CHAPTER 2

Weighted Unconstrained Problems

In this chapter we shall discuss solution procedures of two facility locaton problems in the unconstrained case with the minimax objective. The distance measure chosen for the purpose is rectilinear. Section 2.1.1 deals with the case in which the weights are symmetric and in secton 2.2.1 the weights have been supposed to be asymmetric.

2.1.1 Solution of a weighted one-centre problem under the L_1 norm

The problem considered in this section now follows

$$\min_{P \in R^2} \max_{i \in I} w_i d(P, P_i) \qquad (1)$$

where $P(x, y)$ is a variable point in the plane, $I = \{1,2,\ldots,n\}$, $P_i(a_i, b_i)$ are given points in the plane belonging to a finite set S, w_i are positive weights, and $d(P, P_i)$, the rectilinear distance from P to P_i, is given by

$$d(P, P_i) = | x - a_i | + | y - b_i |.$$

To solve (1) we first obtain the maximum weighted rectilinear distance of the set S from an arbitrarily chosen point P in the plane and then move P such that the weighted rectilinear distance of it from another point in S is the same as the distance between P and the weighted farthest point but greater than the distance between P and any other point of S -- a criterion to be called primal feasibility hereafter. This distance is then gradually diminished by moving the point, maintaining primal feasibility all the while, along a path determined by the pair of points of S until either at least a third point of S is encountered such that all these points of S are equidistant from the moving point P or no such third point at all exists. We have developed the algorithm by translating these ideas for which we would require certain basic concepts such as an equipolygon, a well-behaved point etc. defined in Section 2.1.2. In order to accomplish the task of gradually reducing this distance until the minimum value of the objective function is achieved we have adopted a solution procedure based on the methods of two dimensional analytic coordinate geometry. The strategy for solving the problem has been explained in detail in Section 2.1.3.

As regards the scope of application of the minimax criterion we might consider locating a new facility, say, a polyclinic or a fire station in a large metropolitan area where the objective is to minimize the maximum rectilinear travel distance of a potential user, weighted by some importance factor which is any positive number quantifying the nature of interaction between the facility and the category of user.

In the recent past Love, Morris and Wesolowsky [55] have made an in-depth study of models concerning layout and location of facilities. The equiweighted rectilinear metric problem has been investigated by Francis [34] and Elzinga and Hearn [31] using geometrical properties. By transforming into an equivalent linear programming problem Francis and

White [36] have developed a solution procedure via linear programming for the weighted version of the location problem. The solution procedure developed by them requires α_{ij} and β_{ij} (pp 384-389, [36]) to be calculated for all the demand points to arrive at a conclusion whereas our method, being essentially an iterative one, requires no more investigation if the point of intersection of two equipolygons lies on the rectangle formed by any two of the points defining the equipolygons as a result of which the remaining equipolygons are excluded from further consideration. Wendel, Hurter and Lowe [77] have given efficient algorithms for finding the set of efficient locations with the L_1 norm for the single facility planar location model. Drezner and Wesolowsky [24] have also extensively studied one centre L_p -distance minimax locational problems. Francis, McGinnis and White [37] and Hansen, Peeters and Thisse [44] have given a method-oriented selective survey of the literature and provided a comparison of the different computationally efficient algorithms. Morris [64] has presented an efficient algorithm for solving the constrained multifacility location problems. For the multifacility minimax location problem we refer the reader to the excellent works of, among others, Drezner [21], Aneja, Chandrasekaran and Nair [2], Ko, Lee and Chang [49], Dearing and Francis [18] and Wesolowsky [78].

2.1.2 Procedure

At the outset we shall try to obtain the locus of a point in R^2 from which the weighted rectilinear distances of two given location points are equal.

Let $P_1(a_1, b_1)$ and $P_2(a_2, b_2)$ be any two points in the plane with associated weights w_1 and w_2 and (x, y) any point from which the weighted rectilinear distances of P_1 and P_2 are the same. Without any loss of generality we may assume $a_1 \leq a_2$ as otherwise we may always interchange the labels of the points. Furthermore, let us suppose $w_1 < w_2$. Equating the weighted rectilinear distances of P_1 and P_2 from (x, y) we get

$$w_1(|x - a_1| + |y - b_1|) = w_2(|x - a_2| + |y - b_2|) \qquad (2)$$

Let us rewrite this equation as $F(y) = G(x)$

$$\text{where} \quad F(y) = w_1|y - b_1| - w_2|y - b_2| \qquad (3)$$

$$\text{and} \quad G(x) = w_2|x - a_2| - w_1|x - a_1| \qquad (4)$$

For a given $x = x_0$ we want to find an y such that $F(y) = G(x_0)$. In other words, we want to determine the points of intersection of the curves $u = G(x_0)$ and $u = F(t)$.

Let us now investigate the nature of the functions defined by (3) and (4).

If $b_1 < b_2$ then the function $u = F(t)$ is continuous everywhere, strictly increasing in $(-\infty, b_1)$ and (b_1, b_2) and strictly decreasing in (b_2, ∞) and having its maximum positive value of $w_1(b_2 - b_1)$ at $t = b_2$ associated with the greater weight. Also $F(t) \longrightarrow -\infty$ as $t \longrightarrow \pm\infty$.

If, on the other hand, $b_1 > b_2$ then the above function strictly increases in $(-\infty, b_2)$ and decreases in (b_2, b_1) and (b_1, ∞) but still has the maximum positive value of $w_1(b_1 - b_2)$ at $t = b_2$.

Since the function represented by (4) has a form identical to that given by (3), it immediately follows that the function $u = G(t)$ is continuous, strictly decreasing in $(-\infty, a_1)$ and (a_1, a_2) and strictly increasing in (a_2, ∞), having the minimum negative value of $-w_1(a_2 - a_1)$ at $t = a_2$. Also $G(t) \longrightarrow +\infty$ as $t \longrightarrow \pm\infty$.

Note: If $w_1 > w_2$ then the curves represented by (3) and (4) will simply exchange their respective forms.

As $\max F(t) > 0$ and $G(a_2) < 0$ it follows from the nature of the curve $u = F(t)$ that it will intersect $u = G(a_2)$ at two points. Hence we can conclude that the set of (x, y) satisfying (2) is not void.

In a similar manner it can be deduced that the curves $u = G(t)$ and $u = F(b_2)$ will intersect at exactly two points, say α and β ($> \alpha$), given by

$$\beta = \left[w_2a_2 - w_1a_1 + w_1(b_2 - b_1)\right]/(w_2 - w_1) \quad \text{and}$$

$$\alpha = \begin{cases} \left[w_2a_2 - w_1a_1 - w_1(b_2 - b_1)\right]/(w_2 - w_1) \text{ if } F(b_2) > G(a_1) \\ \left[w_2a_2 + w_1a_1 - w_1(b_2 - b_1)\right]/(w_2 + w_1) \text{ if } F(b_2) \le G(a_1) \end{cases}$$

provided $a_1 \neq a_2$.

If $a_1 = a_2$ then $\beta = a_1 + w_1(b_2 - b_1)/(w_2 - w_1)$

and $\alpha = a_1 - w_1(b_2 - b_1)/(w_2 - w_1)$

If $x < \alpha$ or $x > \beta$ then since $G(x) > F(b_2)$, the curves (3) and

(4) will not intersect at all.

If, on the other hand, we consider any $x_0 \in (\alpha, \beta)$ then there exists an y_0 such that (x_0, y_0) satisfies (2) which implies that the curve $u = F(t)$ intersects $u = G(x_0)$ at two distinct points. At $x = \alpha$ or $x = \beta$ there exists only one value of y, viz. $y = b_2$, such that $F(b_2) = G(\alpha) = G(\beta)$.

From the above discussions we, therefore, have the following lemma.

Lemma 1. The locus of (x, y) as given by (2) is a closed polygon having within it the point associated with the greater weight.

In what follows we shall call this locus the equipolygon of P_1 and P_2 and denote it by EP(1-2). This equipolygon cuts the line joining P_1 and P_2 both internally and externally in the inverse ratio of the weights at the extremities, their coordinates being given by

$$\left((w_1a_1 + kw_2a_2)/(w_1 + kw_2),\ (w_1b_1 + kw_2b_2)/(w_1 + kw_2)\right)$$

where $k = 1$ in the former case and $k = -1$ in the latter.

We state without proof the following corollary to be required in the sequel.

Corollary 1: The weighted rectilinear distance from any point P in R^2 to the location point corresponding to the greater weight is greater than that to the other point when the point P is outside the equipolygon, a closed contour in this case, and vice versa.

To have a closer look into the nature of the equipolygon we divide the entire plane into 9 regions I through IX depending on the position of (x, y) as follows; see figure 1.

I $x \leq a_1$, $y \leq b_i$; II $x \leq a_1$, $b_i \leq y \leq b_j$;

III $x \leq a_1$ $y \geq b_j$; IV $a_1 \leq x \leq a_2$, $y \leq b_i$;

V $a_1 \leq x \leq a_2$, $b_i \leq y \leq b_j$; VI $a_1 \leq x \leq a_2$, $y \geq b_j$;

VII $x \geq a_2$, $y \leq b_i$; VIII $x \geq a_2$, $b_i \leq y \leq b_j$;

IX $x \geq a_2$, $y \geq b_j$;

where $b_i \leq b_j$; $i, j = 1, 2$ and $i \neq j$.

In each of the above regions if the equipolygon is defined then it may be represented as shown below.

