Chapter 7

Conventionality of Simultaneity, Absolute Synchronization and Twin Paradox in the Non-abrupt Turn-around Scenario

7.1 Introduction

This chapter also deals with the ordinary twin paradox but with a difference. Since the advent of SR hundreds of papers on twin paradox have appeared in the literature. Barring a few, most of the authors have asserted that there is no "paradox". Indeed once "paradox" gets resolved the term "paradox" should never be used in the concerned context. Yet almost all the authors dealing with the topic concerning twins continue to refer to it as the twin paradox. There is no point however to give excuses for using the phrase "twin paradox" or raising this sementic issue at the fag end of this thesis. But there is a reason. If the counter-intuitive issue had really long been resolved it would not remain still one of the most enduring puzzles of physics.

As for example in 1918, Einstein came up with the resolution of the paradox, in terms of his own general theory of relativity; on the other hand say about eighty years after Einstein's 1905 paper an article appears in a reputed journal with the provocative title "Twin paradox: A complete treatment from the point of view of each twin"[1] indicating as if so far complete treatment was not available! Some relatively recent papers[2, 3, 4] even challenge SR in the context of the paradox. Unnikrishnan[2] claims to have shown "Logical and physical flaws" in Einstein's analysis and further remarked in some context that "special relativity principle formulated originally for physics in empty space is not valid in the matter filled universe." Some of Unnikrishnan's arguments however was promptly challenged by Gron[5] as it was rightly shown that there was serious error in the former's analysis.

A few years ago (2006) there had been a considerable hype in the media stating for example "...scientist solves Einstein's twin paradox" referring to a paper that appeared in J.of theo.phys. Students of Physics would be puzzled since it would mean that twin paradox was being solved for the first time! What we would mean to say is that in spite of various authors having remarked that clock paradox or twin paradox have long been solved the debate on the issue still continues. Pesic[6] had rightly remarked "... numerous papers continue to illuminate the problem. Alternative situations have emerged that surely would have delighted Einstein".

One of such alternative situations concerns the so-called non-abrupt turn-around scenario where the rocket bound twin B during turn-around gets finitely decelerated. Let us consider this now.

In the twin problem, as said earlier, the traveller Barbara (B) faces some acceleration during turn-around by virtue of which she lies in an accelerated frame, whereas Alex (A), the stay at home one remains in an inertial frame. Since Lorentz transformation (LT) holds only between two inertial frames, one cannot use LT to compute the ages from the perspective of the accelerated observer B. In case of abrupt turn-around however, one can work with LT by assuming that the accelerated motion of B can be decomposed into two uniform motions for her forward and return journeys. But a correction is to be introduced in the calculation of time due to change of simultaneity when B jumps from one inertial frame to another. The main problem arises when one considers a more realistic situation where the acceleration faced by the traveller is finite. Here the question arises can one deal with the situation in the same way when the turn-around is non-abrupt (finite acceleration).

It is often believed that one needs general relativity (GR) to deal with the situation[1, 7, 8, 9]. For example Perrin in his paper "Twin paradox: A complete treatment from the point of view of each twin"[1] set up a round-trip situation assuming finite (proper) acceleration of B and worked through from the perspectives of each twin. In particular the author obtained the time elapsed on the clock of the stay-at-home twin during the periods of acceleration by solving the Einstein's field equation and the geodesic equation in the frame of reference of the traveller twin. The calculation is quite involved and one may have the feeling that the result thus obtained cannot be obtained purely from special relativistic considerations. However we will show below (Sec.7.3) that the A-clock time can be calculated fully in the context of SR provided one synchronizes the coordinate clocks suitably in the frame of reference of the traveller twin.

To understand this synchronization issue it is worthwhile to first discuss briefly the conventionality of simultaneity thesis (CS-thesis)[10, 11, 12, 13, 14, 15, 16, 17] of SR. According to CS-thesis[1], in relativity theory various conventions can be adopted in synchronizing distant clocks in a given inertial frame. Einstein's synchronization convention which asserts that the one-way-speed (OWS) of light is isotropic and is equal to its two-way-speed (TWS) 'c' is just one among these possible modes of synchronization. Now if we can obtain a synchronization scheme where there is no change of simultaneity when one shifts from one inertial frame to another, then in the twin problem the effect due to the change of synchronization of the coordinate clocks of B can be eliminated. In such a mode of synchronization, time should be absolute. The proposed synchronization scheme is known as absolute synchronization. Indeed in a previous chapter we have seen that correction of time due to the change of simultaneity in the traveller's frame during turn-around suggests how naturally absolute synchronization emerges as the best synchronization in the context of acceleration. In the relativistic world if the distant clocks in an inertial frame Σ are synchronized according to absolute synchronization, then the

