interpretations of observations.

The propagation of waves through homogeneous isotropic elastic material of unbounded extension is not a subject of very complexity. The waves are either dilatational or distortional or a combination there of. The picture changes radically as soon as there is a boundary. Interaction of two types of waves occurs, when boundary is present and this interaction presents an inherent difficulty in the solution of elasto-dynamic problems. More over the effect of a free surface on the generation and propagation of waves in elastic medium has been the subject of many investigations ever since the discovery in existence of surface waves by LORD RAYLEIGH.

In general, problems which mostly attract the researchers both theoretical and experimental, in relation to the generation and propagation of waves in an elastic medium may be classified as

i) diffraction of propagating waves through the medium due to any obstacle, cavity or a crack of any shape situated somewhere in the medium;

ii) reflection, refraction and diffraction of propagating waves due to mixed boundary conditions;

iii) wave motion generated due to a punch on some bounded region of the medium;

iv) radiation of waves i.e the wave motions generated due to some fixed external disturbance and propagating away from the source of disturbance;

v) wave motion generated in a medium when a source of disturbance moves along the medium.

Depending on the nature of the source of disturbance, shape of the punch or normal loading on the free surface and the presence of discontinuities in the medium, different complicated problems arise. The solution of these problems need an advance level of sophisticated mathematical techniques.

We present some of the mathematical techniques in short and give references to some of the problems along with their solutions. These may be important and interesting in engineering science, in earthquake engineering, in geophysics and in seismology, and definitely to the mathematicians because of the complicated mathematics involved in the formulation of the problems and in the determination of their solutions.

The dynamic response of an elastic half space due to an external load or a punch on the free surface and also the scattering of elastic waves by a finite crack or a strip inside an elastic medium may be investigated by the use of integral transform techniques.

The integral transform $\bar{f}(\zeta)$ of a function $f(x)$ defined in an interval (a, ∞) is an expression of the form

$$\bar{f}(\zeta) = \int_a^\infty f(x)K(x, \zeta)dx, \qquad (1)$$

where a is real number and ζ is a complex parameter varying over some region D of the complex plane. $K(x,\zeta)$ is called the kernal of the transformation. The transformation (1) becomes particularly useful if it possesses inverse mapping. In that case one can express $f(x)$ in terms of its integral transform by

$$f(x) = \frac{1}{2\pi i}\int_\Gamma \bar{f}(\zeta)M(x,\zeta)d\zeta \quad .$$

Here $M(x,\zeta)$ is a suitable function defined in $a < x < \infty$ and $\zeta \in D$ and is called the kernel of the inverse transform, which is defined for all x in the interval (a,∞). The complex parameter ζ is in the region D while Γ is a suitable path of integration in D.

With the aid of these transforms one can replace some of the independent variables, so that in many cases it is possible to reduce the governing partial differential equations to ordinary differential equation in the transformed space with one less independent variable. If the reduced equation for $\bar{f}$ can be solved, the solution f can be expressed in terms of the inversion integral, which may then be evaluated. The inversion from the transformed space to the space of actual variables usually involves very complicated integrations. In many cases even the numerical integration can not be performed successfully because of the highly oscillatory character of the integrands. [cf. ERINGEN and SUHUBI (1975), chap.7; ACHENBACH (1976), Chap.7]. It is mentioned in the preceeding para that the inversion from the transformed space to the original space of variables involves various complications. In particular, mixed boundary value problems like the dynamic response of a punch on an elas-tic half space and the problems involving the presence of a crack or a strip inside an elastic medium may be reduced to FREDHOLM's integral equation of first kind or to dual integral equations.

Different techniques have been applied by many authors to tackle these type of problems. From these stand point, these problems may be divided into two categories: one for low frequency oscillation of the source or long wave scattering or transmission and the other for high frequency oscillation or short wave scattering or transmisson in the medium. The terms long and short are used in comparison to the region of the source of distrubance or the size of the strip or crack etc. in side the medium to the wave length of distrubance. In case of low frequency oscillations NOBLE's (1963) method of solving dual integral equations, TRATER's (1962) technique for solving dual integral equations, Matched Asymptotic Expansion, and variational principle are found to be very useful whereas in case of high frequency oscillations WEINER-HOPF (NOBLE,1958)technique, or Geometrical Ray Theory are found to be most suitable.

Suppose that a mixed boundary value problem is formulated by suitable integral transform so as to be governed by a set of dual integral equation of the form

$$\begin{aligned} &\int_0^\infty G(p)f(p)J_\nu(rp)\,dp = g(r), \quad 0 < r < 1 \\ \text{and} \quad &\int_0^\infty f(p)J_\nu(rp)\,dp = 0, \quad r > 1 \end{aligned} \tag{2}$$

where G(p), g(r) are the known functions of the variable indicated while J_ν is a Bessel function of the first kind of order ν. We are supposed to determine f(p).

According to NOBLE (1963) ifG(p) can be expressed in the form

$G(p) = p^\gamma[1 + H(p)]$ and that if H(p) tends to zero as $p \to \infty$ then he proved that

$$f(p) = \frac{p^{1-\frac{\gamma}{2}}}{2^{-1+\frac{1}{2}\gamma}\Gamma(\frac{1}{2}\gamma)} \int_0^1 \xi^{1/2}\,\theta(\xi)\,J_{\nu+\frac{1}{2}\gamma}(\xi p)\,d\xi$$

where $\theta(\xi)$ satisfies the FREDHOLM integral equation of the second kind

$$\theta(x)+\frac{1}{\pi}\int_{0}^{1}K(x,\xi)\theta(\xi)\,d\xi = F(x) \qquad (3)$$

with kernal

$$K(x,\xi)\pi(x\xi)^{1/2}\int_{0}^{\infty}pH(p)J_{\nu+\frac{1}{2}\gamma}(xp)J_{\nu+\frac{1}{2}\gamma}(\xi p)\,dp$$

and if $o < \gamma < 2$ then

$$F(x) = x^{-\nu-\frac{1}{2}\gamma+\frac{1}{2}}\int_{0}^{x}g(r)r^{\nu+1}(x^2-r^2)^{-1+\frac{1}{2}\gamma}\,d\gamma .$$

The integral equation (3) can be solved for $\theta(x)$ and consequently $f(p)$ can be determined.

ROBERTSON (1966) considered a circular rigid disc pressed on the free surface of an elastic half space and vibrating on the surface having a smooth contact. The displacement being specified under the disc and the surface outside the disc is assumed to be stress ~~f~~ free. This mixed boundary value problem is reduced to a set of dual integral equations. The solution of this dual integral equations is then reduced to the FREDHOLM integral equation of the second kind, an iterative solution of whi-ch is then obtained for low frequency oscillation of the disc.

In another paper ROBERTSON (1967,b) considered a longitudinal wave harmonic in time to be incident normal to a penny-shaped crack on a semi-infinite elastic solid. This problem is also formulated by a set of dual integral equations. An equivalent FREDHOLM integral equation of the second kind is determined by the use of NOBLE's (1963) method, which is also solved by iterative method assuming low frequency oscillation of the applied stress on the crack surface. GLADWELL (1968) assumed a circular indentor vibrating about one of its diameter. A displacement under the indentor is prescribed and stress out side the indentor is taken to be zero. This mixed boundary value problem is formulated so as to be governed by a set of dual integral equations which is again converted to an equivalent Fredholm integral equation of the second kind by the use of NOBLE's method and the solution is obtained for low frequency oscillation of the indentor.

Almost in a similar way MAL, ANG and KNOPOFF (1968) determined the solution of the problem of diffraction of axisymmetric harmonic elastic waves by a rigid circular disc, MAL (1969) studied the diffraction of elastic waves in presence of a penny-shaped crack inside a semi-infinite medium and SINGH, DHALIWAL and VRBIK (1983) considered the mixed boundary value problem arising out of the interaction of the anti-plane shear waves to an arbitray angle to the moving crack.

WICKHAM (1977) considered the time harmonic vibration of frequency w of a rigid infinite strip in a semi-infinite homogeneous isotropic elastic solid. The motion is forced by prescribed displacement distribution $v_o(x)e^{-iwt}$, normal to the infinite strip $|x|<1, y=o, -\infty<z<\infty$. It is assumed that the tangential stress at the plane $y=o$, is zero and the normal stress is also zero for $|x|>1$. Boundary conditions are formulated as

$$v(x,o) = v_o(x) \quad ; \quad |x|\leq 1$$
$$\tau_{yy}(x,o) = o \quad ; \quad |x|>1$$
$$\tau_{xy}(x,o) = o \quad ; \quad -\infty < x < \infty.$$

In this paper the approach is some what different from the others, which has already been discussed. The author assumed a function $f(x)$

(unknown) to be the normal stress below the strip. Thus $\tau_{yy}(x,o)$ is assumed to be known on $(-\infty, \infty)$. Assuming the normal stress to be known on $y = o$, the integral representation for the potentials ϕ and Ψ are determined by the use of Green's function. with the help of ϕ and Ψ (which involve the unknown function $f(x)$) an integral equation of the first kind for $f(x)$ is obtained, which is then converted to an integral equation of the second kind by NOBLE's method and the iterative solution is obtained for small wave number.

In this connection I could not resist the temptation of referring to a paper by FABRIKANT and SANKAR (1984), where a mixed boundary value problem in a non-homogeneous medium is considered. An asymmetric contact in the form of a circle $\rho = a$ is assumed to be present on the half space. An arbitrary normal displacement is prescribed inside the circle $\rho = a$, while the boundary $Z = 0$ of the half space out side the circle is stress free and the tangential stress vanishes all over the plane $Z=0$. Following ROSTOVTSEV (1964) the solution of the problem is reduced to the solution of a two dimensional integral equation in polar co-ordinates. The authors then proved that it is possible to reduce the integral equation to a sequence of two Abel type integral operator and another operator introduced by them. The inverse of the operators can be found easily and the exact solution of the two dimensional equation is obtained in the closed form.

In all the cases discussed above, dual integral equations are converted to Fredholm's integral equation of the second kind which is then solved iteratirely for low frequency oscillation. But TRANTER's method (1962) of solving dual integral equation is different. If a mixed boundary value problem is formulated as in (2), then according to TRANTER f(p) (which is to be determined) is taken in the form

$$f(p) = p^{1-k} \sum_{m=0}^{\infty} a_m J_{\nu+2m+k}(p), \qquad (4)$$

where m is a positive integer or zero, k is real and positive and the real part of ν is greater than -1. It may be seen that under the conditions stated above the integral

$$\int_0^\infty p^{1-k} J_{\nu+2m+k}(p) J_\nu(rp)dp \qquad (5)$$

converges for both $r > 1$ and $0 < r < 1$ and its value is also given by WATSON (1944). For a choice of f(p) as given in (4) and using the value of the integral (5) it can be proved that the second of the equation (2) is automatically satisfied. The coefficients a_m have to be so chosen that the form of f(p) as given in (4) also satisfies the first equation of (2). Again from WATSON (1944) we have the x value of the integral in (5) to be equal to

$$\frac{r^\nu \Gamma(\nu+m+1)}{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)} \; {}_2F_1(\nu+m+1, 1-k-m; \nu+1; r^2) \qquad (6)$$

when $0 < r < 1$.

The value of f(p) from (4) is substituted in the first equation of (2). After some algebraic simplification with the help of Jacobi polynomial, and using the value of the integral (5) as given in (6), the Hankel inversion formula and also noting that

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a, c-b; c; z). $$

We finally obtain

$$ \sum_{m=0}^{\infty} a_m \int_0^{\infty} G(p) p^{1-2k}\ J_{\nu+2m+k}(p)\ J_{\nu+2n+k}(p)\ dp = E(\nu,n,k) \tag{7} $$

where $\dfrac{2^{k-1}\,\Gamma(\nu+1)\,\Gamma(n+k)}{\Gamma(\nu+n+1)}\,E(\nu,\ n,\ k)$

$$ = \int_0^1 g(r) r^{\nu+1} (1-r^2)^{k-1}\ \mathfrak{F}_n(k+\nu,\ \nu+1, r^2)\, dr. \tag{8} $$

The Jacobi polynomial being defined by

$$ \mathfrak{F}_n(\alpha,\gamma,x) = {}_2F_1(-n,\ \alpha+n;\ \gamma;x). $$

As a special case when $g(r)$ in (7) is replaced by Ar^{ν} (A being constant) and making use of the result [cf.WATSON (1944),p.404], the equation (7) can be written as

$$ a_n + \sum_{m=0}^{\infty} L_{m,n}\ a_m = (2\nu+4n+2k)E(\nu,n,k) \tag{9} $$

where $(2\nu+4n+2k)^{-1}\ L_{m,n}$

$$ = \int_0^{\infty} \left\{ p^{2-2k}\ G(p) - 1 \right\}\ p^{-1}\ J_{\nu+2m+k}(p)\ J_{\nu+2n+k}(p)\ dp. $$

Since Jacobi polynomials satisfy the relations

$$ \mathfrak{F}_0(\alpha,\gamma,x) = 1 \text{ and} $$

$$\int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma}\, J_m(\alpha,\gamma,x)\, J_n(\alpha,\gamma,x)\,dx = 0$$

when $m \neq n$, we have from (8) , for $g(r) = Ar^{\nu}$

$$E(\nu,n,k) = A \quad \frac{\Gamma(\nu+1)}{2^k\Gamma(k+\nu+1)} \quad \text{for } n = 0$$

$$= 0 \qquad\qquad \text{for } n > 0 \tag{10}$$

Therefore by the use of (10), equation (9) takes the form

$$a_0 + \sum_{m=0}^{\infty} L_{m,0}\, a_m = A \; \frac{2^{1-k}\Gamma(\nu+1)}{\Gamma(\nu+k)} \tag{11}$$

$$a_n + \sum_{m=0}^{\infty} L_{m,n}\, a_m = 0 \ , \ n > 0.$$

The iterative solution of the equation (11) is

$$a_n = A\,\frac{2^{1-k}\,\Gamma(\nu+1)}{\Gamma(\nu+k)}\left[\delta_n - c_n + c_n' - c_n'' + \cdots\cdots\cdots\right],$$

where $\delta_n = 0$ for $n > 0$, $\delta_0 = 1$ and

$$C_n = L_{0,n},\ C_n' = \sum_{m=0}^{\infty} L_{m,n}\, C_m,\ C_n'' = \sum_{m=0}^{\infty} L_{m,n}\, C_m' \text{ etc.}$$

With the determination of the constants, the asymptotic solution of the dual integral equations in (2) is determined.

BOSE (1968) considered a rigid circular disc indented in a semi infinite elastic solid which performs small oscillations normal to its plane without loosing contact with the surface of the solid. With the heop of Hankel transform, the problem has been reduced to the solution of dual integral equation and then following the method of TRANTER to the solution of an infinite set of algebraic equations. At present a new technique known as matched asymptotic expansion has been introduced and developed to solve the mixed boundary value problems involving wave propagation in elastic solids due to low frequency vibration of the scatterer.

This method has primarily been used in solving the celebrated Navier Stokes equations of Hydrodynamics. It was developed by PRANDTL (1905) to solve the problem of high speed viscous flow past of a body. Some times this method is found to be convenient to solve scattering and diffraction of elastic waves by cracks and bodies of finite length.

Diffraction and scattering of elastic waves due to the presence of a finite crack or an obstacle of finite dimension can be solved by the method of matched asymptotic expansion in case when the dimension of the scatterer is small compared to the wave length of the propagating waves.

In this method, two expansions are developed simultaneously firstly, an inner expansion valid close to the scatterer and secondly, an outer expansion valid far away from it. The inner expansion is constrained to obey the boundary condition at the surface of the scatterer while the outer expansion generated so as to satisfy the condition at infinity but not at the surface of the scatterer.

To render the problem determinate, it is necessary to use the fact that the inner and outer expansions are different forms of the same function. This leads to the matching of these two expansions in an intermediate region. This makes it possible to derive alternatively the successive terms in each expansion. In this way construction of a uniformly valid composite expansion is done.

In order to explain the mathematical procedure, consider the dynamical equations of elasticity

$$(\lambda+\mu)\,\mathrm{grad}\,\Delta + \mu\,\nabla^2\vec{u} - \rho\,\frac{\partial^2\vec{u}}{\partial t^2} = 0 \tag{12}$$

Putting $\vec{u} = \vec{u}_0\, e^{iwt}$ and dropping the zero subscript one derives

$$(\lambda+\mu)\,\mathrm{grad}\,\Delta + \mu\,\nabla^2\vec{u} + \rho w^2\vec{u} = 0 \tag{13}$$

Introducing a characteristic geometric length l in the problem and putting

$\vec{u}' = \vec{u}/a$, $\vec{r}' = \vec{r}/a$ and finally dropping the primes, the above equation in the dimensionless form can be written as

$$\frac{(\lambda+\mu)}{\rho w^2 a^2}\,\mathrm{grad}\,\Delta + \frac{\mu}{\rho w^2 a^2}\,\nabla^2\vec{u} + \vec{u} = 0 \tag{14}$$

Setting $\vec{u} = \mathrm{grad}\,\Phi + \mathrm{rot}\,\vec{F}$, it is found that Φ and $\vec{F}$ satisfy the equation

$$\nabla^2\Phi + M^2\Phi = 0 \tag{15}$$

$$\nabla^2\vec{F} + m^2\vec{F} = 0 \tag{16}$$

where M and m are two dimensionless numbers defined by $M=\eta m$, and

$$m^2 = \frac{\rho w^2 a^2}{\mu} \quad \text{and} \quad \eta = \sqrt{\mu/(\lambda+2\mu)}\ .$$

The analysis is based on the asumption that M and m are small.

Let an axially symmetric body be depressed by an amount $d_o e^{iwt}$ along its axis of symmetry into the elastic space by an exciting

periodic force. If $\vec{I}$ stands for the unit vector along the axis of symmetry taken as n-axis, then the boundary conditions become
$\vec{u} = d_o \vec{I} e^{iwt}$ at the surface of the body while
$\vec{u} = o$ as $r \to \infty$.

Applying the principle of superposition and using nondimensional unit this is equivalent to a problem with boundary condition
$\vec{u} = o$ at the surface of the body
$\vec{u} = - \frac{d_o}{a} \vec{I}$ as $r \to \infty$.

In order to obtain appropriate inner solution for $\vec{u}$, we assume an expansion of the form

$$\vec{u} = \vec{u}_o + \vec{u}_1 + \ldots\ldots\ldots + \vec{u}_n + \ldots\ldots \tag{17}$$

such that

$$u_n = m^n \operatorname{grad} \varphi_n + M^n \operatorname{rot} \vec{f}_n \tag{18}$$

and

$$\Phi = \varphi_o + M\varphi_1 + M^2\varphi_2 + \ldots\ldots\ldots + M^n\varphi_n \ldots\ldots \tag{19}$$

$$\vec{F} = \vec{f}_o + m\vec{f}_1 + m^2\vec{f}_2 + \ldots\ldots\ldots\, m^n \vec{f}_n + \ldots$$

where φ_n and $\vec{f}_n$ satisfy the equations

$$\nabla^2 \varphi_o = o; \quad \nabla^2 \varphi_1 = o; \quad \nabla^2 \varphi_2 + \varphi_o = o \tag{20}$$

and

$$\nabla^2 \vec{f}_o = o; \quad \nabla^2 \vec{f}_1 = o; \quad \nabla^2 \vec{f}_2 + \vec{f}_o = o$$

The expansion given by (17) is assumed to satisfy the condition at the surface of the body only.

As such these expansions are valid only in the vicinity of the body. Next consider the matched outer expansion. To find the required outer expansion for Φ, we set

$$\bar{x} = Mx\ ,\ \bar{y} = My,\ \bar{z} = Mz.$$

As $M \to 0$, the point (x,y,z) will move to infinity and it is in this neighbourhood that we are interested in finding the appropriate expansion. We Write

$$\Phi = Mh_1(\bar{x},\bar{y},\bar{z})+M^2h_2(\bar{x},\bar{y},\bar{z})+ \ldots\ldots$$

where $\bar{\nabla}^2 h_j + h_j = 0$; $j = 1,2, \ldots\ldots\ldots$

In a similar fashion to find the outer expansion for $\vec{F}$, we set

$$x^* = mx\ ,\ y^* = my,\ z^* = mz \text{ and}$$

$$\vec{F} = m\vec{g}_1(x^*,y^*,z^*) + m^2\vec{g}_2(x^*,y^*,z^*) + \ldots\ldots$$

where $\nabla^{*2}\vec{g}_j + \vec{g}_j = 0$; $j = 1,2, \ldots\ldots\ldots$.

Let us denote the outer expansion for $\vec{u}$ as

$$\vec{u} = \vec{U}_0 + \vec{U}_1 + \ldots\ldots + \vec{U}_n + \ldots\ldots$$

with $\vec{U}_0 = (-d_0/l)\ \vec{I}$ and

$$\vec{U}_n = M^n \operatorname{grad} h_n + m^n \operatorname{rot} \vec{g}_n;\ n = 1,2 \ldots\ldots .$$

We are now in a position to obtain the inner and outer solutions and match them appropriately in order to determine the unknown constants which arise in each of them. This is done using VAN DYKE'S (1964) asymptotic matching principle which amounts to the following:
the p-term inner expansion of (the q-term outer expansion) = the q-term outer expansion of (the p-term inner expansion), where p and q may be taken as any two integers equal or unequal.

Using the method of matched asymptotic expansion the disturbance due to the action of a periodic symmetric force acting on a rigid circular disc attached to the free surface of an elastic half space has been studied by KANWAL (1965). Scattering of SH-waves by a rough half space of arbitrary slope and scattering of Rayleigh waves by a ridge have been studied by SABINA and WILLIS (1975,1977) by the method of matched asymptotic expansion. DUTTA and AKILY (1978) applied M.A.E. to obtain the scattered field when the wave length is large compared with the linear dimension of the inclusion in a half space.

KRIEGSMANN and REISS (1983) assumed that a localized inhomogeneity in the medium acts as a scatterer. An asymptotic expansion which is uniformly valid in space is obtained for low frequency scattering of a plane wave incident on the scatterer. It is assumed that the characteristic length of the scattering region is small compared to the wave length of the incident wave. The method of M.A.E is used in the analysis.

Another method that may be applied to solve the mixed boundary value problem is the variational principle.

Variational techniques have been applied with much success for several years in attacking diffraction and scattering problems in electromagnetic theory. While the power of variational techniques for obtaining approximate solution to problems of elastostatics is well known, similar methods do not appear to have been much in use to solve mixed boundary value problems in elastodynamics. The method of obtaining approximate solution of dual integral equations by variational method was developed by NOBLE (1958-59). Being guided by this method STALLYBRASS IN 1962 developed a variation procedure of solving the socalled punch or contact problems in which a rigid punch or die of arbitrary cross section and with a flat base is forced to oscillate in contact with an elastic medium occupying either a half space, or the infinite region bounded by parallel planes. He has shown that a function can be constructed whose stationary value is proportional to the amplitude of oscillations of the punch, if the boundary conditions of the problem are suitably restricted. It is known that the displacement field U_α generated in an elastic medium occupying a region D, by traction T_i applied to the bounding surface B with a harmonic time dependence e^{-iwt} , can be expressed in the form

$$U_\alpha(P) = \int_B U^i_\alpha(P,Q)\, T_i(Q)\, dA_Q, \quad P\ D+B, \tag{21}$$

where $U^i_\alpha(PQ) = U^\alpha_i(Q,P)$ and $B = B_u + B_T$.

dA_Q is an elemental area at Q , a point on B and $T_i = \tau_{ij}\, n_j$ where τ_{ij} are the components of stress tensor and n_j stands for the components of the outer unit normal to B. The singular functions $U^i_\alpha(P,Q)$ is the Green's function which may be interpreted as the components of displacement in the rectangular cartesion direction x_α at P due to an oscillating unit concentrated surface force in the x_i direction at Q.

The boundary B of the region D is divided into two regions B_u and B_T where (i) on B_u, the boundary conditions are of mixed type (e.g. normal component of displacement vector and tangential component of the surface traction are prescribed).

(ii) on B_T all the components of surface traction T_i will vanish.

$$\text{Let } U_\alpha \Big|_{B_u} = f_\alpha \text{ and } T_i \Big|_{B_u} = g_i ,$$

then using $T_i \big|_{B_T} = 0$, we obtain

$$f_\alpha(Q') = \int_{B_u} u_\alpha^i (Q', Q)\, g_i(Q)\, dA_Q \qquad (22)$$

which provides a vector integral equation for the determination of the unknown components of the surface truction g_i on B_u. Instead of trying to find out an exact solution of this integral equation a functional is constructed which is stationary relative to small variations of the unknown components of g_i about their exact values. It can be shown that the functional

$$F_1(g_i^*) = 2\int_{B_u} g_i^*(Q)\, f_i(Q) dA_Q - \int_{B_u} g_i^*(Q) u_i^*(Q) dA_Q \qquad (23)$$

whbre $u_i^*(Q) = \int_{B_u} u_\alpha^i(Q, Q') g_i^*(Q') dA_Q$,

is stationary with respect to first variations of g_i^* about their correct values, as determined by the integral equation. The essence of the above refermulation is that the errors made by using $F_1[g_i^*]$ in place of $F_1[g_i]$ are of the order of magnitude of the squares of the errors in g_i^* relative to g_i. If therefore, we can arrange that $F_1[g_i]$ is proportional to a quantity of interest, and admissible functions g_i^*, in close proximity to the unknown functions g_i, can be obtained, then $F_1[g_i^*]$ will provide us with a good approximation to $F_1[g_i]$.

For purpose of calculation, it is more convenient to have a scalar invariant functional which may be obtained from (23) by replacing g_i^* by cg_i^* and using the stationary property of $F_1[g_i^*]$ to obtain a value for c, the functional obtained is

$$F_2[g_i^*] = \frac{\left[\int_{B_u} g_i^*(Q) f_i(Q) dA_Q\right]^2}{\int_{B_u} g_i^*(Q) u_i^*(Q) dA_Q} \qquad (24)$$

We find that

$$F_2[\mathcal{G}_1] = F_1[\mathcal{G}_1].$$

In order to illustrate the power of the above variational method, the classical Reissner-Sagoci problem of the forced torsional oscillations of a rigid circular disk attached to an elastic half-space was reconsidered by STALLYBRASS (1962).
Approximation was obtained for sufficiently low values of a certain frequency parameter. The boundary value problem is to determine the solution of the differential equation (omitting the time factor e^{iwt})

$$\nabla^2 u_\varphi - \frac{u_\varphi}{\rho^2} + k^2 u_\varphi = 0 \qquad (25)$$

$$u_\varphi = u_\varphi(\rho, Z), \; k^2 = w^2/c^2$$

subject to the boundary conditions

$$u_\varphi = \beta\rho \;, \; Z = 0 \;, \; \rho \leq a \qquad (26,a)$$

$$\tau_{\varphi Z} = \mu \frac{\partial u_\varphi}{\partial z} = 0, \; Z = 0, \; \rho > a \qquad (26,b)$$

where u_φ is the azimuthal component of displacement and β the amplitude of oscillation of the disk.
Considering general solution of (25) in the form

$$u_\varphi(s, z') = \beta \int_0^\infty \frac{A(\lambda)}{\sqrt{\lambda^2 - \alpha^2}} e^{-Z'\sqrt{\lambda^2-\alpha^2}} \lambda J_1(s\lambda)\, d\lambda$$

where $s = \rho/a$, $Z' = Z/a$, $\alpha = ka$, a non-dimensional frequency parameter, it can be shown that the displacement field $\overset{*}{u}_\varphi(s)$ corresponding to an arbitrary admissible stress $\overset{*}{}_{\!\!\varphi z}(s)$, $s \leq 1$ can be obtained in the form

$$u^*_\varphi(s) = -\frac{a}{\mu}\int_0^1\left[\int_0^\infty \frac{\lambda}{\sqrt{\lambda^2-\alpha^2}} J_1(\lambda s')J_1(\lambda s)\,d\lambda\right] T^*_{\varphi z}(s')s'\,d s' \qquad (27)$$

which is the required relation between displacement and stress. Substituting this relation into the expression for the scalar invariant functional (24) and using the boundary condition (26,a) we get

$$F_2[T^*_{z\varphi}] = 2\pi\mu a^3\beta^2 \frac{\left[\int_0^1 T^*_{\varphi z}(s)s^2\,ds\right]^2}{\int_0^\infty \frac{\lambda}{\sqrt{(\lambda^2-\alpha^2)}}\left[\int_0^1 T^*_{\varphi z}(s)J_1(\lambda s)s\,ds\right]^2 d\lambda} \,. \qquad (28)$$

Now $F_2[T_{z\varphi}] = -2\pi\beta\int_0^a T_{\varphi z}(\rho)\rho^2\,d\rho = M_o\beta$, where M_o is the moment of the forces applied to the disk. Replacing $F_2[T^*_{\varphi z}]$ by $F_2[T_{\varphi z}]$ in (28), we obtain

$$\beta \approx \frac{M_o}{2\pi\mu a^3}\,\frac{\int_0^\infty \frac{\lambda}{\sqrt{(\lambda^2-\alpha^2)}}\left[\int_0^1 T^*_{\varphi z}(s)J_1(\lambda s)\,s\,ds\right]^2 d\lambda}{\left[\int_0^1 T^*_{\varphi z}(s)s^2\,ds\right]^2} \qquad (29)$$

the approximation is in the variational sense.

A natural first approximation of $T^*_{\varphi z}(s)$ for small values of frequency is to use the exact static stress distribution.

We therefore take

$$T^*_{\varphi z}(s) = \frac{s}{\sqrt{(1-s^2)}}$$

which when substituted in (29) gives

$$\beta = \frac{9\,M_o}{16\,\mu a^3}\,I_1 \qquad (30)$$

where $I_1 = \int_0^\infty \frac{1}{\sqrt{(\lambda^2-\alpha^2)}}J_{3/2}(\lambda)\,J_{3/2}(\lambda)\,d\lambda$.

Numerical values of M_o calculated from (30) can be found to be in good agreement with the exact value for low frequency.

The same variational principle has been applied by STALLYBRASS and SCHERER (1976) to solve the problem of forced vertical vibration of a rigid elliptical disk on an elastic half space. The methods discussed above are not applicable in case high frequency oscillations or short wave propagation. In these cases WIENER-HOPF technique finds extensive application in various mixed boundary value problems by means of integral transformations.

As a general case the three part mixed boundary value problems [cf. NOBLE (1958), p.196] may be formulated in complex ζ -plane as

$$e^{i\zeta q}\,F_+(\zeta) + K(\zeta)\,F_1(\zeta) + e^{i\zeta p}\,F_-(\zeta)$$

$$= \frac{A}{\sqrt{2\pi}}\,\frac{e^{i(\zeta - k\cos\theta)q} - e^{i(\zeta - k\cos\theta)p}}{\zeta - k\cos\theta} \qquad (31)$$

where A is a constant and $\zeta = \sigma + i\,t$ and $k = k_1 + ik_2$. The equation (31) holds in the strip $-k_2 < t < k_2$, $F_+(\zeta)$, $F_-(\zeta)$ and $F_1(\zeta)$ are the unknown functions:

$$F_+(\zeta) = \frac{1}{\sqrt{2\pi}} \int_q^\infty f(x)\,e^{i\zeta(x-q)}\,dx\,,$$

$$F_-(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^p f(x)e^{i\zeta(x-p)}\,dx\,, \qquad (32)$$

$$F_1(\zeta) = \frac{1}{\sqrt{2\pi}} \int_p^q f(x)e^{i\zeta x}\,dx\,.$$

$F_+(\zeta)$ is assumed regular in $t > -k_2$, $F_-(\zeta)$ in $t < k_2$ and $F_1(\zeta)$ is an entire function. It is assumed that $K(\zeta)$ is regular in $-k_2 < t < k_2$, and has branch points at $\zeta = \pm k$, on the supposition that we can write $K(\zeta) = K_+(\zeta)\, K_-(\zeta)$; $K_+(-\zeta) = K_-(+\zeta)$ and taking $0 < \theta < \frac{1}{2}\pi$, we get $k_2 \cos\theta > 0$. It will then prove convenient to rearrange (31) so as to apply the Wiener-Hopf technique in a strip $-k_2 < t < k_2 \cos\theta$. Multiply (31) by $\exp(-i\zeta q)\{K_+(\zeta)\}^{-1}$ and rearrange in the form

$$\frac{F_+(\zeta)}{K_+(\zeta)} - \frac{A}{\sqrt{2\pi}}\,\frac{\exp(-ik\cos\theta\, q)}{\zeta - k\cos\theta}\left[\frac{1}{K_+(\zeta)} - \frac{1}{K_+(k\cos\theta)}\right] +$$

$$+ U_+(\zeta) + V_+(\zeta) = -\exp(-i\zeta q)K_-(\zeta)\, F_1(\zeta) - U_-(\zeta) - V_-(\zeta) +$$

$$-(\ ) + \frac{A}{\sqrt{2\pi}}\,\frac{\exp(-ik\cos\theta\, q)}{(\zeta - k\cos\theta)K_+(k\cos\theta)} . \qquad (33)$$

In the above equation we have written

$$U_+(\zeta) + U_-(\zeta) = e^{i\zeta(p-q)}\, F_-(\zeta) \,/\, K_+(\zeta),$$

$$V_+(\zeta) + V_-(\zeta) = A\,(2\pi)^{-1/2}\, e^{i\zeta(p-q) - i k\cos\theta\, p} / \{(\zeta - k\cos\theta)K_+(\zeta)\}.$$

In a similar way, multiply (31) by $\exp(-i\zeta p)\{K_-(\zeta)\}^{-1}$ and rearrange as

$$\frac{F_-(\zeta)}{K_-(\zeta)} + R_-(\zeta) + \frac{A}{\sqrt{2\pi}}\,\frac{e^{-i k\cos\theta\, p}}{(\zeta - k\cos\theta)K_-(\zeta)} - S_-(\zeta)$$

$$= -e^{-i\zeta p}\, K_+(\zeta)\, F_1(\zeta) - R_+(\zeta) + S_+(\zeta) \qquad (34)$$

where $R_+(\zeta) + R_-(\zeta) = e^{i\zeta(q-p)}\, F_+(\zeta) \,/\, K_-(\zeta)$.

$$S_+(\zeta) + S_-(\zeta) = A(2\pi)^{-1/2} e^{i\zeta(q-p)-i k \cos \theta q} / \left\{(\zeta - k \cos \theta) K_-(\zeta)\right\} .$$

The left hand side of (33) and the right hand side of (34) are regular in $t > -k_2$. The other sides are regular in $t < k_2 \cos \theta$. Assume that behaviours at infinity are such that Liouville's Theorem can be applied in the usual way to prove that each side of each equation equals zero.

We introduce the notation

$$F_+(\zeta) - (2\pi)^{-1/2} A e^{-i k \cos \theta q} (\zeta - k \cos \theta)^{-1} = H_+^*(\zeta) \qquad (35,a)$$

$$F_-(\zeta) + (2\pi)^{-1/2} A e^{-i k \cos \theta p} (\zeta - k \cos \theta)^{-1} = H_-(\zeta) \qquad (35,b)$$

where $H_+^*(\zeta)$ has a pole at $\zeta = k \cos \theta$ but otherwise regular in $t > -k_2$. H_- is regular in $t < k_2$ cos. Equating the left hand side of (33) and (34) to zero and using the general decomposition theorem [cf. NOBLE (1958, p. 184] , we obtain after simplification

$$\frac{H_+^*(\zeta)}{K_+(\zeta)} + \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{e^{i\tau(p-q)} H_-(\tau)}{(\tau - \zeta) K_+(\tau)} d\tau +$$

$$+ \frac{A}{\sqrt{2\pi}} \frac{e^{-i k \cos \theta q}}{(\zeta - k \cos \theta) K_+(k \cos \theta)} = 0 \text{ , and}$$

$$\frac{H_-(\zeta)}{K_-(\zeta)} - \frac{1}{2\pi i} \int_{id-\infty}^{id+\infty} \frac{e^{i\tau(q-p)} H_+^*(\tau)}{(\tau - \zeta) K_-(\tau)} d\tau = 0 .$$

In these equations $-k_2 < d < k_2 \cos \theta$, $-k_2 < c < k_2 \cos \theta$. Assuming $0 < \theta < \pi/2$ and taking $d = -c = a$ and taking

$S_+^*(\varrho) = H_+^*(\varrho) + H_-(-\varrho)$ and $D_+^*(\varrho) = H_+^*(\varrho) - H_-(-\varrho)$,

we obtain from the last two equations after some manipulation

$$\frac{S_+^*(\varrho)}{K_+(\varrho)} - \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty}\frac{e^{i\tau(q-p)}\,S_+^*(\tau)}{(\tau+\varrho)K_-(\tau)}\,d\tau +$$

$$+\frac{A}{\sqrt{2\pi}}\,\frac{e^{-i\,k\cos\theta\,q}}{(\varrho-k\cos\theta)K_+(k\cos\theta)} = 0\text{ , and}$$

$$\frac{D_+^*(\varrho)}{K_+(\varrho)} + \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty}\frac{e^{i\tau(q-p)}\,D_+^*(\tau)}{(\tau+\varrho)K_-(\tau)}\,d\tau +$$

$$+\frac{A}{\sqrt{2\pi}}\,\frac{e^{-i\,k\cos\theta\,q}}{(\varrho-k\cos\theta)K_+(k\cos\theta)} = 0.$$

These two equations are of the same type and approximate solution can be determined by the method due to JONES (1952).

A number of problems involving the diffraction of elastic waves by finite cracks or scatterers of finite size, the dimension of which are large compared to the wave length of incident wave have been treated by various authors applying Wiener-Hopf technique.

SHIH-JUNG CHANG (1971) considered the interaction by finite closed crack in an elastic medium of infinite extent when a plane dilatational harmonic wave is incident on a crack. High frequency solution is derived with the help of Wiener-Hopf technique.

WICKHAM (1980) considered the short wave radiation from rigid strip which is forced to perform rectilinear oscillation normal to and in smooth contact with a semi infinite a isotropic elastic solid. The mixed boundary value problem is reduced to Fredholm integral equation by the use of Wiener-Hopf technique.

Small time Reissner-Sagoci problem in a bimaterial elastic half space under an impulsive twist was reconsidered by GEORGE (1983). The problem is reduced to an integral equation. By the use of asymptotic analysis and application of Wiener-Hopf technique the equation is converted to Fredholm integral equation of second kind.

We now present in short another method of solving scattering problems for high frequencies viz. elastodynamic ray theory which can successfully be applied to obtain relatively simple approximations to diffracted fields of elastic waves in presence of cracks or strips of finite width in an elastic medium. Geometrical elastodynamics, Geometrical diffraction theory and uniform asymptotic theory together constitute the elastodynamic ray theory. Elastodynamic ray theory were studied in great details by KARAL and KELLER (1959). The application of ray theory to diffraction by smooth obstacles has also been investigated in some detail by RESENDE (1963).

In analogy with geometrical optics, the simplest theory for diffraction of elastic waves by cracks may be called geometrical elastodynamics (GE). In GE a crack or a strip acts as a screen, which creates a shadow zone of no motion, and zones of reflected waves. The shadow zone is bounded by all rays passing through the source point and the edge of the crack. The geometrical reflections of these rays bound the zone of reflected rays. The displacement field according to GE is of the same order of magnitude as the incident field. The GE field is however physically unrealistic, because of the discontinuities in displacement at the boundaries of the shadow zone and the zone of reflected waves.

A first correction to GE is supplied by the geometrical theory of diffraction (GTD). This correction is valid for $wa/C_L >> 1$ and at points $s/a > 1$ where w is the circular frequency, a is a length dimension of the crack, C_L is the velocity of longitudinal wave and s is the distance from a crack edge. The correction provided by GTD is of the order $(wa/C_L)^{-1/2}$.

Basic to GTD is the fact that the incident body wave when falls on the edge of a crack gives r is e to two form of diffracted L-rays (longitudinal) and T-rays (transverse) as well as a set of R-rays (Rayleigh waves) along the crack faces. The primary diffracted rays are fans of L- and T- rays which are directly generated by an incident ray. For plane longitudinal and transverse waves, which are under arbitrary angles of incidence with a traction free semi infinite crack in an unbounded body, the displacement field due to diffracted body wave rays have been determined by ACHENBACH, GAUTSEN (1976) by asymptotic considerations. The corresponding surface wave rays have been studied by GAUTSEN, ACHENBACH and MCMAKEN (1978). When an R-ray intersects the edge of a crack, ray of reflected surface wave as well as cones of diffracted body wave rays are generated. For a plane incident surface wave, the reflection coefficients have been computed and also the cones of diffracted L-wave and T-wave have been analyzed in detail by ACHENBACH, GAUTSEN and MCMAKEN (1978).

These plane wave results in presence of a traction free semi-infinite crack in an unbounded medium are canonical solutions. In geometrical diffraction theory these canonical solutions are appropriately adjusted to account for curvature of incident wave fronts and curvature of crack edges and for finite dimensions of the crack as discussed by GAUTSEN etal (1978).

Within the context of the GTD theory, the diffracted field at a point observation is comprised of contributions corresponding to '' primary'' diffracted body wave ray, which are directly generated by incident body wave rays, and constributions corresponding to 'secondary' diffracted body wave rays. The latter are generated by surface wave rays travelling along the crack faces. With GE and GTD, the total displacement field is of the form

$$u^t = u^g + u^d$$

where u^g is the field due to geometrical elastodynamics and u^d is the field due to geometrical theory of diffraction.

The result is still not valid at the boundaries of the shadow zone and at the boundaries of the zones of reflected waves. In a further refinement which is called uniform asymptotic theory (UAT), the fields at these boundaries are corrected. Uniform asymptotic theory in case of acoustic edge diffraction has been explained in details by LEWIS and BOERSMA (1969).

A three dimensional ray tracing algorithm is used by LANGSTON and JIA-JULEE (1983) to compute the high frequency response of an SH plane wave incident under several models of the sediment with Duamish River Valley.

Based on ray method expansion, asymptotic method is developed by SHEN (1983) for the solution of linear equations governing compressible viscous flow with free surface.

With this much of discussion on the various methods that are generally found to be useful in dealing with the mixed boundary value problems, we briefly discuss the two problems that are taken up in the first chapter.

In the first problem we have considered the rocking motion of a rigid strip on a semi-infinite elastic medium having a frictionless contact with the medium. A time harmonic displacement distribution $v_0 e^{-i\omega t}$ normal to the strip is prescribed where as the stress out side the strip is zero on the free surface. The mixed boundary problem is reduced to a set of dual integral equations, which is then solved by TRANTER's (1962) technique for low frequency oscillation.

In the second paper we have discussed the response of a semi-infinite elastic solid to a rotatory vibration of indentor over a circular area about a diameter. By the use of Hankel transform the solution of the problem is reduced to the solution of a pair of dual integral equation which is then solved by TRANTER's method.

The normal stress below the disc, total torque and the displacement on the free surface have been determined.

From our experience it appears that though TRANTER's method is no less powerful than the other existing methods for solving dual integral equation involving the solution of mixed boundary value problem, it has not much application in the literature.

Next we would discuss some other methods which have wide application in elastodynamic problems. One such is CAGNIARD's method (1939) which is a powerful technique and enables one to find the solution of the problems of seismic pulses or the wave propagation in an elastic medium. Two media in contact may also be dealt with when the source is in one of these media.

According to DIX (1954), Cagniard's method is not to use the standard Laplace transform inversion formula, but to use a series of transformations to beat the expression for the Laplace transform into the explicit Laplace transform integral, thus enabling one to obtain the derived solution directly out of this integral expression. An advantage of Cagniard's method over the other, is that it permits exact numerical computation of examples, whereas alternate approaches usually give approximations which are good only at large distances from the source.

As an illustration of Cagniard's method, we consider the problem of DIX (1954). It is assumed that there is a source function in a spherical cavity in an infinite medium given by

$$\varphi = \frac{1}{R} H[t - \frac{R}{\alpha}] \qquad (36)$$

for P-waves where $R = (r^2 + z^2)^{1/2}$ and H is Heaviside unit function: $H(\tau) = 0$ for $\tau \leq 0$ and $H(\tau) = 1, \tau > 0$.

If we assume the variations only with the radius r and z, the equation for φ in cylindrical coordinates is :

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial^2 \varphi}{\partial t^2} \qquad (37)$$

where α is the P-wave velocity.

Taking the Laplace transform of (36), substituting $t - R/\alpha = \tau$ and noting that $H(\tau) = 0$, for $\tau < 0$, we have from Laplace transform of (36) and (37)

$$\frac{\bar{\varphi}}{p} = \frac{1}{pR} e^{-pR/\alpha} = \int_0^\infty G(\lambda) J_0(\lambda r)\, e^{-(\lambda^2 + p^2/\alpha^2)^{1/2} z} \, d\lambda \qquad (38)$$

where p in $\bar{\phi}/p$ is included in $G(\lambda)$, and z is chosen positive. Equation (38) holds if

$$G(\lambda) = \lambda / \left[p\left(\lambda^2 + \frac{p^2}{\alpha^2} \right)^{1/2} \right] \tag{39}$$

From (36) and (39) using the substitution : $\lambda = pu$ and $1/\alpha = s$, we have to prove that

$$\frac{e^{-pR/\alpha}}{pR} = \int_0^\infty \frac{u\, J_0(pur)\, \exp\left[-p(u^2+s^2)^{1/2} z\right]}{(u^2+s^2)^{1/2}}\, du$$

$$= \int_0^\infty e^{-pt} A(r,\ z,\ t)\, dt \tag{40}$$

and that A will give the unit step function given by (36). We use the integral expression for $J_0(pur)$ and then change the order of integration to obtain (40) in the following form

$$\frac{2}{\pi} \int_0^{\pi/2} \mathrm{Re} \left[\int_0^\infty \exp\left[- p(iur \cos \theta + az)\right] \frac{u}{a}\, du \right] dW \quad . \tag{41}$$

where $a = (u^2 + s^2)^{1/2}$.

Changing the variable u in (41) by substituting

$$t' = iur \cos w + (u^2 + s^2)^{1/2} z \ , \tag{42}$$

one obtains (41) as

$$\frac{2}{\pi} \mathrm{Re} \int_0^{\pi/2} dW \int_{H_W} e^{-pt'} \frac{u}{a} \frac{\partial u}{\partial t'}\, dt' . \tag{43}$$

From the contour shown in fig (1), we obtain

$$\int_{H_W} = \int_{zs}^{\infty}$$

and (43) becomes

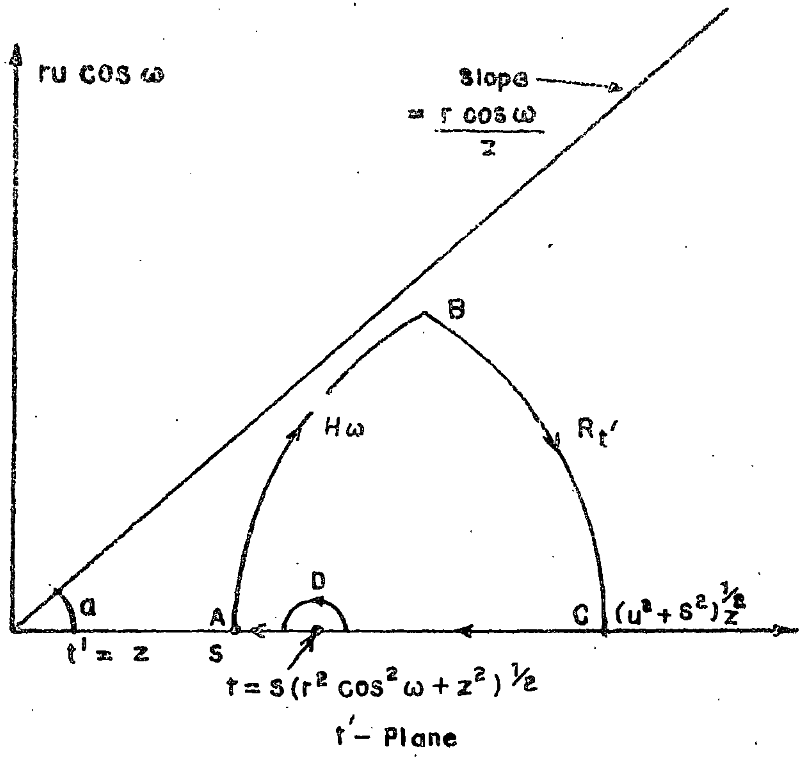

ru cos ω
Slope
= r cos ω / z
B
Hω
R_{t'}
D
a
A
C
(u² + s²)^{½} z
t' = z
s
t = s (r² cos² ω + z²) ½
t' – Plane

Fig. 1.

$$\frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2} dW \int_{zs}^{\infty} e^{-pt'}\,\frac{\partial u}{\partial t'}\,dt'. \qquad (44)$$

Rearranging the order of (44) , we write (44) as

$$\int_{zs}^{\infty} e^{-pt'}\left[\frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2}\frac{u}{a}\,\frac{\partial u}{\partial t'}\,dW\right]dt'. \qquad (45)$$

Comparing (45) with (40), we define A as

$$A(r,z,t') = 0 \qquad \text{for } t' < zs$$

$$= \frac{2}{\pi}\,\mathrm{Re}\int_0^{\pi/2}\frac{u}{a}\,\frac{\partial u}{\partial t'}\,dW \quad \text{for } t' > zs; \qquad (46)$$

and we can say that we have solved our problem, because such an A satisfied the equation (40). Equation (46) is the first form of our solution. We use the substitution (42) to replace the variable w by the variable u and keep t' constant. Then the integral (46) becomes

$$A_{t'>zs} = \frac{2}{\pi}\,\mathrm{Re}\int_{c_{t'}}\frac{u}{a}\,\frac{\partial u}{\partial t'}\,\frac{\partial w}{\partial u}\,du$$

$$= \frac{2}{\pi}\,\mathrm{Im}\int_{c_{t'}}\frac{u\,du}{a\left[u^2r^2+(t'-az)^2\right]^{1/2}} \qquad (47)$$

where $c_{t'}$ is the corresponding integration path in u-plane. Equation (47) [cf. Mathematical aspects of seismology, Elsevier Publishing Co, New - York (1968). Markus Bath, pp.271-272] may now be written as

$$A_{t'>zs} = \frac{1}{i\pi}\int_{c_{t'}+c'_{t'}}\frac{u\,du}{a\left[u^2r^2+(t'-az)^2\right]^{1/2}} \qquad (48)$$

where $c_{t'}' = \bar{c}_{t'}$, i.e the integral is taken along the conjugate path but in reverse order (see fig.2).

To evaluate the integral (48) , this is to be noted that the integrand in (48) has branch points in the u-plane. The four points c, c', Q and Q' which are branch points of the integrand and the branch cuts are shown in the figures 2 and 3. Therefore

$$\int_{c_t+c_{t'}} = \frac{1}{2}\int_{c_t+c_{t'}+D'+D}$$

As there is no pole within the contour in fig.3, so we get,

$$\int_{cc'} = \int_{c_t, c_{t'}', D'D} = \int_{\Gamma}$$

and $A(r,z,t)_{t'>zs}$

$$= \frac{1}{2\pi i}\lim_{\Gamma\to\infty}\int_{\Gamma}\frac{u\,du}{a\left[u^2(r^2+z^2)z^2s^2+t'^2-2t'az\right]^{1/2}} \qquad (49)$$

Putting $u = R_1 e^{i\theta}$, for $0 \leq \theta \leq 2\pi$ and letting $R_1 \to \infty$, the integral is evaluated and from (49) we have

$$A(r,z,t')_{t'>zs} = \frac{1}{(r^2+z^2)^{1/2}} = \frac{1}{R} \qquad (50)$$

This is what we are supposed to prove i.e A is the unit step solution of our problem for $t' > zs$, but we must prove that A is the unit step solution for $t' > Rs$ i.e. we must show that for $zs < t' < Rs$, $A = 0$. We can not have any pulse before the time $t' = Rs = R/\alpha$, which is the time of arrival of the pulse at a distance R from the source.

To prove that $A = 0$, for $zs < t' < Rs$ we put:

$$t' = s\,(r^2c^2 + z^2)^{1/2} \quad (o < c < 1) \tag{51}$$

Then from (42)

$$t' = s(r^2c^2+z^2)^{1/2} = iru \cos w + z(u^2+s^2)^{1/2} \tag{52}$$

Therefore

$$u = \frac{r\ s\ z(c^2-\cos^2 w)^{1/2}}{r^2 \cos^2 w + z^2} - i\,\frac{r\ s \cos w\ (r^2c^2+z^2)^{1/2}}{r^2 \cos^2 w + z^2} \tag{53}$$

$c_{t'}$ starts at $w = o$. For this w

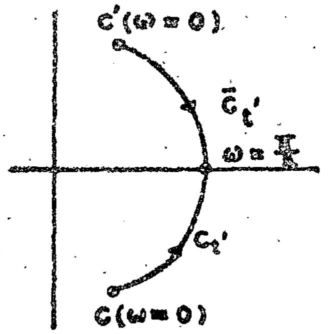
$$u_{w=o} = -i\,\frac{rs}{R^2}\left[(r^2c^2+z^2)^{1/2} - z(1-c^2)^{1/2}\right] \tag{54}$$

The path $c_{t'}$ runs from E down the imaginary u-axis to $\cos w=c$ and after that upto B. At B we have $w=\pi/2$ and from (53), $u=rcs/z$. There is no pole inside the contour shown in fig.5 and we have by Cauchy's integral theorem

$$\int_{c_{t'}} + \int_{c'_{t'}} = \int_{EE'}$$

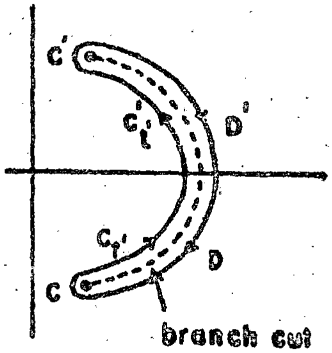
and $$\int_{EE'} \frac{u\,du}{i\pi(u^2+s^2)^{1/2}\left[u^2r^2+\left\{t'-z(u^2+s^2)^{1/2}\right\}^2\right]^{1/2}} = 0$$

This proves that $A = 0$, for $t' < Rs$. Therefore we can replace zs by Rs in equation (48). This method was applied by GHOSH (1964) who considered a torsional radiator in the form of circular disc of finite radius attached to the surface of a semi-infinite isotropic medium. A twisting moment $M\delta(t)$ is applied to the disk. By applying CAGNIARD's (1939) method an exact evaluation



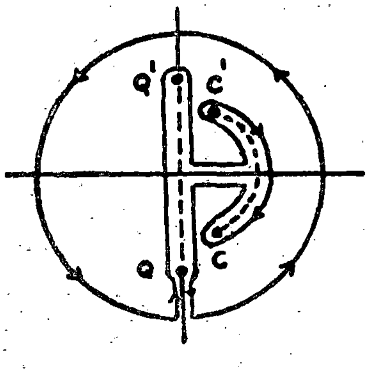
u - plane

Fig. 2.



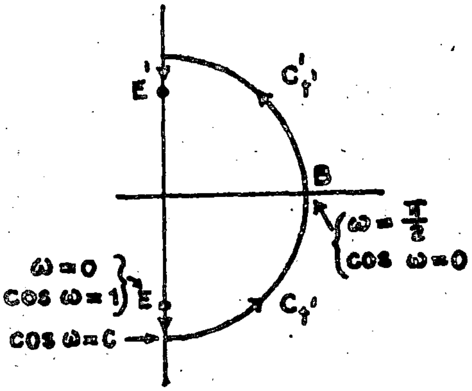
u - plane

Fig. 3.



u - plane

Fig. 4.



u - Plane

Fig. 5.

of the displacement at any point in the medium was made.

EASON and WILSON (1969) considered the displacement produced by a torsional body force situated within an elastic half-space which is bounded to a half-space of different material properties. Using CAGNIARD's transformation, displacement at points of the surface due to impulsive body force acting on the circular region in the interface was worked out in details.

GRAVIN (1956) first applied CAGNIARD's technique in two dimensional case using cartesian coordinates with some modification.

A line source is assumed to be situated at a depth (0,h) below the free surface of an elastic half space. The medium is disturbed by the emission of axially symmetric pulse from the source. The resulting distnrbance at any point of the medium is determined on the medium as a function of time.

The displacement components s_x and s_y along the direction of the coordinate axes are obtained in the following form

$$\bar{s}_x(x,o) = - 2ip^2 \bar{f}(p) \int_{-\infty}^{\infty} F_x(k)\exp[-h\eta_p - ikx]\,dk \qquad (55)$$

and $$\bar{s}_y(x,o) = 2\bar{s}_{oy} + 2\bar{f}(p) \int_{-\infty}^{\infty} F_y(k)\exp[-h\eta_p - ikx]\,dk \qquad (56)$$

where bar denotes the Laplace transform to suppress the time variable t, p is the Laplace transform parameter, $\bar{f}(p)$ is the transform of a time function to be adjusted, $\eta_p = \sqrt{k^2+(p^2/v_p^2)}$; V_p is the P-wave velocity and $\bar{s}_{oy}$ is the transform of the y-displacement that would result at (x,o) in an infinite medium. $F_x(k)$ and $F_y(k)$ are respectively odd and even function of k.

Changing the variable of integration by the substitution

$u = (k\, v_p)/p$, one obtains

$$\bar{s}_{x,y}(x,o) = (o,\ 2\,\bar{s}_{oy}) \mp \frac{4\nu}{V_p} p\bar{f}(p) \left(o,\ \nu\right)\ X$$

$$X\ (I_m,\ R_e) \int_0^\infty G_{x,y}(\nu,u)\ \exp\left\{-p\left[\frac{h}{V_p}\ (u^2+1) + \frac{iux}{V_p}\right]\right\} du \qquad (57)$$

where $\nu = \sqrt{\left[\mu/(\lambda+2\mu)\right]}$, and μ are Lame's constants. A new integration variable is now defined by

$$t = \frac{h}{V_p}\ (u^2+1)^{1/2} + \frac{iux}{V_p}\ ;\ 0 \leq u \leq \infty \qquad (58)$$

This represents a conformal transformation from the u-plane to the t-plane which changes the path of integration and the positions of the singularities. The path of integration in the u-plane along the real axis from zero to infinity is made equivalent to a curve of integration passing through the origin in the t-plane, which by use of Cauchy's theorem and Jordan's lemma is finally reduced to an integral along the real t-axis from h/V_p to ∞. Thus it becomes possible to find Laplace inversion by inspection.

MITRA (1959,a) extended GARVIN's results to the case in which the source is distributed over an area. He (1960) applied GARVIN's method to find the surface displacement due to a time source when the body force is of the form

$$X = H(t)\ \frac{\partial}{\partial x}\left[\ \delta(x)\ \delta(y-h)\right],$$

and $$= H(t)\ \frac{\partial}{\partial y}\left[\ \delta(x)\ \delta(y-h)\right].$$

MITRA (1964), using the modified CAGNIARD's method studied the uniform impulsive pressure acting over a circular portion of the surface of an elastic half space on the assumption that the surface traction on $z = o$ is

$$\tau_{zz} = P\,\delta(t)\ ; \quad 0 \leq r \leq a$$

$$= 0 \qquad ; \quad r > a$$

$$\tau_{zr} = 0$$

Another modification of CAGNIARD's method was developed by DE-HOOP (1959). The integration variable in (55) and (56) are changed by the substitution $u = (k V_P) / p$ like GARVIN. Then again a new integration variable is introduced by the substitution

$t = \frac{h}{V_P}(u^2+1)^{1/2} + i\,\frac{ux}{V_P}$ as in (58), but in this case it is assumed that t is positive and real instead of conformal mapping from u-plane to t-plane as assumed by GARVIN. As a result the path of integration with respect to u which is from $-\infty$ to ∞ along the real axis is deformed to the branch of a hyperbola whose equation is

$$u = \frac{-itx \pm h\left(t^2 - \frac{x^2+z^2}{V_P^2}\right)^{1/2} \left(\frac{x^2+z^2}{V_P^2} < t < \infty.\right)}{\frac{x^2+z^2}{V_P}}\ ;$$

Hence the integration along the real axis in the u-plane may be replaced by the branch of the hyperbolic path. Consequently Laplace inversion can be obtained by inspection.

This modified method of CAGNIARD is found to be more convenient than that of GRAVIN and in recent time this method is widely used in different problems to find Laplace inversion.

GHOSH (1971) applied CAGNIARD's method as modified by DE-HOOP, to obtain displacement in the integral form due to a sudden creation of normal stress discontinuity over a circular area expanding uniformly after creation. ROY (1981) used this technique to find the displacement field due to a transient response of an elastic half-space subject to a uniform normal pressure acting over an elliptic area. MITTAL and SIDHU (1982) using DE-HOOP's version of CAGNIARD's method evaluated surface displacement due to SH-type of waves. PAL (1983) applied modified CAGNIARD's method to find the exact solution of displacement function due to the generation of SH-waves due to a stress discontinuity moving with nonuniform velocity.

Another type of problems that has to be encountered to study the dynamic behavior of an elastic solid is the response of an elastic solid to moving loads. The moving load problems which have been studied may be put into three categories:

i) steady wave motion due to a load moving with constant velocity for all time to come,

ii) transient wave motion due to a load which begins to act at certain instant and then moves with constant velocity, and

iii) transient wave motion due to a load which begins to act at certain instant and then moves in some direction with nonuniform speed.

The steady motion of a line load on the surface of an elastic half-space studied by SNEDDON [1951, cf. page-447-449], COLE and HUTH (1958) and GHOSH and GHOSH (1978) are the typical examples of the first type of problems. The transient problem of a line load, which suddenly appears on the surface and then moves with constant velocity studied by ANG (1960), is of type (ii). As a representative of the third kind of problem we refer to the study of FREUND (1972).

An analytic technique was developed by FREUND (1972) which made it possible to obtain an exact solution of a particular problem in category (iii).
For introducing the technique the author considered a line load in an unbounded elastic solid moving with nonuniform speed in a particular direction. Carterian coordinate system was used. At any time $t = 0$, a line load begins to act along y-axis and the line load moves along x-direction for $t > o$. For any time $t > o$, x-coordinate of the load in given $l(t)$. It is assumed that the function $l(t)$ is continuous, monotone increasing function of time t and that it never acts at a single point for a finite length of time. Under the conditions mentioned the function $l(t)$ is invertible, that is there exists a function $n(x)$ which is the time at which the load acts at x. The functions $l(t)$ and $n(x)$ satisfy the following relations identically

$$l[n(x)] = t; \quad n[l(t)] = x \tag{59}$$

$$\dot{l}[n(x)]n'(x) = 1; \quad n'[l(t)]\dot{l}(t) = 1 \tag{60}$$

where dot denotes time derivative and dash denotes x-derivative. Because of the symmetry with respect to the plane $z = o$, the problem may be looked upon as boundary value problem for the half space $z > o$, with mixed boundary condition on $z = o$. If ϕ and ψ are dilatational and rotational displacement potentials then equations of motion are formulated as

$$\begin{aligned} \varphi_{xx} + \varphi_{zz} - a^2 \varphi_{tt} &= o \\ \psi_{xx} + \psi_{zz} - b^2 \psi_{tt} &= o \end{aligned} \tag{61}$$

where a, b are the dilataional and shear wave slowness. In case of normal loading the boundary conditions to be satisfied by the solution of (61) are

$$\sigma_{zz}(x,o,t) = -\frac{1}{2}\,\delta[x-l(t)];\quad u(x,o,t) = 0 \qquad (62)$$

σ_{ij} is the stress component and u is the displacement component in x-direction.

The solution of the problem is obtained by making use of Laplace transform method. The time variable is first eleminated by application of the transform

$$\hat{\varphi}(x,z,s) = \int_{o}^{\infty} \varphi\,(x,z,t)e^{-st}\,dt. \qquad (63)$$

and next the dependence on x is avoided by taking the transform

$$\Phi(\lambda,z,s) = \int_{-\infty}^{\infty} \hat{\varphi}\,(x,z,s)e^{-s\ x}\,dx. \qquad (64)$$

Applying the transforms on the boundary conditions and keeping in mind the physical condition i.e. the solution of the transformed differential equation should remain bounded as $z \to \infty$, one obtains

$$\Phi(\lambda,z,s) = \frac{1}{\rho s^2}\,A(\lambda,s)\,\exp(-s\alpha z) \qquad (65)$$

$$\Psi(\lambda,z,s) = -(\lambda/\rho s^2\beta)A(\lambda,s)\exp(-s\beta z) \qquad (66)$$

where is the material density and

$$\alpha = (a^2-\lambda^2)^{1/2},\quad \beta = (b^2-\lambda^2)^{1/2}. \qquad (67)$$

The amplitude $A(\lambda,s)$ in (65) and (66) is the double transform of the boundary condition and is derived by making use of the relationship [cf. VANDERPOL and BREMMER (1964), p. 79].

$$\delta[x-l(t)] = n'(x)\, \delta[t-n(x)]. \tag{68}$$

In view of (68)

$$\hat{\sigma}_{zz}(x,o,s) = \frac{1}{2}\, n'(x)\exp\,(-sn(x)\; H[n(x)]$$

and $n(o) = o$, $H[n(x)] = H[x]$. Then, applying the two sided Laplace transform,

$$\Sigma_{zz}(\lambda,o,s) = A(\lambda,s) = -\frac{1}{2}\int_o^{\infty} n'(x)e^{-s\lambda x}\, e^{-sn(x)}\, dx. \tag{69}$$

The transformed solution is thus completely determined, and the potential function φ may be written as the double inversion integral $\varphi(x,z,t) =$

$$= -\frac{1}{4\pi i}\int_{B_1}\frac{1}{2\pi i}\int_{B_2}\frac{1}{s}\int_o^{\infty} n'(\xi)e^{s(\lambda x-\lambda\xi-\alpha z-n(\xi)+t)}\, d\xi\, d\lambda\, ds \tag{70}$$

where B_1 and B_2 are the usual inversion paths for one sided and two sided Laplace transforms. The double integral in (70) is inverted by means of DE-HOOP's (1959) technique.

In a subsequent study of nonuniformly moving line load or a pressure step on a surface of an elastic solid as well as nonuniformly moving dislocation the above method, as shown by FREUND (1973) can be applied.

We now briefly describe the nature of the problems taken up in the second chapter and a problem of third chapter.

The second chapter of the thesis is concerned with problems of eleastic waves due to sources in the form of a ring on the surface of an isotropic elastic half space.

The first problem considered is the response of an elastic half-space to a ring source. The radius of the ring is assumed to increase with a constant velocity c less than that of shear wave velocity. A twisting impulse is prescribed in the form of $P\delta(r-ct)H(t)$. By using Laplace transform, Hankel transform and De-Hoop's version of Cagniard's method, the displacement is determined at any point of the medium in integral form. Exact evalution of displacement just after the arrival of the disturbance and displacement at any point after a sufficiently large time have also been determined.

The second problem is to study the motion produced in an isotropic elastic half-space due to impulsive torsional motion of a circular ring source located on a free surface of a homogeneous as well as in inhomogeneous medium.

The torsional motion is prescribed by $P\delta(r-a)\delta(t)$, a being the radius of the ring. In case of inhomogeneous medium inhomogeneity is prescribed by $\mu = \mu_0(1+\epsilon z)^2$ and $\rho = \rho_0(1+\epsilon z)^2$, where ρ is the material density. The method of approach is the same as that of the previous one. Graphs have been plotted for displacement on free surface as a function of time and the variation in displacement due to the presence of inhomogeneity has also been shown.

In the last problem of the second chapter, exact expressions for displacement in a homogeneous isotropic elastic half-space subjected to an impulsive torsional force over the rim of a nonuniformly expanding ring source on the free surface is obtained by CAGNIARD, DE-HOOP technique. Different wave front surface with their region of existence have been shown. The first motion responses near different wave arrivals have been obtained by a limiting process. The displacement on the free surface as a function of position of the source have been shown by means of graphs.

