CHAPTER - 5

H - FUNCTION

5. Study of some approximate forms of H - Function.

The H-function plays an important role in the theory of Radiative transfer in semi infinite plane parallel atmosphere. For the standard problems in isotropic and anisotropic scattering, the angular distributions of the emergent intensities are expressed in terms of H-function which satisfies the following nonlinear integral equation:

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\psi(\mu')H(\mu')}{\mu + \mu'} d\mu' \qquad (5.A)$$

where the characteristic function $\psi(\mu)$ is an even polynomial in μ and satisfies the following condition

$$\int_0^1 \psi(\mu) d\mu \leq \frac{1}{2} \qquad (5.B)$$

For more general laws of scattering, we get simultaneous non-linear integral equation whose solutions can also be expressed in terms of H-function and the following relation holds

$$\frac{1}{H(\mu,\omega)} = 1 - \mu \int_0^1 \frac{\psi(\mu')}{\mu + \mu'} H(\mu') d\mu' \qquad (5.C)$$

In most cases, the emergent intensity is expressed in terms of H - function. Therefore good approximate forms are often found to be useful. Abu-Shumays (1967), Karanjai (1968 a,b), Karanajai & Sen (1970) , (1971) introduced different approximate forms of H-function.

Burman (1989) extended one approximate form of H - function for isotropic scattering developed by Karanjai and Sen (1970) to the case of anisotropic scattering. Here we will use different approximate forms of H - function developed by Karanjai and Sen (1970,1971) with the general characteristic function given by

$$\psi(\mu) = \omega_1 + \omega_2\mu^2 + \omega_3\mu^4 \tag{5.D}$$

Using this method we have computed the H - function numerically and compared our results with that of Chandrasekhar. The results are found to be correct in most cases upto four decimal places. Also we will find moments of order zero and one and Chandrasekhar's constant p,q and c using the different forms.

5.1. The forms of H - function and the development of the problem:

The approximate forms of H - function considered are:

Form I: $$H(\mu,\omega) = 1 + \frac{a\mu + b\mu^2 + c\mu^3}{A + 2\mu} \quad *$$

Form II: $$H(\mu,\omega) = 1 + \frac{a\mu + b\mu^2 + c\mu^3}{1 + k\mu} \quad *$$

Form III: $$H(\mu,\omega) = 1 + a\mu + b\mu^2 + c\mu^3 \quad *$$

where $A = \sqrt{1-\omega}$ and a,b,c are constants which are to be detemined.

The H - function satisfies the following relations: (Chandrasekhar 1960)

$$\int_0^1 H(\mu)\psi(\mu)d\mu = 1 - \left[1 - 2\int_0^1 \psi(\mu)d\mu\right]^{\frac{1}{2}} \tag{5.1.1}$$

* Karanjai, S. and Sen. M. (1971)

$$\left[1 - 2\int_0^1 \psi(\mu)d\mu\right]^{\frac{1}{2}} + \int_0^1 H(\mu)\psi(\mu)\mu^2 d\mu + \frac{1}{2}\left[\int_0^1 H(\mu)\psi(\mu)\mu\, d\mu\right]^2$$

$$= \int_0^1 \psi(\mu)\mu^2 d\mu \tag{5.1.2}$$

$$\int_0^1 \frac{\psi(\mu)H(\mu)}{1 - k\mu} d\mu = 1 \tag{5.1.3}$$

where k is determined from the transcedental equation

$$1 + \frac{2\omega_3}{3k^2} + \frac{2}{k^2}\left[\omega_2 + \frac{\omega_3}{k^2}\right] = \frac{1}{k}\left[\omega_1 + \frac{1}{k^2}\left(\omega_2 + \frac{\omega_3}{k^2}\right)\right] log\left(\frac{1 + k}{1 - k}\right) \tag{5.1.4}$$

Let us the following notations:

$$P_n = \sum_{r=1}^{3} \frac{\omega_r}{n + 2(r-1)} \tag{5.1.5a}$$

$$x_n = \sum_{r=1}^{n-1} \frac{1}{(n-r)k^r}, \quad n \geq 2 \tag{5.1.5b}$$

$$y_n = \sum_{r=1}^{3} \frac{\omega_r}{k^{n+2(r-1)}} \tag{5.1.5c}$$

$$V = \left[1 - 2 \int_0^1 \psi(\mu) d\mu \right]^{\frac{1}{2}} = \left(1 - 2P_1\right)^{\frac{1}{2}} \tag{5.1.6}$$

We consider different forms of H - function as given in Form I to Form III in Sec. 5.1

5.2 Form I.

5.2.1 First Approximation

Referring to this form of H - function we develop necessary equations to evaluate H - functions. Substituting Form I and (5.D) in Eqs. (5.1.1) , (5.1.2) , (5.1.3) we determine the constants a,b,c.

We denote

$$\frac{1}{2^n} T_n = \int_0^1 \frac{\mu^n}{A + 2\mu} d\mu \tag{5.2.1.1}$$

where T_n s are defined by

$$T_j = \frac{2^j}{j} - A T_{j-1}, \qquad T_1 = 2 - A \log\left(\frac{A + 2}{A} \right) \tag{5.2.1.2}$$

Also,

$$I_j = \int_0^1 \frac{\mu^j}{(A + 2\mu)(1 - k\mu)} d\mu \tag{5.2.1.3}$$

then we can write

$$I_j = \frac{1}{k}\left(I_{j-1} - \frac{T_{j-1}}{2^j} \right) \quad j = 2,3,\ldots\ldots,7 \tag{5.2.1.4a}$$

with

$$I_1 = - \frac{1}{(Ak+2)} \left[\frac{A}{2} log\left(\frac{A+2}{A} \right) + \frac{1}{k} log(1 - k) \right] \tag{5.2.1.4b}$$