[1] A comprehensive review of the thesis is presented in a paper by Anderson,Vetharaniam and Stedman[18]Also one can study the papers of Ref[12, 13, 19]

transformation equation thus generated is given as

$$x = \gamma(x_0 - ut_0),$$
$$t = \gamma^{-1}t_0, \tag{7.1}$$

where u is the relative velocity of Σ with respect to the preferred frame Σ_0 and $\gamma = (1 - u^2/c^2)^{-1/2}$ is the usual Lorentz factor. The transformation Eq.(7.1) is known as the Tangherlini transformation (TT)[12, 13, 14, 20, 21] or often inertial transformation[15].

Coming back to the context of the twin paradox, when the traveller B faces some finite acceleration, she continuously changes her inertial frames. In case of abrupt turn-around however, since as we have already mentioned, one can easily find the correction term that is to be added for the change of simultaneity between these two inertial frames. On the other hand, for finite acceleration one will need to find ways to incorporate the effects due to change of simultaneity in a continuous manner. However any endeavour towards this end is not witnessed in the literature.

We propose to deal with the situation by choosing the absolute synchronization from the onset so that one does away with the need to incorporate corrections due to change of simultaneity that we discussed earlier. Thus, using TT we will be able to recover the expression for the A-clock time (that have been obtained by Perrin[1] in a complicated manner by solving Einstein's field equations and the geodesic equations) from B's perspective.simply from the consideration of SR alone[2]. To our knowledge no such *"complete treatment"* of the paradox involving finite acceleration in the context of SR is available in literature. In this paper we shall present such an analysis. We hope that this work will provide a deeper understanding regarding the subtleties of the problem. Perhaps this will also put a stop to the long controversy

[2]It may be noted that the context of SR does not change for merely choosing a different (coordinate) clock synchronization scheme.

regarding the necessity of the involvement of GR in the resolution of the twin problem. Before presenting our main calculations we shall give in the next section a brief review of Perrin's[1] analysis. Sec.7.4 will be devoted to summarizing our results and conclusions.

7.2 A Brief Review of Perrin's Paper

Robert Perrin in his paper "Twin paradox: A complete treatment from the point of view of each twin" [1] divided the round-trip of the traveller into acceleration phases and uniform velocity phases. He divided the round-trip into six phases as follows:
Phase 1 is the acceleration phase where the traveller leaves the earth with a proper acceleration g until her velocity increases from 0 to u relative to the earth twin.
Phase 2 is the uniform velocity phase where the traveller covers a distance L_0 with the constant velocity u.
Phase 3 is the deceleration phase where the traveller moves with an acceleration '$-g$' and turns around.
Phase 4 is the moment after turn-around where the traveller suffers the same amount of acceleration and travels toward earth with velocity $-u$.
Phase 5 is again the constant velocity phase where the traveller moves toward earth with velocity $-u$.
Phase 6 is the final acceleration phase where the traveller suffers the acceleration g and ends up at rest on the earth.

From special relativistic consideration it has been calculated that the time elapsed on earth for the entire round-trip of the traveller is

$$T = 4\frac{\gamma u}{g} + 2\frac{L_0}{u}. \tag{7.2}$$

The above result is obtained in detail in the next section.

To compute the time recorded in the traveller's clock, the stay-at-home sibling uses the standard time dilation result following LT. This time comes out to be

$$\overline{T} = \frac{2c}{g} ln(\frac{1 + u/c}{1 - u/c}) + 2\gamma^{-1}\frac{L_0}{u}. \tag{7.3}$$

From the point of view of the traveller twin the author similarly divided the trip into six phases and interpreted the acceleration phases as the turning on of a gravitational field along the x-direction. From the perspective of the traveller the time elapsed on her clock is the same as that calculated by the earth-bound observer and that elapsed on earth during the uniform velocity phases are obtained using the length contraction formula. But in order to calculate the time elapsed on earth *during the accelerated segments* of the journey Perrin solved the gravitational field equations and the geodesic equations of motion. He claimed that these calculations cannot be completed without involving GR.

