

Chapter 7

Scaling Property in Cold Compact Stars

7.1 Introduction

In the MIT bag model, scaling property with respect to the bag constant B in the EOS for strange matter has been reported earlier by Witten [25]. Following Witten's [25] work, Benvenuto and Horvath [159] and Zdunik [75] also reported some scaling laws in the EOS considered by them for strange stars. The strange matter EOS formulated by Dey *et al* [42] can be approximated to a linear form as shown by Gondek-Rosińska *et al* [74] which is very similar to the EOS obtained by Zdunik [75] for strange matter in the MIT bag model. The linear nature of these equations of state also provides some scaling laws; analogous to the scaling law with respect to the bag constant B . Thus, scaling property in compact stars like strange stars may be thought, at first sight, as due to the equation of state being a linear one. But, in this chapter, we shall show that there is a type of scaling property in compact stars, which is, in fact, a general feature of a spherical distribution in hydrostatic equilibrium. We demonstrate this by analysing the Tolman-Oppenheimer-Volkoff (TOV) equations and the boundary conditions that

are used while solving these equations. A special class of these scaling property is explicitly exhibited by the solution considered by Mukherjee *et al* [67]. In chapters 4 & 5, we have shown that this solution can be applied to describe strange stars as well as stars whose interior might have other exotic components. Moreover, although for a large value of the parameter λ the EOS becomes almost linear in this model, the same is not true for a smaller value of λ . Scaling property, nevertheless, applies in all the cases. Physical implications of this scaling property in the Vaidya-Tikekar model [63] will be discussed in this chapter.

7.2 Scaling law in strange star EOS

While studying the expected features of compact stars made of u , d and s quarks (strange stars), Witten [25] first pointed out a scaling property in the mass-radius ($M - b$) relation for such stars. The equation of state for these stars, in the MIT Bag model with massless quarks, can be written as

$$p = \frac{1}{3}(\rho - 4B) \quad (7.1)$$

$$\rho = \frac{9}{4}\pi^{2/3}n^{4/3} + B \quad (7.2)$$

where ρ is the energy-density, p the pressure, B the Bag constant (vacuum energy), and $n_B = \frac{1}{3}(n_u + n_d + n_s)$, the baryon number density. The stars exhibit scaling with respect to the Bag constant

$$M(B) = \sqrt{\frac{B'}{B}}M(B') \quad (7.3)$$

$$b(B) = \sqrt{\frac{B'}{B}}b(B') \quad (7.4)$$

where B and B' are two values of the Bag Constant.

Recently, Gondek-Rosińska *et al* [74] presented a linear EOS for strange matter given by,

$$p = a(\rho - \rho_0),$$

which was obtained by linearizing the EOS for strange matter formulated by Dey *et al* [42]. Similar EOS was obtained by Zdunik [75], who considered MIT bag model to formulate strange matter EOS. The linear nature of the EOS also allows to scale all the physical parameters of a star with some powers of ρ_0 for a fixed value of a , analogous to the scaling law with the bag constant B . This has been discussed by Zdunik [75].

Thus in case of strange stars having a linear EOS, one can always identify a scaling law. We find that this scaling behaviour is not restricted to strange stars alone and it may occur in more general cases. It is not even necessary to consider a linear equation of state.

7.3 Standard approach

To demonstrate the scaling law in a more general way, let us consider a spherically symmetric, static star having an interior metric

$$ds^2 = -e^{2\gamma(r)} dt^2 + e^{2\mu(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (7.5)$$

where r is the radial co-ordinate. We have used the conventional units ($c = 8\pi G = 1$).

The standard procedure for determining the $(M - b)$ relation for a star is to make use of the TOV equations given by

$$\frac{dp}{dr} = -(\rho + p) \frac{2M(r) + pr^3}{r^2(1 - \frac{2M(r)}{r})} \quad (7.6)$$

$$\frac{dM}{dr} = \frac{1}{2}\rho r^2 \quad (7.7)$$

For a given EOS ($p = p(\rho)$), macroscopic properties of the star such as mass and radius are determined by integrating the TOV equations numerically, using the boundary conditions (i) $M(0) = 0$, (ii) $p(0) = p(\rho_c)$, (iii) $p(b) = 0$ and (iv) $e^{2\gamma(b)} = 1 - \frac{2M(b)}{b}$, where b is the radius of the star.

