Chapter 1

General Introduction

Scattering is said to be ***coherent*** when radiation is emitted in the same frequency in which it was absorbed. On the other hand, when frequency of the emitted radiation differs from that of the absorbed radiation, it is said to be ***non-coherent*** scattering. The term ***'completely non-coherent scattering'*** is sometimes used to mean that the scattering involves not only a change in the frequency but also a complete redistribution in frequency i.e. scattering in which the frequency of re-emission is uncorrelated with the frequency absorbed. It was pointed out by Thomas,[202] and earlier by Henyey,[84] Unno[207,208] and Edmonds[70] that from practical point of view, purely coherent scattering (in frequency) is not possible in strict sense in stellar atmospheres. We designate the scattering as coherent or non-coherent according to our theoretical consideration of the problem. In the word of Jefferies,[98] "The generalization from coherent to non-coherent scattering made necessary the consideration of the simultaneous transfer of photons at all frequencies in the line along with the parallel recognition that the photons belong to the whole line and no longer to a particular frequency. ...."(vide Athay[13])

When an atom absorbs energy of certain frequency, ν, the probability that the energy will be re-emitted in the same frequency will be maximum if

(i). the atom is at rest

(ii). the atom is in the lowest quantum state

(iii). the atom is in a weak radiation field.

Departure from any of the above three conditions will cause non coherent scattering.

In one of his classical papers entitled "The Formation of Absorption Lines", Eddington[68] quoted," The crucial question is whether light absorbed in one part of a line is re-emitted in precisely the same part of the line. If so, the blackening in this frequency is independent of what is happening in neighbouring frequencies. The alternative is that the re-emission has a probability distribution, and is correlated to, but not determined by, the absorbed frequency. For example, if the process is regarded as one of the transition between two energy levels, which are not sharp but are composed of narrow bands of energy , the atom is not likely to return to the precise spot in the lower level from which it started, and the re-emission will not be the exact reverse of the absorption . In that case the line can only be studied as a whole. Modern attempts to interpret the contours of the absorption lines assume (rightly or wrongly) that there is no such redistribution of frequencies." He further put the remark in this regard as footnote in the same paper,"If the above assumption is untrue, the usual treatment of the line contours is entirely unsound".

Besides the redistribution of the kind mentioned in his quotation, he noticed another departure from the simple case, known as interlocking of lines.

When two or more lower (sub states or) lines in a spectrum posses a common upper state, the atom can be excited to that state by absorption in either lines (i.e. any of the lower sub states); but the re-emission will take place according to the transition probability, regardless of the path by which the excitation was made. Thus the absorption from a certain sub state of the lower state in a certain frequency ν has a non-zero probability of the returning to another lower sub state emitting in frequency different from ν giving rise to non-coherent scattering. Similar situation will arise when the numbers of upper sub states will posses a common lower state. This phenomenon is called interlocking of lines without redistribution. The lines are said to form doublet, triplet, quartet or multiplet according to the number of such interlocked lines viz. two, three, four or many. As for examples, some of the interlocked multiplets are $n^3S - 2^3P$, the triplets of the alkali earths, and $2^2P - 2^2S$, the doublets of the alkali metals, which have the same upper states, and lower states which are not themselves the upper states of strong lines (vide Woolley[227]).

In the Mg triplet, at $5167A$, $5173A$, and $5184A$, taken from Grotrain[80](vide Woolley[227]), there are no other transitions connected with the 2^3S. The 2^3P_1 and 2^3P_3 state are not the upper state of any transitions, but 2^3P_2 is the upper state of a fairly weak line $1^1S - 2^3P_2$ at $4571A$.

The interlocking of one member of a multiplet with the other members may be regarded in the following way. The formation of any one line at any point in the atmosphere is governed by the number of atoms per c.c. of the atmosphere at that point in the two atomic states connected with the line. The atmosphere is not exactly in the thermodynamical equilibrium with radiation, and the number of atoms in any state will depend on the intensities of those lines involving transitions in which it is the upper state, the equilibrium condition being that the total number of transitions, per c.c. per sec., into the state must equal the number of transitions away from it, since there can be no secular increase in the number of atoms in any particular state. There is not, however, detailed balancing in each of the separate transitions (vide Woolley[227]).

The equations of formation of the lines are not independent but contain cross-terms.

The equation for the intensity in a particular frequency of a spectral line might then, in general, contain an infinite set of terms involving the intensities of other frequencies in the same line as well as terms involving the intensities in a finite number of other lines in the same spectrum. Fortunately, these difficulties do not arise in some important cases, namely principal lines in spectra, in which the ground state (or metastable state) is sharp. The reason for this is that the distribution of energy levels within a state depends on the life of the state. The spread of energy in the ground (or metastable) state can be ignored.

After coherent scattering, the next simplest case is that of interlocking of the principle lines, for $p\,(\nu, \nu')$ takes a small number only of the non-zero values. Examples of this are the principal lines of Al, ${}^2S_{\frac{1}{2}} - {}^2P_{\frac{3}{2}}$, at $\lambda 3,962A$ and ${}^2S_{\frac{1}{2}} - {}^2P_{\frac{1}{2}}$, at$\lambda 3,944A$, in which ${}^2P_{\frac{1}{2}}$ is the ground state and ${}^2S_{\frac{3}{2}}$ metastable; and the principal triplet of Mg, ${}^3S_1 - {}^3P_2$ at $\lambda 5,184A$, ${}^3S_1 - {}^3P_1$ at $\lambda 5,173A$ and ${}^3S_1 - {}^3P_0$ at $\lambda 5,167A$.

In this case $^3P_2$ and $^3P_0$ are metastable and $^3P_1$ is linked by an inter combination line to the ground state $^1S_0$.

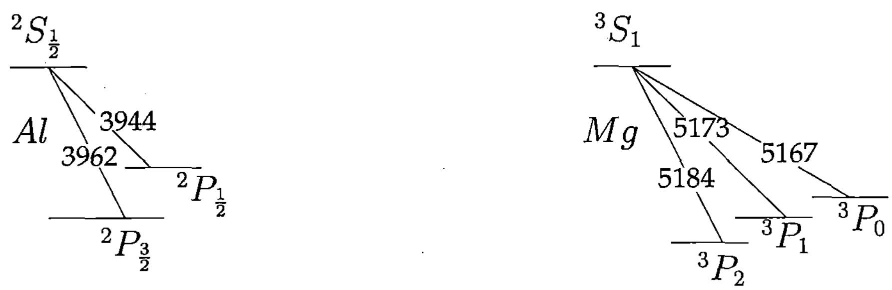

Figure 1.1: Interlocked principal lines of Al and Mg

Interlocking which is the core word of this thesis is associated with **"Transport Theory"** of Astrophysics. Transport theory (Neutron Transport or Radiative Transfer) is such a subject whose study is a must for studying the physics of the distant celestial bodies. Mathematically, it is the underlying physical phenomenon in many astrophysical problems and its study has a great importance as radiation field which is not only the root cause of the change of the structure and dynamics of the medium it propagates through, but also, is practically the only source of information about distant celestial objects serves as an important diagnostic tool in establishing their properties.

A major area of the study of the subject Radiative transfer concerned with the derivation of the distribution function (or the specific Intensity) for a given scattering function from an equation of transfer (usually an integro-differential equation) which is constructed by assuming the physics of the source wherefrom the photons (in case of radiative transfer) or neutrons (in case of neutron transfer) emerge out is known and the scattering laws of the medium through which the photons (or neutrons) proceed are also known, subject to two-point boundary conditions which depend on the nature of the source and the medium.

No one will disagree to admit that an integro-differential equation of transfer with two-point boundary conditions is a very difficult mathematical problem and in many cases it becomes a challenging one. An integro-differential equation of transfer related with interlocking problem is much more difficult. Only a few problems of interlocking have been solved till now. So far interlocking problem has been solved in isotropic medium only. This thesis is the first step in anisotropically scattering media for the case of interlocking problems.

As the notion of plane parallelism is so common to so many stars and other physical situations (vide Collons II[53]), we shall confine our study in determining the solution of the radiative transfer equation for plane parallel atmosphere only.

Before developing the general equation of radiative transfer equation for interlocked multiplet lines in anisotropically scattering medium, we shall write a few words about the phase function which makes a distinction between an equation in isotropic scattering atmosphere and one in anisotropic scattering atmosphere .

Phase Functions :

A phase function $p(\mu, \mu')$ expresses the ratio of energy propagated in direction μ compared to the energy coming from direction μ'. It satisfies two important properties that result directly from physics of light. First, due to the Helmholtz Reciprocity Rule, $p(\mu, \mu')$ is symmetric relative to μ and μ' :

$$\forall\, \mu \in \nu \; and \; \forall\, \mu' \in \nu, \;\; p(\mu, \mu') = p(\mu', \mu) \tag{1.1}$$

Second, due to the Energy Conservation Law, $p(\mu, \mu')$ has to fulfill the normalization condition :

$$\forall\, \mu \in \nu, \;\; \frac{1}{4\pi}\int_{\mu' \in \nu} p(\mu, \mu')\, d\mu' = 1 \tag{1.2}$$

Moreover $p(\mu, \mu')$ is usually symmetric around the incident direction of light and so $p(\mu, \mu')$ depends only on the angle Θ between μ and μ'. Therefore Equation (1.2) can be written :

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} p(\Theta, \gamma)\, sin\Theta d\Theta d\gamma = \frac{1}{2}\int_0^{\pi} p(\Theta)\, sin\Theta d\Theta = 1 \tag{1.3}$$

Finally $t = cos\Theta$ gives the following normalization condition:

$$\int_{-1}^{+1} p(t)\, dt = 2 \tag{1.4}$$

The following are some of the phase functions which we have used in our work:

1. Planetary phase function:

$$p(\cos\Theta) = \varpi_0 (1 + \varpi \cos\Theta) \;\; (-1 \leq \varpi \leq 1)$$

2. Rayleigh phase function :

$$p(\cos\Theta) = \frac{3}{4}\left(1 + \cos^2\Theta\right)$$

3. Pomraning phase function :

$$p\left(\cos\Theta\right) = \frac{3}{4}\left(1 + \lambda\cos^2\Theta\right) \ ; \ \lambda = \frac{5\varpi_0}{5 - 3\varpi_0}$$

4. Three term scattering indicatix:

$$p\left(\cos\Theta\right) = 1 + \varpi_1 p_1\left(\cos\Theta\right) + \varpi_2 p_2\left(\cos\Theta\right)$$

where ϖ_1 and ϖ_2 are constants.

Planck Function:

The functional form of the Planck-function $B_\nu\left(T\right)$ follows immediately from Bose-Einstein quantum statistics which is given by

$$B_\nu\left(T\right) = \left(2h\nu^2/c^2\right)\left[e^{h\nu/kT} - 1\right]^{-1}$$

where $h =$ Planck constant, $k =$ Boltzmamn constant, $\nu =$ frequency, $c =$ speed of light and T is the temperature which, in a model stellar atmosphere in radiative equilibrium, is determined as a function of height τ, in the atmosphere.

Different authors used the different approximate forms of the Planck function $B_\nu\left(T\right)$. Some of them are

(i). **Linear form:**
$B_\nu\left(T\right) = b_0 + b_1\tau$, where b_0 and b_1 are constants.