In the first problems of the last chapter, we have considered a concentrated line load which originates at time $t = 0$ and then moves with uniform velocity along the boundary of an isotropic inhomogeneous medium. It is assumed that the elastic parameters λ and μ and the density ρ of the medium vary according to the law

$$\lambda = \mu = \mu_0(1+\epsilon z)^2 \text{ and } \rho = \rho_0(1+\epsilon z)^2 .$$

DE-HOOP's version of CAGNIARD's method is used to find the displacement components in the integral form. An approximate evaluation of the integrals is worked out near the first arrival of the wave fronts. It is perhaps the first application of Cagniard's method in solving problems in inhomogeneous media.

We now discuss in short the necessity of studying models of the source of earthquake in elasto-dynamics though these may differ from actual phenomena.

The behavior of seismic wave propagation due to the presence of active tectonic belts, fracture or faulting, all come under the purview of elastodynamics. There has been an increasing interest in theoretical study in elastic wave motion. The relation between the earth's deeper structure and some geological formations is of great importance in revealing the genesis of mineral deposits. The rapid increase in the volume and rate of construction-work in seismically active zones makes moreurgent the need for earthquake resistant structures, high dams etc. Most important for the same purpose is the study of earthquake focal parameters and the condition of seismic wave propagation.

The faulting process is, in general, a fracture phenomena. The mechanical energy released at the fault surface is carried to the side by elastic waves propagating through the earth material contained between the source and the site. Each of these phenomena falls within the area of research in continuum mechanics and the latter in elasto dynamics. In addition, the problem of calculation of the response of building foundations to incoming earthquake waves may be regarded as an elastodynamic diffraction problem.

Seismological evidence suggests that earthquakes occur by sudden slippage of earth material across the fault surface, which may be looked upon as pre-existing weak zones of relatively small thickness. Hence if the displacement discontinuity is known across the fault surface, (in fact now it is possible to determine displacement discontinuity after the equilibrium position is restored in the surface after some disturbance) the displacement field in a considerably large area round the fault surface can be determined by the application of representation theorem of elastodynamics.

The displacement produced at any point on a free surface of an elastic half space due to displacement discontinuity across a fault surface can be calculated with the help of Green's function and the representation theorem of elasto-dynamics. The Green's function $G_{ij}(\vec{x}'|\vec{x})$ is cartesian component of displacement in x_j direction produced at a point $\vec{x}'$ of a half space due to an application of a time harmonic force of unit magnitude in x_i-direction at a point $\vec{x}$ on the free surface of a half space. It can be shown that the Fourier transform with respect to time of the displacement at the surface of the half space $x_3 = 0$ due to a prescribed slip on a fault surface S may be expressed in the form [cf. MAL (1972), equation 8]

$$U_m(\vec{x}) = \int_S \left[U_i(\vec{x}')\right]_-^+ T_{ij}^m n_j \, dS$$

where $\left[U_i(\vec{x}')\right]_-^+$ is Fourier transform of the prescribed discontinuity in displacement component U_i across S, n_j is unit normal to S on the positive side and T_{ij}^m is related to G_{ij} through the equation

$$T_{ij}^m(\vec{x}\,|\vec{x}) = c_{ijkl} \frac{\partial}{\partial x_l'} G_{mk} (\vec{x}'|\vec{x})$$

The calculation of the surface displacement can be carried out by taking Fourier transform of the slip function. For a given point $\vec{x}$ and frequency is $T_{ij}^m n_j$ has to be calculated at each point of the fault surface and then integrating along the fault s-urface the product of transformed slip function and $T_{ij}^m n_j$, the displacement components can be determined for different values of w.

Further, it follows from radiation pattern of first motion from earthquakes that a shearing motion occurs at the earthquake focus. To illustrate the mechanism which produces such a motion FOSSUM and FREUND (1975) considered a model of a shallow earthquake focus by a plane shear crack extending at a nonuniform rate under the action of general loading. It is assumed that the crack should remain in one plane. This is in conformity with the field observation which indicates that the directions of seismic fault extension is almost always in the plane of preexisting fault.

In the last problem of the third chapter we have considered a model, where it is assumed that a crack is developed suddenly along a horizontal line at a finite depth below the surface of the earth. The crack is assumed to move along a vertical plane upto the free surface with nonuniform velocity. Assuming the motion to be two dimensional, the surface displacement due to Rayleigh waves produced by nonuniformly moving crack has been determined.

In this connection it may be mentioned that recently, SINGH, MODDIE and HADDOW (1981) have considered the problem of finite length crack propagating with constant velocity in an infinitely long finite width strip when anti-plane shear displacements and stresses are applied to the lateral boundaries of the strip. By employing Fourier transform the solution is reduced to the solution of a pair of dual integral equation, the solutions of whi-ch is obtained directly in a closed form by making use of COOKE's (1970) result.

With this much of review work, we present the thesis chapters.

The notations used in different problems are independent of one another.

CHAPTER I

MIXED BOUNDARY VALUE PROBLEMS

Problem 1 . Harmonic rocking of a rigid strip on
a semi-infinite elastic medium.

Problem 2 . Harmonic rocking of a rigid circular
indentor on an elastic half-space.

HARMONIC ROCKING OF A RIGID STRIP
ON A SEMI INFINITE ELASTIC MEDIUM

INTRODUCTION: To consider the effect of vibrating source of pressure in different form on the surface of elastic medium is almost classical. Perhaps Lamb is the pioneer on this line. The problem considered here is the rocking motion of a rigid strip of infinite length having a smooth contact with elastic medium. Generally this type of problems may be formulated so as to be governed by a set of dual integral equations. This particular problem considered here was considered by Awojobi and Grootenhuis (1965) and Awojobi (1966). They used heuristic technique of successive approximation to solve the dual integral equation. Karasudhi, Keer and Lee (1968) also considered the problem by reducing the governing dual integral equations into a single inhomogeneous Fredholm integral equation of the second kind and then solved the equation by the method of successive approximation for low frequency oscillation. But their final solution involved definite integral which were later numerically evaluated. Recently the same problem was again taken up by Wickham (1977). With the help of suitable Green's function, the mixed boundary value problem is first reduced to a Fredholm integral equation of the first kind involving displacement boundary condition. Using Noble's (1962) method this equation has been reduced to a Fredholm integral equation of the second kind with a Kernal which is small in the low frequency limit. By the use of exact iterative solution of the integral equation of the second kind, a simple explicit long-wave asymptotic formula for the normal stress in terms of the prescribed displacement and dimensionless wave number K has been derived rigorously. Unlike Karasudi et al. the solution does not contain any integral which requires numerical evaluation. But the method of solution is a bit cumbersome.

In this paper we have reconsidered the same problem. The solution of the governing dual integral equations representing the mixed boundary value problem, is reduced to the solution of a set of linear algebraic equations following Tranter (1962) for low frequency oscillation. The asymptotic solution of normal stress below the strip is determined interms of prescribed displacement and the wave number K, where terms involving K^4 and its higher orders are neglected. The value of the reacting couple exerted by the elastic solid on the strip has also been evaluated and they are found to be in agreement with that given by Wickham.

The asymptotic solution obtained by this method is exact in the sense that it does not involve any integral requiring numerical evaluation. It appears that Tranter's technique is no less powerful than the other methods and it is less cumbrous.

FORMULATION OF THE PROBLEM: We consider the rocking vibration of frequency w of a rigid strip having a smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half space $-\infty < x < \infty$, $y \geqslant 0$, $-\infty < z < \infty$. It is assumed that the motion is prescribed by a displacement distribution $v_0 e^{-iwt}$ normal to the finfinite strip $|x| \leqslant x_0$, $y = 0$, $|z| < \infty$, where $v_0 = \phi_0 x$, ϕ_0 being constant and that the tangential components of stress are zero and the normal stress is zero for

$| x | > x_o$, $y = o$, $| z | < \infty$. Thus it follows that the medium under consideration is in a state of dynamic state of plane strain satisfying the two dimensional equations of motion, given by

$$\nabla^2 \phi = \frac{1}{a^2} \frac{\partial^2 \phi}{\partial t^2}, \quad \nabla^2 \psi = \frac{1}{b^2} \frac{\partial^2 \psi}{\partial t^2} \tag{1}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$ a and b are p- and S-wave

velocities in the medium. The scalar potentials ϕ and ψ are associated with the displacement components u and v by the relations

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y} \; (2,a) \text{ and } v = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \qquad (2,b)$$

The stress components are

$$\sigma_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{3,a}$$

$$\sigma_{yy} = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial v}{\partial y} \tag{3,b}$$

where λ and μ are Lame's constants. The boundary conditions are

$$v\,(x,o) = v_o\, e^{-iwt} = \phi_o\, x e^{-iwt}, \quad | x | \leqslant x_o \text{ on } y = o \tag{4,a}$$

$$\sigma_{yy}(x,o) = o, \quad | x | > x_o \text{ on } y = o \text{ , and} \tag{4,b}$$

$$\sigma_{xy}(x,o) = o \text{ every where on } y = o \tag{4,c}$$

To find the solutions of the equations (1) subject to the conditions (4-a,b,c), we make the substitution

$$\left[\varphi,\psi\right] = \left[\Phi,\Psi\right] e^{-iwt}$$

In view of the boundary conditions (4), it follows that Φ and Ψ may be taken in the form

$$\Phi = \frac{x_o^2}{2\pi} \int_{-\infty}^{\infty} A(\xi) e^{-pY} e^{-i\xi x}\, d\xi \qquad (5,a)$$

and $$\Psi = \frac{x_o^2}{2\pi} \int_{-\infty}^{\infty} B(\xi) e^{-sY} e^{-i\xi X}\, d\xi\, , \qquad (5,b)$$

where $X = x/x_o$ and $Y = y/x_o$. p and s in equation (5-a,b) are

$$p(\xi) = \sqrt{\xi^2 - x_o^2 h^2},\ |\xi| > x_o h \text{ and } s(\xi) = \sqrt{\xi^2 - x_o^2 k^2},\ |\xi| > x_o k,$$

where $h = w/a$ and $k = w/b$.

From equation (1) and Φ, Ψ as obtained in (5-a,b), the displacement components and the stress components may be written in the form

$$\begin{aligned}
u &= \frac{x_o}{2\pi} \int_{-\infty}^{\infty} \left[-i\xi A(\xi) e^{-pY} + sB(\xi) e^{-sY}\right] e^{-i\xi x}\, d\xi \\
v &= -\frac{x_o}{2\pi} \int_{-\infty}^{\infty} \left[pA(\xi) e^{-pY} + i\xi B(\xi) e^{-sY}\right] e^{-i\xi x}\, d\xi \\
\sigma_{xy} &= \frac{\mu}{2\pi} \int_{-\infty}^{\infty} \left[2i\xi\, pA(\xi) e^{-pY} - (\xi^2 + s^2) B(\xi) e^{-sY}\right] e^{-i\xi x}\, d\xi \\
\sigma_{yy} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\left\{-\lambda\xi^2 + (\lambda + 2\mu) p^2\right\} A(\xi) e^{-pY} + 2i\mu\xi s B(\xi) e^{-sY}\right] e^{-i\xi x}\, d\xi
\end{aligned} \qquad (6)$$

In the expression given in (6) in subsequent analysis the time factor e^{-iwt} has been omitted. From the boundary condition (4-c), it follows that

$$B(\xi) = 2i\xi\, pA(\xi)/\,(\xi^2 + s^2\,) \tag{7}$$

and the conditions (4-a,b) yield the dual integral equation

$$\int_{-\infty}^{\infty} \frac{x_0^2k^2\ pA(\xi)}{2\xi^2 - x_0^2k^2}\, e^{-i\xi X}\, d\xi = 2\pi\phi_0,\ |\,X\,| < 1, \tag{8-a}$$

and
$$\int_{-\infty}^{\infty} \frac{R(\xi)\ A(\xi)}{2\xi^2 - x_0^2k^2}\, e^{-i\xi X}\, d\xi = 0,\ |\,X\,| > 1, \tag{8-b}$$

where $R(\xi) = (2\xi^2 - x_0^2k^2\,)^2 - 4ps\xi^2$. This is to be noted that $A(\xi)$ is an odd function of ξ, consequently the equations (8-a,b) may be rewritten as

$$\int_0^{\infty} \frac{x_0^2k^2\sqrt{\xi}\ p(\xi)A(\xi)}{2\xi^2 - x_0^2k^2}\, J_{\frac{1}{2}}(\xi X)d\xi = i\phi_0\ \sqrt{2\pi X}\ ,\ |X| \leqslant 1 \tag{9-a}$$

and
$$\int_0^{\infty} \frac{R(\xi)\sqrt{\xi}\ A(\xi)}{2\xi^2 - x_0^2k^2}\, J_{\frac{1}{2}}(\xi X)d\xi = 0\ ,\ |X| > 1 \tag{9-b}$$

To facilitate our analysis we write equations(9-a,b) in the form

$$\int_0^{\infty} [1+H(\xi)]\, D(\xi)\, \xi^{-1}\ J_{\frac{1}{2}}(\xi X)d\xi = i\phi_0\ \sqrt{2\pi X}\ ,|X| \leqslant 1 \tag{10-a}$$

and $$\int_0^\infty D(\xi)\, J_{\frac{1}{2}}(\xi X)\, d\xi = 0, \quad |X| > 1 \tag{10-b}$$

where $$D(\xi) = -(1-\eta)\frac{\sqrt{\xi}\, R(\xi) A(\xi)}{2\xi^2 - x_0^2 k^2}$$

and $$1+H(\xi) = -\frac{\xi x_0^2 k^2\, p(\xi)}{(1-\eta) R(\xi)},$$

$\eta = \lambda/2(\lambda+\mu)$ being Poisson's ratio. It should be noted that $H(\xi) \to 0$ as $x_0 k \to 0$ and $x_0 h \to 0$.

SOLUTION OF THE DUAL INTEGRAL EQUATION: To find the solution of the dual integral equations (10-a,b) following Tranter (1962) we assume

$$D(\xi) = \sqrt{\xi} \sum_{m=0}^{\infty} a_m J_{1+2m}(\xi) \tag{11}$$

So that the equation (10-b) is automatically satisfied. The coefficients a_m are to be so chosen that the form of $D(\xi)$ as assumed in (11) satisfies the equation (10-a). So we must have

$$\sum_{m=0}^{\infty} a_m \int_0^\infty [1+H(\xi)]\, \xi^{-1/2} J_{1+2m}(\xi) J_{\frac{1}{2}}(\xi X)\, d\xi = i\phi_0 \sqrt{2\pi X},\ |X| \leqslant 1 \tag{12}$$

Multiplying equation (12) by $X^{3/2}(1-X^2)^{-1/2}\ \mathcal{F}(1, \frac{3}{2}, X^2)$

where n is a positive integer or zero and $\mathcal{F}_n$ is a Jacobi polynomial of degree n and then integrating with respect to X from 0 to 1, one obtains

$$\sum_{m=0}^{\infty} a_m \int_0^{\infty} [1+H(\xi)]\xi^{-1} J_{1+2m}(\xi)J_{1+2n}(\xi)d\xi = E(\tfrac{1}{2},n,\tfrac{1}{2}) = E_n \text{ (say)}, \qquad (13)$$

where $E_n = 41\phi_o \dfrac{\Gamma(n+\frac{3}{2})}{\Gamma(n+\frac{1}{2})} \int_0^1 x^2(1-x^2)^{-1/2} \mathcal{F}_n(1,\tfrac{3}{2},x^2)dx.$

Using the result that $\mathcal{F}_o(1,\frac{3}{2},x^2) = 1$ we obtain from orthogonality relation of Jacobi polynomial

$$E_n = \tfrac{1}{2}\pi\phi_o \text{ or } 0 \text{ according as } n = o \text{ or } n \neq o. \qquad (14)$$

Since $\int_0^{\infty} \xi^{-1} J_{1+2m}(\xi)J_{1+2n}(\xi)d\xi = o$ for $m \neq n$

$= (2+4n)^{-1}$ for $m = n$,

wo we obtain from (13)

$$a_n + \sum_{m=o}^{\infty} L_{mn}a_m = (2+4n)E_n \qquad (15)$$

where $L_{mn} = (2+4n)\int_0^{\infty} H(\xi)\xi^{-1} J_{1+2m}(\xi)J_{1+2n}(\xi)d\xi \qquad (16)$

Equation (15) gives us an infinite set of algebraic equations for the determination of the coefficients a_m.

Using the generalisation of Neumann's integral [cf.Watson(1958)p.150]

$$J_{1+2m}(\xi)J_{1+2n}(\xi) = \frac{2}{\pi}\int_0^{\pi/2} J_{2+2m+2n}(2\xi\cos\theta)\cos 2(m-n)\theta\, d\theta$$

in the equation (16) and changing the order of integration one obtains

$$L_{mn} = \frac{4(1+2n)}{\pi} \int_0^{\pi/2} A_{mn} \cos 2(m-n)\,\theta \, d\,\theta, \tag{17}$$

where
$$A_{mn} = \int_0^\infty H(\xi)\,\xi^{-1}\, J_{2+2m+2n}(2\xi\cos\theta)\,d\xi$$

$$= -\,I - \frac{1}{2(m+n+1)} \tag{18}$$

and
$$I = \int_0^\infty \frac{x_o^2\,k^2\,p(\xi)}{(1-\eta)R(\xi)}\, J_{2+2m+2n}(2\xi\cos\theta)\,d\xi \tag{19}$$

EVALUATION OF THE INFINITE INTEGRAL BY METHOD OF CONTOUR INTEGRATION: To evaluate the integral I, we put $\xi = x_o hx$ and take $k/h = \tau$, then

$$I = \int_0^\infty \frac{\tau^2 q_1(x)}{(1-\eta)Q(x)}\, J_{2+2m+2n}(2\,x_o hx\cos\theta)\,dx, \tag{20}$$

where $Q(x) = (2x^2 - \tau^2)^2 - 4x^2\, q_1(x)q_2(x)$ and

$$q_1(x) = (x^2-1)^{1/2},\quad q_2(x) = (x^2-\tau^2)^{1/2}.$$

Taking

$$Q_o(x) = (2x^2-\tau^2)^2 + 4x^2\, q_1(x)q_2(x) \text{ and}$$

$$\Delta_o(x) = (2x^2-\tau^2)^4 - 16x^4\, q_1^2(x)q_2^2(x),$$ the integral I may be written in the form

$$I = \int_0^\infty \frac{\tau^2\, q_1(x)Q_o(x)}{(1-\eta)\,\Delta_o(x)}\, J_{2+2m+2n}(2x_o\, hx\cos\theta)\,dx. \tag{21}$$

For our convenience we replace the Bessel function of the first kind by Hankel function given by

$$J_{2+2m+2n}(\ ) = \frac{1}{2}\left[H^{(1)}_{2+2m+2n}(\ ) + H^{(2)}_{2+2m+2n}(\ )\right] ,$$

where $H^{()}_{\nu}$ denotes Hankel function. Consequently, $I = I_1 + I_2$ where I_1 and I_2 are integrals involving $H^{(1)}_{2+2m+2n}(2x_0\ hx\cos\theta)$ and $H^{(2)}_{2+2m+2n}(2x_0\ hx\cos\theta\ )$ respectively.

Thus for I_1 and I_2 we consider the integrals

$$J_{1,2} = \frac{1}{2\,(1-\eta)} \int_{\Gamma_{1,2}} \frac{\tau^2 q_1(z) Q_0(z)}{\Delta_0(z)} H^{(1,2)}_{2+2m+2n}(2x_0\ hz\cos\theta)\,dz$$

where Γ_1 and Γ_2 are the contours in the first and fourth quadrants of the complex z-plane, as shown in the figure 1. Consistent sign of the double valued functions $q_1(z)$ and $q_2(z)$ are shown in the Fig.1. The branch points $(1,0)$ and $(\tau,0)$, the poles $(\tau_j,0)$ for $j = 0,1,2$ which are the zeroes of the function $\Delta_0(z)$ and the origin at which the Hankel function fails to have finite value are all avoided by semi circular indentations, in order to ensure that the integrands are analytic within and on the contour. It is found that the integrals satisfy Jordon's lemma and therefore the contribution to the integrals from the infinitely distant parts of the contours is zero.

After integration round the contours Γ_1 and Γ_2 and adding we obtain finally,

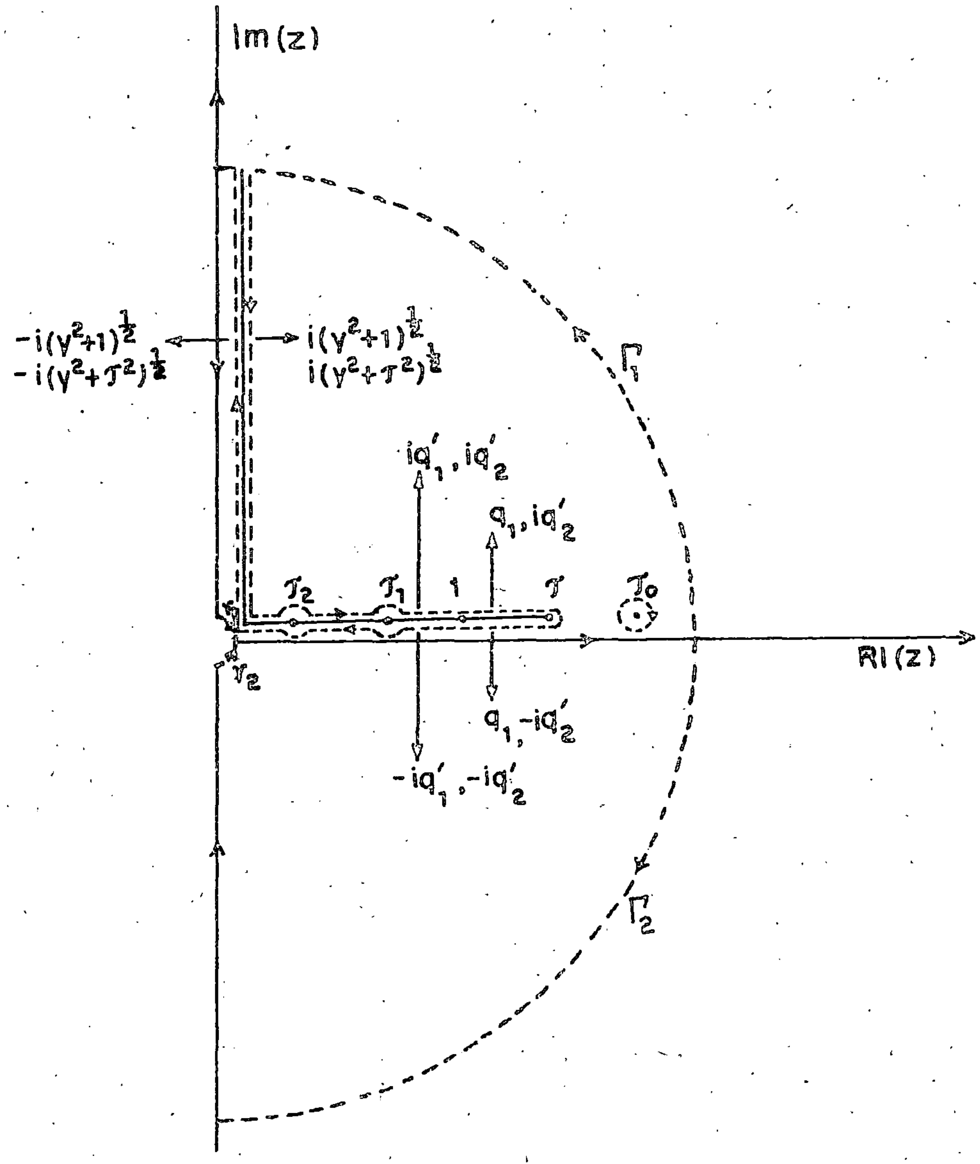


Fig.1- Contours of integration Γ_1 and Γ_2 in the complex z-plane for Poisson's ratio $\eta = 1/4$

$$I = I_1 + I_2$$

$$= \frac{i\pi\tau^2}{1-\eta} D_0 \, H^{(1)}_{2+2m+2n}(2\, x_0\, h\tau_0 \cos\theta) - \frac{i\tau^2}{1-\eta}\int_0^1 M_1(x) H^{(1)}_{2+2m+2n}(2x_0 hx \cos\theta)dx -$$

$$- \frac{i\tau^2}{1-\eta}\int_0^{\tau} M_2(x) H^{(1)}_{2+2m+2n}(2\, x_0\, hx \cos\theta)dx -$$

$$- \frac{\tau^2}{2(1-\eta)} \oint_{\gamma_1} \frac{q_1(z)Q_0(z)}{\Delta_0(z)} H^{(1)}_{2+2m+2n}(2\, x_0\, hz \cos\theta)\, dz -$$

$$- \frac{\tau^2}{2(1-\eta)} \oint_{\gamma_2} \frac{q_1(z)Q_0(z)}{\Delta_0(z)} H^{(2)}_{2+2m+2n}(2\, x_0\, hz \cos\theta)\, dz \qquad (22)$$

In (22) notations used are given below

$D_0 = \dfrac{q_1(\tau_0)(2\tau_0^2 - \tau^2)^2 + 4\tau_0^2\, q_1^2(\tau_0) q_2(\tau_0)}{\Delta_0'(\tau_0)}$, ($\Delta_0'(x)$ is derivative of $\Delta_0(x)$ with respect to x)

$$M_1(x) = \frac{q_1'(x)(2x^2 - \tau^2)^2}{\Delta_0(x)}, \quad (1 - x^2)^{1/2} = q_1'(x), \text{ (say)}$$

$$M_2(x) = \frac{4x^2\, q_1^2(x)\, q_2'(x)}{\Delta_0(x)}, \quad (\tau^2 - x^2)^{1/2} = q_2'(x), \text{ (say).}$$

In order to simplify (22), we make use of the series expansion for $H^{(1,2)}_{2+2m+2n}(z)$ in the form

$$H^{(1,2)}_{2+2m+2n}(z) = \sum_{r=0}^{\infty} (-1)^{r} \left(\frac{z}{2}\right)^{2r+2m+2n+2} \Big/ \left[r!\ \Gamma(r+2m+2n+3)\right] \pm$$

$$\pm \frac{i}{\pi}\Big[\Big\{2 \sum_{r=0}^{\infty} (-1)^{r} \left(\frac{z}{2}\right)^{2r+2m+2n+2} \Big/ \left[r!\ \Gamma(r+2m+2n+3)\right]\Big\} \log\left(\frac{z}{2}\right) -$$

$$- \sum_{r=0}^{2m+2n+1} \left(\frac{z}{2}\right)^{2r-2m-2n-2} \times \frac{(2m+2n+1-r)!}{r!} -$$

$$- \sum_{l=0}^{\infty} (-1)^{l} \left(\frac{z}{2}\right)^{2m+2n+2+2l} \times \frac{\left\{\chi(2m+2n+l+3)+\chi(l+1)\right\}}{l!\ (2m+2n+2+l)!}\Big], \quad (23)$$

where $\chi(\ )$ is Euler's function. When $H^{(1,2)}_{2+2m+2n}(\ )$ occurring on the right hand side of (22) are replaced by the corresponding series mentioned above, it is found that the terms involving $(2\, x_o h)^{2r-2m-2n-2}$ for which $(2r-2m-2n-2) < 0$ on the right hand side of equation (22) vanish. This can be proved by intergrating the complex function

$$f(z) = \frac{T^2}{2(1-\eta)} (2\, x_o h \cos\theta)^{2r-2m-2n-2}\, \frac{q_1(z) Q_o(z)}{\Delta_o(z)}\, z^{2r-2m-2n-2}$$

along the contours Γ_1 and Γ_2 as shown in fig.1. and then subtracting. Again when $2r - 2m - 2n - 2 = 0$, origin is not the singularity of the Hankel functions. In this case the contours considered slightly differ from that of the contours shown in fig.1, Here the identation round the origin is not required and it is found after integration of the function $f(z)$ round the modified contours Γ_1 and Γ_2 and subtracting the sum of the terms on right hand side of (22) which do not involve any power of $(2\,x_0 h)$ is equal to $^{-1}/2(m+n+1)$. Finally when $(2r - 2m - 2n - 2) > 0$, origin is not the singularity of Henkel functions and hence considering the same contour where the indentation at the origin is deleted, it is found that the terms involving $(2x_0 h)^{2r-2m-2n-2}$ on the right hand side of (22) is not zero because the contribution to the integrals from the infinitely distant parts of the contours do not vanish in this case. Thus with the help of (22) and (18) we may write

$$A_{mn} = -\frac{i\,\pi T^2}{1-\eta} D_0 \bar{H}^{(1)}_{2+2m+2n}(2\,x_0 hx\cos\theta) +$$

$$+\frac{i\,T^2}{1-\eta}\int_0^1 M_1(x)\,\bar{H}^{(1)}_{2+2m+2n}(2\,x_0 hx\cos\theta)\,dx +$$

$$+\frac{i\,T^2}{1-\eta}\int_0^T M_2(x)\,\bar{H}^{(1)}_{2+2m+2n}(2\,x_0\,hx\cos\theta)\,dx \qquad (24)$$

where

$$\bar{H}^{(1)}_{2+2m+2n}(z) = \sum_{r=0}^{\infty} (-1)^{r} \left(\frac{z}{2}\right)^{2r+2m+2n+2} \Big/ \left[r!\,\Gamma(r+2m+2n+3)\right] +$$

$$+ \frac{i}{\pi}\left[\left\{2 \sum_{r=0}^{\infty} (-1)^{r} \left(\frac{z}{2}\right)^{2r+2m+2n+2} \Big/ \left[r!\,\Gamma(r+2m+2n+3)\right]\right\} \log\left(\frac{z}{2}\right) -\right.$$

$$- \sum_{r=m+n+2}^{2m+2n+1} \left(\frac{z}{2}\right)^{2r-2m-2n-2} \times \frac{(2m+2n+1-r)!}{r!} -$$

$$\left.- \sum_{l=0}^{\infty} (-1)^{l} \left(\frac{z}{2}\right)^{2m+2n+2+2l} \times \frac{\{\chi(2m+2n+l+3)+\chi(l+1)\}}{l!\,(2m+2n+2+l)!}\right], \qquad (25)$$

for $m, n = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots$

DETERMINATION OF THE COEFFICIENTS a_n : For low frequency of vibration i.e for small (x_0k) and (x_0h) if we neglect terms involving $(x_0k)^4$ and higher order of (x_0k) in the expansion of $\bar{H}^{(1)}_{2+2m+2n}(2x_0hx\cos\theta)$ and $\bar{H}^{(1)}_{2+2m+2n}(2x_0h_0\cos\theta)$ occurring in A_{mn} of (24), we find

$$(1-\eta)A_{00} = \left\{(x_0k)^2\log(x_0k)\right\}B_1\cos^2\theta + (x_0k)^2\cos^2\theta\left\{\left(-\frac{i\pi}{2} + \log\cos\theta -\right.\right.$$

$$\left.\left. -\log\tau - \frac{\chi(3)+\chi(1)}{2}\right)B_1 + B_2\right\} - \frac{(x_0k)^4\log(x_0k)}{3\tau^2}B_3\cos^4\theta\,,$$

$$(1-\eta)A_{10} = (1-\eta)A_{01} = -\frac{(x_0k)^2}{6}B_1\cos^2\theta + \frac{(x_0k)^2\log(x_0k)}{12\tau^2}B_3\cos^4\theta \text{ and}$$

$$\underset{(m+n>1)}{(1-\eta)A^{*}_{mn}} = -\frac{(x_0k)^2}{(m+n)(m+n+1)\cancel{(m+n+2)}}B_1\cos^2\theta, \qquad (26)$$

where $B_1 = D_o \tau_o^2 - \frac{1}{\pi}\int_o^1 M_1(x)x^2\,dx - \frac{1}{\pi}\int_o^\tau M_2(x)\,x^2\,dx$,

$$B_2 = D_o \tau_o^2 \log \tau_o - \frac{1}{\pi}\int_o^1 M_1(x)x^2 \log x\,dx - \frac{1}{\pi}\int_o^\tau M_2(x)x^2 \log x\,dx,$$

and $B_3 = D_o \tau_o^4 - \frac{1}{\pi}\int_o^1 M_1(x)x^4\,dx - \frac{1}{\pi}\int_o^\tau M_2(x)x^4\,dx.$

When A_{oo}, A_{10} etc from (26) are substituted in (17) and then integrated over θ, it is found that

$$(1-\eta)L_{oo} = \left\{K^2 \log K\right\}B_1 + K^2\left\{B_1\left(-\frac{i\pi}{2} + \frac{1}{2} - \log 2\tau - \frac{\chi(3)+\chi(1)}{2}\right) + B_2\right\} - \left\{K^4 \log K\right\}\frac{B_3}{4\tau^2} \qquad (27)$$

$$(1-\eta)L_{10} = -K^2\frac{B_1}{12} + \left\{K^4 \log K\right\}\frac{B_3}{24\tau^2}$$

$L_{01} = 3\,L_{10}$ and $L_{mn} = o$ for $m + n > 1$, where $x_o k = K$ is the dimensionless wave number.

Equation (15) can be solved, iteratively to determine a_n when values of L_{mn} are known. Using the values of L_{mn} given by (27) we obtain a_n in the following form where terms involving K^4 and its higher orders have been neglected

$$a_o = i\pi\phi_o\Big[1 - \frac{1}{1-\eta}\Big\{B_1K^2\log K + \Big[B_1\Big(-\frac{i\pi}{2} + \frac{1}{2} - \log 2\tau - \frac{3-4\nu}{4}\Big) + B_2\Big]K^2 -$$

$$- \frac{B_1^2}{1-\eta}K^4(\log K)^2 - \Big[\frac{B_3}{4\tau^2} + 2\frac{B_1}{1-\eta}\Big\{B_1\Big(-\frac{i\pi}{2} + \frac{1}{2} - \log 2\tau - \frac{3-4\nu}{4}\Big) + B_2\Big\}\Big]K^4\log K\Big\}\Big]$$

$$(28)$$

$$a_1 = \frac{i\pi\,\phi_o}{4(1-\eta)}\Big[B_1K^2 - \Big(\frac{B_3}{2} + \frac{B_1^2}{1-\eta}\Big)K^4\log K\Big]$$

and $a_n = o$ for $n > 2$.

where $\nu = 0.5772157$ .... is the Euler's constant.

RESULTS: With the evaluation of B_i's for $i = 1,2,3$ (which have been shown in the appendix), it follows from (11) that $D(\xi)$ is known which is the solution of the dual integral equations given in (10-a,b).

i) Noting that $A(\xi)$ is an odd function of ξ, with the help of last equation of (6) and (7), normal stress on $y = o$ is given by

$$(\sigma_{yy})_{y=o} = \frac{-1}{\pi}\int_0^\infty\Big[-\lambda x_o^2h^2 + 2\mu(\xi^2 + x_o^2h^2) - \frac{4\mu ps\xi^2}{\xi^2+s^2}A(\xi)\sin\xi X\Big]d\xi, \qquad (29)$$

when $A(\xi)$ is replaced in terms of $D(\xi)$ in (29) one obtains

$$(\sigma_{yy})_{y=o} = \frac{i\mu}{\pi(1-\eta)}\int_0^\infty\Big[(\xi^2+s^2) - \frac{4bs\xi^2}{\xi^2+s^2}\Big]\frac{\xi^2+s^2}{\sqrt{\xi}R(\xi)}D(\xi)\sin\xi X\,d\xi\,.$$

When $D(\xi)$ is replaced by the infinite series given by the equation (11), we have

$$(\sigma_{yy})_{y=0} = \frac{i\mu}{\pi(1-\eta)}\left[a_0 \int_0^\infty J_1(\xi)\sin\xi X \, d\xi + a_1 \int_0^\infty J_3(\xi)\sin\xi X \, d\xi + \ldots\ldots\right]$$

$$= \frac{i\mu}{\pi(1-\eta)}\left[\frac{a_0 X}{\sqrt{1-X^2}} + \frac{a_1(3X-4X^3)}{\sqrt{1-X^2}}\right], |X| < 1$$

because $a_n = 0$, for $n > 2$. Substituting the values of the constants a_0, a_1 we finally obtain

$$(\sigma_{yy})_{y=0} = \frac{\mu\phi_0}{1-\eta}\frac{X}{\sqrt{(1-X^2)}}\Big[1 - \frac{B_1}{1-\eta}K^2\log K + \frac{1}{1-\eta}\Big\{B_1\Big(\frac{i\pi}{2} + \log 2\tau - \nu + 1 - X^2\Big) - B_2\Big\}K^2 +$$

$$+ \frac{B_1^2}{(1-\eta)^2}K^4\log^2 K + \frac{1}{1-\eta}\Big\{\frac{B_1^2}{1-\eta}\Big(-i\pi - \frac{5}{4} - 2\log 2\tau + 2\nu + X^2\Big) +$$

$$+ \frac{2B_1 B_2}{1-\eta} + \frac{B_3}{4}\Big(\frac{1}{\tau^2} - \frac{3}{2} + 2X^2\Big)\Big\}K^4 \log K\Big], \ |X| < 1$$

ii) The value of the reactive couple exerted by the elastic material on the strip is

$$G = 2\int_0^{x_0}(\sigma_{yy})_{y=0}\, x\, dx = V(g_1 - ig_2)$$

where $V = \mu\,\phi_0\,x_0^2\,\pi$,

$$g_1 = -\frac{1}{2(1-\eta)}\Big\{1 - \frac{B_1}{1-\eta}K^2\log K + \frac{1}{1-\eta}\Big[B_1\Big(\log 2\tau + \frac{1}{4} - \nu\Big) - B_2\Big]K^2 + \frac{B_1^2}{(1-\eta)^2}K^4\log^2 K -$$

$$- \frac{1}{1-\eta}\Big[\frac{B_1^2}{1-\eta}\Big(\frac{1}{2} + 2\log 2\tau - 2\nu\Big) - \frac{2B_1 B_2}{1-\eta} - \frac{B_3}{4\tau^2}\Big]K^4 \log K \qquad (30)$$

and

$$g_2 = -\frac{1}{2(1-\eta)}\Big\{-\frac{\pi B_1}{2(1-\eta)}K^2 + \frac{\pi B_1^2}{(1-\eta)^2}K^4 \log K\Big\}.$$

For Poisson's ration $\eta = \frac{1}{4}$, τ^2 is equal to 3. τ_0, τ_1 and τ_2 which are the roots of $\Delta_o(x) = o$ are then given by

$$\tau_o^2 = \frac{3}{2-2\,/\sqrt{3}}\,,\quad \tau_1^2 = \frac{3}{2+2\,/\sqrt{3}} \quad \text{and} \quad \tau_2^2 = 3/4.$$

With these values of τ_i, values of the constants B_1, B_2 and B_3 are determined and they are

$B_1 = - \cdot 28125$, $B_2 = - \cdot 1507729$ and $B_3 = - \cdot 8828124$.

For different values of K, we compare values of $|g_1|$ and $|g_2|$ with corresponding values determined by Wickham (5)

K	Wickham		values from formula (30)	
	$\lvert g_1\rvert$	$\lvert g_2\rvert$	$\lvert g_1\rvert$	$\lvert g_2\rvert$
0.143	.655	.008	.655	.008
0.2578	.638	.026	.639	.024
0.4368	.609	.075	.616	.066

Numerical values calculated from (30) differ from the values obtained by Wickham due to the fact that Wickham's results contain terms upto K^2 where as in our results terms upto ($K^4 \log K$) have been retained.

APPENDIX

Evaluation of B_1, B_2 and B_3.

We have

$$B_1 = D_o \tau_o^2 - \frac{1}{\pi} \int_0^1 M_1(x)x^2 \, dx - \frac{1}{\pi} \int_0^{\tau} M_2(x)x^2 \, dx \qquad \text{(A1)}$$

Splitting the integrands of the integrals of (A1) into partial fractions, we write

$$\int_0^1 M_1(x)x^2 \, dx = \frac{1}{16(1-\tau^2)} \sum_{j=o}^{2} P_j \int_0^1 \frac{q_1'(x)\, x^2}{x^2 - \tau_j^2} \, dx$$

$$= \frac{1}{16(1-\tau^2)} \left[\sum_{j=o}^{2} P_j \int_0^1 q_1'(x)dx + \sum_{j=1}^{2} \tau_j^2 P_j \int_0^1 \frac{q_1'(x)}{x^2 - \tau_j^2} \, dx - \tau_o^2 P_o \int_0^1 \frac{q_1'(x)}{\tau_o^2 - x^2} \, dx \right]$$

$$= \frac{\pi}{32(1-\tau^2)} \left[\sum_{j=o}^{2} P_j \left(\frac{1}{2} - \tau_j^2 \right) + P_o \, \tau_o \left(\tau_o^2 - 1 \right)^{1/2} \right] \qquad \text{(A2)}$$

and $$\int_0^{\tau} M_2(x)x^2 \, dx = \frac{1}{16(1-\tau^2)} \sum_{j=o}^{2} S_j \int_0^{\tau} \frac{q_2'(x)}{x^2 - \tau_j^2} \, x^2 \, dx$$

$$= \frac{1}{16(1-\tau^2)} \left[\sum_{j=o}^{2} S_j \int_0^{\tau} q_2'(x)dx \sum_{j=1}^{2} \tau_j^2 S_j \int \frac{q_2'(x)}{x^2 - \tau_j^2} \, dx - \tau_o^2 S_o \int_0^{\tau} \frac{q_2'(x)}{\tau_o^2 - x^2} \, dx \right]$$

$$= \frac{\pi}{32(1-\tau^2)} \left[\sum_{j=o}^{2} S_j \left(\frac{\tau^2}{2} - \tau_j^2 \right) + S_o \, \tau_o \left(\tau_o^2 - \tau^2 \right)^{1/2} \right], \qquad \text{(A3)}$$

where f denotes the principal value of the integral. Since by partial fraction

$$\frac{(2x^2 - \tau^2)^2}{\Delta_0(x)} = \frac{1}{16(1 - \tau^2)} \sum_{j=0}^{2} \frac{P_j}{x^2 - \tau_j^2} \quad \text{and}$$

$$\frac{4x^2(x^2 - 1)}{\Delta_0(x)} = \frac{1}{16(1 - \tau^2)} \sum_{j=0}^{2} \frac{S_j}{x^2 - \tau_j^2}, \quad \text{so we obtain}$$

after algebraic simplification

$$D_0 \tau_0^2 = \frac{\tau_0^2(2\tau_0^2 - \tau^2)^2 q_1(\tau_0) + 4\tau_0^4 q_2(\tau_0) q_1^2(\tau_0)}{\Delta_0'(\tau_0)}$$

$$= \frac{1}{32(1 - \tau^2)} \left[P_0 \tau_0 q_1(\tau_0) + S_0 \tau_0 q_2(\tau_0)\right]. \qquad (A4)$$

Hence it follows by the use of (A2), (A3) and (A4)

$$B_1 = \frac{1}{32(1 - \tau^2)} \sum_{j=0}^{2} \left[P_j\left(\tau_j^2 - \frac{1}{2}\right) + S_j\left(\tau_j^2 - \frac{\tau^2}{2}\right)\right].$$

Next we consider

$$B_2 = D_0 \tau_0^2 \log \tau_0 - \frac{1}{\pi} \int_0^1 M_1(x) x^2 \log x \, dx - \frac{1}{\pi} \int_0^{\tau} M_2(x) x^2 \log x \, dx$$

By (A4) it follows that

$$D_0 \tau_0^2 \log \tau_0 = \frac{1}{32(1 - \tau^2)} \left[P_0 \tau_0 q_1(\tau_0) + S_0 \tau_0 q_2(\tau_0)\right] \log \tau_0 \qquad (A5)$$

By splitting the integrands of integrals of B_2 as before one obtains,

$$\int_0^1 M_1(x)x^2 \log x \, dx = \frac{1}{16(1-\tau^2)} \sum_{j=0}^{2} P_j \int_0^1 \frac{q_1'(x).x^2 \log x}{x^2 - \tau_j^2} dx$$

$$= \frac{1}{16(1-\tau^2)} \left[\sum_{j=0}^{2} P_j \int_0^1 q_1'(x) \log x \, dx + \sum_{j=1}^{2} \tau_j^2 P_j \int_0^1 \frac{q_1'(x) \log x}{x^2 - \tau_j^2} dx - P_0 \tau_0^2 \int_0^1 \frac{q_1'(x) \log x}{\tau_0^2 - x^2} dx \right]$$

$$= \frac{\pi}{32(1-\tau^2)} \left[\sum_{j=0}^{2} P_j \left\{ \left(\tau_j^2 - \frac{1}{2} \right) \log 2 - \frac{1}{4} \right\} + \sum_{j=1}^{2} P_j \; \tau_j \, q_1'(\tau_j) \tan^{-1} \frac{q_1'(\tau_j)}{\tau_j} - \right.$$

$$\left. - P_0 \; \tau_0 \; q_1(\tau_0) \log \frac{\tau_0 + q_1(\tau_0)}{\tau_0} \right] \qquad \text{(A6)}$$

and $$\int_0^{\tau} M_2(x) x^2 \log x \, dx = \frac{1}{16(1-\tau^2)} \sum_{j=0}^{2} S_j \int_0^{\tau} \frac{q_2'(x) \; x^2 \log x}{x^2 - \tau_j^2} dx$$

$$= \frac{1}{16(1-\tau^2)} \left[\sum_{j=0}^{2} S_j \int_0^{\tau} q_2'(x) \log x \, dx + \sum_{j=1}^{2} \tau_j^2 S_j \int_0^{\tau} \frac{q_2'(x) \log x}{x^2 - \tau_j^2} dx - \right.$$

$$\left. - \tau_0^2 S_0 \int_0^{\tau} \frac{q_2'(x) \log x}{\tau_0^2 - x^2} dx \right]$$

$$= \frac{\pi}{32(1-\tau^2)} \left[\sum_{j=0}^{2} S_j \left\{ \left(\frac{\tau^2}{2} - \tau_j^2 \right) \log \frac{\tau}{2} - \frac{\tau^2}{4} \right\} + \sum_{j=1}^{2} S_j \; \tau_j q_2'(\tau_j) \tan^{-1} \frac{q_2'(\tau_j)}{\tau_j} + \right.$$

$$\left. + S_0 \; \tau_0 \; q_2(\tau_0) \log \tau - S_0 \; \tau_0 \; q_2(\tau_0) \log \frac{\tau_0 + q_2(\tau_0)}{\tau_0} \right] \qquad \text{(A7)}$$

In this case using (A5),(A6) and (A7), we obtain after some algebraic simplification

$$B_2 = \frac{1}{32(1-\tau^2)}\Big[\sum_{j=0}^{2}\Big\{P_j\Big[\frac{1}{4}-(\tau_j^2-\frac{1}{2})\log 2\Big]+S_j\Big[\frac{\tau^2}{4}-(\frac{\tau^2}{2}-\tau_j^2)\log\frac{\tau}{2}\Big]\Big\}-$$

$$-\sum_{j=1}^{2}\Big\{P_j\ \tau_j\, q_1'(\tau_j)\tan^{-1}\frac{q_1'(\tau_j)}{\tau_j}+S_j\ \tau_j\, q_2'(\tau_j)\tan^{-1}\frac{q_2'(\tau_j)}{\tau_j}+$$

$$+P_0\ \tau_0\, q_1(\tau_0)\log(\tau_0+q_1(\tau_0))+S_0\ \tau_0\, q_2(\tau_0)\log\frac{\tau_0+q_2(\tau_0)}{\tau}\Big].$$

Finally to obtain B_3 we proceed as before and we have by (A4)

$$D_0\ \tau_0^4 = \frac{1}{32(1-\tau^2)}\Big[P_0\ \tau_0^3\, q_1(\tau_0) + S_0\ \tau_0^3\, q_2(\tau_0)\Big], \qquad (A8)$$

$$\int_0^1 M_1(x)x^4\,dx = \frac{\pi}{32(1-\tau^2)}\Big[\frac{1}{2}+\sum_{j=0}^{2}P_j\ \tau_j^2\,(\frac{1}{2}-\tau_j^2)+P_0\ \tau_0^3\, q_1(\tau_0)\Big] \qquad (A9)$$

$$\int_0^\tau M_2(x)x^4\,dx = \frac{\pi}{32(1-\tau^2)}\Big[\frac{\tau^4}{2}+\sum_{j=0}^{2}S_j\ \tau_j^2(\frac{\tau^2}{2}-\tau_j^2)+S_0\ \tau_0^3\, q_2(\tau_0)\Big]\cdot \qquad (A10)$$

Consequently, we have from (A8), (A9) and (A10)

$$B_3 = \frac{1}{32(1-\tau^2)}\Big[-\frac{1}{2}(1+\tau^4)+\sum_{j=0}^{2}\tau_j^2\Big\{P_j(\tau_j^2-\frac{1}{2})+S_j(\tau_j^2-\frac{\tau^2}{2})\Big\}\Big].$$

HARMONIC ROCKING OF A RIGID CIRCULAR INDENTOR ON AN ELASTIC HALF-SPACE

INTRODUCTION: Study of elastic waves due to different types of oscillations of an indentor or of a vibrating punch on the surface of a homogeneous, isotropic elastic medium has become almost classical. Robertson (1964) extended the static problem of indentation of a semi-infinite elastic solid by a rigid circular disc solved by Sneddon (1951) and derived the solution due to a forced vertical vibration of the disk for low frequencies. Zakorko and Rostovtsev (1965) considered the effect of sinusoidal load transmitted through a weight less rigid circular punch, where a single equation is used which contained directly the sought for pressure through an unknown function which is then transformed to an integral equation of the second kind and then solved iteratively. Robertson (1967) determined the stress distribution due to impinging on the surface of a penny-shaped crack of a plane longitudinal wave, harmonic in time. He adopted the same technique of reducing the dual integral equation to Fredholm integral equation and solving by the method of iteration. Gladwell (1968) discussed the response of a semi-infinite elastic solid to a rotatory vibration of an indentor, over a circular area about a diameter, on the free surface. There also dual integral equations are converted to Fredholm integral equation of second kind and for low frequency oscillation, the solution is determined by iteration.

In this paper, the authors reconsider the problem of Gladwell (1968). By using Hankel transform the solution of the problem has been reduced to the solution of dual integral equations and then, following the technique of Tranter, to the solutions of an infinite set of algebraic equations. There is not much reference to the method of

Tranter in the literature involving the solution of dual integral equations. The authors find that the method prescribed by Tranter is no less powerful than the other methods used in this connection.

The normal stress below the disk, total torque, and the displacement on the free surface at large distance compared to the wavelength have been determined. Unlike Gladwell (1968), the results derived are exact in the sense that they do not involve any integral. The graphs of the normalised stress below the disc against the normalised distance from the centre of the disc have been plotted.

FORMULATION OF THE PROBLEM AND REDUCTION TO DUAL INTEGRAL EQUATIONS:

We consider a harmonic oscillation of frequency ω of a rigid circular disc of radius r_o about a diameter. The disc having a frictionless contact, is indented on the free surface of a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty < x < \infty$, $-\infty < y < \infty$ and $z > o$, z-axis being drawn into the medium. The indentor is assumed to produce rotatory vibration of amplitude φ_o about one of its diameters which is taken to be the y-axis.

The equations of motion in the cylindrical polar co-ordinates are

$$\nabla^2 \varphi = \frac{1}{a^2} \frac{\partial^2 \varphi}{\partial t^2} \text{ and } \quad \nabla^2 \psi = \frac{1}{b^2} \frac{\partial^2 \psi}{\partial t^2} \qquad (1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$ and φ, ψ are scalar potentials associated with dilatational and rotational parts of the displacement, a and b are P- and S- wave velocities respectively.

The nonvanishing components of displacement are

$$u_r = \frac{\partial\varphi}{\partial r} + \frac{\partial^2\psi}{\partial r\partial z}, \quad u_\theta = \frac{1}{r}\left(\frac{\partial\varphi}{\partial\theta} + \frac{\partial^2\psi}{\partial\theta\partial z}\right),$$

$$u_z = \frac{\partial\varphi}{\theta z} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{b^2}\frac{\partial^2\psi}{\partial t^2} \tag{2}$$

and those of the stress are

$$\sigma_{rz} = \mu\left[2\frac{\partial^2\varphi}{\partial r\partial z} + 2\frac{\partial^3\psi}{\partial r\partial z^2} - \frac{1}{b^2}\frac{\partial^3\psi}{\partial r\partial t^2}\right],$$

$$\sigma_{\theta z} = \mu\left[\frac{2}{r}\frac{\partial^2\varphi}{\partial\theta\partial z} + \frac{2}{r}\frac{\partial^3\psi}{\partial\theta\partial z^2} - \frac{1}{b^2 r}\frac{\partial^3\psi}{\partial\theta\partial t^2}\right], \tag{3}$$

$$\sigma_{zz} = \mu\left[\frac{1}{b^2}\frac{\partial^2\varphi}{\partial t^2} + 2\left(\frac{\partial^2\varphi}{\partial z^2} - \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^3\psi}{\partial z^3} - \frac{1}{b^2}\frac{\partial^3\psi}{\partial z\partial t^2}\right)\right].$$

The boundary conditions to be satisfied on $z = 0$, are

$$u_z = \varphi_0 r \cos\theta \; ; \; r \leq r_0 \tag{4a}$$

$$\sigma_{zz} = 0 \qquad ; \; r > r_0 \tag{4b}$$

$$\sigma_{rz} = 0 = \sigma_{\theta z} \text{ every where on } z = 0. \tag{4c}$$

To find the solutions of the equations(1) subject to the conditions (4), we make the substitution

$$[\varphi, \psi] = [\bar{\Phi}(r,\theta,z), \bar{\Psi}(r, \theta, z)]e^{iwt}.$$

In view of the boundary condition (4a), $\bar{\Phi}$ and $\bar{\Psi}$ are taken in the form

$$\bar{\Phi} = r_0^2 \cos\theta \int_0^\infty \xi A(\xi) J_1(\rho\xi) e^{-pz}\, d\xi \tag{5a}$$

and $\bar{\Psi} = r_o^3 \cos\theta \int_0^\infty \xi\, B(\xi) J_1(\rho\xi) e^{-sZ}\, d\xi$ (5b)

where $\rho = r/r_o$ and $Z = z/r_o$. p and s in equations (5a - b) are

$$p(\xi) = \sqrt{\xi^2 - r_o^2h^2},\quad \xi > r_oh \text{ and } s(\xi) = \sqrt{\xi^2 - r_o^2k^2},\quad \xi > r_ok$$

$$= i\sqrt{r_o^2h - \xi^2},\quad r_oh > \xi \qquad = i\sqrt{r_o^2k^2 - \xi^2},\quad r_ok > \xi$$

where h = w/a and k = w/b. With the help of equations (1), (2) and from $\bar{\Phi}$, Ψ as obtained in (5), the following expressions for the components of displacement and those of required stress components are obtained as

$$u_r = r_o \cos\theta \int_0^\infty \xi \frac{\partial J_1(\rho\xi)}{\partial\rho} \left\{ A(\xi)\, e^{-pZ} - sB(\xi)\, e^{-sZ} \right\} d\xi,$$

$$u_\theta = \frac{r_o^2}{r} \sin\theta \int_0^\infty \xi\, J_1(\rho\xi) \left\{ -A(\xi)\, e^{-pZ} + sB(\xi) e^{-sZ} \right\} d\xi,$$

$$u_z = r_o \cos\theta \int_0^\infty \xi\, J_1(\rho\xi) \left\{ -pA(\xi) e^{-pZ} + \xi^2\, B(\xi) e^{-sZ} \right\} d\xi, \qquad (6)$$

$$\sigma_{rz} = \mu \cos\theta \int_0^\infty \xi \frac{\partial J_1(\rho\xi)}{\partial\rho} \left\{ -2pA(\xi) e^{-pZ} + (2\xi^2 - r_o^2k^2) B(\xi) e^{-sZ} \right\} d\xi,$$

$$\sigma_{\theta z} = \mu \frac{r_o}{r} \sin\theta \int_0^\infty \xi\, J_1(\rho\xi) \left\{ 2pA(\xi) e^{-pZ} - (\xi^2 + s^2) B(\xi) e^{-sZ} \right\} d\xi,$$

$$\sigma_{zz} = \mu \cos\theta \int_0^\infty \xi J_1(\rho\xi) \left\{ A(\xi)\, (\xi^2 + s^2) e^{-pZ} - 2sB(\xi)\, \xi^2\, e^{-sZ} \right\} d\xi.$$

In the expressions given in (6) and in subsequent analysis the time factor e^{-iwt} has been omitted. From the boundary condition (4c) we have

$$B(\xi) = 2\, pA(\xi)/(\xi^2 + s^2) \qquad (7)$$

and the conditions (4,a-b) give rise to the dual integral equations

$$-\frac{1}{(1-\eta_1)}\int_0^\infty \frac{r_0^2k^2\,p(\xi)}{R(\xi)}\,D(\xi)\,J_1(\rho\xi)d\xi = \phi_0\rho\ ,\quad 0\le\rho\le 1 \tag{8}$$

and $$\int_0^\infty D(\xi)J_1(\rho\xi)d\xi = 0,\quad \rho > 1$$

where $R(\xi) = (2\xi^2 - r_0^2k^2)^2 - 4ps\xi^2$,

$$D(\xi) = -(1-\eta_1)\,\frac{A(\xi)R(\xi)}{2\xi^2 - r_0^2k^2}\ , \tag{9}$$

and $\eta_1 = \lambda/2(\lambda+\mu)$ is poisson's ratio, λ and μ being Lame's constants.

Equations (8) can be written in the form

$$\int_0^\infty \left\{1 + H(\xi)\right\}\xi^{-1}\,D(\xi)J_1(\rho\xi)\,d\xi = \phi_0\rho\ ,\quad 0\le\rho\le 1 \tag{10}$$

and $$\int_0^\infty D(\xi)J_1(\rho\xi) = 0,\quad \rho > 1$$

where $1 + H(\xi) = -\dfrac{\xi r_0^2k^2\,p(\xi)}{(1-\eta_1)\,R(\xi)}$ and it can be whown that

$H(\xi) \to 0$ as $r_0h \to 0$ and $r_0k \to 0$.

SOLUTION OF THE DUAL INTEGRAL EQUATIONS: Following Tranter we take

$$D(\xi) = \xi^{1/2} \sum_{m=0}^{\infty} a_m J_{2m+\frac{3}{2}}(\xi), \tag{11}$$

so that the second equation of (10) is automatically satisfied. The coefficients a_m are to be so chosen that the form of $D(\xi)$ as assumed in (11) satisfies the first of equations (10). For such a choice of a_m we must have

$$\sum_{m=0}^{\infty} a_m \int_0^{\infty} [1+H(\xi)]\xi^{-\frac{1}{2}} J_{2m+\frac{3}{2}}(\xi) J_1(\rho\xi) \overset{d\xi}{\wedge} = \phi_0 \rho \quad , 0 \leq \rho \leq 1 \tag{12}$$

multiplying the equation (12) by $\rho^2(1-\rho^2)^{-1/2} \mathcal{F}_n(\frac{3}{2}, 2, \rho^2)$, where n is a positive integer or zero and $\mathcal{F}_n$ is a Jacobi polynomial of degree n and then integrating with respect to ρ from 0 to 1, we obtain

$$\sum_{m=0}^{\infty} a_m \int_0^{\infty} [1+H(\xi)]\xi^{-1} J_{2m+\frac{3}{2}}(\xi) \overset{J_{2n+3/2}(\xi)}{\wedge} d\xi = E(1, n, \tfrac{1}{2}) = E_n \text{ (say)} \tag{13}$$

$$\text{where } E_n = \frac{\sqrt{2}\,\Gamma(2+n)}{\Gamma(2)\,\Gamma(n+\frac{1}{2})} \phi_0 \int_0^1 \rho^3(1-\rho^2)^{-1/2} \mathcal{F}_n(\tfrac{3}{2}, 2, \rho^2)\, d\rho.$$

Noting that $\mathcal{F}_0(\frac{3}{2}, 2, \rho^2) = 1$, we have from orthogonality relation of Jacobi polynomial

$$E_n = \frac{2\sqrt{2}\,\phi_0}{3\sqrt{\pi}} \text{ or } 0 \text{ according as } n = 0 \text{ or } n \neq 0. \tag{14}$$

$$\text{Since } \int_0^{\infty} \xi^{-1} J_{2m+\frac{3}{2}}(\xi) J_{2n+\frac{3}{2}}(\xi)\, d\xi = 0, \quad \text{for } m \neq n$$
$$= (3+4n)^{-1} \quad \text{for } m = n$$

so we have from equation (13)

$$a_n + \sum_{m=0}^{\infty} L_{mn}\, a_m = (3 + 4n)\, E_n \tag{15}$$

where $$L_{mn} = (3 + 4n) \int_0^{\infty} H(\xi)^{-1}\, J_{2m+\frac{3}{2}}(\xi) J_{2n+\frac{3}{2}}(\xi)\, d\xi \tag{16}$$

Equation (15) gives an infinite set of algebraic equations for the determination of the coefficients a_m.

DETERMINATION OF THE COEFFICIENTS a_m : Using the generalisation of Neumann's Integral [cf. Watson (1958) p 150]

$$J_{2m+\frac{3}{2}}(\xi)\, J_{2n+\frac{3}{2}}(\xi) = \frac{2}{\pi} \int_0^{\pi/2} J_{2m+2n+3}(2\xi \cos\alpha) \cos 2(m-n)\alpha \, d\alpha$$

$$= \frac{4}{\pi^2} \int_0^{\pi/2} \cos 2(m-n)\alpha \, d\alpha \int_0^{\pi/2} \sin(2m+2n+3)\beta \, \sin(2\xi\cos\alpha \sin\beta)\, d\beta$$

in the equation (16) and changing the order of integration we obtain

$$L_{mn} = \frac{4(3 + 4n)}{\pi^2} \int_0^{\pi/2} \cos 2(m-n)\alpha \, d\alpha \int_0^{\pi/2} \sin(2m+2n+3)\beta \, d\beta \ \times$$

$$\times \int_0^{\infty} H(\xi)\xi^{-1} \sin(2\xi \cos\alpha \sin\beta)\, d\xi \, . \tag{17}$$

The last integral in (17) may be written in the form

$$\int_0^{\infty} H(\xi)\xi^{-1} \sin(2\xi\cos\alpha \sin\beta)\, d\xi = -\frac{P}{(1-\eta_1)} - \frac{\pi}{2} \tag{18}$$

where $P = \int_0^\infty \frac{r_o^2 k^2 p(\xi)}{R(\xi)} \sin (2\xi \cos \alpha \sin \beta)\, d\xi$. Following the method of Lapwood [cf. Ewing et al (1957) p.49], the integral in (18) is found to be $P = D_o e^{-2\, i\, r_o\, g \cos \alpha \sin \beta}$

$$- \int_0^{r_o h} \frac{r_o^2 k^2 \sqrt{r_o^2 h^2 - \xi^2}\; e^{-i\xi \cos \alpha \sin \beta}}{(2\xi^2 - r_o^2 k^2)^2 + 4\xi^2 \sqrt{r_o^2 h^2 - \xi^2}\sqrt{r_o^2 k^2 - \xi^2}}\, d\xi$$

$$- \int_{r_o h}^{r_o k} \frac{4 r_o^2 k^2\, \xi^2 (\xi^2 - r_o^2 h^2) \sqrt{r_o^2 k^2 - \xi^2}\; e^{-2i\xi \cos \alpha \sin \beta}}{(2\xi^2 - r_o^2 k^2)^4 + 16\, \xi^4 (\xi^2 - r_o^2 h^2)(r_o^2 k^2 - \xi^2)}\, d\xi \qquad (19)$$

where $g = w/c$, c is Rayleigh wave velocity and

$$D_o = \left[\frac{\pi\, r_o^2\, k^2\, p\ (\xi)}{R'(\xi)} \right]_{\xi = r_o g} \qquad (20)$$

Substituting $\xi = r_o\, gx$ in the integrals of (19) and expanding the exponential term in power series, we obtain

$$P = \sum_{l=o}^{\infty} \frac{(-2\, i\, r_o g \cos \alpha \sin \beta)^l}{l!} M_l \qquad (21)$$

where $M_l = D_o - \frac{c^2}{b^2} \int_0^{c/a} \frac{\sqrt{\frac{c^2}{a^2} - x^2}\;\; x^l}{(2x^2 - \frac{c^2}{b^2})^2 + 4x^2 \sqrt{\frac{c^2}{a^2} - x^2}\sqrt{\frac{c^2}{b^2} - x^2}}\, dx -$

$$- 4\, \frac{c^2}{b^2} \int_{c/a}^{c/b} \frac{(x^2 - c^2/a^2)(c^2/b^2 - x^2)^{1/2}\;\; x^{l+2}}{(2x^2 - \frac{c^2}{b^2})^4 + 16 x^4 (x^2 - \frac{c^2}{a^2})(\frac{c^2}{b^2} - x^2)}\, dx \qquad (22)$$

After substitution of the value of P from (21), (18) becomes

$$\int_0^\infty H(\xi)\,\xi^{-1} \sin\,(2\xi \cos\alpha \sin\beta)\,d\xi = \sum_{l=1}^{\infty} (-2ir_o g \cos\alpha \sin\beta)^l \frac{M_l}{l!}, \qquad (23)$$

since $M_o = -\frac{\pi}{2}(1-\eta_1)$, by (A3).