We assume that the energy density of the stellar matter depends on a parameter R , which has the dimension of a length. The energy-density and pressure, expressed in geometrized units of length $^{-2}$, assumed finite inside the star, can be written, without loss of generality, as

$$\rho(r) = \frac{1}{R^2} f(r/R), \quad p(r) = \frac{1}{R^2} g(r/R). \quad (7.8)$$

If we now consider a transformation

$$\tilde{r} = r/R, \quad \tilde{M} = M/R, \quad \tilde{\rho} = \rho R^2, \quad \tilde{p} = p R^2$$

the TOV equations and all the boundary conditions remain invariant. The boundary condition $p(b) = 0$, in particular, becomes $\tilde{p}(\tilde{b}) = 0$, and the first zero of $g(x)$ determines \tilde{b} and hence the radius, $b = \tilde{b}R$. The scaling property thus appears once we can identify or construct a parameter R with the dimension of a length in the EOS. As an example, we consider the case of a polytropic equation of state

$$p(\rho) = K \rho^{1+1/n} = K \rho^\Gamma, \quad (7.9)$$

where n is the polytropic index and Γ is the adiabatic index. Since K is a dimensional quantity, one can construct the parameter $R_0 = K^{\frac{1}{(2\Gamma-2)}}$, with the dimension of a length and we find a scaling behaviour,

$$M(K) = \left(\frac{K'}{K}\right)^{\frac{1}{2\Gamma-2}} M(K')$$

$$b(K) = \left(\frac{K'}{K}\right)^{\frac{1}{2\Gamma-2}} b(K')$$

where K and K' are two values of the parameter K .

7.4 Scaling law in Vaidya-Tikekar [63] model

The scaling property mentioned above is explicitly built in Vaidya and Tikekar [63] model. In particular, we consider the solutions given by Mukherjee *et al* [67] to prove

Distance	$\frac{dp}{d\rho}$	
r (km)	$\lambda = 2$	$\lambda = 100$
0	.3418	.2268
2	.3437	.2296
4	.3484	.2363
6	.3525	.2423
7.7	.3518	.2424

Table 7.1: Variation of $\frac{dp}{d\rho}$ along the radius of a star of $M = 0.88 M_{\odot}$, $b = 7.7 \text{ km}$, for $\lambda = 2$ ($R = 20.224 \text{ km}$) and $\lambda = 100$ ($R = 108.779 \text{ km}$).

our point. A simple way to study the relevant scaling property here is to note that in the model discussed in chapter 2, R has the dimension of a length and to define

$$\begin{aligned} \tilde{\rho} &= \rho R^2, & \tilde{p} &= p R^2, \\ \tilde{M} &= \frac{M}{R}, & \tilde{r} &= \frac{r}{R}, \quad \& \quad \tilde{b} = \frac{b}{R}. \end{aligned}$$

The model can now be written in terms of $\tilde{\rho}$, \tilde{p} , \tilde{M} , \tilde{r} and \tilde{b} , and it becomes independent of R . The elimination of the parameter R from the EOS implies the existence of a homologous family of stars, (for $R > b > 2M$), unless restricted by other physical constraints. The family of stars are represented by \tilde{M} , \tilde{b} , $\tilde{\rho}$ and \tilde{p} . λ here helps to parametrize the equation of state, given implicitly by the equations (2.29) and (2.30) in chapter 2. To see this in a simple way, we can write,

$$\frac{dp}{d\rho} = \frac{d\tilde{p}}{d\tilde{\rho}} = \frac{z(1-z^2)^2(\psi_z/\psi)^2 - (1-z^2)(\psi_z/\psi)}{z(1-z^2)(\lambda+1) + 4z}. \quad (7.10)$$

We have shown in Table 7.1, values of $\frac{dp}{d\rho}$ for different λ . It is seen that at a small λ , the relevant stellar matter has almost a radiation like behaviour which becomes less

stiff as λ increases. Note that the EOS in our model is not exactly linear, at least for a small λ , but the scaling property nevertheless works.