I $x + y = a + b$ where $a = (w_2a_2 - w_1a_1)/(w_2 - w_1)$

and $b = (w_2b_2 - w_1b_1)/(w_2 - w_1)$

II $(w_2 - w_1)x + k(w_1 + w_2)y = (w_2a_2 - w_1a_1) + k(w_1b_1 + w_2b_2)$

III $x - y = a - b$

IV $(w_1 + w_2)x + (w_2 - w_1)y = (w_1a_1 + w_2a_2) + (w_2b_2 - w_1b_1)$

V $x + ky = c + kd$, where $c = (w_2a_2 + w_1a_1)/(w_2 + w_1)$

and $d = (w_2b_2 + w_1b_1)/(w_2 + w_1)$

VI $(w_1 + w_2)x - (w_2 - w_1)y = (w_1a_1 + w_2a_2) - (w_2b_2 - w_1b_1)$

VII $x - y = a - b$

VIII $(w_2 - w_1)x - k(w_1 + w_2)y = (w_2a_2 - w_1a_1) - k(w_1b_1+w_2b_2)$

IX $x + y = a + b$

$$\text{where } k = \begin{cases} 1, & \text{if } b_2>b_1 \\ -1, & \text{if } b_2<b_1 \end{cases}$$

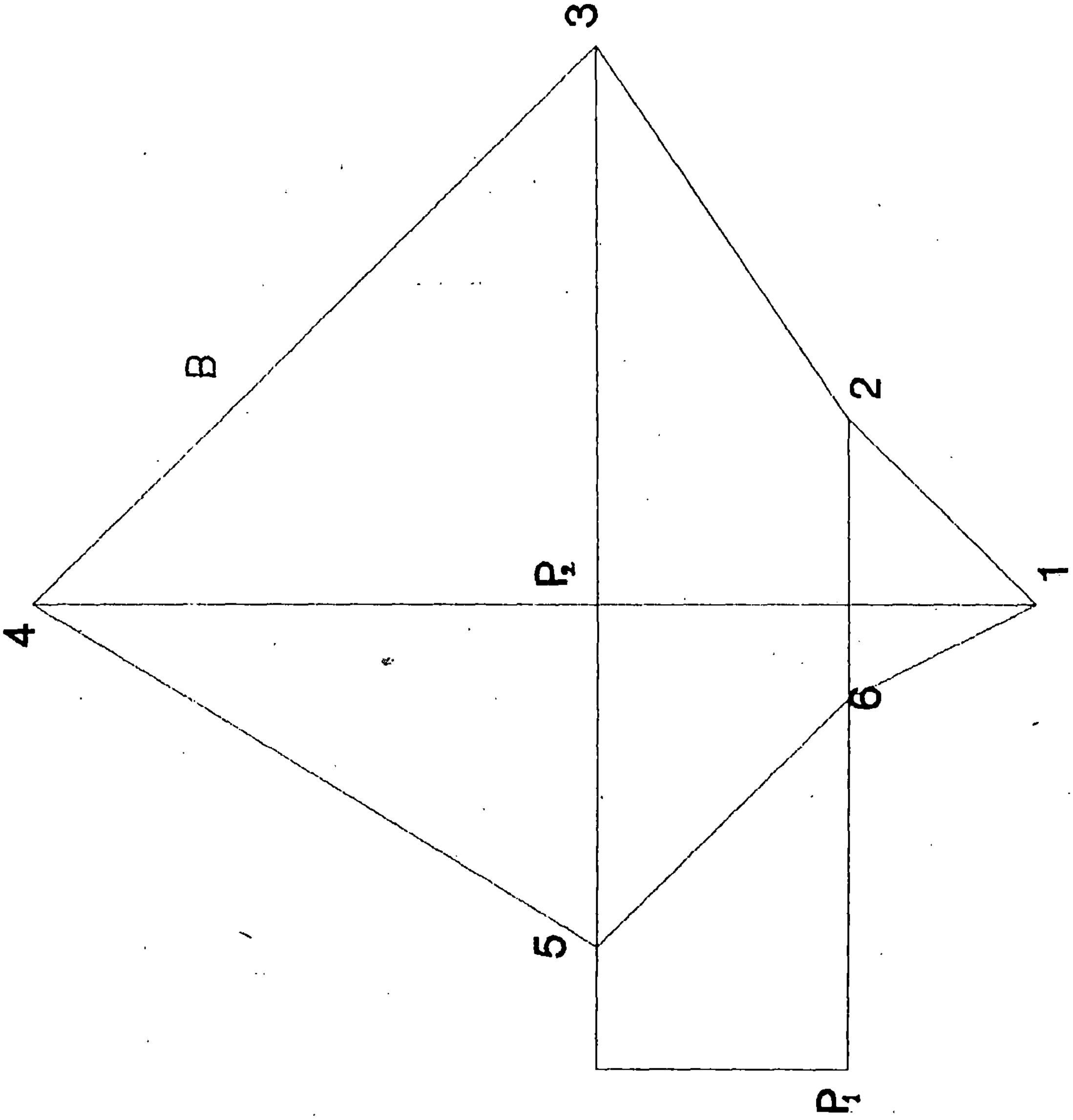


Fig.1

Definition 1. $R\Sigma$ represents the smallest bounding rectangle constructed through the four points of a finite set Σ, having minimum and maximum ordinates and minimum and maximum abscissas respectively, by drawing lines parallel to the x and y axes. When $\Sigma = \{A_1, A_2, \ldots, A_k\}$ then the smallest bounding rectangle is denoted by $R(A_1, A_2, \ldots, A_k)$.

Let us now make a detailed study of the structure of the equipolygon vis-a-vis the regions of definitions.

Let l amd s denote the lengths of adjacent sides of $R(P_1, P_2)$ where $l = |a_2 - a_1|$ and $s = |b_2 - b_1|$. Let $l > s$. If $w_2 > w_1$ and $(w_2/w_1) < (l/s)$ the locus of (x, y) consists of six straight line segments lying in regions IV through IX joined end to end forming a closed polygon as shown in figure 1, whereas if $w_2 > w_1$ and $(w_2/w_1) \geq (l/s)$ it has four segments in regions V, VI, VIII and IX or IV, V, VII and VIII according as $b_1 <$ or $> b_2$; see figure 2. If, on the other hand, $w_1 > w_2$ then the equipolygon comprises line segments belonging to regions I through VI in the former and I, II, IV and V or II, III, V and VI in the latter cases. If $l \leq s$ the regions within which segments forming the equipolygon will lie may be obtained in an analogous manner.

If $b_1 = b_2$ then the locus is a four-sided equipolygon, the sides being located in regions IV, VI, VII and IX or I, III, IV and VI according as $w_1 <$ or $> w_2$. In a similar manner when $a_1 = a_2$ the equipolygon has sides belonging to regions II, III, VIII and IX or I, II, VII and VIII in the respective cases.

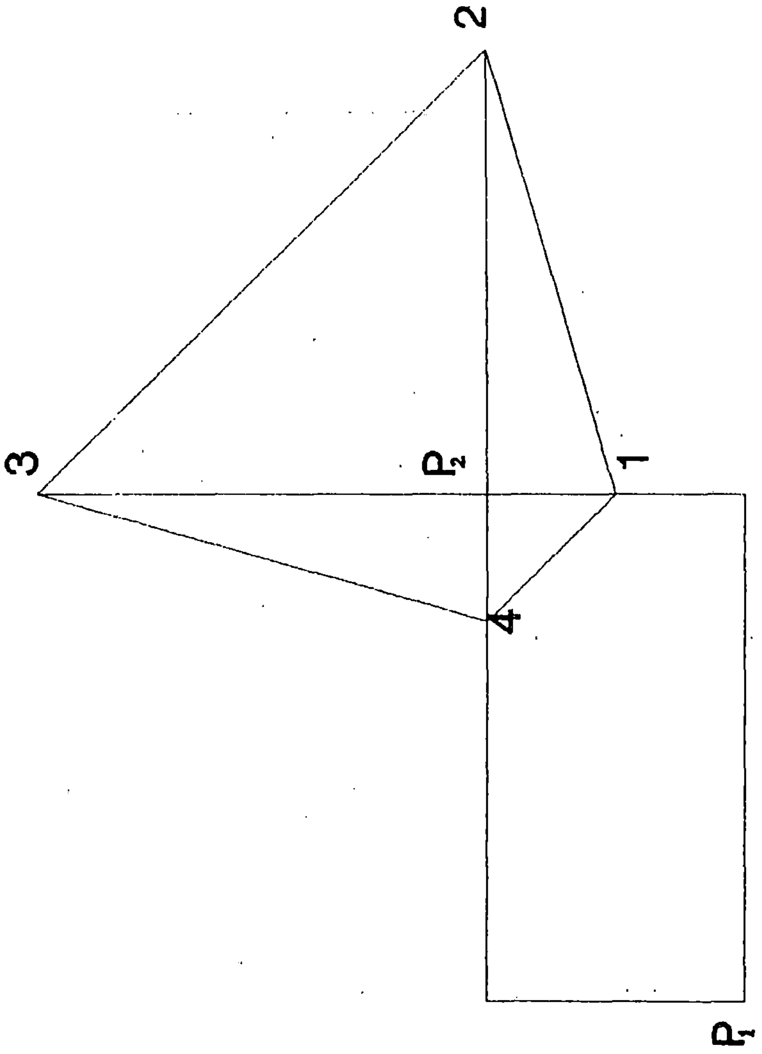


Fig.2

If $w_1 = w_2$ the locus represented by (2) is no longer a closed polygon and consists of a line segment through the centre of the rectangle lying within region V flanked by semi-infinite lines parallel to the y-axis contained in regions IV and VI. This is illustrated in figure 3 in case $l > s$.If $l < s$ then the locus is made up of a line segment in zone V bounded by semi-infinite lines parallel to the x-axis contained in regions II and VIII. When $l = s$ the locus consists of either diagonal together with any two half-rays with vertices at the extremities of this diagonal in regions III and VII or I and IX according as $b_1 <$ or $> b_2$.