Substituting in (17), the result obtained in (23) and then integrating term by term one obtains

$$L_{mn} = (-1)^{m+n} \frac{3\ +4n}{(1-\eta_1)} I_{mn}, \quad \text{where}$$

$$I_{mn} = \sum_{l=1}^{\infty} \frac{(-i)^l\ l!\ (r_o g/2)^l\ M_l}{\Gamma(\frac{1}{2}+m-n+1)\ \Gamma(\frac{1}{2}-m+n+1)\ \Gamma(\frac{1}{2}+m+n+\frac{5}{2})\ \Gamma(\frac{1}{2}-m-n-\frac{1}{2})}. \qquad (24)$$

Using the fact that when $-\frac{1}{2}+m+n+\frac{3}{2}$ is not a negative integer or zero,

$$\frac{1}{\Gamma(\frac{1}{2}-m-n-\frac{1}{2}} = \frac{(-1)^{m+n+1}}{\pi} \cos\frac{1}{2}\pi\ \Gamma(-\frac{1}{2}+m+n+\frac{3}{2}),$$ we write below some particular values of I_{mn} for different values of m,n to calculate L_{mn}.

$$I_{mm} = -\frac{2}{\pi}\left[\frac{2M_2(r_o g)^2}{(4m+5)(4m+3)(4m+1)} - \frac{2i}{3}\frac{M_3(r_o g)^3}{\Gamma(2m+4)\,\Gamma(1-2m)} - \right.$$

$$\left. -\frac{3\pi}{16}\frac{M_4(r_o g)^4}{\Gamma(2m+\frac{9}{2})\,\Gamma(\frac{3}{2}-2m)} + \cdots\cdots\cdots\right]$$

$$I_{m(m+1)} = \frac{2}{\pi}\left[\frac{M_2(r_o g)^2}{(4m+7)(4m+5)(4m+3)} + \frac{4M_4(r_o g)^4}{(4m+9)(4m+7)(4m+5)(4m+3)(4m+1)} + \dots\right]$$

$$I_{m(m+2)} = -\frac{2}{\pi}\frac{M_4(r_o g)^4}{(4m+11)(4m+9)(4m+7)(4m+5)(4m+3)} + \dots\dots .$$

For low frequency oscillation, $r_o h$ and $r_o k$ are small, consequently $r_o g$ is also a small quantity. Retaining terms upto fourth power of $(r_o g)$ in L_{mn}, we obtain with the help of equation (25)

$$L_{oo} = \frac{2}{5}\frac{M_2}{M_o}(r_o g)^2 - \frac{1}{3}\frac{M_3}{M_o}(r_o g)^3 - \frac{6}{35}\frac{M_4}{M_o}(r_o g)^4 ,$$

$$L_{o1} = \frac{1}{15}\frac{M_2}{M_o}(r_o g)^2 + \frac{4}{135}\frac{M_4}{M_o}(r_o g)^4 ,$$

$$L_{1o} = \frac{1}{35}\frac{M_2}{M_o}(r_o g)^2 + \frac{4}{315}\frac{M_4}{M_o}(r_o g)^4 , \tag{26}$$

$$L_{11} = \frac{2}{45}\frac{M_2}{M_o}(r_o g)^2 + \frac{2}{495}\frac{M_4}{M_o}(r_o g)^4 ,$$

$$L_{o2} = \frac{1}{945}\frac{M_4}{M_o}(r_o g)^4 , \quad L_{20} = \frac{1}{3465}\frac{M_4}{M_o}(r_o g)^4$$

The values of the constants a_m are determined from (15) with the help of (14) and (26). Retaining terms upto the fourth power of $(r_o g)$, we find

$$a_o = 2\sqrt{\frac{2}{\pi}}\, \rho_o \left[1 - \frac{2}{5}\frac{M_2}{M_o}(r_o g)^2 + \frac{1}{3}\frac{M_3}{M_o}(r_o g)^3 + \frac{2}{5}\left\{\frac{3}{7}\frac{M_4}{M_o} + \frac{17}{42}\left(\frac{M_2}{M_o}\right)^2\right\}(r_o g)^4\right]$$

$$a_1 = -\frac{2}{15}\sqrt{\frac{2}{\pi}}\, \rho_o \left[\frac{M_2}{M_o}(r_o g)^2 + \frac{4}{9}\left\{\frac{M_4}{M_o} - \left(\frac{M_2}{M_o}\right)^2\right\}(r_o g)^4\right]$$

$$a_2 = -\frac{2}{945}\sqrt{\frac{2}{\pi}}\, \rho_o \left[\frac{M_4}{M_o} - \left(\frac{M_2}{M_o}\right)^2\right](r_o g)^4, \quad a_3 = a_4 = \ldots\ldots = o.$$

RESULTS: i) From the last equation of (6), (7), (9) and (11) the normal stress below the disc is

$$(\sigma_{zz})_{z=0} = -\frac{\mu \cos\theta}{(1-\eta_1)} \sum_{m=0}^{\infty} a_m \int_0^{\infty} \sqrt{\xi}\, J_1(\rho\xi) J_{2m+\frac{3}{2}}(\xi)\, d\xi \tag{27}$$

for $\rho < 1$, the value of the integral in (27) is given by [8, p - 401]

$$\int_0^{\infty} \sqrt{\xi}\, J_1(\rho\xi) J_{2m+\frac{3}{2}}(\xi)\, d\xi$$

$$= \frac{\rho\sqrt{2}\,\Gamma(m+2)}{\Gamma(m+\frac{1}{2})}\, {}_2F_1\left(m+2, \frac{1}{2}-m; 2; \rho^2\right) = \frac{\rho\sqrt{2}\,\Gamma(m+2)}{\Gamma(m+\frac{1}{2})}(1-\rho^2)^{-1/2} \quad X$$

$$X\ {}_2F_1\left(-m, m+\frac{3}{2}; 2; \rho^2\right). \tag{28}$$

Therefore, $$(\sigma_{zz})_{z=0} = 2\mu\cos\theta\,\frac{\phi_o}{M_o}\,\frac{\rho}{\sqrt{(1-\rho^2)}}\left[1-\frac{2}{3}\frac{M_2}{M_o}(1-\frac{\rho^2}{2})(r_og)^2+\right.$$

$$\left.+\frac{1}{3}\frac{M_3}{M_o}(r_og)^3+\frac{2}{45}\left\{(1+4\rho^2-\frac{\rho^4}{2})\frac{M_4}{M_o}+(\frac{13}{2}-4\rho^2+\frac{\rho^4}{2})(\frac{M_2}{M_o})^2\right\}(r_og)^4\right]. \tag{29}$$

ii) Total torque about y-axis is

$$T = -\int_0^{2\pi}\int_0^{r_o}(\sigma_{zz})_{z=0}\,r^2\cos\theta\,dr\,d\theta$$

$$= -\frac{4\pi\mu r_o^3}{3}\,\frac{\phi_o}{M_o}\left[1-\frac{2}{5}\frac{M_2}{M_o}(r_og)^2+\frac{1}{3}\frac{M_3}{M_o}(r_og)^3+\frac{1}{105}\left\{18\frac{M_4}{M_o}+17(\frac{M_2}{M_o})^2\right\}(r_og)^4\right].$$

This result agrees with result obtained by Gladwell [1968].

iii) We have from (6), (7), (9) and (11), the displacement components on the free surface far out side the disc as

$$(u_r)_{z=0} = -\frac{r_o\cos\theta}{(1-\eta_1)}\sum_{m=0}^{\infty}a_m\int_0^{\infty}\xi^{3/2}\,\frac{2\xi^2-r_o^2k^2-2ps}{R(\xi)}\,J_{2m+\frac{3}{2}}(\xi)\quad X$$

$$X\left\{J_0(\rho\xi)-\frac{1}{\rho\xi}J_1(\rho\xi)\right\}d\xi\,, \tag{30}$$

$$(u_\theta)_{z=0} = \frac{r_o^2}{r}\,\frac{\sin\theta}{(1-\eta_1)}\sum_{m=0}^{\infty}a_m\int_0^{\infty}\sqrt{\xi}\,\frac{2\xi^2-r_o^2k^2-2ps}{R(\xi)}\,J_{2m+\frac{3}{2}}(\xi)J_1(\rho\xi)d\xi \tag{31}$$

$$(u_z)_{z=0} = -\frac{r_0 \cos\theta}{(1-\eta_1)} \sum_{m=0}^{\infty} a_m \int_0^{\infty} \sqrt{\xi}\, \frac{r_0^2 k^2\, p}{R(\xi)}\, J_{2m+\frac{3}{2}}(\xi) J_1(\rho\xi) d \quad . \tag{32}$$

The integrals in the above expressions can be evaluated by Lapwood's method as in the appendis. But to obtain more information about bodily P- and S-wave the following method is adopted. To evaluate the integral in (32) i.e.

$$I_z = \int_0^{\infty} \sqrt{\xi}\, \frac{r_0^2 k^2\, p}{R(\xi)}\, J_{2m+\frac{3}{2}}(\xi)\, J_1(\rho\xi) d\xi\, , \tag{33}$$

we write $J_1(\rho\xi)$ in the form

$$J_1(\rho\xi) = \frac{1}{2}\left[H_1^{(1)}(\rho\xi) + H_1^{(2)}(\rho\xi) \right].$$

In the complex $\zeta\,(=\xi + i\eta\,)$- plane, we draw cuts parallel to the imaginary axis joining $\pm\, r_0 h$ with $\pm\,(r_0 h - i\infty)$, $\pm\, r_0 k$ with $\pm\,(r_0 k - i\,\infty)$ making the factors $\sqrt{\zeta - r_0 h}$, $\sqrt{\zeta + r_0 h}$ of $\sqrt{\zeta^2 - r_0^2 h^2}$ and $\sqrt{\zeta - r_0 k}$, $\sqrt{\zeta + r_0 k}$ of $\sqrt{\zeta^2 - r_0^2 k^2}$ single valued. Integrating the part of (33) consisting of $H_1^{(1)}(\rho\xi)$ along the boundary of the fourth quadrant with the cuts, then adding the integrals, we obtain

$$I_z = -\, i\, D_0 \sqrt{}\,(r_0 g)\, J_{2m+\frac{3}{2}}(r_0 g)\, H_1^{(2)}(r_0 g) \, -$$

$$-4ie^{-i\pi/4} \int_0^{\infty} \frac{(r_0 k - i\eta\,)^{5/2}\, r_0^2 k^2 \sqrt{(2r_0 k - i\eta\,)\left\{(r_0 k - i\eta)^2 - r_0^2 h^2\right\}}}{\left[2(r_0 k - i\eta)^2 - r_0^2 k^2\right]^4 + 16\, i\eta (r_0 k - i\eta)^4 (2r_0 k - i\eta)\left[(r_0 k - i\eta)^2 - r^2 h^2\right]} \; X$$

$$X\ J_{2m+\frac{3}{2}}(r_o k - i\eta)\ H_1^{(2)}(rk - i\rho\eta)\ \sqrt{\eta}\ d\eta -$$

$$-ie^{-i\pi/4}\int_o^\infty \frac{\sqrt{(r_o h - i\eta)}\,r_o^2k^2\ \sqrt{(2r_o h - i\eta)}\ \left[2(r_o h - i\eta)^2 - r_o^2k^2\right]^2}{\left[2(r_o h - i\eta)^2 - r_o^2k^2\right]^4 + 16i\eta(r_o h - i\eta)^4(2r_o h - i\eta)\left[(r_o h - i\eta)^2 - r_o^2k^2\right]}\ X$$

$$X\ J_{2m+\frac{3}{2}}(r_o h - i\eta)\ H_1^{(2)}(r\ h - i\rho\eta)\ \sqrt{\eta}\ d\eta \cdot \qquad (34)$$

First term on the right hand side of (34) arises due to pole at (r_og), second and third terms form integrals along loops round the branch cuts at r_ok and r_oh respectively.

Assuming $rk > rh >> 1$, the integrals in (34) can be evaluated asymptotically at points far out side the disc. Hankel functions are expanded asymptotically. Retaining the first term in the expansion of Hankel functions, the integrals arising in (34) are of the form

$$\int_o^\infty \sqrt{\eta}\ G(\eta)\ e^{-\rho\eta}\ d\eta = \frac{\overline{\left| 3/2 \right.}}{\rho^{3/2}}\,G(o) + \frac{\overline{\left| 5/2 \right.}}{\rho^{5/2}}\,G'(o) + \frac{\overline{\left| 7/2 \right.}}{\rho^{7/2}}\,\frac{G''(o)}{2!}$$

Keeping the leading terms again, the asymptotic values of the displacement on the free surface at points far from the centre of the disc is determined and it is found to be

$$(u_z)_{z=o} = \frac{r_o\ \cos\theta}{(1 - \eta_1)}\sum_{m=o}^{\infty} a_m\left[i\ \sqrt{\frac{2}{\pi}}\ D_o e^{-i(rg+\frac{\pi}{4})}\ J_{2m+\frac{3}{2}}(r_og)\ \sqrt{\frac{r_o}{r}}\ +\right.$$

$$+\,4e^{-irk}\,\frac{K^2}{(r_0k)^{3/2}}\,J_{2m+\frac{3}{2}}(r_0k)\left(\frac{r_0}{r}\right)^2+e^{-irh}\,\frac{S(2b^2-a^2)}{aK(r_0k)^{3/2}}\,J_{2m+\frac{3}{2}}(r_0h)\left(\frac{r_0}{r}\right)^2\Bigg]$$

Similarly we obtain

$$(u_r)_{z=0}=-\,\frac{r_0\,\cos\theta}{(1-\eta_1)}\sum_{m=0}^{\infty}a_m\Bigg[\sqrt{\frac{2}{\pi}}\,D_0Q\,e^{-i(rg+\frac{\pi}{4})}J_{2m+\frac{3}{2}}(r_0g)\left(1-\frac{i}{rg}\right)\sqrt{\frac{r_0}{r}}\;-$$

$$-\,2ie^{-irk}\,\frac{K}{(r_0k)^{3/2}}\,J_{2m+\frac{3}{2}}(r_0k)\left(1-\frac{i}{rk}\right)\left(\frac{r_0}{r}\right)^2\;+$$

$$+\,2e^{-irh}\,\frac{S}{(r_0k)^{3/2}}\,J_{2m+\frac{3}{2}}(r_0h)\left(b-\frac{ia}{rk}\right)\left(\frac{r_0}{r}\right)^2\Bigg]$$

and

$$(u_\theta)_{z=0}=\frac{r_0\,\sin\theta}{(1-\eta_1)}\sum_{m=0}^{\infty}a_m\Bigg[i\sqrt{\frac{2}{\pi}}\,e^{-i(rg+\frac{\pi}{4})}\,\frac{D_0Q}{r_0g}\,J_{2m+\frac{3}{2}}(r_0g)\left(\frac{r_0}{r}\right)^{3/2}\;+$$

$$+2e^{-irk}\,\frac{K}{(r_0k)^{5/2}}\,J_{2m+\frac{3}{2}}(r_0k)\left(\frac{r_0}{r}\right)^3+2ie^{-irh}\,\frac{aS}{(r_0k)^{5/2}}\,J_{2m+\frac{3}{2}}(r_0h)\left(\frac{r_0}{r}\right)^3\Bigg]$$

where $K=\dfrac{\sqrt{a^2-b^2}}{a}$, $S=\dfrac{a^3\sqrt{ab\,(a^2-b^2)}}{(2\,b^2-a^2)^3}$ and

$$Q=\frac{a\,(2b^2-c^2)-2b\,\sqrt{a^2-c^2}\,\sqrt{b^2-c^2}}{c^2\sqrt{a^2-c^2}}\;.$$

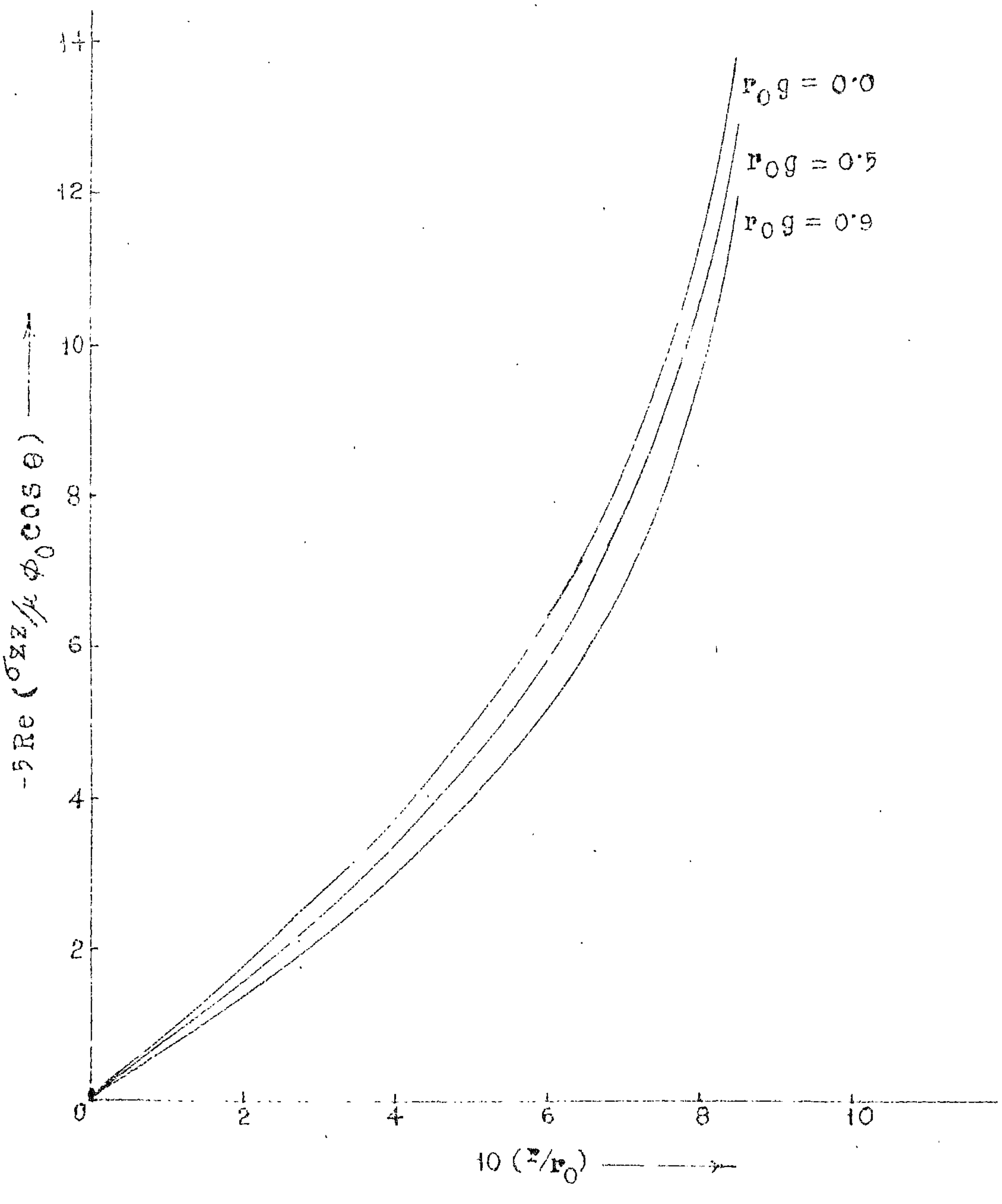
iv) From equation (28)

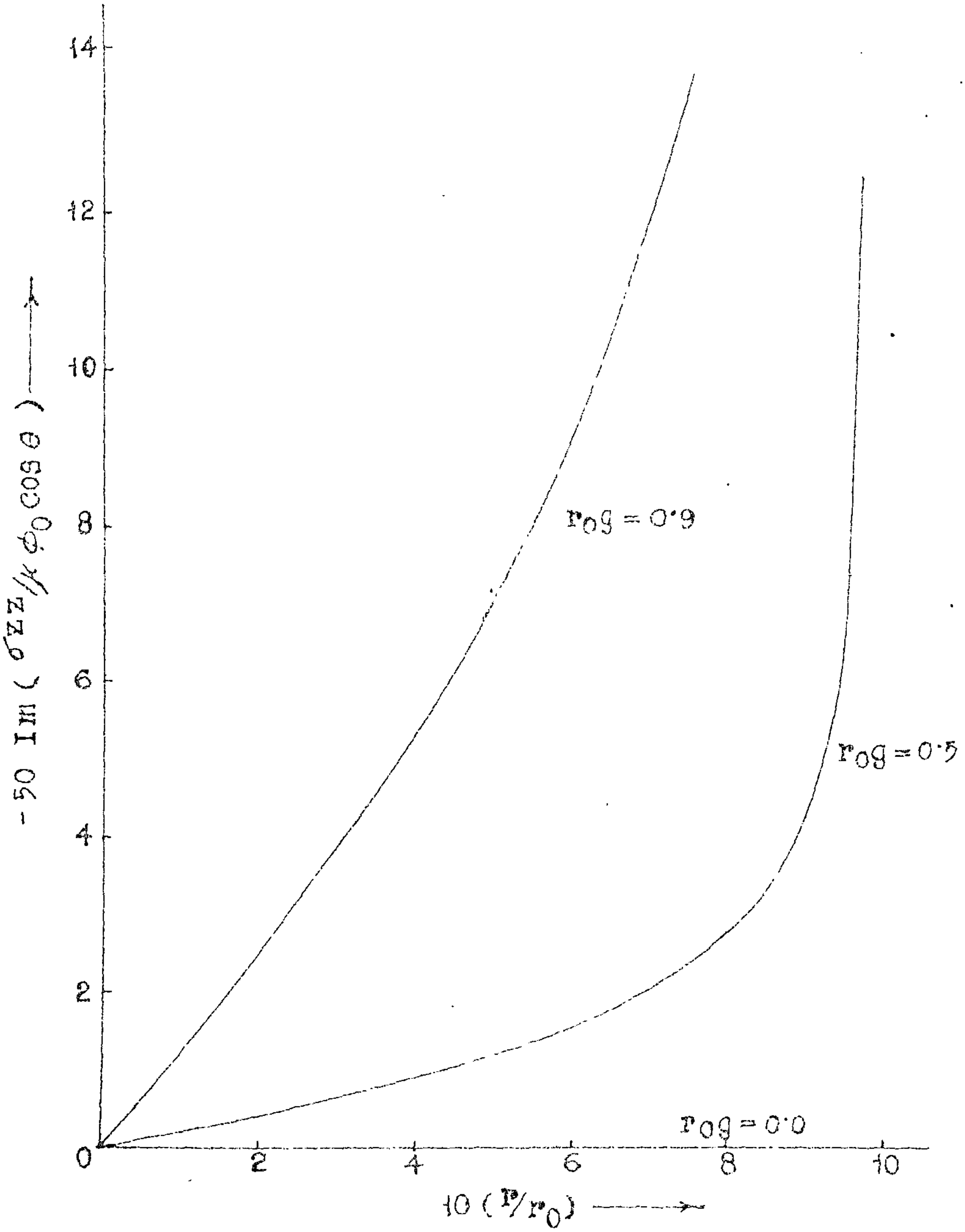
$$\frac{\mathrm{Re}(\sigma_{zz})_{z=0}}{\mu\cos\theta\,\phi_0}=\frac{2}{M_0}\,\frac{\rho}{\sqrt{1-\rho^2}}\Big[1-\frac{2}{3}\frac{M_2}{M_0}\Big(1-\frac{\rho^2}{2}\Big)(r_0g)^2+\frac{2}{45}\Big\{\Big(1+4\rho^2-\frac{\rho^4}{2}\Big)\frac{M_4}{M_0}+$$

$$+\Big(\frac{13}{2}-4\rho^2+\frac{\rho^4}{2}\Big)\Big(\frac{M_2}{M_0}\Big)^2\Big\}(r_0g)^4\Big]$$

and $$\frac{\mathrm{Im}\,(\sigma_{zz})_{z=0}}{\mu\cos\theta\,\phi_0}=\frac{2}{3M_0}\,\frac{\rho}{\sqrt{1-\rho^2}}\,\frac{M_3}{M_0}\,(r_0g)^3$$

are plotted against $\rho(<1)$ for values of r_0g = 0·0, 0·5 and 0·9. It is found that the absolute value of stress just below the disk increases with the increase of distance of the point from the centre of the disc where as at a point below the disc the absolute value of real part of stress decreases and imaginary part of stress increases with the increase of the frequency of oscillation of the disc.


$r_0 g = 0{\cdot}0$
$r_0 g = 0{\cdot}5$
$r_0 g = 0{\cdot}9$
$-5\,\mathrm{Re}\,(\sigma_{zz}/\mu\,\phi_0 \cos\theta)$
14
12
10
8
6
4
2
0
2
4
6
8
10
$10\,(z/r_0)$


$-50\ \mathrm{Im}\ (\sigma_{zz}/\mu\ \phi_0 \cos\theta)$ ⟶
14
12
10
8
6
4
2
0
2
4
6
8
10
$10\ (r/r_0)$ ⟶
$r_0g = 0{\cdot}9$
$r_0g = 0{\cdot}5$
$r_0g = 0{\cdot}0$

APPENDIX

To find the value of M_o, following Lapwood, we consider the function

$$F(\varsigma) = \frac{r_o^2k^2p(\varsigma)}{R(\varsigma)}$$

in the region R_1 and R_2 of the complex ς ($= \xi + i\eta$)- plane bounded by the curves as shown in the Fig. A1. By Cauchy's residue theorem, we have, when the function F( ) is integrated along the curve bounding the region R_1

$$\int_0^\infty F(\xi)d\xi + \int_0^\infty \frac{r_o^2k^2\sqrt{\eta^2+r_o^2h^2}}{R(i\eta)}d\eta - i(1-\eta_1)\frac{\pi}{2} = 0 \qquad (A1)$$

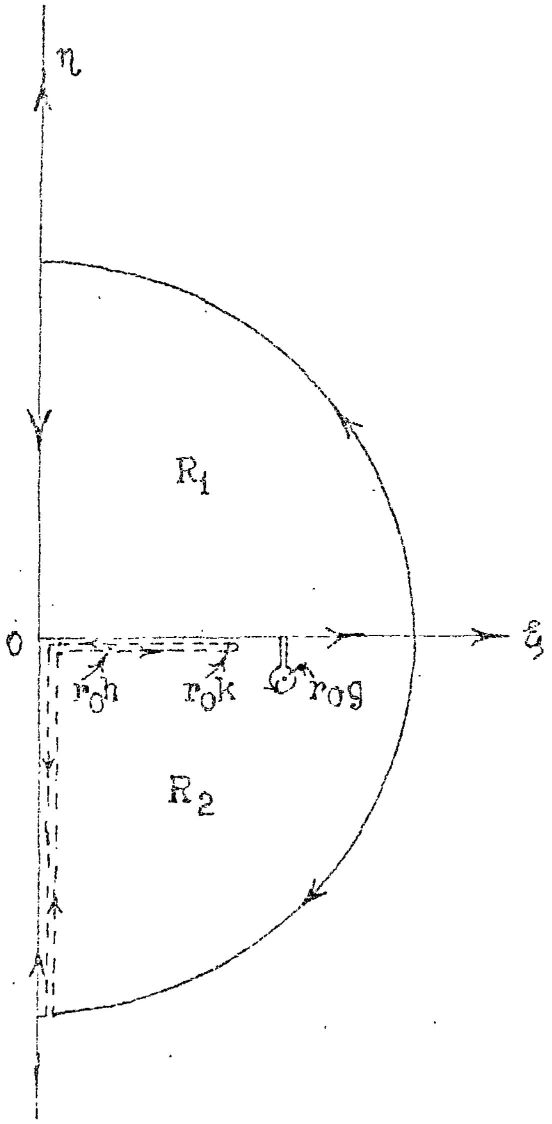
and integrating along the curve bounding the region R_2

$$\int_0^\infty F(\xi)d\xi + \int_0^\infty \frac{r_o^2k^2\sqrt{\eta^2+r_o^2h^2}}{R((-i\eta)}d\eta + i(1-\eta_1)\frac{\pi}{2} -$$

$$- 2i\int_0^{r_oh} \frac{r_o^2k^2\sqrt{r_o^2h^2-\xi^2}}{(2\xi^2-r_o^2k^2)^2+4\xi^2\sqrt{r_o^2h^2-\xi^2}\sqrt{r_o^2k^2-\xi^2}}d\xi -$$

$$-8i\int_{r_oh}^{r_ok}\frac{r_o^2k^2(\xi^2-r_o^2h^2)\sqrt{r_o^2k^2-\xi^2}\quad d\xi}{(2\xi^2-r_o^2k^2)^4+16\xi^4(\xi^2-r_o^2h^2)(\xi^2-r_o^2k^2)} = -2iD_o \qquad (A2)$$

The terms $- i(1-\eta_1)\pi/2$ and $i(1-\eta_1)\pi/2$ occurring on the left hand side of equations (A1) and (A2) respectively are the


η
R₁
0
ξ
r₀h
r₀k
r₀g
R₂

values of the integrals on the large circular arcs (Fig. A1) in the first and fourth quadrants. $-2D_o i$ on the right hand side of (A2) is the contribution from the residue at $(r_o g)$ where D_o is given by (20).

Subtracting (A2) from (A1) and dividing by 2i we obtain

$$-(1-\eta_1)\pi/2 = D_o - \frac{c^2}{b^2}\int_0^{c/a} \frac{\sqrt{\frac{c^2}{b^2} - x^2}\,dx}{\left(2x^2 - \frac{c^2}{b^2}\right)^2 + 4x^2\sqrt{\frac{c^2}{a^2} - x^2}\sqrt{\frac{c^2}{b^2} - x^2}} -$$

$$- \frac{4c^2}{b^2}\int_{c/a}^{c/b} \frac{x^2\left(x^2 - \frac{c^2}{a^2}\right)\sqrt{\frac{c^2}{b^2} - x^2}\,dx}{\left(2x^2 - \frac{c^2}{b^2}\right)^4 + 16x^4\left(x^2 - \frac{c^2}{a^2}\right)\left(\frac{c^2}{b^2} - x^2\right)} \qquad \text{(A3)}$$

$= M_o$, defined in (22).

CHAPTER II

RING SOURCE PROBLEMS

Problem 1. Displacement produced in an
elastic half-space by the
impulsive torsional motion
of a circular ring source.

Problem 2. SH-waves in an elastic half-space
due to a ring source of increasing
radius.

Problem 3. Torsional response of an elastic
half-space to a nonuniformly
expanding ring source.

Displacement Produced in an Elastic Half-Space by the Impulsive Torsional Motion of a Circular Ring Source

INTRODUCTION: At present much attention has been given to problems concerned with wave propagation in homogeneous as well as in inhomogeneous, isotropic, elastic media. Much of this work has been connected with problems of seismological interest, involving wave propagation. The normal loading problem of an elastic half-space was first investigated by Lamb (1904). This type of problem was then investigated by Eason (1964), Mitra (1964), Chakraborty and De (1971) and many others. In fact a class of elastic half-space problems involving an axisymmetric, normally applied, surface load was investigated by Gakenheimer (1971). He assumed that loads suddenly emanate from a point on the surface and expand radially at a constant rate. He used Cagniard's method to evaluate the inverse transforms. This paper has a particular reference to the work by Ghosh (1971) where techniques similar to those adopted here, are used. Many recent studies on elastic wave propagation are due to the work of Cagniard (1962), who developed a particular technique of finding the Laplace inversion, that has been found to be extremely useful in dealing with problems of this type.

The type of disturbing force considered in this paper is impulsive in time and acts over the circumference of a circular region of

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Published in Pure and Applied Geophysics, vol.119.P.102,(1980/81)

constant radius on the free surface of a semi-infinite, isotropic, elastic half-space. The effect of the inhomogeneity of the medium on the disturbance produced is determined in the integral form, whereas the displacement in the case of a homogeneous medium is determined exactly. The displacement at any point on the free surface is evaluated numerically and the graphs are drawn to show how the vibration of a point in the medium is affected due to the inhomogeneity of the medium, which enters into the expression for displacement through the factor ϵ.

Case I: Homogeneous medium

FORMULATION OF THE PROBLEM: Let (r,θ,z) be the cylindrical polar co-ordinates, z-axis being directed into the isotropic elastic medium, the plane boundary being $z=0$ with the origin at the centre of the ring source $r=a$, $z=0$.

The displacement is calculated at points inside and on the free surface of the medium, subject to the condition that the half-space is initially at rest and that the displacement remains bounded even for large values of z. For torsional motion of the ring all quantities depend on r, z and the time t, the only non-zero component of the displacement vector is the component v along the direction of θ increasing. The relevant non-vanishing stress components are

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \qquad (1)$$

and

$$\tau_{\theta z} = \mu \frac{\partial v}{\partial z} \qquad (2)$$

where μ is Lame's constant. The only non-zero equation of motion is

$$\frac{\partial}{\partial r}(\tau_{r\theta}) + \frac{\partial}{\partial z}(\tau_{\theta z}) + 2\frac{\tau_{r\theta}}{r} = \rho\frac{\partial^2 v}{\partial t^2} \qquad (3)$$

where ρ is the density of the material, assumed constant. The boundary condition is

$$\tau_{\theta z} = P\delta(r - a)\delta(t) \text{ at } z = o, \qquad (4)$$

where P is a constant, a is the radius of the ring source and $\delta(t)$ is Dirac's delta function.

Using (1) and (2) the equation (3) can be written in the form

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\ \frac{\partial^2 v}{\partial t^2} \qquad (5)$$

where $\beta = \sqrt{(\mu/\rho)}$ is the shear wave velocity.

METHOD OF SOLUTION: We define for all positive real values of s the Laplace transform $f_1(r,z,s)$ of a function $f(r,z,t)$ by the relation

$$f_1(r,z,s) = \int_0^\infty f(r,z,t)e^{-st}\,dt. \qquad (6)$$

Applying the Laplace transform (6) to the equation (5) we obtain

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right) + \frac{\partial^2 v_1}{\partial z^2} = \frac{s^2 v_1}{\beta^2}\,. \qquad (7)$$

Define the Hankel transform $v_2(\xi,z,s)$ of $v_1(r,z,s)$ by the equation

$$v_2(\xi,z,s) \int_0^\infty rJ_1(\xi r)v_1(r,z,s)dr, \qquad (8)$$

where J_1 is a Bessel function.

Multiplying the equation (7) by $rJ_1(\xi r)$ and integrating with respect to r from o to ∞ we get,

$$\frac{d^2v_2}{dz^2} = \left(\xi^2 + \frac{s^2}{\beta^2}\right)v_2 \,. \qquad (9)$$

The general solution of this equation which remains bounded as $z \to +\infty$ is

$$v_2 = A \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]. \qquad (10)$$

where A is to be determined from the boundary conditions,

$\tau_{\theta z_1} = P\delta(r - a)$ at $z = o$, where $\tau_{\theta z_1}$ is the Laplace transform of $\tau_{\theta z}$. From the Hankel transform $(\tau_{\theta z_1})_2$ of $\tau_{\theta z_1}$, we obtain by using (2)

$$(\tau_{\theta z_1})_2 = \mu\frac{dv_2}{dz} = PaJ_1(\xi a) \text{ at } z = o.$$

On $z = o$, $v_2 = A$ and $dv_2/dz = - A(\xi^2 + s^2/\beta^2)^{1/2}$.

Using these relations we get

$$A = -\frac{Pa}{\mu}\,\frac{J_1(\xi a)}{(\xi^2+s^2/\beta^2)^{1/2}} \,.$$

Substituting the value of A in (10) and inverting the Hankel transform (8), we obtain

$$v_1(r,z,s) = -\frac{Pa}{\mu}\int_0^\infty \frac{\xi J_1(\xi a)J_1(\xi r)}{(\xi^2+s^2/\beta^2)^{1/2}} \exp\left[-z\left(\xi^2+\frac{s^2}{\beta^2}\right)^{1/2}\right]d\xi. \qquad (11)$$

From a well-known result (Watson (1966),p.358)

$$J_1(\xi r)J_1(\xi a) = \frac{1}{\pi}\int_0^\pi J_0(\xi R)\cos\phi\, d\phi\ .$$

and

$$J_0(\xi R) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi R\sin\psi}\, d\psi \qquad \text{(Erdelyi(1953).p.14)}$$

where $R = \sqrt{(r^2+a^2-2ar\cos\phi)}$, we obtain

$$\frac{2\pi^2\mu v_1}{Pa} = -\int_0^\pi I_1\cos\phi\, d\phi \qquad (12)$$

where

$$I_1 = \int_0^{2\pi}\int_0^\infty \frac{\xi\exp\left[-z(\xi^2+s^2/\beta^2)^{1/2} + i\xi R\sin\psi\right]}{(\xi^2+s^2/\beta^2)^{1/2}}\, d\xi\, d\psi$$

If we put $p = \xi\sin\psi$ and $q = \xi\cos\psi$ in I_1, then

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\left[-z(p^2+q^2+s^2/\beta^2)^{1/2} + iRp\right]}{(p^2+q^2+s^2/\beta^2)^{1/2}}\, dp\, dq\,. \qquad (13)$$

To find the inversion of I_1, we adopt Cagniard's technique as modified by De Hoop(1959). Accordingly in (13), we put $p = ms$ and $q = ns$, then

$$I_1 = 2 \int_0^{\infty} dn \int_{-\infty}^{\infty} \frac{s \exp\left\{- s\left[z(m^2+n^2+1/\beta^2)^{1/2} - iRm\right]\right\}}{(m^2 + n^2 + 1/\beta^2)^{1/2}} \, dm \qquad (14)$$

In the above integral the path of integration with respect to m is the real axis (Fig.1) which is deformed in such a way that $- iRm + z(m^2 + n^2 + 1/\beta^2)^{1/2} = t$, where t is real and positive. The deformed path of integration is the branch Γ of a hyperbola whose equation is

$$m = \frac{iRt \pm z\left[t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)\right]^{1/2}}{z^2 + R^2}, \quad \left\{(z^2+R^2)(n^2+1/\beta^2)\right\}^{1/2} < t < \infty.$$

In the course of deformation of the path of integration it is essential to know the singularities of the function $s/(m^2+n^2+1/\beta^2)^{1/2}$ in the m-plane which are the branch points $\pm\, i(n^2+1/\beta^2)^{1/2}$.

Since the hyperbolic path Γ does not cross any of the singularities during its deformation, it is possible by virtue of Cauchy's theorem and Jordon's lemma, to replace the integration along the real m-axis by an integration along the hyperbolic path Γ.

We assume

$$m_+ = \frac{iRt + z\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2 + R^2}$$

and

$$m_- = \frac{iRt - z\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2 + R^2}$$

then

$$\frac{dm_\pm}{dt} = \frac{iR\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2} \pm zt}{(z^2+R^2)\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}$$

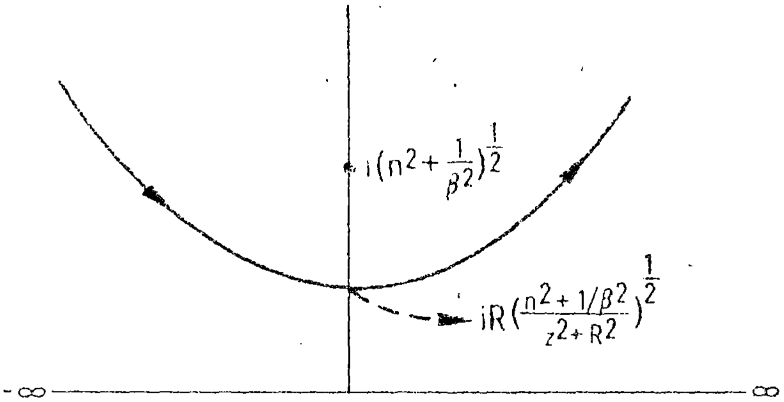

Figure 1
Paths of integration in the complex *m*-plane.

The point where Γ cuts the imaginary axis is given by

$$t = \left\{(z^2 + R^2)(n^2 + 1/\beta^2)\right\}^{1/2}$$

and the point is

$$m = \frac{iR(n^2+1/\beta^2)^{1/2}}{(z^2+R^2)^{1/2}}$$

which is below the branch point $i(n^2+1/\beta^2)^{1/2}$. Hence (14) can be written as

$$I_1=2\int_0^\infty dn \int_{\left\{(z^2+R^2)(n^2+1/\beta^2)\right\}^{1/2}}^{\infty} se^{-st}\left[\frac{1}{(m_+^2+n^2+1/\beta^2)^{1/2}}\frac{dm_+}{dt} - \frac{1}{(m_-^2+n^2+1/\beta^2)^{1/2}}\frac{dm_-}{dt}\right]dt \quad (15)$$

Now using the fact that $m_- = -\bar{m}_+$ and $dm_-/dt = -(d\bar{m}_+/dt)$ where $\bar{m}$ is the complex conjugate of m, (15) can be written as

$$I_1=4\int_0^\infty dn \int_{\left\{(z^2+R^2)(n^2+1/\beta^2)\right\}^{1/2}}^{\infty} se^{-st}\, \mathrm{Re}\left[\frac{(dm_+/dt)}{(m_+^2+n^2+1/\beta^2)^{1/2}}\right]dt$$

Changing the order of integration, we get,

$$I_1=4\int_{(z^2+R^2)^{1/2}/\beta}^{\infty} se^{-st}\,dt \int_0^{\left[t^2/(z^2+R^2)-1/\beta^2\right]^{1/2}} \mathrm{Re}\left[\frac{dm_+/dt}{(m_+^2+n^2+1/\beta^2)^{1/2}}\right]dn \quad (16)$$

Now,

$$\mathrm{Re}\left[\frac{(dm_+/dt)}{(m_+^2+n^2+1/\beta^2)^{1/2}}\right] = \frac{1}{\left\{t^2-(z^2+R^2)(n^2+1/\beta^2)\right\}^{1/2}}$$

Substituting this result in (16), we obtain

$$I_1=4\int_{\frac{1}{\beta}(z^2+R^2)^{1/2}}^{\infty} se^{-st}\,dt\int_0^{\left[t^2/(z^2+R^2)-1/\beta^2\right]^{1/2}}\frac{dn}{\left[t^2-(z^2+R^2)(n^2+1/\beta^2)\right]^{1/2}}$$

$$=\frac{2\pi}{(z^2+R^2)^{1/2}}\int_{\left[(z^2+R^2)^{1/2}\right]/\beta}^{\infty} se^{-st}\,dt.$$

Hence the Laplace inversion of I_1 is

$$I=\frac{2\pi}{(z^2+R^2)^{1/2}}\frac{d}{dt}\left[H\left\{t-\frac{(z^2+R^2)^{1/2}}{\beta}\right\}\right]$$

$$=\frac{2\pi}{(z^2+R^2)^{1/2}}\;\delta\left[t-\frac{(z^2+R^2)^{1/2}}{\beta}\right] \qquad (17)$$

Therefore the Laplace inversion of (12) by using the Laplace inversion of I_1 as given in (17) is

$$v(r,z,t)=-\frac{Pa}{\pi\mu}\int_0^{\pi}\frac{\delta\left[t-\dfrac{(z^2+r^2+a^2-2ra\cos\phi)^{1/2}}{\beta}\right]}{(z^2+r^2+a^2-2ra\cos\phi)^{1/2}}\cos\phi\,d\phi. \quad (18)$$

To evaluate the above integral we put

$$(z^2 + r^2 + a^2 - 2ra \cos\phi)^{1/2} = \beta\theta,$$

then $\frac{d\phi}{d\theta} = \frac{\beta^2\theta}{r\, a \sin\phi}$,

$$\sin\phi = \frac{1}{2ra}\left[2(\beta^2\theta^2 - z^2)(r^2+a^2) - (\beta^2\theta^2 - z^2)^2 - (r^2-a^2)^2\right]^{1/2}$$

and (18) can be written as

$$v(r,z,t) = -\frac{P\beta}{\pi\mu r}\int_{\frac{1}{\beta}\{z^2+(r-a)^2\}^{1/2}}^{\frac{1}{\beta}\{z^2+(r+a)^2\}^{1/2}} \frac{(z^2+r^2+a^2-\beta^2\theta^2)\,\delta(t-\theta)}{\left[2(r^2+a^2)(\beta^2\theta^2-z^2)-(r^2-a^2)^2-(\beta^2\theta^2-z^2)^2\right]^{1/2}}\,d\theta$$

$$= \frac{P\beta}{\pi\mu r}\,\frac{\beta^2 t^2 - z^2 - r^2 - a^2}{\left\{2(r^2+a^2)(\beta^2 t^2 - z^2) - (r^2-a^2)^2 - (\beta^2 t^2 - z^2)^2\right\}^{1/2}}$$

for $\frac{1}{\beta}\left\{z^2 + (r-a)^{2\ 1/2}\right. < t < \frac{1}{\beta}\left\{z^2 + (r+a)^2\right\}^{1/2}$ (19)

Case II: Inhomogeneous Medium

FORMULATION OF THE PROBLEM: In this case the same problem of torsional motion of a semi-infinite elastic medium due to the presence of a ring source $r = a$, on the free surface $z = 0$ as in Case I is considered. The only difference is that the medium under consideration is inhomogeneous in nature, the coefficient of rigidity and the density of the medium are assumed to be

$$\mu = \mu_0(1+\epsilon z)^2 \quad \text{and} \quad \rho = \rho_0(1+\epsilon z)^2. \tag{20}$$

Here also the non-vanishing stress components and the non-zero equations of motion are the same as in Case I, given by the equations (1),(2) and (3).

METHOD OF SOLUTION: Firstly we put $\bar{v} = (1 + \epsilon z)v$ in the equations (1),(2) and (3). The transformed equations are

$$\tau_{r\theta} = \mu_0(1 + z)\left(\frac{\partial\bar{v}}{\partial r} - \frac{\bar{v}}{r}\right),$$

$$\tau_{\theta z} = \mu_0\left\{(1 + \epsilon z)\frac{\partial\bar{v}}{\partial z} - \epsilon\bar{v}\right\} \tag{21}$$

and

$$\frac{\partial^2\bar{v}}{\partial r^2} + \frac{1}{r}\left(\frac{\partial\bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) + \frac{\partial^2\bar{v}}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2\bar{v}}{\partial t^2} \tag{22}$$

where $\beta = \sqrt{(\mu_0 / \rho_0)}$.

Taking the Laplace transform of the equation with respect to t, we obtain

$$\frac{\partial^2\bar{v}_1}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{v}_1}{\partial r} - \left(\frac{1}{r^2} + \frac{s^2}{\beta^2}\right)\bar{v}_1 + \frac{\partial^2\bar{v}_1}{\partial z^2} = 0 \tag{23}$$

where s is the Laplace transform parameter which is real and positive. Taking the Hankel transform of the equation (23) we have

$$\frac{d^2\bar{v}_2}{dz^2} = \left(\xi^2 + \frac{s^2}{\beta^2}\right)\bar{v}_2\,. \tag{24}$$

The general solution of this equation which remains bounded for large values of z is

$$\bar{\bar{v}}_2 = B \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]. \tag{25}$$

Applying the Hankel transform and the Laplace transform on the boundary condition

$$\tau_{\theta z} = \mu_0\left[(1+\epsilon z)\frac{\partial \bar{\bar{v}}}{\partial z} - \epsilon \bar{\bar{v}}\right] = P\delta(r-a)\delta(t)$$

and using (25), the value of B is found to be

$$B = -\frac{PaJ_1(\xi a)}{\mu\left\{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}\right\}}$$

Substituting this value of B in (25), it follows that

$$\bar{\bar{v}}_2 = -\frac{PaJ_1(\xi a)}{\mu_0\left\{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}\right\}} \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]. \tag{26}$$

Taking the Hankel inversion of (26) , we have

$$\bar{v}_1 = -\frac{Pa}{\mu_0}\int_0^\infty \frac{\xi J_1(\xi a)J_1(\xi r)}{\left\{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}\right\}} \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]d\xi\,, \tag{27}$$

Now,

$$\int_0^\infty \exp\left[-k\left\{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}\right\}\right]dk = \frac{1}{\epsilon + (\xi^2 + s^2/\beta^2)^{1/2}}$$

Using the above result,(27) is written as

$$\bar{v}_1 = -\frac{Pa}{\mu_0}\int_0^\infty e^{-\epsilon k}\, dk \int_0^\infty \xi J_1(\xi a) J_1(\xi r) \exp\left[-(z+k)\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right] d\xi \qquad (28)$$

We now replace $J_1(\xi a) J_1(\xi r)$ of (28) by the integral, which was used to modify equation (11). Finally we get

$$\bar{v}_1 = -\frac{Pa}{2\pi^2 \mu_0}\int_0^\infty e^{-\epsilon k}\, dk \int_0^\pi I_2 \cos\phi \, d\phi \, , \qquad (29)$$

where

$$I_2 = \int_0^{2\pi} d\psi \int_0^\infty \xi \exp\left[-(z+k)\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2} + i\xi R \sin\psi\right] d\xi .$$

Assuming $p = \xi \sin\psi$ and $q = \xi \cos\psi$, it follows that

$$I_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-(z+k)(p^2+q^2+s^2/\beta^2)^{1/2} + i\, Rp\right] dp . dq$$

$$= 2\int_0^\infty dn \int_{-\infty}^\infty s^2 \exp\left[-s\left\{(z+k)\left(m^2+n^2+\frac{1}{\beta^2}\right)^{1/2} - iRm\right\}\right] dm. \qquad (30)$$

where , $p = ms$ and $q = ns$.

As in Case I, here also the path of integration with respect to m which is the real axis is deformed such that

$- iRm + (z+k)(m^2 + n^2 + 1/\beta^2)^{1/2} = t$, whre t is real and positive. The deformed path is a branch Γ_1 of a hyperbola the equation of which is

$$m = \frac{iRt \pm (z+k)\left[t^2 - \{(z+k)^2 + R^2\}(n^2 + 1/\beta^2)\right]^{1/2}}{(z+k)^2 + R^2},$$

$$\{(z+k)^2 + R^2\}^{1/2}\left(n^2 + \frac{1}{\beta^2}\right)^{1/2} < t < \infty.$$

Noting that the point where Γ_1 cuts the imaginary axis is

$$m = \frac{iR(n^2 + 1/\beta^2)^{1/2}}{\{(z+k)^2 + R^2\}^{1/2}}$$

when

$$t = \{(z+k)^2 + R^2\}^{1/2}\left(n^2 + \frac{1}{\beta^2}\right)^{1/2},$$

one gets from the equation (30)

$$I_2 = 4 \int_0^\infty dn \int_{\{(z+k)^2+R^2\}^{1/2}(n^2+1/\beta^2)^{1/2}}^{\infty} s^2 e^{-st}\, \mathrm{Re}\left(\frac{dm_+}{dt}\right) dt.$$

Changing the order of integration, we obtain

$$I_2 = 4(z+k)/\{(z+k)^2 + R^2\}^{3/2} \int_{\frac{1}{\beta}\{(z+k)^2+R^2\}^{1/2}}^{\infty} ts^2 e^{-st}\, dt \times$$

$$\times \int_0^{\left[t^2/\{(z+k^2)+R^2\}-(1/\beta^2)\right]^{1/2}} \frac{dn}{\left[t^2/\{(z+k)^2+R^2\} - 1/\beta^2 - n^2\right]^{1/2}}$$

$$= \frac{2\pi(z + k)}{\left\{(z + k)^2 + R^2\right\}^{3/2}} \int_{\left[\left\{(z+k)^2+R^2\right\}^{1/2}\right]/\beta}^{\infty} t s^2 e^{-st}\, dt. \qquad (31)$$

Hence the Laplace inversion of (31) is

$$I = 2\pi(z+k)/\left\{(z+k)^2+R^2\right\}^{3/2} \frac{d^2}{dt^2}\left[t\, H\left\{t - \frac{1}{\beta}\left[(z+k)^2+R^2\right]^{1/2}\right\}\right]$$

$$= \frac{2\pi(z+k)}{\left\{(z+k)^2+R^2\right\}^{3/2}}\left\{2\delta\left[t-\left(\frac{(z+k)^2+R^2}{\beta^2}\right)^{1/2}\right]+t\delta'\left[t-\left(\frac{(z+k)^2+R^2}{\beta^2}\right)^{1/2}\right]\right\}.$$

Taking the Laplace transform of (29) and using the value of I, it is found that

$$\bar{\bar{v}} = -\frac{Pa}{\pi\mu_0}\int_0^{\infty}(z+k)Je^{-\epsilon k}\, dk. \qquad (32)$$

where

$$J = \int_0^{\pi}\left\{2\delta\left[t-\left(\frac{(z+k)^2+R^2}{\beta^2}\right)^{1/2}\right]+t\delta'\left[t-\left(\frac{(z+k)^2+R^2}{\beta^2}\right)^{1/2}\right]\right\} \times$$

$$\times \frac{\cos\phi}{\left\{(z+k)^2+R^2\right\}^{3/2}}\, d\phi.$$

To evaluate the above integral we put

$$l = \frac{1}{\beta}\left\{(z+k)^2 + R^2\right\}^{1/2},$$

then

$$J = \int_{(1/\beta)\left\{(z+k)^2+(r-a)^2\right\}^{1/2}}^{(1/\beta)\left\{(z+k)^2+(r+a)^2\right\}^{1/2}} \left\{2\delta(t-\ell)+t\delta'(t-\ell)\right\} \frac{\cos\phi}{\ell^3\beta^3}\,\frac{d\phi}{d\ell}\,d\ell\,,$$

where

$$\frac{d\phi}{d\ell} = \frac{\beta^2\ell}{ra\sin\phi} \quad \text{and} \quad \cos\phi = \frac{(z+k)^2+r^2+a^2-\beta^2\ell^2}{2ra}.$$

Substituting these values, we get

$$J = \frac{1}{ra\beta}\int_{(1/\beta)\left\{(z+k)^2+(r-a)^2\right\}^{1/2}}^{(1/\beta)\left\{(z+k)^2+(r+a)^2\right\}^{1/2}} f(\ell,k)\left[2\delta(t-\ell)+t\delta'(t-\ell)\right]d\ell, \qquad (33)$$

where

$$f(\ell,k)=\frac{(z+k)^2+r^2+a^2-\beta^2\ell^2}{\ell^2\left[2(r^2+a^2)\left\{\beta^2\ell^2-(z+k)^2\right\}-(r^2-a^2)^2-\left\{\beta^2\ell^2-(z+k)^2\right\}^2\right]^{1/2}}$$

and it is to be remembered that δ' is the derivative of the Dirac's δ-function with respect to t. Integrating (33), we get

$$J = \frac{1}{ra\beta}\left[2f(t,k)-tf(\ell_1,k)\delta(t-\ell_1)+tf(\ell_2,k)\delta(t-\ell_2)+tf'(t,k)\right] \qquad (34)$$

where

$$\ell_1=\frac{1}{\beta}\left\{(z+k)^2+(r+a)^2\right\}^{1/2}, \quad \ell_2=\frac{1}{\beta}\left\{(z+k)^2+(r-a)^2\right\}^{1/2} \quad \ell_2 \leq t \leq \ell_1.$$

It is to be noted that if t does not belong to $[\ell_2, \ell_1]$ then the integrand in (33) is zero, consequently $J = 0$.

Substituting the value of J in (32), we get

$$\bar{v} = -\frac{P}{\pi\mu c\beta r}\int_0^\infty (z+k)e^{-\epsilon k}\Big[2f(t,k) - tf(\ell_1,k)\delta(t-\ell_1) + tf(\ell_2,k)\delta(t-\ell_2) + tf'(t,k)\Big]dk. \qquad (35)$$

Now, $\ell_2 \leqslant t \leqslant \ell_1$ implies that

$$\{\beta^2t^2 - (r+a)^2\}^{1/2} - z \leqslant k \leqslant \{\beta^2t^2 - (r-a)^2\}^{1/2} - z. \qquad (36)$$

In evaluating the integral (35), the following sub-cases are to be considered, keeping in mind that k satisfies (36) and that $k \geqslant o$

i) If $\{\beta^2t^2 - (r-a)^2\}^{1/2} - z < o$, that is if $\beta t < \{z^2 + (r-a)^2\}^{1/2}$ then, t does not belong to $[\ell_2, \ell_1]$, so $J = 0$. Consequently $\bar{v} = o$. This is in accordance with the physical condition of the problem because a disturbance cannot reach a point Q (Fig.2) before the time $(1/\beta)\{z^2 + (r-a)^2\}^{1/2}$, which is the time of arrival of the disturbance at the point Q from the nearest point of the ring source.

ii) $\{\beta^2t^2 - (r+a)^2\}^{1/2} - z < o < \{\beta^2t^2 - (r-a)^2\}^{1/2} - z$, that is,

$$\{z^2 + (r-a)^2\}^{1/2} < \beta t < \{z^2 + (r+a)^2\}^{1/2}.$$

In this case (35) takes the form

$$\bar{v} = \frac{-P}{\pi \mu_o \beta r} \int_0^{\{\beta^2 t^2 - (r-a)^2\}^{1/2} - z} (z+k) e^{-\epsilon k} \Big[2f(t,k) - tf(l_1,k)\delta(t-l_1) + tf(l_2,k)\delta(t-l_2) + tf'(t,k) \Big] dk. \tag{37}$$

The integrand of (37) is considered as a generalized function, so the finite part of the integral (37) is retained (JONES(1966),p.89)and we get

$$\bar{v} = \frac{P\beta}{\pi r \mu_o} \frac{\beta^2 t^2 - z^2 - r^2 - a^2}{\left[2(r^2+a^2)(\beta^2 t^2 - z^2) - (r^2-a^2)^2 - (\beta^2 t^2 - z^2)^2\right]^{1/2}}$$

$$+ \frac{P\beta\epsilon}{\pi r \mu_o} \int_0^{\{\beta^2 t^2 - (r-a)^2\}^{1/2} - z}$$

$$\times \frac{\{(z+k)^2 + r^2 + a^2 - \beta^2 t^2\} e^{-\epsilon k}\, dk}{\left[2(r^2+a^2)\{\beta^2 t^2 - (z+k)^2\} - (r^2-a^2)^2 - \{\beta^2 t^2 - (z+k)^2\}^2\right]^{1/2}}$$

Hence

$$v = \frac{P\beta}{\pi r \mu_o (1+\epsilon z)} \frac{\beta^2 t^2 - z^2 - r^2 - a^2}{\left[2(r^2+a^2)(\beta^2 t^2 - z^2) - (r^2-a^2)^2 - (\beta^2 t^2 - z^2)^2\right]^{1/2}}$$

$$+ \frac{P\beta\epsilon}{\pi r \mu_o (1+ z)} \int_0^{\{\beta^2 t^2 - (r-a)^2\}^{1/2} - z}$$

$$\times \frac{\{(z+k)^2 + r^2 + a^2 - \beta^2 t^2\} e^{-\epsilon k}\, dk}{\left[2(r^2+a^2)\{\beta^2 t^2 - (z+k)^2\} - (r^2-a^2)^2 - \{\beta^2 t^2 - (z+k)^2\}^2\right]^{1/2}} \tag{38}$$

In (38) if we put $\epsilon = 0$, we get the same result that we have determined in (19) of Case I.

iii) If $\left\{\beta^2 t^2 - (r+a)^2\right\}^{1/2} - z > o$, that is if $\beta t > \left\{z^2 + (r+a)^2\right\}^{1/2}$ then

$$v = \frac{P\beta\epsilon}{\pi r/\mu_0(1+\epsilon z)} \int_{\left\{\beta^2 t^2 - (r+a)^2\right\}^{1/2} - z}^{\left\{\beta^2 t^2 - (r-a)^2\right\}^{1/2} - z}$$

$$\times \frac{\left\{(z+k)^2 + r^2 + a^2 - \beta^2 t^2\right\} e^{-\epsilon k}\, dk}{\left[2(r^2+a^2)\left\{\beta^2 t^2 - (z+k)^2\right\} - (r^2-a^2)^2 - \left\{\beta^2 t^2 - (z+k)^2\right\}^2\right]^{1/2}} \qquad (39)$$

It is interesting to note that in the case of a homogeneous medium there is no displacement at a point Q (Fig.2) after the time $t = (1/\beta)\left\{z^2 + (r+a)^2\right\}^{1/2}$, which is the time required by the disturbance to reach the point Q directly from the farthest point on the ring source from the point Q. But in the case of an inhomogeneous medium the disturbance reaches a point Q even after the time $t = (1/\beta)\left\{z^2 + (r+a)^2\right\}^{1/2}$ which is the maximum time required by a direct wave to reach the point Q from the farthest point on the source for the point Q. This is due to the fact that in the case of an inhomogeneous medium the region $z > o$ may be considered as an assembly of an infinite number of thin layers of material of infinitesimal thickness of continuously varying density and coefficient of rigidity. That is why the disturbance, which reaches the point Q after successive reflection and refraction in different layers of the medium, arrives at Q after the time $\beta t = \left\{z^2 + (r+a)^2\right\}^{1/2}$. The disturbance comes continuously after the time $\beta t = \left\{z^2 + (r+a)^2\right\}^{1/2}$ with decreasing intensity.

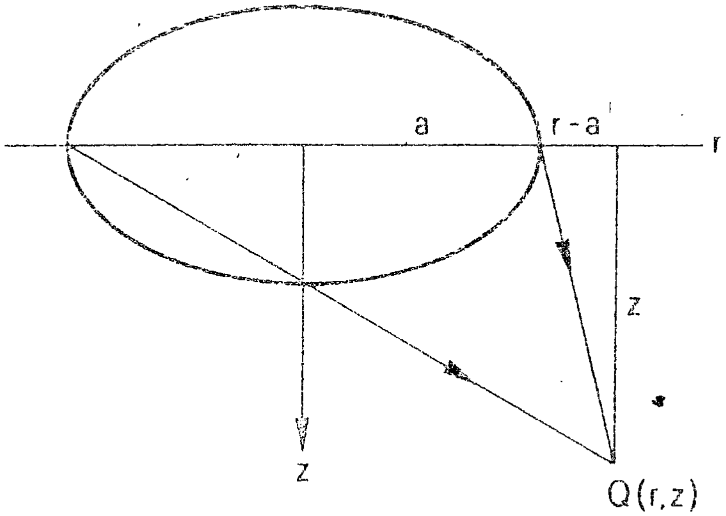

Figure 2
Arrival of the direct wave to Q from the nearest and the farthest point of the source.

Numerical solution on the free surface z = o

In order to obtain the displacement on the free surface we make the substitution

$$\left[2(r^2+a^2)(\beta^2t^2-k^2)-(r^2-a^2)^2-(\beta^2t^2-k^2)^2\right]^{1/2} = 2ra\sin\theta$$

which transforms the equations (38) and (39) to the forms given by

$$\frac{v\pi\mu_o a}{P\beta} = d = d_1 + d_2$$

where

$$d_1 = \frac{a}{r}\,\frac{\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1}{\left[2\left(\dfrac{r^2}{a^2}+1\right)\dfrac{\beta^2t^2}{a^2}-\left(\dfrac{r^2}{a^2}-1\right)^2-\dfrac{\beta^4t^4}{a^4}\right]^{1/2}}$$

$$d_2 = \epsilon a\int_0^{\cos^{-1}A}\frac{\cos\theta\left\{\exp -\epsilon a\left[2\frac{r}{a}\cos\theta+\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1\right]^{1/2}\right\}}{\left[2\frac{r}{a}\cos\theta+\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1\right]^{1/2}}\,d\theta. \tag{40}$$

$$A = \frac{r^2+a^2-\beta^2t^2}{2ra},\quad r-a<\beta t<r+a,$$

$$\frac{v\pi\mu_o a}{P\beta} = d' = \epsilon a\int_0^{\pi}\frac{\cos\theta\,\exp\left\{-\epsilon a\left[2\frac{r}{a}\cos\theta+\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1\right]^{1/2}\right\}}{\left[2\frac{r}{a}\cos\theta+\dfrac{\beta^2t^2}{a^2}-\dfrac{r^2}{a^2}-1\right]}\,d\theta, \tag{41}$$

for $\beta t > r + a$ respectively.

If $\epsilon a = o$, then from (40) it follows that $d = d_1$, which corresponds to the displacement inhomogeneous medium. The integrals in (40) and (41) giving the displacements d and d' have been numerically evaluated for different values of ϵa at different points on the free surface and are presented in Tables 1-4 for different values of $\beta t/a$.

Concluding remarks

From Tables 1-4 it is found that the difference in the values of the displacement at any point corresponding to $\epsilon a = o$ and $\epsilon a = 10$ gradually diminishes with the

Table 1

$r/a = 2.\ (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\epsilon a = o$	d when $\epsilon a = 1$	d when $\epsilon a = 10$
1.2	-0.97596	-0.32841	-0.50851
1.4	-0.58468	-0.08456	-0.41435
1.6	-0.38490	0.00497	-0.31149
1.8	-0.24498	0.05256	-0.21268
2.0	-0.12909	0.08585	-0.11623
2.2	-0.02001	0.11644	-0.01716
2.4	0.09676	0.15276	0.09355
2.6	0.24498	0.20795	0.23612
2.8	0.50411	0.32902	0.48230

Table 2

$r/a = 10,\ (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\epsilon a = 0$	d when $\epsilon a = 1$	d when $\epsilon a = 10$
9.2	-0.14221	-0.00782	-0.13509
9.4	-0.08155	-0.00314	-0.08070
9.6	-0.04927	-0.00063	-0.04911
9.8	-0.02559	0.00324	-0.02556
10.0	-0.00500	0.00868	-0.00500
10.2	0.01537	0.01588	0.01537
10.4	0.03834	0.02557	0.03833
10.6	0.06901	0.03990	0.06900
10.8	0.12546	0.06799	0.12542

Table 3

$r/a = 50,\ (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\epsilon a = 0$	d when $\epsilon a = 1$	d when $\epsilon a = 10$
49.2	-0.02700	-0.00822	-0.02699
49.4	-0.01525	-0.00693	-0.01525
49.6	-0.00894	-0.00474	-0.00894
50.0	-0.00020	0.00028	-0.00020
50.2	0.00387	0.00318	0.00387
50.4	0.00851	0.00665	0.00851
50.6	0.01475	0.01130	0.01475
50.8	0.02633	0.01944	0.02633

Table 4

$r/a = 2, (\beta t/a) > (r/a) + 1$

$\beta t/a$	d' when $\epsilon a = 1$	d' when $\epsilon a = 10$
3.2	−0.17211	d' is of the order of 10^{-7}
3.4	−0.07793	
3.6	−0.04250	
3.8	−0.02533	
4.0	−0.01593	
4.2	−0.01040	
4.4	−0.00697	
4.6	−0.00477	
4.8	−0.00332	

When $r = 10a$ or $\epsilon a = 10$, d' is very small.

increase in the value of r/a. This is also apparent from the expression for d_2 in (40) because the exponential term

$$\exp\left\{ - \epsilon a \left[2 \frac{r}{a} \cos \theta + \frac{\beta^2 t^2}{a^2} - \frac{r^2}{a^2} - 1 \right]^{1/2} \right\}$$

in the integrand for large values of r/a decreases rapidly with the increase in value of ϵa.

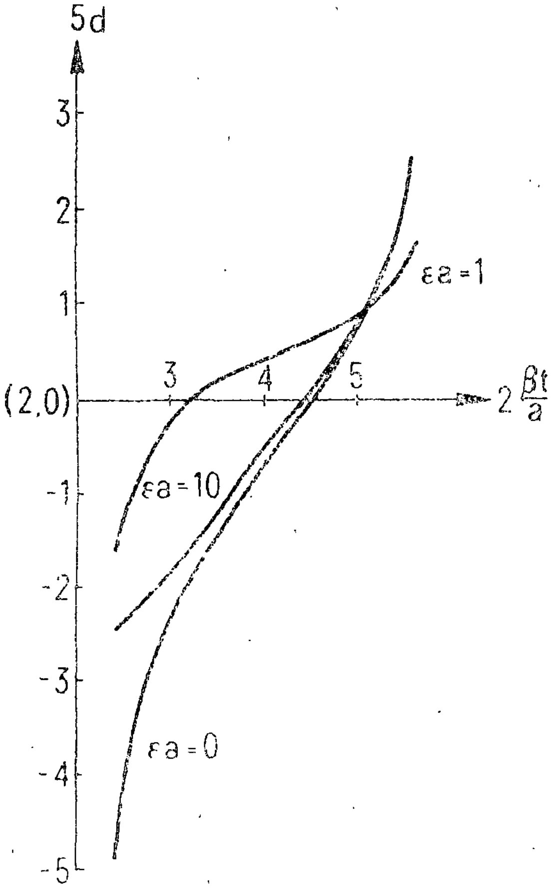

Figure 3

$r = 2a$, variation in displacement near the source for $\varepsilon a = 0, 1, 10$.

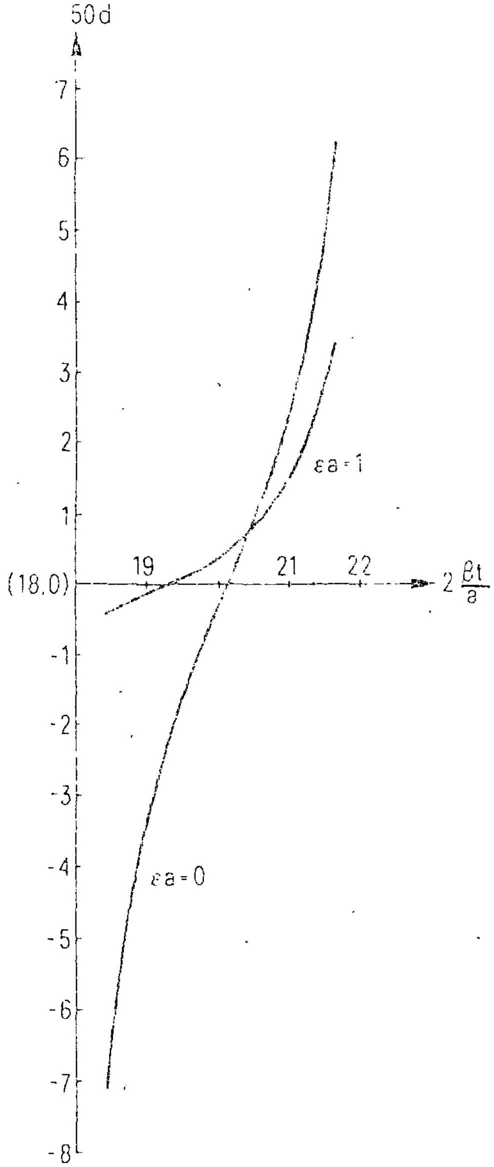

Figure 4
$r = 10a$, variation in displacement at a moderate distance from the source for $\varepsilon a = 0, 1$.