Now from (5.1.1) , (5.1.2) , (5.1.3) we obtain the following equations:

$$a = \xi_1 + \eta_1 c \tag{5.2.1.5a}$$

$$b = \xi_2 + \eta_2 c \tag{5.2.1.5b}$$

where

$$\xi_1 = \frac{M_1}{L_1}, \qquad \xi_2 = \frac{M_2}{L_1} \tag{5.2.1.6a}$$

$$\eta_1 = \frac{N_1}{L_1}, \qquad \eta_2 = \frac{N_2}{L_1} \tag{5.2.1.6b}$$

$$M_1 = L\delta_2 - S\alpha_2, \qquad M_2 = S\alpha_1 - L\delta_1 \tag{5.2.1.7a}$$

$$N_1 = \alpha_2\delta_3 - \delta_2\alpha_3, \qquad N_2 = \delta_3\alpha_1 - \alpha_3\delta_1 \tag{5.2.1.7b}$$

$$L_1 = \alpha_1\delta_2 - \delta_1\alpha_2, \qquad L = 1 - V - P_1 , \tag{5.2.1.7c}$$

$$S = 1 + \omega_2 x_3 + \omega_3 x_5 + y_1 log(1 - k) \tag{5.2.1.7d}$$

Here c is determined from the following quadratic equation

$$\zeta_1 c^2 + \zeta_2 c + \zeta_3 = 0 \tag{5.2.1.8}$$

where

$$\zeta_1 = \frac{\Re_1^2}{2}, \quad \zeta_2 = \Re_1 \Re_2 + \Re_3, \quad \zeta_3 = \Re_4 + \frac{\Re_2^2}{2} - P_3 \tag{5.2.1.9}$$

$$\Re_1 = \beta_1 \eta_1 + \beta_2 \eta_2 + \beta_3 \tag{5.2.1.10a}$$

$$\Re_2 = \beta_1 \xi_1 + \beta_2 \xi_2 + P_2 \tag{5.2.1.10b}$$

$$\Re_3 = V(\eta_1 \gamma_1 + \eta_2 \gamma_2 + \gamma_3) \tag{5.2.1.10c}$$

$$\Re_4 = V(P_3 + \xi_1 \gamma_1 + \xi_2 \gamma_2) \tag{5.2.1.10d}$$

$$\alpha_n = \sum_{r=1}^{3} \frac{\omega_r}{2^{2r+n-1}} T_{2r+n-2}; \quad n = 1,2,3 \tag{5.2.1.11a}$$

$$\beta_n = \sum_{r=1}^{3} \frac{\omega_r}{2^{2r+n-1}} T_{2r+n-1}, \quad n = 1,2,3 \tag{5.2.1.11b}$$

$$\gamma_n = \sum_{r=1}^{3} \frac{\omega_r}{2^{2r+n-1}} T_{2r+n}, \quad n = 1,2,3 \tag{5.2.1.11c}$$

$$\delta_n = \sum_{r=1}^{3} \omega_r I_{2r+n-2}, \quad n = 1,2,3 \tag{5.2.1.11d}$$

5.2.2 Second Approximation:

Here we will determine the H- function from (5.C) based on the computed values of a, b, c. Using the approximate form of H-function (form I) and the Equation (5.D) in the Equation (5.C) we obtain the following equation :

$$\frac{1}{H(\mu,\omega)} = 1 - \mu\left[Z + E_1 a + E_2 b + E_3 c\right] \tag{5.2.2.1}$$

where

$$Z = \sum_{r=1}^{3} \omega_r D_{2r-1} \tag{5.2.2.2}$$

$$E_n = \sum_{r=1}^{3} \omega_r F_{2r+n-2} \quad , n = 1,2,3 \tag{5.2.2.3}$$

where

$$F_p = \frac{T_p}{2^p} - \mu F_{p-1} \tag{5.2.2.4a}$$

with

$$F_1 = -\frac{1}{2\mu - A}\left[\frac{A}{2} log\left(\frac{A+2}{A}\right) - \mu\, log\left(\frac{1+\mu}{\mu}\right)\right] \tag{5.2.2.4b}$$

and finally ,

$$D_j = \frac{1}{j-1} - \mu D_{j-1} \quad j = 2,3,\ldots,8 \tag{5.2.2.5a}$$

with

$$D_1 = log\left(\frac{1+\mu}{\mu}\right) \tag{5.2.2.5b}$$

The values H - function can be computed from (5.2.2.1) for different ω and μ. For different ω we get different k which can be obtained from the transcedental equation by applying the Newton-Raphson method.

5.2.3. Applications and Results

We apply the method to two different cases:

Case I. The phase function is planetary:

$$p(\mu,\mu') = \omega(1 + xcos(\theta))$$

Here the characteristic function is:

$$\psi(\mu,\omega) = \frac{1}{2}\omega[1 + x(1-\omega)\mu^2], \quad x = 1.$$

then

$$\omega_1 = \frac{\omega}{2}, \quad \omega_2 = \frac{x\omega(1-\omega)\mu^2}{2}, \quad \omega_3 = 0$$

Case II. The phase function is isotropic.

Here

$$p(\mu,\mu') = \frac{\omega}{2}$$

and

$$\psi(\mu) = \frac{\omega}{2}$$

so that

$$\omega_1 = \frac{\omega}{2}, \quad \omega_2 = 0, \quad \omega_3 = 0$$

Tables 5.1.I and 5.1.II are shown below for **Case I** and **Case II** respectively.