From his quite involved GR analysis he showed that the time elapsed on earth (as calculated by the traveller) during the acceleration phases comes out to be

$$t_I + t_{III} = 2\frac{\gamma u}{g} + \frac{L_0 u}{c^2}, \tag{7.4}$$

where t_I and t_{III} are the time elapsed on earth during the acceleration phases I and III respectively.

With this value for the earth-clock time, perspective of both the twins matches, thus resolving the paradox. In the next section we shall show that the time elapsed on the earth clock can be obtained from special relativistic considerations alone by solving some simple equations.

7.3 Tangherlini Transformation and the Twin Paradox

As mentioned in Sec.7.1 clock synchronization is a matter of convention and different conventions yield different sets of transformation equations. Selleri suggested that in general the space-time transformation between the preferred frame Σ_0 and an arbitrary frame Σ with relative velocity u is given as

$$\begin{aligned} x &= \gamma(x_0 - ut_0), \\ t &= \gamma^{-1}t_0 + \epsilon(x_0 - ut_0). \end{aligned} \tag{7.5}$$

which represents a set of theories equivalent to SR. The free parameter ϵ depends on the simultaneity convention adopted in the frame Σ and is a function of the relative velocity u. For different values of ϵ we get different sets of transformation equations. For example, for standard synchrony

$$\epsilon = -\beta\gamma/c = -u\gamma/c^2. \tag{7.6}$$

For this value of ϵ Eq.(7.5) reduces to LT. From Eq.(7.5) one can also obtain the OWS' of light along the positive (c'_+) and negative (c'_-) x-directions in the frame Σ as

$$\begin{aligned} \tfrac{1}{c'_+} &= \tfrac{1}{c} - \left[\tfrac{u}{c^2} + \epsilon\gamma^{-1}\right], \\ \tfrac{1}{c'_-} &= \tfrac{1}{c} + \left[\tfrac{u}{c^2} + \epsilon\gamma^{-1}\right] \end{aligned} \tag{7.7}$$

Putting the value of ϵ given by Eq.(7.6) one can easily check that the OWS' of light in Σ comes out to be the same as its TWS c, as is demanded by the standard synchrony.

For $\epsilon = 0$ Eq.(7.5) reduces to

$$\begin{aligned} x &= \gamma(x_0 - ut_0), \\ t &= \gamma^{-1}t_0, \end{aligned} \tag{7.8}$$

which is the same as Eq.(7.1). So for $\epsilon = 0$ we get the Tangherlini transformation (TT) representing the relativistic world with absolute synchrony.

The inverse of TT can be calculated as

$$\begin{aligned} x_0 &= \gamma^{-1}(x + \gamma^2 ut) = \gamma^{-1}(x - vt), \\ t_0 &= \gamma t, \end{aligned} \tag{7.9}$$

where v is the relative velocity of Σ_0 with respect to Σ which is given as

$$|v| = \left|-\gamma^2 u\right| \tag{7.10}$$

To obtain the A-clock time from the perspective of B we consider that the stay-at-home twin (A) is attached to the frame Σ_0 and the traveller (B) to Σ. We follow closely the approach suggested by Perrin[1] and hence divide the round-trip similarly into six phases—two uniform velocity and four acceleration phases.

Below we shall compute the ages from the perspective of both the twins A and B step-wise.

Step 1:

Before calculating the ages from the individual perspectives let us first write down the time dilation formulas and the length contraction formulas.

Time Dilation Formulas:

The transformation equations (7.1) and (7.9) suggests that the twin A attached to the preferred frame observes time dilation whereas the other twin B observes the time to be contracted. So from the perspective of A we get a time dilation formula (TDF) but from the perspective of B we obtain a time contraction formula (TCF), which are

$$TDF: \quad \Delta\tau_B(A) = \gamma^{-1}\Delta t_A(A), \tag{7.11}$$

$$TCF: \quad \Delta\tau_A(B) = \gamma\Delta t_B(B). \tag{7.12}$$

In the above $\Delta\tau_X(Y)$, (where X and Y stands for both A or B) denotes the time recorded in X-clock for two events which occurred at the position of X, as inferred by another observer Y, drawn from Y's coordinate clock record $\Delta t_Y(Y)$ and the relevant TDF (in case Y represents A) or TCF (in case Y represents B). Also the interval $\Delta\tau_X(Y)$ is based on a single clock measurement and hence it refers to the proper time of X.. So the τ-symbol stands for the proper time. The interval $\Delta t_Y(Y)$, as mentioned earlier, is the coordinate time of Y and it gives the difference in clock readings for the same set of events (which occurred in X-frame) measured by two spatially separated (synchronized) coordinate clocks stationary with respect to the frame of reference attached to Y. However for the round trip of the observer X, only a single clock (of Y) is required to measure the interval $\Delta t_Y(Y)$, but even then it still represents the coordinate time, so t-symbol is used instead of τ.