In our subsequent discussion we will make use of the following definitions and notations.

Definition 2. With respect to any point $(h, k) \in R^2$ we denote by $L(h, k; a_i, b_i)$ the path consisting of two line segments joining (h, k) to (a_i, k) and (a_i, k) to (a_i, b_i).

Definition 3. Γ_{ij} will represent the portion of EP(i-j) intercepted by the boundaries of $R(P_i, P_j)$.

Definition 4. M_{ij} denotes a portion of EP(i-j) from any point upto the nearest $\partial R(P_i, P_j)$ along the direction of descent. This is illustrated in figure 1 where M_{12} consists of the line segments B4 and 45 of EP(1-2).

Definition 5. Q will be called a well-behaved point with respect to suffixes i, j $\in$ I, i $\neq$ j, if

$w_i d(Q, P_i) = w_j d(Q, P_j) \geq w_k d(Q, P_k)$, for all $k \in I - \{i,j\}$.

Definition 6. A direction of descent is one by moving along

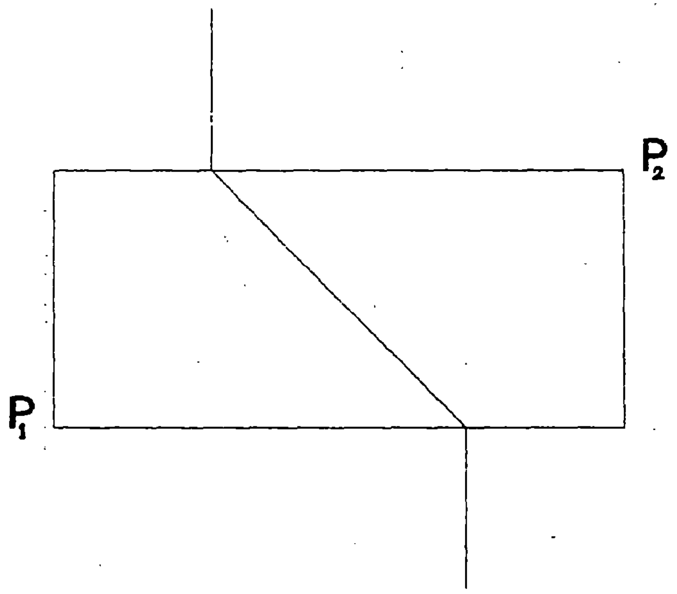


Fig.3

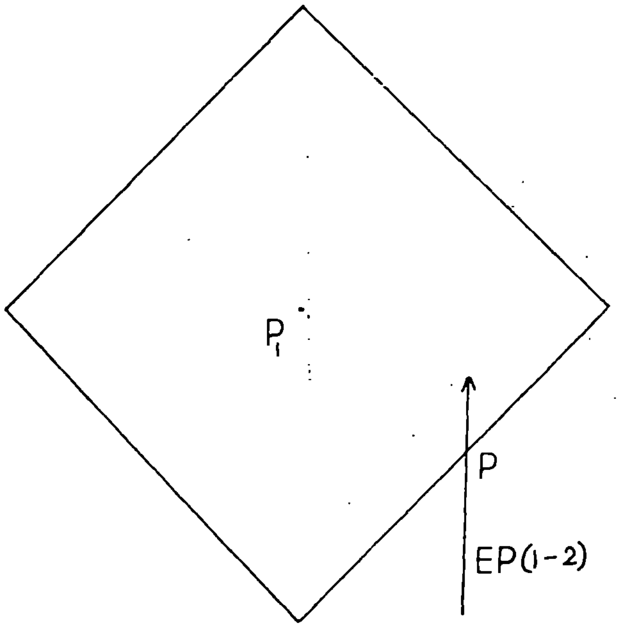


Fig 4

which the value of the objective function does not increase.

To bring the ideas contained in the last definition into a sharp focus let A, B be the consecutive vertices of the edge AB of an equipolygon EP(1-2) and P be any point on AB. Then the direction PA will be called the direction of descent at P with respect to P_1, P_2 if $d(U, P_1) \leq d(P, P_1)$, U being any point on the line segment AP.

2.1.3. Solution of the problem

Our algorithm is based on the movement in the direction of descent along an edge of an equipolygon. It is, therefore, proper now to introduce a rule for its determination.

Determination of the Direction of Descent

Let P be any point on an edge of an equipolygon EP(1-2). Let us construct a diamond [36] through P centred at either of the points P_1, P_2. Then the portion of the edge of EP(1-2) directed towards the diamond gives the direction of descent as shown in Figure 4.

For unequal weights the orientations of the directions of descent of the equipolygon vis - a - vis the different regions are given below. See figure 1 for an illustration.

Region	Orientation
IV	from 1 to 6
VI	from 4 to 5
VII	from 2 to 1
VIII	from 3 to 2
IX	from 3 to 4 or 4 to 3

In case the weights are equal the direction of descent will be towards the longer side of the rectangle $R(P_1, P_2)$.

Let us now discuss the strategy for solving problem (1). The optimal objective value with respect to any two points A and B occurs in R(A,B). Moreover, RS contains all $R(P_i,P_j)$, P_i, $P_j \in S$. Consequently, the minimax solution of problem (1) lies on RS.

At a particular iteration if there exists a well-behaved point $\notin R(P_1, P_2)$ on an edge of an equipolygon EP(1-2), the well-behaved point associated with the immediately succeeding iteration may be obtained as follows.

We consider the intersection of the direction of descent of this edge of EP(1-2) with the equipolygons formed by P_1 or P_2 and each of the other (n-2) points and find the one nearest to a well-behaved point. If none exists then the extreme point of the edge of EP(1-2) serves as the next well-behaved point. Mathematically, if the end points of an edge of EP(1-2) through P be denoted by A, B such that $\vec{PA}$ is the direction of descent with respect to P_1, P_2 then $w_1 d(U_i, P_1) = w_i d(U_i, P_i)$, $i \in I - \{1, 2\}$, provided U_i is on PA and such an U_i exists. Let $PU = \min_i \{\|PU_i\|\}$. We choose U as the well-behaved point for the next iteration in case such an U is available; otherwise, we choose A.

We next prove the following lemma to be subsequently required for developing our algorithm.

Lemma 2. Let us consider the two given location points P_i, $i = 1, 2$ and find the greater of the weighted rectilinear distances of these from any point (g, h). If the path corresponding to the maximum distance be represented by $L(g,h;\ a_i, b_i)$ then there exists a point $P \in L(g,\ h;\ a_i,\ b_i)$ such that the weighted rectilinear distances of P_i, $i = 1, 2$ from P are equal.

Proof: If the weighted distances of P_1, P_2 from (g, h) be equal nothing remains to be proved. Consider, therefore, the situation in which the two are not the same and the greater distance corresponds to P_1. We define a function $f(x, y)$ as follows :

$$f(x,\ y) = f_1(x,\ y) - f_2(x,\ y) \text{ where}$$

$$f_i(x,\ y) = w_i(|x - a_i| + |y - b_i|).$$

Clearly, $f(x, y)$ is a continuous function, $f(g, h) > 0$ and $f(a_1, b_1) = - f_2(a_1, b_1) < 0$. Hence the proof of the lemma is complete.

We have the following corollary, the proof of which, being obvious, is left out.

Corollary 2. Every equipolygon $EP(k-i)$, where P_k is the weighted farthest point from (g, h), $i \in I - \{k\}$, intersects $L(g,h;\ a_k, b_k)$ at least once.

Suppose that we are moving along the direction of descent of the equipolygon $EP(i-j)$, $w_i \geq w_j$ along A_1A and on reaching A, let $w_i d(A, P_i) = w_j d(A, P_j) = w_k d(A, P_k)$. Then since $w_k d(A_1, P_k) < w_i d(A_1, P_i) = w_j d(A_1, P_j)$ we conclude that the equipolygons $EP(i-k)$ and $EP(j-k)$ passing through A will have some portions at least of them on either side of A_1A in the neighbourhood of A. In particular, it is obvious that a portion of the boundary of EP(i-k) will be within EP(i-j) irrespective of the magnitudes of w_i and w_k, and that of EP(j-k) will be outside EP(i-j) as shown in figure 5.

It is to be noted further that if the point A be outside $R(P_i, P_j, P_k)$ then A cannot be a minimax location for, any movement from A towards the nearest $\partial R(P_i, P_j, P_k)$ along the direction perpendicular to the boundary will cause the objective function value to diminish. The direction of movement is given by the following criterion.

Criterion C

The rule of selecting one from three or more equipolygons meeting at a non-optimal point : Let P(g, h) be the point from which the weighted rectilinear distances of $(a_i, b_i) \in S_1 \subseteq S$, the cardinality of S_1 being r (≥ 3), are equal but those of points $\in S \setminus S_1$ are less. We take any three points of S_1. Since P is not an optimal point, one of the following possibilities must be true.

1. All three points lie on the same side of $x = g$ but any two of them lie on one side and the third on the other of $y = h$.
2. All three points lie on the same side of $y = h$ while any two of them lie on one side and the third on the other of $x = g$.