Table 5.1.I

μ	ω = .1	ω = .2	ω = .3	ω = .4	ω = .5	ω = .6
.05	1.008922	1.018211	1.027930	1.038159	1.049009	1.060643
.10	1.014462	1.029702	1.045854	1.063092	1.081657	1.101906
.15	1.018783	1.038754	1.060126	1.083175	1.108286	1.136034
.20	1.022349	1.046286	1.072103	1.100185	1.131073	1.165575
.25	1.025384	1.052738	1.082436	1.114977	1.151062	1.191744
.30	1.028018	1.058369	1.091511	1.128056	1.168872	1.215266
.35	1.030337	1.063351	1.099582	1.139760	1.184917	1.236624
.40	1.032400	1.067803	1.106832	1.150328	1.199496	1.256167
.45	1.034252	1.071815	1.113393	1.159941	1.212829	1.274158
.50	1.035926	1.075456	1.119371	1.168737	1.225092	1.290802
.55	1.037449	1.078779	1.124845	1.176826	1.236421	1.306264
.60	1.038842	1.081826	1.129883	1.184297	1.246931	1.320681
.65	1.040122	1.084634	1.134538	1.191224	1.256714	1.334165
.70	1.041303	1.087230	1.138855	1.197668	1.265850	1.346812
.75	1.042395	1.089639	1.142871	1.203681	1.274404	1.358703
.80	1.043411	1.091881	1.146618	1.209308	1.282433	1.369910
.85	1.044356	1.093974	1.150124	1.214585	1.289988	1.380492
.90	1.045240	1.095933	1.153412	1.219547	1.297111	1.390504
.95	1.046067	1.097771	1.156502	1.224221	1.303840	1.399993
1.00	1.046843	1.099498	1.159413	1.228634	1.310207	1.409001

Table 5.1.I (Cont.)

μ	ω = .7	ω = .8	ω = .9	ω = .925	ω = .950	ω = .975
.05	1.073329	1.087572	1.104637	1.109753	1.115546	1.122618
.10	1.124423	1.150310	1.182331	1.192189	1.203519	1.217641
.15	1.167358	1.204037	1.250537	1.265143	1.282128	1.303628
.20	1.205016	1.251919	1.312633	1.332031	1.354810	1.384024
.25	1.238763	1.295436	1.370158	1.394393	1.423099	1.460340
.30	1.269408	1.335459	1.424001	1.453109	1.487859	1.533415
.35	1.297494	1.372569	1.474741	1.508749	1.549644	1.603775
.40	1.323412	1.407184	1.522793	1.561717	1.608840	1.671781
.45	1.347458	1.439619	1.568465	1.612314	1.665734	1.737693
.50	1.369865	1.470126	1.612005	1.660775	1.720547	1.801712
.55	1.390822	1.498907	1.653607	1.707290	1.773457	1.863995
.60	1.410484	1.526132	1.693440	1.752018	1.824610	1.924668
.65	1.428983	1.551944	1.731641	1.795093	1.874130	1.983838
.70	1.446428	1.576464	1.768332	1.836629	1.922123	2.041595
.75	1.462917	1.599800	1.803617	1.876729	1.968682	2.098017
.80	1.478532	1.622043	1.837591	1.915480	2.013888	2.153171
.85	1.493345	1.643277	1.870336	1.952963	2.057814	2.207118
.90	1.507422	1.663575	1.901927	1.989249	2.100526	2.259912
.95	1.520818	1.683001	1.932431	2.024403	2.142084	2.311601
1.00	1.533586	1.701615	1.961909	2.058484	2.182542	2.362231

Table 5.1.II

μ	ω = .2	ω = .3	ω = .4	ω = .5	ω = .6	ω = .7
.05	1.016037	1.024758	1.034099	1.044192	1.055255	1.067639
.10	1.025598	1.039810	1.055273	1.072274	1.091277	1.113034
.15	1.032921	1.051471	1.071881	1.094605	1.120368	1.150364
.20	1.038895	1.061064	1.085675	1.113351	1.145091	1.182551
.25	1.043932	1.069212	1.097482	1.129539	1.166662	1.210982
.30	1.048273	1.076274	1.107784	1.143773	1.185799	1.236479
.35	1.052073	1.082487	1.116900	1.156451	1.202977	1.259587
.40	1.055438	1.088014	1.125050	1.167853	1.218536	1.280697
.45	1.058446	1.092975	1.132398	1.178188	1.232727	1.300105
.50	1.061156	1.097460	1.139070	1.187615	1.245747	1.318038
.55	1.063614	1.101541	1.145162	1.196261	1.257750	1.334681
.60	1.065856	1.105273	1.150752	1.204228	1.268863	1.350184
.65	1.067910	1.108703	1.155906	1.211597	1.279190	1.364672
.70	1.069801	1.111868	1.160674	1.218439	1.288817	1.378250
.75	1.071548	1.114798	1.165101	1.224812	1.297818	1.391008
.80	1.073169	1.117521	1.169225	1.230765	1.306256	1.403024
.85	1.074675	1.120059	1.173076	1.236339	1.314186	1.414364
.90	1.076081	1.122430	1.176682	1.241573	1.321653	1.425087
.95	1.077395	1.124651	1.180068	1.246497	1.328699	1.435245
1.00	1.078628	1.126737	1.183252	1.251139	1.335360	1.444883

Table 5.1.II (Cont.)