Length contraction formulas:

Similarly from the set of equations (7.1) and (7.9) we obtain two length contraction formulas, more correctly contraction from the perspective of A and dilation from that of B which are

$$LCF1: \quad L_B(A) = \gamma^{-1} L_B(B), \tag{7.13}$$

$$LCF2: \quad L_A(B) = \gamma L_A(A). \tag{7.14}$$

Where $L_A(A)$ and $L_B(B)$ are the rest lengths of rods in the frames of A (i.e Σ_0) and B (i.e Σ) respectively. The lengths $L_B(A)$ and $L_A(B)$ are the corresponding lengths noted by the observers in the other frames (A and B respectively). In the twin problem the only distance of interest is that of the point of turn-around from A which is clearly a rest length in A i.e. $L_A(A) = L_0$ (say). Hence the relevant LCF is that given by Eq.(7.14), which is

$$L = \gamma L_0, \tag{7.15}$$

with $L = L_A(B)$.

Perspective of A:

Step 2:

Phase 1, as said before, is the acceleration phase where the acceleration of Σ-frame is

$$g_u = \frac{du}{dt_0} = (1 - \frac{u^2}{c^2})^{3/2} g, \tag{7.16}$$

or

$$dt_A(A) = dt_0 = \frac{du}{(1 - \frac{u^2}{c^2})^{3/2} g}. \tag{7.17}$$

The phase 1 ends when B attains a velocity u with respect to A, hence on integrating the right hand side from velocity 0 to u we can get the total time elapsed in A-clock during phase 1

$$\Delta t_A(A)_1 = \frac{1}{g} \int_0^u \frac{du}{(1 - \frac{u^2}{c^2})^{3/2}} = \frac{\gamma u}{g} \tag{7.18}$$

Note that in the notation $\Delta t_A(A)_1$, the subscript 1 denotes the phase 1.

Now the entire journey is composed of four such acceleration phases, i.e, phases 1,3,4 and 6. Hence considering all these phases the total time elapsed is

$$4\Delta t_A(A)_1 = 4\frac{\gamma u}{g}. \tag{7.19}$$

Step 3:

The remaining two phases i.e 2 and 5 are the uniform velocity phases where the twin B traverses a distance L_0 with the constant velocity u. Hence the total time elapsed during these phases is

$$2\Delta t_A(A)_2 = 2\frac{L_0}{u}. \tag{7.20}$$

Step 4:

Considering all the six phases the total time measured in A-clock comes out to be

$$\Delta t_A(A) = 4\frac{\gamma u}{g} + 2\frac{L_0}{u}. \tag{7.21}$$

Step 5:

The time recorded in B-clock as observed by A can be obtained by applying the TDF given by Eq.(7.11). For the uniform velocity phases this formula can be used directly but for the acceleration phases the speed is not constant and hence the TDF is differentially true. For the acceleration phases therefore we have

$$dt_B(A) = \gamma^{-1} dt_A(A), \tag{7.22}$$

On integrating the above equation we obtain the time recorded in B-clock during each acceleration phase as

$$\Delta t_B(A)_1 = \int_0^u \gamma^{-1} dt_A(A) = \int_0^u \frac{du}{(1-\frac{u^2}{c^2})g} = \frac{c}{2g} ln(\frac{1+u/c}{1-u/c}). \tag{7.23}$$

Step 6:

For the uniform velocity phases the calculation is simple and the time recorded in B-clock as computed by A is

$$\Delta t_B(A)_2 = \gamma^{-1} \Delta t_A(A)_2 = \gamma^{-1}\frac{L_0}{u}. \tag{7.24}$$

Step 7:

The total time recorded in B -clock as observed by A thus comes out to be

$$\Delta t_B(A) = 4\Delta t_B(A)_1 + 2\Delta t_B(A)_2 = \frac{2c}{g} ln(\frac{1+u/c}{1-u/c}) + 2\gamma^{-1}\frac{L_0}{u}. \tag{7.25}$$