As an example let us assume that each of b_i, b_j, $b_k > h$ and $a_i < g$, a_j, $a_k > g$. We retain (a_i, b_i) and consider the directions of descent of EP(i-j) and EP(i-k) at P. We next construct a diamond with respect to $P_i(a_i, b_i)$ passing through P. Of the two equipolygons EP(i-j) and EP(i-k) the one

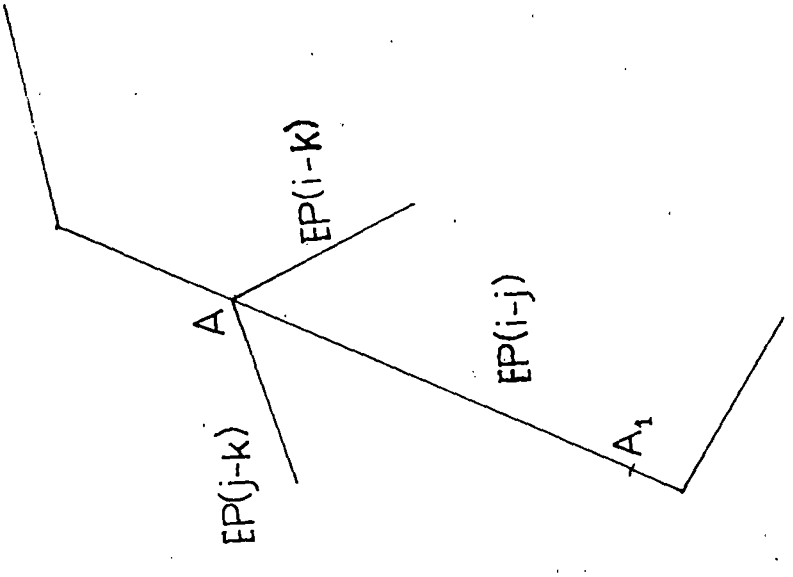


Fig. 5

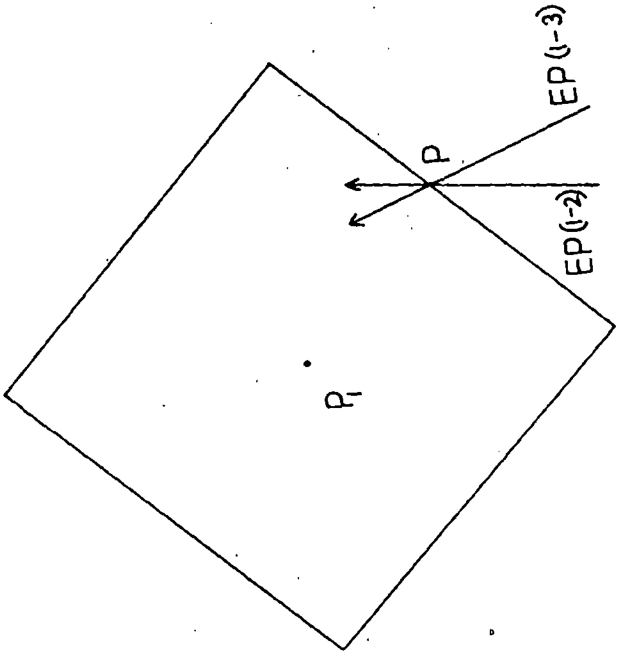


Fig 6

having a smaller angle of inclination with the side of the diamond passing through P will be selected for the subsequent iteration. Figure 6 illustrates the case of EP(1-2) being chosen.

In the algorithm that follows we assume that the movement always takes place along the direction of descent of an equipolygon in such a way that primal feasibility condition is never violated.

2.1.4. Algorithm

Step 0. (Initialisation Step)

Take any extreme point $(\bar{x}, \bar{y}) \in \partial RS$ and calculate the maximum weighted rectilinear distance of it from the set S. Let this maximum occur for i = 1. Now determine a point A on $L(\bar{x}, \bar{y}; a_1, b_1)$ such that

$$\max_{A \in L(\bar{x}, \bar{y} : a_1, b_1)} \left\{ w_1 d(A, P_1) = w_i d(A, P_i) \;\middle|\; i \in I - \{1\} \right\}$$

holds. Let $P_2 \in S$ be the point satisfying the above; go to step 1.

Step 1. If $A \in \partial R(P_1, P_2)$ or if we can move upto a point $B \in \partial R(P_1, P_2)$ in such a way that any point $P \in M_{12}$ is a well-behaved point with respect to indices 1, 2 in which case $A \longleftarrow B$, then go to step 3(a). Else go to step 2.

Step 2. For a point $P \in M_{12}$ such that $w_j d(P, P_j) = w_1 d(P, P_1)$, $j \in I - \{1, 2\}$ and the distance of P from A measured along M_{12} is a minimum, if $P \in R(P_1, P_2, P_j)$ then go to step 3(b).

Else apply criterion C to determine the direction of next movement, A ←— P, rename the corresponding points defining the equipolygon as P_1 and P_2 and go to step 1.

Step 3(a). If any other equipolygon intersects Γ_{12} at P then any point $\in$ AP is a possible minimax location. Otherwise, any point $\in \Gamma_{12}$ will be a required location.

Step 3(b). From P we follow the path along Γ_{kl} satisfying criterion C, kl being any combination of 1, 2 and j, until we obtain the point of intersection Q of Γ_{kl} and another equipolygon, in which case any point $\in$ PQ is a possible minimax location. Otherwise, any point $\in$ the stretch of Γ_{kl} from P to the point of intersection of Γ_{kl} and the boundary of $R(P_k, P_l)$ situated along the direction of descent is a required location.

2.1.5. Analysis of time complexity

Let us take two location points P_i and P_j the weighted rectilinear distances of which from some (h, k) are equal. Let the edge of EP(i-j) containing (h, k) meet $R(P_i, P_j)$ at (p, q). We next determine if the point of intersection of the segment joining (h, k) and (p, q), and some other EP(i-l) closest to (h, k), belongs to this segment.

Our algorithm chooses one of the corner points of RS as a starting solution. Taking the right hand bottom corner as the initial choice, (h, k) may initially belong to one of the zones IV, VII or VIII with respect to P_i and P_j. If it is in zone VII, since two equipolygons in this zone, being parallel, cannot intersect each other, it cannot remain there in the following iterations, thus signalling the failure to obtain an (h_1, k_1). If, on the other hand, the non-optimal point (h, k) belongs to zone IV initially, it is then obvious that (h_1, k_1) also belongs to zone IV. By the very definition of zone IV, if (h_1, k_1) be non-optimal then the ordinates of the three points will be greater than k_1 whereas two of them will be on the one side and the third on the other of $x = h_1$. **Criterion C** determines which one of the two points on the same side of $x = h_1$ is to be retained in the next iteration. Thus each of the retained points generating the next iteration has ordinate greater than k_1 while continuing to remain on either side of $x = h_1$ indicating that (h_1, k_1) still belongs to zone IV with respect to the updated points. If more than three points have the same weighted distance from (h_1, k_1) then by repeated application of **Criterion C** one can conclude that if the iteration at some stage be restricted to zone IV it will continue to remain so until optimality is achieved. On the other hand, if no (h_1, k_1) belonging to the line segment terminated by (h, k) and (p, q) exists then any one of the zones I or VII may have to be traversed. By the same token we may draw a parallel between the arguments given for zone IV and those for zone VIII.

The procedure developed by us is based on two equipolygons intersecting at most once in a given zone. If

i_1, i_2 be the indices corresponding to a well-behaved point in a particular non-optimal iteration, then these indices are not required to be considered any further until optimal solution is obtained. Thus in the worst case the algorithm has a $O(n^2)$ time complexity. Had we tried to solve the LP formulation of the present problem via simplex method it wouldn't have been possible for us to conclude beforehand the order of polynomial time complexity.

2.1.6. Numerical Example

Let us find the solution to the problem considered by Francis and White [36] by making all $g_i = 0$. The location points are (3, 3), (3, 6), (6, 3) and (7, 8) with associated weights 2. 3, 4 and 2 respectively. We construct the rectangle RS and take a point $(3, 8) \in \partial RS$. The weighted farthest point is found to be $P_1 = (6, 3)$. We next find the point A to be $(6, 6) \in L(3, 8; 6, 3)$. We designate the point (3, 3) as P_2 following step 0 of the algorithm. Clearly $A \notin R(P_1, P_2)$. Moving along EP(1-2) we reach the point $P = \left(\frac{81}{14}, \frac{75}{14} \right)$ from which the weighted rectilinear distances of P_1, P_2 and $P_3 = (3, 6)$ are equal. Since $P \in R(P_1, P_2, P_3)$ applying **criterion C** we can move along the direction of descent of EP(1-3) upto the point $T = \left(\frac{36}{7}, \frac{33}{7} \right)$ from which the weighted rectilinear distance of each of P_1, P_2 and $P_4 = (7, 8)$ is the same. Hence by step 3(b), any point $\in PT$ is a required minimax location.

2.1.7. Computational Experience

No. of points	Frequency of convergence in		
	1 iteration	2 iterations	3 iterations
500	18	5	2
550	19	5	1
600	17	5	3
650	17	7	1
700	18	6	1
750	19	5	1
800	15	9	1
850	18	6	1
900	18	7	0
950	19	6	0
1000	21	2	2

Table 1

This section deals with the computational test of the algorithm. With this end in view we developed the Pascal code of the algorithm. Three sets of random numbers corresponding to x_i, y_i and w_i were generated 25 times for a given n ($500 \leq n \leq 1000$) over a rectangle RS chosen in advance having unequal adjacent sides by employing standard Turbo Pascal Procedure Randomize and Function Random. n was next allowed to vary and the same procedure repeated . Random data generation technique was resorted to on account of non-availability of actual data required for large size problems. It is interesting to note that in no case the algorithm required more than three iterations to converge. The results of computation are summarised in Table 1 above.