μ	ω = .80	ω = .85	ω = .90	ω = .925	ω = .950	ω = .975
.05	1.081997	1.090337	1.099962	1.105538	1.111946	1.119988
.10	1.138958	1.154393	1.172585	1.183327	1.195915	1.211975
.15	1.186840	1.208965	1.235465	1.251342	1.270181	1.294618
.20	1.228877	1.257413	1.292055	1.313066	1.338263	1.371410
.25	1.266590	1.301305	1.343949	1.370093	1.401740	1.443893
.30	1.300881	1.341565	1.392073	1.423340	1.461512	1.512932
.35	1.332347	1.378806	1.437039	1.473410	1.518159	1.579074
.40	1.361420	1.413468	1.479284	1.520730	1.572094	1.642698
.45	1.388426	1.445885	1.519141	1.565624	1.623623	1.704082
.50	1.413621	1.476321	1.556869	1.608345	1.672984	1.763440
.55	1.437212	1.504988	1.592680	1.649099	1.720373	1.820942
.60	1.459370	1.532064	1.626751	1.688056	1.765949	1.876728
.65	1.480239	1.557698	1.659230	1.725363	1.809848	1.930913
.70	1.499939	1.582016	1.690247	1.761143	1.852187	1.983599
.75	1.518577	1.605131	1.719912	1.795507	1.893069	2.034871
.80	1.536243	1.627138	1.748324	1.828550	1.932582	2.084806
.85	1.553017	1.648123	1.775571	1.860358	1.970809	2.133470
.90	1.568971	1.668161	1.801730	1.891008	2.007822	2.180933
.95	1.584166	1.687319	1.826872	1.920570	2.043865	2.227242
1.00	1.598659	1.705659	1.851059	1.949105	2.078461	2.272448

5.3. Form II.

5.3.1 First approximation :

As in form I we will obtain the similar set of equations. We follow the same procedure that we have done in case of form I. We denote

$$\vartheta_n = \int_0^1 \frac{\mu^n}{1 + k\mu} d\mu \tag{5.3.1.1}$$

We define

$$\vartheta_n = \frac{1}{k}\left(\frac{1}{n} - \vartheta_{n-1}\right) \tag{5.3.1.2a}$$

where

$$\vartheta_1 = \frac{1}{k}(1 - T), \quad T = \frac{1}{k} log(1 + k) \tag{5.3.1.2b}$$

$$R_n = \int_0^1 \frac{\mu^n}{1 - k^2\mu^2} d\mu \tag{5.3.1.3}$$

By the same procedure we obtain the similar equations (from 5.3.1.1 to 5.3.1.1.) but with different values

$$a = \xi_1 + \eta_1 c \tag{5.3.1.4a}$$

$$b = \xi_2 + \eta_2 c \tag{5.3.1.4b}$$

$$\xi_1 = \frac{M_1}{L_1}, \qquad \xi_2 = \frac{M_2}{L_1} \tag{5.3.1.5a}$$

$$\eta_1 = \frac{N_1}{L_1}, \qquad \eta_2 = \frac{N_2}{L_1} \tag{5.3.1.5b}$$

$$M_1 = L\delta_2 - S\alpha_2, \qquad M_2 = S\alpha_1 - L\delta_1 \tag{5.3.1.6a}$$

$$N_1 = \alpha_2\delta_3 - \delta_2\alpha_3, \qquad N_2 = \delta_3\alpha_1 - \alpha_3\delta_1 \tag{5.3.1.6b}$$

$$L_1 = \alpha_1\delta_2 - \delta_1\alpha_2, \qquad L = 1 - V - P_1 \tag{5.3.1.6c}$$

$$S = 1 + \omega_2 x_3 + \omega_3 x_5 + y_1 \log(1-k) \tag{5.3.1.6d}$$

Here c is determined from the following quadratic equation

$$\zeta_1 c^2 + \zeta_2 c + \zeta_3 = 0 \tag{5.3.1.7}$$

where

$$\zeta_1 = \frac{\Re_1^2}{2}, \quad \zeta_2 = \Re_1\Re_2 + \Re_3, \quad \zeta_3 = \Re_4 + \frac{1}{2}\Re_2^2 - P_3 \tag{5.3.1.8}$$

$$\Re_1 = \beta_1\eta_1 + \beta_2\eta_2 + \beta_3 \tag{5.3.1.9a}$$

$$\Re_2 = \beta_1\xi_1 + \beta_2\xi_2 + P_2 \tag{5.3.1.9b}$$

$$\Re_3 = V(\eta_1\gamma_1 + \eta_2\gamma_2 + \gamma_3) \tag{5.3.1.9c}$$

$$\Re_4 = V(P_3 + \xi_1\gamma_1 + \xi_2\gamma_2) \tag{5.3.1.9d}$$

$$\alpha_n = \sum_{r=1}^{3} \frac{\omega_r}{2^{2r+n-1}} T_{2r+n-2} \tag{5.3.1.10a}$$

$$\beta_n = \sum_{r=1}^{3} \frac{\omega_r}{2^{2r+n-1}} T_{2r+n-1} \tag{5.3.1.10b}$$

$$\gamma_n = \sum_{r=1}^{3} \frac{\omega_r}{2^{2r+n-1}} T_{2r+n} \tag{5.3.1.10c}$$

$$\delta_n = \sum_{r=1}^{3} \omega_r I_{2r+n-2} \tag{5.3.1.10d}$$

Since a,b,c are known these will be substituted in the equation (5.C) which will give us the approximate values of H - function.

5.3.2. Second Approximation:

From equation (5.C) using the values of a,b,c we may write

$$\frac{1}{H(\mu,\omega)} = 1 - \mu\left[Z + E_1 a + E_2 b + E_3 c\right] \qquad (5.3.2.1)$$

where

$$Z = \sum_{r=1}^{3} \omega_r D_{2r-1} \qquad (5.3.2.2)$$

$$E_n = \sum_{r=1}^{3} \omega_r F_{2r+n-2} \quad n = 1,2,3 \qquad (5.3.2.3)$$

and

$$F_j = I_{j-1} - \mu F_{j-1}, \quad \text{with} \quad F_1 = \frac{1}{1 - k\mu}\left(T - \mu D_1\right), \qquad (5.3.2.4)$$

As in Section 5.2.3 the method is applied in two cases taking different values of ω and μ and the results are shown in Tables 5.2.I and 5.2.II respectively.