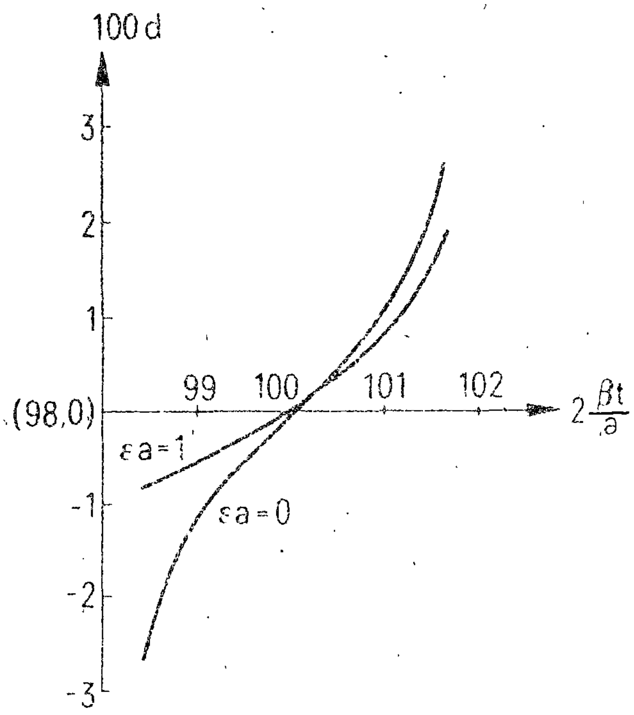

Figure 5
$r = 50a$, variation in displacement at a large distance from the source for $\varepsilon a = 0, 1$.

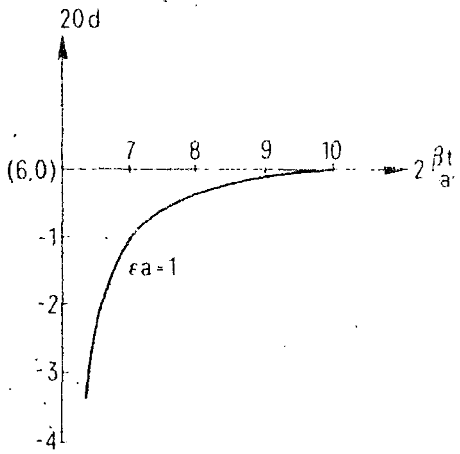

Figure6
$r = 2a$, variation in displacement after the maximum time required by a direct wave to arrive from the farthest point of the source when $\varepsilon a = 1$

SH-WAVES IN AN ELASTIC HALF SPACE DUE TO A RING SOURCE OF INCREASING RADIUS.

INTRODUCTION: The torsional vibration of an elastic half space due to a surface force which is periodic in time was first considered by Reissner (1937). Reissner and Sagoci (1944) determined the distribution of the stresses in the interior of a semi-infinite, homogenous isotropic elastic material due to a periodic shear stresses applied in an axially symmetric manner to a circular area of the plane surface by means of a rigid disk, the torsional displacement being prescribed under the disk. Verma (1957) discussed the static distribution of stresses and displacement when shearing stress is prescribed on the circumfeience of a circle on the plane boundary. Datta (1961) discussed the corresponding problem when shearing stress decreases exponentially with time. Ghosh (1964) exactly evaluated the displacement at any point of the medium when a twisting moment in the form $M\delta(t)$ is applied to the disk by following Cagniard (1939) and Dix (1954). Ghosh (1971) also discussed the axisymmetric problem of propagation of a stress discontinuity over a circular region by using Cagniard's (1939) method as modified by De-Hoop (1959). In the present paper the author determines the displacement in the integral form due to a ring source which increases steadily when the twisting impulse is prescribed by $P\delta(r-ct)H(t)$, where δ,H are two dimensional delta function and Heaviside function respectively, and then the exact evaluation of the displacement is determined after the first arrival of the shear wave and, the displacement at any point for large values of the time t.

Published in Ind. J. Mech. Math., Vol.XVIII, No.,2 P.74 (1980)

FORMULATION OF THE PROBLEM:

The isotropic, elastic, semi infinite medium is supposed to occupy the region $z > 0$. We choose cylindrical polar co-ordinates (r, θ, z) with the z-axis directed into the medium, the plane boundary being $z = 0$ with origin at the centre of the source. The displacement is calculated at points inside the medium assuming that the half space is, initially, at rest and that the displacement remains bounded even as $z \to +\infty$. Since the motion is symmetrical about z-axis for torsional motion of the ring source, all quantities depend on r, z and the time t. The only non-vanishing component of the displacement vector is the component v along the direction of θ increasing. Hence the non-vanishing stress components are

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \text{ and } \quad \tau_{\theta z} = \mu \frac{\partial v}{\partial z} \tag{1}$$

where μ is the coefficient of rigidity. The only non-zero equation of motion is

$$\frac{\partial}{\partial r}(\tau_{r\theta}) + \frac{\partial}{\partial z}(\tau_{\theta z}) + 2\frac{\tau_{r\theta}}{r} = \rho \frac{\partial^2 v}{\partial t^2} \tag{2}$$

where ρ is the density of the medium, assumed constant. The boundary condition

$$\tau_{\theta z} = P\,\delta(r - ct)\,H(t) \text{ at } z = 0 \tag{3}$$

c, P being constant H is the Heaviside function and δ is the two dimensional delta function given by

$$2\pi \int_0^\infty \delta(r)\, r\,dr = 1.$$

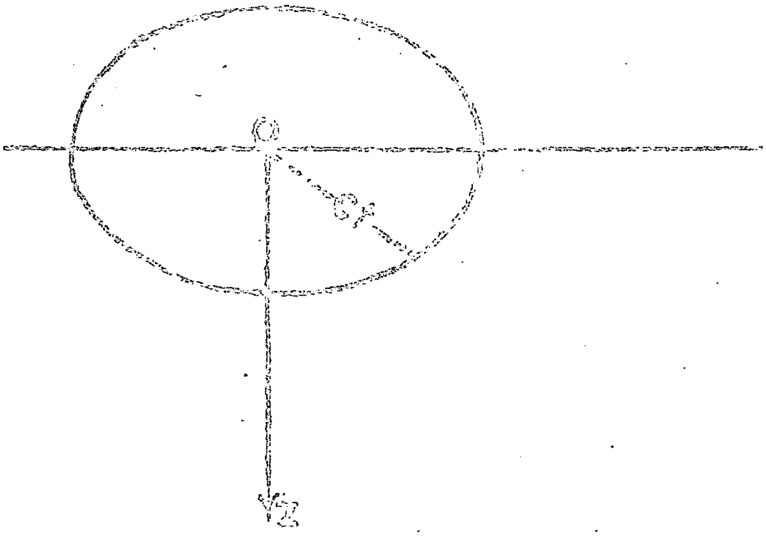

Fig.1 Co-ordinates system in the medium.

SOLUTION: Wedefine for all positive real values of s, the Laplace transform $f_1(r, z, s)$ of a function $f(r, s, t)$by

$$f_1(r,z,s) \int_0^\infty e^{-st} f(r,z,t)dt \qquad (4)$$

Substituting the value of $\tau_{r\theta}$ and $\tau_{\theta z}$ in equation (2) and then applying the Laplace transform (4), we obtain

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right) + \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{\beta^2} s^2 v_1 \qquad (5)$$

where $\beta = \sqrt{(\mu/p)}$ is the shear wave velocity.

Defining v_2 by the equation.

$$v_2(\xi,z,s) = \int_0^\infty r\, J_1(\xi r) v_1(r,z,s) dr \qquad (6)$$

and then multiplying the equation (5) by $rJ_1(\xi r)$ and integrating with respect to r from o to ∞, we get

$$\frac{d^2 v_2}{dz^2} = (\xi^2 + s^2/\beta^2) v_2 \qquad (7)$$

Taking ξ real, the general solution of the equation (7) which remains bounded for large values of z, is

$$v_2 = A \exp\left[-z(\xi^2 + s^2/\beta^2)^{1/2}\right]. \qquad (8)$$

The Laplace transform of $\tau_{\theta z}$ is

$$(\tau_{\theta z})_1 = P\int_0^\infty e^{-st}\,\delta(r-ct)H(t)dt$$

$$= \frac{P}{2\pi cr}\,e^{-sr/c}$$

It's Hankel transform is

$$(\tau_{\theta z})_2 = \frac{P}{2\pi c}\int_0^\infty e^{-sr/c}\,J_1(\xi r)dr$$

$$= \frac{P}{2\pi c}\left[1 - \frac{s}{c}\left(\xi^2 + \frac{s^2}{c^2}\right)^{-1/2}\right]$$

[See Erdelyi, et al 1964, p. 19].

Noting that on $z = o$,

$\frac{dv_2}{dz} = -A(\xi^2 + s^2/\beta^2)^{1/2}$ and using the boundary condition,

we get

$$A = -\frac{P}{2\pi/\mu\xi c}\;\frac{\left[1 - \frac{s}{c}\left(\xi^2 + \frac{s^2}{c^2}\right)^{-1/2}\right]}{(\xi^2 + s^2/\beta^2)^{1/2}}.$$

Substituting this value of A in (8) and inverting the Hankel transform (6), we obtain

$$v_1 = -\frac{P}{2\pi/\mu c\xi}\int_0^\infty \frac{1 - \frac{s}{c}\left(\xi^2 + \frac{s^2}{c^2}\right)^{-1/2}}{(\xi^2+s^2/\beta^2)^{1/2}}\,J_1(\xi r)e^{-z(\xi^2+s^2/\beta^2)^{1/2}}\,d\xi \quad (9)$$

Now,

$$J_1(\xi r)= \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi r\sin\psi}(\cos\psi-i\sin\psi)\,d\psi .$$ (See Erdelyi, A. etal 1953 P.14)

Substituting this value of $J_1(\xi r)$ in (9) and putting $p=\xi\sin\psi$ and $q=\xi\cos\psi$, we get

$$v_1=\frac{-P}{4\pi^2\mu c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(q-ip)\left\{\left(p^2+q^2+\frac{s^2}{c^2}\right)^{1/2}-\frac{s}{c}\right\}}{(p^2+q^2+s^2/c^2)^{1/2}(p^2+q^2+s^2/\beta^2)^{1/2}}\;X$$

$$X\;\frac{e^{-z(p^2+q^2+s^2/\beta^2)^{1/2}+irp}}{(p^2+q^2)}\,dp\,dq .$$

To find the inversion of v_1, we put

$p=ms$ and $q=ns$ in the above integral, then we have

$$v_1=\frac{-P}{4\pi^2\mu c}\int_{-\infty}^{\infty}dn\int_{-\infty}^{\infty}\frac{(m-in)\left\{(m^2+n^2+1/c^2)^{1/2}-1/c\right\}e^{-s\left[z(m^2+n^2+1/\beta^2)^{-1/2}-irm\right]}}{(m^2+n^2+1/c^2)^{1/2}(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}\,dm$$

$$=\frac{iP}{2\pi^2\mu c}\int_{0}^{\infty}dn\int_{-\infty}^{\infty}\frac{m\left\{(m^2+n^2+1/c^2)^{1/2}-1/c\right\}e^{-s\left[z(m^2+n^2+1/\beta^2)^{1/2}-irm\right]}}{(m^2+n^2+1/c^2)^{1/2}(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}\,dm \tag{10}$$

In the integral of the equation (10), the path of integration with respect to m is the real axis which is deformed in such a way that

$$-irm+z(m^2+n^2+1/\beta^2)^{1/2}=t \tag{11}$$

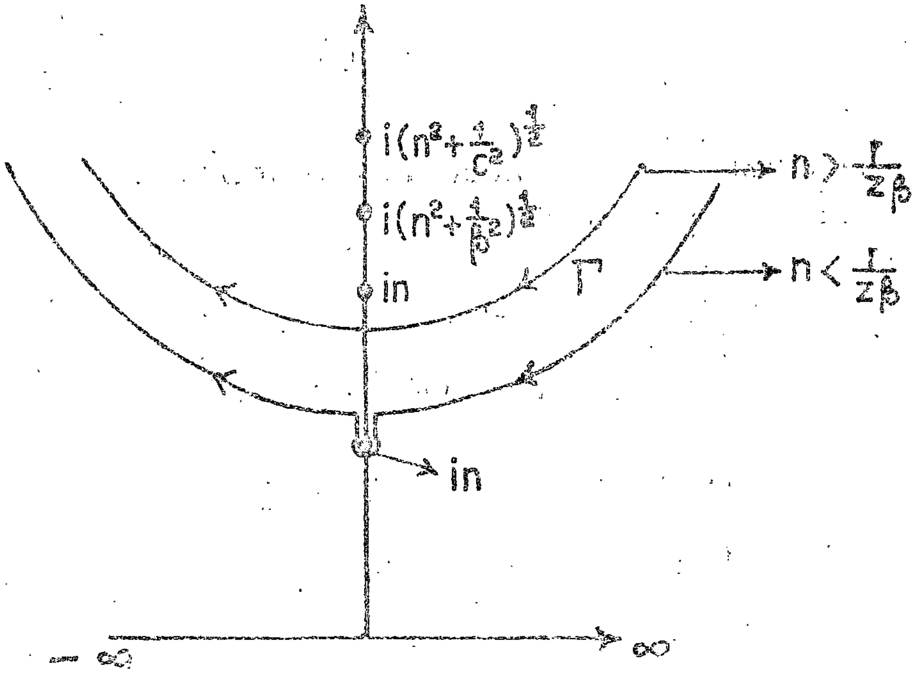

Fig. 2 Path of integration in the complex m-plane.

where t is real and positive. The deformed path of integration is the branch Γ of a hyperbola whose equation is

$$m=\frac{irt\pm z\left[t^2-(z^2+r^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2+r^2}\,,\quad \left\{(z^2+r^2)\left(n^2+\frac{1}{\beta^2}\right)\right\}^{1/2}<t<\infty$$

We write

$$m_{\pm}=\frac{irt\pm z\left[t^2-(z^2+r^2)(n^2+1/\beta^2)\right]^{1/2}}{z^2+r^2}\,,$$

$$K(m,n)=\frac{m\left\{(m^2+n^2+1/c^2)^{1/2}-1/c\right\}}{(m^2+n^2+1/c^2)^{1/2}(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}$$

and we obtain finally the expression of v_1 in the form

$$v_1=\frac{-P}{\pi^2/\mu c}\int_{1/\beta(z^2+r^2)^{1/2}}^{\infty}e^{-st}\,dt\int_0^{\sqrt{\gamma}}\mathrm{Im}\left[K(m_+,n)\frac{dm_+}{dt}\right]dn.$$

where $\gamma=\{t^2/(z^2+r^2)\}-\beta^{-2}$.

Taking the Laplace inversion, we get

$$v=\frac{-P}{\pi^2/\mu c}H\left[t-\beta^{-1}(z^2+r^2)^{1/2}\right]\int_0^{\sqrt{\gamma}}\mathrm{Im}\left[K(m_+,n)\frac{dm_+}{dt}\right]dn \qquad (12)$$

APPROXIMATE EVALUATION OF THE DISPLACEMENT:

Case1. <u>Displacement after the first arrival</u>.

To integrate $\displaystyle\int_0^{\sqrt{\gamma}}\mathrm{Im}\left[K(m_+,n)\frac{dm_+}{dt}\right]dn$ (13)

we put $n=\sqrt{\gamma}\sin\alpha$ and $t_\beta=\beta^{-1}(z^2+r^2)^{1/2}$,

which is the time taken by the shear wave to reach the point(r,θ,z).

The integral (13) after the substitution takes the form

$$\int_0^{\pi/2} \operatorname{Im}\left[K(m_+, n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right]d\alpha. \tag{14}$$

Now, $\operatorname{Im}\left[K(m_+, n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right] = \frac{\beta}{r}\left[\frac{\beta(z^2+r^2)^{1/2}}{\beta^2(z^2+r^2)-c^2r^2\ ^{1/2}} - 1\right]$

as $t \to t_\beta$.

Hence from (12) , we obtain

$$v = \frac{-P\beta}{2\pi\mu cr}\left\{\frac{\beta(z^2+r^2)^{1/2}}{\beta^2(z^2+r^2)-c^2r^2\ ^{1/2}} - 1\right\} H(t-t_\beta),$$

which is the displacement at any point (r,z) just after the arrival of the disturbance. It is interesting to note that the displacement due to the first arrival of the disturbance at any point of the z-axis is zero which is also expected from the physical stand point. It is to be noted that the displacement at any point on the free surface $z=0$ varies inversely as r.

Case 2. Displacement after sufficiently large time when $z \neq 0$.

In this case, $\operatorname{Im}\left[K(m_+, n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right]$

$$= \frac{(z^2+r^2)^{1/2}}{t}\,\frac{r\{z^2\sin^2\alpha-(r^2+z^2)\cos^2\alpha\}}{(z^2+r^2\cos^2\alpha)^2} -$$

$$- \frac{(z^2+r^2)^{3/2}}{ct^2}\cdot\frac{rz\{z^2-3(r^2+z^2)\cos^2\alpha+r^2\cos^4\alpha\}}{(z^2+r^2\cos^2\alpha)^3}\ .$$

The terms containing $1/t^3$ and higher orders are neglected. After the above substitution (14) takes the following form

$$\frac{r(z^2+r_1^2)^{1/2}}{t}\int_0^{\pi/2}\frac{z^2\sin^2\alpha-(r^2+z^2)\cos^2\alpha}{(z^2+r^2\cos^2\alpha)^2}\,d\alpha\;-$$

$$-\frac{rz(z^2+r^2)^{3/2}}{ct^2}\int_0^{\pi/2}\frac{z^2-3(r^2+z^2)\cos^2\alpha+r^2\cos^4\alpha}{(z^2+r^2\cos^2\alpha)^3}\,d\alpha \qquad (15)$$

The first integral of (15) is zero, hence for the large value of the time t compared to t_β, the displacement is given by

$$v=-Pr(4z^2+5r^2)/(4\pi\mu c^2t^2z^2).$$

In this case the displacement at any point varies inversely as t^2. Also this is to be noted that the displacement increase with the increase of r when t is very large, which is in conformity with the physical condition because the radius of the ring source after large time t is infinitely large.

Case 3. Displacement at the free surface.

In this case taking $z=o$, we obtain from Eq. (10)

$$v_1=\frac{iP}{2\pi^2\mu c}\left[\int_0^\infty dn\int_{-\infty}^{\infty}\frac{me^{isrm}}{(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}\;-\right.$$

$$\left.-\frac{1}{c}\int_0^\infty dn\int_{-\infty}^{\infty}\frac{me^{isrm}}{(m^2+n^2+1/c^2)^{1/2}(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}\right]dm.$$

(16)

The path of integration in the complex m-plane is the real axis, which is deformed in such a way that

$-irm=t$, where t is real and positive. Taking the integral over the deformed path we get

$$v_1 = \frac{Pr}{\pi^2 \mu C}\left[\int_0^\infty dn \int_{r(n^2+1/\beta^2)^{1/2}}^{\infty} \frac{te^{-st}}{\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)}\, dt - \right.$$

$$\left. -\frac{r}{c}\left\{\int_0^\infty dn \int_{r(n^2+1/\beta^2)^{1/2}}^{r(n^2+1/c^2)^{1/2}} \frac{te^{-st}}{\left\{r^2(n^2+1/c^2)-t^2\right\}^{1/2}\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)}\, dt\right.\right]$$

Changing the order of integration. We obtain

$$v_1 = \frac{Pr}{\pi^2 \mu C}\left[\int_{r/\beta}^\infty te^{-st}\, dt \int_0^{(t^2/r^2-1/\beta^2)^{1/2}} \frac{dn}{\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)} - \right.$$

$$-\frac{r}{c}\left\{\int_{r/\beta}^{r/c} te^{-st}\, dt \int^{(t^2/r^2-1/\beta^2)^{1/2}} \frac{dn}{\left\{r^2(n^2+1/c^2)-t^2\right\}^{1/2}\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)} + \right.$$

$$\left.\left. + \int_{r/c}^\infty te^{-st}\, dt \int_{(t^2/r^2-1/c^2)^{1/2}}^{(t^2/r^2-1/\beta^2)^{1/2}} \frac{dn}{\left\{r^2(n^2+1/c^2)-t^2\right\}^{1/2}\left\{t^2-r^2(n^2+1/\beta^2)\right\}^{1/2}(t^2-r^2n^2)}\right.\right]$$

Taking Laplace inversion of the above integral, we finally obtain.

$$v = \frac{P\beta}{2\pi\mu c r} H(t-r/\beta) - \frac{Pt\beta^3}{\pi^2\mu c r^2 (\beta^2 - c^2)^{1/2}}$$

$$X\left\{[H(t-r/\beta)-H(t-r/c)]\Pi(n,R)+H(t-r/c).\Pi(n,R,\bar{\phi})\right\}.$$

where

$\Pi(R,n)$ is the complete elliptic integral of 3rd kind,

$\Pi(R,n,\bar{\phi})$ is the elliptic integral of 3rd kind,

$R^2 = \dfrac{c^2(t^2\beta^2-r^2)}{r^2(\beta^2-c^2)}$, $n^2 = (t^2\beta^2 - r^2)/r^2$ and

$$\bar{\phi} = \cos^{-1}\left[\beta/c\ (c^2t^2-r^2)/(\beta^2t^2-r^2)^{1/2}\right].$$

TORSIONAL RESPONSE OF AN ELASTIC HALF SPACE TO A NONUNIFORMLY EXPANDING RING SOURCE.

INTRODUCTION: The study of the dynamic behaviour of an elastic solid under various forms of moving loads and torsional pressure has been gaining importance day by day. This is because of their importance in seismology, structural design and under ground exploration.

Gakenheimer (1971) in one of his papers presented in details the problem of a load emanating from a point on the surface and then expanding radially at a constant rate. He considered the cases when the loads are disk-shaped or ring-shaped and the expanding rates are supper seismic, transseismic and sub seismic. Almost at the same time Ghosh (1971) also considered the problem of propagation of a stress discontinuity over an expanding circular region with a constant velocity which is less than the shear wave velocity of the medium. Freund (1973) considered the non uniformly moving line load as well as point load. Stronge (1970) discussed the problem of an accelerating line load in an acoustic half space. The nonuniform pressure distribution problem applied to an elastic half space over a circular zone are discussed by Brock (1980) and by Roy (1979). Almost a same type of problem has been considered by Aggarwal and Ablow (1965). There it was assumed that circularly symmetric load spreads out from a point

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Published in ZAMM, vol. 63, p. 7346 (1983)

on an acoustic half space with decelerating speed. Ghosh (1980/81) determined exactly the displacement produced by SH-type of waves when a torsional force is prescribed over a circular region on the free surface of a homogeneous isotropic medium and that in the integral form in case of a non homogeneous medium.

In the present paper, the displacement at any point (r,z) in the semi infinite medium is determined in the integral form by prescribing a time dependent torsional force over the rim of a circular zone. The ring is assumed to expand in an arbitrary manner with time. It is found that the displacement field contains besides the usual SH-waves, contribution from conical waves which arise due to the motion of the source. The region of conical waves which depend on the nature of the motion of the source and the initial speed of expansion of the source are investigated in details. Different wave front surfaces are located and first motion responses near different wave arrivals have been obtained.

Finally numerical evaluation of the displacement on the free surface has been made for a decelerating ring source whose radius at time t is of the form $h(t)=At^{1/2}$. Displacements at points on the free surface for different position of the source have been shown by means of graphs.

FORMULATION OF THE PROBLEM: Consider a homogeneous isotropic elastic half space on the free surface of which a ring source producing SH-type of waves is expanding with non uniform velocity. (r,θ,z) are the cylindrical polar co-ordinates, z-axis being directed into the medium and the plane boundary being $z = 0$.

The origin of co-ordinates is at the centre of the ring $r = h\,(t)$, $z = o$. The ring is assumed to expand with uniform acceleration or with deceleration and an impulsive torque applied to the ring is prescribed.

The displacement is determined in the integral form at any point inside and on the free surface of the medium, subject to the condition that the half-space is initially at rest and that the displacements remain bounded for large values of z. For torsional motion of the ring all quantities depend on r, z and the time t. We assume that $h(t)$ is non negative and monotone increasing function. The only non-zero component of the displacement vector is the component v along the direction of θ increasing. The relevant non vanishing stress components are

$$\tau_{r\theta} = \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \text{ and } \quad \tau_{\theta z} = \mu \frac{\partial v}{\partial z} \qquad (1\ a,b)$$

where μ is the Lame's constant. The non zero equation of the displacement field is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\,\frac{\partial^2 v}{\partial t^2} \qquad (2)$$

where β is the shear wave velocity. The boundary condition of the motion is

$$\tau_{\theta z} = -\,P\,\delta\left[h(t) - r\right] H(t), \quad z = o \qquad (3)$$

Where P is a constant, $\delta(\ )$ is Dirac's delta function, $H(\ )$ is the Heaviside step function and $h(t)$ is the radius of the ring at time t. Initial conditions of the motion are given by

$$h(t) = o, \ t = o \text{ and } \mathring{h}(t) > o, \ t > o \tag{4}$$

where dot denotes the time derivative.

METHOD OF SOLUTION: We define Laplace transform $f_1(r,z,p)$ of the function $f(r,z,t)$ by

$$f_1(r,z,p) = \int_0^\infty \exp(-pt)\, f(r,z,t)dt \tag{5}$$

where p is real and positive and Hankel transform $f_2(\xi,z,p)$ of $f_1(r,z,p)$ by

$$f_2(\xi,z,p) = \int_0^\infty r\, J_1\, (\xi r)\, f_1(r,z,p)dr \tag{6}$$

where J_n is the Bessel function of the first kind of order n.

Applying Laplace and Hankel transforms, to the equation (2) successively we obtain

$$\frac{d^2 v_2}{dz^2} - k^2\, v_2 = o \tag{7}$$

where $k^2 = \xi^2 + (p^2 / \beta^2)$.

The solution of the equation (7) which remains bounded as $z \to +\infty$ is

$$v_2 = K \exp(-kz) \tag{8}$$

The value of the constant K is determined, by using the condition (4), the equation (8) and the Hankel transform of the Laplace transform of the equation (3) It is found to be

$h(t) = 0$, and

$$K = \frac{p}{\mu} \int_0^\infty \frac{1}{k} h(T) J_1(\xi h(T)) \exp(-pT) dT . \qquad (9)$$

Substituting the value of K in (8) and then taking Hankel's inversion one gets

$$v_1 = \frac{p}{\mu} \int_0^\infty h(T) \exp(-pT) \int_0^\infty \frac{\xi}{k} J_1(\xi r) J_1(\xi h(T)) \exp(-kz) d\xi \, dT . \qquad (10)$$

LAPLACE INVERSION: In this section the Laplace inverse transform is evaluated by Cagniard's technique.

We make use of the following results

$$J_1(\xi h(T)) J_1(\xi r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} J_0(\xi S) \cos \phi \, d\phi \quad \text{and}$$

$$J_0(\xi S) = \frac{1}{2\pi} \int_0^{2\pi} \exp(i\xi S \cos u) du, \text{ where}$$

$S = (r^2 + h^2(T) - 2r\, h(T) \cos \phi)^{1/2}$, to obtain the equation (10) as

$$v_1 = \frac{p}{4\pi^2 \mu} \int_0^\infty h(T) \exp(-pT) \int_{-\pi}^{\pi} I \cos \phi \, d\phi \, dT, \qquad (11)$$

where

$$I = \int_0^\infty \int_0^{2\pi} \frac{\xi}{k} \exp(i\xi S \cos(\psi - u) - kz) d\xi \, du \qquad (12)$$

and ψ is any constant angle.

In (12), we put

$\alpha' = \xi \cos u$ and $\beta' = \xi \sin u$, then substitute

$\alpha' = w \cos\psi - q \sin\psi$ and $\beta' = w \sin\psi + q \cos\psi$ and finally replace w by wp and q by qp to obtain (12) in the form

$$I = p \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left[-p\left\{-iwS + (w^2+q^2+1/\beta^2)^{1/2}\, z\right\}\right]}{(w^2+q^2+1/\beta^2)^{1/2}}\, dw\, dq \tag{13}$$

Equation (13) is well known form for determining the Laplace inversion of a function by applying Cagniard's technique as modified by De-Hoop. Substituting $t = -iwS + z(w^2+q^2+1/\beta^2)^{1/2}$ in (13) where t is real and positive, the Laplace inversion of (13) is found to be equal to

$$G(t) = 4\frac{d}{dt}\left\{ H\left[t - \;/\beta\right] \int_{0}^{(t^2/\rho^2 - 1/\beta^2)^{1/2}} Rl\left[\frac{1}{(w_+^2 + q^2 + 1/\beta^2)^{1/2}} \frac{dw_+}{dt}\right] dq \right\} \tag{14}$$

where $\rho^2 = z^2 + S^2$ and

$$w_+ = \frac{iSt + z\left\{t^2 - \;{}^2(q^2 + 1/\beta^2)\right\}^{1/2}}{\rho^2}$$

Applying convolution theorem on (11), the Laplace inversion of v_1 is obtained in the form

$$v = \frac{p}{4\pi^2\mu} \int_0^{\infty} h(T)\, dT \int_{-\pi}^{\pi} \cos\phi\, d\phi \int_0^{t} \delta(u - T)\, G(t-u)\, du,$$

which when simplified takes the form

$$v = \frac{p}{\pi\mu} \int_0^{t} h(T)\, dT \int_0^{\pi} \frac{\cos\phi}{\rho}\, \delta\left(t - T - \rho/\beta\right) d\phi . \tag{15}$$

Integrating over ϕ , we obtain

$$v=\frac{P\beta}{\pi r\mu}\int_{0}^{t}\left\{H\left[t-\tau-\frac{\sqrt{z^{2}+(r-h(\tau)\ )^{2}}}{\beta}\right]-H\left[t-\tau-\frac{\sqrt{z^{2}+(r+h(\tau)\ )^{2}}}{\beta}\right]\right\}Q(\tau)d\tau$$

(16)

where

$$Q(\tau)=\frac{z^{2}+r^{2}+h^{2}(\tau)-\beta^{2}(t-\tau)^{2}}{\left[\left\{z^{2}+(r+h(\tau)\ )^{2}-\beta^{2}(t-\tau)^{2}\right\}\left\{\beta^{2}(t-\tau)^{2}-z^{2}-(r-h(\tau))^{2}\right\}\right]^{1/2}}\ .$$

To facilitate our discussion, equation (16) is written in an alternative form,

$$v=\frac{P\beta}{\pi r\mu}\int_{0}^{t-z/\beta}\left\{H\left[r-h(\tau)+\sqrt{\beta^{2}(t-\tau)^{2}-z^{2}}\right]H\left[h(\tau)-\sqrt{\beta^{2}(t-\tau)^{2}-z^{2}}\right]+\right.$$

$$+H\left[r+h(\tau)-\sqrt{\beta^{2}(t-\tau)^{2}-z^{2}}\right]H\left[-h(\tau)+\sqrt{\beta^{2}(t-\tau)^{2}-z^{2}}\right]-$$

$$\left.-H\left[r-h(\tau)-\sqrt{\beta^{2}(t-\tau)^{2}-z^{2}}\right]\right\}Q\ (\tau)\ d\tau\ . \qquad (17)$$

The region of support for τ - integration is bounded by the curves:

I : $\quad r = h\ (\tau) + \sqrt{\beta^{2}(t-\tau)^{2}-z^{2}}\ ;\ 0 < \tau < t-z/\beta .$ (18)

II : $\quad r = h\ (\tau) - \sqrt{\beta^{2}(t-\tau)^{2}-z^{2}}\ ;\ \tau_{0} < \tau < t-z/\beta .$ (19)

III: $\quad r = -h\ (\tau) + \sqrt{\beta^{2}(t-\tau)^{2}-z^{2}}\ ;\ 0 < \tau < \tau_{0} .$ (20)

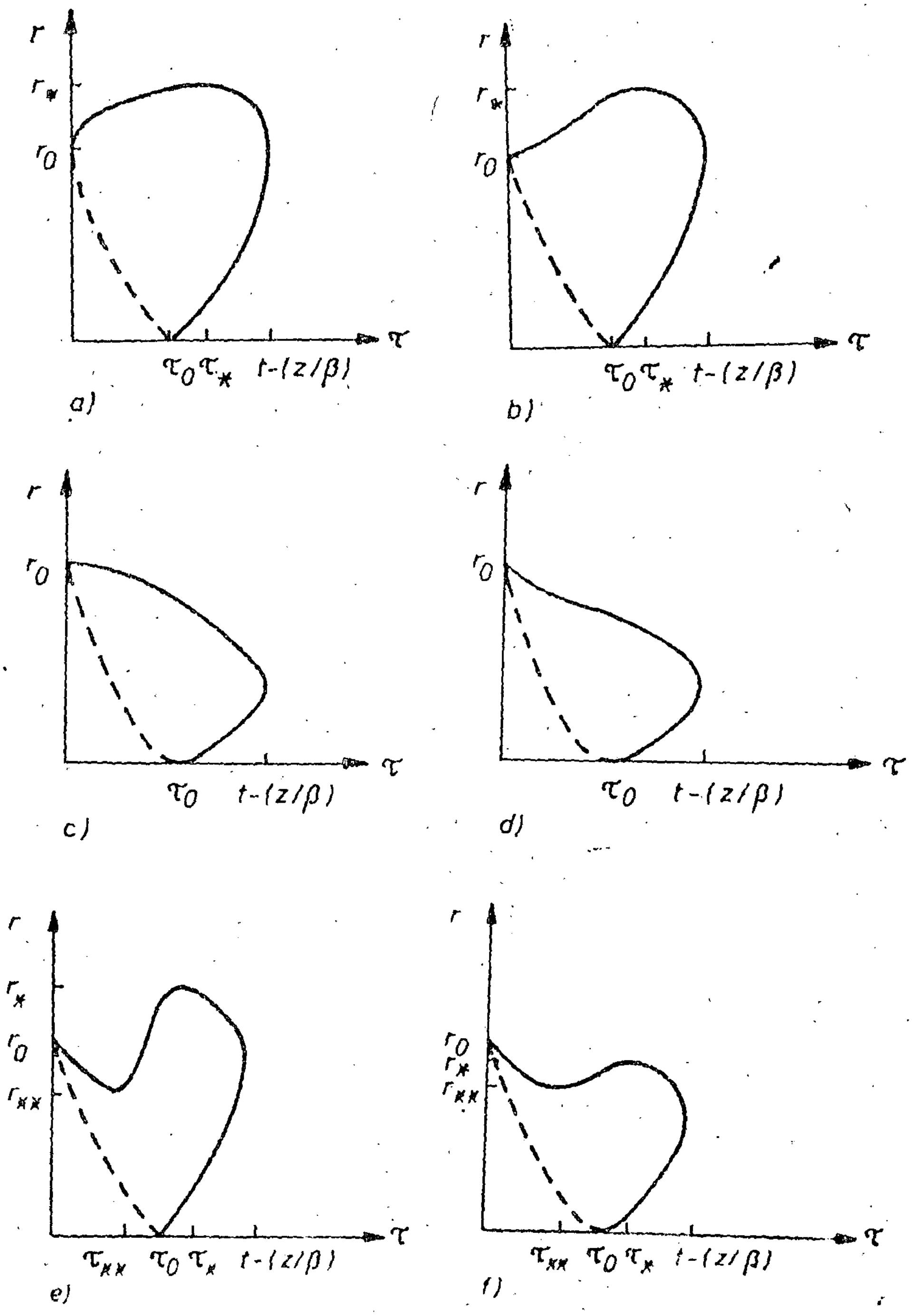

Fig. 1. Region of support for τ-integration for fixed z and t. (a – b) when a single maximum exists i.e. the starting velocity $h(0)$ is such that $(\partial r/\partial \tau) > 0$; ($c$ – d) the case when no extremum exists; (e – f) when a maximum(r_*) and a minimum (r_{**}) both exist.

The region of τ integration for $Q(\tau)$ bounded by the curves I, II and III are shown in the figs.1(a - f) and the following remarks can be made about them.

It is to be noted that the curves II and III are monotone increasing and decreasing in their respective region of existence viz. $(\tau_o,\ t - z/\beta)$ and (o, τ_o) where

$$h(\tau_o) = \left\{\beta^2(t-\tau_o)^2 - z^2\right\}^{1/2}$$

The curve I has extremum where

$$\frac{\partial r}{\partial \tau} = \dot{h}(\tau) - \frac{\beta^2(t-\tau)}{\left\{\beta^2(t-\tau)^2 - z^2\right\}^{1/2}} \tag{21}$$

vanishes and

$$\frac{\partial^2 r}{\partial \tau^2} = \ddot{h}(\tau) - \frac{\beta^2 z^2}{\left\{\beta^2(t-\tau)^2 - z^2\right\}^{3/2}} \tag{22}$$

does not vanish.

We consider the different cases that arise due to non uniform increase of the ring source. Let the source increase with uniform acceleration $\ddot{h}(\tau) > o$. In this case, if the initial velocity $\dot{h}(o)$ of the source be such that $\left(\frac{\partial r}{\partial \tau}\right)_o > o$, then since $\left(\frac{\partial r}{\partial \tau}\right)_o > o$ and

$\left(\frac{\partial r}{\partial \tau}\right)_{t-z/\beta} < 0$, the curve I has only one maximum at $r = r_*$ because $\frac{\partial^2 r}{\partial \tau^2}$ either changes sign once from positive to negative or remains negative throughout in $(0, t-z/\beta)$. The corresponding cases are shown in Fig.1(a-b). Next let the initial velocity $\dot{h}(0)$ of the source be such that $\left(\frac{\partial r}{\partial \tau}\right)_0 < 0$. In this case, if $\left(\frac{\partial^2 r}{\partial \tau^2}\right)_0 < 0$, $\left(\frac{\partial^2 r}{\partial \tau^2}\right)$ will be negative throughout the interval $(0, t-z/\beta)$; the curve I then corresponds to Fig.1(c), since both $\left(\frac{\partial r}{\partial \tau}\right)_0$ and $\left(\frac{\partial r}{\partial \tau}\right)_{t-z/\beta}$ are negative. But if $\left(\frac{\partial^2 r}{\partial \tau^2}\right)_0$ be positive then $\frac{\partial^2 r}{\partial \tau^2}$ changes sign once from positive to negative in the interval $(0, t-z/\beta)$. Hence in this case the curve I has either no extremum which corresponds to Fig.1(d) or there is a maximum proceded by a minimum which is shown in Fig.1 (e,f). Finally, in case of a decelerating motion of the source i.e when $\ddot{h}(\tau)$(not necessarily a constant) < 0, throughout the interval, the curve has either only one maximum if $\left(\frac{\partial r}{\partial \tau}\right)_0 > 0$ as in Fig.1(a) or no extremum as in Fig.1(c) when $\left(\frac{\partial r}{\partial \tau}\right)_0 < 0$.

We consider the curves I and II together. Their combined equation is

$$(r-h(\tau))^2 = \beta^2(t-\tau)^2 - z^2 \tag{23}$$

For figures 1(c,d), τ is a single valued function of r. For the figs.1(a,b) τ may be a double valued function whereas for figs, 1(e,f), τ may be a triple valued function of r. Taking the equations (18) and (19) together, the values of τ are designated as $\tau = \tau_1$, $\tau = (\tau_1, \tau_2)$ and $\tau = (\tau_1, \tau_2, \tau_3)$ where $\tau_1 > \tau_2 > \tau_3$ depending on whether τ is single, double or triple valued function of r. In (20) r is a monotone decreasing function of τ, so the corresponding value of τ is designated as $\tau = \tau_4$.

With the above values of the roots of the equations (18)-(20) and from a close examination of the different figs. 1(a-f), the displacement produced by the SH-type of waves is given by
$v = v^1 + v^2 + v^3$, where

$$
\begin{aligned}
v^1 &= BH(r_o - r)\, I(Q(\tau);\ \tau_4, \tau_1) \\
v^2 &= B\Big[H(r - r_o) - G(r - \max(r_*, r_o))\Big] I(Q(\tau);\ \tau_2, \tau_1) \\
v^3 &= B\Big[G(r - \min(r_*, r_o)) - G(r - \min(r_{**}, r_o))\Big] I(Q(\tau);\ \tau_3, \tau_2)
\end{aligned}
\tag{24}
$$

and

$$
G(r - \max(r_*, r_o)) = \begin{array}{l} H(r - r_*) \text{ if } r_* = \max(r_*, r_o) \\ H(r - r_o) \text{ if } r_o = \max(r_*, r_o) \\ \qquad\qquad \text{or } r_* \text{ does not exist.} \end{array}
$$

$$
G(r - \min(r_*, r_o)) = \begin{array}{l} H(r - r_*) \text{ if } r_* = \min(r_*, r_o) \\ H(r - r_o) \text{ if } r_o = \min(r_*, r_o) \\ \qquad\qquad \text{or } r_* \text{ does not exist.} \end{array}
$$

Similar meaning is attached to the symbol
$G(r - \min(r_{**}, r_o))$. 'B' has been written for $\frac{P\beta}{\pi r/\mu}$.

$r_o = \sqrt{\beta^2 t^2 - z^2}$ is the value of r at $\tau = o$ and

$$I(F(\tau);\ a, b) = \int_a^b F(\tau)\, d\tau .$$

WAVE FRONT ANALYSIS: In this section we locate and analyse the nature of the wave fronts.

The nature of the wave front changes due to non uniform expansion of the source and also it depends on the inital velocity $\dot{h}(o)$ ($=u_o$) of expansion of the source. We consider decelerating and accelerating expansion of the source for different initial velocities.

Case of decleration:

i) let $\dot{h}(o) = u_o < \beta$.

From (21) and (22) , $(\frac{\partial r}{\partial \tau})_o$ is negative for all z and $\frac{\partial^2 r}{\partial \tau^2}$ is also negative as $\ddot{h}(\tau)$ is negative. So the curve I in ($o, t-z/\beta$) is such that r decreases with the increase of τ . This corresponds to the region of integration as depicted in fig.1(c) and consequently the wave front is of the form as shown in fig.2(a).

ii) u_o($> \beta$) is finite.

It follows from (21), $(\frac{\partial r}{\partial \tau})_o$ is positive for $o < z < z_2$ and negative for $z > z_2$ where z_2 is obtained from

$u_o = \beta^2 t/(\beta^2 t^2 - z_2^2)^{1/2}$ i.e $z_2 = \beta t\ (1 - \beta^2/u_o^2)^{1/2}$ and further $\frac{\partial^2 r}{\partial \tau^2}$ is negative as $\ddot{h}(\tau)$ is negative. Therefore the region of

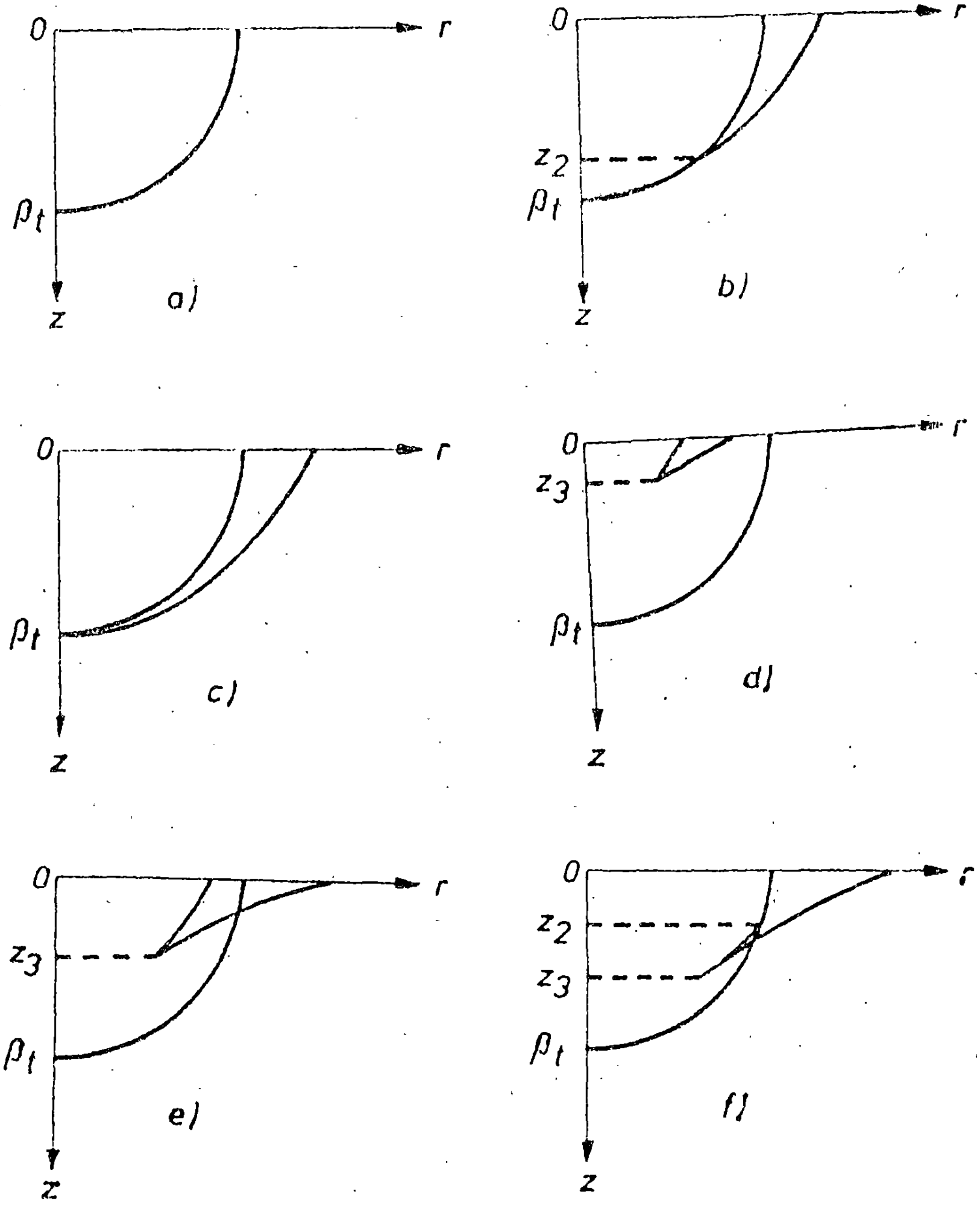

Fig. 2. Different types of wave fronts, at particular admissible values of time and position, which arise due to non uniformity and initial velocity of expansion of the ring source

integrations for $o < z < z_2$ and for $z > z_2$ correspond to the regions shown in the figs.1(a) and 1(c) respectively and consequently the wave front is given by the fig.2(b).

iii) u_o is infinitely large.
From (21) and (22), it follows that $(\frac{\partial r}{\partial \tau})_o$ is positive for all z and $(\frac{\partial^2 r}{\partial \tau^2})$ is negative for all z and for all τ . Hence the region of integration is fig.1(a) and the corresponding wave front is as shown in fig.2(c).

Case of acceleration:
i) We assume that the ring source expands with uniform acceleration f and starts with the velocity $u_o(= \dot{h}(o))$. First let $u_o < \beta$; then $(\partial r/\partial \tau)_o$ is negative for all z and $(\partial^2 r/\partial \tau^2)_o$ is positive for $o < z < z_1$ and negative for $z > z_1$, where z_1 is to be determined from the condition $(\partial^2 r/\partial \tau^2)_o = o$. For $z > z_1$, $(\partial^2 r/\partial \tau^2)$ is negative for τ in $(o, t-z/\beta)$. Consequently the region of integration is the fig.1(c). On the other hand if z lies in (o, z_1) then $(\partial^2 r/\partial \tau^2)$ is first positive and then negative as τ increases in $(o, t-z/\beta)$, so in this case the region of integration is either fig.1(d) or fig.1(e or f).
By using (22), z_1 is determined from the equation

$$f = \beta^2 z_1^2 / \beta^2 t^2 - z_1^2)^{3/2} \tag{25}$$

It is to be noted that $z_1 = o$ when $f = o$ and z_1 is a monotone increasing function of f. Further, in (o, z_1), $(\partial r/\partial \tau)$ may have two zeroes or there is no zero in the region $o < < t - z/\beta)$, depending on the value of z. The condition that $(\partial r/\partial \tau)$ may have two zeroes is, $o < z < z_3$, where

$$z_3 = \beta \left(\frac{u_0+ft}{f} \right) \left\{ 1 - \frac{\beta^{2/3}}{(u_0+ft)^{2/3}} \right\}^{3/2} \qquad \text{for } u_0 + ft > \beta$$

$$= 0 \qquad \text{for } u_0 + ft < \beta.$$

It can be shown further that $z_3 < z_1$. Hence for $0<z<z_3$ the region of integration is fig.1(e or f) and for $z_3 < z < z_1$, the region of integration is fig.1(d). Therefore for accelerating source with initial velocity $u_0 < \beta$, the wave front is of the form as shown in fig.2(a) if the observation time be such that $(u_0 + ft) \leq \beta$ and for $(u_0 + ft) > \beta$ the wave front is like the figures as in 2(d) or 2(e) according as the position of the source at the observation time is inside or out side the characteristic surface $r^2 + z^2 + \beta^2 t^2$.

ii) Next let $u_0 > \beta$ from (21) we have $(\partial r/\partial \tau)_0$ is positive for $0 < z < z_2$ and is negative for $z > z_2$ where z_2 is given by

$$z_2 = \beta t(1 - \beta^2/u_0^2)^{1/2} \qquad (26)$$

Also $(\partial^2 r/\partial \tau^2)_0$ is positive for $0 < z < z_1$ and is negative for $z > z_1$, where z_1 is given by (25). So for $0 < z < z_1$, $(\partial^2 r/\partial \tau^2)$ is first positive and then negative in $0 \leqslant \tau < t - z/\beta$. We consider the case for $z_1 < z_2$ first. In this case

$$\beta^2 z_1^2/(\beta^2 t^2 - z_1^2)^{3/2} < \beta^2 z_2^2 / (\beta^2 t^2 - z_2^2)^{3/2},$$

since $\beta^2 z^2/(\beta^2 t^2 - z^2)^{3/2}$ is a monotone increasing function of z. Using (25) and (26), we obtain

$$\beta^2 (u_0 + ft) / u_0^3 < 1 \qquad (27)$$

Under the condition obtained in (27) the region of integration is like that of the fig.1(b) in the range $o < z < z_1$ and for $z_1 < z < z_2$, the region of integration is of the type as shown in fig.1(a). For $z_2 < z < \beta t$, the region of integration, is shown in fig.1(c). Therefore for $u_o > \beta$ and for the relation given in (27), it follows that the nature of the wave front is of the type as shown in fig.2(b).

Finally, we study the case when $z_1 > z_2$ i.e when $\beta^2(u_o+ft)/u_o^3 > 1$. Here for $o < z < z_2$ the region of integration is as in fig.1(b). Since z_3 is always less than z_1, so for $z_2 < z < z_3$ the region of integration is like fig.1(e or f) and for $z_3 < z < z_1$ the region of integration is like that as shown in fig.1(d). Fig.1(c) represents the region of integration for $z_1 < z < \beta t$. Accordingly the wave front takes the shape of the fig.2(f).

FIRST MOTION RESPONSES: The expression for the displacement as given in (24) is in the form of integrals over finite ranges. As such, computation of displacement for a given model can be done with the high power computer. However some idea about the nature of displacement at the time of the first arrival of wave fronts can be obtained by limiting process following Stronge(1970) The displacement field just after arrival time of the characteristic surface $r = r_*$ is from (24),

$$v = B\ I(Q\ (T);\ T_2,\ T_1) \qquad (28)$$

where as just before the arrival time the displacement is given by $v = o$.

To evaluate (28) near $r = r_*$, we put $r = r_* - \Delta r$ and $\tau = \tau_* + \theta$ in equations (18) and (19) . Using Taylor's expansion in the neighbourhood of (τ_*, r_*) and with the help of equations (21) and (22) we find the limits of integration of equation (28) in the new variable θ as

$$\theta_{1,2} = \pm \frac{\sqrt{2\Delta r}\left\{\beta^2(t-\tau_*)^2 - z^2\right\}^{3/4}}{\left[\beta^2 z^2 - \ddot{h}(\tau_*)\left\{\beta^2(t-\tau_*)^2 - z^2\right\}^{3/2}\right]^{1/2}} \tag{29}$$

The same procedure is followed to determine in the neighbourhood of $(\tau_*\ ,\ r_*)$, the value of Q which is found to be

$$Q(\tau_*+\theta) = \frac{\left[r_* h(\tau_*)\left\{\beta^2(t-\tau_*)^2 - z^2\right\}\right]^{1/2}}{\left[\theta^2\left\{\ddot{h}(\tau_*)(\beta^2(t-{}_*)^2-z^2)^{3/2} - \beta^2 z^2\right\} + 2\Delta r\left\{\beta^2(t-\tau_*)^2-z^2\right\}^{3/2}\right]^{1/2}} \tag{30}$$

where the lowest term in θ and Δr are retained. The value of the integral (28) after substituting the value of Q from (30) and the limits of integration for the new variable θ as obtained in (29) is found to be

$$\frac{p\,\beta}{r\,\mu}\left[\frac{r_*\,h(\tau_*)\left\{\beta^2\,(t-\tau_*)^2 - z^2\right\}}{\beta^2 z^2 - \ddot{h}(\tau_*)\left\{\beta^2(t-\tau_*)^2 - z^2\right\}^{3/2}}\right]^{1/2},$$ which is the displacement at the first arrival of the wave front given by $r = r_*$.

To find the displacement at the first arrival of the wave front given by $r = r_{**}$, we define $Q(\tau)$ in the neighbourhood of (τ_{**}, r_{**}) and out side the region of integration by

$$Q(\tau) = \frac{z^2 + r^2 + h^2(\tau) - \beta^2 (t-\tau)^2}{\left\{z^2+(r+h(\tau))^2 - \beta^2(t-\tau)^2\right\}^{1/2} \left|\left\{\beta^2(t-\tau)^2 - z^2 - (r-h(\tau))^2\right\}^{1/2}\right|} \tag{31}$$

and put $r = r_{**} + \Delta r$ and $\tau = \tau_{**} + \theta$. Following the same procedure as done in case of $r = r_*$, the displacement at the first arrival of the wave surface $r = r_{**}$ is found to be

$$\frac{p\beta}{r\mu}\left[\frac{r_{**}\, h(\tau_{**})\left\{\beta^2 (t-\tau_{**})^2 - z^2\right\}}{\ddot{h}(\tau_{**})\left\{\beta^2(t-\tau_{**})^2 - z^2\right\}^{3/2} - \beta^2 z^2}\right]^{1/2}.$$

The displacement at a point due to the first arrival of the wave fronts $r = r_*$ and $r = r_{**}$ simultaneously is also determined. At this point the wave fronts $r = r_*$ and $r = r_{**}$ form a cusp (cf.fig.2(d,e,f). In this case this is to be noted that at the cusp $r = r_* = r_{**} = \bar{r}$ (say) and $(\partial r/\partial\tau) = (\partial^2 r/\partial\tau^2) = o$ where as $(\partial^3 r/\partial\tau^3) \neq o$. Hence it follows from equation (24) that the displacement due to first arrival of this wave front at $r = \bar{r}$ is

$$\frac{p\beta}{\pi r\mu}\left[I(Q(\tau);\ \bar{\tau}, \tau_1) + I(Q(\tau); \tau_2, \bar{\tau})\right] \tag{32}$$

where $\bar{\tau} = \tau_{*} = \tau_{**}$ and τ_1, τ_2 are the two values of τ close to $\bar{\tau}$ and correspond to the points lying on either side of $(\bar{\tau}, \bar{r})$ on the curve I and II together.

To evaluate the integrals in (32), $\tau = \bar{\tau} + \theta$ and $r = \bar{r} - \Delta r$ are put in the first integral where as $\tau = \bar{\tau} - \theta$ and $r = \bar{r} + \Delta r$ are put into the second integral of (32). Also this is to be remembered that outside the region of integration in the neighbourhood of $(\bar{\tau}, \bar{r})$, $Q(\tau)$ is defined as in (31).

After the above mentioned substitution in (28) and retaining the lowest order term of θ and Δr one gets the displacement due to first arrival at $r = \bar{r}$ as

$$\frac{2P\beta}{\pi \bar{r} \mu} \left[\frac{3\bar{r} h(\bar{\tau}) \{\beta^2 (t - \bar{\tau})^2 - z^2\}^2}{3\beta^4 z^2 (t - \bar{\tau}) - \ddot{h}^{\cdot}(\bar{\tau}) \{\beta^2 (t - \bar{\tau})^2 - z^2\}^{5/2}} \right]^{1/2} \int_0^a \frac{d\theta}{\sqrt{(a^3 - \theta^3)}} \qquad (33)$$

where

$$a^3 = \frac{6\Delta r (\bar{r} - h(\bar{\tau}))^5}{3\beta^4 z^2 (t - \bar{\tau}) - \ddot{h}^{\cdot}(\bar{\tau}) (\bar{r} - h(\bar{\tau}))^5} .$$

By substituting $\theta^3 = a^3 \sin^2 \alpha$, the integral in (33) is evaluated. The displacement due to first arrival of the wave front $r = \bar{r}$ is found to be

$$\frac{2^{5/6} P\beta}{3^{2/3} \pi \bar{r} \mu (\Delta r)^{1/6}} \frac{\bar{r}^{1/2} h^{1/2}(\bar{\tau}) \{\beta^2 (t - \bar{\tau}) - z^2\}^{7/12}}{\left[3\beta^4 z^2 (t - \bar{\tau}) - \ddot{h}^{\cdot}(\bar{\tau}) \{\beta^2 (t - \bar{\tau})^2 - z^2\}^{5/2}\right]^{1/3}} B\left(\frac{1}{3}, \frac{1}{2}\right)$$

where $B(m,n)$ is the Beta function.

It is interesting to note that in this case the displacement due to first arrival at this point is infinitely large due to the presence of the factor $(\Delta r)^{1/6}$ in the denominator.

Finally we consider the characteristic surface $r^2 + z^2 = \beta^2 t^2$ which corresponds to a disturbance initiated at the origin when the torque is first applied at $\tau = 0$. This disturbance spreads out from the origin with a velocity equal to β. To find the displacement due to the first arrival of this surface, following Aggarwal and Ablow (1965) let us consider the curve

$$\Gamma \; : \; r = \left\{\beta^2 (t - \tau)^2 - z^2\right\}^{1/2} \tag{34}$$

and the lines

$$l_1 \; : \; r = \sqrt{\beta^2 t^2 - z^2} - \epsilon_2 \tag{35}$$

$$l_2 \; : \; r = \sqrt{\beta^2 t^2 - z^2} + \epsilon_1 \tag{36}$$

where ϵ_1 and ϵ_2 are very small positive quantities.

Then to the first order of ϵ_1 and ϵ_2

$$l_1 \times \Gamma \equiv \frac{\epsilon_2 \sqrt{\beta^2 t^2 - z^2}}{\beta^2 t} = \tau' \text{ (say)}$$

$$l_1 \times \text{III} \equiv \frac{\epsilon_2 \sqrt{\beta^2 t^2 - z^2}}{\dot{h}(0) \sqrt{\beta^2 t^2 - z^2} + \beta^2 t} = \tau_4 \text{ (say)}$$

and $$l_2 \times \text{I} \equiv \frac{\epsilon_1 \sqrt{\beta^2 t^2 - z^2}}{\dot{h}(0) \sqrt{\beta^2 t^2 - z^2} - \beta^2 t} = \tau_2 \text{ (say)};$$

ϵ_1, ϵ_2 are such that $\tau_2 < \tau'$ and tends to zero as $t \to \rho_0/\beta$, where $\rho_0 = \sqrt{r^2 + z^2}$. Then it follows immediately that $I(Q(\tau);\ \tau_4, \tau') \to 0$ and $I(Q(\tau);\ \tau_2, \tau') \to 0$ as $t \to \rho_0/\beta$. Also $\{I(Q(\tau);\ \tau', \tau_1') - I(Q(\tau);\ \tau', \tau_1)\} \to 0$ as $t \to \rho_0/\beta$, where $\tau_1' \equiv l_1 \times I$ and $\tau_1 \equiv l_2 \times I$ are the values of τ which correspond to the points on the right of τ'. From this it follows that the displacement is continuous across the characteristic surface $\rho_0 = \beta t$, showing that the displacement due to the first arrival of the characteristic surface $r^2 + z^2 = \beta^2 t^2$ is zero.

SURFACE DISPLACEMENT: In this section surface displacement has been determined numerically for a particular type of nonuniformly moving surface. We consider a decelerating ring source whose radius $h(\tau)$ at any time τ is assumed to be $h(\tau) = A\tau^{1/2}$. The displacement at any point $(r, 0)$ at the time of observation t is determined.

According to the position of the source the following three possible cases are considered.

i) Radius $h(\tau)$ of the ring coinciding with the rim of the conical wave front and moving with it so that $\beta t < h(t)$.

ii) $\beta t < h(t) < r_*$

iii) $h(t) < \beta t$.

To determine the displacement on the free surface, we put $z = 0$ in the function $Q(\tau)$ of equation (24) and the variable of integration τ is changed to T, by substituting $T = \tau/t$.

$Q(T)$ is then obtained in the form

$$Q(Tt)=R(T)=\frac{\frac{r^2}{\beta^2 t^2}+\frac{A^2}{\beta^2 t}T-(1-T)^2}{\left[\left\{\left(\frac{r}{\beta t}+\frac{A}{\beta}\sqrt{\frac{T}{t}}\right)^2-(1-T)^2\right\}\left\{(1-T)^2-\left(\frac{r}{\beta t}-\frac{A}{\beta}\sqrt{\frac{T}{t}}\right)^2\right\}\right]^{1/2}}$$

on a close examination of the regions of integration as shown in Fig.1, the displacement v, in case of (i) is given by

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\ I(R(T)\ ;\ T_1,\ T_2)\ \text{for}\ o < r < \beta t$$

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\ I(R(T)\ ;\ T_1,\ T_2)\ \text{for}\ \beta t < r < h\ (t)$$

The displacement in case of (ii) is given by

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\ I(R(T)\ ;\ T_1,\ T_2)\ \text{for}\ r < \beta\ t.$$

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\ I(R(T)\ ;\ T_1,\ T_2)\ \text{for}\ \beta t < r < h\ (t)$$

$$\frac{\mu v}{P}=\frac{\beta t}{\pi r}\ I(R(T)\ ;\ T_1,\ T_3)\ \text{for}\ h\ (t) < r < r_*$$

and the displacement in case (iii) is

$$\frac{\mu}{P}v=\frac{\beta t}{\pi r}\ I(R(T);\ T_1,\ T_2)\ \text{for}\ o < r < h\ (t)$$

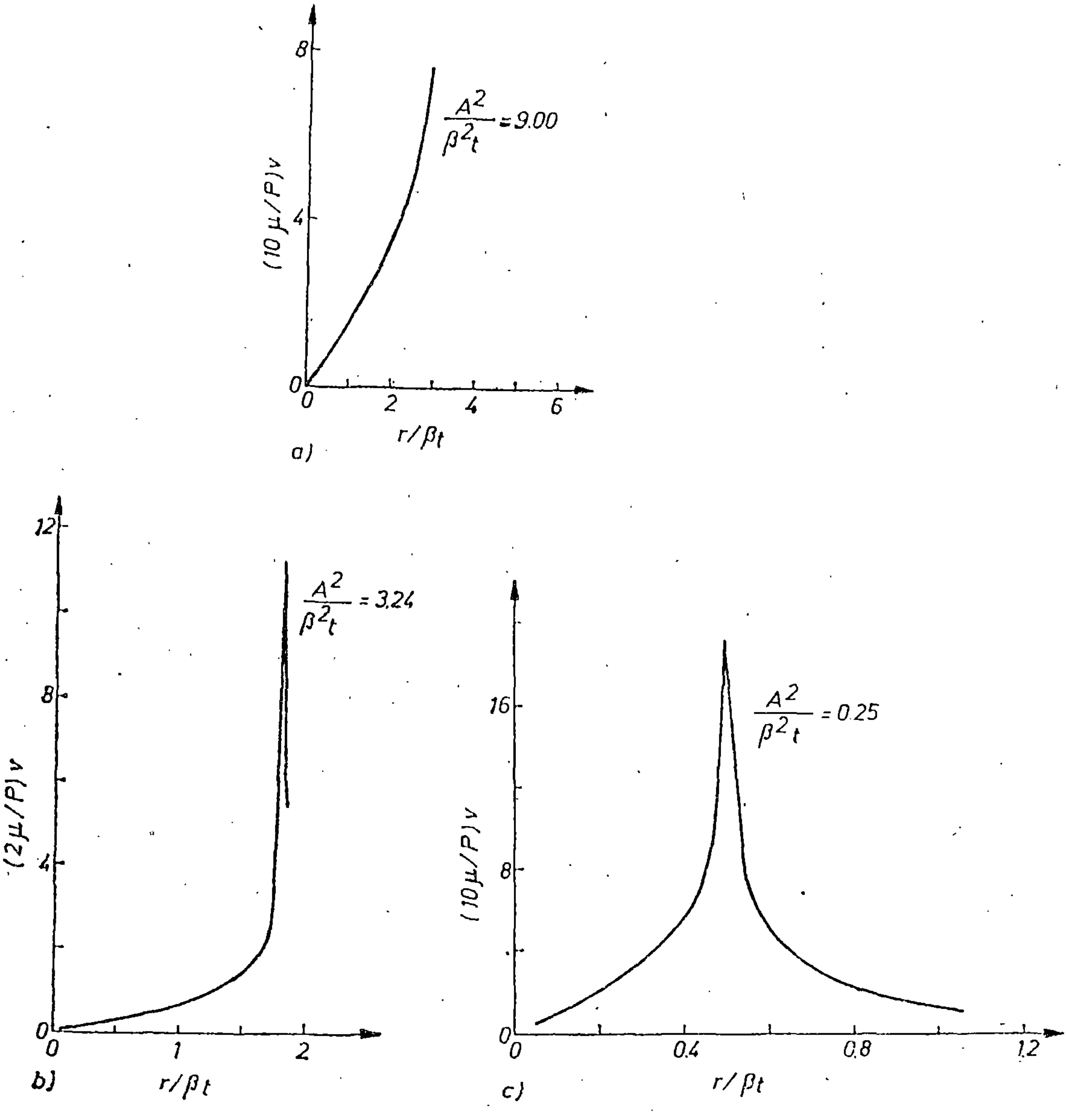

Fig. 3. Graphs showing $(\mu/P)\,v$ versus $(r/\beta t)$ when $z = 0$.

(a), (b), (c) correspond to the cases (i), (ii) and (iii) respectively

$$\frac{\mu}{P} v = \frac{\beta t}{\pi r} I (R (T) ; T_1, T_3) \text{ for } h (t) < r < \beta t$$

$$\frac{\mu}{P} v = \frac{\beta t}{\pi r} I (R (T); T_1, T_3) \text{ for } \beta t < r < r_*$$

where

$$T_1 = 1 - \frac{r}{\beta t} + \frac{1}{2} \frac{A^2}{\beta^2 t} - \frac{1}{2} \frac{A}{\beta \sqrt{t}} \sqrt{\frac{A^2}{\beta^2 t} + 4 \left(1 - \frac{r}{\beta t} \right)}$$

$$T_2 = 1 + \frac{r}{\beta t} + \frac{1}{2} \frac{A^2}{\beta^2 t} - \frac{1}{2} \frac{A}{\beta \sqrt{t}} \sqrt{\frac{A^2}{\beta^2 t} + 4 \left(1 + \frac{r}{\beta t} \right)}$$

$$T_3 = 1 - \frac{r}{\beta t} + \frac{1}{2} \frac{A^2}{\beta^2 t} + \frac{1}{2} \frac{A}{\beta \sqrt{t}} \sqrt{\frac{A^2}{\beta^2 t} + 4 \left(1 - \frac{r}{\beta t} \right)}$$

All the above integrals are numerically evaluated and the graphs are plotted by specifying admissible values of $(r/\beta t)$ against $(\mu /P)v$.