(ii). **Non-linear forms:**

(a) **Exponential form:**
$B_\nu\left(T\right) = b_0 + b_1 e^{-\beta\tau}$, where β, b_0 and b_1 are constants.

(b) $B_\nu(T) = b_0 + b_1\tau + b_2E_2(\tau)$ where b_0, $b_1$and b_2 are constants and $E_2(\tau)$ is the function

$$E_2(\tau) = \int_1^\infty \frac{e^{-\tau x}}{x^2} dx$$

1.1 Development of the equation:-

Woolley and Stibbs[229] applied the theory of absorption lines by coherent scattering to the case of interlocking without redistribution to deduce the equation of transfer for interlocked triplets in the Milne-Eddigton model and they solved the problem by Eddington approximation method by making some assumptions which are stated below:

(I). No distribution in frequency takes place other than due to interlocking;

(II). The lines are so closed together that variations of the continuous absorptions coefficient and of the Planck-function with wave-length may be neglected. This also means that the lower states are nearly equal in excitation potential and they have the same classical damping constant. Then the ratios of the line absorption co-efficients to the continuous absorption co-efficients are proportional to the transition probabilities for spontaneous emission from the upper states to the respective lower states;

(III). The ratio of the line absorption co-efficients to the continuous absorption co-efficients are independent of the depth.;

(IV). The Planck-function and the co-efficient, which is introduced to allow for the thermal emission associated with the absorption, are independent of the both frequency and depth.

With same logic and assumption, we attempt here to give the derivation of the general form of the equations of radiative transfer for the case of interlocking of multiplets of order m.

Let ν_i, $(i = 1, 2, \cdots \cdots, m)$ be the central frequencies of the m number of multiplet lines and when a quantum of frequencies $\nu_i + \Delta\nu_i$, $(i = 1, 2, \cdots \cdots, m)$ is absorbed the energy of the upper state will be $E_0 + h\Delta\nu$ and the subsequent re-emission is any of $\nu_i + \Delta\nu_i$, $(i = 1, 2, \cdots \cdots, m)$ These 'm' frequencies are interlocked with each other but with no other frequencies. For each value of $\Delta\nu$ there are 'm' simultaneous equations of which the r^{th} interlocked line is:

$$\cos\vartheta \frac{dI_r(z,\vartheta,\phi)}{\rho dz} = (k_r + l_r) I_r(z,\vartheta,\phi) + (k_r + \varepsilon_r l_r) B(\nu_r, T) + (1-\varepsilon_r) \sum_{s=1}^{m} p(r,s) l_r J_r \tag{1.5}$$

where ϑ denotes the polar angle which the direction considered makes with the outward normal to an element of area $d\sigma$ (across which the dE_ν amount of radiant energy in the frequency interval $(\nu, \nu + \Delta\nu)$ is transported), ϕ the azimuthal angle referred to a suitably chosen $x-$axis and ρ is the density of the material through which a pencil of radiation is propagated.

Introducing the normal optical thickness

$$\tau = \int_z^\infty k\rho dz$$

measured , in terms of the scattering co-efficients k, from the boundary inward

and

$$\mu = \cos\vartheta$$

we have

$$\mu \frac{dI_r(\tau,\mu,\phi)}{d\tau} = (k_r + l_r) I_r(\tau,\mu,\phi) + (k_r + \varepsilon_r l_r) B(\nu_r, T) + (1 - \varepsilon_r) \sum_{s=1}^{m} p(r,s) l_r J_r \tag{1.6}$$

We now have to evaluate the quantities $p(r,s)$. To do this we note that the number of transitions per c.c.. from the m number of lower states to a band of the upper states lying within $E_0 + h\Delta\nu$ to $E_0 + h(\Delta\nu + \delta\nu)$ is

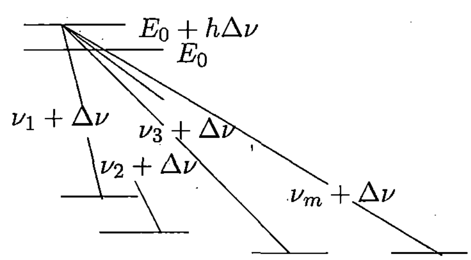

Figure 1.2: The Formation of m number of Interlocked lines with a common upper state

$$\rho dx \delta\nu \left\{ \sum_{s=1}^{m} \frac{l(\nu_s + \Delta\nu) J(\nu_s + \Delta\nu)}{h(\nu_s + \Delta\nu)} \right\}$$

This must be equal to the number of transitions leaving the upper sub-state into the m-lines per c.c. per sec.

Let the population of the upper state with energies between $E_0 + h\Delta\nu$ and $E_0 + h(\Delta\nu + \delta\nu)$ be $N_u(\Delta\nu)\delta\nu$; then the number of transition is

$$N_u(\Delta\nu)\delta\nu \{A_{u1} + A_{u2} + \cdots + A_{um}\}$$

The secular equilibrium of the sub-state gives

$$N_u(\Delta\nu)\,\delta\nu\,\{A_{u1}+A_{u2}+\cdots +A_{um}\} = \rho dx\left\{\sum_{s=1}^{m}\frac{l(\nu_s+\Delta\nu)\,J(\nu_s+\Delta\nu)}{h(\nu_s+\Delta\nu)}\right\} \tag{1.7}$$

The energy emitted in the first line is $N_uA_{u1}h(\nu_1+\Delta\nu)$, and similarly for the other lines. Accordingly

$$p(r,s) = \frac{\nu_r+\Delta\nu}{\nu_s+\Delta\nu}\cdot\frac{A_{ur}}{A_{u1}+A_{u2}+\cdots A_{um}} \tag{1.8}$$

The equations can be written more simply if we suppose that the 'm' number of lines are so close together that we may ignore differences in the frequencies,
and

$$k_1 = k_2 = \cdots = k_m = k$$

and we take

$$B(\nu_1,T) = B(\nu_2,T) = \cdots = B(\nu_m,T) = B(\nu,T) = B_\nu(T)$$

So, the expression for $p(r,s)$ may be written as

$$p(r,s) = \frac{A_{ur}}{\sum\limits_{s=1}^{m} A_{us}} \tag{1.9}$$

Since p does not involve $'s'$ we set $p(r,s) = \alpha_r$,

we notice that

$$\sum_{r=1}^{m}\alpha_r = 1$$

Then with

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_m = \varepsilon,$$

the equations can be written as

$$\mu \frac{dI_r(\tau,\mu,\phi)}{d\tau} = (k + l_r)\, I_r(\tau,\mu,\phi)$$
$$+ (k + \varepsilon l_r)\, B_\nu(T) + (1-\varepsilon)\sum_{s=1}^{m} p(r,s)\, l_r J_r \qquad (1.10)$$

In these equations references to a particular $\Delta\nu$ have been omitted for the sake of clarity.

From Boltmann's equation we have

$$\frac{N_1}{q_1} = \frac{N_2}{q_2} = \cdots = \frac{N_m}{q_m} \qquad (1.11)$$

where the q_i's are the statistical weights.

Now, we have

$$\eta_\nu = \frac{l_\nu}{k}$$

where l_ν= line absorption coefficient and k = the coefficient of continuous absorption.

But, the line absorption coefficient l_ν is related with the concentration N of the atoms forming line absorption is as follows:

$$l_\nu = \frac{N\alpha(\nu)_D}{\rho}$$

where $\alpha(\nu)_D$ is the atomic line absorption co-efficient modified by Doppler effect due to thermal motion of the atom and ρ is the density of the atmosphere.

So, we can write from above

$$\eta_\nu = \frac{N\alpha(\nu)_D}{k\rho} \qquad (1.12)$$

From the equations (1.11) and (1.12), we obtain $\eta_n = const. \times q_n f$, the oscillator strength f being related to the downward transition probability, namely

$$f = \frac{1}{3} \cdot \frac{q_u}{q_n} \cdot \frac{A_{un}}{\gamma_n},$$

where γ_n is the classical damping constant $8\pi^2 e^2 \nu_0^2 / 3mc^3$, where m is the mass of photon. Since γ is the same for m numbers of lines,

$$\eta_1 / A_{u1} = \eta_2 / A_{u2} = \cdots = \eta_m / A_{um}$$

for all $\Delta\nu$, and from the equation (1.9), which defines α_n,we obtain

$$\frac{\eta_1}{\alpha_1} = \frac{\eta_2}{\alpha_2} = \cdots = \frac{\eta_m}{\alpha_m} \tag{1.13}$$

Hence

$$\alpha_r \left(\sum_{s=1}^{m} \eta_s \right) = \left(\sum_{s=1}^{m} \alpha_s \right) \eta_r = \eta_r \tag{1.14}$$

which gives

$$\alpha_r = \eta_r / \sum_{s=1}^{m} \eta_s \tag{1.15}$$

and

$$\sum_{r=1}^{m} \alpha_r = 1 \,, \qquad r = 1, 2, \cdots, m \tag{1.16}$$

and the equation (1.6)becomes

$$\mu \frac{dI_r(\tau, \mu, \phi)}{d\tau} = (1 + \eta_r) I_r(\tau, \mu, \phi) - (1 + \varepsilon \eta_r) B_\nu(T) + (1 - \varepsilon) \alpha_r \sum_{s=1}^{m} \eta_s J_s \tag{1.17}$$

But the source function $J_s(\vartheta, \phi)$ or equivalently $J_s(\mu, \phi)$ is given by Chandrasekhar[45]

$$J_s = \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} p(\mu, \phi; \mu', \phi') I_s(\tau, \mu, \phi)\, d\mu' d\phi', \quad \mu = \cos\vartheta \tag{1.18}$$

where integration is taken over all directions (ϑ', ϕ')

So, the equation of transfer becomes

$$\begin{aligned}\mu\frac{dI_r(\tau, \mu, \phi)}{d\tau} &= (1+\eta_r) I_r(\tau, \mu, \phi) - (1+\varepsilon\eta_r) B_\nu(T) \\ &+ (1-\varepsilon)\alpha_r \left\{\sum_{s=1}^{m}\eta_s\frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} p(\mu, \phi; \mu', \phi') \times \right. \\ &\left. \times I_s(\tau, \mu, \phi)\, d\mu' d\phi'\right\}\end{aligned} \tag{1.19}$$

It is evident that for the type of the problem we have formulated, solutions of the equation of transfer must be sought which exhibits axial symmetry about z-axis. The intensity and the source function must therefore be azimuth independent, and the equation of transfer (1.19) becomes

$$\begin{aligned}\mu\frac{dI_r(\tau, \mu)}{d\tau} &= (1+\eta_r) I_r(\tau, \mu) - (1+\varepsilon\eta_r) B_\nu(T) \\ &-\frac{1}{2}(1-\varepsilon)\alpha_r\left\{\sum_{s=1}^{m}\eta_s\int_{-1}^{+1} p(\mu, \mu') I_s(\tau, \mu)\, d\mu'\right\}\end{aligned} \tag{1.20}$$

In the above equation (1.20), τ denotes the optical depth and $\eta_r = l_r/k$, l_r denotes the absorption co-efficient for the rth interlocked line and k denotes the continuous absorption which is supposed to be constant for each line. ε, the co-efficient, which is introduced to allow for thermal emission associated with the line absorption, and $B_\nu(T)$,the Planck-function, are considered to be constant for each line.