Table 5.2.I

μ	ω = .1	ω = .2	ω = .3	ω = .4	ω = .5	ω = .6
.05	1.008917	1.018190	1.027880	1.038065	1.048850	1.060392
.10	1.014454	1.029670	1.045779	1.062950	1.081417	1.101530
.15	1.018773	1.038715	1.060033	1.083001	1.107994	1.135578
.20	1.022339	1.046242	1.071998	1.099989	1.130745	1.165062
.25	1.025373	1.052689	1.082322	1.114763	1.150705	1.191188
.30	1.028006	1.058317	1.091390	1.127829	1.168493	1.214677
.35	1.030324	1.063296	1.099455	1.139522	1.184521	1.236007
.40	1.032386	1.067747	1.106700	1.150082	1.199084	1.255527
.45	1.034238	1.071757	1.113252	1.159687	1.212405	1.273498
.50	1.035912	1.075397	1.119231	1.168476	1.224657	1.290125
.55	1.037435	1.078718	1.124703	1.176559	1.235977	1.305573
.60	1.038828	1.081765	1.129738	1.184025	1.246478	1.319977
.65	1.040107	1.084571	1.134391	1.190948	1.256254	1.333450
.70	1.041287	1.087166	1.138705	1.197388	1.265383	1.346087
.75	1.042380	1.089574	1.142720	1.203398	1.273931	1.357969
.80	1.043395	1.091816	1.146650	1.209021	1.281955	1.369166
.85	1.044340	1.093908	1.149969	1.214296	1.289505	1.379741
.90	1.045224	1.095968	1.153256	1.219254	1.296623	1.389745
.95	1.046051	1.097704	1.156345	1.223926	1.303347	1.399227
1.00	1.046807	1.099431	1.159255	1.228336	1.309711	1.408229

Table 5.2.I (Cont.)

μ	ω = .7	ω = .8	ω = .9	ω = .925	ω = .950	ω = .975
.05	1.072950	1.087011	1.103811	1.108140	1.114530	1.121481
.10	1.123857	1.149481	1.181120	1.190854	1.202044	1.216002
.15	1.166674	1.203037	1.249086	1.263548	1.280369	1.301682
.20	1.204248	1.250880	1.311016	1.330256	1.352856	1.381868
.25	1.237932	1.294228	1.368420	1.392487	1.421004	1.458033
.30	1.268528	1.334183	1.422170	1.451103	1.485657	1.530994
.35	1.296574	..371238	1.472838	1.506666	1.547359	1.601266
.40	1.322459	1.405807	1.520831	1.559571	1.606490	1.669201
.45	1.346477	1.438205	1.566455	1.610117	1.663331	1.735057
.50	1.368861	1.468680	1.609954	1.658536	1.718100	1.799031
.55	1.389797	1.497433	1.651523	1.705017	1.770973	1.861276
.60	1.409441	1.524634	1.691326	1.749715	1.822096	1.921918
.65	1.427923	1.550423	1.729501	1.792763	1.871590	1.981062
.70	1.445354	1.574924	1.766170	1.834277	1.919560	2.038797
.75	1.461829	1.598242	1.801435	1.874357	1.966100	2.095200
.80	1.477431	1.620469	1.835391	1.913090	2.011288	2.150338
.85	1.492233	1.641688	1.868119	1.950557	2.055200	2.204272
.90	1.506299	1.661972	1.899695	1.986828	2.097898	2.257054
.95	1.519686	1.681385	1.930185	2.021969	2.139444	2.308733
1.00	1.532444	1.699986	1.959650	2.056038	2.179891	2.359355

Table 5.2.II

μ	ω = .2	ω = .3	ω = .4	ω = .5	ω = .6	ω = .7
.05	1.016019	1.024716	1.034016	1.044050	1.055026	1.067283
.10	1.025570	1.039746	1.055148	1.072059	1.090932	1.112502
.15	1.032887	1.051393	1.071728	1.094343	1.119949	1.149718
.20	1.038856	1.060976	1.085502	1.113055	1.144618	1.181825
.25	1.043890	1.069116	1.097293	1.129217	1.166148	1.210950
.30	1.048229	1.076172	1.107584	1.143431	1.185253	1.235644
.35	1.052026	1.082380	1.116690	1.156092	1.202405	1.258712
.40	1.055389	1.087903	1.124832	1.167480	1.217942	1.279791
.45	1.058396	1.092860	1.132174	1.177804	1.232114	1.299170
.50	1.061105	1.097342	1.138840	1.187221	1.245118	1.317079
.55	1.063562	1.101421	1.144926	1.195858	1.257108	1.333702
.60	1.065803	1.105151	1.150513	1.203817	1.268208	1.349186
.65	1.067856	1.108579	1.155662	1.211179	1.278524	1.363657
.70	1.069747	1.111742	1.160427	1.218015	1.288141	1.377221
.75	1.071494	1.114671	1.164851	1.224382	1.297133	1.389965
.80	1.073113	1.117393	1.168972	1.230330	1.305563	1.401968
.85	1.074619	1.119929	1.172821	1.235900	1.313485	1.413296
.90	1.076025	1.122299	1.176425	1.241129	1.320945	1.424009
.95	1.077338	1.124519	1.179808	1.246049	1.327984	1.434157
1.00	1.078578	1.126603	1.182990	1.250688	1.334640	1.443785

Table 5.2.II (Cont.)