The scaling law discussed so far can also be expressed in terms of the central density of the star. In section 2.2.4, we obtained central density as

$$\rho_c = \frac{3(\lambda + 1)}{R^2}. \quad (7.11)$$

Inserting the expression for R in the above mentioned scaling formulae, we can rewrite the scaling formulae, as

$$\tilde{\rho} = \rho \left(\frac{\rho_c}{h} \right), \quad \tilde{p} = p \left(\frac{\rho_c}{h} \right), \quad \tilde{M} = M \left(\frac{\rho_c}{h} \right)^{1/2}, \quad \tilde{b} = \left(\frac{\rho_c}{h} \right)^{1/2},$$

where, $h = 3(\lambda + 1)$.

Thus if mass \tilde{M}_1 and radius \tilde{b}_1 of a star are known for a central density ρ_{c1} , then for a central density ρ_{c2} the rescaled mass \tilde{M}_2 and radius \tilde{b}_2 can be written as

$$\tilde{M}_1 = \left(\frac{\rho_{c1}}{\rho_{c2}} \right)^{1/2} \tilde{M}_2$$

$$\tilde{b}_1 = \left(\frac{\rho_{c1}}{\rho_{c2}} \right)^{1/2} \tilde{b}_2.$$

Thus the star \tilde{M}_2, \tilde{b}_2 will also be described by this model with the same λ .

The scaling property can be generalized easily to the case of a charged sphere too, as mentioned in chapter 3. This happens because all field equations and boundary conditions in this model can be given in terms of z , and the reduced quantities $\tilde{\rho} = \rho R^2$, $\tilde{p} = p R^2$, $\tilde{\sigma} = \sigma R^2$, $\tilde{M} = \frac{M}{R}$, $\tilde{b} = \frac{b}{R}$. In other words, the model is fully described in terms of $\tilde{\rho}$, \tilde{p} , $\tilde{\sigma}$, \tilde{M} , and \tilde{b} and the parameter R does not appear anywhere. It is obvious that the expressions (3.18)-(3.26) in chapter 3 and boundary conditions have the required scaling behaviour. Thus we observe that the class of charged stars described in this model have a similar scaling behaviour. Note that the electric field also needs a scaling here.

The expressions for ρ and p in chapter 2, may be obtained in many ways. In Bag model, one has to include the contribution of vacuum energy density in the expressions for ρ and p and for a strange star the equation of state is a linearized relation between ρ and p [74]. In chapter 4, we have shown that our model can be used to reconstruct strange matter EOS. We have also shown in chapter 5 that a quark-diquark EOS can also be constructed by this model [162]. It appears that scaling behaviour has a physical relevance and one may expect to find a large class of cold stars more compact than ordinary neutron stars.

7.5 Discussions

We have noted that a cold compact star should have a scaling property, whenever one can identify a parameter with the dimension of a length. The result does not depend on the specific form of the equation of state. It is rather a general feature of a spherical distribution that is in hydrostatic equilibrium. We have also shown that our model exhibits this scaling property explicitly. For example, if we have a star with mass M , radius b , and charge q , there may also be a star with mass, radius and charge given by ξM , ξb and ξq , where ξ is a constant. Making a contact with the physical value of any one of these reduced quantities, one determines the appropriate value of ξ and hence the physical values of other quantities. Thus the scaling follows once we can identify a suitable dimensional parameter in the equation of state. Different parameters may be available in different cases but the results may be trivial in some cases. Also, the scaling behaviour will be restricted within a range determined by physical constraints (e.g. $R > b > 2M$ as well as $\frac{dp}{d\rho} < 1$). The nature of interaction among the constituents of the stellar matter may impose further constraints. Thus the scaling behaviour will perhaps be realistic only for scaling parameters ($\sim \frac{b}{\bar{b}}$) whose values are close to 1, so that the underlying physical processes and interactions do not become different from those in the unscaled case.