1.1.1 Boundary Conditions:

The boundary conditions for solving the equation (1.20)

$$I_r(0,-\mu)=0;\;(0<\mu\leq 1) \tag{1.21}$$

$$I_r(\tau,\mu)\cdot\; e^{-\tau\mu}\rightarrow 0 \qquad as\;\;\tau\rightarrow\infty$$

i.e.

$I_r(\tau,\mu)$ is at most linear in τ as τ tends to infinity (1.22)

Another form of the equation (1.20):

In the equation(1.17),

$$\alpha_r\left\{\sum_{s=1}^{m}\eta_s J_s\right\}=\alpha_r\left\{\sum_{s=1}^{m}\eta_s\left(J_s-B\right)\right\}=\eta_r\left\{\sum_{s=1}^{m}\alpha_s\left(J_s-B\right)\right\}+\eta_r B$$

The equation (1.17) now take the form

$$\mu\frac{dI_r\left(\tau,\mu,\phi\right)}{d\tau}=\left(1+\eta_r\right)\left\{I_r\left(\tau,\mu,\phi\right)-B\right\} -\left(1-\varepsilon\right)\eta_r\left\{\sum_{s=1}^{m}\alpha_s\left(J_s-B\right)\right\} \tag{1.23}$$

The form (1.23) of the equation of transfer of interlocking lines for the case of triplet is used by Woolley and Stibbs[229] for obtaining the solution by Eddington's approximation. This form is used nowhere in this thesis.

1.2 Discrete-ordinate method.

The method used in this thesis was first extensively used by Chandrasekhar for solving a problem of stellar and atmospheric radiation and is popularly known as **Chandrasekhar's discrete-ordinate method.**

The method begins with the replacement of the source function $S_\nu(\tau_\nu, \mu)$ of the radiative transfer equation

$$\mu \frac{dI_\nu(\tau_\nu, \mu)}{d\tau_\nu} = I_\nu(\tau_\nu, \mu) - S_\nu(\tau_\nu, \mu) \tag{1.24}$$

by the mean intensity J, given by

$$S_\nu(\tau_\nu, \mu) = J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu, \mu')\, d\mu' \tag{1.25}$$

converting the resulting equation into an integro-differential equation

$$\mu \frac{dI_\nu(\tau_\nu, \mu)}{d\tau_\nu} = I_\nu(\tau, \mu) - \frac{1}{2}\int_{-1}^{+1} I_\nu(\tau_\nu, \mu')\, d\mu' \tag{1.26}$$

It is noted that the integro-differential equation (1.26) is converted in terms of specific intensity I_ν alone which, a function of two variables μ and τ, appears differentiated with respect to τ and integrated over μ.

In this method, the radiation field and as such the total solid angle is divided into a finite number of discrete-ordinates (directions). The intensity of each discrete ordinate represents the whole intensity of the corresponding small section of solid angle. The integral in the integro-differential equation associated with any problem of radiative or heat or neutron transfer is replaced by a quadrature, such as Gaussian, Lobatto, or Chebyshev. The radiative transfer equations for the set of discrete directions are then solved and the solution set is used to construct the solution of the main problem.

Chandrasekhar divided the radiation field into 2n number of discrete-ordinates in the directions μ_i, $(i = \pm 1, \cdots, \pm n)$, sub diving the interval $[-1, +1]$ of μ into 2n points μ_{-n}, $\mu_{-(n-1)}$, $\cdots$, μ_{-1}, μ_{+1}, $\cdots$, μ_{n-1}, μ_n so that the points are the 2n non-zero zeros of the even Legendre's polynomial $P_{2n}(\mu)$. Chandrasekhar replaced the definite integral of the integro-differential equation of radiative transfer by the Gaussian sums of numerical quadrature as follows:

$$\int_{-1}^{+1} I(\tau, \mu')\, d\mu' = \sum_{j=-n}^{j=+n} a_j I(\tau, \mu_j) \tag{1.27}$$

where μ_j's with $(j = \pm 1, \pm 2, \cdots, \pm n)$ are the $2n$ zeros of the Legendre polynomial $P_{2n}(\mu)$ of order $2n$ and the $a_j's$ are the weight factors, given by

$$a_j = \frac{1}{P'_{2n}(\mu_j)} \int_{-1}^{+1} \frac{P_{2n}(\mu)}{\mu - \mu_j} d\mu \tag{1.28}$$

The quadrature constants a_j and μ_j can be chosen in a variety of ways. Wick[224] suggested that the best choice for constants are those of the Gaussian quadrature formula. Gauss himself has shown that for a given number of divisions the best representation of an integral is obtained when the spacing of the division points is symmetrical about the mid-point of the range of integration, the interval being divided according to the zeros μ_j of the Legendre polynomial $P_{2n}(\mu)$.

Furthermore, μ_i's and a_j's follow the following properties:

$$\mu_i = -\mu_i \tag{1.29a}$$

$$a_j = a_{-j} \tag{1.29b}$$

by Abramowitz and Stegun[5] (vide Peraiah[159])

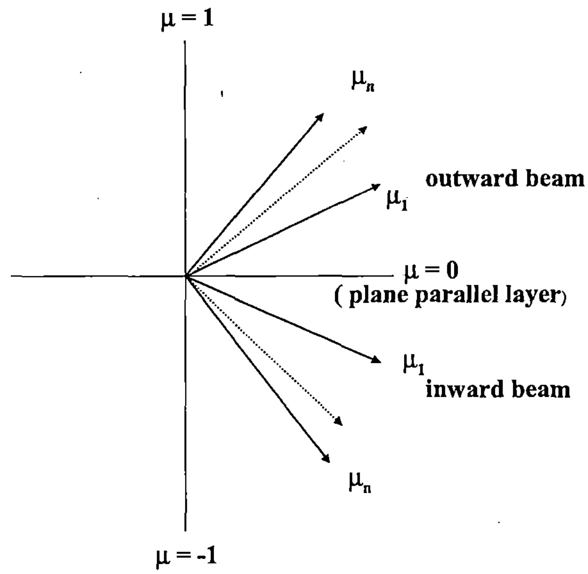

Figure 1.3: Gaussian points μ_j's

It is noticed that the Gaussian sum would give exact values of the integral, if $I(\mu)$ could be written down as a polynomial in μ of degree less than or equal to $4n-1$.

He then discretized the differential equation so formed into 2n ordinary differential equations along 2n directions μ_i, $(i = \pm 1, \cdots, \pm n)$ and solved each individual equations and combined them to form the solution of the desire integro-differential equation.

On the other hand, some workers, like Siewert,[186] Barichello and Siewert,[20,21] forming the 2N ordinary differential equations in the same manner as Chandrasekhar, expressed them into a matrix form. They, in lieu of taking the points μ_i, $(i = \pm 1, \cdots, \pm n)$ as the zeros of an even Legendre polynomial of order 2n, assume that the points as the eigen value of a matrix. This method is identified by a few authors as **matrix form of discrete ordinate method**.

There are also some other authors who formed the ordinary differential equations for each direction μ_i by dividing the interval $[-1, +1]$ of μ of the definite integral involved in the integro-differential equation following Chandrasekhar's method, but they used neither Chandrasekhar's form nor matrix form of the method. The equations so formed are known as **discrete ordinate equations**.

Here we shall now present a solution of the simplest form of a transfer equation which Chandrasekhar solved by his discrete-ordinate method.

1.2.1 Basic Radiative Transfer Equation, the Boundary Conditions and the use of Discrete Ordinate Method:

1.2.1.1 Basic Radiative Transfer Equation:

Simplest form of a transfer equation which Chandrasekhar solved by his discrete-ordinate method is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2} \int_{-1}^{+1} I(\tau, \mu') \, d\mu' \tag{1.30}$$

where I is the intensity, μ is the cosine of the incident angle ϑ made by the incident ray coming from any star to the surface at which intensity I is calculated with the outward normal and τ is the normal optical depth, given by, $\tau = \int_z^\infty k\rho dz$, z being the linear distances normal to the plane of stratification of a plane parallel atmosphere.

1.2.1.2 Boundary Conditions for Solving the Transfer Equation (1.30) :

$$I\,(0,-\mu)=0\ ,\ \ (\,0<\mu\leq 1) \tag{1.31a}$$

and $\quad I_r(\tau,\mu)\cdot\ e^{-\tau\mu}\rightarrow 0$

i.e. $\quad I_r(\tau,\mu)$ is at most linear in τ as τ tends to infinity (1.31b)

1.2.2 Chandrasekhar's solution:

1.2.2.1 Use of Discrete-ordinate Method to Solve the Equation (1.30), Subject to the Boundary Conditions (1.31a) and (1.31b):

Dividing the radiation field into 2n streams in the direction μ_i, $(i=\pm1,\ \cdots,\pm n)$, we can replace the equation of transfer (1.30) by the system of 2n linear differential equations:

$$\mu_i\frac{dI_i}{d\tau}=I_i-\frac{1}{2}\sum_j a_jI_j,\quad (i=\pm1,\cdots,\pm n) \tag{1.32}$$

where the μ_i's, $(i=\pm1,\cdots,\pm n\ and\ \mu_{-i}=-\mu_i)$ are the zeros of the Legendre Polynomial $P_{2n}\,(\mu)$ and the a_j's $(j=\pm1,\cdots,\pm n$ and $a_{-j}=a_i)$ are the corresponding Gaussian Weights. Further I_i is used for $I\,(\tau,\mu_i)$.

First we observe that the equation (1.32) admits a solution of the form:

$$I_i=g_ie^{-k\tau},\ \ (i=\pm1,\cdots,\pm n\ ) \tag{1.33}$$

where g_i and k are constants.

Introducing the equation (1.33) in the equation (1.32), we obtain a relation:

$$g_i\left(1+\mu_i k\right)=\frac{1}{2}\sum_j a_j g_j,\quad (i=\pm 1,\cdots,\pm n\ ) \tag{1.34}$$

Hence,

$$g_i=\frac{\text{constant}}{\left(1+\mu_i k\right)},\qquad ,(i=\pm 1,\cdots,\pm n\ ) \tag{1.35}$$

where the 'constant' is independent of i . Substituting the foregoing form in the equation(1.34) , we obtain the characteristic equation:

$$1=\frac{1}{2}\sum_j \frac{a_j}{\left(1+\mu_j k\right)} \tag{1.36}$$

Remembering that $\mu_{-i} = -\mu_i$ and $a_{-j} = a_j$, we can write the characteristic equation(1.36) in the form:

$$1=\frac{1}{2}\sum_{j=1}^{n} \frac{a_j}{\left(1-\mu_j^2 k^2\right)} \tag{1.37}$$

which has two roots, each equal to zero, because $\sum\limits_{j=1}^{n} a_j = 1$ and other $(2n-2)$ non zero distinct roots.