μ	ω = .80	ω = .85	ω = .900	ω = .925	ω = .950	ω = .975
.05	1.081456	1.089669	1.099131	1.104606	1.110915	1.118798
.10	1.138155	1.153405	1.171365	1.181965	1.194390	1.210262
.15	1.185870	1.207776	1.234002	1.249713	1.268363	1.292586
.20	1.227789	1.256082	1.290424	1.311254	1.336245	1.369161
.25	1.265414	1.299870	1.342193	1.368146	1.399577	1.441488
.30	1.299635	1.340049	1.390222	1.421290	1.459238	1.510409
.35	1.331046	1.377224	1.435112	1.471279	1.515800	1.576460
.40	1.360073	1.411832	1.477297	1.518534	1.569666	1.640012
.45	1.387040	1.444204	1.517102	1.563374	1.621113	1.701339
.50	1.412202	1.474602	1.554787	1.606050	1.670454	1.760650
.55	1.435764	1.503235	1.590561	1.646765	1.717804	1.818113
.60	1.457896	1.530282	1.624600	1.685690	1.763347	1.873866
.65	1.478741	1.555889	1.657051	1.722967	1.807218	1.928025
.70	1.498420	1.580184	1.688041	1.758722	1.849532	1.980687
.75	1.517038	1.603277	1.717684	1.793062	1.890391	2.031940
.80	1.534687	1.625265	1.746075	1.826084	1.929885	2.081859
.85	1.551445	1.646231	1.773302	1.857873	1.968094	2.130512
.90	1.567384	1.666252	1.799443	1.888506	2.005091	2.177958
.95	1.582565	1.685394	1.824568	1.918051	2.040940	2.224255
1.00	1.597044	1.703719	1.848740	1.946572	2.075702	2.269452

5.4. Form III.

5.4.1 First Approximation.

As in previous two cases we use Form III [Burman (1989)] and equation (5.D) in equations (5.2.1), (5.2.2) and (5.2.3) to determine desired constants a,b,c. We denote

$$Q_1 = \omega_1 x_2 + \omega_2 x_4 + \omega_3 x_6 + y_2 log(1-k) \tag{5.4.1.1a}$$

$$Q_2 = \omega_1 x_3 + \omega_2 x_5 + \omega_3 x_7 + y_3 log(1-k) \tag{5.4.1.1b}$$

$$Q_3 = \omega_2 x_3 + \omega_3 x_5 + y_4 log(1-k) \tag{5.4.1.1c}$$

$$Q_4 = -\left[1 + \omega_2 x_3 + \omega_2 x_4 + \omega_3 x_6 + y_2 log(1-k)\right] \tag{5.4.1.1d}$$

$$L = 1 - V - P_1, \quad L_1 = \frac{L}{P_2}, \quad L_2 = \frac{P_3}{P_2}, \quad L_3 = \frac{P_4}{P_2} \tag{5.4.1.2a}$$

$$B_1 = \frac{Q_1 L_3 - Q_3}{Q_2 - Q_1 L_2}, \quad B_2 = \frac{Q_4 - Q_1 L_1}{Q_2 - Q_1 L_2} \tag{5.4.1.2b}$$

$$V_1 = L_2 B_1 + L_3, \quad V_2 = L_1 - L_2 B_2 \tag{5.4.1.2c}$$

$$b = B_1 c + B_2, \tag{5.4.1.3}$$

$$a = -A_1 c + A_2 , \tag{5.4.1.4}$$

where c is obtained from the following quadratic equation

$$\zeta_1 c^2 + \zeta_2 c + \zeta_3 = 0 \tag{5.4.1.5}$$

with

$$\zeta_1 = \frac{S_1^2}{2} \tag{5.4.1.6a}$$

$$\zeta_2 = V\left(-V_1 P_4 + B_1 P_5 + P_6\right) + S_1 S_2 \tag{5.4.1.6b}$$

$$\zeta_3 = \frac{S_2^2}{2} + V\left(P_3 + V_2 P_4 + B_2 P_5\right) - P_3 \tag{5.4.1.6c}$$

$$S_1 = -V_1 P_3 + B_1 P_4 + P_5 \tag{5.4.1.7a}$$

$$S_2 = P_2 + V_2 P_3 + B_2 P_4 \tag{5.4.1.7b}$$

5.4.2. Second approximation.

We use the known values of a,b,c in equation (5.D) to determine the second approximate form of H - function. The process is exactly similar to those in previous two forms and we have from Burman (1989) the following notations

$$\frac{1}{H(\mu,\omega)} = 1 - \mu\left[\sum_{i=1}^{7}\frac{y_i}{7-i+1} + y_8 log\left(\frac{1+\mu}{\mu}\right)\right] \tag{5.4.2.1}$$

where

$$y_1 = c\omega_3,\ y_2 = b\omega_3 - \mu y_1 \tag{5.4.2.2}$$

$$y_3 = c\omega_2 + a\omega_3 - \mu y_2,\ y_4 = b\omega_2 + \omega_3 - \mu y_3 \tag{5.4.2.3}$$

$$y_5 = c\omega_1 + a\omega_2 - \mu y_4,\ y_6 = b\omega_1 + \omega_2 - \mu y_5 \tag{5.4.2.4}$$