CHAPTER III

MOVING SOURCE PROBLEMS.

Problem 1. Displacement due to a uniformly moving line load over the plane boundary of an inhomogeneous elastic half-space.

Problem 2. Rayleigh waves due to a nonuniformly propagating dip-slip fault.

DISPLACEMENT DUE TO A UNIFORMLY MOVING LINE LOAD OVER THE PLANE BOUNDARY OF AN INHOMOGENEOUS ELASTIC HALF-SPACE.

INTRODUCTION: Since the publication of the classical paper by LAMB (1904) the problem of line and point sources in homogeneous media has attracted the attention of many investigators. But the corresponding problems for inhomogeneous media have not been discussed by many authors as yet. The problem of wave propagation in an inhomogeneous medium is important to geophysicists, because any realistic model of the Earth must take into account the continuous change in the elastic properties of the material in the vertical direction. Since the mathematical treatment of a complicated model is extremely difficult and since the approximation to such a problem does not lead to any worth while solution, so some simplifying assumptions are usually made. WILSON (1942) studied the propagation of surface waves in a semi-infinite medium, assuming the density to be constant and the coefficient of rigidity to be varying exponentially with depth. STONELEY (1934), however, considered the transmission of RAYLEIGH waves in a heterogeneous medium in which the rigidity varies linearly with depth. The field due to a point source in an inhomogeneous isotropic medium in which density is constant but the bulk modulus varies with depth according to the law $\lambda = \lambda_0(1+\epsilon z)^2$ has been considered by SINGH (1967).

In the present paper, considering an elastic medium in which the elastic parameters λ, μ and density ρ vary according to the law $\lambda = \mu = \mu_0 (1 + \epsilon z)^2$ and $\rho = \rho_0(1 + \epsilon z)^2$, the transient problem

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Published in Gerlands Beitr, Leipzig 87 (1978)4, S.317 - 332

for a two-dimensional line load moving with uniform velocity v over the surface of the non-homogeneous semi-infinite medium is studied. The ground motion excited by the moving surface load occurs, for example, from nuclear blasts and from shock waves generated by supersonic aircrafts. These practical problems have been formulated mathematically by a two-dimensional normal line load which is suddenly created at $t = o$ and moves subsequently with uniform velocity along the free surface. The method of solutions involves the use of the integral transform and CAGNIARD's (1962) method as modified by DE HOOP (1959). The application of CAGNIARD's method in the solution of transient problems in inhomogeneous media does not seem to have been discussed earlier.

This steadily moving line load problem, where t varies from $-\infty$ to ∞, has been solved by CHAKRAVARTY and DE (1971) following the method of COLE AND HUTH (1958). Of course, the transient solution for a point load moving over the surface of a homogeneous isotropic half-space had been thoroughly discussed by GAKENHEIMER and MIKLOWITZ (1969). An exact solution of the buried uniformly moving line-source problem has also been obtained by MITRA (1959).

FORMULATION OF THE PROBLEM: The inhomogeneous semi-infinite medium is supposed to occupy the region $z > o$ as shown in Fig.1. The x-axis is taken along the free surface, whereas the z-axis points vertically downwards into the medium. A concentrated line load, which is assumed to originate on the free surface at the origin at time $t = o$, moves with uniform velocity $v(v < \alpha, \beta)$ along the positive direction of the x-axis.

The equations of motion for a non-homogeneous medium in the absence of body forces are

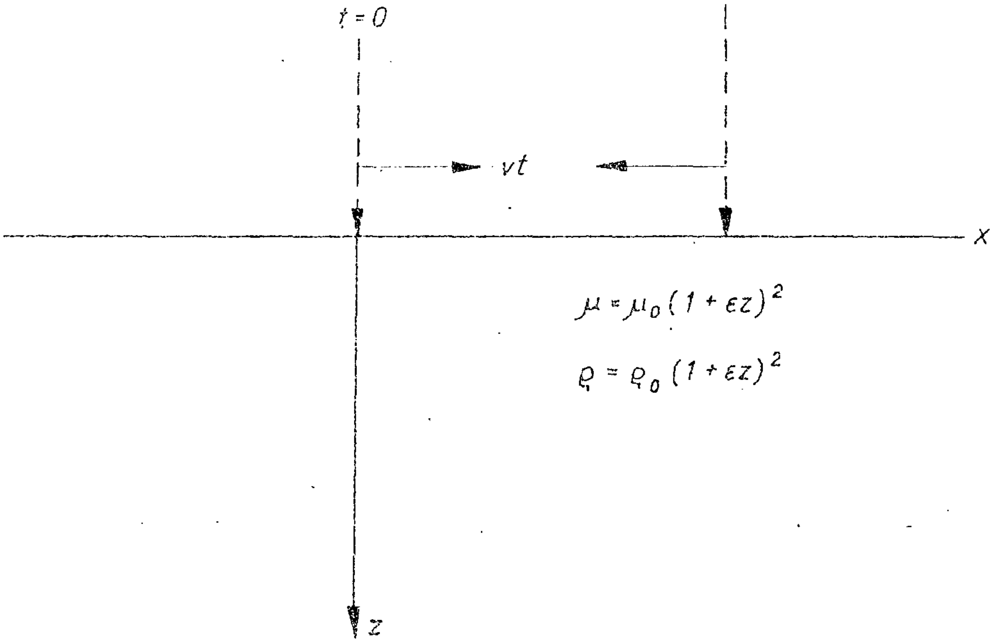

Fig. 1

$$\frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] = \rho \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial w}{\partial z}\right] = \rho \frac{\partial^2 w}{\partial t^2} \tag{2}$$

u and w are the displacement components in the x- and z-directions, λ, μ are LAME's constants and ρ is the density of the medium. It is assumed that

$$\lambda = \mu = \mu_0(1 + \epsilon z)^2, \quad \rho = \rho_0(1 + \epsilon z)^2, \tag{3}$$

such that the velocity of propagation is independent of z. The equations (1) and (2) have to be solved subject to the boundary conditions

$$\begin{aligned} \tau_{xz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 && \text{at } z = 0, \\ \tau_{zz} &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial w}{\partial z} = -P\delta(x - vt) && \text{at } z = 0, t > 0. \end{aligned} \tag{4}$$

$\delta(x - vt)$ is DIRAC's delta function.

FORMAL SOLUTION: In order to solve the equations (1) and (2), we make the substitution

$$U = u(1 + \epsilon z) \text{ and } W = v(1 + \epsilon z). \tag{5}$$

This transforms the equations (1) and (2) into the forms

$$3\,\frac{\partial^2 U}{\partial x^2} + 2\,\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial z}\right) + \frac{\partial^2 U}{\partial z^2} = \frac{\rho_0}{\mu_0}\,\frac{\partial^2 U}{\partial t^2} \tag{6}$$

and

$$\frac{\partial^2 W}{\partial x^2} + 2\,\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial z}\right) + 3\frac{\partial^2 W}{\partial z^2} = \frac{\rho_0}{\mu_0}\,\frac{\partial^2 W}{\partial t^2}\,. \tag{7}$$

We introduce the FOURIER transform over x defined by

$$f_1(p,z,t) = \int_{-\infty}^{\infty} f(x,z,t)e^{ipx}\,dp$$

and then take the LAPLACE transform over t defined by

$$\bar{f}_1(p,z,s) = \int_{0}^{\infty} f_1(p,z,t)e^{-st}\,dt.$$

The equations (6) and (7) after these transformations take the forms

$$\left(\frac{d^2}{dz^2} - 3k_1^2\right)\bar{U}_1 = 2ip\,\frac{d\bar{W}_1}{dz} \tag{8}$$

and

$$\left(3\,\frac{d^2}{dz^2} - k_2^2\right)\bar{W}_1 = 2ip\,\frac{d\bar{U}_1}{dz}\,, \tag{9}$$

where

$$k_1^2 = p^2 + \frac{s^2}{\alpha^2}\,, \quad k_2^2 = p^2 + \frac{s^2}{\beta^2}\,, \quad \alpha^2 = 3\beta^2 = \frac{\lambda+2\mu}{\rho}$$

Using the conditions that the displacement components vanish as z approaches ∞ , the solutions of (8) and (9) are

$$\bar{U}_1 = Ae^{-k_1 z} + Be^{-k_2 z} , \tag{10}$$

$$\bar{W}_1 = \frac{1}{ip}\left(Ak_1 e^{-k_1 z} + \frac{p^2 B}{k_2} e^{-k_2 z}\right). \tag{11}$$

Using (5), the above equations become

$$\bar{u}_1 = \frac{1}{1+\epsilon z}\left(Ae^{-k_1 z} + Be^{-k_2 z}\right) \tag{12}$$

and

$$\bar{w}_1 = \frac{1}{ip(1+\epsilon z)}\left(Ak_1 e^{-k_1 z} + \frac{p^2 B}{k_2} e^{-k_2 z}\right). \tag{13}$$

A and B have to be determined from the conditions

$$\frac{d\bar{u}_1}{dz} = ip\bar{w}_1 \text{ and } -ip\mu_o \bar{u}_1 + 3\mu_o \frac{d\bar{w}_1}{dz} = \frac{P}{ipv-s} \text{ on } z = o,$$

which are obtained by taking first the FOURIER and then the LAPLACE transform on both sides of equations (4). It is found that

$$A = \frac{ipP(\epsilon k_2 + k_2^2 + p^2)}{\mu_o(ipv-s)(k_1k_2-p^2)f(p)} , \quad B = -\frac{ipP(2k_1k_2 + \epsilon k_2)}{\mu_o(ipv-s)(k_1k_2-p^2)f(p)} ,$$

where

$$f(p) = (p^2 - 3k_1k_2) - 3\epsilon(k_1 + k_2) - 3\epsilon^2 .$$

Substituting the values of A and B in (12) and (13) and taking FOURIER inversion, we get

$$\bar{u} = \frac{iP}{2\pi\mu_0(1+\epsilon z)} \int_{-\infty}^{\infty} \frac{p(\epsilon k_2 + k_2^2 + p^2)e^{-k_1 z} - pk_2(2k_1+\epsilon)e^{-k_2 z}}{(ipv - s)(k_1 k_2 - p^2)f(p)} e^{-ipx}\, dp \qquad (14)$$

and

$$\bar{w} = \frac{P}{2\pi\mu_0(1+\epsilon z)} \int_{-\infty}^{\infty} \frac{k_1(\epsilon k_2 + k_2^2 + p^2)e^{-k_1 z} - p^2(2k_1+\epsilon)e^{-k_2 z}}{(ipv - s)(k_1 k_2 - p^2)f(p)} e^{-ipx}\, dp \qquad (15)$$

LAPLACE INVERSIONS: We assume

$$\bar{u} = \frac{P}{2\pi\mu_0(1+\epsilon z)}(I_1 - I_2), \qquad (16)$$

where
$$I_1 = \int_{-\infty}^{\infty} \frac{ip(k_2 + k_2^2 + p^2)e^{-k_1 z - ipx}}{(ipv-s)(k_1 k_2 - p^2)f(p)}\, dp$$

and
$$I_2 = \int_{-\infty}^{\infty} \frac{ipk_2(2k_1+\epsilon)\, e^{-k_2 z - ipx}}{(ipv - s)(k_1 k_2 - p^2)f(p)}\, dp.$$

To find the inversions of I_1 and I_2, we adopt CAGNIARD's technique as modified by DEHOOP (1959). Accordingly, we put $p = -sh$ in I_1, which then reduces to the form

$$I_1 = \int_{-\infty}^{\infty} \frac{ih(\epsilon k_2' + sk_2'^2 + sh^2)e^{-s(k_1' z - ihx)}}{(ihv + 1)\phi(h)\psi(s,h)}\, dh, \qquad (17)$$

where

$$k_1'^2 = h^2 + \frac{1}{\alpha^2}, \quad k_2'^2 = h^2 + \frac{1}{\beta^2},$$

$$\phi(h) = (3k_1'k_2' - h^2)(h^2 - k_1'k_2') = -(h^4 - 4h^2 k_1'k_2' + 3k_1'k_2').$$

It has to be noted that $\phi(h)=0$ is the RAYLEIGH wave velocity equation corresponding to the homogeneous medium with $\lambda = \mu_0 = \mu$ and

$$\psi(s,h)=s^2 + \frac{3s\epsilon(k_1' + k_2')}{3k_1'k_2' - h^2} + \frac{3\epsilon^2}{3k_1'k_2' - h^2} = (s-\epsilon m_1)(s-\epsilon m_2),$$

where $$m_{1,2} = \frac{-3(k_1' + k_2') \pm [9(k_1'-k_2')^2+12h^2]^{1/2}}{2(3k_1'k_2' - h^2)};$$

m_1 and m_2 are both negative. Breaking up $(\epsilon k_2' + sk_2'^2+sh^2)/\psi(s,h)$ into partial fractions, the equation (17) can be written as

$$I_1 = \int_{-\infty}^{\infty} \frac{ih(1-ihv)e^{-s(k_1'z-ihx)}}{(1 + h^2v^2)\phi(h)} \left(\frac{M}{s-\epsilon m_1} + \frac{N}{s-\epsilon m_2} \right) dh; \qquad (18)$$

similarly,

$$I_2 = \int_{-\infty}^{\infty} \frac{ih(1-ihv)e^{-s(k_2'z-ihx)}}{(1 + h^2v^2)\phi(h)} \left(\frac{S}{s-\epsilon m_1} + \frac{T}{s-\epsilon m_2} \right) dh. \qquad (19)$$

In (18) and (19),

$$M \equiv M(h) = \frac{k_2'+m_1(k_2'^2+h^2)}{m_1 - m_2}, \quad N \equiv N(h) = \frac{k_2'+m_2(k_2'^2+h^2)}{m_2 - m_1}$$

and $$S \equiv S(h) = \frac{k_2'+2m_1k_1'k_2'}{m_1 - m_2}, \quad T \equiv T(h) = \frac{k_2'+2m_2k_1'k_2'}{m_2 - m_1}.$$

First let us consider the integral

$$\int_{-\infty}^{\infty} \frac{ih(1-ihv)e^{-s(k_1'z-ihx)}}{(1 + h^2v^2)\phi(h)} \frac{M}{s-\epsilon m_1} dh, \qquad (20)$$

which occurs in equation (18). In this integral the path of integration with respect to h, which is the real axis, is deformed in such a way that

$$z\left(h^2 + \frac{1}{\alpha^2} \right)^{1/2} - ihx = q,$$

where q is real and positive. The deformed path of integration is the branch Γ_1 (Fig.2) of a hyperbola, whose equation is

$$h = \frac{iqx \pm z\left(q^2 - \frac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2 + z^2}, \quad \frac{(x^2 + z^2)^{1/2}}{\alpha} < q < \infty.$$

In the course of deformation of the path of integration it is essential to know all the singularities of $M/[(1 + h^2v^2)\phi(h)]$ in the h-plane, which are the poles at $\pm(i/u)$, $\pm(i/v_R)$ and the branch points at $\pm(i/\alpha)$ and $\pm(i/\beta)$, where v_R is the RAYLEIGH WAVE velocity corresponding to the homogeneous medium when $\lambda = \mu = \mu_0$.

Since the hyperbolic path Γ_1 does not cross any of the singularities during its deformation, it is possible by virtue of CAUCHY's theorem and JORDAN'S lemma to replace the integration along the real h-axis by an integration along the hyperbolic path Γ_1. We write

$$h_+ = \frac{iqx+z\left(q^2 - \frac{x^2+z^2}{\alpha^2} \right)^{1/2}}{x^2 + z^2}, \quad h_- = \frac{iqx-z\left(q^2 - \frac{x^2+z^2}{\alpha^2} \right)}{x^2 + z^2};$$

then

$$\frac{dh_\pm}{dq} = \frac{ix\left(q^2 - \frac{x^2+z^2}{\alpha^2} \right)^{1/2} \pm qz}{(x^2 + z^2)\left(q^2 - \frac{x^2 + z^2}{\alpha^2} \right)^{1/2}}$$, writing M_+ and M_- for $M(h_+)$ and $M(h_-)$

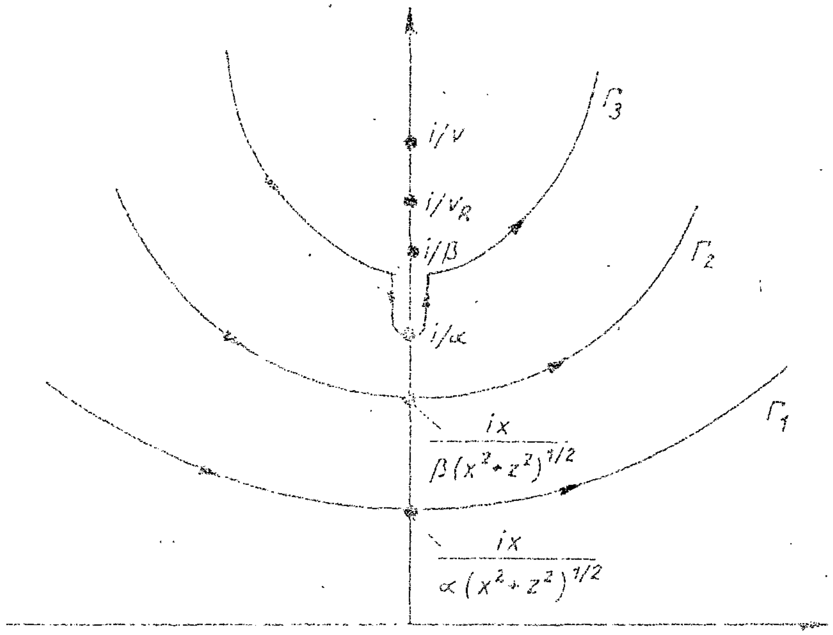
Γ_3
i/v
i/v_R
i/β
Γ_2
i/α
Γ_1
$\frac{ix}{\beta(x^2+z^2)^{1/2}}$
$\frac{ix}{\alpha(x^2+z^2)^{1/2}}$

Fig. 2

and using the facts that

$$h_- = -\bar{h}_+, \quad \frac{dh_-}{dq} = -\left(\frac{d\bar{h}_+}{dq}\right), \quad m_{1-} = \bar{m}_{1+}, \quad M_- = \bar{M}_+,$$

where $\bar{h}$ is the complex conjugate of h, the expression (20) takes the form

$$\int_{\frac{1}{\alpha}\cdot(x^2+z^2)^{1/2}}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+M_+}{(1+h_+^2v^2)\phi(h_+)}\;\frac{e^{-sq}}{s-\epsilon m_{1+}}\;\frac{dh_+}{dq}\right]dq\; +$$

$$+\int_{\frac{1}{\alpha}\cdot(x^2+z^2)^{1/2}}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2M_+}{(1+h_+^2v^2)\phi(h_+)}\;\frac{e^{-sq}}{s-\epsilon m_{1+}}\;\frac{dh_\pm}{dq}\right]dq.$$

Using the convolution theorem, the LAPLACE inversion of the above integral is

$$\int_0^t d\tau \int_{t_\alpha}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+M_+e^{\epsilon(t-\tau)m_{1+}}}{(1+h_+^2v^2)\phi(h_+)}\;\frac{dh_\pm}{dq}\right]\delta(\tau-q)dq\; +$$

$$+\int_0^t d\tau \int_{t_\alpha}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2M_+e^{\epsilon(t-\tau)m_{1+}}}{(1+h_+^2v^2)\phi(h_+)}\;\frac{dh_\pm}{dq}\right]\delta(\tau-q)dq,$$

where $t_\alpha = (x^2+z^2)^{1/2}/\alpha$ is the arrival time of P-waves. By use of the properties of the δ-function, the above integrals can be written as

$$H(t-t_\alpha)\left[\int_{t_\alpha}^{t} -2\,\mathrm{Im}\left\{\frac{h_+ M_+\, e^{\epsilon(t-\tau)m_{1+}}}{(1+h_+^2v^2)\phi(h_+)}\;\frac{dh_+}{d\tau}\right\}d\tau\;+\right.$$

$$\left.+\int_{t_\alpha}^{t} 2v\,\mathrm{Re}\left\{\frac{h_+^2 M_+\, e^{\epsilon(t-\tau)m_{1+}}}{(1+h_+^2v^2)\phi(h_+)}\;\frac{dh_+}{d\tau}\right\}d\tau\right]. \qquad (21)$$

It should be noted that in the integrand of the above integral q has been replaced by every where.

In a similar manner the LAPLACE inversion of the other part of I_1 in (18) can be determined. It is found to be a similar expression as the expression in (21) except that M_+ and m_{1+} have to be replaced by N_+ and m_{2+} respectively. Thus the LAPLACE inversion of I_1 is

$$H(t-t_\alpha)\int_{t_\alpha}^{t} -2\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left\{M_+\, e^{\epsilon(t-\tau)m_{1+}} + N_+\, e^{\epsilon(t-\tau)m_{2+}}\;\frac{dh_+}{d\tau}\right\}\right]d\tau\;+$$

$$+\,H(t-t_\alpha)\int_{t_\alpha}^{t} 2v\,\mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left\{M_+\, e^{\epsilon(t-\tau)m_{1+}} + N_+\, e^{\epsilon(t-\tau)m_{2+}}\;\frac{dh_+}{d\tau}\right\}\right]d\tau. \qquad (22)$$

Next we shall calulate the LAPLACE inversion of I_2 that occurs in (19). As before here also define

$$z\left(h^2+\frac{1}{\beta^2}\right)^{1/2} - ihx = r\,,$$

where r is real and positive.

So,

$$h_{\pm} = \frac{irx \pm z\left(r^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2} . \tag{23}$$

Case 1: A path along which r is real and non-negative is the hyperbolic path Γ_2 (Fig.2) represented parmetrically by the above equation with $r > (x^2+z^2)^{1/2}/\beta$, provided the path where it cuts the imaginary axis, viz. $h = ix/\beta(x^2+z^2)^{1/2}$, lies below the branch point i/α, which occurs when $x < \beta z/(\alpha^2-\beta^2)^{1/2}$. In this case the path Γ_2 (Fig.2) does not cross any of the singularities during the deformation. Following the same procedure as that done in case of I_1, the LAPLACE inversion of I_2 is found to be

$$H(t-t_\beta)\int_{t_\beta}^{t} -2\,\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left\{S_+\,e^{\epsilon(t-\tau)m_{1+}} + T_+\,e^{\epsilon(t-\tau)m_{2+}}\right\}\frac{dh_+}{d\tau}\right]d\tau\ +$$

$$+\ H(t-t_\beta)\int_{t_\beta}^{t} 2v\,\mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left\{S_+\,e^{\epsilon(t-\tau)m_{1+}} + T_+\,e^{\epsilon(t-\tau)m_{2+}}\right\}\frac{dh_+}{d\tau}\right]d\tau, \tag{24}$$

where $t_\beta = (x^2+z^2)^{1/2}/\beta$ is the arrival time of S-waves, and h_+ occurring in the above expression is obtained by replacing r by in the expression for h_+ as given in (23).

Case 2: If $x > \beta z/(\alpha^2-\beta^2)^{1/2}$, the point $ix/\beta(x^2+z^2)^{1/2}$ lies above the branch point i/α. Therefore, the path of integration in the h-plane has to be deformed to the path Γ_3 (Fig.2) round the branch point i/α as shown in Fig.2.

We consider the integral

$$\int_{-\infty}^{\infty} \frac{ih(1 - ihv)S\, e^{-s(k_2' z - ihx)}}{(1 + h^2v^2)\phi(h)(s - \epsilon m_1)}\, dh \tag{25}$$

occurring in I_2 of equation (19). Here too we put

$$z\left(h^2 + \frac{1}{\beta^2}\right)^{1/2} - ihx = r.$$

On the two finite straight line portions of the path Γ_3, h is given by

$$h_{\pm} = \pm\eta + \frac{i\left\{rx - z\left(\frac{x^2+z^2}{\beta^2}\right)^{1/2}\right\}}{x^2 + z^2},$$

where finally η should be made to tend to zero, and on the remaining portions of the path of Γ_3,

$$h_{\pm} = \frac{irx \pm z\left(r^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2 + z^2}.$$

On the straight line portions of the path Γ_3, r varies from $r = t_{\alpha\beta}$ to $r = t_\beta$, where $t_{\alpha\beta} = x/\alpha + z(1/\beta^2 - 1/\alpha^2)^{1/2}$ is the arrival time of PS-waves. The expression in (25) can then be written in the form

$$\int_{t_{\alpha\beta}}^{t_\beta} \left[\frac{ih_+(1-ih_+v)S_+}{(1+h_+^2v^2)\phi(h_+)(s-\epsilon m_{1+})}\,\frac{dh_+}{dr} - \frac{ih_-(1-ih_-v)}{(1+h_-^2v^2)\phi(h_-)(s-\epsilon m_{1-})}\,\frac{dh_-}{dr}\right] e^{-sr}\, dr\, +$$

$$+ \int_{t_\beta}^{\infty} \left[\frac{ih_+(1-ih_+v)S_+}{(1+h_+^2v^2)\phi(h_+(s-\epsilon m_{1+})}\,\frac{dh_+}{dr} - \frac{ih_-(1-ih_-v)}{(1+h_-^2v^2)\phi(h_-)(s-\epsilon m_{1-})}\,\frac{dh_-}{dr}\right] e^{-sr}\, dr. \tag{26}$$

Noting that $h_- = -\bar{h}_+$, $dh_-/dr = -(d\bar{h}_+/dr)$ and $S_- = \bar{S}_+$ on the path Γ_3, the expression (26) takes the form

$$\int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\left[\frac{h_+S_+}{(1+h_+^2v^2)\phi(h_+)}\,\frac{e^{-sr}}{s-\epsilon m_{1+}}\,\frac{dh_\pm}{dr}\right]dr + \int_{t_{\alpha\beta}}^{t_\beta} 2v\,\mathrm{Re}\left[\frac{h_+^2S_+}{(1+h_+^2v^2)\phi(h_+)}\right. \times$$

$$\times \left.\frac{e^{-sr}}{s-\epsilon m_{1+}}\,\frac{dh_\pm}{dr}\right]dr + \int_{t_\beta}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+S_+}{(1+h_+^2v^2)\phi(h_+)}\,\frac{e^{-sr}}{s-\epsilon m_{1+}}\,\frac{dh_\pm}{dr}\right]dr +$$

$$+ \int_{t_\beta}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2S_+}{(1+h_+^2v^2)\phi(h_+)}\,\frac{e^{-sr}}{s-\epsilon m_{1+}}\,\frac{dh_\pm}{dr}\right]dr. \tag{27}$$

To transform the other integral of I_2 occurring in (19), a similar procedure is applied, and finally I_2 in (19) takes the following form:

$$I_2 = \int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\left[\frac{h_+e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_\pm}{dr}\right]dr +$$

$$+ \int_{t_\beta}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_\pm}{dr}\right]dr +$$

$$+ \int_{t_{\alpha\beta}}^{t_\beta} 2v\,\mathrm{Re}\left[\frac{h_+^2e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_\pm}{dr}\right]dr +$$

$$+ \int_{t_\beta}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_\pm}{dr}\right]dr. \tag{28}$$

It must be remembered that the value of h_+ when r lies in $[t_{\alpha\beta}, t_\beta]$ has to be taken as

$$h_+ = \frac{i\left[rx - z\left(\frac{x^2+z^2}{\beta^2} - r^2\right)^{1/2}\right]}{x^2+z^2},$$

and for r lying in $[t_\beta, \infty)$,

$$h_+ = \frac{irx + z\left(r^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}.$$

Next the LAPLACE inversion of I_2 in (28) has to be calculated. By applying the convolution theorem, the LAPLACE inversion of the first integral of I_2 in (28) is found to be

$$\int_0^t d\tau \int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\left[\frac{h_+ e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]\delta(\tau - r)\,dr =$$

$$= \int_0^t d\tau \int_{t_{\alpha\beta}}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+ e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]\delta(\tau - r)\,dr -$$

$$- \int_0^t d\tau \int_{t_\beta}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+ e^{-sr}}{(1+h_+^2v^2)\phi(h_+)}\left(\frac{S_+}{s-\epsilon m_{1+}} + \frac{T_+}{s-\epsilon m_{2+}}\right)\frac{dh_+}{dr}\right]\delta(\tau - r)\,dr,$$

and it takes the following form when the δ-function property is used:

$$H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -2\,\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}} + T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right]d\tau -$$

$$-H(t-t_\beta)\int_{t_\beta}^{t} -2\,\mathrm{Im}\left[\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}} + T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right]d\tau. \tag{29}$$

It can be shown that the last term of (29) is cancelled with the LAPLACE inversion of the second integral in (28).

Similarly, the LAPLACE inversion of the other integrals of (29) can be determined, and finally , after simplification, we get the LAPLACE inversion of I_2 as

$$H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -2\ \mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right]d\tau -$$

$$- H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} 2v\ \mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right]d\tau\ . \qquad (30)$$

Combining the results of the inverse LAPLACE transforms of I_1 and I_2 from (22) and (24) it follows that

$$u(x,z,t) = \frac{P}{\pi\mu_0(1+\epsilon z)}\ \mathrm{X}$$

$$\mathrm{X}\left[H(t-t_\alpha)\int_{t_\alpha}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-\tau)m_{1+}}+N_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\ +\right.$$

$$+ H(t-t_\alpha)\int_{t_\alpha}^{t} v\ \mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-\tau)m_{1+}}+N_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\ -$$

$$- H(t-t_\beta)\int_{t_\beta}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\ -$$

$$\left.- H(t-t_\beta)\int_{t_\beta}^{t} v\ \mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\right] \qquad (31)$$

for $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, and when $x > \beta z/(\alpha^2 - \beta^2)^{1/2}$, from (22) and (30) it follows that

$$u(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)} \times$$

$$\times\Bigg[H(t-t_\alpha)\int_{t_\alpha}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}(M_+e^{\epsilon(t-\tau)m_{1+}} + N_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+H(t-t_\alpha)\int_{t_\alpha}^{t} v\ \mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}(M_+e^{\epsilon(t-\tau)m_{1+}} + N_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_\pm}{d\tau}\right\}d\tau -$$

$$-H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}(S_+e^{\epsilon(t-\tau)m_{1+}} + T_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_\pm}{d\tau}\right\}d\tau -$$

$$-H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} v\ \mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}(S_+e^{\epsilon(t-\tau)m_{1+}} + T_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_\pm}{d\tau}\right\}d\tau\Bigg]. \tag{32}$$

Carrying on a similar procedure as done for the evaluation of the displacement along the x-direction, the expression for the displacement along the z-direction can also be determined from (13) and is found to be equal to

$$w(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)} \times$$

$$\times\Bigg[H(t-t_\alpha)\int_{t_\alpha}^{t} \mathrm{Re}\left\{\frac{k'_{1+}}{(1+h_+^2v^2)\phi(h_+)}(M_+e^{\epsilon(t-\tau)m_{1+}} + N_+e^{\epsilon(t-\tau)m_{2+}})\frac{dh_\pm}{d\tau}\right\}d\tau +$$

$$+ H(t-t_\alpha)\int_{t_\alpha}^{t} v\ \mathrm{Im}\left\{\frac{k'_{1\pm}h_+}{(1+h_+^2v^2)\phi(h_+)}(M\,e^{\epsilon(t-\tau)m_{1+}} + N\,e^{\epsilon(t-\tau)m_{2+}})\frac{dh_\pm}{d\tau}\right\}d\tau +$$

$$+ H(t-t_\beta)\int_{t_\beta}^{t}\mathrm{Re}\left\{\frac{h_+^2}{k'_{2+}(1+h_+^2v^2)\phi(h_+)}\left(S_+\, e^{\epsilon(t-\tau)m_{1+}} + T_+\, e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\ +$$

$$+ H(t-t_\beta)\int_{t_\beta}^{t} v\ \mathrm{Im}\left\{\frac{h_+^3}{k'_{2+}(1+h_+^2\, v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\Bigg]$$

(33)

for $x < \beta z/(\alpha^2-\beta^2)^{1/2}$, and if $x > \beta z/(\alpha^2-\beta^2)^{1/2}$,

$$w(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)}\ \mathrm{X}$$

$$\mathrm{X}\Bigg[H(t-t_\alpha)\int_{t_\alpha}^{t}\mathrm{Re}\left\{\frac{k'_{1+}}{(1+h_+^2v^2)\phi(h_+)}\left(M_+ e^{\epsilon(t-\tau)m_{1+}} + N_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\ +$$

$$+ H(t-t_\alpha)\int_{t_\alpha}^{t} v\ \mathrm{Im}\left\{\frac{k'_{1+}h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M_+\ e^{\epsilon(t-\tau)m_{1+}} + N_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\ +$$

$$+ H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t}\mathrm{Re}\left\{\frac{h_+^2}{k'_{2+}(1+h_+^2v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\ +$$

$$+ H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} v\ \mathrm{Im}\left\{\frac{h_+^3}{k'_{2+}(1+h_+^2v^2)\phi(h_+)}\left(S_+ e^{\epsilon(t-\tau)m_{1+}} + T_+ e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}d\tau\Bigg]$$

(34)

where $k'_{1+} = (h_+^2 + \frac{1}{\alpha^2})^{1/2}$ and $k'_{2+} = (h_+^2 + \frac{1}{\beta^2})^{1/2}$.

It should be remembered that in the first two integrals of the equations (31), (32), (33) and (34)

$$h_+ = \frac{i\tau x + z\left(\tau^2 - \frac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2+z^2}, \quad t_\alpha \leqq \tau \leqq t,$$

and in the last two integrals of those equations

$$h_+ = \frac{i\tau x + z\left(\tau^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}, \quad t_\beta \leqq \tau \leqq t,$$

where as in the last two integrals of (32) and (34)

$$h_+ = \frac{i\left[\tau x - z\left(\frac{x^2+z^2}{\beta^2} - \tau^2\right)^{1/2}\right]}{x^2+z^2}, t_{\alpha\beta} \leqq \tau \leqq t_\beta .$$

WAVE FRONT EXPANSION: The wave forms of the solutions given in (31) to (34) are evaluated by approximate estimation of the above integrals in the neighbourhood of the time of the first arrival of the different waves. To facilitate this evaluation we put $= A + a$, where A is the lower limit of the integrals in question and a varies from 0 to t-A. Then when $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, from (31) we get

$$u(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)} \times$$

$$\times\Bigg[H(t-t_\alpha)\int_o^{t-t_\alpha} -\operatorname{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-t_\alpha-a)\,m_{1+}} +\right.\right.$$

$$\left.\left. +N_+e^{\epsilon(t-t_\alpha-a)\,m_{2+}}\right)\frac{dh_+}{da}\right\}da + H(t-t_\alpha)\int_o^{t-t_\alpha} v\operatorname{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-t_\alpha a)\,m_{1+}} +\right.\right.$$

$$\left.\left. + N_+ e^{\epsilon(t-t_\alpha-a)} \frac{dh}{da}\right\}da +$$

$$+H(t-t_\beta)\int_o^{t-t_\beta}\operatorname{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-t_\beta-a)\,m_{1+}} +T_+e^{\epsilon(t-t_\beta-a)\,m_{2+}}\right)\frac{dh_+}{da}\right\}da +$$

$$+H(t-t_\beta)\int_o^{t-t_\beta} -v\operatorname{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-t_\beta-a)\,m_{1+}} +T_+e^{\epsilon(t-t_\beta-a)\,m_{2+}}\right)\frac{dh_+}{da}\right\}da\Bigg].$$

(35)

For $x > \beta z/(\alpha^2-\beta^2)^{1/2}$,

$$u(x,z,t) = \frac{P}{\pi\mu_o(1+\epsilon z)} \times$$

$$\times\Bigg[H(t-t_\alpha)\int_o^{t-t_\alpha} -\operatorname{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-t_\alpha-a)\,m_{1+}} +N_+e^{\epsilon(t-t_\alpha-a)\,m_{2+}}\right)\frac{dh_+}{da}\right\}da +$$

$$+H(t-t_\alpha)\int_o^{t-t_\alpha} v\operatorname{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(M_+e^{\epsilon(t-t_\alpha-a)\,m_{1+}} +N_+e^{\epsilon(t-t_\alpha-a)\,m_{2+}}\right)\frac{dh_+}{da}\right\}da +$$

$$+H(t-t_{\alpha\beta})\int_0^{t-t_{\alpha\beta}}\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-t_{\alpha\beta}-a)\,m_{1+}}+T_+e^{\epsilon(t-t_{\alpha\beta}-a)\,m_{2+}}\right)\frac{dh_+}{da}\right\}da\;+$$

$$+H(t-t_{\alpha\beta})\int_0^{t-t_{\alpha\beta}}-v\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\phi(h_+)}\left(S_+e^{\epsilon(t-t_{\alpha\beta}-a)\,m_{1+}}+T_+e^{\epsilon(t-t_{\alpha\beta}-a)\,m_{2+}}\right)\frac{dh_+}{da}\right\}da\Bigg].$$

$$(36)$$

A similar type of expressions for $w(x,z,t)$ can be written by substituting $\tau = A + a$ in the equations (33) and (34). For approximate evaluation of the integrals (35) and (36) just after the arrival of the corresponding wave fronts it has to be noted that

$$e^{\epsilon(t-A-a)\,m_{1+}} \to 1,\quad e^{\epsilon(t-A-a)\,m_{2+}} \to 1 \text{ and } a \to 0 \text{ as } t \to A,$$

where A is the arrival time of a typical wave front. So, when $A = t_\alpha$, using the facts that

$$h_+ \to \frac{ix}{\alpha(x^2+z^2)^{1/2}},\quad h_+^2 \to \frac{-x^2}{\alpha^2(x^2+z^2)}$$

and

$$\frac{(M_+ + N_+)}{(1+h_+^2v^2)\phi(h_+)} \to$$

$$\to \frac{(x^2+z^2)^2\,\alpha^4\left\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\right\}}{\left\{x^2(\alpha^2-v^2)+\alpha^2z^2\right\}\left[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4\beta x^2z\left\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\right\}^{1/2}\right]}$$

as $a \to 0$ and that

$$\frac{dh_+}{da} = \frac{\left(h_+^2 + \frac{1}{\alpha^2}\right)^{1/2}}{\left\{2\frac{(x^2+z^2)^{1/2}}{\alpha} + a\right\}^{1/2}} \; \frac{1}{a^{1/2}} ,$$

the first two integrals of the equations (35) and (36) just after the arrival of P-waves can approximately be evaluated to the form

$$u(x,z,t) = -\frac{\sqrt{2}PH(t-t_\alpha)xz\alpha^{3/2}(x^2+z^2)^{1/4}\left\{vx+\alpha(x^2+z^2)^{1/2}\right\}\left\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\right\}(t-t_\alpha)^{1/2}}{\pi/\mu_0(1+\epsilon z)\; x^2(\alpha^2-v^2)+\alpha^2z^2\; \left[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4x^2z\beta \times \times\left\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\right\}^{1/2}\right]}.$$

Similarly, the approximate value of w just after the arrival of P-waves is given by

$$w(x,z,t) = \frac{\sqrt{2}PH(t-t_\alpha)z^2\alpha^{3/2}(x^2+z^2)^{1/4}\left\{vx+\alpha(x^2+z^2)^{1/2}\right\}\left\{x^2(\alpha^2-2\beta^2)+\alpha^2z^2\right\}(t-t_\alpha)^{1/2}}{\pi/\mu_0(1+\epsilon z)\left\{x^2(\alpha^2-v^2)+\alpha^2z^2\right\}\left[\beta^2x^2(3z^2-x^2)-3\alpha^2z^2(x^2+z^2)-4x^2z\beta \times \times\left\{x^2(\alpha^2-\beta^2)+\alpha^2z^2\right\}^{1/2}\right]}.$$

The same method is applied for approximate evaluation of u and w just after the arrival of S-waves. It should be remembered in this case that

$$A = t_\beta, \qquad h_+ = \frac{i\tau x + z\left(\tau^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}, \; t_\beta < \tau < t.$$

The effects of u_1 and w_1 on the displacement components u and v due to S-waves just after their arrival are found to be

$$u_1(x,z,t)= \frac{2\sqrt{2}PH(t-t_\beta)xz^2\alpha\beta^{3/2}(x^2+z^2)^{1/4}\left\{vx+\beta(x^2+z^2)^{1/2}\right\}\left\{\beta^2z^2-x^2(\alpha^2-\beta^2)\right\}^{1/2}(t-t_\beta)^{1/2}}{\pi/\mu_0(1+\epsilon z)\left\{x^2(\beta^2-v^2)+\beta^2z^2\right\}\left[\alpha^2x^2(3z^2-x^2)-3\beta^2z^2(x^2+z^2)-4x^2z\alpha \times \left\{\beta^2z^2-x^2(\alpha^2-\beta^2)\right\}^{1/2}\right]},$$

$$w_1(x,z,t)= -\frac{2\sqrt{2}PH(t-t_\beta)x^2z\alpha\beta^{3/2}(x^2+z^2)^{1/4}\left\{vx+\beta(x^2+z^2)^{1/2}\right\}\left\{\beta^2z^2-x^2(\alpha^2-\beta^2)\right\}^{1/2}(t-t_\beta)^{1/2}}{\pi/\mu_0(1+\epsilon z)\left\{x^2(\beta^2-v^2)+\beta^2z^2\right\}\left[\alpha^2x^2(3z^2-x^2)-3\beta^2z^2(x^2+z^2)-4x^2z\alpha \times \left\{\beta^2z^2-x^2(\alpha^2-\beta^2)\right\}^{1/2}\right]}$$

for $0 < x < \beta z/(\alpha^2-\beta^2)^{1/2}$, and in order to obtain the displacement due to S-wave arrival in the region $x > \beta z/(\alpha^2-\beta^2)^{1/2}$, we write last two integrals of (32) in the form

$$\int_{t_{\alpha\beta}}^{t_\beta}\left[-\mathrm{Im}\left\{\frac{h_+}{(1+v^2h_+^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh}{d\tau}\right\}+\right.$$
$$\left.+v\mathrm{Re}\left\{\frac{h_+^2}{(1+v^2h_+^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}\right]d\tau+$$
$$+\int_{t_\beta}^{t}\left[-\mathrm{Im}\left\{\frac{h_+}{(1+v^2h_+^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}+\right.$$
$$\left.+v\mathrm{Re}\left\{\frac{h_+^2}{(1+v^2h_+^2)\phi(h_+)}\left(S_+e^{\epsilon(t-\tau)m_{1+}}+T_+e^{\epsilon(t-\tau)m_{2+}}\right)\frac{dh_+}{d\tau}\right\}\right]d\tau \quad (37)$$

This is to be remembered that for $t_\beta < \tau < t$

$$h_+ = \frac{1}{x^2+z^2}\left[i\tau x + z\left(\tau^2 - \frac{x^2+z^2}{\beta^2}\right)^{1/2}\right]$$

The displacement due to the first arrival of S-wave is obtained from the last integral of (37) as $t \to t_\beta + 0$ and this is found to be

$$u_1(x,z,t) = \frac{-8\sqrt{2}\, PH(t-t_\beta)\,\alpha^2\beta^{3/2}\,x^3z^3(x^2+z^2)^{1/4}}{\pi\mu_0(1+\epsilon z)\left\{x^2(\beta^2-v^2)+\beta^2z^2\right\}} \times$$

$$\times \frac{\left\{x^2(\alpha^2-\beta^2)-\beta^2z^2\right\}\left\{\beta(x^2+z^2)^{1/2}+vx\right\}(t-t_\beta)^{1/2}}{\left\{x^2\alpha^2(3z^2-x^2)-3z^2\beta^2(x^2+z^2)\right\}^2+16\alpha^2x^4z^2\left\{x^2(\alpha^2-\beta^2)-\beta^2z^2\right\}}$$

Adopting the similar procedure for the last two integrals of (34) we have, for $t \to t_\beta + 0$ in the region $x > \beta z/(\alpha^2-\beta^2)^{1/2}$

$$w_1(x,z,t) = \frac{8\sqrt{2}\, PH(t-t_\beta)\alpha^2\beta^{3/2}\,x^4z^2\,(x^2+z^2)^{1/4}}{\pi\mu_0(1+\epsilon z)\left\{x^2(\beta^2-v^2)+\beta^2z^2\right\}} \times$$

$$\times \frac{\left\{x^2(\alpha^2-\beta^2)-\beta^2z^2\right\}\left\{\beta(x^2+z^2)^{1/2}+vx\right\}(t-t_\beta)^{1/2}}{\left\{x^2\alpha^2(3z^2-x^2)-3z^2\beta^2(x^2+z^2)\right]^2+16\alpha^2x^4z^2\left\{x^2(\alpha^2-\beta^2)-\beta^2z^2\right\}}$$

In the region $x > \beta z/(\alpha^2-\beta^2)^{1/2}$, PS-waves exist and arrive earlier than S-waves.

We approximately calculate the last two integrals of (36), which will give the effect of u_2 on the displacement components u due to PS-waves just after their arrival. In this case

$$A = t_{\alpha\beta},\ h_+ = \frac{i\left[\tau x - z\left(\frac{x^2+z^2}{\beta^2} - \tau^2\right)\right]}{x^2+z^2},\ t_{\alpha\beta} \leq \tau \leq t_\beta\ .$$

Then

$$h_+ \to i\left[\frac{1}{\alpha} + \frac{\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2}}{x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - \frac{z}{\alpha}}\, a\right],$$

$$\frac{S + T}{(1+h_+^2 v^2)\phi(h_+)} \to \frac{i\ 2\sqrt{2}\ \alpha^6\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{3/4} a^{1/2}}{(\alpha^2 - v^2)\left\{\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z\right\}^{1/2}}$$

and

$$\frac{dh_+}{da} \to \frac{i\alpha\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2}}{\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z},$$

where in these expressions terms containing higher order of a are neglected because $a \to 0$ as $t \to t_{\alpha\beta}$, and we get

$$u_2(x,z,t) = \frac{4\sqrt{2}PH(t-t_{\alpha\beta})\alpha^5\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{5/4}}{3\pi/^{\mu} o(1+z)\ (\alpha - v)\left[\alpha x\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)^{1/2} - z\right]^{3/2}}\,(t-t_{\alpha\beta})^{3/2}\ .$$

Similarly, the effect on the displacement component w just after the arrival of PS-waves is given by

$$w_2(x,z,t) = -\frac{4\sqrt{2}\,PH(t-t_{\alpha\beta})\alpha^4\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)^{3/4}}{3\pi\mu_o(1+z\epsilon)(\alpha-v)\left[\alpha x\left(\frac{1}{\beta^2}-\frac{1}{\alpha^2}\right)^{1/2}-z\right]^{3/2}}\,(t-t_{\alpha\beta})^{3/2}$$

We now find out the effects of u_3 and w_3 on the displacement components u and w in the neighbourhood of the point C(Fig.3), where S- and PS-waves arrive at the same time. In this case $t_\beta = t_{\alpha\beta}$ and

$$u_3(x,z,t) = -\frac{4\,PH\,(t-t_{\alpha\beta})\,2^{3/4}\,\alpha^{1/4}\,\beta^{3/2}\,z^{5/2}}{3\pi\mu_o(1+\epsilon z)\,x^{13/4}\,(\alpha-v)}\,(t-t_{\alpha\beta})^{3/4},$$

$$w_3(x,z,t) = \frac{4\,PH\,(t-t_{\alpha\beta})\,2^{3/4}\,\alpha^{1/4}\,\beta^{3/2}\,z^{3/2}}{3\pi\mu_o(1+\epsilon z)\,x^{9/4}\,(\alpha-v)}\,(t-t_{\alpha\beta})^{3/4}.$$

CONCLUDING REMARKS: It is found from the integrals (31) to (34) that the effect of inhomogeneity enters into the expressions for u and v through the factors $e^{\epsilon(t-T)\,m_1}$ and $e^{\epsilon(t-T)\,m_2}$ in the corresponding integrands. So, if these two factors are absent, and that is so if $\epsilon = 0$, a parallel case for a homogeneous medium is obtained.

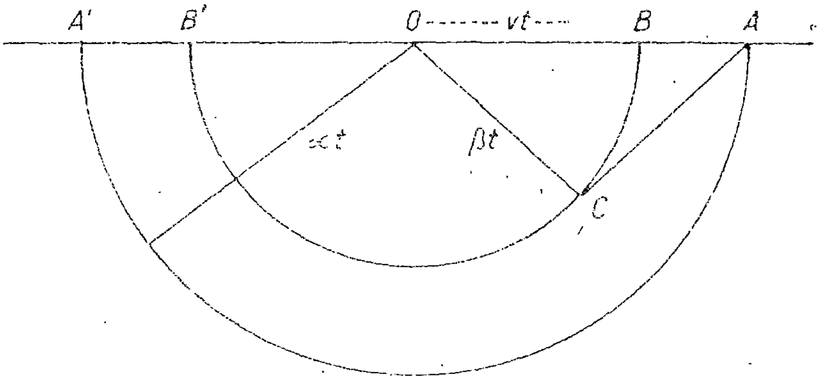

Fig. 3

Also, it is interesting to note that in the neighbourhood of points just after the arrival of the different wave fronts the displacement components are independent of ϵ ,i.e., at any point, the effect of the first arrival of wave fronts on the displacement components is the same for homogeneous as well as for inhomogeneous media. But as time goes on, ϵ occurring in the exponential terms of the integrals (31) to (34) for u and w will have its effect, and consequently, the amplitude of the wave fronts will decay exponentially with time due to inhomogeneity of the medium.

RAYLEIGH WAVES DUE TO NONUNIFORMLY PROPAGATING DIP-SLIP FAULT

INTRODUCTION: The study of dynamic crack propagation is very important in geophysics and in earthquake engineering science. In geophysics it is desirable to formulate the earthquake source in terms of physical parameters and to study the long period waves over a large distance and for a long time. Also in structural engineering it is essential to know the nature of surface waves covering a large distance. At aparticular place the ground motion produced by the earthquake is a very complicated function of the nature of propagation of the crack and the geological properties of the place as well. Most of the known solutions of the moving crack are restricted by the assumption of constant velocity of propagation, which is not in general expected. Mal (1972) discussed Rayleigh wave propagation by a finite fault moving with constant velocity. He represented the shear failure by a jump in the tangential components of displacement across the fault surface. Achenbach and Abo-Zeno (1972) analysed the wave motions generated by a vertical strike slip fault on which motion is opposed by a frictional shear stress and which is assumed to increase linearly with depth. Freund (1973) discussed wave motions as expected in case of a nonuniformly expanding line load. Fossum and Freund (1975) considered a model in which a plane strain shear crack moves from rest at a nonuniform rate under the action of general loading. First motion response of an elastic half space due to a

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Published in Indian J. pure appl. Math., 14(8):994-1006,1983.

nonuniformly moving dislocation by Cagniard De-Hoop technique is determined by Roy (1978). In a recent paper Markenscoff and Clifton (1981) analyzed the motion of an edge dislocation starting from rest and moving thereafter nonuniformly on its slip plane by means of Laplace transform, where the inversion of the transform is accomplished by Cagniard De-Hoop method.

In the present paper an idealised earthquake model is considered. A fault break along a horizontal line at a finite depth below the free surface is assumed to appear suddenly and to move vertically upward with nonuniform motion upto the free surface. A discontinuity in components of displacement across the fault break is prescribed. The displacement components on the free surface due to Rayleigh waves are determined for nonuniform motion of the crack.

To find the solution of the problem the technique developed by Knopoff and Gilbert with appropriate modification is used. The technique is found to be extrememely powerful for tackling such type of boundary value problems. Ghosh (1972) applied the method to show the possibility of attenuation of microseismic waves due to the presence of an upward folding of the ocean bottom into the liquid. Following Knopoff and Gilbert, the moving crack is replaced by a set of virtual sources located at the fault surface HO. The displacement on the free surface is written as the sum of the contribution of these sources with the aid of suitable Green's function representation theorem.

Three particular cases of nonuniform motion of the crack are considered. Horizontal and vertical components of surface displacements due to Rayleigh waves produced by the propagating crack are determined and shown by means of graphs.

In the mathematical and physical structure of wave propagation phenomenon, the model assumed here is although over simplified, yet it brings forth some major features which are usually present in the ground motion.

FORMULATION OF THE PROBLEM AND SOLUTION: The origin of the coordinate frame (x,y) is at the epicentre o. It is assumed that a crack suddenly appearing at the focus H moves vertically upwards upto the free surface o with a nonuniform speed. The length of the crack measured from H at any time t is $h(t)$, which is assumed to be strictly monotonic increasing function of time t.

The Fourier transform $\bar{f}(x,y,w)$ of the function $f(x,y,t)$ is defined by

$$\bar{f}(x,y,w) = \int_{-\infty}^{\infty} f(x,y,t)\, e^{iwt}\, dt \tag{1}$$

Let $G_n^m(x,y|x_0,y_0)$, $(m,n = (x,y))$ be the component of Green's function $\vec{G}^m(x,y|x_0,y_0)$ at the point (x,y) in the direction of n due to a point source of force in m-direction and situated at (x_0,y_0). If now $u(x,y)$ and $v(x,y)$ be the displacement components along x and y directions respectively and $P_{xx}^{(u,v)}$, $P_{xy}^{(u,v)}$ and $P_{yy}^{(u,v)}$ be the stress components then their Fourier transforms defined by (1) satisfy the following differential equations.

$$\frac{\partial \bar{P}_{xx}^{(u,v)}}{\partial x} + \frac{\partial \bar{P}_{xy}^{(u,v)}}{\partial y} + \rho w^2 \bar{u}(x,y) = 0 \tag{2}$$

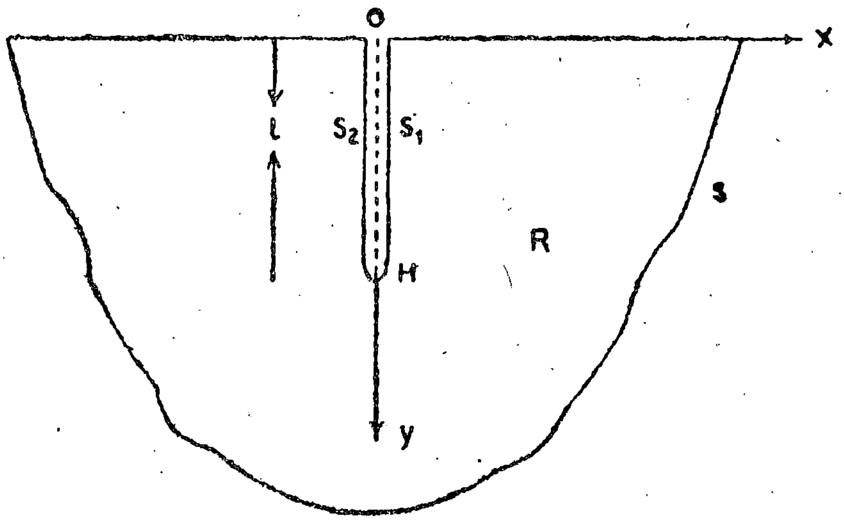


FIG. 1. Geometry of dip-slip fault.

$$\frac{\partial \bar{P}_{xy}^{(u,v)}}{\partial x} + \frac{\partial \bar{P}_{yy}^{(u,v)}}{\partial y} + \rho w^2 \;\bar{\bar{v}}\,(x,y) = 0 \qquad (3)$$

$$\frac{\partial \bar{P}_{xx}}{\partial x}\left[\vec{G}^{x}(x,y|x_o,y_o)\right] + \frac{\partial \bar{P}_{xy}}{\partial y}\left[\vec{G}^{x}(x,y|x_o,y_o)\right] + \rho w^2\;\bar{G}^{x}_{x}(x,y|x_o,y_o) =$$

$$= -\,\delta(x-x_o)\;\delta\,(y-y_o) \qquad (4)$$

$$\frac{\partial \bar{P}_{xy}}{\partial x}\left[\vec{G}^{x}(x,y|x_o,y_o)\right] + \frac{\partial \bar{P}_{yy}}{\partial y}\left[\vec{G}^{x}(x,y|x_o,y_o)\right] + \rho w^2\;\bar{G}^{x}_{y}(x,y|x_o,y_o) = 0$$

$$(5)$$

$$\frac{\partial \bar{P}_{xx}}{\partial x}\left[\vec{G}^{y}(x,y|x_o,y_o)\right] + \frac{\partial \bar{P}_{xy}}{\partial y}\left[\vec{G}^{y}(x,y|x_o,y_o)\right] + \;{}^{2}\;\bar{G}^{y}_{x}(x,y|x_o,y_o) = 0$$

$$(6)$$

$$\frac{\partial \bar{P}_{xy}}{\partial x}\left[\vec{G}^{y}(x,y|x_o,y_o)\right] + \frac{\partial \bar{P}_{yy}}{\partial y}\left[\vec{G}^{y}(x,y|x_o,y_o)\right] + \rho w^2\;\bar{G}^{y}_{y}(x,y|x_o,y_o) =$$

$$= -\,\delta(x-x_o)\;\delta(y-y_o) \qquad (7)$$

where ρ is the density of the material and $\delta(\;)$ is Dirac's delta function.

Multiply equation (2) by $\bar{G}_x^x(x,y|x_o,y_o)$ and (4) by $\bar{u}(x,y)$ and subtract the latter from the former. Also multiply equation (3) by $\bar{G}_y^x(x,y|x_o,y_o)$ and (5) by $\bar{v}(x,y)$ and subtract the latter from the former. These two resulting equations are then added and integrated over the region R to yield the following equation.

$$\iint_R \Big\{ \frac{\partial}{\partial x}\Big[\bar{G}_x^x(x,y|x_o,y_o)\ \bar{P}_{xx}(u,v) + \bar{G}_y^x(x,y|x_o,y_o)\ \bar{P}_{xy}(u,v) -$$

$$- \bar{u}\ \bar{P}_{xx}[\vec{G}^x(x,y|x_o,y_o)] - \bar{v}\ \bar{P}_{xy}[\vec{G}_y^x(x,y|x_o,y_o)]\Big] +$$

$$+ \frac{\partial}{\partial y}\Big[\bar{G}_x^x(x,y|x_o,y_o)\ \bar{P}_{xy}(u,v) + \bar{G}_y^x(x,y|x_o,y_o)\ \bar{P}_{yy}(u,v) -$$

$$- \bar{u}\ \bar{P}_{xy}[\vec{G}^x(x,y|x_o,y_o)] - \bar{v}\ \bar{P}_{yy}[\vec{G}^x(x,y|x_o,y_o)]\Big]\Big\} dR = \bar{u}(x_o,y_o) \tag{8}$$

For details of the analysis to obtain the equation (8), we refer to the paper of Ghosh (1972).

Applying Green's theorem, the integral in (8) over the region R is converted to an integral over the curves S, S_1, S_2 (shown in Fig.1.) bounding the region R and We have

$$\int_{S+S_1+S_2} \bar{G}_x^x(x,y|x_o,y_o)\ \bar{P}_{nx}(u,v) + \bar{G}_y^x(x,y|x_o,y_o)\ \bar{P}_{ny}(u,v) -$$

$$-\bar{u}\ \bar{P}_{nx}[\vec{G}^x(x,y|x_o,y_o)] - \bar{v}\ \bar{P}_{ny}[\vec{G}^x(x,y|x_o,y_o)]\ ds = \bar{u}(x_o,y_o), \tag{9}$$

n is the direction of the outward normal at ds.

Since the stresses due to (u,v) are zero on the free surfaces S, S_1, S_2 and the stresses due to Green's function are also zero on the free surface S, so we obtain from (9)

$$\int_o^1 \left\{ [\bar{u}]\ \bar{P}_{xx}\ [\vec{G}^x(o,y|x_o,y_o)] + [\bar{v}]\ \bar{P}_{xy}\ [\vec{G}^x(o,y|x_o,y_o)] \right\} dy =$$

$$= \bar{u}(x_o,y_o) \qquad (10)$$

Where $[u]$ and $[v]$ represent the jump discontinuity in displacement components across the crack HO and HO = ℓ is the length of the crack. Since we are considering a dip-slip fault, so there is no displacement discontinuity along x-direction across the fault surface. Consequently $[u] = o$. Also, as we are interested in surface displacement only, so the equation (10) reduces to the form

$$\int_o^1 [\bar{v}]\ \bar{P}_{xy}\ [\vec{G}^x(o,y|x_o,o)]\,dy = \bar{u}\ (x_o,\ o) \qquad (11)$$

Considering the equations (2), (3), (6) and (7) and following the same procedure, we get

$$\int_o^1 [\bar{v}]\ \bar{P}_{xy} [\vec{G}^y(o,y|x_o,o)]\,dy = \bar{v}\ (x_o,o) \qquad (12)$$

The Fourier transforms of the Green's functions are

$$\bar{G}^x_x(x,y|x_0,0)= \frac{1}{2\pi/\mu}\int_{-\infty}^{\infty}\frac{\nu_2\, e^{-i\ (x-x_0)}}{R(\xi)}\left[2\xi^2\, e^{-\nu_1 y}-(2\xi^2-k_2^2)e^{-\nu_2 y}\right]d\xi,$$

$$\bar{G}^x_y(x,y|x_0,0)= \frac{-i}{2\pi/\mu}\int_{-\infty}^{\infty}\frac{\xi\, e^{-i\xi(x-x_0)}}{R(\xi)}\left[2\nu_1\ \nu_2\ e^{-\nu_1 y}\ (2\xi^2- k_2^2\ )e^{-\nu_2 y}\right]d\xi, \quad (13)$$

$$\bar{G}^y_x(x,y|x_0,0)= \frac{-i}{2\pi/\mu}\int_{-\infty}^{\infty}\frac{\xi\, e^{-i\xi(x-x_0)}}{R(\xi)}\left[(2\xi^2-k_2^2)e^{-\nu_1 y}-2\nu_1\ \nu_2\ e^{-\nu_2 y}\right]d\xi,$$

$$\bar{G}^y_y(x,y|x_0,0)= \frac{-1}{2\pi/\mu}\int_{-\infty}^{\infty}\frac{\nu_1\, e^{-i\xi(x-x_0)}}{R(\xi)}\left[(2\xi^2-k_2^2)e^{-\nu_1 y}-2\xi^2 e^{-\nu_2 y}\right]d\xi.$$

Here $k_1^2 = w^2/\alpha^2$, $k_2^2 = w^2/\beta^2$, $\nu_1 = \sqrt{\xi^2 - k_1^2}$, $\nu_2 = \sqrt{\xi^2 - k_2^2}$ and $R(\xi) = 4\xi^2\ \nu_1\ \nu_2 - (\xi^2 + \nu_2^2\ )^2$; α, β are respectively P-wave and S-wave velocities. The values of $\nu_1(\xi)$ and $\nu_2(\xi)$ are to be so chosen that with such values the expression for the displacement decay exponentially as $y \to \infty$ for real $\nu_1(\xi)$ and $\nu_2(\xi)$.

The Fourier transform of the stress $P_{xy}\left[\bar{G}^x(0,y|x_0,0)\right]$ on the line of faulting HO is given by

$$\bar{P}_{xy}[\bar{G}^x(0,y|x_0,0)]= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi x_0}}{R(\xi)}\left[(2\xi^2-k_2^2)e^{-\nu_2 y}-4\xi^2\nu_1\ \nu_2\ e^{-\nu_1 y}\right]d\xi. \quad (14)$$

Since we want to determine the surface displacement due to Rayleigh waves, we need to determine the value of the integral in (14) for Rayleigh pole contribution only, for which we refer to Mal and Knopoff (1968). The Rayleigh pole contribution to the integral (14) is evaluated by following the method prescribed by Lapwood (1949) and $\bar{P}_{xy}[\vec{G}^x(o,y|x_o,o)]$ is found to be

$$iwA_1\left\{\exp\left(-\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2 y}\right)-\exp\left(-\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2 y}\right)\right\}\exp\left(i\frac{wx_o}{c_R}\right),\ \text{for } w>o$$

$$\text{and } iwA_1\left\{\exp\left(\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2 y}\right)-\exp\left(\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2 y}\right)\right\}\exp\left(i\frac{wx_o}{c_R}\right),\text{for } w<o \tag{15}$$

$$\text{where } A_1=\frac{1}{4c_R}\,\frac{\sqrt{(\alpha^2-c_R^2)(\beta^2-c_R^2)}}{2\frac{\alpha}{\beta}(2\beta^2-c_R^2)-\beta^2\left(\frac{\alpha^2-c_R^2}{\beta^2-c_R^2}\right)^{1/2}-\alpha^2\left(\frac{\beta^2-c_R^2}{\alpha^2-c_R^2}\right)^{1/2}-2\sqrt{(\alpha^2-c_R^2)(\beta^2-c_R^2)}} \tag{16}$$

and c_R is the Rayleigh wave velocity.

Similarly, the contribution from the Rayleigh pole to $\bar{P}_{xy}[\vec{G}^y(o,y|x_o,o)]$ is found to be

$$B_1 w\left\{\exp\left(-\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2 y}\right)-\exp\left(-\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2 y}\right)\right\}\exp\left(i\frac{wx_o}{c_R}\right),\ \text{for } w>o \tag{17}$$

$$\text{and } -B_1 w\left\{\exp\left(\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2 y}\right)-\exp\left(\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2 y}\right)\right\}\exp\left(i\frac{wx_o}{c_R}\right),\text{for } w<o$$

where

$$B_1 = \frac{1}{2\beta c_R} \frac{(2\beta^2 - c_R^2)\sqrt{\alpha^2 - c_R^2}}{\frac{2\alpha}{\beta}(2\beta^2 - c_R^2) - \beta^2\left(\frac{\alpha^2 - c_R^2}{\beta^2 - c_R^2}\right)^{1/2} - \alpha^2\left(\frac{\beta^2 - c_R^2}{\alpha^2 - c_R^2}\right)^{1/2} - 2\sqrt{(\alpha^2 - c_R^2)(\beta^2 - c_R^2)}} \tag{18}$$

The discontinuity in displacement along the line of faulting at any time t and at a depth y below the free surface is assumed to be

$$\begin{aligned}[v] &= DH[h(t) - (1 - y)]\,[H(y) - H(y - 1)] \\ &= DH[t - r(y)]\,[H(y) - H(y - 1)]\end{aligned} \tag{19}$$

H( ) is Heaviside step function and $r(y) = h^{-1}(1-y)$, which is the inverse function of h and it exists as h(t) is strictly monotonic increasing function. Fourier transform of the equation (19) is given by

$$\begin{aligned}[\bar{v}] &= D[H(y) - H(y - 1)] \int_{-\infty}^{\infty} H[t - r(y)]\, e^{iwt}\, dt \\ &= D[H(y) - H(y - 1)] \int_{r(y)}^{\infty} e^{iwt} dt = D[H(y) - H \\ &\quad (y - 1)]\left(\pi\delta(w) + \frac{i}{w}\right) e^{iwr(y)}\end{aligned} \tag{20}$$

Putting the value of $[\bar{v}]$ from (20) in (11) and then taking Fourier inversion of (11) and changing the order of integration one obtains

$$u(x_o,o)= \frac{D}{2\pi}\int_o^1 dy \int_{-\infty}^{\infty} e^{iw(r(y)-t)}(\pi\,\delta(w)+\frac{1}{w})\bar{P}_{xy}[\,\vec{G}^x(o,y|x_o,o)\,]dw.$$

Substituting the value of $\bar{P}_{xy}$ from (15) in the above equation we get

$$u(x_o,o)= -\frac{A_1 D}{\pi}\int_o^1 dy \int_o^{\infty}\Big\{\Big[\exp(-\frac{w}{\alpha c_R}\sqrt{\alpha^2-c_R^2}\,y) -$$

$$- \exp(-\frac{w}{\beta c_R}\sqrt{\beta^2-c_R^2}\,y)\Big]\cos(r(y)-t+\frac{x_o}{c_R})w\Big\}dw$$

or

$$\frac{u(x_o,o)}{D} = -\frac{A_1 c_R}{\pi}\int_o^1\Bigg\{\frac{\alpha\sqrt{\alpha^2-c_R^2}\,y}{(\alpha^2-c_R^2)\,y^2+\alpha^2\,(c_R\,r(y)-t\,c_R+x_o)^2} -$$

$$-\frac{\beta\sqrt{\beta^2-c_R^2}\,y}{(\beta^2-c_R^2)\,y^2+\beta^2\,(c_R\,r(y)-t\,c_R+x_o)^2}\Bigg\}dy \qquad (21)$$

Similarly, we obtain

$$\frac{v(x_o,o)}{D} = -\frac{B_1 c_R}{\pi}\int_o^1\Bigg\{\frac{\alpha^2(c_R\,r(y)-t\,c_R+x_o)}{(\alpha^2-c_R^2)\,y^2+\alpha^2(c_R\,r(y)-tc_R+x_o)^2} -$$

$$-\frac{\beta^2(c_R\,r(y)-t\,c_R+x_o)}{(\beta^2-c_R^2)\,y^2+\beta^2(c_R\,r(y)-t\,c_R+x_o)^2}\Bigg\}dy \qquad (22)$$

DIFFERENT CASES OF NONUNIFORM CRACK SPEED: In this section we determine the Rayleigh wave displacement on the free surface for different nonuniform motions of the vertical crack.