With $(2n-2)$ distinct non-zero roots $\pm\ k_\alpha$, $\alpha = 1,2,\cdots,n-1$, we can establish a relation with the zeros $\pm\ \mu_i$, $i = 1,2,\cdots,n$ of the Legendre Polynomial $P_{2n}\left(\mu\right)$ which is

$$k_1\cdots k_{n-1}\cdot\mu_1\cdots\mu_n=\frac{1}{\sqrt{3}} \tag{1.38}$$

With these distinct non-zero roots which are numerically greater than 1, we can show that the general solution of the system of equation (1.32) is of the form:

$$I_i=b\left(\sum_{\alpha=1}^{n-1}\frac{L_\alpha e^{-k_\alpha \tau}}{1+\mu_i k}+\sum_{\alpha=1}^{n-1}\frac{L_{-\alpha} e^{+k_\alpha \tau}}{1+\mu_i k}+\tau+\mu_i+Q\right), \tag{1.39}$$

$$(i=\pm 1,\pm 2,\cdots,\pm n)$$

where $L_{\pm\alpha}$ and b, which can be connected with the flux of radiation πF normal to the plane of stratification of the plane-parallel scattering atmosphere by the relation

$$F = \frac{4}{3}b \tag{1.40}$$

and Q, satisfying the relation,

$$Q = \sum_{i=1}^{n} \mu_i - \sum_{\alpha=1}^{n-1} \frac{1}{k_\alpha} \tag{1.41}$$

are arbitrary constants of integration.

We have already mentioned the two boundary conditions (1.31a) and (1.31b) for this problem. By virtue of the boundary condition (1.31b) which gives that none of the I_i's increase more rapidly than e^τ as $\tau \to \infty$, we omit all the terms in $e^{+k_\alpha \tau}$, thus leaving

$$I_i = b\left(\sum_{\alpha=1}^{n-1} \frac{L_\alpha e^{-k_\alpha \tau}}{1+\mu_i k} + \tau + \mu_i + Q\right), \tag{1.42}$$

$$(i = \pm 1, \pm 2, \cdots, \pm n)$$

Next, the boundary condition (1.31a) implies that there is no radiation incident on $\tau = 0$. The absence of any radiation in the directions $-1 \leq \mu < 0$ at $\tau = 0$ gives in our present approximation that

$$I_i = 0 \text{ at } \tau = 0 \text{ and } i = 1, 2. \cdots, n \tag{1.43}$$

Hence, by (1.42), we get

$$\sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1 - \mu_i k_\alpha} - \mu_i + Q = 0 \quad (i = 1, 2, \cdots, n) \tag{1.44}$$

which are the n relations which determine the n constants of integration L_α, $(\alpha = 1, 2, \cdots, n)$ and Q . The constant b is left arbitrary and is related to the assigned constant net flux of the radiation through the atmosphere given by (1.40).

1.2.2.2 Closed form of emergent intensity $I(0,\mu)$

Letting

$$S(\mu) = \sum_{\alpha=1}^{n-1} \frac{L_\alpha}{1-\mu k} - \mu + Q\,, \tag{1.45}$$

the boundary conditions (1.31b) can be expressed as:

$$S(\mu_i) = 0 \quad , \quad i = 1, 2, \cdots, n \tag{1.46}$$

and the angular distribution of the emergent radiation is expressible as

$$I(0,\ \mu) = \frac{3}{4} F S(-\mu) \tag{1.47}$$

Now, multiplying $S(\mu)$ by $R(\mu)$, given by

$$R(\mu) = \prod_{\alpha=1}^{n-1} (1 - k_\alpha \mu) \tag{1.48}$$

we get a polynomial in μ of degree n which vanishes for $\mu = \mu_i$, helping us to conclude that the polynomial $S(\mu)\,R(\mu)$ must be identical with the polynomial $P(\mu)$, given by,

$$P(\mu) = \prod_{i=1}^{n} (\mu - \mu_i) \tag{1.49}$$

and so, the co-efficients of each term of the two polynomials must coincide and therefore, we shall get

$$S(\mu)\,R(\mu) = (-1)^n k_1 k_2 \cdots k_{n-1} P(\mu)$$

producing the relation:

$$S(-\mu) = k_1 k_2 \cdots k_{n-1} \mu_1 \cdots \mu_{n-1} H(\mu) \tag{1.50}$$

where

$$H(\mu) = \frac{1}{\mu_1 \cdots \mu_{n-1}} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha=1}^{n-1} (1 + k_\alpha \mu)} \tag{1.51}$$

So, from the equation (1.47), using the equation (1.50), we can write

$$I(0,\ \mu) = \frac{3}{4} F\, k_1 k_2 \cdots k_{n-1} \mu_1 \cdots \mu_{n-1} H(\mu) \tag{1.52}$$

Therefore, using the equation (1.38), we can express the emergent radiation in terms of H-function $H(\mu)$ as

$$I(0,\ \mu) = \frac{\sqrt{3}}{4} F H(\mu) \tag{1.53}$$

1.3 Works done so far

1.3.1 Works done on discrete-ordinate method

Discrete-ordinates method for the radiative transfer and the neutron transport is not a new, but has a long history. Though the method is an old one, even then it doesn't lose the importance. In the language of Atanacković-Vukmanović[12] mentioned in an invited review paper of radiative transfer ,"In 1940s and 1950s several powerful methods for solving RT problems were developed: the method of discrete ordinates by Chandrasekhar, Ambarzumian's method based on the invariance principle, Sobolev's escape probability method (1957), etc. Their importance is twofold: on one hand, they are the bases of many modern techniques and, on the other, their "exact" solutions to simplified transfer problems serve as a reliable test of accuracy of new numerical methods."

Many workers worked on this method time to time. Among them, Chandrasekhar is noteworthy. Some authors gave the identity of the discrete ordinate method, which is well known as **Chandrasekhar's discrete ordinate method**, by using the term **Wick-Chandrasekhar's**

discrete ordinate method, though neither Wick nor Chandrasekhar presented the method first.

The method was first brought to the transport theory from the Kinetic theory of gases as developed by Joule (vide Peraiah[159]) in a rather primitive form (two parallel and opposite intensities) by Schuster[178] and Schwarzsohild,[180] and Milne[147] (vide Kourganoff and Busbridge[129]).

In Kinetic theory of gases , the molecules in a box are presumed to be moving in three equal pairs of streams, parallel to length, breadth, and depth of the box in which the gas is situated and directed opposite direction to each other. (vide Peraiah[159]). Same treatment was done by Schuster and Schwarzsohild in transport theory. The transfer equation in plane parallel stratification

$$\mu\frac{dI\left(\tau,\mu\right)}{d\tau}=I\left(\tau,\mu\right)-\frac{1}{2}\int_{-1}^{+1}I\left(\tau,\mu'\right)d\mu'$$

is replaced by a pair of equations for I_+ and I_-, the outward and inward intensities, thus

$$+\frac{1}{2}\frac{dI_+}{d\tau}=I_+-\frac{1}{2}\left(I_++I_-\right) \quad (1.54a)$$

$$-\frac{1}{2}\frac{dI_-}{d\tau}=I_--\frac{1}{2}\left(I_++I_-\right) \quad (1.54b)$$

(vide Chandrasekhar[45] and Peraiah[159]).

The factor $\frac{1}{2}$ on the LHS is chosen arbitrarily as in the kinetic theory of gases.(vide Peraiah[159])

The method was generalized first by Wick[224] (vide Kourganoff and Busbridge[129] and Woolley and Stibbs[229]), in connection with a diffusion problem, by replacing the integral of the equation of transfer(1.24) by the Gaussian sums of numerical quadrature as in equation(1.27).

Chandrasekhar[45] developed one dimensional mathematical models of radiative transfer and discussed the time independent problems at a length by using this technique.

Considering the theory of diffuse reflection and transmission by a plane-parallel atmosphere of finite optical thickness under conditions of (I) isotropic scattering with an albedo $\tilde{\omega}_0 \leq 1$, (II) scattering in accordance with Rayleigh's Phase function (II) scattering in accordance with the phase function $\lambda\,(1 + x cos\theta)$, and (V) Rayleigh scattering with proper allowance for the polarization of radiation field, Chandrasekhar[43] showed it is possible to eliminate the constants of integration (which are twice as many as in the case of semi infinite atmospheres) and expressed the solutions for the reflected and transmitted radiations in closed forms in general n^{th} approximation. He also showed a pair of functions $X(\mu)$ and $Y(\mu)$ which depends only on the roots of a characteristic equation and the optical thickness of the atmosphere play the same basic role in the theory as $H(\mu)$ does in the theory of semi-infinite atmospheres making possible the passage to the limit of infinite approximation and the determination of the exact laws of diffuse reflection and transmission.

Chandrasekhar[45] applied his method of discrete ordinates first to solve the transfer equation for coherent scattering in the stellar atmosphere with Planck's function as a linear function of optical depth (viz.,$B_\nu\,(T) = b_0 + b_1\tau$).

He discussed the equations of Radiative transfer for an electron scattering atmosphere and gave the solution of the equation by his method of discrete ordinates (vide Chandrasekhar[45] and Woolley and Stibbs[229]).

A moving atmosphere, postulated for Cephied variables, is sometimes suspected of giving rise to irregular asymmetries and displacement of lines in spectra of supergaints. Underhill[206] applied the Chandrasekhar's theory, as explained in Chandrasekhar's paper[40] and his successive papers, of transfer of radiation through a Schuster- Schwarzschild model atmosphere to formulate the problem of a uniformly expanding atmosphere and applied conveniently the Chandrasekhar's discrete ordinate method to solve the transfer equation for the case.

Rybicki[173] wrote in a review paper about Chandrasekhar's works on the method of discrete ordinates, " Chandrasekhar's numerical comparison of low order results with the exact analytical result

$$H(\mu) = (1+\mu)\exp\left\{-\frac{\mu}{\pi}\int_0^{\pi/2}\frac{\log\left[(1-\phi\cot\phi)/\sin^2\phi\right]}{\cos^2\phi+\mu^2\sin^2\phi}d\phi\right\} \quad (1.55)$$

convinced him that the approximation results of the discrete ordinate method would converge to the exact results in the limit $n \rightarrow \infty$. Only much later was this convergence proved mathematically" (vide Anselone[10,11]). He also wrote,"Chandrasekhar also applied the method of discrete ordinates to the problem of diffuse reflection, in which radiation is incident on the medium at angle μ_0, and one is required to find the radiation emergent at angle μ. This relationship is given in terms of a scattering function $S(\mu, \mu_0)$. It had previously been shown by Hopf [Equation (191), of Hopf[86]] that the scattering function is related to the above H-function (1.55) by means of the relation

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S(\mu,\mu_0) = \varpi_0 H(\mu)H(\mu_0) \quad (1.56)$$

Thus, the function $S(\mu, \mu_0)$ of two variables, can be simply expressed in terms of a single variable, the same H-function that appears in the solution for the radiative equilibrium problem.

When Chandrasekhar applied the discrete ordinate method to the semi-infinite diffuse reflection problem (cf §26 of Chandrasekhar[45]), he found a result of the same form as equation (1.56), where the H-function

$$H(\mu) = \frac{\prod_{i=1}^{n}(1+\mu/\mu_i)}{\prod_{\alpha=1}^{n}(1+\mu k_\alpha)} \tag{1.57}$$

were precisely the same as that for the discrete ordinate solution (1.57) to the radiative equilibrium problem".

Rybicki[173] further wrote,"For the case of a finite medium, besides the diffuse scattering function $S(\mu,\mu_0)$ there is also a diffuse transmission function $T(\mu,\mu_0)$ to be determined. These functions satisfy the extended relations, given by Ambarstsumian[8]

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) S(\mu,\mu_0) = \varpi_0 H(\mu) H(\mu_0) \tag{1.58a}$$

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) S(\mu,\mu_0) = \varpi_0 H(\mu) H(\mu_0) \tag{1.58b}$$

where the functions * X and Y were the solutions to certain functional equations. After seeing these forms in Ambartsumian's paper, Chandrasekhar[43] was able to put the discrete ordinate solution for this problem into the same form, where the X- and Y- functions were also expressible in closed form in terms of the roots of the characteristic function. ....".