$$y_7 = a\omega_1 - \mu y_6,\ y_8 = \omega_1 - \mu y_7 \tag{5.4.2.5}$$

The results of two cases are given Table 5.3.I and Table 5.3.II respectively

Table 5.3.I

μ	ω = .1	ω = .2	ω = .3	ω = .4	ω = .5	ω = .6
.05	1.008923	1.018212	1.027925	1.038136	1.048952	1.060530
.10	1.014464	1.029706	1.045852	1.063067	1.081585	1.101753
.15	1.018786	1.038761	1.060127	1.083151	1.108209	1.135864
.20	1.022353	1.046296	1.072109	1.100166	1.130997	1.165396
.25	1.025389	1.052751	1.082446	1.114962	1.150987	1.191562
.30	1.028023	1.058384	1.091524	1.128045	1.168800	1.215083
.35	1.030343	1.063367	1.099599	1.139753	1.184850	1.236441
.40	1.032406	1.067822	1.106852	1.150326	1.199431	1.255985
.45	1.034259	1.071836	1.113416	1.159942	1.212769	1.273978
.50	1.035933	1.075478	1.119396	1.168741	1.225034	1.290624
.55	1.037457	1.078802	1.124874	1.176833	1.236367	1.306089
.60	1.038850	1.081851	1.129914	1.184307	1.246880	1.320508
.65	1.040130	1.084659	1.134570	1.191237	1.256667	1.333994
.70	1.041311	1.087256	1.138889	1.197684	1.265805	1.346644
.75	1.042404	1.089666	1.142907	1.203700	1.274361	1.358537
.80	1.043420	1.091909	1.146655	1.209328	1.282394	1.369746
.85	1.044365	1.094003	1.150163	1.214608	1.289951	1.380330
.90	1.045249	1.095962	1.153452	1.219571	1.297076	1.390344
.95	1.046076	1.097801	1.156544	1.224248	1.303807	1.399835
1.00	1.046853	1.099529	1.159456	1.228662	1.310176	1.408844

Table 5.3.I (Cont.)

μ	ω = .7	ω = .8	ω = .9	ω = .925	ω = .950	ω = .975
.05	1.073127	1.087233	1.104088	1.109138	1.114845	1.121856
.10	1.124143	1.149835	1.181559	1.191324	1.202539	1.216591
.15	1.167037	1.203486	1.249638	1.264137	1.280989	1.302419
.20	1.204672	1.251321	1.311653	1.330935	1.353571	1.382715
.25	1.238405	1.294807	1.369124	1.393236	1.421792	1.458966
.30	1.269041	1.334809	1.422928	1.451909	1.486504	1.531996
.35	1.297122	1.371904	1.473641	1.507519	1.548254	1.602323
.40	1.323036	1.406508	1.521672	1.560464	1.607425	1.670305
.45	1.347080	1.438935	1.567329	1.611043	1.664301	1.736201
.50	1.369487	1.469437	1.610857	1.659491	1.719100	1.800208
.55	1.390444	1.498214	1.652451	1.705997	1.771999	1.862481
.60	1.410106	1.525436	1.692276	1.750717	1.823144	1.923148
.65	1.428605	1.551245	1.730472	1.793786	1.872658	1.982314
.70	1.446052	1.575763	1.767158	1.835318	1.920647	2.040068
.75	1.462541	1.599097	1.802440	1.875414	1.967202	2.096488
.80	1.478157	1.621340	1.836411	1.914162	2.012406	2.151641
.85	1.492971	1.642573	1.869154	1.951643	2.056330	2.205587
.90	1.507049	1.662870	1.900743	1.987927	2.099042	2.258382
.95	1.520447	1.682296	1.931245	2.023080	2.140600	2.310072
1.00	1.533215	1.700909	1.960722	2.057160	2.181057	2.360704

Table 5.3.II

μ	ω = .2	ω = .3	ω = .4	ω = .5	ω = .6	ω = .7
.05	1.016032	1.024765	1.034092	1.044157	1.055168	1.067460
.10	1.025592	1.039826	1.055272	1.072236	1.091165	1.112790
.15	1.032916	1.051495	1.071888	1.094571	1.120248	1.150088
.20	1.038890	1.061096	1.085690	1.113322	1.144970	1.182258
.25	1.043928	1.069250	1.097504	1.129517	1.166543	1.210680
.30	1.048270	1.076318	1.107814	1.143758	1.185683	1.236171
.35	1.052070	1.082536	1.116935	1.156442	1.202865	1.259277
.40	1.055435	1.088067	1.125091	1.167850	1.218428	1.280387
.45	1.058444	1.093031	1.132444	1.178191	1.232624	1.299795
.50	1.061155	1.097520	1.139120	1.187624	1.245648	1.317729
.55	1.063614	1.101604	1.145216	1.196274	1.257656	1.334373
.60	1.065856	1.105339	1.150811	1.204245	1.268773	1.349878
.65	1.067911	1.108771	1.155968	1.211619	1.279104	1.364368
.70	1.069802	1.111938	1.160739	1.218465	1.288735	1.377948
.75	1.071550	1.114871	1.165170	1.224841	1.297740	1.390708
.80	1.073170	1.117596	1.169296	1.230797	1.306181	1.402726
.85	1.074677	1.120136	1.173150	1.236375	1.314113	1.414068
.90	1.076083	1.122508	1.176759	1.241612	1.321583	1.424792
.95	1.077398	1.124731	1.180146	1.246538	1.328632	1.434952
1.00	1.078631	1.126818	1.183332	1.251183	1.335296	1.444592

Table 5.3.II (Cont.)

μ	ω = .80	ω = .85	ω = .90	ω = .925	ω = .950	ω= .975
.05	1.081679	1.089916	1.099402	1.104877	1.111193	1.119130
.10	1.138514	1.153802	1.171796	1.182396	1.194829	1.210785
.15	1.186328	1.208279	1.234546	1.250256	1.268915	1.293242
.20	1.228323	1.256667	1.291054	1.311881	1.336882	1.369916
.25	1.266010	1.300520	1.342892	1.368840	1.400279	1.442320
.30	1.300282	1.340753	1.390977	1.422039	1.459994	1.511303
.35	1.331737	1.377974	1.435914	1.472073	1.516600	1.577406
.40	1.360801	1.412622	1.478139	1.519367	1.570503	1.641001
.45	1.387801	1.445029	1.717979	1.564240	1.622008	1.702364
.50	1.412992	1.475456	1.555694	1.606945	1.671351	1.761707
.55	1.436581	1.504118	1.591496	1.647686	1.718726	1.819197
.60	1.458737	1.531189	1.625559	1.686634	1.764290	1.874974
.65	1.479603	1.556819	1.658032	1.723932	1.808180	1.929154
.70	1.499303	1.581135	1.689043	1.759706	1.850512	1.981835
.75	1.517939	1.604247	1.718704	1.794064	1.891387	2.033105
.80	1.535605	1.626252	1.747113	1.827102	1.930896	2.083039
.85	1.552379	1.647235	1.774357	1.858906	1.969118	2.131706
.90	1.568333	1.667271	1.800513	1.889552	2.006127	2.179165
.95	1.583528	1.686429	1.825652	1.919111	2.041988	1.225474
1.00	1.598021	1.704767	1.849837	1.947644	2.076761	2.270682