2.1.8 Sensitivity Analysis

At early stages of problem formulation some factors may be overlooked and in many practical situations data may not be known in advance exactly. Sensitivity analysis takes care of these factors and updates the current optimal solution without performing the expensive task of resolving the problem from scratch. Let us now see how these ideas can be implemented to obtain the current optimal solution from the previous solution. Let us consider the following instances.

(i) Introducing a new demand point

(ii) Removing an existing demand point

(iii) Changing the weight associated with a given demand point.

Regarding case (i) if the weighted distance of the recently added point be less than or equal to the optimum objective value calculated at both the extremities of the stretch (prior to insertion of the demand point) constituting the set of optimum solutioms then the solution set remains the same as before; else if this distance be less than the optimum value of the objective function obtained at one extremity only then we have to recalculate the stretch; else we choose either extremity of the stretch as $(\bar{x}, \bar{y})$ and repeat the algorithm described in sec. 2.1.4 after making allowance for this additional point.

As regards case (ii) if the deleted point be not an active demand point then the set of optimal solutions remains unchanged; else we choose any end point of the stretch as the starting point $(\bar{x}, \bar{y})$ and proceed in accordance with the directions indicated in sec. 2.1.4.

In case (iii) if altering the weight associated with an existing demand point destroys primal feasibility at either end point of the stretch then we follow a procedure similar to case (ii) to restore primal feasibility.

As an illustration let us introduce a new demand point at (5, 2) with associated weight 5 in addition to the four already existing ones considered in sec. 2.1.6 to throw light on the observations made above. As the weighted rectilinear distance of the newly added point from both the end points of the stretch is greater than the previous optimal solution violating primal feasibility, we follow the procedure mentioned in (i) to get the new stretch extending from (5.05, 4.24) to (5.00, 4.29). The stretch belongs to the smallest rectangle formed by the fourth and the newly introduced points whereas in the original problem the stretch is the line segment from (5.14, 4.71) to (5.79, 5.36) lying within the smallest rectangle constructed with the second and third demand points.

2.2.1 Solution of an asymmetric rectilinear distance minimax location problem

In this section we consider a minimax location problem using a rectilinear measure of distance lacking symmetry so that with each demand point is associated four different weights corresponding respectively to the main four directions. This lack of symmetry is typical of rush hour traffic where the speeds towards and away from the commercial centre of a metropolitan city are different. There are other practical situations also where distance between two points is not a symmetric function - for an air craft, for example, flying in the presence of steady wind flowing in one direction only the speeds in the direction of current and opposite to it are different. Again, for motion on an inclined plane the upslope speed is different from the downslope speed.

Presently we would discuss briefly how the above mentioned model may be gainfully applied to the tea industry. The northern part of West Bengal abounds in tea gardens. The Terai region of Darjeeling district and the Dooars region of Jalpaiguri district account respectively for approximately 50 and 250 gardens. Tea is one of the chief agricultural produce earning foreign exchange. It is conventionally grown at a place where there is no waterlogging despite abundant rainfall. For this reason it is natural that the sub-Himalayan region of West Bengal should be selected for tea plantations. But, as a matter of fact, most gardens employ conventional methods for growing and plucking of tea. For an increase in the yield as also for the protection and sustenance of the plants, it requires, among other things, pruning, knife cleaning and depilation, chilling, light hoeing, trimming, spraying of pesticides and weed killers, plucking out of creepers, infilling, construction and maintenance of drains etc. As an example, by simply improving on the existing drainage system the Trihanna Tea Estate increased the yield by about 12.5% (the international market value of this extra yield being estimated at $ 0.2 million). To check soil erosion and prevent water from accumulating, each garden develops its own drainage system depending on its topography. The gardens are situated on an inclined plane extending northward from the base. For an incline of less than 1 in 50 the usual practice is to construct north south drains interspersed at regular intervals with east west ones while for inclines exceeding it, contour drains are preferred. The places from which the above operations of plant treatment etc. are being carried out, to be called the facility point hereafter, are not normally located in accordance with the prescriptions suggested by facility layout and location models. The Panighata Tea Estate in the lower Terai offers a case in point (total area 6.17 hectares, area under cultivation 4.25 hectares of an irregular shape and an elevated northern side, maximum width 2.5 kms). The facility point is located at one end of the garden while the other end is 4 kms apart. Our research was motivated by the problem of locating the facility point so that the maximum distance from it to a plant - a demand point - is minimised. Moreover, for a considerable time of the year there is a steady westwind blowing over the gardens. Thus on both counts the minimax criterion involving asymmetric L_1 metric for a single facility is the most appropriate one for the declared objective.

Dykstra et al. [30] consider the cost of log harvesting in which the logs are displaced from 'prebunching sites' by means of helicopters. Hodgson et al. [46] and Chen [13] have proposed solutions to the p-centroid location problem of log harvesting on an inclined plane. Drezner and Wesolowsky [25] have provided an efficient algorithm for an asymmetric rectilinear distance minimax location problem while Chakrabarty and Chaudhuri [10] have given a geometrical solution procedure for the constrained two-dimensional minimax problem using weighted rectilinear norm. Tamir [73] has given a complexity bound improvement for the 1-centre rectilinear asymmetric distance location problem in the plane.

For a survey of selected locational literature we would like to refer to the excellent works of, among others, Francis et al. [37], Hansen et al. [44] and Love et al. [55].

A plethora of publications dealing with the minimax criterion for the unweighted as well as the weighted rectilinear metric is available (Francis [34], Elzinga and Hearn [31], Wesolowsky [78], Morris [64], Francis et al. [38], Hansen et al. [43], Drezner [21] and Batta et al. [4]. But scant little attention has been given to the asymmetric L_1- distance location problems. Ours is an effort to fill this gap. We have developed an iterative technique based on a well-defined stopping criterion.

The solution procedure which we have developed, at first finds a point P from which the weighted rectilinear distance of at least two points of the set S of given location points are equal while the weighted distances of P from the remaining points of S are greater. The point P is now moved until optimality is reached by maintaining the above mentioned property of distance to be called primal feasibility hereafter. All this is accomplished with the help of plane analytic geometry.

Throughout our discussion we shall use a single letter, or a juxtaposed pair of letters, subscripted or otherwise, in bold face Roman type to denote vectors.

2.2.2. Problem formulation

Let us consider the following problem:

$$\min_{(x,\, y) \in \mathbf{R}^2} \quad \max_{i \in I} \quad \mathbf{W}_i^{*} \mathbf{Z}_i \tag{1}$$

where (x, y), a variable point, denotes the proposed location of the facility point, I = {1, 2,...,n}, $P_i(x_i, y_i)$

are the existing location points belonging to the set

$S = \{ P_i(x_i, y_i): i \in I \}$,

$Z_i = \begin{bmatrix} |x-x_i| \\ |y-y_i| \end{bmatrix}$ and W_i^* represents the transpose of $W_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix}$,

u_i and v_i being given by

$$u_i = \begin{cases} u_i^- \text{ if } x < x_i \\ u_i^+ \text{ if } x > x_i \end{cases} ; \quad v_i = \begin{cases} v_i^- \text{ if } y < y_i \\ v_i^+ \text{ if } y > y_i \end{cases} .$$

We have not attempted to define u_i, v_i when $x = x_i$ or $y = y_i$ since they have no contribution towards the objective owing to the fact that $|x - x_i| = 0$ for $x = x_i$ and $|y - y_i| = 0$ for $y = y_i$.

We shall use the following notations throughout our discussion :

$$d(P, P_i) = |x - x_i| + |y - y_i|$$

$$\rho(P, P_i) = W_i^* Z_i = u_i|x - x_i| + v_i|y - y_i|$$

To start with let us assume that $u_i^+ > u_i^-$, $v_i^+ > v_i^-$, although such an assumption is not at all necessary to develop our algorithm.

Using the approach on page 227 of Francis et al. [38] we can translate the above problem into the following linear program (LP):

Minimise z subject to

$$u_i^+ (x - x_i) + v_i^+ (y - y_i) \le z$$

$$u_i^- (x_i - x) + v_i^+ (y - y_i) \le z$$

$$u_i^+ (x - x_i) + v_i^- (y_i - y) \le z$$

$$u_i^- (x_i - x) + v_i^- (y_i - y) \le z$$

for all $i \in I$.

2.2.3. Solution of the problem

To solve problem (1) we have developed a simple geometrical approach. For a problem of moderate size with $n \le 20$, say, (1) may be solved by using the ruler and the compass. But when the problem size is large our method can be easily implemented on a PC.

We shall first obtain the locus of (x, y) satisfying $W_1^* Z_1 = W_2^* Z_2$ which implies

$$u_1 |x - x_1| + v_1 |y - y_1| = u_2 |x - x_2| + v_2 |y - y_2| \qquad (2)$$

or what is the same thing as $F(y) = G(x)$ (3)

where $F(y) = v_1 |y - y_1| - v_2 |y - y_2|$ (4)

and $G(x) = u_2 |x - x_2| - u_1 |x - x_1|$ (5)

Without any loss of generality we may always assume $x_1 < x_2$. For, if otherwise, we might call the point with the smaller x-coordinate (x_1, y_1). For convenience, let us for the time being consider $y_1 < y_2$. It can be easily shown by direct substitution that the point

$$\left(\left[u_2^- x_2 + u_1^+ x_1\right] \Big/ \left[u_2^- + u_1^+\right], \left[v_2^- y_2 + v_1^+ y_1\right] \Big/ \left[v_2^- + v_1^+\right]\right)$$

$\in \left\{(x, y): x_1 \le x \le x_2, y_1 \le y \le y_2\right\}$ lies on the locus represented by equation (2) with $u_1 = u_1^+$, $v_1 = v_1^+$, $u_2 = u_2^-$ and $v_2 = v_2^-$, implying that the equation has a solution. Clearly, W_1 may be greater than, equal to, or less than W_2.