Step 8:

From Eqs. (7.21) and (7.25) the differential aging from the point of view of A can be calculated as

$$\delta t(A) = 2\frac{L_0}{u}(1-\gamma^{-1}) + 4\frac{\gamma u}{g} - \frac{2c}{g} ln(\frac{1+u/c}{1-u/c}). \tag{7.26}$$

Perspective of B:

Step 9:

From the point of view of the traveller B the journey can be similarly divided into six phases. The phase I is as before the acceleration phase. From the principle of equivalence one may interpret this as the turning on of a gravitational field in the $-x$ direction until the earth-bound twin attains a velocity $-v$. During this phase the acceleration experienced by the earth-bound twin is

$$g_v = \frac{dv}{dt} = (1 + \frac{4v^2}{c^2})^{1/2} g, \tag{7.27}$$

or

$$dt_B(B) = dt = \frac{dv}{(1 + \frac{4v^2}{c^2})^{1/2} g}. \tag{7.28}$$

Hence the time elapsed in the B-clock $\Delta t_B(B)_I$ can be obtained by integrating the right hand side between speeds 0 and v, thus giving

$$\Delta t_B(B)_I = \frac{1}{g} \int_0^v \frac{dv}{(1 + \frac{4v^2}{c^2})^{1/2}} = \frac{c}{2g} \ln \left[\frac{2v}{c} + \left(1 + \frac{4v^2}{c^2} \right)^{1/2} \right]. \tag{7.29}$$

Replacing v by $\gamma^2 u$ in the above equation gives

$$\Delta t_B(B)_I = \frac{c}{2g} ln(\frac{1 + u/c}{1 - u/c}). \tag{7.30}$$

which is the same as $\Delta t_B(A)_1$ given by Eq.(7.23).

Step 10:

On the completion of phase I, phase II starts where A moves with a constant velocity $-v$ and covers a distance $L = \gamma L_0$. The time in B-clock during this phase is thus

$$\Delta t_B(B)_{II} = \frac{L}{|-v|} = \frac{\gamma L_0}{\gamma^2 u} = \frac{\gamma^{-1} L_0}{u}, \tag{7.31}$$

where we have made use of Eq.(7.10).

Step 11:

During phase III, the gravitational field is again turned on as in phase I but now in the opposite sense unless A comes to rest and reverses his direction. The return trip of A is same as the outward trip with two acceleration and one uniform velocity phases. The total time elapsed in B-clock (considering all these phases) thus comes out to be

$$\Delta t_B(B) = 4\frac{c}{2g} ln(\frac{1+u/c}{1-u/c}) + 2\frac{\gamma^{-1}L_0}{u} = \Delta t_B(A). \tag{7.32}$$

Step 12:

In this step we shall compute the time elapsed in A-clock as observed by B during the acceleration phase I (or during each acceleration phase). Note that during these phases, as said earlier, a gravitational field is turned on, so Perrin[1] claimed that GR is required for the calculations to be performed. In contrast we shall do the calculation using SR alone.

To obtain this time assuming absolute synchronization we use the TCF (Eq.(7.12)). During phase I this time is

$$\Delta t_A(B)_I = \int_0^v \gamma dt_B(B) = \frac{1}{g}\int_0^v \frac{dv}{\sqrt{1-u^2/c^2}\sqrt{1+4v^2/c^2}} \tag{7.33}$$

Solving the equation and writing v in terms of u, we get

$$\Delta t_A(B)_I = \frac{\gamma u}{g}. \tag{7.34}$$

Step 13:

The time elapsed in A-clock during the uniform velocity phase is

$$\Delta t_A(B)_{II} = \gamma \Delta t_B(B)_{II} = \frac{L_0}{u}. \tag{7.35}$$

Step 14:

The total time elapsed in A-clock as calculated by B is thus

$$\Delta t_A(B) = 4\Delta t_A(B)_I + 2\Delta t_A(B)_{III} = 4\frac{\gamma u}{g} + 2\frac{L_0}{u} = \Delta t_A(A). \tag{7.36}$$

On comparison of Eqs.(7.32)and (7.36) we can see that the differential aging from the perspective of B comes out to be the same as that from the perspective of A

$$\delta t(B) = \delta t(A). \tag{7.37}$$

The perspectives of both the twins match, thus dissolving the paradox completely.