He further pointed out that "The transfer problems Chandrasekhar considered had already been simplified by making a number of physical assumptions and approximations; e.g., plane-parallel geometry, coherent scattering, and single-scattering albedo independent of depth. In a sense, choosing discrete angles is just one more simplifying

* Ambartsumian used the notation ϕ and ψ for these functions

approximation, on a par with the others made. Then the crucial questions to ask are whether these simplified equations are of practical use, can increase our mathematical or physical understanding, or satisfy some criterion of mathematical beauty. I believe for the discrete ordinates method the answer is 'yes' to each of these questions.

As to the practicality of the method, remember that Chandrasekhar was not acting solely as a mathematical physicist, but as an astrophysicist attempting to find answers to practical problems in stellar atmospheres and planetary atmospheres......... The discrete ordinate method gave him highly accurate solutions in a completely straightforward way.

As to the method's relation to mathematical and physical understanding, Chandrasekhar was obviously delighted when he continually found many results of this method that were perfectly consistent with exact requirements of the theory. Many known analytically exact results are obeyed precisely to all orders of approximation in the discrete ordinate method, for example, the Hopf-Bronstein relation $J(0) = \sqrt{3}F/4$ in the Milne problem, the structure of the diffuse scattering and transmission functions as given in equations (1.56) and (1.58a,1.58b), and their reciprocity relations. Often he was able to determine the form of the exact solutions only after he had solved the discrete ordinate equations first. These circumstances convinced Chandrasekhar that the discrete ordinate method was more than just a convenient numerical method; it also preserved essential mathematical and physical characteristics of the problem being investigated."

The above discussion noted from the Rybicki's review paper[173] gives us a clear picture and the usefulness about the Chandrasekhar discrete ordinate method.

The method of discrete ordinates developed by Wick[224] and Chandrasekhar[45] constitutes a powerful technique for the solution of transfer problems. The accuracy of the method is strongly dependent on the particular choice of finite stream quadrature formula used to represent continuous radiation field. Chandrasekhar's[45] use of a Gaussian quadrature formula was criticized by Kourganoff and Pecker[130] and Kourganoff and Busbridge.[129]

Kourganoff and Pecker[130] produced a paper on the choice of numerical integration formulae in the solution of the integro-differential equations of transfer (radiation, neutrons) by the "method of discrete-ordinates". In their paper they commented that the method of Gaussian subdivisions and the characteristic roots used by Chandrasekhar in his treatment of radiative equilibrium is not necessarily the most effective one. To establish their comment they performed the calculations of the 1st and 4th approximations in the "standard problem" of isotropic diffusion in the Newton-Cotes and Tchebycheff formulae and found that the former gives more accurate results in the 2nd and 4th approximations than were obtained by Chandrasekhar. This result was explained by discussing the distribution of the Gaussian subdivisions.

Sykes[201] obtained highly accurate results from the discrete ordinate method by splitting the interval and fitting the Gaussian formula separately to the upward and downward hemispherical stream i.e. separately over the ranges $(-1, 0)$ and $(0, 1)$.

Kourganoff[128] extended Chandrasekhar's limiting process on the n^{th} approximation obtained by the method of discrete beams for emergent intensity to functions describing the internal state of the atmosphere. He showed how, by a suitable interpretation of the constant of integration , the solutions in the n^{th} approximation for the source function $\Im\tau$ can also be transformed into a form suitable for effecting the passage to infinite approximation and giving the exact $\Im\tau$ for the problem of isotropic scattering in a gray atmosphere and for a line formation in a non gray atmosphere.

Sen[181] solved the equation of transfer of radiation in a spherically scattering symmetric atmosphere for non-conservative isotropic scattering by the method of Chandrasekhar in which the integrals are replaced by corresponding Gaussian sums, and first approximation results have been fitted to two boundary conditions, one of no incident radiation and other of a very weak radiation field penetrating from outside.

Sen[182] solved the problem of softening of radiation by multiple compton scattering in stellar atmosphere containing free electrons in the first approximation (in Chandrasekhar's method of solution by Gaussian approximation) by the method of trigonometric series and calculated the intensity distribution at the outer surface by retaining the first- and second-order terms of Taylor's expansion of scattering intensity.

Horak[89] considered the transfer of radiation by a plane parallel atmosphere containing a uniform distribution of emission sources for the cases: (a) scattering according to the Rayleigh phase function and (b)Rayleigh scattering and derived the exact expression for the emergent intensity for finite atmospheres. He also gave the solution in n^{th} approximation for any depth τ and calculated polarization for the atmosphere of optical thickness $\tau = 0.20$.

King[120] developed radiative transfer theory for band-absorbing semi-infinite atmospheres possessing line structure by extending the formalism of Chandrasekhar to an integration over frequency space as well as over μ space and obtained the solution of the monochromatic equation of transfer for a plane-parallel atmosphere in local thermodynamic equilibrium with the aid of the method of discrete ordinates .

Under radiative equilibrium the emission co-efficient of a narrow band-absorbing gas is equal to the frequency integral of the average intensity and the band-absorption co-efficient. King[119] developed a two-point Gaussian quadrature formula for the frequency integration, by using the Elsasser band of equally spaced, equally intense, Lorentz-broadened lines as the absorption model. By the use of that formula, he extended the Chandrasekhar method of discrete-ordinates to include an integration over frequency space as well as μ-space.

Krook[131] translated the Wick-Chandrasekhar method of discrete ordinate into an equivalent moment procedure for the approximate solution of the equation of transfer for a plane-stratified gray atmosphere in radiative equilibrium and used it to discuss the relation between various methods that are also based on infinitely approximating to the angular distribution of intensity.

Lenoble,[138] applying the Chandrasekhar's method of approximations to the diffusion of radiation from sun and sky in a plane homogeneous scattering layer of large particles (fog or sea), gave notation and principle of the method and established the equations which were applied to two cases: (1) for sky radiation only and no solar radiation; (2) for an infinite layer with solar radiation only.

Piotrowski[161,162] (vide Busbridge and Orchard[32]) found the asymptotic solution for the phase function $1+\varpi\cos\theta$ using the method of discrete ordinates as developed by Chandrasekhar[41,45]

King[121] derived the exact form of the source function for a finite gray atmosphere in radiative equilibrium (planetary thermal problem) forming the solution in n^{th} approximation by using the method of discrete ordinates.

King[122] developed the transport theory of non-gray atmosphere of finite thickness and treated this planetary thermal problem by using both an invariance attack and the method of discrete ordinates.

Lenoble[139] used the Chandrasekhar method to calculate the illumination

(I) in the sea, for a uniform sky and for the sun at 59^0 from the zenith, assuming the absorption co-efficient to be half the scattering co-efficient and

(II) below, a mist of large drops, there being no absorption and a uniform sky.

Lenoble[140] gave the equation governing the penetration, summarizing Chandrasekhar's approximate method of resolution, applied it to a layer of haze and to the sea and discussed the approximations.

Jefferies and Thomas[99] obtained an algebraic solution for the depth variation of the source function $S_L(\tau)$ for resonance and strong subordinate lines by using Eddington approximation and the method of discrete ordinates.

Sen and Lee[183] solved the problem of broadening of spectral lines by the Doppler effect due to thermal motion of electrons in an axially symmetric, plane-parallel electron atmosphere, scattering according to Rayleigh's phase matrix and taking into account the polarization of the radiation field in the first approximation of Chandrasekhar's method of discrete ordinates.

King and Florance[123] demonstrated the physical basis underlying Sykes' choice of a double-Gauss method in providing the optimum fit of the kernel in the Schwartzchild-Milne integral equation by an exponential function series.

Abhyankar[1] presented a numerical method for computing absorption-line profiles in a plane-parallel stratified moving atmosphere by extending the Rottenberg's[171] idea of dividing the atmosphere into many thin layers, of course not for the spherical layers as done by Rottenberg, but for plane parallel layers and retaining the equation of transfer in its discrete ordinate form.

King, Sillars and Harrison[124] expressed the Hopf q-function in the equilibrium gray-atmosphere problem in the discrete-ordinate approximation to attain its extreme accurate value.

Hummer[94,95] (vide Rybicki and Hummer[174]) solved the radiative transfer equation for spectral line formation by non-coherent scattering in inhomogeneous plane-parallel media by using a generalization of Chandrasekhar's discrete ordinate method.

Avrett and Hummer[14] used the generalization of the Wick-Chandrasekhar discrete-ordinate method in the theory of line formation to find the expression for the source function $S(\tau)$

Samuelson[176] extended the method of discrete ordinates to describe the steady-state distribution of thermal radiation and the corresponding depth-dependent thermal structure of a plane parallel semi-infinite particulate medium in radiative equilibrium.

Samuelson[177] used the method of discrete ordinates to investigate the outgoing thermal radiation field at the top of cloudy atmospheres as a function of the scattering and thermal properties of the atmosphere.

Black[27] applied the method of discrete ordinates to calculate the diffusely reflected and transmitted spectral line profiles for uniform non coherent scattering media onto which radiation of frequency near that of a resonance line of the medium incident.

Blerkom and Hummer[28] obtained numerical solutions of high accuracy for the ionization balance in an isothermal, plane-parallel hydrogen model nebulae of various optical thickness, using generalization of the Wick-Chandrasekhar discrete-ordinate method .

Rybicki and Hummer[174] gave the discrete-ordinate representation of the radiative transfer equation for spectral line formation by non-coherent scattering in inhomogeneous plane-parallel media casting it into matrix form and derived the Reccati-transformation for finite atmosphere.

Assuming uniform velocity in each layer, Kulander[130] solved the Eddington approximation to transfer equation by a discrete ordinate method for a semi-infinite, isothermal atmosphere with a constant density of particles having only two discrete energy level.

The DOM, described by Chandrasekhar in 1950,[45] has been deeply studied by Lathrop and Carlson[34] (vide Joseph, Coelho, Cuenot and Hafi[101]) in 60-70s and by Truelove, Fiveland and Jamaluddin in the 80s (vide Joseph, Coelho, Cuenot and Hafi[101]). Significant improvements have been achieved in the last decade aiming at the reduction of the ray effects and false scattering, more accurate quadratures and the extension to complex geometries (vide Joseph Coelho Cuenot and Hafi[101])

Hummer[96] used a generalized discrete ordinate method to obtain accurate numerical solutions of the line transfer problem in which the scattering is described by a redistribution function.

Code[47] solved the time-dependent equation of radiative transfer for a plane-parallel isotropic scattering medium by the method of discrete ordinates.

Be[23] developed a method for solving the one-dimensional multigroup transport equation in a homogeneous semi-infinite medium with anisotropic scattering and used a variational treatment to enable the method to be applied to finite slabs where only the emergent angular fluxes are of interest. He showed the method which is not limited by any restriction on the number of spatial mesh intervals used to be competitive in computing time with conventional discrete ordinate techniques.