Case 1. Here it is assumed that $h(t) = ct$, where c is the constant velocity of propagation of the fault and we have

$$t = \frac{l-y}{c} = r(y) \; . \tag{23}$$

Substituting the value of $r(y)$ from (23) in (21) and (22) and integrating the resulting equation one gets

$$\frac{u(x_o,o)}{D} = - \frac{AG}{4\pi P} \frac{c^2}{c_R^2} \left[\frac{A}{B} \left\{ \frac{1}{2} \ln \frac{A^2+X^2}{T^2} + \frac{c_R}{cA} \tan^{-1} \frac{A}{X} \right\} - \right.$$

$$\left. - \frac{G}{H} \left\{ \frac{1}{2} \ln \frac{G^2+X^2}{T^2} + \frac{c_R}{cG} \tan^{-1} \frac{G}{X} \right\} \right] \tag{24}$$

$$\frac{v(x_o,o)}{D} = - \frac{A(1+G^2)}{2\pi P} \frac{c}{c_R} \left[\frac{1}{B} \left\{ \frac{cA}{c_R} \tan^{-1} \frac{A}{X} - \frac{1}{2} \ln \frac{A^2+X^2}{T^2} \right\} - \right.$$

$$\left. - \frac{1}{H} \left\{ \frac{cG}{c_R} \tan^{-1} \frac{G}{X} - \frac{1}{2} \ln \frac{G^2+X^2}{T^2} \right\} \right] \tag{25}$$

where $A = \sqrt{1 - \frac{c_R^2}{\beta^2} \frac{\beta^2}{\alpha^2}}$, $B = 1 + \frac{c^2}{c_R^2} \left(1 - \frac{c_R^2}{\beta^2} \cdot \frac{\beta^2}{\alpha^2}\right)$, $G = \sqrt{1 - \frac{c_R^2}{\beta^2}}$,

$H = 1 + \frac{c^2}{c_R^2}\left(1 - \frac{c_R^2}{\beta^2}\right)$, $X = \frac{x_o}{l} - \frac{t\, c_R}{l}$, $T = \frac{c_R}{c} - \frac{t\, c_R}{l} + \frac{x_o}{l}$ and

$P = 2\,(1 + G^2) - \frac{A}{G} - \frac{G}{A} - 2\,AG.$

$T = 0$ implies $t = \frac{x_o}{c_R} + \frac{l}{c}$ which is the time taken to reach the point (x_o, o) by Rayleigh wave, which is emitted from the epicentre O when the crack reaches the free surface and $X = 0$ implies $t = x_o/c_R$ which is the time taken to reach the point (x_o, o) by the Rayleigh wave generated at H as soon as the crack appears at H.

Case 2. In this case it is assumed that the crack starts to move vertically upward with a finite velocity a and has a retardation b. Here at a time t after the formation of the crack

$$h(t)\ (\leq l) = at - \frac{1}{2} bt^2, \text{ so that}$$

$$r(y) = \frac{2}{a} \frac{l(1 - z)}{1 + \sqrt{1 - \frac{2bl}{a^2}(1 - z)}} \tag{26}$$

Substituting the value of $r(y)$ in (21) and (22) we obtain

$$\frac{u(x_o, o)}{D} = -\frac{AG}{\pi P} \int_o^1 \left[\frac{Az}{Az^2 + \left\{ \frac{c_R}{a} \frac{2(1 - z)}{1 + \sqrt{1 - 2F(1 - z)}} + X \right\}^2} - \right.$$

$$\left. - \frac{Gz}{G^2 z^2 + \left\{ \frac{c_R}{a} \frac{2(1 - z)}{1 + \sqrt{1 - 2F(1 - z)}} + X \right\}^2} \right] dz, \tag{27}$$

$$\frac{v(x_o, o)}{D} = -\frac{A}{\pi P} \frac{c_R}{\beta^2} \left(1 - \frac{\beta^2}{\alpha^2}\right)\left(1 - \frac{c_R^2}{2\beta^2}\right) \times$$

$$\times \int_{o}^{1} \frac{\left(2\frac{c_R}{a}\,\frac{1-z}{1+\sqrt{(1-2F(1-z))}}+X\right)z^2\,dz}{\left\{A^2z^2+\left(2\frac{c_R}{a}\,\frac{1-z}{1+\sqrt{1-2F(1-z)}}+X\right)^2\right]\left[G^2z^2+\left(2\frac{c_R}{a}\,\frac{1-z}{1+\sqrt{1-2F(1-z)}}+X\right)^2\right\}}$$

(28)

where $F = \frac{bl}{a^2}$ and the other constants have the same values mentioned earlier. It may be noted that the integrands in equation (27) and (28) have a singularity at $z = o$ provided

$$2\frac{c_R}{a}\,\frac{1}{1+\sqrt{1-2F}} + X = 0\ ,$$ which implies that

$$t = \frac{x_o}{c_R} + \frac{2l}{a\left(1+\sqrt{1-\frac{2bl}{a^2}}\right)}\ .$$

This is the time to reach the point (x_o,o) by the Rayleigh wave emitted from the epicentre O just after the arrival of the crack at this point.

Case 3. Finally let the crack at a depth l below the free surface, start to move vertically upward with infinitely large velocity which gradually decays with time. Accordingly h(t) is taken in the form

$h(t) = D_1\sqrt{t}$ where D_1 is a constant.

or $D_1\sqrt{t} = l - y$

Therefore $r(y) = [\,(l - y)\,/\,D_1\,]^2$ (29)

As before substituting the value of $r(y)$ in (21) and (22) and making a change of variable of the integration , we have

$$\frac{u(x_o,o)}{D} = -\frac{K^2AG}{4\pi P}\int_o^1\left[\frac{A}{z^4-4z^3+(K^2A^2+2M+4)^2\,z^2-4Mz+M^2}\right.$$

$$\left.-\frac{G}{z^4-4z^3+(K^2G^2+2M+4)^2\,z^2-4\,Mz+M^2}\right]zdz, \qquad (30)$$

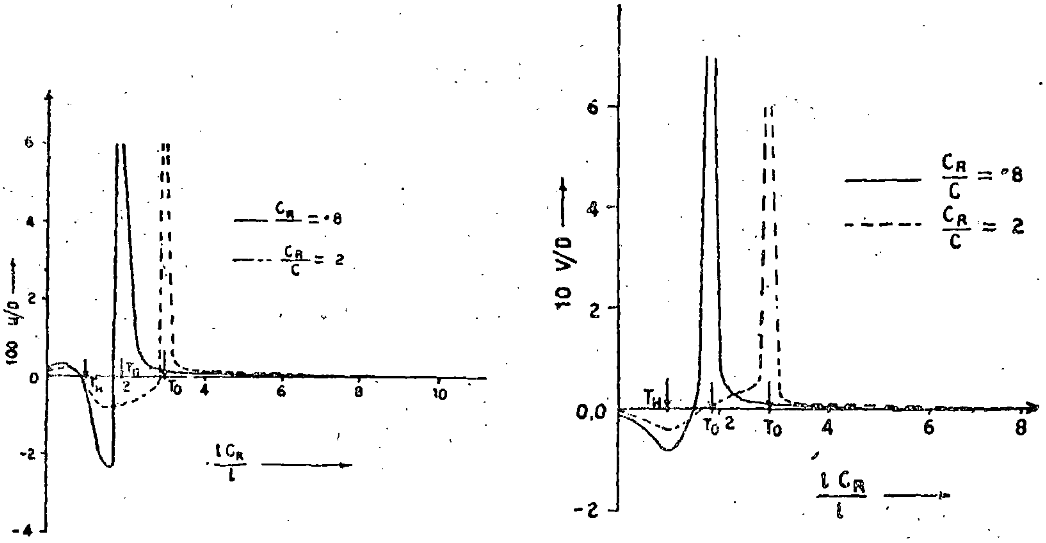
$$\frac{v(x_o,o)}{D} = \frac{K^3Ac_R^2}{\pi P\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\left(1-\frac{c_R^2}{2\beta^2}\right)\,X$$

$$X\int_o^1\frac{z^2(z^2-2z+M)\,dz}{\left\{K^2A^2z^2+(z^2-2z+M)^2\right\}\left\{K^2G^2z^2+(z^2-2z+M)^2\right\}} \qquad (31)$$

where $K^2=D_1^2/lc_R$, $M=1-(tc_R/l)(D_1^2/lc_R)+(x_o/l)(D_1^2/lc_R)$ and the other constants have same values as mentioned before. Again, the integrands in equation (30) and (31) are singular at $z=o$ if $M=0$. This corresponds to $t=(x_o/c_R)+(l^2/D_1^2)$ which is the time of arrival at (x_o,o) of the Rayleigh wave which is generated at o just after the arrival of the crack on the free surface.

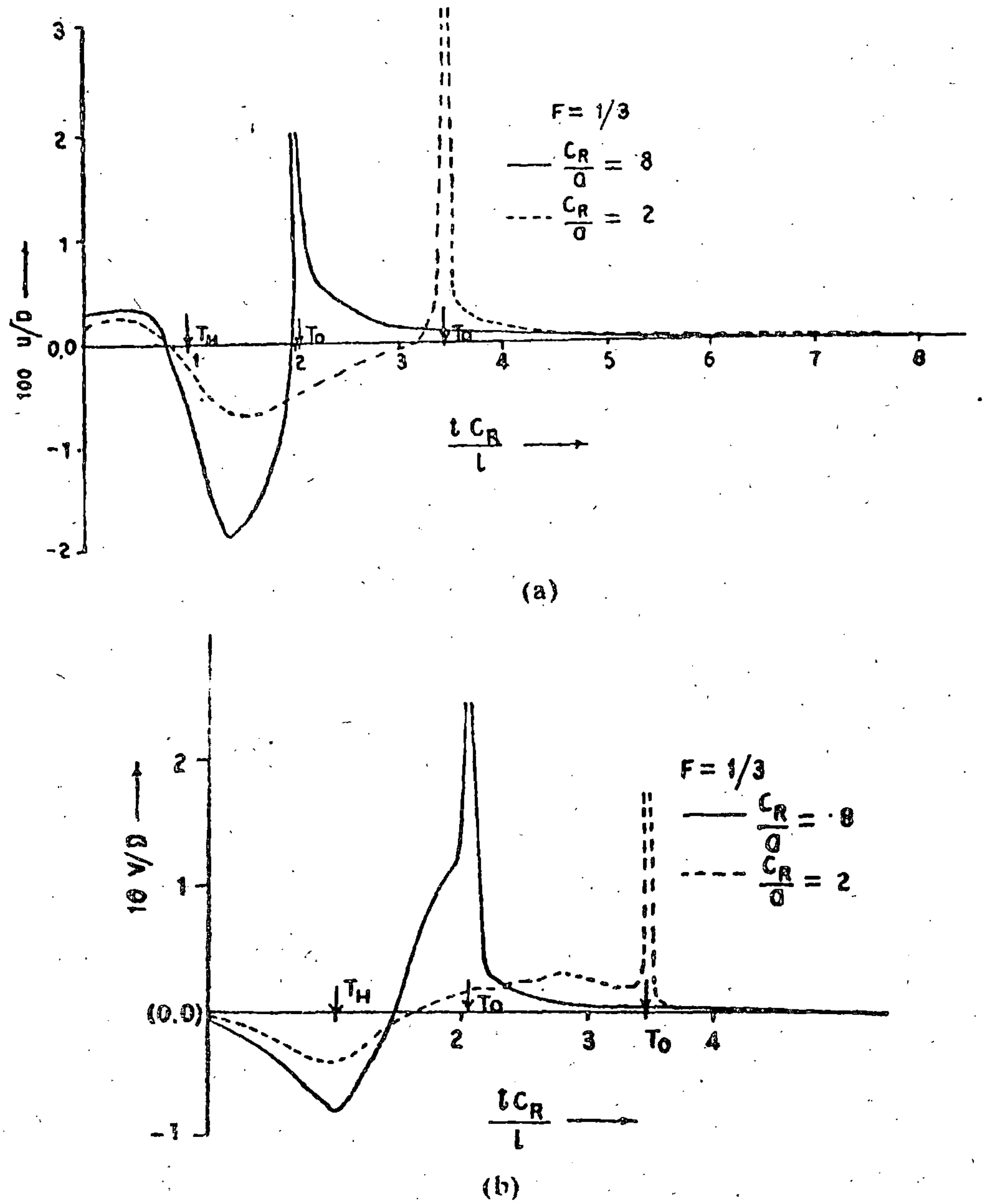
NUMERICAL RESULTS AND CONCLUSION: When the earth material is under tension or compression in a direction parallel to the free surface, shear failure occurs on a fault plane. In general, this failure moves with nonuniform speed. Numerical computations are carried out for poisson solid ($\alpha/\beta = \sqrt{3}$) and for $c_R/\beta = {}^{\cdot}9194$. The quantities A, B, G, H, X, T, F, K, M, F defined in section 3 are all dimensionless.

Figures 2, 3 and 4 show the variation of components of displacement with time. The dimensionless displacement components are plotted against dimensionless time. T_H and T_o indicated in the figures correspond to the arrival times at (x_o, o) of the Rayleigh waves from the focus H and the fault break at o.
Figs.2(a-b) correspond to the case 1 of section 3 where the constant velocity of propagation of the crack ($c_R/c = {}^{\cdot}8$ and 2) is assumed. Figs.3(a-b) correspond to the case where the crack starts with a finite velocity a and has a retardation b. Here also two cases $c_R/a = {}^{\cdot}8$ and 2 with the assumption that $F = 1/3$, are considered. Figs.4(a-b) depicts the case 3 where the initial velocity of crack propagation is assumed to be infinitely large and K is taken to be equal to 1.
From equations (21) and (22) it may be noted that $u(x_o, o)/D$ and $v(x_o, o)/D$ are functions $(x_o/1)-(c_Rt/1)$. Therefore in all computational works, without any loss of generality $x_o/1$ has been taken to be equal to 1, because any change in value of $x_o/1$ will merely cause a shifting of the graphs along the direction of $c_Rt/1$.
We find that in each case the strongest ground motion occurs at $T_o = c_Rt/1$ which correspond to the arrival time of Rayleigh waves from the surface break at O. Also it is found that though the nature of the graphs in three different cases differ between T_H and T_o but their natures are almost the same after the arrival of

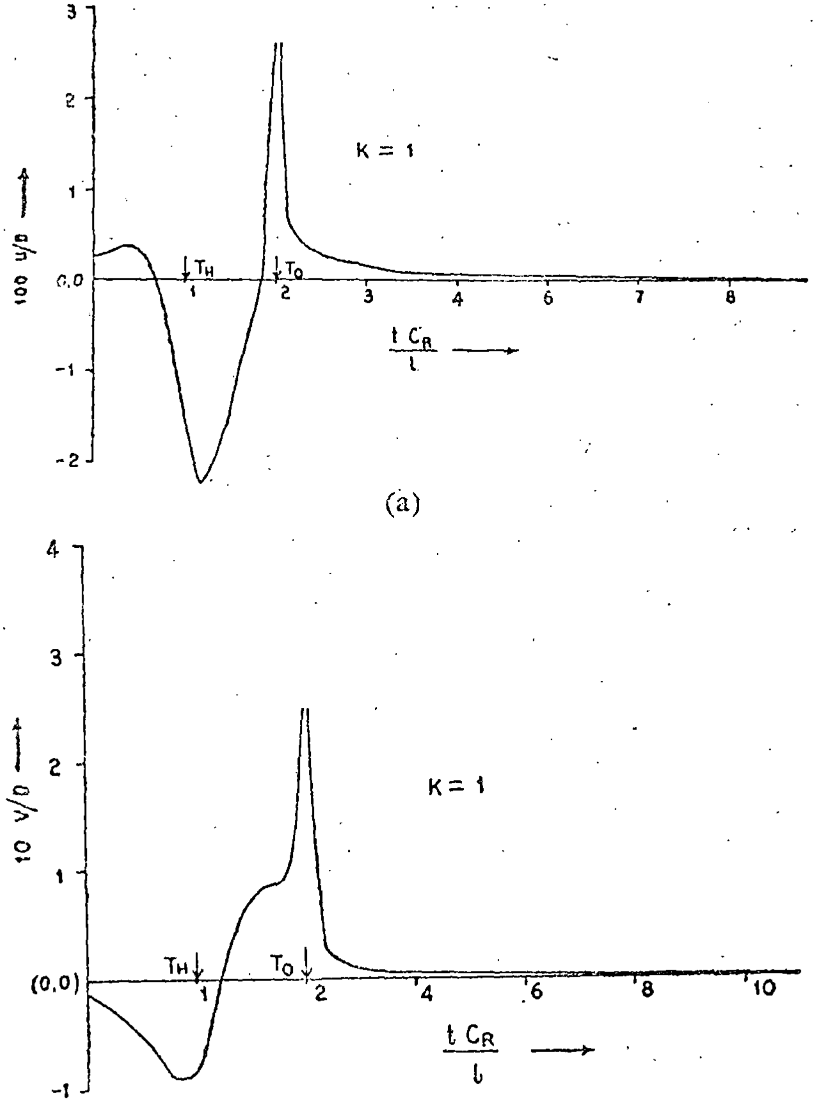


(a) (b)

FIGS. 2(a, b). Horizontal and vertical components of displacement versus time. Case of crack propagation with uniform velocity.



FIGS. 3(a, b). Horizontal and Vertical components of displacement versus time. Case of crack propagation with retardation and finite starting velocity.



(a)

(b)

FIGS. 4(a, b). Horizontal and Vertical components of displacement versus time. Case of crack propagation with retardation and infinite starting velocity.

Rayleigh waves from the surface break. This may be explained from the fact that the main contribution to the ground motion due to Rayleigh wave is from a small portion of the fault near the surface after the arrival of Rayleigh wave from the surface break. So the contribution from the details of crack initiation becomes insignificant after T_o.

It may be mentioned in this connection that though contribution of Rayleigh wave to the ground motion is significant at large distances from the epicentre, the effect of body waves near the epicentral region can not be ignored. This effect may be incorporated if we consider in addition to the contribution from Rayleigh pole, the contribution from the branch line integrals arising from the evaluation of stresses due to Green's function.

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Displacement Produced in an Elastic Half-space by the Impulsive Torsional Motion of a Circular Ring Source

By MUKTIMOY GHOSH[1])

Summary – In this paper the problem of disturbance in an elastic semi-infinite medium due to the torsional motion of a circular ring source on the free surface of a medium are studied. Two cases, when the medium is either homogeneous or inhomogeneous, are treated. In order to solve the problem, the Laplace transform and the Hankel transform and the Laplace inversion by Cagniard's method as modified by DE HOOP (1959) are applied. Finally, the integrals for displacement are evaluated numerically. The displacement on the free surface as a function of time is shown by means of graphs, in the case of both a homogeneous and an inhomogeneous medium, indicating clearly the variation in displacement due to the presence of an inhomogeneity.

Key words: Theoretical seismology; Torsional ring source; Cagniard–de Hoop transformation.

1. Introduction

At present much attention has been given to problems concerned with wave propagation in homogeneous as well as in inhomogeneous, isotropic, elastic media. Much of this work has been connected with problems of seismological interest, involving wave propagation. The normal loading problem of an elastic half-space was first investigated by LAMB (1904). This type of problem was then investigated by EASON (1964), MITRA (1964), CHAKRABORTY and DE (1971) and many others. In fact a class of elastic half-space problems involving an axisymmetric, normally applied, surface load ~~is~~ was investigated by GAKENHEIMER (1971). He assumed that loads suddenly emanate from a point on the surface and expand radially at a constant rate. He used Cagniard's method to evaluate the inverse transforms. This paper has a particular reference to the work by GHOSH (1971) where techniques similar to those adopted here, are used. Many recent studies on elastic wave propagation are due to the work of CAGNIARD (1962), who developed a particular technique of finding the Laplace inversion, that has been found to be extremely useful in dealing with problems of this type.

The type of disturbing force considered in this paper is impulsive in time and acts over the circumference of a circular region of constant radius on the free surface of a semi-infinite, isotropic, elastic half-space. The effect of the inhomogeneity

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of the medium on the disturbance produced is determined in the integral form, whereas the displacement in the case of a homogeneous medium is determined exactly. The displacement at any point on the free surface is evaluated numerically and the graphs are drawn to show how the vibration of a point in the medium is affected due to the inhomogeneity of the medium, which enters into the expression for displacement through the factor ε.

Case I: Homogeneous medium

2. Formulation of the problem

Let (r, θ, z) be the cylindrical polar co-ordinates, z-axis being directed into the isotropic elastic medium, the plane boundary being $z = 0$ with the origin at the centre of the ring source $r = a$, $z = 0$.

The displacement is calculated at points inside and on the free surface of the medium, subject to the condition that the half-space is initially at rest and that the displacement remains bounded even for large values of z. For torsional motion of the ring all quantities depend on r, z and the time t, the only non-zero component of the displacement vector is the component v along the direction of θ increasing. The relevant non-vanishing stress components are

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \tag{1}$$

and

$$\tau_{\theta z} = \mu \frac{\partial v}{\partial z} \tag{2}$$

where μ is Lame's constant. The only non-zero equation of motion is

$$\frac{\partial}{\partial r}(\tau_{r\theta}) + \frac{\partial}{\partial z}(\tau_{\theta z}) + 2\frac{\tau_{r\theta}}{r} = \rho\frac{\partial^2 v}{\partial t^2} \tag{3}$$

where ρ is the density of the material, assumed constant. The boundary condition is

$$\tau_{\theta z} = P\delta(r - a)\delta(t) \tag{4}$$

where P is a constant, a is the radius of the ring source and $\delta(t)$ is Dirac's delta-function.

Using (1) and (2) the equation (3) can be written in the form

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} \tag{5}$$

where $\beta = \sqrt{(\mu/\rho)}$ is the shear wave velocity.

3. *Method of solution*

We define for all positive real values of s the Laplace transform $f_1(r, z, s)$ of a function $f(r, z, t)$ by the relation

$$f_1(r, z, s) = \int_0^\infty f(r, z, t)\, e^{-st}\, dt. \tag{6}$$

Applying the Laplace transform (6) to the equation (5) we obtain

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v_1}{\partial r} - \frac{v_1}{r}\right) + \frac{\partial^2 v_1}{\partial z^2} = \frac{s^2 v_1}{\beta^2}. \tag{7}$$

Define the Hankel transform $v_2(\xi, z, s)$ of $v_1(r, z, s)$ by the equation

$$v_2(\xi, z, s) = \int_0^\infty r J_1(\xi r) v_1(r, z, s)\, dr, \tag{8}$$

where J_1 is a Bessel function.

Multiplying the equation (7) by $rJ_1(\xi r)$ and integrating with respect to r from 0 to ∞ we get,

$$\frac{d^2 v_2}{dz^2} = \left(\xi^2 + \frac{s^2}{\beta^2}\right) v_2. \tag{9}$$

The general solution of this equation which remains bounded as $z \to +\infty$ is

$$v_2 = A \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right], \tag{10}$$

where A is to be determined from the boundary conditions, $\tau_{\theta z_1} = P\delta(r - a)$, where $\tau_{\theta z_1}$ is the Laplace transform of $\tau_{\theta z}$. From the Hankel transform $(\tau_{\theta z_1})_2$ of $\tau_{\theta z_1}$, we obtain by using (2)

$$(\tau_{\theta z_1})_2 = \mu \frac{dv_2}{dz} = PaJ_1(\xi a)$$

on $z = 0$, $v_2 = A$ and $dv_2/dz = -A(\xi^2 + s^2/\beta^2)^{1/2}$.

Using these relations in equation (10) we get

$$A = -\frac{Pa}{\mu} \frac{J_1(\xi a)}{(\xi^2 + s^2/\beta^2)^{1/2}}.$$

Substituting the value of A in (10) and inverting the Hankel transform (8), we obtain

$$v_1(r, z, s) = -\frac{Pa}{\mu} \int_0^\infty \frac{\xi J_1(\xi a) J_1(\xi r)}{(\xi^2 + s^2/\beta^2)^{1/2}} \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right] d\xi. \tag{11}$$

From a well-known result (WATSON (1966), p. 358)

$$J_1(\xi r)J_1(\xi a) = \frac{1}{\pi}\int_0^{\overset{\pi}{\cancel{\infty}}} J_0(\xi R)\cos\phi\, d\phi,$$

and

$$J_0(\xi R) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi R\sin\psi}\, d\psi \qquad \text{(ERDELYI (1953), p. 14)}$$

where $R = \sqrt{(r^2 + a^2 - 2ar\cos\phi)}$, we obtain

$$\frac{2\pi^2\mu v_1}{Pa} = -\int_0^{\pi} I_1 \cos\phi\, d\phi \tag{12}$$

where

$$I_1 = \int_0^{2\pi}\int_0^{\infty} \frac{\xi\exp\left[-z(\xi^2 + s^2/\beta^2)^{1/2} + i\xi R\sin\psi\right]}{(\xi^2 + s^2/\beta^2)^{1/2}}\, d\xi\, d\psi.$$

If we put $p = \xi\sin\psi$ and $q = \xi\cos\psi$ in I_1, then

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\left[-z(p^2 + q^2 + s^2/\beta^2)^{1/2} + iRp\right]}{(p^2 + q^2 + s^2/\beta^2)^{1/2}}\, dp\, dq. \tag{13}$$

To find the inversion of I_1, we adopt Cagniard's technique as modified by DE HOOP (1959). Accordingly in (13), we put $p = ms$ *and* $q = ns$, then

$$I_1 = 2\int_0^{\infty} dn \int_{-\infty}^{\infty} \frac{s\exp\{-s[z(m^2 + n^2 + 1/\beta^2)^{1/2} - iRm]\}}{(m^2 + n^2 + 1/\beta^2)^{1/2}}\, dm. \tag{14}$$

In the above integral the path of integration with respect to m is the real axis (Fig. 1) which is deformed in such a way that $-iRm + z(m^2 + n^2 + 1/\beta^2)^{1/2} = t$, where t is real and positive. The deformed path of integration is the branch Γ of a hyperbola whose equation is

$$m = \frac{iRt \pm z[t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)]^{1/2}}{z^2 + R^2}, \qquad \{(z^2 + R^2)(n^2 + 1/\beta^2)\}^{1/2} < t < \infty.$$

In the course of deformation of the path of integration it is essential to know

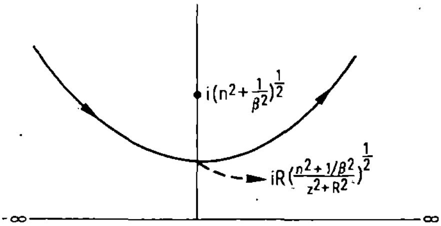

Figure 1
Paths of integration in the complex m-plane.

the singularities of the function $s/(m^2 + n^2 + 1/\beta^2)^{1/2}$ in the m-plane, which are the branch points $\pm i(n^2 + 1/\beta^2)^{1/2}$.

Since the hyperbolic path Γ does not cross any of the singularities during its deformation, it is possible by virtue of Cauchy's theorem and Jordon's lemma, to replace the integration along the real m-axis by an integration along the hyperbolic path Γ.

We assume

$$m_+ = \frac{iRt + z[t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)]^{1/2}}{z^2 + R^2}$$

and

$$m_- = \frac{iRt - z[t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)]^{1/2}}{z^2 + R^2}.$$

The point where Γ cuts the imaginary axis is given by

$$t = \{(z^2 + R^2)(n^2 + 1/\beta^2)\}^{1/2}$$

and the point is

$$m = \frac{iR(n^2 + 1/\beta^2)^{1/2}}{(z^2 + R^2)^{1/2}}$$

which is below the branch point $i(n^2 + 1/\beta^2)^{1/2}$. Hence (14) can be written as

$$I_1 = 2\int_0^\infty dn \int_{\{(z^2 + R^2)(n^2 + 1/\beta^2)\}^{1/2}}^\infty s\,e^{-st}\left[\frac{1}{(m_+^2 + n^2 + 1/\beta^2)^{1/2}}\frac{dm_+}{dt} - \frac{1}{(m_-^2 + n^2 + 1/\beta^2)^{1/2}}\frac{dm_-}{dt}\right]dt. \tag{15}$$

Now using the fact that $m_- = -\bar{m}_+$ and $dm_-/dt = -(d\bar{m}_+/dt)$ where $\bar{m}$ is the complex conjugate of m, (15) can be written as

$$I_1 = 4\int_{(z^2 + R^2)^{1/2}/\beta}^\infty s\,e^{-st}\,dt \int_0^{[t^2/(z^2 + R^2) - 1/\beta^2]^{1/2}} Rl\left[\frac{dm_+/dt}{(m_+^2 + n^2 + 1/\beta^2)^{1/2}}\right]dn. \tag{16}$$

Now,

$$Rl\left[\frac{(dm_+/dt)}{(m_+^2 + n^2 + 1/\beta^2)^{1/2}}\right] = \frac{1}{\{t^2 - (z^2 + R^2)(n^2 + 1/\beta^2)\}^{1/2}}.$$

Substituting this result in (16), we obtain

$$I_1 = \frac{2\pi}{(z^2 + R^2)^{1/2}}\int_{[(z^2 + R^2)^{1/2}]/\beta}^\infty s\,e^{-st}\,dt.$$

Hence the Laplace inversion of I_1 is

$$I = \frac{2\pi}{(z^2 + R^2)^{1/2}}\frac{d}{dt}\left[H\left\{t - \frac{(z^2 + R^2)^{1/2}}{\beta}\right\}\right]$$

$$= \frac{2\pi}{(z^2 + R^2)^{1/2}}\delta\left[t - \frac{(z^2 + R^2)^{1/2}}{\beta}\right]. \tag{17}$$

Therefore the Laplace inversion of (12) by using the Laplace inversion of I_1 as given in (17) is

$$v(r, z, t) = -\frac{Pa}{\pi\mu}\int_0^{\pi} \frac{\delta\left[t - \dfrac{(z^2 + r^2 + a^2 - 2ra\cos\phi)^{1/2}}{\beta}\right]}{(z^2 + r^2 + a^2 - 2ra\cos\phi)^{1/2}}\cos\phi\, d\phi. \tag{18}$$

To evaluate the above integral we put

$$(z^2 + r^2 + a^2 - 2ra\cos\phi)^{1/2} = \beta\theta,$$

then

$$v(r, z, t) = \frac{P\beta}{\pi\mu r}\,\frac{\beta^2 t^2 - z^2 - r^2 - a^2}{\{2(r^2 + a^2)(\beta^2 t^2 - z^2) - (r^2 - a^2)^2 - (\beta^2 t^2 - z^2)^2\}^{1/2}}$$

for

$$\frac{\{z^2 + (r - a)^2\}^{1/2}}{\beta} < t < \frac{\{z^2 + (r + a)^2\}^{1/2}}{\beta}. \tag{19}$$

Case II: Inhomogeneous Medium

4. Formulation of the problem

In this case the same problem of torsional motion of a semi-infinite elastic medium due to the presence of a ring source $r = a$, on the free surface $z = 0$ as in Case I is considered. The only difference is that the medium under consideration is inhomogeneous in nature, the coefficient of rigidity and the density of the medium are assumed to be

$$\mu = \mu_0(1 + \varepsilon z)^2 \qquad \text{and} \qquad \rho = \rho_0(1 + \varepsilon z)^2. \tag{20}$$

Here also the non-vanishing stress components and the non-zero equations of motion are the same as in Case I, given by the equations (1), (2) and (3).

5. Method of solution

Firstly we put $\bar{v} = (1 + \varepsilon z)v$ in the equations (1), (2) and (3). The transformed equations are

$$\begin{aligned} \tau_{r\theta} &= \mu_0(1 + \varepsilon z)\left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right), \\ \tau_{\theta z} &= \mu_0\left\{(1 + \varepsilon z)\frac{\partial \bar{v}}{\partial z} - \varepsilon\bar{v}\right\} \end{aligned} \tag{21}$$

and

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) + \frac{\partial^2 \bar{v}}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 \bar{v}}{\partial t^2} \tag{22}$$

where $\beta = \sqrt{(\mu_0/\rho_0)}$.

Taking the Laplace transform of the equation with respect to t, we obtain

$$\frac{\partial^2 \bar{v}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}_1}{\partial r} - \left(\frac{1}{r^2} + \frac{s^2}{\beta^2}\right)\bar{v}_1 + \frac{\partial^2 \bar{v}_1}{\partial z^2} = 0 \tag{23}$$

where s is the Laplace transform parameter which is real and positive. Taking the Hankel transform of the equation (23) we have

$$\frac{d^2 \bar{v}_2}{dz^2} = \left(\xi^2 + \frac{s^2}{\beta^2}\right)\bar{v}_2. \tag{24}$$

The general solution of this equation which remains bounded for large values of z is

$$\bar{v}_2 = B \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]. \tag{25}$$

Applying the Hankel transform and the Laplace transform on the boundary condition

$$\tau_{\theta z} = \mu_0\left[(1 + \varepsilon z)\frac{\partial \bar{v}}{\partial z} - \varepsilon \bar{v}\right] = P\delta(r - a)\delta(t)$$

and using (25), the value of B is found to be

$$B = -\frac{PaJ_1(\xi a)}{\mu_0\{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}\}}.$$

Substituting this value of B in (25), it follows that

$$\bar{v}_2 = -\frac{PaJ_1(\xi a)}{\mu_0\{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}\}} \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right]. \tag{26}$$

Taking the Hankel inversion of (26), we have

$$\bar{v}_1 = -\frac{Pa}{\mu_0}\int_0^\infty \frac{\xi J_1(\xi a)J_1(\xi r)}{\{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}\}} \exp\left[-z\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right] d\xi. \tag{27}$$

Now,

$$\int_0^\infty \exp\left[-k\{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}\}\right] dk = \frac{1}{\varepsilon + (\xi^2 + s^2/\beta^2)^{1/2}}.$$

Using the above result, (27) is written as

$$\bar{v}_1 = -\frac{Pa}{\mu_0}\int_0^\infty e^{-\varepsilon k}\, dk \int_0^\infty \xi J_1(\xi a)J_1(\xi r) \exp\left[-(z + k)\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2}\right] d\xi. \tag{28}$$

We now replace $J_1(\xi a)J_1(\xi r)$ of (28) by the integral, which was used to modify equation (11). Finally we get

$$\bar{v}_1 = -\frac{Pa}{2\pi^2\mu_0}\int_0^\infty e^{-\varepsilon k}\, dk \int_0^\pi I_2 \cos\phi\, d\phi, \tag{29}$$

where

$$I_2 = \int_0^{2\pi} d\psi \int_0^{\infty} \exp\left[-(z+k)\left(\xi^2 + \frac{s^2}{\beta^2}\right)^{1/2} + i\xi R \sin\psi\right] d\xi.$$

Assuming $p = \xi \sin\psi$ and $q = \xi \cos\psi$, it follows that

$$I_2 = 2\int_0^{\infty} dn \int_{-\infty}^{\infty} s^2 \exp\left[-s\left\{(z+k)\left(m^2 + n^2 + \frac{1}{\beta^2}\right)^{1/2} - iRm\right\}\right] dm, \tag{30}$$

where, $p = ms$ and $q = ns$.

As in Case I, here also the path of integration with respect to m which is the real axis is deformed such that $-iRm + (z+k)(m^2 + n^2 + 1/\beta^2)^{1/2} = t$, where t is real and positive. The deformed path is a branch Γ_1 of a hyperbola the equation of which is

$$m = \frac{iRt \pm (z+k)[t^2 - \{(z+k)^2 + R^2\}(n^2 + 1/\beta^2)]^{1/2}}{(z+k)^2 + R^2},$$

$$\{(z+k)^2 + R^2\}^{1/2}\left(n^2 + \frac{1}{\beta^2}\right)^{1/2} < t < \infty.$$

Noting that the point where Γ_1 cuts the imaginary axis is

$$m = \frac{iR(n^2 + 1/\beta^2)^{1/2}}{\{(z+k)^2 + R^2\}^{1/2}}$$

when

$$t = \{(z+k)^2 + R^2\}^{1/2}\left(n^2 + \frac{1}{\beta^2}\right)^{1/2},$$

one gets from the equation (30)

$$\begin{aligned} I_2 &= 4\int_0^{\infty} dn \int_{\{(z+k)^2 + R^2\}^{1/2}(n^2 + 1/\beta^2)^{1/2}}^{\infty} s^2 e^{-st} RI\left(\frac{dm_+}{dt}\right) dt \\ &= \frac{2\pi(z+k)}{\{(z+k)^2 + R^2\}^{3/2}} \int_{[\{(z+k)^2 + R^2\}^{1/2}]/\beta}^{\infty} ts^2 e^{-st}\, dt. \end{aligned} \tag{31}$$

Hence the Laplace inversion of (31) is

$$\begin{aligned} I = \frac{2\pi(z+k)}{\{(z+k)^2 + R^2\}^{3/2}} \Biggl\{ 2\delta\left[t - \left(\frac{(z+k)^2 + R^2}{\beta^2}\right)^{1/2}\right] \\ + t\delta'\left[t - \left(\frac{(z+k)^2 + R^2}{\beta^2}\right)^{1/2}\right]\Biggr\}. \end{aligned}$$

On substitution of this value of I in (29), it is found that

$$\bar{v} = -\frac{Pa}{\pi\mu_0} \int_0^{\infty} (z+k) J e^{-\varepsilon k}\, dk. \tag{32}$$

where

$$J=\int_0^{\pi}\left\{2\delta\left[t-\left(\frac{(z+k)^2+R^2}{\beta^2}\right)^{1/2}\right]+t\delta'\left[t-\left(\frac{(z+k)^2+R^2}{\beta^2}\right)^{1/2}\right]\right\}$$
$$\times\frac{\cos\phi}{\{(z+k)^2+R^2\}^{3/2}}\,d\phi.$$

To evaluate the above integral we put

$$l=\frac{1}{\beta}\{(z+k)^2+R^2\}^{1/2},$$

then

$$J=\int_{(1/\beta)\{(z+k)^2+(r-a)^2\}^{1/2}}^{(1/\beta)\{(z+k)^2+(r+a)^2\}^{1/2}}\{2\delta(t-l)+t\delta'(t-l)\}\frac{\cos\phi}{l^3\beta^3}\frac{d\phi}{dl}\,dl,$$

where

$$\frac{d\phi}{dl}=\frac{\beta^2 l}{ra\sin\phi}\quad\text{and}\quad\cos\phi=\frac{(z+k)^2+r^2+a^2-\beta^2l^2}{2ra}.$$

Substituting these values, we get

$$J=\frac{1}{ra\beta}\int_{(1/\beta)\{(z+k)^2+(r-a)^2\}^{1/2}}^{(1/\beta)\{(z+k)^2+(r+a)^2\}^{1/2}}f(l,k)[2\delta(t-l)+t\delta'(t-l)]\,dl,\tag{33}$$

where

$$f(l,k)=\frac{(z+k)^2+r^2+a^2-\beta^2l^2}{l^2[2(r^2+a^2)\{\beta^2l^2-(z+k)^2\}-(r^2-a^2)^2-\{\beta^2l^2-(z+k)^2\}^2]^{1/2}}$$

and it is to be remembered that δ' is the derivative of the Dirac's δ-function with respect to t. Integrating (33), we obtain

$$J=\frac{1}{ra\beta}[2f(t,k)-tf(l_1,k)\delta(t-l_1)+tf(l_2,k)\delta(t-l_2)+tf'(t,k)]\tag{34}$$

where

$$l_1=\frac{1}{\beta}\{(z+k)^2+(r+a)^2\}^{1/2},\quad l_2=\frac{1}{\beta}\{(z+k)^2+(r-a)^2\}^{1/2}\quad l_2<t<l_1.$$

It is to be noted that if t does not belong to (l_2, l_1) then the integrand in (33) is zero, consequently $J=0$.

Substituting the value of J in (32), we get

$$\bar{v}=-\frac{P}{\pi\mu_0\beta r}\int_0^{\infty}(z+k)\,e^{-\varepsilon k}[2f(t,k)-tf(l_1,k)\delta(t-l_1)$$
$$+tf(l_2,k)\delta(t-l_2)+tf'(t,k)]\,dk.\tag{35}$$

Now, $l_2<t<l_1$ implies that

$$\{\beta^2t^2-(r+a)^2\}^{1/2}-z\leq k\leq\{\beta^2t^2-(r-a)^2\}^{1/2}-z.\tag{36}$$

In evaluating the integral (35), the following sub-cases are to be considered, keeping in mind that k satisfies (36) and that k is positive.

i) If $\{\beta^2t^2 - (r - a)^2\}^{1/2} - z < 0$, that is, if $\beta t < \{z^2 + (r - a)^2\}^{1/2}$ then, t does not belong to (l_2, l_1), so $J = 0$. Consequently $\bar{v} = 0$. This is in accordance with the physical condition of the problem because a disturbance cannot reach a point Q (Fig. 2) before the time $(1/\beta)\{z^2 + (r - a)^2\}^{1/2}$, which is the time of arrival of the disturbance at the point Q from the nearest point of the ring source.

ii) $\{\beta^2t^2 - (r + a)^2\}^{1/2} - z < 0 < \{\beta^2t^2 - (r - a)^2\}^{1/2} - z$, that is,

$$\{z^2 + (r - a)^2\}^{1/2} < \beta t < \{z^2 + (r + a)^2\}^{1/2}.$$

In this case (35) takes the form

$$\bar{v} = -\frac{P}{\pi\mu_0\beta r}\int_0^{\{\beta^2t^2-(r-a)^2\}^{1/2}-z} (z + k)\, e^{-\varepsilon k}[2f(t, k) - tf(l_1, k)\delta(t - l_1) + tf(l_2, k)\delta(t - l_2) + tf'(t, k)]\, dk. \tag{37}$$

The integrand of (37) is considered as a generalized function, so the finite part of the integral (37) is retained (JONES (1966), p. 89) and we get

$$\bar{v} = \frac{P\beta}{\pi r\mu_0}\frac{\beta^2t^2 - z^2 - r^2 - a^2}{[2(r^2 + a^2)(\beta^2t^2 - z^2) - (r^2 - a^2)^2 - (\beta^2t^2 - z^2)^2]^{1/2}}$$

$$+ \frac{P\beta\varepsilon}{\pi r\mu_0}\int_0^{\{\beta^2t^2-(r-a)^2\}^{1/2}-z}$$

$$\times \frac{\{(z + k)^2 + r^2 + a^2 - \beta^2t^2\}\, e^{-\varepsilon k}\, dk}{[2(r^2 + a^2)\{\beta^2t^2 - (z + k)^2\} - (r^2 - a^2)^2 - \{\beta^2t^2 - (z + k)^2\}^2]^{1/2}}$$

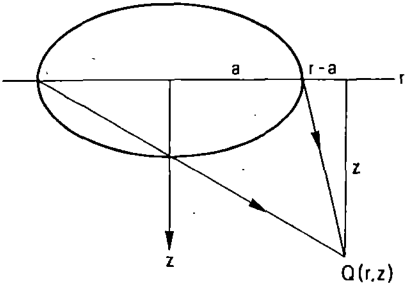

Figure 2
Arrival of the direct wave to Q from the nearest and the farthest point of the source.

Hence

$$v = \frac{P\beta}{\pi r \mu_0(1 + \varepsilon z)} \frac{\beta^2 t^2 - z^2 - r^2 - a^2}{[2(r^2 + a^2)(\beta^2 t^2 - z^2) - (r^2 - a^2)^2 - (\beta^2 t^2 - z^2)^2]^{1/2}}$$
$$+ \frac{P\beta\varepsilon}{\pi r \mu_0(1 + \varepsilon z)} \int_0^{\{\beta^2 t^2 - (r - a)^2\}^{1/2} - z}$$
$$\times \frac{\{(z + k)^2 + r^2 + a^2 - \beta^2 t^2\} e^{-\varepsilon k} \, dk}{[2(r^2 + a^2)\{\beta^2 t^2 - (z + k)^2\} - (r^2 - a^2)^2 - \{\beta^2 t^2 - (z + k)^2\}^2]^{1/2}}. \tag{38}$$

In (38) if we put $\varepsilon = 0$, we get the same result that we have determined in (19) of Case I.

iii) If $\{\beta^2 t^2 - (r + a)^2\}^{1/2} - z > 0$, that is if $\beta t > \{z^2 + (r + a)^2\}^{1/2}$ then

$$v = \frac{P\beta\varepsilon}{\pi r \mu_0(1 + \varepsilon z)} \int_{\{\beta^2 t^2 - (r - a)^2\}^{1/2} - z}^{\{\beta^2 t^2 - (r - a)^2\}^{1/2} - z}$$
$$\times \frac{\{(z + k)^2 + r^2 + a^2 - \beta^2 t^2\} e^{-\varepsilon k} \, dk}{[2(r^2 + a^2)\{\beta^2 t^2 - (z + k)^2\} - (r^2 - a^2)^2 - \{\beta^2 t^2 - (z + k)^2\}^2]^{1/2}}. \tag{39}$$

It is interesting to note that in the case of a homogeneous medium there is no displacement at a point Q (Fig. 2) after the time $t = (1/\beta)\{z^2 + (r + a)^2\}^{1/2}$, which is the time required by the disturbance to reach the point Q directly from the farthest point on the ring source ~~from~~ to the point Q. But in the case of an inhomogeneous medium the disturbance reaches a point Q even after the time $t = (1/\beta)\{z^2 + (r + a)^2\}^{1/2}$ which is the maximum time required by a direct wave to reach the point Q from the farthest point on the source ~~from~~ to the point Q. This is due to the fact that in the case of an inhomogeneous medium the region $z > 0$ may be considered as an assembly of an infinite number of thin layers of material of infinitesimal thickness of continuously varying density and coefficient of rigidity. That is why the disturbance, which reaches the point Q after successive reflection and refraction in different layers of the medium, arrives at Q after the time $\beta t = \{z^2 + (r + a)^2\}^{1/2}$. The disturbance comes continuously after the time $\beta t = \{z^2 + (r + a)^2\}^{1/2}$ with decreasing intensity.

6. *Numerical solution on the free surface $z = 0$*

In order to obtain the displacement on the free surface we make the substitution

$$[2(r^2 + a^2)(\beta^2 t^2 - k^2) - (r^2 - a^2)^2 - (\beta^2 t^2 - k^2)^2]^{1/2} = 2ra \sin \theta$$

which transforms the equations (38) and (39) to the forms given by

$$\frac{v \pi \mu_0 a}{P\beta} = d = d_1 + d_2$$

where

$$d_1 = \frac{a}{r}\,\frac{\dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1}{\left[2\left(\dfrac{r^2}{a^2}+1\right)\dfrac{\beta^2 t^2}{a^2} - \left(\dfrac{r^2}{a^2}-1\right)^2 - \dfrac{\beta^4 t^4}{a^4}\right]^{1/2}},$$

$$d_2 = \varepsilon a \int_0^{\cos^{-1} A} \frac{\cos\theta \exp\left\{-\varepsilon a\left[2\dfrac{r}{a}\cos\theta + \dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1\right]^{1/2}\right\}}{\left[2\dfrac{r}{a}\cos\theta + \dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1\right]^{1/2}}\,d\theta, \tag{40}$$

$$A = \frac{r^2 + a^2 - \beta^2 t^2}{2ra}, \qquad r - a < \beta t < r + a,$$

and

$$\frac{v\pi\mu_0 a}{P\beta} = d' = \varepsilon a \int_0^{\pi} \frac{\cos\theta \exp\left\{-\varepsilon a\left[2\dfrac{r}{a}\cos\theta + \dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1\right]^{1/2}\right\}}{\left[2\dfrac{r}{a}\cos\theta + \dfrac{\beta^2 t^2}{a^2} - \dfrac{r^2}{a^2} - 1\right]^{1/2}}\,d\theta, \tag{41}$$

for $\beta t > r + a$ respectively.

If $\varepsilon a = 0$, then from (40) it follows that $d = d_1$, which corresponds to the displacement inhomogeneous medium. The integrals in (40) and (41) giving the displacements d and d' have been numerically evaluated for different values of εa at different points on the free surface and are presented in Tables 1–4 for different values of $\beta t/a$.

Concluding remarks

From Tables 1–4 it is found that the difference in the values of the displacement at any point corresponding to $\varepsilon a = 0$ and $\varepsilon a = 10$ gradually diminishes with the

Table 1

$r/a = 2$, $(r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\varepsilon a = 0$	d when $\varepsilon a = 1$	d when $\varepsilon a = 10$
1.2	−0.97596	−0.32841	−0.50851
1.4	−0.58468	−0.08456	−0.41435
1.6	−0.38490	0.00497	−0.31149
1.8	−0.24498	0.05256	−0.21268
2.0	−0.12909	0.08585	−0.11623
2.2	−0.02001	0.11644	−0.01716
2.4	0.09676	0.15276	0.09355
2.6	0.24498	0.20795	0.23612
2.8	0.50411	0.32902	0.48230

Table 2

$r/a = 10, (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\varepsilon a = 0$	d when $\varepsilon a = 1$	d when $\varepsilon a = 10$
9.2	-0.14221	-0.00782	-0.13509
9.4	-0.08155	-0.00314	-0.08070
9.6	-0.04927	-0.00063	-0.04911
9.8	-0.02559	0.00324	-0.02556
10.0	-0.00500	0.00868	-0.00500
10.2	0.01537	0.01588	0.01537
10.4	0.03834	0.02557	0.03833
10.6	0.06901	0.03990	0.06900
10.8	0.12546	0.06799	0.12542

Table 3

$r/a = 50, (r/a) - 1 < (\beta t/a) < (r/a) + 1$

$\beta t/a$	d when $\varepsilon a = 0$	d when $\varepsilon a = 1$	d when $\varepsilon a = 10$
49.2	-0.02700	-0.00822	-0.02699
49.4	-0.01525	-0.00693	-0.01525
49.6	-0.00894	-0.00474	-0.00894
49.8	-0.00428	-0.00232	-0.00428
50.0	-0.00020	0.00028	-0.00020
50.2	0.00387	0.00318	0.00387
50.4	0.00851	0.00665	0.00851
50.6	0.01475	0.01130	0.01475
50.8	0.02633	0.01944	0.02633

Table 4

$r/a = 2, (\beta t/a) > (r/a) + 1$

$\beta t/a$	d' when $\varepsilon a = 1$	d' when $\varepsilon a = 10$
3.2	-0.17211	d' is of the order of 10^{-7}
3.4	-0.07793	
3.6	-0.04250	
3.8	-0.02533	
4.0	-0.01593	
4.2	-0.01040	
4.4	-0.00697	
4.6	-0.00477	
4.8	-0.00332	

When $r = 10a$ or $\varepsilon a = 10$, d' is very small.

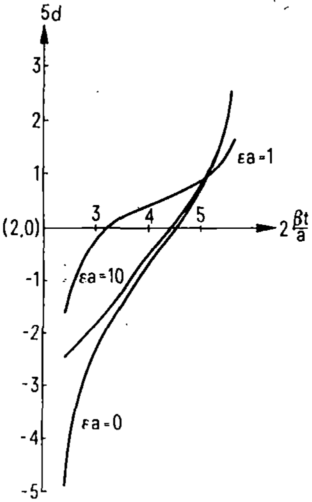

Figure 3
$r = 2a$, variation in displacement near the source for $\varepsilon a = 0, 1, 10$.

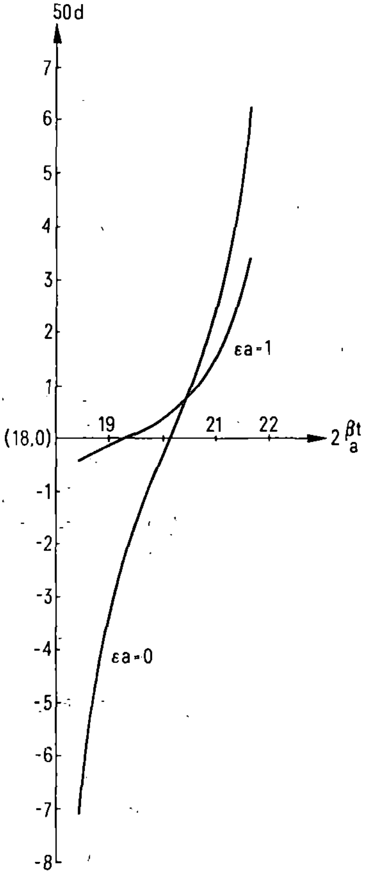

Figure 4
$r = 10a$, variation in displacement at a moderate distance from the source for $\varepsilon a = 0, 1$.

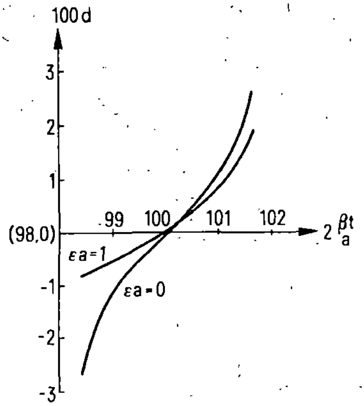

Figure 5
$r = 50a$, variation in displacement at a large distance from the source for $\varepsilon a = 0, 1$.

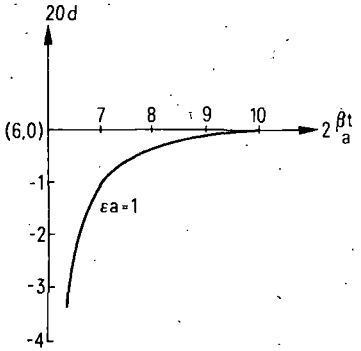

Figure 6
$r = 2a$, variation in displacement after the maximum time required by a direct wave to arrive from the farthest point of the source when $\varepsilon a = 1$.

increase in the value of r/a. This is also apparent from the expression for d_2 in (40) because the exponential term

$$\exp\left\{-\varepsilon a\left[2\frac{r}{a}\cos\theta + \frac{\beta^2 t^2}{a^2} - \frac{r^2}{a^2} - 1\right]^{1/2}\right\}$$

in the integrand for large values of r/a decreases rapidly with the increase in value of εa.

Acknowledgment

I take this opportunity of thanking Prof. M. L. Ghosh, D.Sc. for his valuable suggestions at every stage in preparing this paper.

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(Received 14th March 1979)

SH-WAVES IN AN ELASTIC HALF SPACE DUE TO A RING SOURCE OF INCREASING RADIUS.

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ABSTRACT : It is assumed that the radius of a ring source on free surface is increasing with constant velocity c which is less than the shear wave velocity. Following Cagniard's method as modified by De-Hoop, the displacement produced at any point has been determined in the integral form, from which the displacement at any point just after the arrival of the disturbance has been evaluated. The displacement at any point has also been calculated after sufficiently large time.

1. INTRODUCTION : The torsional vibration of an elastic half space due to a surface force which is periodic in time was first considered by Reissner (1937). Reissner and Sagoci (1944) determined the distribution of the stresses in the interior of a semi-infinite, homogenous isotropic elastic material due to a periodic shear stresses applied in an axially symmetric manner to a circular area of the plane surface by means of a rigid disk, the torsional displacement being prescribed under the disk. Verma (1957) discussed the static distribution of stresses and displacement when shearing stress is prescribed on the circumference of a circle on the plane boundary. Datta (1961) discussed the corresponding problem when shearing stress decreases exponentially with time. Ghosh (1964) exactly evaluated the displacement at any point of the medium when a twisting moment in the form $M_0\delta(t)$ is applied to the disk by following Cagniard (1939) and Dix (1954). Ghosh (1971) also discussed the axisymmetric problem of propagation of a stress discontinuity over a circular region by using Cagniard's (1939) method as modified by De-Hoop (1959). In the present paper the author determines the displacement in the integral form due to a

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ring source which increases steadily when the twisting impulse is prescribed by $P\delta(r-ct)\,H(t)$, where δ, H are two dimensional delta function and Heaviside function respectively, and then the exact evaluation of the displacement is determined after the first arrival of the shear wave and, the displacement at any point for large values of the time t.

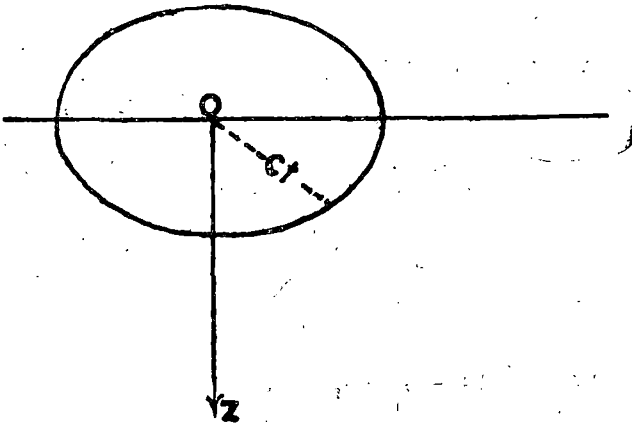

Fig. 1. Co-ordinates system in the medium.

2. FORMULATION OF THE PROBLEM :

The isotropic, elastic, semi-infinite medium is supposed to occupy the region $z \geqslant 0$. We choose cylindrical polar co-ordinates (r, θ, z) with the z-axis directed into the medium, the plane boundary being $z=0$ with origin at the centre of the source. The displacement is calculated at points inside the medium assuming that the half space is, initially, at rest and that the displacement remains bounded even as $z \rightarrow +\infty$. Since the motion is symmetrical about z-axis for torsional motion of the ring source, all quantities depend on r, z and the time t. The only non-vanishing component of the displacement vector is the component v along the direction of θ increasing. Hence the non-vanishing stress components are

$$\tau_{r\theta}=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) \quad \text{and} \quad \tau_{\theta z}=\mu\frac{\partial v}{\partial z} \tag{1}$$

where μ is the coefficient of rigidity. The only non-zeros equation of motion is

$$\frac{\partial}{\partial r}(\tau_{r\theta})+\frac{\partial}{\partial z}(\tau_{\theta z})+2\frac{\tau_{r\theta}}{r}=\rho\frac{\partial^2 v}{\partial t^2} \tag{2}$$

where ρ is the density of the medium, assumed constant. The boundary condition is

$$\tau_{\theta z}=P\ \delta\ (r-ct)\ H\ (t) \qquad \text{at } z=0 \tag{3}$$

c, P being constant H is the Heaviside function and δ is the two dimensional delta function given by

$$2\pi\int_0^{\infty} \delta\ (r)\ rdr=1.$$

3. **SOLUTION** ; We define for all positive real values of s, the Lapiaee transform f_1 (r, z, s) of a function f (r, z, t) by

$$f_1\ (r, z, s)=\int_0^{\infty} e^{-st}\ f\ (r, z, t)\ dt \tag{4}$$

Substituting the values of $\tau_{r\theta}$ and $\tau_{\theta z}$ in equation (2) and then applying the L-alapce transform (4), we obtain

$$\frac{\partial^2 v_1}{\partial r^2}+\frac{1}{r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\frac{\partial^2 v_1}{\partial z^2}=\frac{1}{\beta^2}\ s^2\ v_1 \tag{5}$$

where $\beta=\sqrt{}\ (\mu/\rho)$ is the shear wave velocity.

Defining v_2 by the equation.

$$v_2\ (\xi, z, s)=\int_0^{\infty} r\ J_1\ (\xi r)\ v_1\ (r, z, s)\ dr \tag{6}$$

and then multiplying the equation (5) by $rJ_1\ (\xi r)$ and integrating with respect to r from o to ∞, we get

$$\frac{d^2 v_2}{dz^2}=\left(\xi^2+s^2/\beta^2\right)\ v_2 \tag{7}$$

Taking ξ real, the general solution of the equation (7) which remains bounded for large values of z, is

$$v_2=A\ \exp\left[-z\ (\xi^2+s^2/\beta^2)^{1/2}\right] \tag{8}$$

The Laplace transform of $\tau_{\theta z}$ is

$$(\tau_{\theta z})_1=P\int_0^{\infty} e^{-st}\ \delta\ (r-ct)\ H\ (t)\ dt$$

$$=\frac{P}{2\pi cr}\ e^{-sr/c}$$

It's Hankel transform is

$$(\tau_{\theta z})_2=-\frac{P}{2\pi c}\int_0^{\infty} e^{-sr/c} J_1(\xi r)\,dr$$

$$=\frac{P}{2\pi\xi c}\left[1-\frac{s}{c}\left(\xi^2+\frac{s^2}{c}\right)^{-1/2}\right] \quad \text{[See Erdelyi et al 1964, p19]}$$

Noting that on $z=0$,

$\frac{dv_2}{dz}=-A\,(\xi^2+s^2/\beta^2)^{1/2}$ and using the boundary condition,

we get

$$A=-\frac{P}{2\pi\mu\xi c}\,\frac{\left[1-\frac{s}{c}\left(\xi^2+\frac{s^2}{c^2}\right)^{-1/2}\right]}{(\xi^2+s^2/\beta^2)^{1/2}}$$

Substituting this value of A in (8) and inverting the Hankel transform (6), we obtain

$$v_1=-\frac{P}{2\pi\mu c}\int_0^{\infty}\frac{1-\frac{s}{c}\left(\xi^2+\frac{s^2}{c^2}\right)^{-1/2}}{(\xi^2+s^2/\beta^2)^{1/2}}\,J_1(\xi r)\,e^{-z(\xi^2+s^2/\beta^2)^{1/2}}\,d\xi \qquad (9)$$

Now,

$$J_1(\xi r)=\frac{1}{2\pi}\int_0^{2\pi} e^{i\xi r\sin\psi}(\cos\psi-i\sin\psi)\,d\psi$$

(See Erdelyi et al 1953, p.14)

Substituting this value of $J_1(\xi r)$ in (9) and putting

$p=\xi\sin\psi$ and $q=\xi\cos\psi$, we get

$$v_1=-\frac{P}{4\pi^2\mu c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(q-ip)\left\{\left(p^2+q^2+\frac{s^2}{c^2}\right)^{1/2}-\frac{s}{c}\right\}}{(p^2+q^2+(s^2/c^2))^{1/2}\,(p^2+q^2+(s^2/\beta^2))^{1/2}}$$

$$\times\frac{e^{-z(p^2+q^2+s^2/\beta^2)^{1/2}+irp}}{(p^2+q^2)}\,dp\,dq.$$

To find the inversion of v_1, we put

$p=ms$ and $q=ns$ in the above integral, then we have

$$v_1=\frac{iP}{2\pi^2\mu c}\int_0^{\infty}dn\int_{-\infty}^{\infty}\frac{m\left\{(m^2+n^2+1/c^2)^{1/2}-1/c\right\}e^{-s\{z(m^2+n^2+1/\beta^2)^{1/2}-irm\}}}{(m^2+n^2+1/c^2)^{1/2}\,(m^2+n^2+1/\beta^2)^{1/2}\,(m^2+n^2)}\,dm$$

(10)

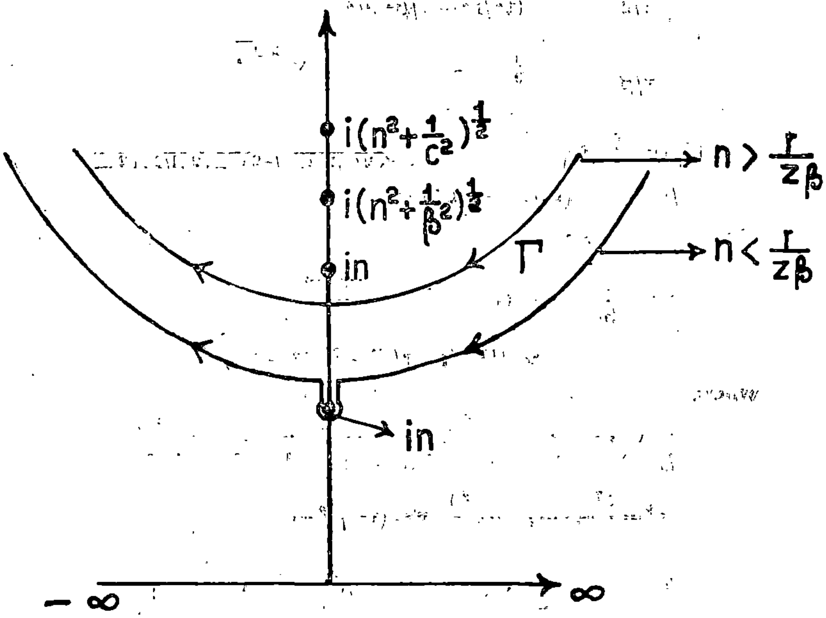

Fig--- 2

Fig 2. Path of integration in the complex m-plane.

Using the usual Cagniard-De-Hoop (1959) transformation given by

$$-imr+z\,(m^2+n^2+1/\beta^2)^{1/2}=t \tag{11}$$

where t is real and positive we obtain finally the expression of v_1 in the form

$$v_1=\frac{-P}{\pi^2\mu c}\int_0^\infty dn\int e^{-st}\ \mathrm{Im}\left[K\,(m_+,\,n)\,\frac{dm_+}{dt}\right]dt.$$

$$\{(z^2+r^2)\,(n^2+1/\beta^2)\}^{1/2}$$

where $m_+=\dfrac{irt+z\,[t^2-(z^2+r^2)\,(n^2+1/\beta^2)]^{1/2}}{z^2+r^2}$

Next change in order of integration leads to

$$v_1=\frac{-P}{\pi^2\mu c}\int_{\beta^{-1}(z^2+r^2)^{1/2}}^{\infty}{}^{-st}\,dt.\int_0^{\sqrt{\gamma}}\mathrm{Im}\left[K(m_+,\,n)\,\frac{dm_+}{dt}\right]dn.$$

where $\gamma=t^2/(z^2+r^2)-\beta^{-2}$

Taking the Laplace inversion, we get

$$v=\frac{-P}{\pi^2\mu c}\,H\left[t-\beta^{-1}(z^2+r^2)^{1/2}\right]\int_0^{\sqrt{\gamma}}\operatorname{Im}\left[K(m_+,n)\frac{dm_+}{dt}\,dn.\right. \tag{12}$$

4. APPROXIMATE EVALUATION OF THE DISPLACEMENT :

Case 1. Displacement after the first arrival.

To integrate $\displaystyle\int_0^{\sqrt{\gamma}}\operatorname{Im}\left[K(m_+,n)\frac{dm_+}{dt}\right]dn.$ (13)

we put $n=\sqrt{\gamma}\sin\alpha$ and $t_s=\beta^{-1}(z^2+r^2)^{1/2}$,

which is the time taken by the shear wave to reach the point (r, θ, z).