7.4 Summary

The present analysis shows that the whole calculation can be performed considering SR alone. Even for finite acceleration, during the acceleration phases one can easily obtain the A-clock time fully in the context of SR if the clocks are synchronized following absolute synchronization convention. To our knowledge the proposed analysis is the first one to treat the twin problem involving finite acceleration lying in the realm of SR. This treatment thus makes it evident that for the *complete treatment* of the twin problem whose origin lies in SR only special relativistic effects are sufficient, any reference to GR in the context of the ordinary twin paradox is redundant. The above treatment also shows that the twin problem does not arise at all if the clocks are synchronized according to absolute synchrony. Hence it reveals clearly the fact that at the heart of the standard twin problem lies the adopted synchronization convention, i.e the Einstein synchrony since, as we have seen, in the relativistic world if the clocks are synchronized according the standard synchrony then there develops a contradiction between the predictions of the twins, but if in the same relativistic world the clocks are synchronized following absolute synchrony the contradiction vanishes. As said earlier, in the ordinary twin problem, the rocket-bound sibling is subjected to a turn-around acceleration due to which she shifts from one inertial frame to another. For the clocks synchronized according to Einstein synchrony, the simultaneity being relative, a change in synchrony occurs at this point of turn around, and hence while computing the ages the twin B has to take proper care of this effect, in addition to the time dilation effect[3]. In our earlier works we have shown that the paradox gets resolved if the effect linked with relativity of simultaneity is taken care of. It has been shown there that the reason behind this desynchronization is that the clocks in the first frame of B refuse to be synchronized in her second frame *automatically.* The clocks if left to themselves define absolute synchrony; any other synchrony (with $\epsilon \neq 0$ in Eq.(7.5)) can be achieved only through human intervention i.e artificially. Selleri therefore called absolute synchrony as *"nature's choice"*. The twin paradox thus arises when B misses out this desynchronization effect while calculating the ages. But for the clocks synchronized according to absolute synchrony the desynchronization and hence the paradox does not arise. So this work of ours makes it clear that the twin problem is an artifact of Einstein's mode of synchronization.

[3] The stay-at-home twin A being in an inertial frame can correctly calculate the ages considering only the time dilation effect.

References

[1] R. Perrin, *Am. J. Phys.* **47**(4), 317-319 (1979).

[2] C. S. Unnikrishnan, *Current science* **89**(12), 2009-2015 (2005).

[3] K. Hazra, *Current science* **95**(6), 707-708 (2008).

[4] Subash Kak *Int. J. Th. Phys.* **46**(5), 1424-1430 (2007).

[5] Ø. Gron, *Current science* **95**(6), 707-708 (2008).

[6] P. Pesic, *Euro. J. Phys.* **24**, 585-589 (2003).

[7] , *Eur. J. Phys.* **27**, 885-889 (2006).

[8] C. Møller, *The Theory of Relativity* (Oxford University Press, Oxford, 1952).

[9] Y. Shadmi, *Phys. Educ.* **20**, 33-38 (1985).

[10] H. Reichenbach, *The Philosophy of Space and Time* (Dover, New York, 1958).

[11] A. Grünbaum, *Philosophical Problems of Space and Time* 1st edn (Knopf, New York, 1963).

[12] S. K. Ghosal, D. Mukhopadhyay, and Papia Chakraborty, *Eur. J. Phys.* **15**, 21-28 (1994).

[13] T. Sjödin, *Il Nuovo Cimento B* **51**, 229-245 (1979).

[14] S. K. Ghosal, Papia Chakraborty, and D. Mukhopadhyay, *Europhys. Lett.* **15**(4), 369-374 (1991).

[15] F. Selleri, *Found. Phys.* **26**, 641-664 (1996).

[16] A. Winnie, *Phil. Sci.* **37**, 81-99 (1970).

[17] R. Mansouri, and R. U. Sexl, *Gen. Rel. Grav.* **8**, 497-513 (1977).

[18] R. Anderson, I. Vetharaniam and G. E. Stedman, *Phys. Rep.* **295**, 93-180 (1998).

[19] M. Redhead and T. A. Debs, *Am. J. Phys.* **64**(4), 384-392 (1996).

[20] S. K. Ghosal, K. K. Nandi; and P. Chakraborty, *Z. Naturforsch* **46**a, 256-258 (1991).

[21] F. R. Tangherlini, *Nuovo cimento suppl.* **20**, 1-86 (1961).