Considering the problem of the radiation field in a plane-parallel multilayer system, whose outer boundary is irradiated by parallel rays, Barkov[22] studied the case of non-isotropic radiation and gave a formal solution of the problem and using this, constructed the spatial angular distribution function of the intensity of the diffused radiation by applying the method of discrete ordinates.

Liou[142] developed theoretically the discrete-ordinate method for radiative transfer introduced originally by Chandrasekhar and verified numerically for use in solving the transfer of both solar and thermal infrared radiation through cloudy and hazy atmospheres.

Liou[143] derived explicitly the analytic equations in closed forms for the cases of two-stream and four-stream approximation from the exact solutions provided by Liou.[142]

Hansen and Travis[81] said in their review paper that an advantage of the discrete ordinate method is that it yields the internal field as well as the reflection and transmission. A disadvantage is that considerable algebra is required prior to numerical computations. However, at least for azimuth-independent term, the discrete ordinate method can give rather accurate results (within a percent or so) already for n = 3 or 4, so it is efficient procedure when accuracies of that order are sufficient (cf Weinman and Guetter,[221]Liou[142]). Liou[143] has given a quasi-analytic solution for n=2 (4-stream approximation) which might be sufficiently accurate for computations of the flux in many applications.

Roux and Smith[170] approximated the equation for one-dimensional, axisymmetric radiative transfer in an absorbing, emitting, and isotropically scattering medium by the method of discrete ordinates. Homogeneous and particular solutions are derived from the discrete ordinate form of the radiative transport equation.

Using a discrete ordinates method, Cram[55] solved the radiative transfer equation in a gray atmosphere subject to a specific distribution of mechanical heating and determine the resulting changes in LTE and non LTE conditions.

Nelson Jr. and Victory Jr.[157] compared the Nyström discrete ordinates method and interpolatory discrete ordinates method used in linear transport equation in the simple case of monoenergetic transport in azimuthally symmetric one dimensional slab geometry.

Zasova and Ustinov[233] applied the method of discrete ordinates, by developing it for making applicable to an inhomogeneous atmosphere of large optical depth, to the solution of the transfer equation in the case of an inhomogeneous planetary atmosphere.

Discussing the difficulties inherent in the conventional numerical implementation of the discrete ordinate method (following the Chandrasekhar's prescription) for solving the radiative transfer equation, Stamnes and Swanson[197] developed a matrix formulation to overcome the difficulties. Stamnes and Dale[196] extended the method to compute the full azimuthal dependence of the intensity.

Khalil, Shultis and Lesste[118] developed a plan systematic, gray model of coal particle suspension to test the accuracy of the low-order discrete-ordinates and flux method and of the differential approximation for calculating the radiant energy transport in multiply scattering and heat generating media bounded by diffusely reflecting surfaces and compared the results obtained by these three approximate techniques with those computed by a high order discrete-ordinates method.

The standard discrete-ordinates method is a deterministic(non stochastic) method for solving the linearized Bolzmann transport equation. It is commonly applied to neutron and photon transport problems. Finding its applicability superior to Monte Carlo methods for one dimensional problems in electron transport, Morel and Wienke[151] reviewed briefly the history of discrete-ordinates electron transport methods, described the state-of-art at that time and suggested directions of further works.

One of the dominant numerical approximation methods for the integro-differential equation for neutron transport is the discrete ordinate method. In this method one collocates the equation at preselected angular directions which are the quadrature points of the integral scattering term (the "discrete ordinates"), and then solves the resulting linear hyperbolic system by a variety of difference schemes. The problem with this hybrid collocation difference procedure lies in connecting the spatial differencing with the angular collocation. The possible mismatch can lead to distortions in the angular flux solution. This have led to numerous meliorative procedures, and an extensive literature. The method of collocation is well established among the numerical approximation methods for ordinary and partial differential equations and integral equations. Despite its very limited application in integro-differential equations, Grossman[79] felt it seemed of interest to apply a full collocation scheme to neutron transport equation; building in the required continuity and coupling between space and direction variables through suitable multidimensional spline basis functions and showed how this is done for a simple mono energetic one dimensional form of the neutron transport equation indicating its possible extensions of the method.

Mengüc and Viskanta[146] examined critically the accuracy of the two-flux, spherical harmonic and discrete ordinates method for predicting radiative transfer in a planar highly-forward scattering and absorbing medium.

Karp[117] showed that similar relations like the azimuth-averaged component of the intensity computed from the spherical harmonic method for solving the equation of the radiative transfer is 'exact' at Gaussian quadrature points holds for higher terms in the Fourier expansion of the intensity, but that result is 'exact' at the zeros of the associated Legendre polynomials. The relationship between discrete ordinates and spherical harmonics methods follows from the discussion. A discrete ordinates quadrature scheme, based on the zeros of the associated Legendre polynomials was shown to maintain the correspondence of the methods for those problems as well as providing a better set of points than the other methods in use.

Bergmann, Houf and Incropera[25] performed calculations, based on discrete-ordinate forward scattering and three-flux methods of solving the equation of transfer, to determine the effect of the scattering distribution, which had been systematically varied by changing the asymmetry factor used in the Heney-Greenstein form of the phase function, on radiative transfer in absorbing-scattering liquid which is irradiated across an air interface.

Larsen[136] introduced a parameter ϵ into the discrete ordinates equations in such a way that as ϵ tends to zero, the solution of these equations tends to the solution of the standard diffuse equation and then studied the behaviour of the spatial differencing scheme of the discrete ordinates equations for fixed spatial and angular meshes, in the limit as ϵ tends to zero.

Abhyankar and Bhatia[2] gave a definition of the effective depth of line formation which incorporates its dependence on the angle of emergence as well as on the position of the line and obtained the solution for isotropic scattering in the third approximation of discrete ordinates for various points on the disc of a planet viewed at different phase angles.

Marshak[145] studied the one dimensional transport equation in slab geometry with periodic boundary conditions , reduced it to the integral equation of the Peierls type and estimated the spectral radius of the integral operator. He analyzed the discrete ordinates algorithm for estimating the solution.

Nakajima and Tanka[156] presented matrix formulations for the discrete ordinate and matrix operator methods for solving the transfer of solar radiation in plane-parallel scattering atmosphere introducing eigenspace transformations of the symmetric matrices into the method of Stames and Swanson instead of using the decomposition of an asymmetric matrix. They gave the representations of the reflection and transmission matrices in the matrix operator method and the solutions of the discrete-ordinates method for inhomogeneous sublayers through the addition technique of the matrix operator method.

Nakajima and Tanka showed that the algebraic eigenvalue problem occurring in the discrete-ordinate and matrix operator methods can be reduced to finding eigenvalues and eigenvectors of the product of two symmetric matrices, one of which is positive definite. Stamnes Tsay and Nakajima[198] showed that cholesky decomposition of this positive definite matrix can be used to convert the eigenvalue problem into one involving a symmetric matrix and established, by a careful comparison of Nakajima and Tanka procedure, Cholesky decomposition method of Stamnes Tsay and Nakajima and the original procedure of Stamnes and Swanson, that Stamnes and Swanson prescription is still the most accurate because it avoids round-off errors due to matrix multiplications needed to symmetrize the matrix in the two other procedures.

Stamnes, Tsay, Wiscombe, and Jayaweera[199] summarized an advanced, thoroughly documented, and quite general purpose discrete ordinate algorithm for time-independent transfer calculations in vertically inhomogeneous, nonisothermal, plane-parallel media and made some progresses, in both formulation and numerical solution, in the algorithm.

Myneri, Asrar and Kanemasu[154] discussed a finite element discrete ordinates method for solving the radiative transfer equation in non-rotationally invariant scattering media and the application of the method to the leaf canopy problem.

Cefus and Larsen[36] describing the non-linear "quasi diffusion" method developed by Gol'din and the "second moment" method proposed by Lewis and Miller for obtaining iterative solutions of discrete-ordinate problems, showed that the methods reduce to almost the same linear method for a special class of problems and performed a Fourier stability analysis of the two methods for these special problems.

Yavuz and Larsen[232] proposed a spatial domain decomposition method for modifying the Source Iteration (SI) and Diffusion Synthetic

Acceleration (DSA) algorithms for solving discrete ordinates problems which consists of subdividing the spatial domain of the problem and performing the transport sweeps independently on each subdomain, has the advantage of being parallelizable because the calculations in each subdomain can be performed on separate processors.

Ben Jaffel and Vidal-Madjar[24] modified the discrete ordinate method developed by Wehrse[219] and Schmidt and Wehrse[179] for the resolution of the radiative transfer equation and showed that the construction of a quasi analytical solution to the corresponding matrix diagonalization problem reduces the time calculation and allows the use of more dense discrete frequency and angle grids.

Viik[211] solved a vector equation of the radiative transfer for conservative as well as non-conservative planetary atmospheres using the method of discrete ordinates.

Viik[212] solved another vector equation of the radiative transfer for non-conservative homogeneous plane parallel planetary atmosphere using the method of discrete ordinates.

Gouttebroze[78] extended the discrete ordinate method of Wick-Chandrasekhar to the case of radiative transfer equation of infinitely long cylinders, in Eddington approximation, by replacing the exponentials by modified Bessel's functions.

Wang[218] gave a systematic extensions of Chandrasekhar's work to three dimensions including discussions of specular and diffused parts, reciprocity, solutions and approximations.

Tsay and Stamnes[205] verified a reliable and efficient discrete ordinate method for multiple scattering, radiative transfer calculations in vertically inhomogeneous, non-isothermal atmospheres in local thermodynamic equilibrium.

Helliwell, Sullivan Macdonald and Voss[83] developed a finite difference discrete-ordinate iterative method to solve the three dimensional radiative transfer equation which is applicable to a volume of ocean with position dependent volumetric absorption and scattering coefficients. Input quantities include Sun position and sky radiation distribution, scattering phase function and absorbing, reflecting or emitting objects within the ocean volume. A solution of the one dimensional radiative transfer equation was used to provide boundary values for the 3D solutions.

Larsen[135] showed the distributional solutions of the transport equation to be a certain weak limit of regular solutions of the discrete ordinates equations as N, the order of the angular quadrature set, tends to infinity.

Viik[213] described a method based on the method of discrete ordinates by Chandrasekhar[45] to calculate the $X-$, $Y-$ and $H-$ matrices for molecular scattering in a homogeneous plane parallel atmosphere.

Ganguly, Allen and Victory, Jr.[76] suggested a new approach to discrete-ordinates neutron transport in plane geometry.

Jin and Levermore[100] studied the discrete ordinate method in these limits and found formulae for the resulting diffusion equation and its boundary conditions .

Karanjai and Deb[111] obtained the solution of a transfer equation for coherent scattering in a stellar atmosphere with Planck's function as a nonlinear function of optical depth (viz., $B_\nu(T) = b_0 + b_1 e^{-\beta\tau}$) by the method of discrete ordinates originally due to Chandrasekhar.[45]

Kobayashi[126] presented the discrete-ordinate solutions for a multidimensional radiative transfer equation for a collimated source and to demonstrate the effect of atmospheric heterogeneity on radiative flux.

Kylling[133] solved the transfer equation for normal waves in finite, inhomogeneous and plane-parallel megnetoactive media using discrete ordinate method developed by Chandrasekhar[45] as well as Stammes, Tsay, Wiscombe, Jayaweera[199]

Wehrse and Hof[220] studied the transfer of gamma rays by means of numerically stable method that yields all emergent intensities as well as the energy converted to heat which were determined by solving the equations with the discrete-ordinate-matrix-exponential method.