The integral (13) after the substitution takes the form

$$\int_0^{\pi/2}\operatorname{Im}\left[K(m_+,n)\frac{dm_+}{dt}\frac{dn}{d\alpha}\right]d\alpha. \tag{14}$$

Now, $\operatorname{Im}\left[K(m_+,n)\dfrac{dm_+}{dt}\dfrac{dn}{d\alpha}\right]=\dfrac{\beta}{r}\left[\dfrac{(z^2+r^2)^{1/2}}{\{\beta^2(z^2+r^2)-c^2r^2\}^{1/2}}-1\right]$

as $t\to t_\beta$.

Hence from (12), we obtain

$$v=\frac{-P\beta}{2\pi\mu cr}\left\{\frac{\beta(z^2+r^2)^{1/2}}{\{\beta^2(z^2+r^2)-c^2r^2\}^{1/2}}-1\right\}H(t-t_s),$$

which is the displacement at any point (r, z) just after the arrival of the disturbance. It is interesting to note that the displacement due to the first arrival of the disturbance at any point of the z-axis is zero which is also expected from the physical stand point. It is to be noted that the displacement at any point on the free surface $z=o$, varies inversely as r.

Case 2. Displacement after sufficiently large time when $z\neq o$.

In this case, $\operatorname{Im}\left[K(m_+,n)\dfrac{dm_+}{dt}\dfrac{dn}{d\alpha}\right]$

$$=\frac{(z^2+r^2)^{1/2}}{t}\;\frac{r\{z^2\sin^2\alpha-(r^2+z^2)\cos^2\alpha\}}{(z^2+r^2\cos^2\alpha)^2}-\frac{(z^2+r^2)^{3/2}}{ct^2}.$$

$$-\frac{(z^2+r^2)^{3/2}}{ct^2}\cdot\frac{rz\{z^2-3(r^2+z^2)\cos^2\alpha+r^2\cos^4\alpha\}}{(z^2+r^2\cos^2\alpha)^3}.$$

The terms containing $1/t^3$ and higher orders are neglected. After the above substitution (14) takes the following form

$$\frac{r\,(z^2+r^2)^{1/2}}{t}\int_0^{\pi/2}\frac{z^2\sin^2\alpha-(r^2+z^2)\cos^2\alpha}{(z^2+r^2\cos^2\alpha)^2}\,d\alpha-$$

$$-\frac{rz(z^2+r^2)^{3/2}}{ct^2}\int_0^{\pi/2}\frac{z^2-3\,(r^2+z^2)\cos^2\alpha+r^2\cos^4\alpha}{(z^2+r^2\cos^2\alpha)^3}\,d\alpha \qquad (15)$$

The first integral of (15) is zero, hence for the large value of the time t the displacement is given by

$$v=-Pr\,(4z^2+5r^2)/4\pi\mu\,c^2\,t^2\,z^2.$$

In this case the displacement at any point varies inversely as t^2. Also this is to be noted that the displacement increases with the increase of r when t is very large, which is in conformity with the physical condition because the radius of the ring source after large time t is infinitely large.

Case 3. Displacement at the free surface.

In this case taking $z=0$, we obtain from Eq. (10)

$$v_1=\frac{iP}{2\pi^2\mu c}\left[\int_0^{\infty}dn\int_{-\infty}^{\infty}\frac{m e^{isrm}}{(m^2+n^2+1/\beta^2)^{1/2}(m^2+n^2)}-\right.$$

$$\left.-\frac{1}{c}\int_0^{\infty}dn\int_{-\infty}^{\infty}\frac{m e^{isrm}}{(m^2+n^2+1/c^2)^{1/2}\,(m^2+n^2\;1/\beta^2)^{1/2}\,(m^2+n^2)}\right]dm.$$

$$(16)$$

The path pf integration in the complex m-plane is the real axis, which is deformed in such a way that

$-irm=t.$ where t is real and positive. Taking the integral over the deformed path we get

$$v_1=\frac{Pr}{\pi^2\mu c}\left[\int_0^{\infty}dn\int_{r(n^2+1/\beta^2)^{1/2}}^{\infty}\frac{te^{-st}}{\{t^2-r^2(n^2+1/\beta^2)\}^{1/2}\,(t^2-r^2n^2)}dt\right.$$

$$\left.+\frac{r}{c}\int_0^{\infty}dn\int_{r\,(n^2+1/\beta^2)^{1/2}}^{r\,(n^2+1/c^2)^{1/2}}\frac{te^{-st}}{\{r^2(n^2+1/c^2)-t^2\}\{t^2-r^2(n^2+1/\beta^2)\}^{1/2}(t^2-r^2n^2)}\,dt\right]$$

Changing the order of integration. we obtain

$$v_1=\frac{Pr}{\pi^2\mu c}\left[\int_{r/\beta}^{\infty}te^{-st}\,dt\int_0^{(t^2/r^2-1/\beta^2)^{1/2}}\frac{dn}{\{t^2-r^2\,(n^2+1/\beta^2)\}^{1/2}\,(t^2-r^2\,n^2)}\right.$$

$$+\frac{r}{c}\left\{\int_{r/\beta}^{r/c} te^{-st}\,dt\int_{0}^{(t^2/r^2-1/\beta^2)^{1/2}}\frac{dn}{\{r^2(n^2+1/c^2)-t^2\}^{1/2}\{t^2-r^2(n+1/\beta^2)\}^{1/2}(t^2-r^2n^2)}\right.$$

$$\left.+\int_{r/c}^{\infty} te^{-st}\,dt\int_{(t^2/r^2-1/c^2)^{1/2}}^{(t^2/r^2-1/\beta^2)^{1/2}}\frac{dn}{\{r^2(n^2+1/c^2)-t^2\}^{1/2}\{t^2-r^2(n^2+1/c^2)\}\{(t^2-r^2n^2)}\right\}\Bigg]$$

Taking Laplace inversion of the above integral, we finally obtain,

$$v=\frac{P\beta}{2\pi\mu cr}\,H\,(t-r/\beta)+\frac{Pt\,\beta^3}{\pi^2\mu cr^2(\beta^2-c^2)^{1/2}}$$

$$\times\,\{[H\,(t-r/\beta)-H\,(t-r/c)\,\Pi\,(R, R)+H\,(t-r/c).\,\Pi\,(R, R, \Phi)\}.$$

where.

$\Pi\,(R, R)$ is the complete elliptic integral of 3rd kind,

$\Pi\,(R, R, \Phi)$ is the elliptic integral of 3rd kind,

$$R^2=\frac{c^2\,(t^2\,\beta^2-r^2)}{r^2\,(\beta^2-c^2)}\quad n=(t^2\,\beta^2-r^2)/r^2,$$

$$\Phi=\mathrm{Cos}^{-1}\left[\beta/c\,\{(c^2\,t^2-r^2)/(\beta^2\,t^2-r^2)\}^{1/2}\right].$$

ACKNOWLEDGEMENT : The author expressess his greatfulness to Prof. M. L. Ghosh, D. Sc. for his guidence and suggestions in preparing this paper and is also grateful to the referee for suggesting some improvement of the paper.

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Received 17051978

ZAMM 63, 621–629 (1983)

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Torsional Response of an Elastic Half Space to a Nonuniformly Expanding Ring Source

Es werden exakte Ausdrücke für die Verschiebung in einem homogenen isotropen elastischen Halbraum in integraler Form bestimmt, der impulsförmigen Torsionskräften ausgesetzt ist, die sich über den Rand einer sich ungleichförmig ausdehnenden Schallquelle auf einer freien Oberfläche ausbreiten. Es werden sowohl positiv als auch negativ beschleunigte Ausbreitungen der Schallquelle betrachtet. Mit Hilfe des Verfahrens von Cagniard De-Hoop wird die analytische Lösung in integraler Form bestimmt. Es werden unterschiedliche Wellenfrontflächen und ihr Existenzbereich dargestellt. Die Reaktionen der ersten Bewegung beim Eintreffen verschiedener Wellen werden durch einen Grenzwertprozeß bestimmt. Die Verschiebungen auf der freien Oberfläche werden für verschiedene Positionen der Schallquelle auch numerisch berechnet und grafisch dargestellt.

Exact expressions for displacement in a homogeneous isotropic elastic half-space subjected to an impulsive torsional force spreading over the rim of a nonuniformly expanding ring source on the free surface are obtained in integral form. Both accelerating and decelerating expansion of the source have been considered. The analytic solution, in integral form, is obtained by the Cagniard De-Hoop technique. Different wave front surfaces with their region of existence have been shown. The first motion responses near different wave arrivals have been determined by a limiting process. The displacements on the free surface for different positions of the source have also been evaluated numerically and have been shown by graphs.

Для перемещений в однородном изотропном упругом полупространстве под действием удераной силы кручения, распределённой за краями неравномерно распространяющегося кольцевого источника на свободной поверхности, получены точные выражения в интегральной форме. Рассмотрено ускоренное и замедленное распространение источника. Аналитическое решение в интегральной форме получено методом Каниарда Де-Хупа. Показаны различные поверхности фронта волн и области их существования. Реакции первого движения вблизи различных волн определены предельным переходом. Перемещения на свободной поверхности для различных положений источника подсчитаны численно и представлены графически.

1. Introduction

The study of the dynamic behaviour of an elastic solid under various forms of moving loads and torsional pressure has been gaining importance day by day. This is because of their importance in seismology, structural design and underground exploration.

Gakenheimer [1] in one of his papers presented in details the problem of a load emanating from a point on the surface and then expanding radially at a constant rate. He considered the cases when the loads are disk-shaped or ring-shaped and the expanding rates are super-seismic, transeismic and sub-seismic. Almost at the same time Ghosh [2] also considered the problem of propagation of a stress discontinuity over an expanding circular region with a constant velocity which is less than the shear wave velocity of the medium. Freund [3] considered the non uniformly moving line load as well as point load. Stronge [4] discussed the problem of an accelerating line load in an acoustic half space. The non uniform pressure distribution problem applied to an elastic half space over a circular zone are discussed by Brock [5] and by Roy [6]. Almost a same type of problem has been considered by Aggarwal and Ablow [7]. There it was assumed that circularly symmetric load spreads out from a point on an acoustic half-space with decelerating speed. Ghosh [8] determined exactly the displacement produced by SH-type of waves when a torsional force is prescribed over a circular region on the free surface of a homogeneous isotropic medium and that in the integral form in case of a non homogeneous medium.

In the present paper, the displacement at any point (r, z) in the semi-infinite medium is determined in the integral form by prescribing a time dependent torsional force over the rim of a circular zone. The ring is assumed to expand in an arbitrary manner with time. It is found that the displacement field contains besides the usual SH-waves, contribution from conical waves which arise due to the motion of the source. The region of conical waves which depend on the nature of the motion of the source and the initial speed of expansion of the source are investigated in details. Different wave front surfaces are located and first motion responses near different wave arrivals have been obtained.

Finally numerical evaluation of the displacement on the free surface has been made for a decelerating ring source whose radius at time t is of the form $h(t) = At^{1/2}$. Displacements at points on the free surface for different position of the source have been shown by means of graphs.

2. Formulation of the Problem

Consider a homogeneous isotropic elastic half space on the free surface of which a ring source producing SH-type of waves is expanding with non-uniform velocity. (r, θ, z) are the cylindrical polar co-ordinates, z-axis being directed into the medium and the plane boundary being $z = 0$. The origin of co-ordinates is at the centre of the ring $r = h(t)$, $z = 0$. The ring is assumed to expand with uniform acceleration or with deceleration and an impulsive torque applied to the ring is prescribed.

The displacement is determined in the integral form at any point inside and on the free surface of the medium, subject to the condition that the half-space is initially at rest and that the displacements remain bounded for large values of z. For torsional motion of the ring all quantities depend on r, z and the time t. We assume that $h(t)$ is non negative and monotone increasing function. The only non-zero component of the displacement vector is the component v along the direction of θ increasing. The relevant non vanishing stress components are

$$\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \quad \text{and} \quad \tau_{\theta z} = \mu\frac{\partial v}{\partial z}, \tag{1 a, b}$$

where μ is the Lamé's constant. The nonzero equation of the displacement field is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} \tag{2}$$

where β is the shear wave velocity. The boundary condition of the motion is

$$\tau_{\theta z} = -P\delta[h(t) - r]\,H(t)\,, \qquad z = 0 \tag{3}$$

where P is a constant, $\delta(\ )$ is Dirac's delta function, $H(\ )$ is the Heaviside step function and $h(t)$ is the radius of the ring at time t. Initial conditions of motion are given by

$$h(t) = 0\,, \qquad t = 0 \quad \text{and} \quad \dot{h}(t) > 0\,, \qquad t > 0 \tag{4}$$

where dot denotes the time derivative.

3. Method of solution

We define Laplace transform $f_1(r, z, p)$ of the function $f(r, z, t)$ by

$$f_1(r, z, p) = \int_0^\infty \exp(-pt)\, f(r, z, t)\, \mathrm{d}t \tag{5}$$

where p is real and positive and Hankel transform $f_2(\xi, z, p)$ of $f_1(r, z, p)$ by

$$f_2(\xi, z, p) = \int_0^\infty rJ_1(\xi r)\, f_1(r, z, p)\, \mathrm{d}r \tag{6}$$

where J_n is the Bessel function of the first kind of order n.

Applying Laplace and Hankel transforms, to the equation (2) successively we obtain

$$\frac{\mathrm{d}^2 v_2}{\mathrm{d}z^2} - k^2 v_2 = 0 \tag{7}$$

where $k^2 = \xi^2 + (p^2/\beta^2)$.

The solution of the equation (7) which remains bounded as $z \to +\infty$ is

$$v_2 = K \exp(-kz)\,. \tag{8}$$

The value of the constant K is determined, by using the condition (4), the equation (8) and the Hankel transform of the Laplace transform of the equation (3). It is found to be

$$K = \frac{P}{\mu}\int_0^\infty \frac{1}{k}\, h(\tau)\, J_1(\xi h(\tau)) \exp(-p\tau)\, \mathrm{d}\tau\,. \tag{9}$$

Substituting the value of K in (8) and then taking Hankel's inversion one gets

$$v_1 = \frac{P}{\mu}\int_0^\infty h(\tau)\exp(-p\tau)\int_0^\infty \frac{\xi}{k} J_1(\xi r)\, J_1(\xi h(\tau)) \exp(-kz)\, \mathrm{d}\xi\, \mathrm{d}\tau\,. \tag{10}$$

4. Laplace Inversion

In this section the Laplace inverse transform is evaluated by Cagniard's technique.

We make use of the following results

$$J_1(\xi h(\tau))\, J_1(\xi r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} J_0(\xi S)\cos\varphi\, \mathrm{d}\varphi\,, \qquad J_0(\xi S) = \frac{1}{2\pi}\int_0^{2\pi} \exp(i\xi S\cos u)\, \mathrm{d}u$$

where

$$S = \left(r^2 + h^2(\tau) - 2rh(\tau)\cos\varphi\right)^{1/2},$$

to obtain the equation (10) as

$$v_1 = \frac{P}{4\pi^2\mu}\int\limits_0^\infty h(\tau)\exp(-p\tau)\int\limits_{-\pi}^{\pi} I\cos\varphi\,\mathrm{d}\varphi\,\mathrm{d}\tau\,, \tag{11}$$

where

$$I = \int\limits_0^\infty\int\limits_0^{2\pi}\frac{\xi}{k}\exp\left(i\xi S\cos(\psi - u) - kz\right)\mathrm{d}\xi\,\mathrm{d}u \tag{12}$$

and ψ is any constant angle.

In (12), we put

$$\alpha' = \xi\cos u\,, \qquad \beta' = \xi\sin u\,,$$

then substitute $\alpha' = w\cos\psi - q\sin\psi$ and $\beta' = w\sin\psi + q\cos\psi$ and finally replace w by wp and q by qp to obtain (12) in the form

$$I = p\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{\exp\left[-p\{-iwS + (w^2 + q^2 + 1/\beta^2)^{1/2}z\}\right]}{(w^2 + q^2 + 1/\beta^2)^{1/2}}\,\mathrm{d}w\,\mathrm{d}q\,. \tag{13}$$

Equation (13) is the well known form for determining the Laplace inversion of a function by applying Cagniard's technique as modified by De-Hoop. Substituting $t = -iwS + z(w^2 + q^2 + 1/\beta^2)^{1/2}$ in (13) where t is real and positive, the Laplace inversion of (13) is found to be equal to

$$G(t) = 4\frac{\mathrm{d}}{\mathrm{d}t}\left\{H[t - \varrho/\beta]\int\limits_0^{(t^2/\varrho^2 - 1/\beta^2)^{1/2}} Rl\left[\frac{1}{(w_+^2 + q^2 + 1/\beta^2)^{1/2}}\,\frac{\mathrm{d}w_+}{\mathrm{d}t}\right]\mathrm{d}q\right\} \tag{14}$$

where $\varrho^2 = z^2 + S^2$ and

$$w_+ = \frac{iSt + z\{t^2 - \varrho^2(q^2 + 1/\beta^2)\}^{1/2}}{\varrho^2}.$$

Applying the convolution theorem on (11), the Laplace inversion of v_1 is obtained in the form

$$v = \frac{P}{4\pi^2\mu}\int\limits_0^\infty h(\tau)\,\mathrm{d}\tau\int\limits_{-\pi}^{\pi}\cos\varphi\,\mathrm{d}\varphi\int\limits_0^t \delta(u - \tau)\,G(t - u)\,\mathrm{d}u\,,$$

which when simplified takes the form

$$v = \frac{P}{\pi\mu}\int\limits_0^t h(\tau)\,\mathrm{d}\tau\int\limits_0^\pi\frac{\cos\varphi}{\varrho}\,\delta(t - \tau - \varrho/\beta)\,\mathrm{d}\varphi\,. \tag{15}$$

Integrating over φ, we obtain

$$v = \frac{P\beta}{\pi r\mu}\int\limits_0^t\left\{H\left[t - \tau - \frac{\sqrt{z^2 + \left(r - h(\tau)\right)^2}}{\beta}\right] - H\left[t - \tau - \frac{\sqrt{z^2 + \left(r + h(\tau)\right)^2}}{\beta}\right]\right\}Q(\tau)\,\mathrm{d}\tau \tag{16}$$

where

$$Q(\tau) = \frac{z^2 + r^2 + h^2(\tau) - \beta^2(t - \tau)^2}{\left[\{z^2 + \left(r + h(\tau)\right)^2 - \beta^2(t - \tau)^2\}\,\{\beta^2(t - \tau)^2 - z^2 - \left(r - h(\tau)\right)^2\}\right]^{1/2}}.$$

To facilitate our discussion, equation (16) is written in an alternative form,

$$\begin{aligned} v = \frac{P\beta}{\pi r\mu}\int\limits_0^{t - z/\beta} &\{H\left[r - h(\tau) + \sqrt{\beta^2(t - \tau)^2 - z^2}\right]H\left[h(\tau) - \sqrt{\beta^2(t - \tau)^2 - z^2}\right] + \\ &+ H\left[r + h(\tau) - \sqrt{\beta^2(t - \tau)^2 - z^2}\right]H\left[-h(\tau) + \sqrt{\beta^2(t - \tau)^2 - z^2}\right] - \\ &- H\left[r - h(\tau) - \sqrt{\beta^2(t - \tau)^2 - z^2}\right]\}\,Q(\tau)\,\mathrm{d}\tau\,. \end{aligned} \tag{17}$$

The region of support for τ-integration is bounded by the curves:

$$\text{I:}\quad r = h(\tau) + \sqrt{\beta^2(t - \tau)^2 - z^2}; \qquad 0 < \tau < t - z/\beta\,, \tag{18}$$

$$\text{II:}\quad r = h(\tau) - \sqrt{\beta^2(t - \tau)^2 - z^2}; \qquad \tau_0 < \tau < t - z/\beta\,, \tag{19}$$

$$\text{III:}\quad r = -h(\tau) + \sqrt{\beta^2(t - \tau)^2 - z^2}; \qquad 0 < \tau < \tau_0\,. \tag{20}$$

The region of τ integration for $Q(\tau)$ bounded by the curves I, II and III are shown in the figs. 1(a—f) and the following remarks can be made about them. It is to be noted that the curves II and III are monotone increasing and decreasing in their respective region of existence viz. $(\tau_0, t - z/\beta)$ and $(0, \tau_0)$ where

$$h(\tau_0) = \{\beta^2(t - \tau_0)^2 - z^2\}^{1/2} .$$

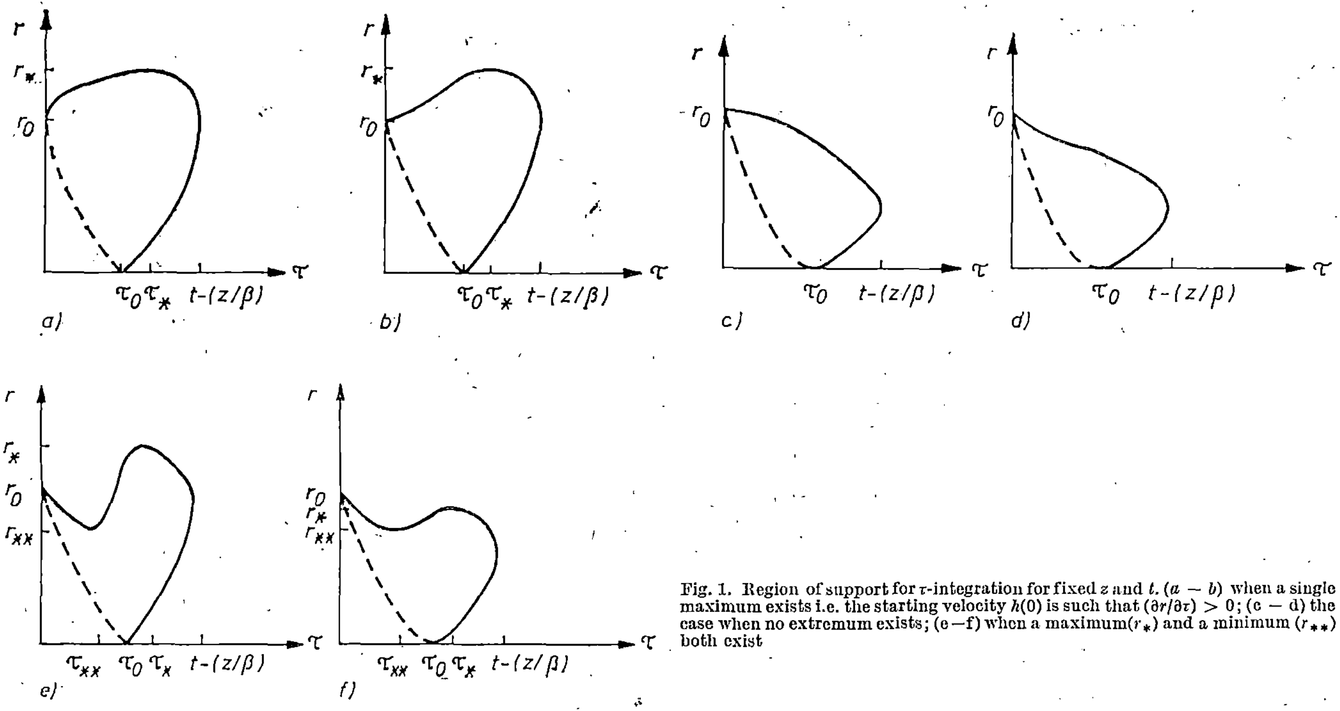

Fig. 1. Region of support for τ-integration for fixed z and t. (a — b) when a single maximum exists i.e. the starting velocity $\dot{h}(0)$ is such that $(\partial r/\partial\tau) > 0$; (c — d) the case when no extremum exists; (e—f) when a maximum(r_*) and a minimum (r_{**}) both exist

The curve I has an extremum where

$$\frac{\partial r}{\partial\tau} = \dot{h}(\tau) - \frac{\beta^2(t - \tau)}{\{\beta^2(t - \tau)^2 - z^2\}^{1/2}} \tag{21}$$

vanishes and

$$\frac{\partial^2 r}{\partial\tau^2} = \ddot{h}(\tau) - \frac{\beta^2 z^2}{\{\beta^2(t - \tau)^2 - z^2\}^{3/2}} \tag{22}$$

does not vanish.

We consider the different cases that arise due to non uniform increase of the ring source. Let the source increase with uniform acceleration $\ddot{h}(\tau) > 0$. In this case, if the initial velocity $\dot{h}(0)$ of the source be such that $\left(\frac{\partial r}{\partial\tau}\right)_0 > 0$, then since $\left(\frac{\partial r}{\partial\tau}\right)_0 > 0$ and $\left(\frac{\partial r}{\partial\tau}\right)_{t-z/\beta} < 0$, the curve I has only one maximum at $r = r_*$ because $\frac{\partial^2 r}{\partial\tau^2}$ either changes sign once from positive to negative or remains negative throughout in $(0, t - z/\beta)$. The corresponding cases are shown in Fig. 1 (a—b). Next let the initial velocity $\dot{h}(0)$ of the source be such that $\left(\frac{\partial r}{\partial\tau}\right)_0 < 0$. In this case, if $\left(\frac{\partial^2 r}{\partial\tau^2}\right)_0 < 0$, $\left(\frac{\partial^2 r}{\partial\tau^2}\right)$ will be negative throughout the interval $(0, t - z/\beta)$; the curve I then corresponds to Fig. 1 (c), since both $\left(\frac{\partial r}{\partial\tau}\right)_0$ and $\left(\frac{\partial r}{\partial\tau}\right)_{t-z/\beta}$ are negative. But if $\left(\frac{\partial^2 r}{\partial\tau^2}\right)_0$ be positive, then $\frac{\partial^2 r}{\partial\tau^2}$ changes sign once from positive to negative in the interval $(0, t - z/\beta)$. Hence in this case the curve I has either no extremum which corresponds to Fig. 1 (d) or there is a maximum preceded by a minimum which is shown in Fig. 1 (e, f). Finally, in case of decelerating motion of the source i.e. when $\ddot{h}(\tau)$ (not necessarily a constant) < 0, throughout the interval, the curve has either only one maximum if $\left(\frac{\partial r}{\partial\tau}\right)_0 > 0$ as in Fig. 1 (a) or no extremum as in Fig. 1 (c) when $\left(\frac{\partial r}{\partial\tau}\right)_0 < 0$.

We consider the curves I and II together. Their combined equation is

$$(r - h(\tau))^2 = \beta^2(t - \tau)^2 - z^2 . \tag{23}$$

For figures 1 (c, d), τ is a single valued function of r. For the figs. 1 (a, b), τ may be a double valued function whereas for figs. 1 (e, f), τ may be triple valued function of r. Taking the equations (18) and (19) together, the values of τ are designated as $\tau = \tau_1$, $\tau = (\tau_1, \tau_2)$ and $\tau = (\tau_1, \tau_2, \tau_3)$ where $\tau_1 > \tau_2 > \tau_3$ depending on whether τ is single, double or triple valued function of r. In (20) r is a monotone decreasing function of τ, so the corresponding value of τ is designated as $\tau = \tau_4$.

With the above values of the roots of the equations (18)—(20) and from a close examination of the different figs. 1 (a—f), the displacement produced by the SH-type of waves is given by $v = v^1 + v^2 + v^3$, where

$$\left.\begin{aligned} v^1 &= BH(r_0 - r)\, I\big(Q(\tau); \tau_4, \tau_1\big)\,, \\ v^2 &= B[H(r - r_0) - G\big(r - \max(r_*, r_0)\big)]\, I\big(Q(\tau); \tau_2, \tau_1\big)\,, \\ v^3 &= B[G\big(r - \min(r_*, r_0)\big) - G\big(r - \min(r_{**}, r_0)\big)]\, I\big(Q(\tau); \tau_3, \tau_2\big) \end{aligned}\right\} \tag{24}$$

and

$$G\big(r - \max(r_*, r_0)\big) = \begin{cases} H(r - r_*) & \text{if } \; r_* = \max(r_*, r_0) \\ H(r - r_0) & \text{if } \; r_0 = \max(r_*, r_0) \quad \text{or } r_* \text{ does not exist}\,. \end{cases}$$

$$G\big(r - \min(r_*, r_0)\big) = \begin{cases} H(r - r_*) & \text{if } \; r_* = \min(r_*, r_0) \\ H(r - r_0) & \text{if } \; r_0 = \min(r_*, r_0) \quad \text{or } r_* \text{ does not exist}\,. \end{cases}$$

Similar meaning is attached to the symbol

$G\big(r - \min(r_{**}, r_0)\big)$. B has been written for $\dfrac{P\beta}{\pi r \mu}$. $r_0 = \sqrt{\beta^2 t^2 - z^2}$ is the value of r at $\tau = 0$ and

$$I\big(F(\tau); a, b\big) = \int_a^b F(\tau)\, \mathrm{d}\tau\,.$$

5. Wave Front Analysis

In this section we locate and analyse the nature of the wave fronts.

It is known that wave front is a surface $\varphi(r, z, t) = 0$ which is a characteristic of the differential equation (2) and also satisfy the eikonal equation $\varphi_r^2 + \varphi_z^2 - \beta^{-2}\varphi_t^2 = 0$ [9].

The nature of the wave front changes due to non-uniform expansion of the source and also it depends on the initial velocity $\dot{h}(0)$ $(= u_0)$ of expansion of the source. We consider decelerating and accelerating expansion of the source for different initial velocities.

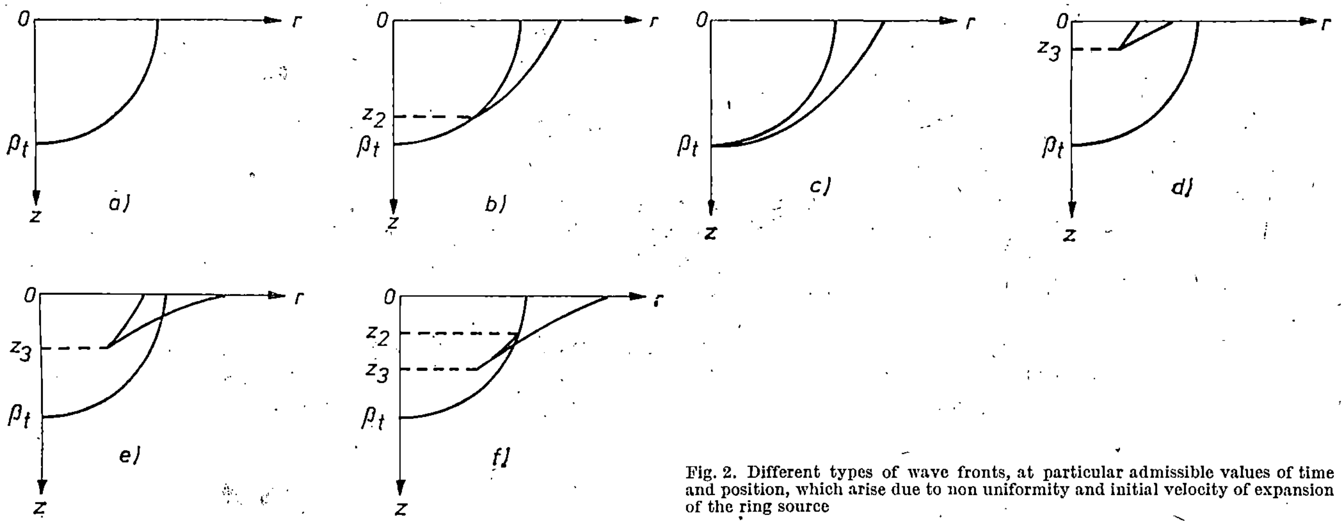

Fig. 2. Different types of wave fronts, at particular admissible values of time and position, which arise due to non uniformity and initial velocity of expansion of the ring source

Case of Deceleration

i) Let $\dot{h}(0) = u_0 < \beta$.

From (21) and (22), $\left(\dfrac{\partial r}{\partial \tau}\right)_0$ is negative for all z and $\dfrac{\partial^2 r}{\partial \tau^2}$ is also negative as $\ddot{h}(0)$ is negative. So the curve I in $(0, t - z/\beta)$ is such that r decreases with the increase of τ. This corresponds to the region of integration as depicted in fig. 1 (d) and consequently the wave front is of the form as shown in fig. 2 (a).

ii) u_0 $(> \beta)$ is finite.

It follows from (21), $\left(\dfrac{\partial r}{\partial \tau}\right)_0$ is positive for $0 < z < z_2$ and negative for $z > z_2$ where z_2 is obtained from $z_2 = \beta t(1 - \beta^2/u_0^2)^{1/2}$, therefore the region of integrations for $0 < z < z_2$ and for $z > z_2$ correspond to the regions shown in the figs. 1 (a) and 1 (c) respectively and consequently the wave front is given by the fig. 2 (b).

iii) u_0 is infinitely large.

From (21) and (22), it follows that $\left(\frac{\partial r}{\partial \tau}\right)_0$ is positive for all z and $\left(\frac{\partial^2 r}{\partial \tau^2}\right)$ is negative for all z and for all τ. Hence the region of integration is fig. 1 (a) and the corresponding wave front is as shown in fig. 2 (c).

Case of Acceleration

i) We assume that the ring source expands with uniform acceleration f and starts with the velocity u_0 $(= \dot{h}(0))$. First let $u_0 < \beta$; then $(\partial r/\partial \tau)_0$ is negative for all z and $(\partial^2 r/\partial \tau^2)_0$ is positive for $0 < z < z_1$ and negative for $z > z_1$, where z_1 is to be determined from the condition $(\partial^2 r/\partial \tau^2)_0 = 0$. For $z > z_1$, $(\partial^2 r/\partial \tau^2)$ is negative for τ in $(0, t - z/\beta)$. Consequently the region of integration is the fig. 1 (c). On the other hand if z lies in $(0, z_1)$ then $(\partial^2 r/\partial \tau^2)$ is first positive and then negative as τ increases in $(0, t - z/\beta)$, so in this case the region of integration is either fig. 1 (d) or fig. 1 (e or f).

By using (22), z_1 is determined from the equation

$$f = \beta^2 z_1^2/(\beta^2 t^2 - z_1^2)^{3/2} . \tag{25}$$

It is to be noted that $z_1 = 0$ when $f = 0$ and z_1 is a monotone increasing function of f. Further, in $(0, z_1)$, $(\partial r/\partial \tau)$ may have two zeroes or there is no zero in the region $0 < \tau < t - z/\beta$, depending on the value of z. The condition that $(\partial r/\partial \tau)$ may have two zeroes is $0 < z < z_3$, where

$$z_3 = \begin{cases} \beta\left(\dfrac{u_0 + ft}{f}\right)\left\{1 - \dfrac{\beta^{2/3}}{(u_0 + ft)^{2/3}}\right\}^{3/2} & \text{for} \quad u_0 + ft > \beta \\ 0 & \text{for} \quad u_0 + ft \leqq \beta . \end{cases}$$

It can be shown further that $z_3 < z_1$. Hence for $0 < z < z_3$ the region of integration is fig. 1 (e or f) and for $z_3 < z < z_1$, the region of integration is fig. 1 (d). Therefore for accelerating source with initial velocity $u_0 < \beta$, the wave front is of the form as shown in fig. 2 (a) if the observation time be such that $(u_0 + ft) \leqq \beta$ and for $(u_0 + ft) > \beta$ the wave front is like the figures as in 2 (d) or 2 (e) according as the position of the source at the observation time is inside or outside the characteristic surface $r^2 + z^2 = \beta^2 t^2$.

ii) Next let $u_0 > \beta$; from (21) we have $(\partial r/\partial \tau)_0$ is positive for $0 < z < z_2$ and is negative for $z > z_2$ where z_2 is given by

$$z_2 = \beta t(1 - \beta^2/u_0^2)^{1/2} . \tag{26}$$

Also $(\partial^2 r/\partial \tau^2)_0$ is positive for $0 < z < z_1$ and is negative for $z > z_1$, where z_1 is given by (25). So for $0 < z < z_1$, $(\partial^2 r/\partial \tau^2)$ is first positive and then negative in $0 \leqq \tau < (t - z/\beta)$. We consider the case for $z_1 < z_2$ first. In this case

$$\beta^2 z_1^2/(\beta^2 t^2 - z_1^2)^{3/2} < \beta^2 z_2^2/(\beta^2 t^2 - z_2^2)^{3/2} ,$$

since $\beta^2 z^2/(\beta^2 t^2 - z^2)^{3/2}$ is a monotone increasing function of z. Using (25) and (26), we obtain

$$\beta^2(u_0 + ft)/u_0^3 < 1 . \tag{27}$$

Under the condition obtained in (27), the region of integration is like that of the fig. 1 (b) in the range $0 < z < z_1$ and for $z_1 < z < z_2$, the region of integration is of the type as shown in fig. 1 (a). For $z_2 < z < \beta t$, the region of integration is shown in fig. 1 (c). Therefore for $u_0 > \beta$ and for the relation given in (27), it follows that the nature of the wave front is of the type as shown in fig. 2 (b).

Finally, we study the case when $z_1 > z_2$ i.e. when $\beta^2(u_0 + ft)/u_0^3 > 1$.

Here for $0 < z < z_2$ the region of integration is as in fig. 1 (b). Since z_3 is always less than z_1, so for $z_2 < z < z_3$ the region of integration is like fig. 1 (e or f) and for $z_3 < z < z_1$ the region of integration is like that as shown in fig. 1 (d). Fig. 1 (c) represents the region of integration for $z_1 < z < \beta t$. Accordingly the wave front takes the shape of the fig. 2 (f).

6. First Motion Responses

The expression for the displacement as given in (24) is in the form of integrals over finite ranges. As such, computation of displacement for a given model can be done with the high power computer. However some idea about the nature of displacement at the time of the first arrival of wave fronts can be obtained by a limiting process following Stronge [4].

The displacement field just after arrival time of the characteristic surface $r = r_*$ is from (24),

$$v = BI\big(Q(\tau); \tau_2, \tau_1\big) \tag{28}$$

whereas just before the arrival time the displacement is given by $v = 0$.

To evaluate (28) near $r = r_*$, we put $r = r_* - \Delta r$ and $\tau = \tau_* + \theta$ in equations (18) and (19). Using Taylor's expansion in the neighbourhood of (τ_*, r_*) and by help of equations (21) and (22) we find the limits of integration of equation (28) in the new variable θ as

$$\theta_{1,2} = \pm \frac{\sqrt{2\Delta r}\ \{\beta^2(t - \tau_*)^2 - z^2\}^{3/4}}{[\beta^2 z^2 - \ddot{h}(\tau_*)\ \{\beta^2(t - \tau_*)^2 - z^2\}^{3/2}]^{1/2}} . \tag{29}$$

The same procedure is followed to determine in the neighbourhood of (τ_*, r_*), the value of Q which is found to be

$$Q(\tau_* + \theta) = \frac{[r_* h(\tau_*)\,\{\beta^2(t-\tau_*)^2 - z^2\}]^{1/2}}{[\theta^2\{\ddot{h}(\tau_*)\,(\beta^2(t-\tau_*)^2 - z^2)^{3/2} - \beta^2 z^2\} + 2\Delta r\{\beta^2(t-\tau_*)^2 - z^2\}^{3/2}]^{1/2}} \tag{30}$$

where the lowest terms in θ and Δr are retained. The value of the integral (28) after substituting the value of Q from (30) and the limits of integration for the new variable θ as obtained in (29) is found to be

$$\frac{P\beta}{r\mu}\left[\frac{r_* h(\tau_*)\,\{\beta^2(t-\tau_*)^2 - z^2\}}{\beta^2 z^2 - \ddot{h}(\tau_*)\,\{\beta^2(t-\tau_*)^2 - z^2\}^{3/2}}\right]^{1/2},$$

which is the displacement at the first arrival of the wave front given by $r = r_*$.

To find the displacement at the first arrival of the wave front given by $r = r_{**}$, we define $Q(\tau)$ in the neighbourhood of (τ_{**}, r_{**}) and outside the region of integration by

$$Q(\tau) = \frac{z^2 + r^2 + h^2(\tau) - \beta^2(t-\tau)^2}{\{z^2 + (r + h(\tau))^2 - \beta^2(t-\tau)^2\}^{1/2}\,|\{\beta^2(t-\tau)^2 - z^2 - (r - h(\tau))^2\}^{1/2}|} \tag{31}$$

and put $r = r_{**} + \Delta r$ and $\tau = \tau_{**} + \theta$. Following the same procedure as done in case of $r = r_*$, the displacement at the first arrival of the wave surface $r = r_{**}$ is found to be

$$\frac{P\beta}{r\mu}\left[\frac{r_{**} h(\tau_{**})\,\{\beta^2(t-\tau_{**})^2 - z^2\}}{\ddot{h}(\tau_{**})\,\{\beta^2(t-\tau_{**})^2 - z^2\}^{3/2} - \beta^2 z^2}\right]^{1/2}.$$

The displacement at a point due to the first arrival of the wave fronts $r = r_*$ and $r = r_{**}$ simultaneously, is also determined. At this point wave fronts $r = r_*$ and $r = r_{**}$ from a cusp (cf. fig. 2 (d, e, f). In this case this is to be noted that at the cusp $r = r_* = r_{**} = \bar{r}$ (say) and $(\partial r/\partial\tau) = (\partial^2 r/\partial\tau^2) = 0$ where as $(\partial^3 r/\partial\tau^3)) \neq 0$. Hence it follows from equation (24) that the displacement due to first arrival of this wave front at $r = \bar{r}$ is

$$\frac{P\beta}{\pi r\mu}[I(Q(\tau); \bar{\tau}, \tau_1) + I(Q(\tau); \tau_2, \bar{\tau})] \tag{32}$$

where $\bar{\tau} = \tau_* = \tau_{**}$ and τ_1, τ_2 are the two values of τ close to $\bar{\tau}$ and correspond to the points lying on either side of $(\bar{\tau}, \bar{r})$ on the curve and I and II together.

To evaluate the integrals in (32), $\tau = \bar{\tau} + \theta$ and $r = \bar{r} - \Delta r$ are put in the first integral where as $\tau = \bar{\tau} - \theta$ and $r = \bar{r} + \Delta r$ are put into the second integral of (32). Also this is to be remembered that outside the region of integration in the neighbourhood of $(\bar{\tau}, \bar{r})$, $Q(\tau)$ is defined as in (31).

After the above mentioned substitution in (28) and retaining the lowest order term of θ and Δr, one gets the displacement due to first arrival at $r = \bar{r}$ as

$$\frac{2P\beta}{\pi\bar{r}\mu}\left[\frac{3\bar{r}h(\bar{\tau})\,\{\beta^2(t-\bar{\tau})^2 - z^2\}^2}{3\beta^4 z^2(t-\bar{\tau}) - \dddot{h}(\bar{\tau})\,\{\beta^2(t-\bar{\tau})^2 - z^2\}^{5/2}}\right]^{1/2}\int\limits_0^a \frac{\mathrm{d}\theta}{\sqrt{(a^3 - \theta^3)}} \tag{33}$$

where

$$a^3 = \frac{6\Delta r(\bar{r} - h(\bar{\tau}))^5}{3\beta^4 z^2(t-\bar{\tau}) - \dddot{h}(\bar{\tau})\,(\bar{r} - h(\bar{\tau}))^5}.$$

By substituting

$$\theta^3 = a^3 \sin^2\alpha\,,$$

the integral in (33) is evaluated. The displacement due to first arrival of the wave front $r = \bar{r}$ is found to be

$$\frac{2^{5/6}P\beta}{3^{2/3}\pi\bar{r}\mu(\Delta r)^{1/6}}\,\frac{\bar{r}^{1/2}h^{1/2}(\bar{\tau})\,\{\beta^2(t-\bar{\tau})^2 - z^2\}^{7/12}}{[3\beta^4 z^2(t-\bar{\tau}) - \dddot{h}(\bar{\tau})\,\{\beta^2(t-\bar{\tau})^2 - z^2\}^{5/2}]^{1/3}}\,B\left(\frac{1}{3}, \frac{1}{2}\right)$$

where $B(m, n)$ is the Beta function.

It is interesting to note that in this case the displacement due to first arrival at this point is infinitely large due to the presence of the factor $(\Delta r)^{1/6}$ in the denominator.

Finally we consider the characteristic surface $r^2 + z^2 = \beta^2 t^2$ which corresponds to a disturbance initiated at the origin when the torque is first applied at $\tau = 0$. This disturbance spreads out from the origin with a velocity equal to β. To find the displacement due to the first arrival of this surface, following Aggarwal and Ablow [7] let us consider the curve

$$\Gamma\colon\quad r = \{\beta^2(t-\tau)^2 - z^2\}^{1/2} \tag{34}$$

and the lines

$$l_1\colon\quad r = \sqrt{(\beta^2 t^2 - z^2)} - \varepsilon_2; \qquad l_2\colon\quad r = \sqrt{(\beta^2 t^2 - z^2)} + \varepsilon_1 \tag{35), (36}$$

where ε_1 and ε_2 are very small positive quantities. Then to the first order of ε_1 and ε_2,

$$(l_1 \times I) \equiv \frac{\varepsilon_2 \sqrt{\beta^2 t^2 - z^2}}{\beta^2 t} = \tau'(\text{say}),$$

$$(l_1 \times \text{III}) \equiv \frac{\varepsilon_2 \sqrt{\beta^2 t^2 - z^2}}{\dot{h}(0) \sqrt{(\beta^2 t^2 - z^2)} + \beta^2 t} = \tau_4(\text{say}),$$

$$(l_2 \times \text{I}) \equiv \frac{\varepsilon_1 \sqrt{\beta^2 t^2 - z^2}}{\dot{h}(0) \sqrt{(\beta^2 t^2 - z^2)} - \beta^2 t} = \tau_2(\text{say});$$

ε_1, ε_2 are such that $\tau_2 < \tau'$ and tends to zero as $t \to \varrho_0/\beta$, where $\varrho_0 = \sqrt{(r^2 + z^2)}$. Then it follows immediately that

$$I(Q(\tau); \tau_4, \tau') \to 0 \quad \text{and} \quad I(Q(\tau); \tau_2, \tau') \to 0 \quad \text{as} \quad t \to \varrho_0/\beta\,.$$

Also $I(Q(\tau); \tau', \tau_1') - I(Q(\tau); \tau', \tau_1) \to 0$ as $t \to \varrho_0/\beta$, where $\tau_1' \equiv (l_1 \times I)$ and $\tau_1 \equiv (l_2 \times I)$ are the values of τ which correspond to the points on the right of τ'. From this it follows that the displacement is continuous across the characteristic surface $\varrho_0 = \beta t$, showing that the displacement due to the first arrival of the characteristic surface $r^2 + z^2 = \beta^2 t^2$ is zero.

7. Surface Displacement

In this section surface displacement has been determined numerically for a particular type of nonuniformly moving surface. We consider a decelerating ring source whose radius $h(\tau)$ at any time τ is assumed to be $h(\tau) = A\tau^{1/2}$. The displacement at any point $(r, 0)$ at the time of observation t is determined.

According to the position of the source the following three possible cases are considered.

i) Radius $h(\tau)$ of the ring coinciding with the rim of the conical wave front and moving with it so that $\beta t < h(t)$.

ii) $\beta t < h(t) < r_*$.

iii) $h(t) < \beta t$.

To determine the displacement on the free surface, we put $z = 0$ in the function $Q(\tau)$ of equation (24) and the variable of integration τ is changed to T, by substituting $T = \tau/t$. $Q(\tau)$ is then obtained in the form

$$Q\,(Tt) = R(T) = \frac{\dfrac{r^2}{\beta^2 t^2} + \dfrac{A^2}{\beta^2 t} T - (1 - T)^2}{\left[\left\{\left(\dfrac{r}{\beta t} + \dfrac{A}{\beta}\sqrt{\dfrac{T}{t}}\right)^2 - (1 - T)^2\right\}\left\{(1 - T)^2 - \left(\dfrac{r}{\beta t} - \dfrac{A}{\beta}\sqrt{\dfrac{T}{t}}\right)^2\right\}\right]^{1/2}}$$

on a close examination of the regions of integration as shown in Fig. 1, the displacement v, in case of (i) is given by

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad 0 < r < \beta t\,,$$

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad \beta t < r < h(t)\,.$$

The displacement in case of (ii) is given by

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad r < \beta t\,,$$

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad \beta t < r < h(t)\,,$$

$$\frac{\mu v}{P} = \frac{\beta t}{\pi r} I(R(T); T_1, T_3) \quad \text{for} \quad h(t) < r < r_*$$

and the displacement in case (iii) is

$$\frac{\mu}{P} v = \frac{\beta t}{\pi r} I(R(T); T_1, T_2) \quad \text{for} \quad 0 < r < h(t)\,,$$

$$\frac{\mu}{P} v = \frac{\beta t}{\pi r} I(R(T); T_1, T_3) \quad \text{for} \quad h(t) < r < \beta t\,,$$

$$\frac{\mu}{P} v = \frac{\beta t}{\pi r} I(R(T); T_1, T_3) \quad \text{for} \quad \beta t < r < r_*$$

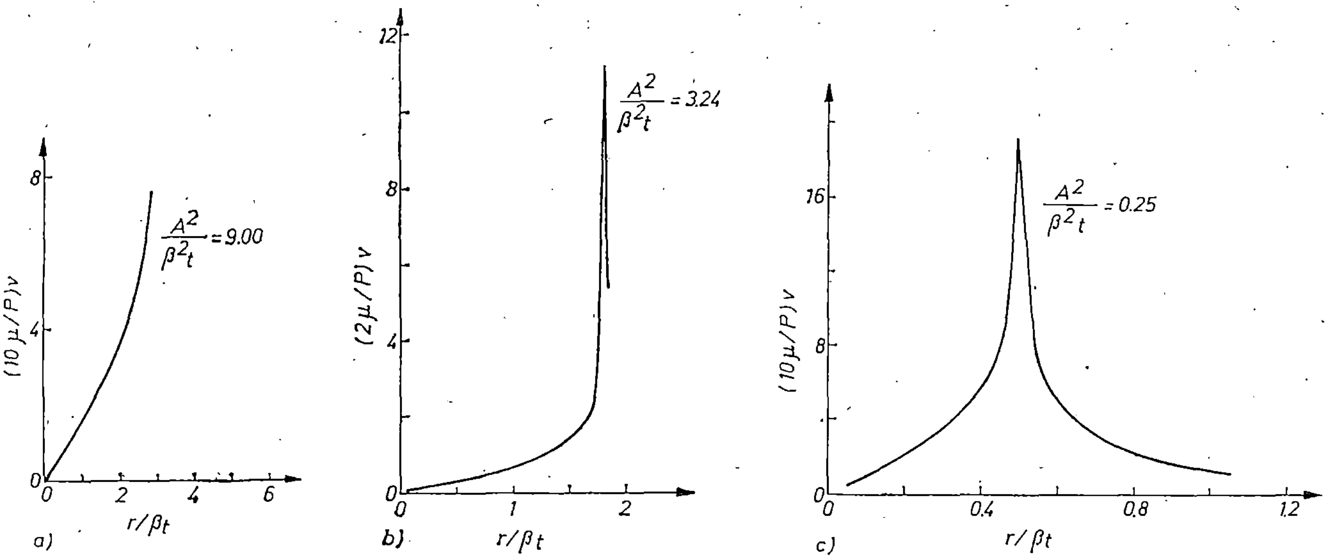

Fig. 3. Graphs showing $(\mu/P)\,v$ versus $(r/\beta t)$ when $z = 0$. (a), (b), (c) correspond to the cases (i), (ii) and (iii) respectively

where

$$T_1 = 1 - \frac{r}{\beta t} + \frac{1}{2}\frac{A^2}{\beta^2 t} - \frac{1}{2}\frac{A}{\beta\sqrt{t}}\sqrt{\frac{A^2}{\beta^2 t} + 4\left(1 - \frac{r}{\beta t}\right)},$$

$$T_2 = 1 + \frac{r}{\beta t} + \frac{1}{2}\frac{A^2}{\beta^2 t} - \frac{1}{2}\frac{A}{\beta\sqrt{t}}\sqrt{\frac{A^2}{\beta^2 t} + 4\left(1 + \frac{r}{\beta t}\right)},$$

$$T_3 = 1 - \frac{r}{\beta t} + \frac{1}{2}\frac{A^2}{\beta^2 t} + \frac{1}{2}\frac{A}{\beta\sqrt{t}}\sqrt{\frac{A^2}{\beta^2 t} + 4\left(1 - \frac{r}{\beta t}\right)}.$$

All the above integrals are numerically evaluated and the graphs are plotted by specifying admissible values of $(r/\beta t)$ against $(\mu/P)\,v$.

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Received 1982 April 28

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Gerlands Beitr. Geophysik, Leipzig 87 (1978) 4, S. 317–332

Displacement Due to a Uniformly Moving Line Load over the Plane Boundary of an Inhomogeneous Elastic Half-Space

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(With 3 figures)

Summary

A concentrated line load originating at $t = 0$ at the origin of co-ordinates moves with uniform velocity along the boundary of an isotropic inhomogeneous medium. Following CAGNIARD's method as modified by DE HOOP, the displacement components u and w are determined in the integral form. *Finally, an approximate evaluation of the integrals is worked out near the first arrival of* the wave fronts.

Zusammenfassung

Eine konzentrierte Linienbelastung, die zum Zeitpunkt $t = 0$ am Koordinatenursprung einsetzt, bewegt sich mit gleichförmiger Geschwindigkeit über die freie Oberfläche eines isotropen, inhomogenen Mediums. Mit Hilfe der Methode von CAGNIARD in ihrer DE HOOPschen Abwandlung werden die Verrückungskomponenten u und w in ihrer Integraldarstellung bestimmt. Schließlich werden die Integrale für die Zeit um das erste Auftreffen der Wellenfronten näherungsweise berechnet.

1. Introduction

Since the publication of the classical paper by LAMB [6] the problem of line and point sources in homogeneous media has attracted the attention of many investigators. But the corresponding problems for inhomogeneous media have not been discussed by many authors as yet. The problem of wave propagation in an inhomogeneous medium is important to geophysicists, because any realistic model of the Earth must take into account the continuous change in the elastic properties of the material in the vertical direction. Since the mathematical treatment of a complicated model is extremely difficult and since the approximation to such a problem does not lead to any worth while solution, so some simplifying assumptions are usually made. WILSON [10] studied the propagation of surface waves in a semi-infinite medium, assuming
he density to be constant and the coefficient of rigidity to be varying exponentially
ith depth. STONELEY [9], however, considered the transmission of RAYLEIGH waves
a heterogeneous medium in which the rigidity varies linearly with depth. The field
to a point source in an inhomogeneous isotropic medium in which density is
stant but the bulk modulus λ varies with depth according to the law $\lambda = \lambda_0(1 + \varepsilon z)^2$
been considered by SINGH [8].

the present paper, considering an elastic medium in which the elastic parameters
d density ϱ vary according to the law $\lambda = \mu = \mu_0(1 + \varepsilon z)^2$ and $\varrho = \varrho_0(1 + \varepsilon z)^2$,

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the transient problem for a two-dimensional line load moving with uniform velocity v over the surface of the non-homogeneous semi-infinite medium is studied. The ground motion excited by the moving surface load occurs, for example, from nuclear blasts and from shock waves generated by supersonic aircrafts. These practical problems have been formulated mathematically by a two-dimensional normal line load which is suddenly created at $t = 0$ and moves subsequently with uniform velocity along the free surface. The method of solutions involves the use of the integral transform and Cagniard's [1] method as modified by De Hoop [4]. The application of Cagniard's method in the solution of transient problems in inhomogeneous media does not seem to have been discussed earlier.

This steadily moving line load problem, where t varies from $-\infty$ to ∞, has been solved by Chakravarty and De [2] following the method of Cole and Huth [3]. Of course, the transient solution for a point load moving over the surface of a homogeneous isotropic half-space has been thoroughly discussed by Gakenheimer and Miklowitz [5]. An exact solution of the buried uniformly moving line-source problem has also been obtained by Mitra [7].

2. Formulation of the problem

The inhomogeneous semi-infinite medium is supposed to occupy the region $z > 0$ as shown in Fig. 1. The x-axis is taken along the free surface, whereas the z-axis points vertically downwards into the medium. A concentrated line load, which is assumed to originate on the free surface at the origin at time $t = 0$, moves with uniform velocity v ($v < \alpha, \beta$) along the positive direction of the x-axis.

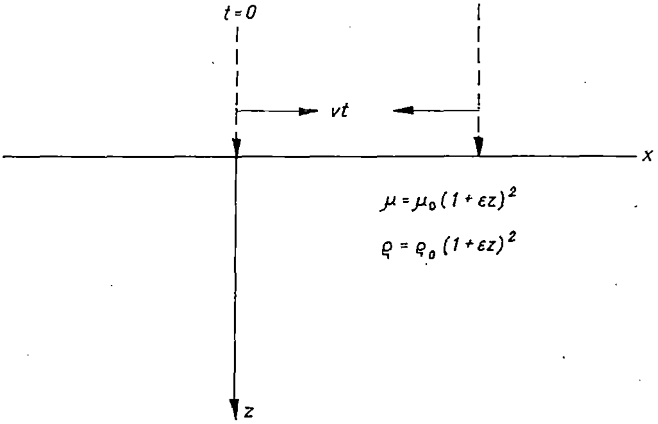

Fig. 1

The equations of motion for a non-homogeneous medium in the absence of bc forces are

$$\frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]=\varrho\frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}\right]=\varrho\frac{\partial^2 w}{\partial t^2}.$$

u and w are the displacement components in the x- and z-directions, λ, μ are LAMÉ's constants and ϱ is the density of the medium. It is assumed that

$$\lambda = \mu = \mu_0(1 + \varepsilon z)^2 , \qquad \varrho = \varrho_0(1 + \varepsilon z)^2 , \tag{3}$$

such that the velocity of propagation is independent of z. The equations (1) and (2) have to be solved subject to the boundary conditions

$$\left.\begin{aligned} \tau_{xz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 && \text{at} \quad z = 0 , \\ \tau_{zz} &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} = -P\delta(x - vt) && \text{at} \quad z = 0 , \quad t > 0 . \end{aligned}\right\} \tag{4}$$

$\delta(x - vt)$ is DIRAC's delta function.

3. Formal solution

In order to solve the equations (1) and (2), we make the substitution

$$U = u(1 + \varepsilon z) \quad \text{and} \quad V = v(1 + \varepsilon z) . \tag{5}$$

This transforms the equations (1) and (2) into the forms

$$3\frac{\partial^2 U}{\partial x^2} + 2\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial z}\right) + \frac{\partial^2 U}{\partial z^2} = \frac{\varrho_0}{\mu_0}\frac{\partial^2 U}{\partial t^2} \tag{6}$$

and

$$\frac{\partial^2 W}{\partial x^2} + 2\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial z}\right) + 3\frac{\partial^2 W}{\partial z^2} = \frac{\varrho_0}{\mu_0}\frac{\partial^2 W}{\partial t^2} . \tag{7}$$

We introduce the FOURIER transform over x defined by

$$f_1(p, z, t) = \int_{-\infty}^{\infty} f(x, z, t)\, e^{ipx}\, dp$$

and then take the LAPLACE transform over t defined by

$$\bar{f}_1(p, z, s) = \int_0^{\infty} f_1(p, z, t)\, e^{-st}\, dt .$$

The equations (6) and (7) after these transformations take the forms

$$\left(\frac{d^2}{dz^2} - 3k_1^2\right)\bar{U}_1 = 2ip\frac{d\bar{W}_1}{dz} \tag{8}$$

and

$$\left(3\frac{d^2}{dz^2} - k_2^2\right)\bar{W}_1 = 2ip\frac{d\bar{U}_1}{dz} , \tag{9}$$

where

$$k_1^2 = p^2 + \frac{s^2}{\alpha^2}, \qquad k_2^2 = p^2 + \frac{s^2}{\beta^2}, \qquad \alpha^2 = 3\beta^2 = \frac{\lambda + 2\mu}{\varrho} .$$

Using the conditions that the displacement components vanish as z approaches ∞, the solutions of (8) and (9) are

$$\bar{U}_1 = A\, e^{-k_1 z} + B\, e^{-k_2 z}\,, \tag{10}$$

$$\bar{W}_1 = \frac{1}{ip}\left(A k_1\, e^{-k_1 z} + \frac{p^2 B}{k_2}\, e^{-k_2 z}\right). \tag{11}$$

Using (5), the above equations become

$$\bar{u}_1 = \frac{1}{1+\varepsilon z}\,(A\, e^{-k_1 z} + B\, e^{-k_2 z}) \tag{12}$$

and

$$\bar{w}_1 = \frac{1}{ip(1+\varepsilon z)}\left(A k_1\, e^{-k_1 z} + \frac{p^2 B}{k_2}\, e^{-k_2 z}\right). \tag{13}$$

A and B have to be determined from the conditions

$$\frac{d\bar{u}_1}{dz} = ip\,\bar{w}_1 \quad \text{and} \quad -ip\mu_0 \bar{u}_1 + 3\mu_0 \frac{d\bar{w}_1}{dz} = \frac{P}{ipv - s} \quad \text{on} \quad z = 0\,,$$

which are obtained by taking first the Fourier and then the Laplace transform on both sides of equations (4). It is found that

$$A = \frac{ipP(\varepsilon k_2 + k_2^2 + p^2)}{\mu_0(ipv - s)\,(k_1 k_2 - p^2)\, f(p)}\,, \qquad B = -\,\frac{ipP(2k_1 k_2 + \varepsilon k_2)}{\mu_0(ipv - s)\,(k_1 k_2 - p^2)\, f(p)}\,,$$

where

$$f(p) = (p^2 - 3k_1 k_2) - 3\varepsilon(k_1 + k_2) - 3\varepsilon^2\,.$$

Substituting the values of A and B in (12) and (13) and taking Fourier inversion, we get

$$\bar{u} = \frac{iP}{2\pi\mu_0(1+\varepsilon z)} \int\limits_{-\infty}^{\infty} \frac{p(\varepsilon k_2 + k_2^2 + p^2)\, e^{-k_1 z} - p k_2(2k_1 + \varepsilon)\, e^{-k_2 z}}{(ipv - s)\,(k_1 k_2 - p^2)\, f(p)}\, e^{-ipx}\, dp \tag{14}$$

and

$$\bar{w} = \frac{P}{2\pi\mu_0(1+\varepsilon z)} \int\limits_{-\infty}^{\infty} \frac{k_1(\varepsilon k_2 + k_2^2 + p^2)\, e^{-k_1 z} - p^2(2k_1 + \varepsilon)\, e^{-k_2 z}}{(ipv - s)\,(k_1 k_2 - p^2)\, f(p)}\, e^{-ipx}\, dp\,. \tag{15}$$

4. Laplace inversions

We assume

$$\bar{u} = \frac{P}{2\pi\mu_0(1+\varepsilon z)}\,(I_1 - I_2)\,, \tag{16}$$

where

$$I_1 = \int\limits_{-\infty}^{\infty} \frac{ip(\varepsilon k_2 + k_2^2 + p^2)\, e^{-k_1 z - ipx}}{(ipv - s)\,(k_1 k_2 - p^2)\, f(p)}\, dp$$

and

$$I_2 = \int_{-\infty}^{\infty} \frac{ipk_2(2k_1 + \varepsilon)\, e^{-k_2 z - ipx}}{(ipv - s)\,(k_1 k_2 - p^2)\, f(p)}\, dp\,.$$

To find the inversions of I_1 and I_2, we adopt Cagniard's technique as modified by De Hoop [4]. Accordingly, we put $p = -sh$ in I_1, which then reduces to the form

$$I_1 = \int_{-\infty}^{\infty} \frac{ih(\varepsilon k_2' + s k_2'^2 + s h^2)\, e^{-s(k_1' z - ihx)}}{(ihv + 1)\, \Phi(h)\, \Psi(s, h)}\, dh\,, \tag{17}$$

where

$$k_1'^2 = h^2 + \frac{1}{\alpha^2}\,, \qquad k_2'^2 = h^2 + \frac{1}{\beta^2}\,,$$

$$\Phi(h) = (3k_1'k_2' - h^2)\,(h^2 - k_1'k_2') = -(h^4 - 4k_1'k_2' + 3k_1'k_2')\,.$$

It has to be noted that $\Phi(h) = 0$ is the Rayleigh wave velocity equation corresponding to the homogeneous medium with $\lambda = \mu_0 = \mu$ and

$$\Psi(s, h) = s^2 + \frac{3s\varepsilon(k_1' + k_2')}{3k_1'k_2' - h^2} + \frac{3\varepsilon^2}{3k_1'k_2' - h^2} = (s - \varepsilon m_1)\,(s - \varepsilon m_2)\,,$$

where

$$m_{1,2} = \frac{-3(k_1' + k_2') \pm [9(k_1' - k_2')^2 + 12h^2]^{1/2}}{2(3k_1'k_2' - h^2)}\,;$$

m_1 and m_2 are both negative. Breaking up $(\varepsilon k_2' + s k_2'^2 + s h^2)/\Psi(s, h)$ into partial fractions, the equation (17) can be written as

$$I_1 = \int_{-\infty}^{\infty} \frac{ih(1 - ihv)\, e^{-s(k_1' z - ihx)}}{(1 + h^2 v^2)\, \Phi(h)} \left(\frac{M}{s - \varepsilon m_1} + \frac{N}{s - \varepsilon m_2}\right) dh\,; \tag{18}$$

similarly,

$$I_2 = \int_{-\infty}^{\infty} \frac{ih(1 - ihv)\, e^{-s(k_2' z - ihx)}}{(1 + h^2 v^2)\, \Phi(h)} \left(\frac{S}{s - \varepsilon m_1} + \frac{T}{s - \varepsilon m_2}\right) dh\,. \tag{19}$$

In (18) and (19),

$$M = \frac{k_2' + m_1(k_2'^2 + h^2)}{m_1 - m_2}\,, \qquad N = \frac{k_2' + m_2(k_2'^2 + h^2)}{m_2 - m_1}$$

and

$$S = \frac{k_2' + 2m_1 k_1' k_2'}{m_1 - m_2}\,, \qquad T = \frac{k_2' + 2m_2 k_1' k_2'}{m_2 - m_1}\,.$$

First let us consider the integral

$$\int_{-\infty}^{\infty} \frac{ih(1 - ihv)\, e^{-s(k_1' z - ihx)}}{(1 + h^2 v^2)\, \Phi(h)}\, \frac{M}{s - \varepsilon m_1}\, dh\,, \tag{20}$$

which occurs in equation (18). In this integral the path of integration with respect to h, which is the real axis, is deformed in such a way that

$$z\left(h^2+\frac{1}{\alpha^2}\right)^{1/2}-ihx=q\,,$$

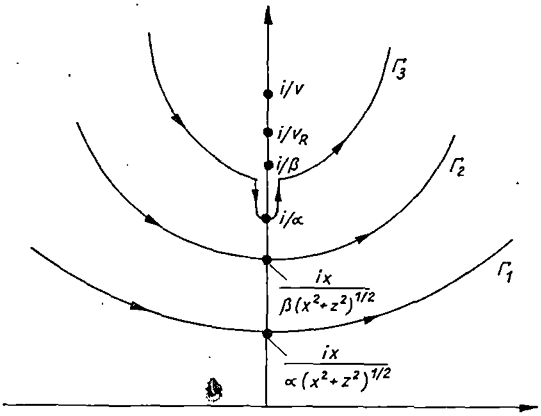

Fig. 2

where q is real and positive. The deformed path of integration is the branch Γ_1 (Fig. 2) of a hyperbola, whose equation is

$$h=\frac{iqx\pm z\left(q^2-\dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2+z^2}\,,\qquad \frac{(x^2+z^2)^{1/2}}{\alpha}<q<\infty\,.$$

In the course of deformation of the path of integration it is essential to know all the singularities of $M/[(1+h^2v^2)\,\Phi(h)]$ in the h-plane, which are the poles at $\pm(i/u)$, $\pm(i/v_R)$ and the branch points at $\pm(i/\alpha)$ and $\pm(i/\beta)$, where v_R is the Rayleigh wave velocity corresponding to the homogeneous medium when $\lambda=\mu=\mu_0$.