Case $W_1 > W_2$. The function $t = G(x)$ is continuous everywhere, strictly increasing in $(-\infty, x_1)$ and strictly decreasing in (x_1, x_2) and (x_2, ∞), having attained the maximum positive value of $u_2^-(x_2 - x_1)$ at $x = x_1$. Moreover, $G(x) \longrightarrow -\infty$ as $x \longrightarrow \pm\infty$. See figure 1.

Since the functions $F(y)$ and $-G(x)$ have identical forms, it follows immediately that $t = F(y)$ is continuous, decreasing in $(-\infty, y_1)$ and increasing in (y_1, y_2) and (y_2, ∞) with the minimum negative value of $v_2^-(y_1 - y_2)$ at $y=y_1$. Furthermore, $F(y) \longrightarrow \infty$ as $y \longrightarrow \pm\infty$. Refer to figure 2.

If $F(y_1) > G(x_2)$ then there exists no y satisfying $F(y)=G(x)$ for any $x \notin [\alpha_{11}, \alpha_{12}]$ where

$$\alpha_{11} = \left[v_2^-(y_1 - y_2) + u_1^- x_1 - u_2^- x_2\right] \Big/ \left[u_1^- - u_2^-\right]$$

and $$\alpha_{12} = \left[v_2^-(y_2 - y_1) + u_1^+ x_1 + u_2^- x_2\right] \Big/ \left[u_1^+ + u_2^-\right]$$

whereas if $F(y_1) < G(x_2)$ then there is no y that corresponds to values of $x \notin [\gamma_{11}, \gamma_{12}]$ such that $F(y) = G(x)$ where

$$\gamma_{11} = \alpha_{11}$$

and $$\gamma_{12} = \left[v_2^-(y_2 - y_1) + u_1^+ x_1 - u_2^+ x_2\right] \Big/ \left[u_1^+ - u_2^+\right]$$

in view of the increasing nature of $G(x)$ in $(-\infty, x_1)$ and its decreasing nature in (x_1, ∞). We can therefore conclude that the locus represented by (2) lies wholly within two straight lines parallel to the y-axis.

On the other hand, if $G(x_1) < F(y_2)$ we cannot get an x such that $G(x) = F(y)$ for values of $y \notin [\beta_{11}, \beta_{12}]$, β_{11} and

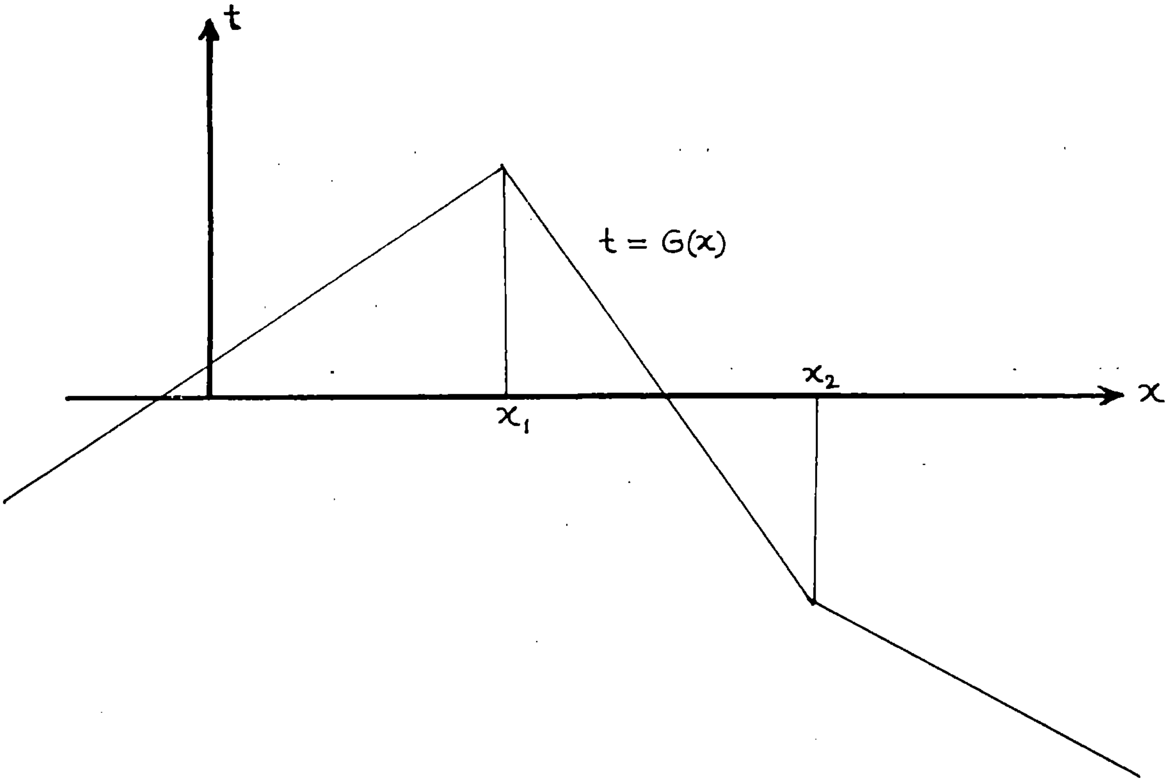


Fig 1

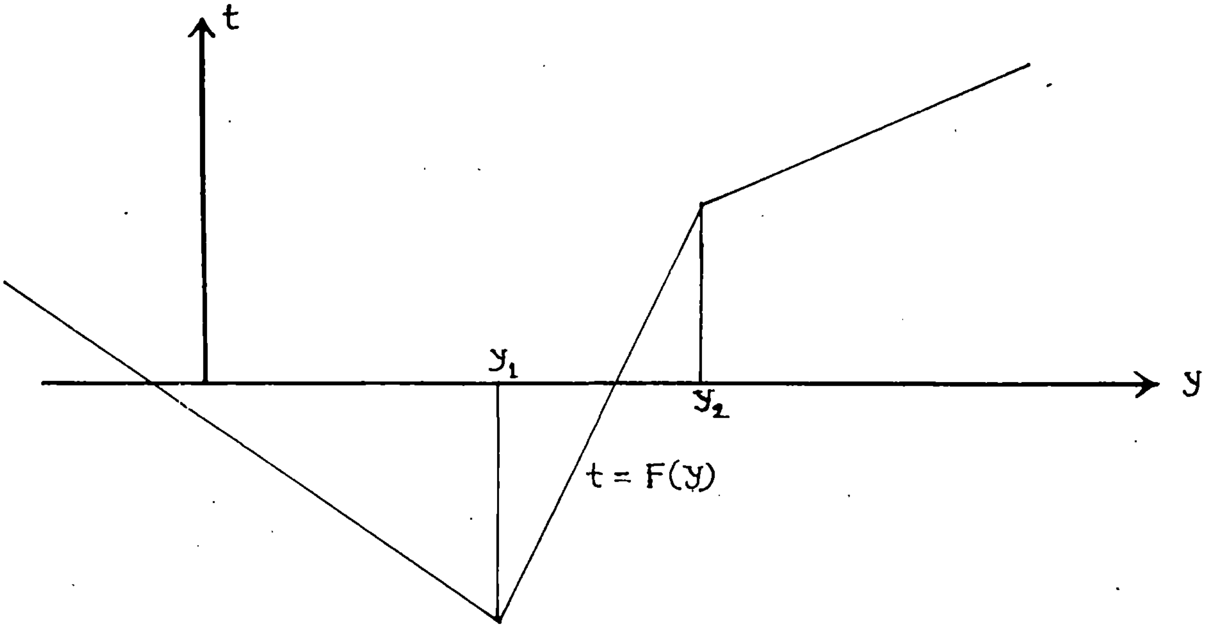


Fig 2

β_{12} being given by

$$\beta_{11} = \left[u_2^-(x_2 - x_1) - v_1^- y_1 + v_2^- y_2\right] \Big/ \left[v_2^- - v_1^-\right]$$

and $\beta_{12} = \left[u_2^-(x_2 - x_1) + v_1^+ y_1 + v_2^- y_2\right] \Big/ \left[v_1^+ + v_2^-\right]$

while if $G(x_1) > F(y_2)$ we could find no x such that $G(x) = F(y)$ holds for any $y \notin [\delta_{11}, \delta_{12}]$, δ_{11} and δ_{12} being given by

$$\delta_{11} = \beta_{11}$$

and $\delta_{12} = \left[u_2^-(x_2 - x_1) + v_1^+ y_1 - v_2^+ y_2\right] \Big/ \left[v_1^+ - v_2^+\right]$

owing to the decreasing nature of $F(y)$ in $(-\infty, y_1)$ and its increasing nature in (y_1, ∞). From the above it readily follows that the locus given by (2) remains wholly within two straight lines parallel to the x-axis.

Since the curve represented by (2) is bounded in both the x and y directions we can immediately conclude that (2) symbolises a bounded curve.