Weng[223] established a theory for discretizing the vector integral differential radiative equation in which phase matrix was derived from averaging the scattering matrix over poly disperse particles and then making a linear transformation of the averaged scattering matrix according to spherical trigonometry. The phase matrix and radiative vector in the vector radiative transfer equation were both expanded into Fourier-cosine and sine series. The complete set of solutions for the discrete matrix equations for cosine and sine modes of the radiative vectors was obtained by solving for the eigenvalues and eigenvectors and particular solutions. The integration co-efficients in the solutions were determined through the continuity conditions at vertically layered interface and the top and bottom boundaries.

Weng[222] applied a multi-layer discrete ordinates method for vector radiative transfer in vertically inhomogeneous, emitted and scattering atmosphere and compared the upwelling radiance from the vector radiative transfer model already established by himself with Chandrasekhar's analytic solutions for a conservative Rayleigh scattering atmosphere.

Shibata and Uchiyama[185] so incorporated the thermal infrared radiation with the discrete ordinate method that it becomes usable in climate models.

Yavuz[231] proposed a simplified discrete-ordinates (S_N) method completely free from all spatial truncation errors for the solution of one-group and isotropic source plane-geometry transport problems with an arbitrary anisotropic scattering of order $L\,(= N - 1)$. The method is based on the expansion of the angular flux in spherical harmonic (P_{N-1}) solutions. The analytic expression for the angular flux for each discrete-ordinates direction depends on the exponential functions, arbitrary constants and interior source.

Barichello and Siewert[18] established the equivalence between the discrete ordinates method and the spherical harmonics method in the works concerning steady-state radiative transfer calculations in plane-parallel media. i.e. established that the choice for a quadrature scheme for the discrete ordinates method as the zeros of the associated Legendre polynomials and the use of generalized Mark boundary conditions in spherical harmonics method for standard radiative transfer problems without the imposed restriction of the azimuthal symmetry give the identical result for the radiation intensity.

Barichello and Siewert[20] used the discrete ordinate method to develop the solution to a class of non-gray problems in the theory of radiative transfer. The model considered allows for scattering with completely frequency redistribution (completely non-coherent scattering) and continuum absorption. Some numerical aspects and the use of this discrete ordinates solution were discussed. The classical X and Y functions were also computed by using the solution.

Barichello and Siewert[19] used the discrete-ordinates method to develop a solution to a class of polarization problems in the theory of radiative transfer.

Mitra and Churnside[149] estimate the optical signal for an oceanographic lidar from the one-dimensional transient (time dependent) radiative transfer equation using the discrete ordinates method.

Viik[215] presented accurate numerical solution for both the internal and external radiation field of a nonconservative plane-parallel semi-infinite Rayleigh-Cabannes scattering atmosphere using the method of discrete -ordinates of Chandrasekhar.

Sharp and Allen[184] solved the time-dependent transport equation for both rod and plane geometries using the discrete-ordinate method.

Barichello, Garcia and Siewert[16] developed a full-range orthogonality relation and used it to construct the infinite-medium Green's function for a general form of the discrete ordinates approximation to the transport equation in plane geometry. The Green's function is then used to define a particular solution that is required in the solution of inhomogeneous version of the discrete ordinates equations.

Siewert[187] used a discrete ordinate method along with elementary numerical Linear Algebra technique to establish an accurate solution for all components in a Fourier representation of the Stokes vector basic to the scattering of polarized light.

Siewert[188] used a discrete ordinate method along with elementary numerical Linear Algebra technique to establish an efficient and accurate solution to a class of multigroup transport problems for which up scattering is an important aspect of the model. The problems considered are defined for finite-plane parallel media, and anisotropic scattering from any group to any group is included in the formulation.

Galinsky[74] modified the forward discrete ordinate method, based on an expansion of the direct beam source term, similar to the gradient correction method used already by Galinsky[75] for diffusion approximation to include effects of a weak inhomogeneity of a medium.

Elaloufi, Carminati and Greffet[73] solved the time-dependent radiative transfer equation in the space-frequency domain by using a standard discrete-ordinate method to study the propagation of light pulses through scattering media.

Aboughantous[4] revisited the structure of the discrete ordinates set with a new approach and built a new set based on Gauss-Legendre (GL) quadrature. The new set comprises only positive direction cosines for all specific intensities. The new set of discrete ordinates enabled transcribing the transfer equation into a complete set of equations i.e. into the set of equations which is closed (N equations in N unknowns) and conservative (the solution satisfies the conservation relation).

A non-physical feature of the transfer equation in spherical geometry is that it is singular at the center of the sphere. This feature is intrinsic to the transfer equation in its native form as an abstract mathematical equation in spherical geometry. He cured this problem by an appropriate transformation of the frame of reference.

The solutions for the discrete ordinates equations and the diffusion equations are presented in two forms: continuous in r and end-points form, and tested quantitatively. The end-points solution is particularly attractive in numerical computations in optically thick media.

Spurr, Kurosu and Chance[195] carried out an internal perturbation analysis of the complete discrete ordinate solution in a plane-parallel multi-layered multiply-scattering atmosphere.

The discrete ordinates method fails in treating specular reflection at the boundary because the quadratures on the sphere do not assume any analytic representation of a function under integration and, therefore, the intensity of the specularly reflected beam is undetermined. Rukolaine and Yuferev[172] presented a new approach to the construction of quadrature schemes to solve this problem.

Spurr[194] solved the radiative transfer equation in a multi-layer multiply-scattering atmosphere using discrete ordinate method and evaluated explicitly all the partial derivatives of the intensity field.

Lemonnier and Dez[137] derived the radiative transfer equation (RTE) in both conservative and non-conservative forms for a plane slab made of an absorbing-emitting material with a continuous transverse variation of the refractive index. The RTE was set in a form which displays an angular redistribution term analogous to what appears in curvilinear media with uniform index. Numerical solutions were provided by means of discrete ordinates method.

Barichello, Rodrigues and Siewert[17] used a discrete ordinate method along with Hermite cubic splines and Newton's method to solve a class of coupled nonlinear radiation-conduction heat transfer problems in a solid cylinder.

Ray effects and false scattering are two major sources of inaccuracy of the discrete ordinates method. High order schemes may reduce false scattering, and the modified discrete ordinates method may mitigate ray effects. Although the origin of the two errors is different, there is an interaction between them, since they tend to compensate each other. Coelho[48] showed that decreasing of one of the errors while keeping the other unchanged in the standard discrete ordinates method may decrease the solution accuracy because the compensation effect disappears and the modified discrete ordinates method does not decrease ray effects caused by sharp gradients of the temperature of the medium. He proposed a new version that successfully mitigates ray effects in that case.

Coelho[49] applied the discrete ordinates and discrete transfer methods to the numerical simulation of radiative heat transfer from non-gray gases in three-dimensional enclosures.

Qin, Jupp and Box[166] extended an accurate and efficient algorithm, the discrete ordinate method, to solve the radiative transfer problem of plane parallel scattering atmosphere illuminated by a parallel beam, an idealized case of the sun, from above the atmosphere so that radiative problem of more general sources such as parallel surface source that illuminated with a parallel beam in any direction and any vertical position, and general surface sources that illuminate continuously in a hemisphere, can be solved.

Lacroix, Parent, Asllanaj, and Jeandel[134] solved the radiative heat transfer equation (RTE) using a S_8 quadrature and a discrete ordinate method.

Collin, Boulet, Lacroix and Jeandel[52] used 2-D discrete ordinate method, formulated by Lacroix, Parent, Asllanaj, and Jeandel,[134] to solve the radiative transfer equation to stimulate the radiation propagation from the heat source through water spray curtains.

van Oss and Spurr[209] derived the homogeneous and particular solutions for the general discrete-ordinate model, noting especially the factor of 2 reduction that allows analytic solutions to be written down for the 4=6 stream cases. The equation of radiative transfer is solved for a vertically inhomogeneous atmosphere by assuming a division into a number of optically uniform adjacent sub-layers.

Silva, Andraud, Charron, Stout and Lafait[56] presented the model based on the resolution of the radiative transfer equation by the discrete ordinate method in steady state domain.

Chalhoub[37] used the discrete-ordinates method to solve radiative transfer problems, in plane-parallel media and presented a generalized analytical discrete-ordinates model for solving single and multi-region problems in which internal sources, reflecting and emitting boundaries, incident distribution of radiation on each surface and a beam incident on one surface are included.

Coelho[50] proposed a new modified discrete ordinates method (NMDOM) to overcome the shortcomings of the standard discrete ordinate method (SDOM) and modified discrete ordinate method (MDOM). The standard discrete ordinates method suffers from two major sources of inaccuracy, the ray effects and false scattering. False scattering were significantly reduced using high order discretization schemes, while ray effects originated from abrupt changes of wall temperatures were mitigated by modified discrete ordinates method (MDOM).

Qin, Box and Jupp[167] presented two methods that can be used to derive the particular solution of the discrete-ordinate method for an arbitrary source in a plane-parallel atmosphere, which allows us to solve the transfer equation 1218 % faster in the case of a single beam source and is even faster for the atmosphere thermal emission source.

An equation of radiative transfer is more accurate than a diffusion equation for the widely employed frequency-domain case. Ren, Abdoulaev, Bal, and Hielscher[168] presented an algorithm by discretizing the equation of radiative transfer by a combination of discrete-ordinate and finite-volume methods that provides a frequency-domain solution of the equation of radiative transfer for heterogeneous media of arbitrary shape to present two numerical simulations.

Biological tissue is a turbid medium that both scatters and absorbs photons. An accurate model for the propagation of photons through tissue can be adopted from transport theory, and its diffusion approximation can be applied to predict the imaging signal around the biological tissue (vide Cong, Wang and Wang[54]).The use of short pulse laser for minimally invasive detection scheme has become an indispensable tool in the technology arsenal of modern medicine and biomedical engineering. Trevedi, Basu and Mitra[204] used a time-resolved technique to detect tumors/ inhomogeneities in tissue by measuring transmitted and reflected scattered temporal optical signals when a short pulse laser source is incident on tissue phantoms and validated the experimental measurements obtained by a parametric study involving different scattering and absorption coefficients of tissue phantoms and inhomogeneities, size of inhomogeneity as well as the detector position with a numerical solution of the transient radiative transport equation obtained by using discrete ordinates method.

Considering the processes of the solar radiation extinction in deep layers of the Venus atmosphere in a wavelength range from 0.44 to 0.66 μm and using the spectra of the solar radiation scattered in the atmosphere of Venus at various altitudes above the planetary surface measured by the Venera-11 entry probe in December 1978 as observational data, Maiorov, Ignat'ev, Moroz, Zasova, Moshkin, Khatuntsev and Ekonomov[144] solved the problem of the data analysis by selecting an atmospheric model applying the discrete-ordinate method in calculations.

Elaloufi and Arridge[72] used the discrete ordinate method to solve the radiative transfer equation (RTE) for slab geometry, taking into account rigorously the interfaces. The important role of interfaces in the ballistic regime and also the diffuse regime were underlined.

Radiative transfer theory considers radiation in turbid media and is used in a wide range of applications. Edström[70] outlined a problem formulation and a solution method for the radiative transfer problem in multilayer scattering and absorbing media using discrete ordinate model geometry.