Since the hyperbolic path Γ_1 does not cross any of the singularities during its deformation, it is possible by virtue of Cauchy's theorem and Jordan's lemma to replace the integration along the real h-axis by an integration along the hyperbolic path Γ_1. We write

$$h_+=\frac{iqx+z\left(q^2-\dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2+z^2}\,,\qquad h_-=\frac{iqx-z\left(q^2-\dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}}{x^2+z^2}\,;$$

then

$$\frac{dh_\pm}{dq}=\frac{ix\left(q^2-\dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}\pm qz}{(x^2+z^2)\left(q^2-\dfrac{x^2+z^2}{\alpha^2}\right)^{1/2}}\,.$$

Using the facts that

$$h_- = -\bar{h}_+ , \qquad \frac{dh_-}{dq} = -\left(\frac{d\bar{h}_+}{dq}\right), \qquad m_{1-} = \bar{m}_{1+} , \qquad M_- = \bar{M}_+ ,$$

where $\bar{h}$ is the complex conjugate of h, the expression (20) takes the form

$$\int\limits_{\frac{(x^2+z^2)^{1/2}}{\alpha}}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+ M_+}{(1+h_+^2 v^2)\,\Phi(h_+)}\,\frac{e^{-sq}}{s-\varepsilon m_{1+}}\,\frac{dh_+}{dq}\right] dq \; +$$

$$+ \int\limits_{\frac{(x^2+z^2)^{1/2}}{\alpha}}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2 M_+}{(1+h_+^2 v^2)\,\Phi(h_+)}\,\frac{e^{-sq}}{s-\varepsilon m_{1+}}\,\frac{dh_+}{dq}\right] dq \,.$$

Using the convolution theorem, the LAPLACE inversion of the above integral is

$$\int\limits_0^t d\tau \int\limits_{t_\alpha}^{\infty} -2\,\mathrm{Im}\left[\frac{h_+ M_+\, e^{\varepsilon m_{1+}(t-\tau)}}{(1+h_+^2 v^2)\,\Phi(h_+)}\,\frac{dh_+}{dq}\right] \delta(\tau - q)\, dq \; +$$

$$+ \int\limits_0^t d\tau \int\limits_{t_\alpha}^{\infty} 2v\,\mathrm{Re}\left[\frac{h_+^2 M_+\, e^{\varepsilon m_{1+}(t-\tau)}}{(1+h_+^2 v^2)\,\Phi(h_+)}\,\frac{dh_+}{dq}\right] \delta(\tau - q)\, dq \,,$$

where $t_\alpha = (x^2+z^2)^{1/2}/\alpha$ is the arrival time of P-waves. By use of the properties of the δ-function, the above integrals can be written as

$$H(t-t_\alpha)\left[\int\limits_{t_\alpha}^{t} -2\,\mathrm{Im}\left\{\frac{h_+ M_+\, e^{\varepsilon m_{1+}(t-\tau)}}{(1+h_+^2 v^2)\,\Phi(h_+)}\,\frac{dh_+}{d\tau}\right\} d\tau \; + \right.$$

$$\left. + \int\limits_{t_\alpha}^{t} 2v\,\mathrm{Re}\left\{\frac{h_+^2 M_+\, e^{\varepsilon m_{1+}(t-\tau)}}{(1+h_+^2 v^2)\,\Phi(h_+)}\,\frac{dh_+}{d\tau}\right\} d\tau \right]. \tag{21}$$

It should be noted that in the integrand of the above integral q has been replaced by τ everywhere.

In a similar manner the LAPLACE inversion of the other part of I_1 in (18) can be determined. It is found to be a similar expression as the expression in (21) except that M_+ and m_{1+} have to be replaced by N_+ and m_{2+} respectively. Thus the LAPLACE inversion of I_1 is

$$H(t-t_\alpha)\int\limits_{t_\alpha}^{t} -2\,\mathrm{Im}\left[\frac{h_+}{(1+h_+^2 v^2)\,\Phi(h_+)}\left\{M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}\right\}\frac{dh_+}{d\tau}\right] d\tau \; +$$

$$+ H(t-t_\alpha)\int\limits_{t_\alpha}^{t} 2v\,\mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2 v^2)\,\Phi(h_+)}\left\{M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}\right\}\frac{dh_+}{d\tau}\right] d\tau \,. \tag{22}$$

Next we shall calculate the LAPLACE inversion of I_2 that occurs in (19). As before here also we define

$$z\left(h^2+\frac{1}{\beta^2}\right)^{1/2}-ihx=r\,,$$

where r is real and positive. So,

$$h_\pm=\frac{irx+z\left(r^2-\dfrac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}\,. \tag{23}$$

Case 1: A path along which r is real and non-negative is the hyperbolic path Γ_2 (Fig. 2) represented parametrically by the above equation with $r > (x^2+z^2)^{1/2}/\beta$, provided the path where it cuts the imaginary axis, viz. $h = ix/\beta(x^2+z^2)^{1/2}$, lies below the branch point i/α, which occurs when $x < \beta z/(\alpha^2-\beta^2)^{1/2}$. In this case the path Γ_2 (Fig. 2) does not cross any of the singularities during the deformation. Following the same procedure as that done in case of I_1, the LAPLACE inversion of I_2 is found to be

$$H(t-t_\beta)\int\limits_{t_\beta}^{t}-2\,\mathrm{Im}\left[\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\left\{S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)}\right\}\frac{dh_+}{d\tau}\right]d\tau\;+$$
$$+\;H(t-t_\beta)\int\limits_{t_\beta}^{t}2v\,\mathrm{Re}\left[\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\left\{S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)}\right\}\frac{dh_+}{d\tau}\right]d\tau\,, \tag{24}$$

where $t_\beta = (x^2+z^2)^{1/2}/\beta$ is the arrival time of S-waves, and h_+ occurring in the above expression is obtained by replacing r by τ in the expression for h_+ as given in (23).

Case 2: If $x > \beta z/(\alpha^2-\beta^2)^{1/2}$, the point $ix/\beta(x^2+z^2)^{1/2}$ lies above the branch point i/α. Therefore, the path of integration in the h-plane has to be deformed to the path Γ_3 (Fig. 2) round the branch point i/α as shown in Fig. 2.

We consider the integral

$$\int\limits_{-\infty}^{\infty}\frac{ih(1-ihv)\,S}{(1+h^2v^2)\,\Phi(h)}\,\frac{e^{-s(k_2'z-ihx)}}{s-\varepsilon m_1}\,dh \tag{25}$$

occurring in I_2 of equation (19). Here too we put

$$z\left(h^2+\frac{1}{\beta^2}\right)^{1/2}-ihx=r\,.$$

On the two finite straight line portions of the path Γ_3, h is given by

$$h_\pm=\pm\eta+\frac{i\left\{rx-z\left(\dfrac{x^2+z^2}{\beta^2}-r^2\right)^{1/2}\right\}}{x^2+z^2}\,,$$

where finally η should be made to tend to zero, and on the remaining portions of the path of Γ_3,

$$h_\pm=\frac{irx\pm z\left(r^2-\dfrac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}\,.$$

On the straight line portions of the path Γ_3, r varies from $r = t_{\alpha\beta}$ to $r = t_\beta$, where $t_{\alpha\beta} = x/\alpha + z(1/\beta^2 - 1/\alpha^2)^{1/2}$ is the arrival time of PS-waves. The expression in (25) can then be written in the form

$$\int_{t_{\alpha\beta}}^{t_\beta}\left[\frac{ih_+(1-ih_+v)\,S_+}{(1+h_+^2v^2)\,\Phi(h_+)\,(s-\varepsilon m_{1+})}\,\frac{dh_+}{dr}-\frac{ih_-(1-ih_-v)}{(1+h_-^2v^2)\,\Phi(h_-)\,(s-\varepsilon m_{1-})}\,\frac{dh_-}{dr}\right]e^{-sr}dr\;+$$

$$+\int_{t_\beta}^{\infty}\left[\frac{ih_+(1-ih_+v)\,S_+}{(1+h_+^2v^2)\,\Phi(h_+)\,(s-\varepsilon m_{1+})}\,\frac{dh_+}{dr}-\frac{ih_-(1-ih_-v)}{(1+h_-^2v^2)\,\Phi(h_-)\,(s-\varepsilon m_{1-})}\,\frac{dh_-}{dr}\right]e^{-sr}\,dr. \tag{26}$$

Noting that $h_- = -\bar{h}_+$, $dh_-/dr = -(d\bar{h}_+/dr)$ and $S_- = \bar{S}_+$ on the path Γ_3, the expression (26) takes the form

$$\int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\;\frac{h_+S_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{e^{-sr}}{s-\varepsilon m_{1+}}\,\frac{dh_+}{dr}\,dr+\int_{t_{\alpha\beta}}^{t_\beta} 2v\,\mathrm{Re}\;\frac{h_+^2S_+}{(1+h_+^2v^2)\,\Phi(h_+)}\;\times$$

$$\times\;\frac{e^{-sr}}{s-\varepsilon m_{1+}}\,\frac{dh_+}{dr}\,dr+\int_{t_\beta}^{\infty} -2\,\mathrm{Im}\;\frac{h_+S_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{e^{-sr}}{s-\varepsilon m_{1+}}\,\frac{dh_+}{dr}\,dr\;+$$

$$+\int_{t_\beta}^{\infty} 2v\,\mathrm{Re}\;\frac{h_+^2S_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,\frac{e^{-sr}}{s-\varepsilon m_{1+}}\,\frac{dh_+}{dr}\,dr\,. \tag{27}$$

To transform the other integral of I_2 occurring in (19), a similar procedure is applied, and finally I_2 in (19) takes the following form:

$$I_2=\int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1+h_+^2v^2)\;\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}dr\;+$$

$$+\int_{t_\beta}^{\infty} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1+h_+^2v^2)\;\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}dr\;+$$

$$+\int_{t_{\alpha\beta}}^{t_\beta} 2v\,\mathrm{Re}\left\{\frac{h_+^2\,e^{-sr}}{(1+h_+^2v^2)\;\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}dr\;+$$

$$+\int_{t_\beta}^{\infty} 2v\,\mathrm{Re}\left\{\frac{h_+^2\,e^{-sr}}{(1+h_+^2v^2)\;\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}dr\,. \tag{28}$$

It must be remembered that the value of h_+ when r lies in $[t_{\alpha\beta}, t_\beta]$ has to be taken as

$$h_+=\frac{i\left\{rx-z\left(\dfrac{x^2+z^2}{\beta^2}-r^2\right)^{1/2}\right\}}{x^2+z^2},$$

and for r lying in $[t_\beta, \infty)$,

$$h_+=\frac{irx+z\left(r^2-\dfrac{x^2+z^2}{\beta^2}\right)^{1/2}}{x^2+z^2}.$$

Next the LAPLACE inversion of I_2 in (28) has to be calculated. By applying the convolution theorem, the LAPLACE inversion of the first integral of I_2 in (28) is found to be

$$\int_0^t d\tau \int_{t_{\alpha\beta}}^{t_\beta} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1+h_+^2v^2)\,\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}\delta(\tau-r)\,dr =$$

$$=\int_0^t d\tau \int_{t_{\alpha\beta}}^{\infty} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1+h_+^2v^2)\,\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}\delta(\tau-r)\,dr -$$

$$-\int_0^t d\tau \int_{t_\beta}^{\infty} -2\,\mathrm{Im}\left\{\frac{h_+\,e^{-sr}}{(1+h_+^2v^2)\,\Phi(h_+)}\left(\frac{S_+}{s-\varepsilon m_{1+}}+\frac{T_+}{s-\varepsilon m_{2+}}\right)\frac{dh_+}{dr}\right\}\delta(\tau-r)\,dr,$$

and it takes the following form when the δ-function property is used:

$$H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -2\,\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\left(S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)}\right)\frac{dh_+}{d\tau}\right\}d\tau -$$

$$-\,H(t-t_\beta)\int_{t_\beta}^{t} -2\,\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\left(S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)}\right)\frac{dh_+}{d\tau}\right\}d\tau\,. \tag{29}$$

It can be shown that the last term of (29) is cancelled with the LAPLACE inversion of the second integral in (28).

Similarly, the LAPLACE inversion of the other integrals of (29) can be determined, and finally, after simplification, we get the LAPLACE inversion of I_2 as

$$H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} -2\,\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\left(S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)}\right)\frac{dh_+}{d\tau}\right\}d\tau -$$

$$-\,H(t-t_{\alpha\beta})\int_{t_{\alpha\beta}}^{t} 2v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\left(S_+\,e^{\varepsilon m_{1+}(t-\tau)}+T_+\,e^{\varepsilon m_{2+}(t-\tau)}\right)\frac{dh_+}{d\tau}\right\}d\tau\,. \tag{30}$$

Combining the results of the inverse LAPLACE transforms of I_1 and I_2 from (22) and (24) it follows that

$$u(x,z,t)=\frac{P}{\pi\mu_0(1+\varepsilz)}\times$$

$$\times\Bigg[H(t-t_\alpha)\int_{t_\alpha}^{t} -\,\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\left(M_+\,e^{\varepsilon m_{1+}(t-\tau)}+N_+\,e^{\varepsilon m_{2+}(t-\tau)}\right)\frac{dh_+}{d\tau}\right\}d\tau +$$

$$+\,H(t-t_\alpha)\int_{t_\alpha}^{t} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\left(M^+\,e^{\varepsilon m_{1+}(t-\tau)}+N_+\,e^{\varepsilon m_{2+}(t-\tau)}\right)\frac{dh_+}{d\tau}\right\}d\tau -$$

$$
\begin{aligned}
&- H(t - t_\beta) \int_{t_\beta}^{t} -\mathrm{Im} \left\{ \frac{h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+\, e^{\varepsilon m_{1+}(t-\tau)} + T_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau - \\
&- H(t - t_\beta) \int_{t_\beta}^{t} v\, \mathrm{Re} \left\{ \frac{h_+^2}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+\, e^{\varepsilon m_{1+}(t-\tau)} + T_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau \Bigg].
\end{aligned}
\tag{31}
$$

for $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, and when $x > \beta z/(\alpha^2 - \beta^2)^{1/2}$, from (22) and (30) it follows that

$$
\begin{aligned}
u(x, z, t) &= \frac{P}{\pi \mu_0 (1 + \varepsilon z)} \times \\
&\times \Bigg[H(t - t_\alpha) \int_{t_\alpha}^{t} -\mathrm{Im} \left\{ \frac{h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(M_+\, e^{\varepsilon m_{1+}(t-\tau)} + N_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau + \\
&+ H(t - t_\alpha) \int_{t_\alpha}^{t} v\, \mathrm{Re} \left\{ \frac{h_+^2}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(M_+\, e^{\varepsilon m_{1+}(t-\tau)} + N_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau - \\
&- H(t - t_{\alpha\beta}) \int_{t_{\alpha\beta}}^{t} -\mathrm{Im} \left\{ \frac{h_+}{(1 + {}^{2} \quad \Phi(h_+)} \left(S_+\, e^{\varepsilon m_{1+}(t-\tau)} + T_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau - \\
&- H(t - t_{\alpha\beta}) \int_{t_{\alpha\beta}}^{t} v\, \mathrm{Re} \left\{ \frac{h_+^2}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+\, e^{\varepsilon m_{1+}(t-\tau)} + T_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau \Bigg].
\end{aligned}
\tag{32}
$$

Carrying on a similar procedure as done for the evaluation of the displacement along the x-direction, the expression for the displacement along the z-direction can also be determined from (13) and is found to be equal to

$$
\begin{aligned}
w(x, z, t) &= \frac{P}{\pi \mu_0 (1 + \varepsilon z)} \times \\
&\times \Bigg[H(t - t_\alpha) \int_{t_\alpha}^{t} \mathrm{Re} \left\{ \frac{k'_{1+}}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(M_+\, e^{\varepsilon m_{1+}(t-\tau)} + N_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau + \\
&+ H(t - t_\alpha) \int_{t_\alpha}^{t} v\, \mathrm{Im} \left\{ \frac{k'_{1+} h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} \left(M_+\, e^{\varepsilon m_{1+}(t-\tau)} + N_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau + \\
&+ H(t - t_\beta) \int_{t_\beta}^{t} \mathrm{Re} \left\{ \frac{h_+^2}{k'_{2+} (1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+\, e^{\varepsilon m_{1+}(t-\tau)} + T_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau + \\
&+ H(t - t_\beta) \int_{t_\beta}^{t} v\, \mathrm{Im} \left\{ \frac{h_+^3}{k'_{2+} (1 + h_+^2 v^2)\, \Phi(h_+)} \left(S_+\, e^{\varepsilon m_{1+}(t-\tau)} + T_+\, e^{\varepsilon m_{2+}(t-\tau)}\right) \frac{dh_+}{d\tau} \right\} d\tau \Bigg]
\end{aligned}
\tag{33}
$$

for $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, and if $x > \beta z/(\alpha^2 - \beta^2)^{1/2}$:

$$w(x, z, t) = \frac{P}{\pi\mu_0(1 + \varepsilon z)} \times$$

$$\times \Bigg[H(t - t_\alpha) \int_{t_\alpha}^{t} \operatorname{Re} \left\{ \frac{k'_{1+}}{(1 + h_+^2 v^2)\, \Phi(h_+)} (M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_\alpha) \int_{t_\alpha}^{t} v \operatorname{Im} \left\{ \frac{k'_{1+} h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} (M_+ e^{\varepsilon m_{1+}(t-\tau)} + N_+ e^{\varepsilon m_{2+}(t-\tau)}) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_{\alpha\beta}) \int_{t_{\alpha\beta}}^{t} \operatorname{Re} \left\{ \frac{h_+^2}{k'_{2+}(1 + h_+^2 + v^2)\, \Phi(h_+)} (S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}) \frac{dh_+}{d\tau} \right\} d\tau +$$

$$+ H(t - t_{\alpha\beta}) \int_{t_{\alpha\beta}}^{t} v \operatorname{Im} \left\{ \frac{h_+^3}{k'_{2+}(1 + h_+^2 v^2)\, \Phi(h_+)} (S_+ e^{\varepsilon m_{1+}(t-\tau)} + T_+ e^{\varepsilon m_{2+}(t-\tau)}) \frac{dh_+}{d\tau} \right\} d\tau \Bigg]. \tag{34}$$

It should be remembered that in the first two integrals of the equations (31), (32), (33) and (34)

$$h_+ = \frac{i\tau x + z\left(\tau^2 - \dfrac{x^2 + z^2}{\alpha^2}\right)^{1/2}}{x^2 + z^2}, \qquad t_\alpha \leqq \tau \leqq t,$$

and in the last two integrals of those equations

$$h_+ = \frac{i\tau x + z\left(\tau^2 - \dfrac{x^2 + z^2}{\beta^2}\right)^{1/2}}{x^2 + z^2}, \qquad t_\beta \leqq \tau \leqq t,$$

where as in the last two integrals of (32) and (34)

$$h_+ = \frac{i\left\{\tau x - z\left(\dfrac{x^2 + z^2}{\beta^2} - \tau^2\right)^{1/2}\right\}}{x^2 + z^2}, \qquad t_{\alpha\beta} \leqq \tau \leqq t_\beta .$$

5. Wave front expansion

The wave forms of the solutions given in (31) to (34) are evaluated by approximate estimation of the above integrals in the neighbourhood of the time of the first arrival of the different waves. To facilitate this evaluation we put $\tau = A + a$, where A is the lower limit of the integrals in question and a varies from 0 to $t - A$. Then when $x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, from (31) we get

$$u(x, z, t) = \frac{P}{\pi\mu_0(1 + \varepsilon z)} \times$$

$$\times \Bigg[H(t - t_\alpha) \int_{0}^{t - t_\alpha} - \operatorname{Im} \left\{ \frac{h_+}{(1 + h_+^2 v^2)\, \Phi(h_+)} (M_+ e^{\varepsilon m_{1+}(t - t_\alpha - a)} + \right.$$

$$\left. + N_+ e^{\varepsilon m_{2+}(t - t_\alpha - a)}) \frac{dh_+}{da} \right\} da +$$

$$\begin{aligned}
&+H(t-t_\alpha)\int\limits_0^{t-t_\alpha} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(M_+\,e^{\varepsilon m_{1+}(t-t_\alpha-a)}+\right.\\
&\qquad\qquad\left.+\,N_+\,e^{\varepsilon m_{2+}(t-t_\alpha-a)})\,\frac{dh_+}{da}\right\}da\,+\\
&+H(t-t_\beta)\int\limits_0^{t-t_\beta}\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-t_\beta-a)}+\right.\\
&\qquad\qquad\left.+\,T_+\,e^{\varepsilon m_{2+}(t-t_\beta-a)})\,\frac{dh_+}{da}\right\}da\,+\\
&+H(t-t_\beta)\int\limits_0^{t-t_\beta}-v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-t_\beta-a)}+\right.\\
&\qquad\qquad\left.\left.+\,T_+\,e^{\varepsilon m_{2+}(t-t_\beta-a)}\,\frac{dh_+}{da}\right\}da\right].
\end{aligned}\tag{35}$$

For $x > \beta z/(\alpha^2-\beta^2)^{1/2}$,

$$\begin{aligned}
u(x,z,t) &= \frac{P}{\pi\mu_0(1+\varepsilon z)}\times\\
&\times\left[H(t-t_\alpha)\int\limits_0^{t-t_\alpha}-\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(M_+\,e^{\varepsilon m_{1+}(t-t_\alpha-a)}+\right.\right.\\
&\qquad\qquad\left.N_+\,e^{\varepsilon m_{2+}(t-t_\alpha-a)})\,\frac{dh_+}{da}\right\}da\,+\\
&+H(t-t_\alpha)\int\limits_0^{t-t_\alpha} v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(M_+\,e^{\varepsilon m_{1+}(t-t_\alpha-a)}+\right.\\
&\qquad\qquad\left.N_+\,e^{\varepsilon m_{2+}(t-t_\alpha-a)})\,\frac{dh_+}{da}\right\}da\,+\\
&+H(t-t_{\alpha\beta})\int\limits_0^{t-t_{\alpha\beta}}\mathrm{Im}\left\{\frac{h_+}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-t_{\alpha\beta}-a)}+\right.\\
&\qquad\qquad\left.+\,T_+\,e^{\varepsilon m_{2+}(t-t_{\alpha\beta}-a)})\,\frac{dh_+}{da}\right\}da\,+\\
&+H(t-t_{\alpha\beta})\int\limits_0^{t-t_{\alpha\beta}}-v\,\mathrm{Re}\left\{\frac{h_+^2}{(1+h_+^2v^2)\,\Phi(h_+)}\,(S_+\,e^{\varepsilon m_{1+}(t-t_{\alpha\beta}-a)}+\right.\\
&\qquad\qquad\left.\left.+\,T_+\,e^{\varepsilon m_{2+}(t-t_{\alpha\beta}-a)})\,\frac{dh_+}{da}\right\}da\right].
\end{aligned}\tag{36}$$

A similar type of expressions for $w(x,z,t)$ can be written by substituting $\tau = A + a$ in the equations (33) and (34). For approximate evaluation of the integrals (35) and (36) just after the arrival of the corresponding wave fronts it has to be noted that

$$e^{\varepsilon m_{1+}(t-A-a)}\to 1\,,\qquad e^{\varepsilon m_{2+}(t-A-a)}\to 1\quad\text{and}\quad a\to 0\quad\text{as}\quad t\to A\,,$$

where A is the arrival time of a typical wave front. So, when $A = t_\alpha$, using the facts that

$$h_+ \to \frac{ix}{\alpha(x^2 + z^2)^{1/2}}, \qquad h_+^2 \to \frac{-x^2}{\alpha^2(x^2 + z^2)}$$

and

$$\frac{(M_+ + N_+)}{(1 + h_+^2 v^2)\,\Phi(h_+)} \to$$
$$\to \frac{(x^2 + z^2)^2 \alpha^4 \{x^2(\alpha^2 - 2\beta^2) + \alpha^2 z^2\}}{\{x^2(\alpha^2 - v^2) + \alpha^2 z^2\}[\beta^2 x^2(3z^2 - x^2) - 3\alpha^2 z^2(x^2 + z^2) - 4\beta x^2 z\{x^2(\alpha^2 - \beta^2) + \alpha^2 z^2\}^{1/2}]}$$

as $a \to 0$ and that

$$\frac{dh_+}{da} = \frac{\left(h_+^2 + \dfrac{1}{\alpha^2}\right)^{1/2}}{\left\{2\dfrac{(x^2 + z^2)^{1/2}}{\alpha} + a\right\}^{1/2}} \frac{1}{a^{1/2}},$$

the first two integrals of the equations (35) and (36) just after the arrival of P-waves can approximately be evaluated to the form

$$u(x, z, t) =$$
$$= -\frac{\sqrt{2} P H(t - t_\alpha)\, x z \alpha^{3/2}(x^2 + z^2)^{1/4}\{vx + \alpha(x^2 + z^2)^{1/2}\}\{x^2(\alpha^2 - 2\beta^2) + \alpha^2 z^2\}(t - t_\alpha)^{1/2}}{\pi\mu_0(1 + \varepsilon z)\{x^2(\alpha^2 - v^2) + \alpha^2 z^2\}[\beta^2 x^2(3z^2 - x^2) - 3\alpha^2 z^2(x^2 + z^2) - 4x^2 z\beta \times \times \{x^2(\alpha^2 - \beta^2) + \alpha^2 z^2\}^{1/2}]}.$$

Similarly, the approximate value of w just after the arrival of P-waves is given by

$$w(x, z, t) =$$
$$= \frac{\sqrt{2}\, P H(t - t_\alpha)\, z^2 \alpha^{3/2}(x^2 + z^2)^{1/4}\{vx + \alpha(x^2 + z^2)^{1/2}\}\{x^2(\alpha^2 - 2\beta^2) + \alpha^2 z^2\}(t - t_\alpha)^{1/2}}{\pi\mu_0(1 + \varepsilon z)\{x^2(\alpha^2 - v^2) + \alpha^2 z^2\}[\beta^2 x^2(3z^2 - x^2) - 3\alpha^2 z^2(x^2 + z^2) - 4x^2 z\beta \times \times \{x^2(\alpha^2 - \beta^2) + \alpha^2 z^2\}^{1/2}]}.$$

The same method is applied for approximate evaluation of u and w just after the arrival of S-waves. It should be remembered in this case that

$$A = t_\beta, \qquad h_+ = \frac{i\tau x + z\left(\tau^2 - \dfrac{x^2 + z^2}{\cancel{\alpha^2}\beta^2}\right)^{1/2}}{x^2 + z^2}, \qquad t_\beta < \tau < t.$$

The effects of u_1 and w_1 on the displacement components u and v due to S-waves just after their arrival are found to be

$$u_1(x, z, t) =$$
$$= \frac{2\sqrt{2}\, P H(t - t_\alpha)\, x z^2 \alpha \beta^{3/2}(x^2 + z^2)^{1/4}\{vx + \beta(x^2 + z^2)^{1/2}\}\{\beta^2 z^2 - x^2(\alpha^2 - \beta^2)\}^{1/2}(t - t_\beta)^{1/2}}{\pi\mu_0(1 + \varepsilon z)\{x^2(\beta^2 - v^2) + \beta^2 z^2\}[\alpha^2 x^2(3z^2 - x^2) - 3\beta^2 z^2(x^2 + z^2) - 4x^2 z\alpha \times \times \{\beta^2 z^2 - x^2(\alpha^2 - \beta^2)\}^{1/2}]},$$

$$w_1(x, z, t) =$$
$$= -\frac{2\sqrt{2}\, P H(t - t_\beta)\, x^2 z \alpha \beta^{3/2}(x^2 + z^2)^{1/4}\{vx + \beta(x^2 + z^2)^{1/2}\}\{\beta^2 z^2 - x^2(\alpha^2 - \beta^2)\}^{1/2}(t - t_\beta)^{1/2}}{\pi\mu_0(1 + \varepsilon z)\{x^2(\beta^2 - v^2) + \beta^2 z^2\}[\alpha^2 x^2(3z^2 - x^2) - 3\beta^2 z^2(x^2 + z^2) - 4x^2 z\alpha \times \times \{\beta^2 z^2 - x^2(\alpha^2 - \beta^2)\}^{1/2}]}$$

for $0 < x < \beta z/(\alpha^2 - \beta^2)^{1/2}$, and in the region $x > \beta z/(\alpha^2 - \beta^2)^{1/2}$, PS-waves exist and arrive earlier than S-waves.

We approximately calculate the last two integrals of (36), which will give the effect of u_2 on the displacement components u due to PS-waves just after their arrival. In this case

$$A = t_{\alpha\beta}\,, \qquad h_+ = \frac{i\left\{\tau x - z\left(\dfrac{x^2+z^2}{\beta^2} - \tau^2\right)^{1/2}\right\}}{x^2+z^2}\,, \qquad t_{\alpha\beta} \leqq \tau \leqq t_\beta\,.$$

Then

$$h_+ \to i\left[\frac{1}{\alpha} + \frac{\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{1/2}}{x\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{1/2} - \dfrac{z}{\alpha}}\, a\right],$$

$$\frac{S+T}{(1+h_+^2 v^2)\,\Phi(h_+)} \to -\frac{i\, 2\sqrt{2}\,\alpha^6\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{3/4} a^{1/2}}{(\alpha^2 - v^2)\left\{\alpha x\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{1/2} - z\right\}^{1/2}}$$

and

$$\frac{dh_+}{da} \to \frac{i\alpha\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{1/2}}{\alpha x\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{1/2} - z}\,,$$

where in these expressions terms containing higher order of a are neglected because $a \to 0$ as $t \to t_{\alpha\beta}$, and we get

$$u_2(x, z, t) = \frac{4\sqrt{2}PH(t - t_{\alpha\beta})\,\alpha^5\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{5/4}}{3\pi\mu_0(1+\varepsilon z)\,(\alpha - v)\left\{\alpha x\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{1/2} - z\right\}^{3/2}}\,(t - t_{\alpha\beta})^{3/2}\,.$$

Similarly, the effect on the displacement component w just after the arrival of PS-waves is given by

$$w_2(x, z, t) = -\frac{4\sqrt{2}\,PH(t - t_{\alpha\beta})\,\alpha^4\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{3/4}}{3\pi\mu_0(1+\varepsilon z)\,(\alpha - v)\left\{\alpha x\left(\dfrac{1}{\beta^2} - \dfrac{1}{\alpha^2}\right)^{1/2} - z\right\}^{3/2}}\,(t - t_{\alpha\beta})^{3/2}\,.$$

We now find out the effects of u_3 and w_3 on the displacement components u and w in the neighbourhood of the point C (Fig. 3), where S- and PS-waves arrive at the same time. In this case $t_\beta = t_{\alpha\beta}$ and

$$u_3(x, z, t) = -\frac{4PH(t - t_{\alpha\beta})}{3\pi\mu_0(1+\varepsilon z)}\,\frac{2^{3/4}\alpha^{1/4}\beta^{3/2}z^{5/2}}{x^{13/4}(\alpha - v)}\,(t - t_{\alpha\beta})^{3/4}\,,$$

$$w_3(x, z, t) = \frac{4PH(t - t_{\alpha\beta})}{3\pi\mu_0(1 + \varepsilon z)} \frac{2^{3/4}\alpha^{1/4}\beta^{3/2}z^{3/2}}{x^{9/4}(\alpha - v)} (t - t_{\alpha\beta})^{3/4} .$$

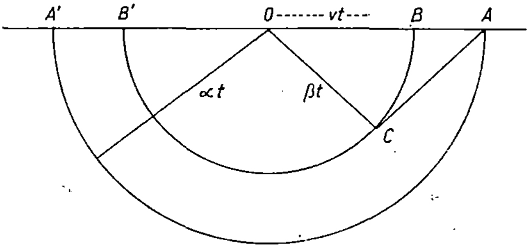

Fig. 3

6. Concluding remarks

It is found from the integrals (31) to (34) that the effect of inhomogeneity enters into the expressions for u and v through the factors $e^{\varepsilon m_1(t-\tau)}$ and $e^{\varepsilon m_2(t-\tau)}$ in the corresponding integrands. So, if these two factors are absent, and that is so if $\varepsilon = 0$, a parallel case for a homogeneous medium is obtained.

Also, it is interesting to note that in the neighbourhood of points just after the arrival of the different wave fronts the displacement components are independent of ε, i.e., at any point, the effect of the first arrival of wave fronts on the displacement components is the same for homogeneous as well as for inhomogeneous media. But as time goes on, ε occurring in the exponential terms of the integrals (31) to (34) for u and w will have its effect, and consequently, the amplitude of the wave fronts will decay exponentially with time due to inhomogeneity of the medium.

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Indian J. pure appl. Math., 14(8) : 994–1006, August 1983

RAYLEIGH WAVES DUE TO NONUNIFORMLY PROPAGATING DIP-SLIP FAULT

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(*Received* 6 *October* 1982)

It is assumed that a crack is developed suddenly along a horizontal line at a finite depth below the surface of the earth which is assumed to be an isotropic homogeneous elastic medium. The crack moves along a vertical plane upto the free surface. Assuming the motion to be two dimensional the surface displacement due to Rayleigh waves produced by nonuniformly moving crack has been determined by using Green's function representation theorem and following the technique developed by Knopoff and Gilbert (1959). For different types of fault propagation, the displacement components derived in integral form are numerically evaluated and are shown by means of graphs which may be of interest in earthquake engineering.

1. INTRODUCTION

The study of dynamic crack propagation is very important in geophysics and in earthquake engineering science. In geophysics it is desirable to formulate the earthquake source in terms of physical parameters and to study the long period waves over a large distance and for a long time. Also in structural engineering it is essential to know the nature of surface waves covering a large distance. At a particular place the ground motion produced by the earthquake is a very complicated function of the nature of propagation of the crack and the geological properties of the place as well. Most of the known solutions of the moving crack are restricted by the assumption of constant velocity of propagation, which is not in general expected. Mal (1972) discussed Rayleigh wave propagation by a finite fault moving with constant velocity. He represented the shear failure by a jump in the tangential components of displacement across the fault surface. Achenbach and Abo-Zeno (1972) analyzed the wave motions generated by a vertical strike slip fault on which motion is opposed by a frictional shear stress and which is assumed to increase linearly with depth. Freund (1973) discussed wave motions as expected in case of a nonuniformly expanding line load. Fossum and Freund (1975) considered a model in which a plane strain shear crack moves from rest at a nonuniform rate under the action of general loading. First motion response of an elastic half space due to a nonuniformly moving dislocation by Cagniard De-Hoop technique is determined by Roy (1978). In a recent paper Markenscoff and Clifton (1981) analyzed the motion of an edge dislocation starting from rest and moving thereafter nonuniformly on its slip plane by means of Laplace transform, where the inversion of the transform is accomplished by Cagniard De-Hoop method.

In the present paper an idealised earthquake model is considered. A fault break along a horizontal line at a finite depth below the free surface is assumed to appear suddenly and to move vertically upward with nonuniform motion upto the free surface. A discontinuity in components of displacement across the fault break is prescribed. The displacement components on the free surface due to Rayleigh waves are determined for nonuniform motion of the crack.

To find the solution of the problem the technique developed by Knopoff and Gilbert with appropriate modification is used. The technique is found to be extremely powerful for tackling such type of boundary value problems. Ghosh (1972) applied the method to show the possibility of attenuation of microseismic waves due to the presence of an upward folding of the ocean bottom into the liquid. Following Knopoff and Gilbert, the moving crack is replaced by a set of virtual sources located at the fault surface HO. The displacement on the free surface is written as the sum of the contribution of these sources with the aid of suitable Green's function representation theorem.

Three particular cases of nonuniform motion of the crack are considered. Horizontal and vertical components of surface displacements due to Rayleigh waves produced by the propagating crack are determined and shown by means of graphs.

In the mathematical and physical structure of wave propagation phenomenon, the model assumed here is although over simplified, yet it brings forth some major features which are usually present in the ground motion.

2. FORMULATION OF THE PROBLEM AND SOLUTION

The origin of the co-ordinate frame (x, y) is at the epicentre 0. It is assumed that a crack suddenly appearing at the focus H moves vertically upwards upto the free surface 0 with a nonuniform speed. The length of the crack measured from H at any time t is $h(t)$, which is assumed to be strictly monotonic increasing function of time t.

The Fourier transform $\bar{f}(x, y, \omega)$ of the function $f(x, y, t)$ is defined by

$$\bar{f}(x, y, \omega) = \int_{-\infty}^{\infty} f(x, y, t)e^{i\omega t}\,dt \qquad \ldots(1)$$

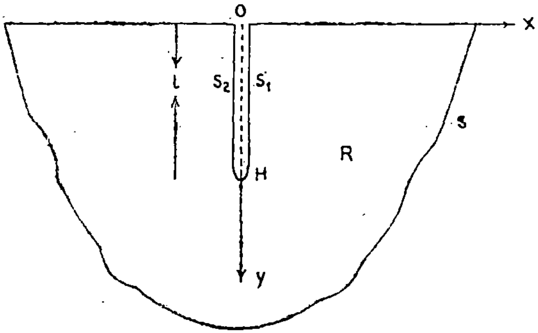


FIG. 1. Geometry of dip-slip fault.

Let $G_n^m(x, y \mid x_0, y_0)$, $(m, n = (x, y))$ be the component of Green's function $G^m(x, y \mid x_0, y_0)$ at the point (x, y) in the direction of n due to a point source of force in m-direction and situated at (x_0, y_0). If now $u(x, y)$ and $v(x, y)$ be the displacement components along x and y directions respectively and $P_{xx}^{(u,v)}$, $P_{xy}^{(u,v)}$ and $P_{yy}^{(u,v)}$ be the stress components then their Fourier transforms defined by (1) satisfy the following differential equations:

$$\frac{\partial \overline{P}_{xx}^{(u,v)}}{\partial x} + \frac{\partial \overline{P}_{xy}^{(u,v)}}{\partial y} + \rho\omega^2 \overline{u}(x, y) = 0 \qquad \ldots(2)$$

$$\frac{\partial \overline{P}_{xy}^{(u,v)}}{\partial x} + \frac{\partial \overline{P}_{yy}^{(uv)}}{\partial y} + \rho\omega^2 \overline{v}(x, y) = 0 \qquad \ldots(3)$$

$$\frac{\partial \overline{P}_{xx}^{[G^x(x,y \mid x_0, y_0)]}}{\partial x} + \frac{\partial \overline{P}_{xy}^{[G^x(x,y \mid x_0,y_0)]}}{\partial y} + \rho\omega^2 \overline{G}_x^x(x, y \mid x_0, y_0)$$
$$= -\delta(x - x_0)\,\delta(y - y_0) \qquad \ldots(4)$$

$$\frac{\partial \overline{P}_{xy}^{[G^x(x,y \mid x_0, y_0)]}}{\partial x} + \frac{\partial \overline{P}_{yy}^{[G^x(x,y \mid x_0,y_0)]}}{\partial y} + \rho\omega^2 \overline{G}_y^x(x, y \mid x_0, y_0) = 0 \qquad \ldots(5)$$

$$\frac{\partial \overline{P}_{xx}^{[G^y(x'y \mid x_0, y_0)]}}{\partial x} + \frac{\partial \overline{P}_{xy}^{[G^y(x'y \mid x_0, y_0)]}}{\partial y} + \rho\omega^2 \overline{G}_x^y(x, y \mid x_0, y_0) = 0 \ldots(6)$$

$$\frac{\partial \overline{P}_{xy}^{[G^y(x'y \mid x_0, y_0)]}}{\partial x} + \frac{\partial \overline{P}_{yy}^{[G^y(x'v \mid x_0, y_0)]}}{\partial y} + \rho\omega^2 \overline{G}_y^y(x'y \mid x_0, y_0)$$
$$= -\delta(x - x_0)\,\delta(y - y_0) \qquad \ldots(7)$$

where ρ is the density of the material and $\delta(\ )$ is Dirac's delta function.

Multiply eqn. (2) by $\bar{G}_x^x(x, y \mid x_0, y_0)$ and (4) by $\bar{u}(x, y)$ and subtract the latter from the former. Also multiply eqn. (3) by $\bar{G}_y^x(x, y \mid x_0, y_0)$ and (5) by $\bar{v}(x, y)$ and subtract the latter from the former. These two resulting equations are then added and integrated over the region R to yield the following equation:

$$\iint_R \Big\{ \frac{\partial}{\partial x}\Big[\bar{G}_x^x(x, y \mid x_0, y_0)\, \bar{P}_{xx}(u, v) + \bar{G}_y^x(x, y \mid x_0, y_0)\, \bar{P}_{xy}(u, v)$$

$$- \bar{u}\bar{P}_{xx}\Big[\mathrm{G}^x(x, y \mid x_0, y_0) \Big] - \bar{v}\,\bar{P}_{xy}\Big[\mathrm{G}_y^x(x, y \mid x_0, y_0) \Big]\Big]$$

$$+ \frac{\partial}{\partial y}\Big[\bar{G}_x^x(x, y \mid x_0, y_0) \bar{P}_{xy}(u, v) + \bar{G}_y^x(x, y \mid x_0, y_0)\, \bar{P}_{yy}(u, v)$$

$$- \bar{u}\bar{P}_{xy}\Big[\mathrm{G}^x(x, y \mid x_0, y_0) \Big] - \bar{v}\,\bar{P}_{yy}\Big[\mathrm{G}^x(x, y \mid x_0, y_0) \Big]\Big]\Big\}$$

$$\times\, dR = \bar{u}\,(x_0, y_0). \qquad \ldots(8)$$

For details of the analysis to obtain eqn. (8), we refer to the paper of Ghosh (1972).

Applying Green's theorem, the integral in (8) over the region R is converted to an integral over the curves S, S_1, S_2 (shown in Fig. 1) bounding the region R and we have

$$\int_{S+S_1+S_2} \Big\{ \bar{G}_x^x(x, y \mid x_0, y_0)\, P_{nx}(u, v) + \bar{G}_y^x(x, y \mid x_0, y_0)\, \bar{P}_{ny}(u, v)$$

$$- \bar{u}\bar{P}_{nx}[\mathrm{G}^x(x, y \mid x_0, y_0)] - \bar{v}\,\bar{P}_{ny}\Big[\mathrm{G}^x x, y \mid x_0, y_0) \Big]\Big\}\, ds = \bar{u}(x_0, y_0). \qquad \ldots(9)$$

Since the stresses due to (u, v) are zero on the free surfaces S, S_1, S_2 and the stresses due to Green's function are also zero on the free surface S, so we obtain from (9)

$$\int_0^l \{[\bar{u}]\; P_{xx}[\mathrm{G}^x(0, y \mid x_0, y_0)] + [\bar{v}]\bar{P}_{xy}\,[\mathrm{G}^x(0, y \mid x_0, y_0)]\}\, dy = \bar{u}(x_0, y_0) \qquad \ldots(10)$$

where $[u]$ and $[v]$ represent the jump discontinuity in displacement components across the crack HO and $HO = l$ is the length of the crack. Since we are considering a dip-slip fault, so there is no displacement discontinuity along x-direction across the fault surface. Consequently $[u] = 0$. Also, as we are interested in surface displacement only, so the equation (10) reduces to the form

$$\int_0^l [\bar{v}]\; \bar{P}_{xy}\,[\mathrm{G}^x(0, y \mid x_0, 0)]\, dy = \bar{u}(x_0, 0). \qquad \ldots(11)$$

Considering the equations (2), (3), (6) and (7), following the same procedure, we get

$$\int_0^l [v]\, \bar{P}_{xy}\, [G^y(0, y \mid x_0, 0)]\, dy = \bar{v}\, (x_0, 0). \qquad \ldots(12)$$

The Fourier transforms of the Green's functions are

$$\bar{G}_x^x(x, y \mid x, 0) = \frac{1}{\mu\ \pi} \int_{-\infty}^{\infty} \frac{\nu_2\ e^{-i\xi(x-x_0)}}{R(\xi)} \left[2\xi^2 e^{-\nu_1 y} - (2\xi^2 - k_2^2)\, e^{-\nu_2 y} \right] d\xi$$

$$\bar{G}_y^x(x, y \mid x_0, 0) = \frac{-i}{2\pi\mu} \int_{-\infty}^{\infty} \frac{\xi\ e^{-i\xi(x-x_0)}}{R(\xi)} \left[2\nu_1\nu_2 e^{-\nu_1 y} - (2\xi^2 - k_2^2) e^{-\nu_2 y} \right] d\xi$$

$$\bar{G}_x^y(x, y \mid x_0, 0) = \frac{-i}{2\pi\mu} \int_{-\infty}^{\infty} \frac{\xi\ e^{-i\xi(x-x_0)}}{R(\xi)} \left[(2\xi^2 - k_2^2)\, e^{-\nu_1 y} - 2\nu_1\nu_2\, e^{-\nu_2 y} \right] d\xi$$

$$\bar{G}_y^y(x, y \mid x_0, 0) = \frac{-1}{2\pi\mu} \int_{-\infty}^{\infty} \frac{\nu_1\ e^{-i\xi(x-x_0)}}{R(\xi)} \left[(2\xi^2 - k_2^2)\, e^{-\nu_1 y} - 2\xi^2\, e^{-\nu_2 y} \right] d\xi. \qquad \ldots(13)$$

Here $k_1^2 = \omega^2/\alpha^2$, $k_2^2 = \omega^2/\beta^2$, $\nu_1 = \sqrt{\xi^2 - k_1^2}$, $\nu_2 = \sqrt{\xi^2 - k_2^2}$ and $R(\xi) = 4\xi^2\nu_1\nu_2 - (\xi^2 + \nu_2^2)^2$; α, β are respectively P-wave and S-wave velocities. The values of $\nu_1(\xi)$ and $\nu_2(\xi)$ are to be so chosen that with such values the expression for the displacement decay exponentially as $y \to \infty$ for real $\nu_1(\xi)$ and $\nu_2(\xi)$.

The Fourier transform of the stress $P_{xy}[G^x(0, y \mid x_0, 0)]$ on the line of faulting HO is given by

$$\bar{P}_{xy}[G^x(0, y \mid x_0, 0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi x_0}}{R(\xi)} \left[(2\xi^2 - k_2^2)\, e^{-\nu_2 y} - 4\xi^2\nu_1\nu_2\, e^{-\nu_1 y} \right] d\xi. \qquad \ldots(14)$$

Since we want to determine the surface displacement due to Rayleigh waves, we need to determine the value of the integral in (14) for Rayleigh pole contribution only, for which we refer to Mal and Knopoff (1968). The Rayleigh pole contribution to the integral (14) is evaluated by following the method prescribed by Lapwood (1949) and $\bar{P}_{xy}[G^x(0, y \mid x_0, 0)]$ is found to be

$$\left.\begin{aligned}
&i\omega A_1\left\{\exp\left(-\frac{\omega}{\alpha c_R}\sqrt{\alpha^2-c_R^2\,y}\right)-\exp\left(-\frac{\omega}{\beta c_R}\sqrt{\beta^2-c_R^2\,y}\right)\right\}\\
&\qquad\times\exp\left(i\frac{\omega x_0}{c_R}\right)\text{ for }\omega>0\\
&\text{and}\\
&i\omega A_1\left\{\exp\left(\frac{\omega}{\alpha c_R}\sqrt{\alpha^2-c_R^2 y}\right)-\exp\left(\frac{\omega}{\beta c_R}\sqrt{\beta^2-c_R^2\,y}\right)\right\}\\
&\qquad\times\exp\left(i\frac{\omega x_0}{c_R}\right),\text{ for }\omega<0
\end{aligned}\right\}\quad\dots(15)$$

where

$$A_1=\frac{1}{4c_R}$$

$$\times\frac{\sqrt{\left(\alpha^2-c_R^2\right)\left(\beta^2-c_R^2\right)}}{\frac{2\alpha}{\beta}\left(2\beta^2-c_R^2\right)-\beta^2\left(\frac{\alpha^2-c_R^2}{\beta^2-c_R^2}\right)^{1/2}-\alpha^2\left(\frac{\beta^2-c_R^2}{\alpha^2-c_R^2}\right)^{1/2}-2\sqrt{\left(\alpha^2-c_R^2\right)\left(\beta^2-c_R^2\right)}}\quad\dots(16)$$

and c_R is the Rayleigh wave velocity.

Similarly, the contribution from the Rayleigh pole to $\overline{P}_{xy}[G^y(0, y \mid x_0, 0)]$ is found to be

$$\left.\begin{aligned}
&B_1\omega\left\{\exp\left(-\frac{\omega}{\alpha c_R}\sqrt{\alpha^2-c_R^2\,y}\right)-\exp\left(-\frac{\omega}{\beta c_R}\sqrt{\beta^2-c_R^2\,y}\right)\right\}\\
&\qquad\times\exp\left(i\frac{\omega x_0}{c_R}\right),\text{ for }\omega>0\\
&\text{and}\\
&-B_1\omega\left\{\exp\left(\frac{\omega}{\alpha c_R}\sqrt{\alpha^2-c_R^2\,y}\right)-\exp\left(\frac{\omega}{\beta c_R}\sqrt{\beta^2-c_R^2\,y}\right)\right\}\\
&\qquad\times\exp\left(i\frac{\omega x_0}{c_R}\right),\text{ for }\omega<0
\end{aligned}\right\}\quad\dots(17)$$

where

$$B_1=\frac{1}{2\beta c_R}$$

$$\times\frac{\left(2\beta^2-c_R^2\right)\sqrt{\alpha^2-c_R^2}}{\frac{2\alpha}{\beta}\left(2\beta^2-c_R^2\right)-\beta^2\left(\frac{\alpha^2-c_R^2}{\beta^2-c_R^2}\right)^{1/2}-\alpha^2\left(\frac{\beta^2-c_R^2}{\alpha^2-c_R^2}\right)^{1/2}-2\sqrt{\left(\alpha^2-c_R^2\right)\left(\beta^2-c_R^2\right)}}.\quad\dots(18)$$

The discontinuity in displacement along the line of faulting at any time t and at a depth y below the free surface is assumed to be

$$\begin{aligned}[v] &= DH[h(t) - (l - y)]\,[H(y) - H(y - l)] \\ &= DH[t - r(y)]\,[H(y) - H(y - l)] \qquad \ldots(19)\end{aligned}$$

$H(\ )$ is Heaviside step function and $r(y) = h^{-1}(l-y)$, which is the inverse function of h and it exists as $h(t)$ is strictly monotonic increasing function. Fourier transform of eqn. (19) is given by

$$\overline{[v]} = D[H(y) - H(y - l)] \int_{-\infty}^{\infty} H[t - r(y)] e^{i\omega t}\, dt$$

$$= D[H(y) - H(y-l)] \int_{r(y)}^{\infty} e^{i\omega t}\, dt = D[H(y) - H(y - l)] \times \left(\pi\delta(\omega) + \frac{i}{\omega}\right) e^{i\omega r(y)}. \qquad \ldots(20)$$

Putting the value of $\overline{[v]}$ from (20) in (11) and then taking Fourier inversion of (11) and changing the order of integration one obtains

$$u(x_0, 0) = \frac{D}{2\pi} \int_0^l dy \int_{-\infty}^{\infty} e^{i\omega(r(y)-t)} \left(\pi\delta(\omega) + \frac{i}{\omega}\right) \bar{P}_{xy}\, [G^x(0, y \mid x_0, 0)]\, d\omega.$$

Substituting the value of $\bar{P}_{xy}$ from (15) in the above equation we get

$$u(x_0, 0) = -\frac{A_1 D}{\pi} \int_0^l dy \int_0^{\infty} \left\{\left[\exp\left(-\frac{\omega}{\alpha c_R}\sqrt{\alpha^2 - c_R^2}\, y\right) - \exp\left(-\frac{\omega}{\beta c_R}\sqrt{\beta^2 - c_R^2}\, y\right)\right] \cos\left(r(y) - t + \frac{x_0}{c_R}\right) \omega \right\} d\omega$$

or

$$\frac{u(x_0, 0)}{D} = -\frac{A_1 C_R}{\pi} \int_0^l \left\{ \frac{\alpha\sqrt{\alpha^2 - c_R^2}\, y}{(\alpha^2 - c_R^2)\, y^2 + \alpha^2 (c_R r(y) - t c_R + x_0)^2} - \frac{\beta\sqrt{\beta^2 - c_R^2}\, y}{(\beta^2 - c_R^2)\, y^2 + \beta^2 (c_R\, r(y) - t c_R + x_0)^2} \right\} dy. \qquad \ldots(21)$$

Similarly, we obtain

$$\frac{v(x_0, 0)}{D} = -\frac{B_1 C_R}{\pi} \int_0^l \left\{ \frac{\alpha^2 (c_R\, r(y) - t c_R + x_0)}{(\alpha^2 - c_R^2) y^2 + \alpha^2 (c_R\, r(y) - t c_R + x_0)^2} \right.$$

(*equation continued on p.* 1001)

$$- \frac{\beta^2(c_R\, r(y) - tc_R + x_0)}{(\beta^2 - c_R^2)y^2 + \beta^2(c_R\, r(y) - tc_R + x_0)^2} \Big\} \, dy. \qquad \ldots(22)$$

3. Different Cases of Nonuniform Crack Speed

In this section we determine the Rayleigh wave displacement on the free surface for different nonuniform motions of the vertical crack.

Case 1—Here it is assumed that $h(t) = ct$, where c is the constant velocity of propagation of the fault and we have

$$t = \frac{l - y}{c} = r(y) . \qquad \ldots(23)$$

Substituting the value of $r(y)$ from (23) in (21) and (22) and integrating the resulting equation one gets

$$\frac{u(x_0, 0)}{D} = - \frac{AG}{4\pi p} \frac{c^2}{C_R^2} \left[\frac{A}{B} \left\{ \frac{1}{2} \ln \frac{A^2 + X^2}{T^2} + \frac{c_R}{c_A} \tan^{-1} \frac{A}{X} \right\} - \frac{G}{H} \left\{ \frac{1}{2} \ln \frac{G^2 + X^2}{T^2} + \frac{c_R}{c_G} \tan^{-1} \frac{G}{X} \right\} \right] \qquad \ldots(24)$$

$$\frac{v(x_0, 0)}{D} = - \frac{A(1 + G^2)}{2\pi P} \frac{c}{c_R} \left[\frac{1}{B} \left\{ \frac{c_A}{c_R} \tan^{-1} \frac{A}{X} - \frac{1}{2} \ln \frac{A^2 + X^2}{T^2} \right\} - \frac{1}{H} \left\{ \frac{c_G}{c_R} \tan^{-1} \frac{G}{X} - \frac{1}{2} \ln \frac{G^2 + X^2}{T^2} \right\} \right] \qquad \ldots(25)$$

where

$$A = \sqrt{1 - \frac{c_R^2}{\beta^2} \frac{\beta^2}{\alpha^2}}, \; B = 1 + \frac{c^2}{c_R^2}\left(1 - \frac{c_R^2}{\beta^2} \cdot \frac{\beta^2}{\alpha^2}\right), \; G = \sqrt{1 - \frac{c_R^2}{\beta^2}}$$

$$H = 1 + \frac{c^2}{c_R^2}\left(1 - \frac{c_R^2}{\beta^2}\right), X = \frac{x_0}{l} - \frac{tc_R}{l}, \; T = \frac{c_R}{c} - \frac{tc_R}{l} + \frac{x_0}{l}$$

and

$$P = 2(1 + G^2) - \frac{A}{G} - \frac{G}{A} - 2AG.$$

$T = 0$ implies $t = \frac{x_0}{c_R} + \frac{l}{c}$ which is the time taken to reach the point $(x_0, 0)$ by Rayleigh wave, which is emitted from the epicentre 0 when the crack reaches the free surface and $X = 0$ implies $t = x_0/c_R$ which is the time taken to reach the point $(x_0, 0)$ by the Rayleigh wave generated at H as soon as the crack appears at H.

Case 2—In this case it is assumed that the crack starts to move vertically upward with a finite velocity a and has a retardation b. Here at a time t after the formation of the crack

$$h(t) \; (\leqslant l) = at - \frac{1}{2} bt^2$$

so that

$$r(y) = \frac{2}{a}\,\frac{1(1-z)}{1+[1-(2bl/a^2)(1-z)]^{1/2}}. \qquad \ldots(26)$$

Substituting the value of $r(y)$ in (21) and (22) we obtain

$$\frac{u(x_0,0)}{D} = -\frac{AG}{\pi P}\int_0^1\left[\frac{Az}{Az^2+\left\{\dfrac{c_R}{a}\,\dfrac{2(1-z)}{1+[1-2F(1-z)]^{1/2}}+X\right\}^2} - \frac{Gz}{G^2z^2+\left\{\dfrac{c_R}{a}\,\dfrac{2(1-z)}{1+[1-2F(1-z)]^{1/2}}+X\right\}^2}\right]dz \qquad \ldots(27)$$

$$\frac{v(x_0,0)}{D} = -\frac{A}{\pi P}\,\frac{c_R}{\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\left(1-\frac{c_R^2}{2\beta^2}\right)$$

$$\times \int_0^1\left(2\,\frac{c_R}{a}\,\frac{1-z}{1+(1-2F(1-z))^{1/2}}+X\right)z^2\,dz \Big/$$

$$\left\{A^2z^2+\left(2\,\frac{c_R}{a}\,\frac{1-z}{1+[1-2F(1-z)]^{1/2}}+X\right)^2\right\}$$

$$\times\left\{G^2z^2+\left(2\,\frac{c_R}{a}\,\frac{1-z}{1+[1-2F(1-z)]^{1/2}}+X\right)^2\right\} \qquad \ldots(28)$$

where $F = bl/a^2$ and the other constants have the same values mentioned earlier. It may be noted that the integrands in equations (27) and (28) have a singularity at $z = 0$ provided

$$2\frac{c_R}{a}\,\frac{1}{1+\sqrt{1-2F}}+X = 0, \text{ which implies that}$$

$$t = \frac{x_0}{c_R}+\frac{2l}{a(1+\sqrt{1-(2bl/a^2)})}.$$

This is the time to reach the point $(x_0, 0/$ by the Rayleigh wave emitted from the epicentre 0 just after the arrival of the crack at this point.

Case 3— Finally let the crack at a depth 1 below the free surface, start to move vertically upward with infinitely large velocity which gradually decays with time. Accordingly $h(t)$ is taken in the form

$h(t) = D_1\sqrt{t}$ where D_1 is a constant.

or

$$D_1\sqrt{t} = 1 - y.$$

Therefore

$$r(y) = [(l-y)/D_1] \qquad \ldots(29)$$

As before substituting the value of $r(y)$ in (21) and (22) and making a change of variable of the integration, we have

$$\frac{u(x_0, 0)}{D} = -\frac{K^2AG}{4\pi P}\int_0^1\left[\frac{A}{z^4-4z^3+(K^2A^2+2M+4)^2 z^2-4Mz+M^2}\right.$$

$$\left.-\frac{G}{z^4-4z^3+(K^2G^2+2M+4)^2 z^2-4Mz+M^2}\right]zdz \qquad \ldots(30)$$

$$\frac{v(x_0, 0)}{D} = \frac{K^3Ac_R^2}{\pi P\beta^2}\left(1-\frac{\beta^2}{\alpha^2}\right)\left(1-\frac{c_R^2}{2\beta^2}\right)$$

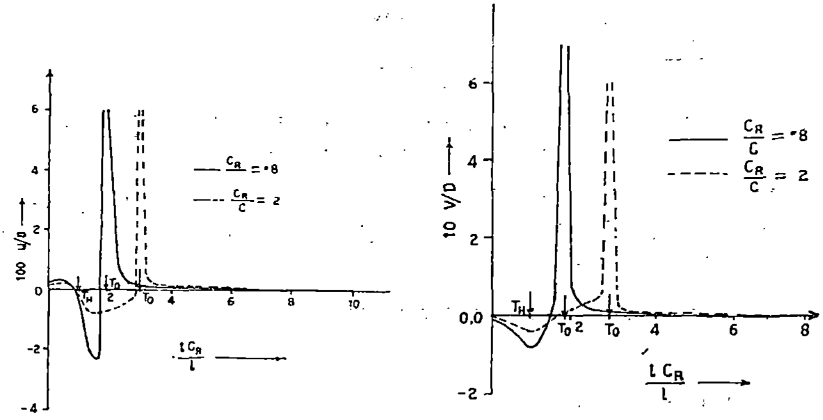
$$\times\int_0^1\frac{z^2(z^2-2z+M)dz}{\{K^2A^2z^4+(z^2-2z+M)^2\}\{K^2G^2z^4+(z^2-2z+M)^2\}} \qquad \ldots(31)$$

where $K^2 = D_1^2/lc_R$, $M = 1-(tc_R/l)(D_1^2/lc_R)+(x_0/l)(D_1^2/lc_R)$ and the other constants have same values as mentioned before. Again, the integrands in eqns. (30) and (31) are singular at $z = 0$ if $M = 0$. This corresponds to $t = (x_0/c_R)+(l^2/D_1^2)$ which is the time of arrival at $(x_0, 0)$ of the Rayleigh wave which is generated at 0 just after the arrival of the crack on the free surface.

4. NUMERICAL RESULTS AND CONCLUSION

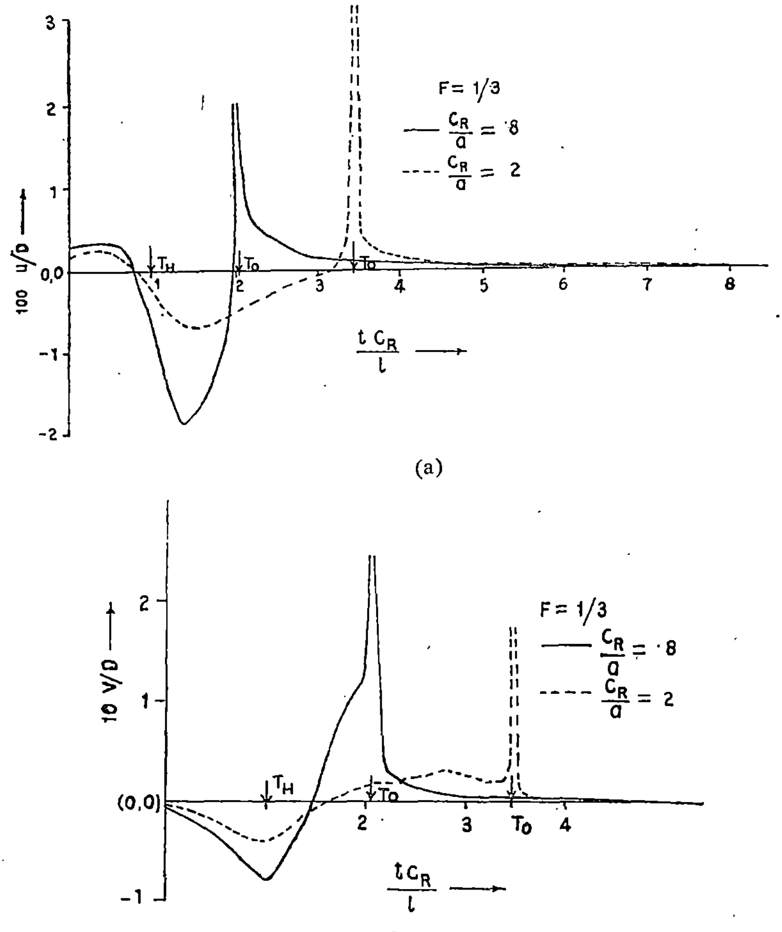
When the earth material is under tension or compression in a direction parallel to the free surface, shear failure occurs on a fault plane. In general, this failure moves with nonuniform speed. Numerical computations are carried out for poisson solid ($\alpha/\beta = \sqrt{3}$) and for $c_R/\beta = 0.9194$. The quantities A, B, G, H, X, T, P, K, M, F defined in section 3 are all dimensionless.

Figures 2, 3 and 4 show the variation of components of displacement with



(a) (b)

FIGS. 2(a, b). Horizontal and vertical components of displacement versus time. Case of crack propagation with uniform velocity.



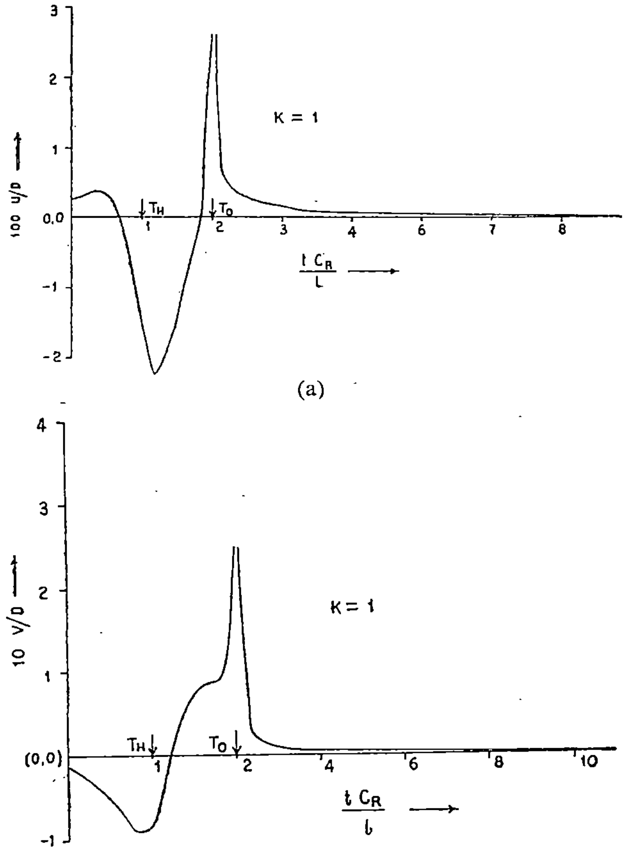
(a)

(b)

FIGS. 3(a, b). Horizontal and Vertical components of displacement versus time. Case of crack propagation with retardation and finite starting velocity.

time. The dimensionless displacement components are plotted against dimensionless time T_H and T_0 indicated in the figures correspond to the arrival times at $(x_0, 0)$ of the Rayleigh waves from the focus H and the fault break at 0. Figures 2(a, b) correspond to the case 1 of section 3 where the constant velocity of propagation of the crack ($c_R/c = 0.8$ and 2) is assumed. Figures 3(a, b) correspond to the case where the crack starts with a finite velocity a and has a retardation b. Here also two cases $c_R/a = 0.8$ and 2 with the assumption that $F = 1/3$, are considered. Figures 4(a, b) depicts the case 3 where the initial velocity of crack propagation is assumed to be infinitely large and K is taken to be equal to 1.

From eqns. (21) and (22) it may be noted that $u(x_0, 0)/D$ and $v(x_0, 0)/D$ are



(a)

(b)

FIGS. 4(a, b). Horizontal and Vertical components of displacement versus time. Case of crack propagation with retardation and infinite starting velocity.

functions (x_0/l) — $(c_R t/l)$. Therefore in all computational works, without any loss of generality x_0/l has been taken to be equal to 1, because any change in value of x_0/l will merely cause a shifting of the graphs along the direction of $c_R t/l$.

We find that in each case the strongest ground motion occurs at $T^0 = c_R t/l$ which correspond to the arrival time of Rayleigh waves from the surface break at 0. Also it is found that though the nature of the graphs in three different cases differ between T_H and T_0 but their natures are almost the same after the arrival of Rayleigh waves from the surface break. This may be explained from the fact that the main contribution to the ground motion due to Rayleigh wave is from a small portion of the fault near the surface after the arrival of Rayleigh wave from the surface break. So the contribution from the details of crack initiation becomes insignificant after T_0. It may be mentioned in this connection that though contribution of Rayleigh wave to the ground motion is significant at large distances from the epicentre, the effect of body waves near the epicentral region cannot be ignored. This effect may be incorporated if we consider in addition to the contribution from Rayleigh pole, the contribution from the branch line integrals arising from the evaluation of stresses due to Green's function.

ACKNOWLEDGEMENT

The authors express their thanks to Mr C. R. Nayek and to Mr R. K. Samanta for their cooperation in computational work.

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