Discussion of results:

We find that for the Form I,Case I the results are very good for values of ω from .1 to .4 where they are correct upto four decimal places for all values of μ. For values from $\omega = .5$ to $\omega = .8$ the results are correct upto three decimal places and from $\omega = .9$ to $\omega = .975$ the values of H - functions are correct upto two decimal places and in between they are correct upto three places but there are no definite patterns and toward the bottom of the table they slightly deviate away from Chandarsekhar's values. Case II gives a little different picture,for example $\omega = .2$ they computed values are correct upto 4 places and from $\omega = .3$ to $\omega =.85$ they are correct upto three places and from ω=.9 to ω=.975 they follow the same pattern as that of Case I.

In case of Form II, when applied to Case I we find that the computed values are correct upto 4 places for ω=.1 and ω=.2 and from ω=.3 to ω=.7 they are correct upto three places after that they almost follow the same pattern as in Case I,Form I but in this case the deviation is slightly more than that Case. Case II reveals that the computed values follow almost the same pattern as in Case I,Form I, for example from ω=.85 to ω=.975 some of the computed values correct upto three decimal places and some of them are correct upto two places but they don't follow any definite pattern.

Form III when applied to Case I the results obtained are found to be correct upto 4 places from ω=.1 to ω=.4. Again from ω=.5 to ω=.975 the computed values are correct upto three places but in this case the results are much more consistent than the two previous forms. For the Case II, the results are correct upto 4 places from ω=.2 to ω=.4 but from ω=.5 to ω=.975 most of the values are correct upto three places, some are correct upto 4 places and some are correct upto two places and yet again we do not find any definite pattern.

The forms I ,II and III thus give quite good results for practical purposes. They can be safely used in calculating intensities and residual intensities from solutions of transfer equation with various phase functions.

5.5. Calculation of Moments and Chandrasekhar constants q and c

The following Tables 5.5.I to 5.5.III give the moments α_0, α_1 and constants q and c [Chandrasekhar (1960)] for different values of ω. These are calculated by using three forms of H-functions , viz., form I,II,and III given sec 5.1.

Table 5.5.I (Form I)

ω	α_0	α_1	q	c
.100	1.032487	.519596	.948991	.024655
.200	1.068474	.541376	.895704	.048491
.300	1.108717	.565829	.839638	.071264
.400	1.154183	.593625	.780068	.092614
.500	1.206334	.625762	.715905	.111997
.600	1.267485	.663824	.645417	.128533
.700	1.341642	.710573	.565584	.140661
.800	1.436867	.771593	.470308	.145154
.900	1.574763	.861961	.343222	.133130
.925	1.623260	.894300	.300912	.124461
.950	1.683628	.934952	.249655	.110872
.975	1.767481	.992124	.180698	.087396

Table 5.5.II (Form II)

ω	α_0	α_1	q	c
.100	1.032161	.519568	.948975	.024653
.200	1.067866	.541348	.895643	.048485
.300	1.107885	.565838	.839513	.071254
.400	1.153186	.593707	.779866	.092602
.500	1.205236	.625957	.715624	.111988
.600	1.266366	.664174	.645068	.128531
.700	1.340605	.711106	.565197	.140670
.800	1.436041	.772309	.469943	.145177
.900	1.574298	.862760	.342975	.133157
.925	1.622908	.895070	.300715	.124487
.950	1.683393	.935647	.249516	.110893
.975	1.767367	.992662	.180625	.087409

Table 5.5.III (Form III)

ω	α_0	α_1	q	c
.100	1.032717	.519678	.949003	.024659
.200	1.068816	.541506	.895738	.048505
.300	1.109032	.565976	.839686	.071286
.400	1.154389	.593778	.780110	.092642
.500	1.206390	.625930	.715919	.112029
.600	1.267384	.664031	.645385	.128567
.700	1.341419	.710848	.565501	.140695
.800	1.436598	.771961	.470189	.145187
.900	1.574568	.862390	.343118	.133156
.925	1.623105	.894714	.300826	.124483
.950	1.683523	.935332	.249593	.110890
.975	1.767428	.992389	.180664	.087403

Now for comparisons sake we give the following table of Chandrasekhar (1960) which gives different moments and the constants.

ω	α_0	α_1	q	c
.100	1.032729	0.519588	.949003	0.024654
.200	1.068832	0.541348	0.89574	0.048491
.300	1.109034	0.565767	0.839686	0.07126
.400	1.154378	0.593541	0.780108	0.092605
.500	1.206366	0.625686	0.715914	0.111984
.600	1.267352	0.663798	0.645375	0.12852
.700	1.341368	0.710639	0.565482	0.140649
.800	1.436535	0.771792	0.470161	0.145147
.900	1.574492	0.862276	0.343079	0.133123
.925	1.623024	0.89462	0.30078	0.124451
.950	1.683484	0.935277	0.249569	0.110873
.975	1.767379	0.99238	0.180632	0.087387