Furthermore, for any $x \in (\alpha_{11}, \alpha_{12})$ or $(\gamma_{11}, \gamma_{12})$ there exists exactly two distinct values of y which, however, coincide in case x is equal to either end point forming the interval. Arguing similarly it can be shown that for any $y \in (\beta_{11}, \beta_{12})$ or $(\delta_{11}, \delta_{12})$ we can have exactly two values of x, which are coincident when y is equal to either end point forming the interval. The above reasoning clearly demonstrates that the locus of (x, y) is a closed curve consisting of several straight line segments described around (x_1, y_1), the point associated with the greater weight. The number of line segments constituting the curve will be given shortly. We shall hereafter call this locus the *equipolygon* of the given pair of points, as shown in figure 3. For simplicity we denote the equipolygon of two points A and B with associated weights W_A and W_B by E_{AB}.

We next state a lemma the proof of which, being obvious, is left out.

Lemma 1. Let $W_A > W_B$. If P lies outside E_{AB} then $\rho(P, A) > \rho(P, B)$ else $\rho(P, A) \leq \rho(P, B)$.

In addition to the definitions and notations already given we shall make use of the following in the sequel.

i) R(A, B) denotes the rectangle with A and B as opposite vertices and sides parallel to the axes of coordinates.

ii) By $\partial R(A,B)$ we shall mean the boundary of the region R(A,B).

iii) L(P, A) denotes an L-shaped curve consisting of two line segments — one through P parallel to the x-axis and the other through A parallel to the y-axis — meeting at a point, provided both the coordinates of P and A are different.

N.B. In case x or y coordinates of P and A are equal L(P, A) degenerates into a straight line segment.

iv) Γ_{AB} is the stretch T_1T_2 of E_{AB} cut off by $\partial R(A, B)$ where T_1 is the extremity of Γ_{AB} first of all encountered, maintaining primal feasibility, and T_2 is the other

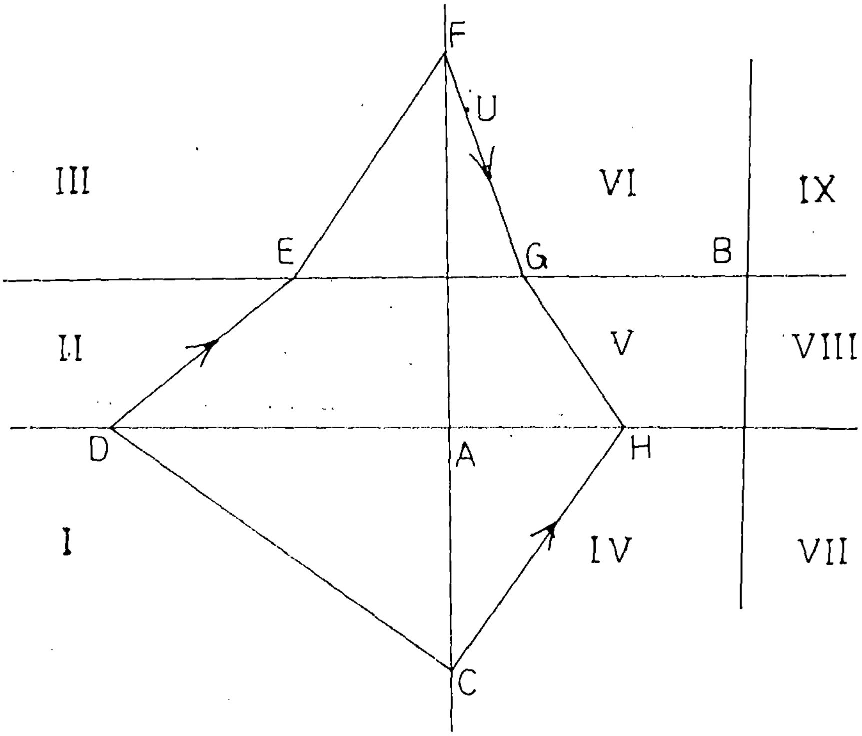


Fig 3

extremity.

v) For any point $Q \in \Gamma_{AB}$ we define $D_{AB}(Q)$ by

$D_{AB}(Q) = u_A^+ v_B^k - u_B^- v_A^l$, where $k = -1$, $l = +1$ in case $y_A \leq y_B$ and $k = +1$, $l = -1$ in case $y_A > y_B$, y_A and y_B standing for the ordinates of A and B respectively.

vi) Let $\rho(Q, P_i) = \rho(Q, P_j) > \rho(Q, P_k)$ where $i, j \in I$ and $k \in I \setminus \{i, j\}$. If Q is not an optimal solution and moving along $E_{P_iP_j}$ does not violate the condition of primal feasibility then the points P_i and P_j will be called the *Dominating points* at Q.

We shall next deliberate upon the number of extreme points the closed and bounded equipolygon representing the locus can have. Let $r = (x_2 - x_1)/(y_2 - y_1)$ denote the ratio of the lengths of the adjacent sides of $R(P_1, P_2)$. If $r \in \left[v_2^- \big/ u_1^+ ,\ v_1^+ \big/ u_2^- \right]$ which corresponds to the case $G(x_1) < F(y_2)$ and $F(y_1) > G(x_2)$ the equipolygon consists of four corner points given by (α_{11}, y_1), (x_1, β_{11}), (α_{12}, y_1) and (x_1, β_{12}). If r coincides with either end point then one of the two vertices of the rectangle R not occupied by a location point will be a corner point.

If, on the other hand, $r > v_1^+ \big/ u_2^-$ or $r < v_2^- \big/ u_1^+$ corresponding to $G(x_1) > F(y_2) \rightarrow F(y_1) > G(x_2)$ or $F(y_1) < G(x_2) \rightarrow G(x_1) < F(y_2)$ respectively, then the equipolygon has six extreme points which are (α_{11}, y_1), (x_1, δ_{11}),

(α_{12}, y_1), (γ_2, y_2), (x_1, δ_{12}) and (γ_1, y_2) or (γ_{11}, y_1), (x_1, β_{11}), (x_2, δ_1), (γ_{12}, y_1), (x_2, δ_2) and (x_1, β_{12}), as the case may be, where

$$\gamma_1 = \left[v_1^+(y_2 - y_1) + u_1^- x_1 - u_2^- x_2\right] \Big/ \left[u_1^- - u_2^-\right]$$

$$\gamma_2 = \left[v_1^+(y_1 - y_2) + u_1^+ x_1 + u_2^- x_2\right] \Big/ \left[u_1^+ + u_2^-\right]$$

$$\delta_1 = \left[u_1^+(x_2 - x_1) + v_1^- y_1 - v_2^- y_2\right] \Big/ \left[v_1^- - v_2^-\right]$$

and $$\delta_2 = \left[u_1^+(x_1 - x_2) + v_1^+ y_1 + v_2^- y_2\right] \Big/ \left[v_1^+ + v_2^-\right]$$

Note: By taking $y_1 > y_2$ results similar to the above with obvious changes at appropriate places may be obtained.

Case $W_1 < W_2$. The equipolygon, in this case, encloses the point (x_2, y_2) instead and results analogous to the preceding will have been found.

Case $W_1 = W_2$. The equipolygon here degenerates into an open polygon with only two extreme points which are respectively

(1) (α_2, y_2) and (α_1, y_1) when $G(x_1) > F(y_2) \Rightarrow G(x_2) < F(y_1)$

(2) (x_2, β_2) and (x_1, β_1) when $G(x_2) > F(y_1) \Rightarrow G(x_1) < F(y_2)$

(3) (x_1, β_1) and (α_1, y_1) when $G(x_1) < F(y_2)$ and $G(x_2) < F(y_1)$

where

$$\alpha_1 = \left[v_1^-(y_2 - y_1) + u_1^+ x_1 + u_1^- x_2\right] \Big/ \left[u_1^+ + u_1^-\right]$$

$$\alpha_2 = \left[v_1^+(y_1 - y_2) + u_1^+ x_1 + u_1^- x_2\right] \Big/ \left[u_1^+ + u_1^-\right]$$

$$\beta_1 = \left[u_1^-(x_2 - x_1) + v_1^+ y_1 + v_1^- y_2\right] \Big/ \left[v_1^+ + v_1^-\right]$$

and $\beta_2 = \left[u_1^+(x_1 - x_2) + v_1^+ y_1 + v_1^- y_2\right] \Big/ \left[v_1^+ + v_1^-\right]$

and for $G(x_1) = F(y_2)$ an extreme point is (x_1, y_2) while for $G(x_2) = F(y_1)$ the corresponding extreme point is (x_2, y_1).

We construct the *smallest rectangle* - to be called SR hereafter - containing all the points of S by drawing lines parallel to the coordinate axes through four properly chosen points having respectively maximum and minimum abscissas and ordinates.

A point P lying outside SR cannot be the optimal location in the unconstrained case as a movement through P perpendicular to the nearest boundary of SR and towards it will cause the objective function value to diminish. Consequently, we shall have to seek the required solution within SR and with this end in view we shall concentrate on Γ_{AB}. But in the constrained case, besides an active boundary of the constrained region, an edge of the equipolygon lying outside SR may have to be considered in order to determine the optimal location.

The theorem stated below, which serves to find the direction of movement when three or more equipolygons coincide at a non-optimal point, will be used in developing our algorithm.

Theorem 1. While moving along an edge of E_{AB} and maintaining primal feasibility let a point G be obtained such that $\rho(G, A) = \rho(G, B) = \rho(G,C)$, $C \in S\setminus\{A, B\}$. By drawing lines through G parallel to the coordinate axes it is easily seen that if G is non-optimal, the point occupying the same quadrant with respect to G as the latest entrant C, is to be dropped.