Pimenta de Abreu[160] derived nonstandard layer-edge conditions for efficient solution of multislab atmospheric radiative transfer problems. Defining a local radiative transfer problem on the lowermost layer of a multislab model atmosphere , he considered a standard discrete ordinates version of this local problem.

Mishra Roy and Misra[148] suggested a new quadrature scheme to make discrete ordinate method computationally more attractive by making the complicated mathematics for determination of direction cosines and weights simple and lucid.

Zorzano, Mancho and Vázquez[234] considered the radiation transfer problem in the discrete-ordinate, plane-parallel approach and introduced two benchmark problems with exact known solutions and show that for strongly non-homogeneous media the homogeneous layers approximation can lead to errors of 10% in the estimation of intensity.

Pozzo, Brandi, Giombi, Baltanás and Cassano[165] determined volumetric optical properties (spectral absorption, scattering and extinction coefficients) of differently expanded narrow-path fluidized beds (FB) of photocatalyst obtained by plasma-CVD deposition of titania onto quartz sand, relavent for photoreactor design purposes by using an unidirectional and unidimensional model for solution of the radiative transfer equation(RTE). They used two simplified approaches: a Kubelka-Munk type of solution by which the RTE was transformed into a pair of ordinary differential equations and a discrete ordinate method by which the complete RTE was transformed into an algebraic system .

Hua, Flamant, Lu and Gauthier[93] developed a model to predict the bed-to-wall radiative heat transfer coefficient in the upper dilute zone of circulating fluidized bed (CFB) combustors and solved the radiative transfer equation by the discrete ordinate method.

Li[141] developed an easy-to-use and comprehensive method, named multi-rays method, on the basis of discrete ordinates scheme with (an) infinitely small weight(s) to calculate total, direct and medium intensities in arbitrary specified directions. In doing this, for each of the specified directions, three identical discrete directions with infinitely small weights are employed to represent the three intensities.

Chalhoub[38] used discrete ordinates method to solve uncoupled multi-wavelength radiative transfer problems in multi-region plane-parallel media. They presented a generalized analytical discrete-ordinates formulation that includes internal sources, as well as reflecting and emitting boundaries, incident distribution of radiation on each surface and a beam incident on one surface, as boundary conditions.

Trabelsi, Sghaier and Sifaoui[203] used a modified discrete ordinates method in a spherical participating media. By breaking up the radiative intensity into two components of which one component was traced back to the enclosure's source, called direct intensity and the other component was rather traced back to the contribution of the medium itself, called diffused intensity, they solved the diffuse RTE numerically using discrete ordinates method.

Klose, Ntziachristos and Hielscher[125] applied the ERT (equation of radiative transfer)-based forward model for light propagation in biological tissue using a finite-difference discrete-ordinates method.

An, Ruan, Qi and Liu[9] proposed a finite element method for simulation of radiative heat transfer in absorbing, emitting and anisotropic scattering. They developed the simulation on the basis of discrete ordinates method and the theories of finite element.

Box and Qin[29] presented an extension to the standard discrete-ordinate method (DOM) to consider generalized sources including: beam sources which can be placed at any (vertical) position and illuminate in any direction, thermal emission from the atmosphere and angularly distributed sources which illuminate from a surface as continuous functions of zenith and azimuth angles.

Banerjee, Ogale and Mitra[15] experimentally determined the information content of lightning optical emissions through clouds in the laboratory and they compared the experimental results with a transient radiative transfer formulation solved using the discrete ordinate method.

To perform a comprehensive experimental and numerical analysis of the shortpulse laser interaction with a tissue medium with the goal of tumor-cancer diagnostics, Pal, Basu, Mitra, and Vo-Dinh[158] formulated a numerical model using the discrete ordinates technique for solving the radiative transport equation associated with the problem.

Rozanov and Kokhanovsky[171] converted Siewert's[187] form of vector radiative transfer equation for a homogeneous isotropic symmetric plane-parallel light scattering slab to a nice form in which the discrete ordinate technique was used comfortably to solve.

Spurr, Haan, van Oss and Vasilko[193] demonstrated that the discrete-ordinate radiative transfer(RT) equations may be solved analytically in a multi-layer multiple scattering atmosphere in the presence of rotational Raman scattering (RRS) treated as a first-order perturbation

Coelho[51] presented a comparison of discretization schemes required to evaluate the radiation intensity at the cell faces of a control volume in differential solution methods of the radiative transfer equation and compared several schemes developed using the normalized variable diagram and the total variation diminishing formalisms along with essentially non-oscillatory schemes and genuinely multidimensional schemes. The calculations were carried out using the discrete ordinates method, but the analysis is found to be equally valid for the finite-volume method.

Kokhanovsky[127] carried out the numerical calculations of the halo brightness and contrast using the discrete ordinate method of the integro-differential radiative transfer equation solution for a typical phase function of crystalline clouds exhibiting halo at 22 and $46A^0$.

Abhiram, Deiveegan, Balaji and Venkateshan[3] presented a multilayer differential discrete ordinate method to solve the radiative transfer equation for an absorbing, emitting and scattering inhomogeneous plane parallel medium.

1.3.2 Works done on interlocking problems.

From the observational point of view, Houtgast[92] first noticed the importance of non coherent scattering for the interpretation of strong absorption lines. He showed that the behaviour of strong fraunhorfer lines across the Sun's disc can only be interpreted under the assumption of non coherent scattering. Spitzer[192] discussed the general characteristics of non coherency and concluded from physical arguments that non coherent scattering is important in stellar atmosphere and coherent scattering is comparatively rare there in. Theoretical treatments of the problem for the case of interlocking have been given by Spitzer[191] and Woolley.[228]

Eddington[68] derived the general method of calculating the contour of an absorption line when the number of atoms in the upper state has been disturbed by interlocking.

Woolley[226] discussed a case of two interlocked absorption lines. A direct solution was made of the simultaneous differential equations, obtained by making simplifying approximations as nearly as possible similar to those ordinarily made in the treatment of principle lines, and it was found that the widths of the lines were not appreciably affected by the interlocking.

Woolley[227] considered the case of triplet (or doublet) of lines and made the conclusion that the measurement of line width at the points for which $\frac{H'_\nu}{H} > \frac{7}{10}$, where $H = \frac{1}{4\pi}\int J(\theta)cos\theta d\omega$ and $H'_\nu = \frac{1}{4\pi}\int J(\theta)'cos\theta d\omega$ in which $J(\theta)'d\nu$ is the flow of radiation of frequency ν to $\nu + d\nu$ within the line in a direction making angle θ with the outward direction, $J(\theta)d\nu$ is the corresponding flow just outside the line and $d\omega$ is the element of solid angle, can be interpreted exactly as if the lines were not interlocked with each other.

Woolley and Stibbs[229] considered the problem of interlocking without redistribution in details and gave the integro-differential equations for triplets along with an approximate solution obtained by applying Eddington's approximation. To illustrate the effect of interlocking,they calculate the quantity

$$\frac{1}{2}\omega = \int_{\eta=\infty}^{0} (1 - r)\, d\eta^{-\frac{1}{2}}$$

for doublet and triplet lines in a region of the spectrum where $b = \frac{3}{2}a$, and with $\epsilon = 0$ and drew conclusion that interlocking has an effect on the curve of growth(the relation between the equivalent width and the number of oscillators) which should be appreciable but not markedly so.

Considering the linear form of Planck-function, Busbridge and Stibbs[33] solved the radiative transfer equation for interlocked multiplets by the method of principal of invariance governing the law of diffuse reflection with a slight modification and calculated three hypothetical line profiles for doublets.

Busbridge[30] obtained the solution for interlocked multiplet lines by a mathematical method which was obtained by Busbridge and Stibbs[33] by the principle of invariance.

Miyamoto[150] investigated abnormally high residual intensities and very large Doppler core widths of Infrared Ca II multiplets in solar spectrum. The characteristic features of this multiplets are the metastability of the lower level and the strong interlocking with the resonance H and K lines through the upper level. By virtue of the metastability of the lower level, the nature of the line formation was found closer to absorption rather than scattering. This being combined with the strong interlocking with resonance line, explains an abnormally high residual intensities.

Siewert and Özişik[190] developed a matrix form of the equations of transfer of the lines for the interlocking multiplets of order N from the equation of transfer of Busbridge and Stibbs[33] for the interlocking multiplets of order k by making a suitable substitution and produced a rigorous solution to the equation of transfer for interlocking doublets by the use of the normal modes and the methods of solution introduced by Siewert & Zweifel.[189]

Karanjai[103] profitably used the approximate form of H-function[102] to minimize to a great extent the computational labour that involves in the calculation of H-function

An exact solution of the equation of transfer was given by Das Gupta[59] by his modified form of Wiener-Hopf technique.

Deb[65] used the following various approximate forms of the H-function, studied by Karanjai[102] and Karanjai and Sen[116] to calculate the value of H-function and residual intensities for doublets as well as triplet lines.

Dasgupta and Karanjai[63] solved the radiative transfer equation for interlocked multiplets without redistribution with the Planck-function, linear in τ, by applying Sobolev's probabilistic method.

Chamberlain and Wallace[39] also studied the case of a multiplet with common lower state with the assumption of monochromatic scattering in each line.

Nagirner and Shneivais[155] analyzed the formation of lines with a common upper level in a semi-infinite medium and used an analytical method developed for two-level atom to study the problem of radiation transfer with the assumptions of a Boltzmann distribution of atoms over sublevels of the lower level and of complete frequency distribution of a radiation within each line. They expressed the intensity of the radiation in the lines through the H-function, obtained the asymptotic and approximate equation for the H-function for the Doppler and Voigt coefficients of absorption and calculated the Doppler H-function estimating the accuracy of the asymptotic forms for the case of the 0.1 resonance triplet.

Das Gupta[61] also obtained an exact solution of the transfer equation with the Planck-function, linear in τ, for non-coherent scattering arising from interlocking principal lines by Laplace transform and the Wiener-Hopf technique using a new representation of the H-function obtained by Das Gupta.[60]

Karanjai and Barman[107] solved same problem by using the extension of the method of discrete-ordinates.

Karanjai[105] calculated Mg b line contours with the solution obtained by Dasgupta and Karanjai[63] and showed that his calculated lines have a good agreement with the observation.

Karanjai and Karanjai[114] solved the equation of transfer for interlocked multiplets with the Planck function as a non-linear function of optical depth following the method used by Das Gupta.[61] They considered two non-linear forms of Planck function viz.,

(a). an exponential atmosphere,(vide Degl'Innocenti,[66] equation(1.11)),

(b). an atmosphere (vide Busbridge[30]) in which
$B_\nu(\tau) = B(\tau) = b_0 + b_1\tau + b_2E_2(\tau)$

Deb, Biswas and Karanjai[64] solved the radiative transfer equation for interlocked multiplets with non-linear Planck-function by using the extension of the method of discrete-ordinates and Deb and Karanjai[110] solved the same problem with the help of the the method of Busbridge and Stibbs.[33]

Mukherjee and Karanjai[152] used the double-ordinate spherical harmonic method presented by Wilson and Sen[225] to solve the equation of radiative transfer in the Milne-Eddington model for interlocked doublets. Solutions have been obtained in the first and second approximation in a particular case $r = 1$.