It is to be noted further that when the number of equipolygons meeting at a point is more than three, by a repeated application of the above the number of dominating points can always be reduced to two.

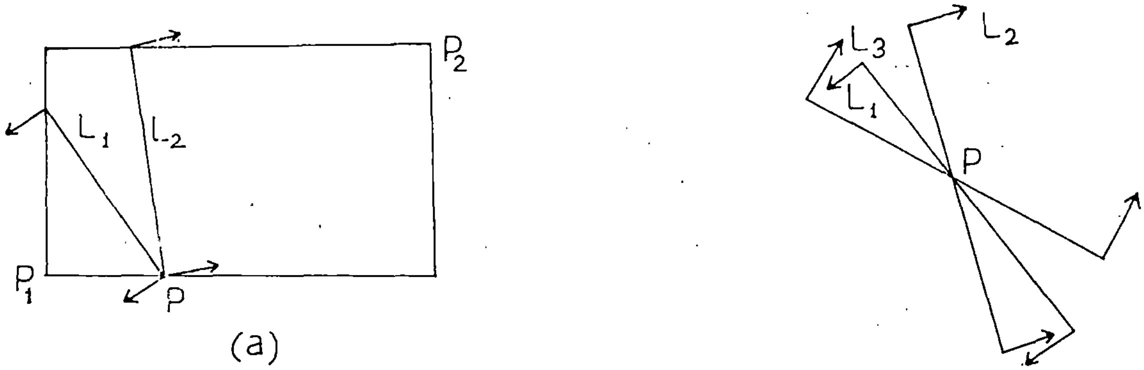
The proof of the theorem follows as a direct consequence of lemma 1.

We now state a criterion which will be useful in determining the optimal solution(s). The proof of the criterion is trivial and is, therefore, omitted.

Stopping criteria (SC)

Let L_i be the isoline of P_i through some point P of the equipolygon $E_{P_iP_j}$. Also let H_i be the half-space defined by L_i containing P_i. We define the cone $\mathscr{C}$ by $\mathscr{C} = \bigcap_{1}^{k} H_i$, where k denotes the cardinality of the set of isolines passing through P and having the same level value . Then, clearly, P is the vertex of $\mathscr{C}$. If $\Sigma = \mathscr{C} \cap R(P_i, P_j) = \{P\}$ we have a unique optimum at P as shown in figure 4. Refer to figure 5 for the case when P is not optimal.

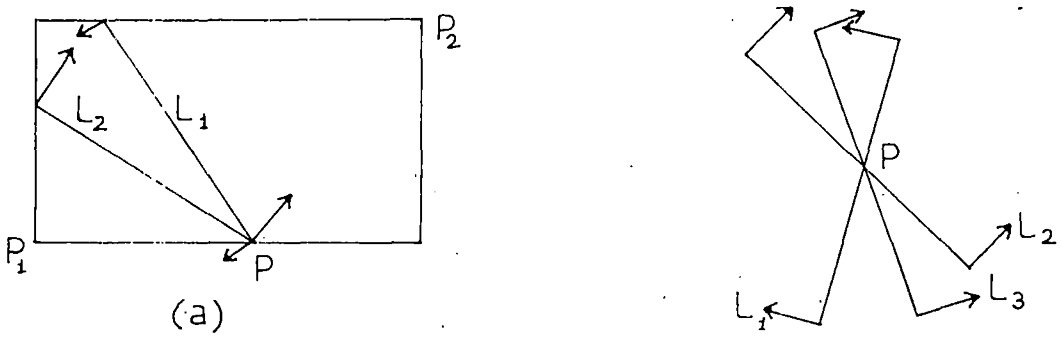
If, however, Σ degenerates into a line then we obtain a stretch comprising the set of optimal points one end of which is clearly P.



(a) (b)

Unique Optimum at P

Fig 4



(a) (b)

No Optimum at P

Fig 5

Before presenting the algorithm of the current problem we introduce a definition to be required afterwards.

The weighted rectilinear distance from any point Q on E_{AB} to the fixed point A is a positive quantity. Also the distance function, being necessarily continuous, must possess a lower bound which is attained by the function at least once. Let the lower bound (or when there are more than one, that lower bound which, after maintaining primal feasibility, comes first) correspond to the point we call ME_{AB}. There exists a direction from Q along which this distance monotonically decreases as Q approaches ME_{AB}. Or it may so happen that at some point P in between Q and ME_{AB} the condition of primal feasibility may be violated. In this latter case we leave E_{AB} and move from P to U to ME_{AB}, U being the point of intesection of E_{AB} and the isoline having the smaller gradient (vide appendix). The path from Q to ME_{AB} consisting of sides of E_{AB} or a combination of sides of E_{AB} and an isoline, maintaining primal feasibility, will be denoted by $S_{AB}(Q)$.

2.2.4. Algorithm

Step 0. Select any extreme point $P \in \partial SR$ as the starting point and find the location point P_i for which $W_i^* PP_i$ is a maximum. Denote P_i by A and obtain a point $Q \in L(P, A)$ such that

$$\left\{ W_i^* \, QP_i = W_A^* \, QA \; : \; P_i \in S \setminus \{A\} \text{ and } d(P,\, Q) \text{ is minimum} \right\}.$$

Without any loss of generality the point P_i satisfying the above may be denoted by B. Go to step 1.

Step 1.

```
If a point $Q_1 \in S_{AB}(Q)$ exists such that $\rho(Q_1,A) = \rho(Q_1,P_i)$,
    $P_i \in S \setminus \{A,B\}$ then go to step 2
else go to step 3.
```

Step 2.

```
If $Q_1$ satisfies SC then
   if $D_{AB}(Q_1) \neq 0$ then $Q_1$ is the unique optimal solution
   else
     if $Q \in \Gamma_{AB}$ then the stretch $QQ_1$ is the set of optimal
         solutions
     else the stretch $T_1Q_1$ of $\Gamma_{AB}$ constitutes the set of
         optimal solutions
else determine the points for the next iteration using
     theorem 1, rename these points as A and B, $Q \longleftarrow Q_1$ and
     go to step 1.
```

Step 3.

```
If $D_{AB}(T_1) = 0$ then
   if $Q \not\in \Gamma_{AB}$ then the whole of $\Gamma_{AB}$ comprises the solution set
   else the stretch $QT_2$ gives the set of optimal solutions
else if $T_1$ satisfies SC then $T_1$ is the unique optimal point
     else $T_2$ is the unique optimal point.
```

2.2.5. Numerical example

We now illustrate the working of the algorithm by means of an example. Let the demand points be all located on a rough inclined plane of inclination 10° having coefficient of friction $\mu = 0.3$, supposed uniform. The x-axis is taken to be horizontal and the y-axis upwards along the line of greatest slope. The forces necessary to overcome the combined effect of gravity and friction on a body of weight W in the upslope and downslope directions are approximately 0.48W and 0.12W respectively.

P_i	x_i	y_i	U_i^-	U_i^+	V_i^-	V_i^+
P_1	8	4	0.8	1.2	0.12	0.48
P_2	3	3	1.6	2.4	0.24	0.96
P_3	9	5	0.6	0.9	0.09	0.36
P_4	4	2	1.2	1.8	0.18	0.72
P_5	6	3	3.2	4.8	0.48	1.92
P_6	5	1	0.4	0.6	0.06	0.24
P_7	3	6	2.4	3.6	0.36	1.44
P_8	5	7	2.8	4.2	0.42	1.68
P_9	7	8	1.6	2.4	0.24	0.96
P_{10}	4	5	2.0	3.0	0.3	1.20

Table 1

We have chosen a model which depends on the velocity of wind in the horizontal direction. Thus if the wind blows steadily from east to west we may take the perturbed values of the forces in the x-increasing and x-decreasing directions to be 1.2W and 0.8W respectively. In the calculations that follow we retain figures correct to 3 places of decimal. Let the position and the weights in the four principal directions - West, East, South and North - associated with each existing location point be given as in table 1 above.

Let us take $P = (3, 8) \in \partial SR$ as the starting point. Using step 0 of the algorithm we can easily find $A = (6, 3)$, $B = (3, 6)$ and $Q = (5.400, 8.000)$. By step 1 it immediately follows that $P_i = (3, 3)$ and $Q_1 = (5.280, 6.300)$. By step 2, since Q_1 does not satisfy SC, $Q \leftarrow Q_1$, drop the point A, and $A \leftarrow P_i$. By two successive iterations of steps 1 and 2, we finally obtain $Q_1 = (4.254, 3.003)$. As Q_1 clearly satisfies SC and, moreover, $D_{AB}(Q_1) \neq 0$, Q_1 is the unique optimal point and the corresponding objective value = 5.597.

Computational experience

To develop the Pascal code of the algorithm we have randomly generated six vectors with n components each - two for position and four for associated directional weights - by means of standard Pascal procedure **Randomize** and function **Random**. This has been repeated for values of n between 500 and 1000 over a preselected rectangular region with unequal contiguous sides, giving rise to 500 random samples of varying sizes. Interestingly enough, in all cases the algorithm required at most three iterations to converge. As actual data for problems with an n of the stated size is not readily available, we had to be content with random data.

2.2.6. Operation count

Since the objective value strictly decreases with each iteration the same pair of points, once excluded at a particular iteration for a given zone, will never recur and consequently, prevention of cycling is guaranteed. Furthermore, we are in a position to employ the information, currently generated, for future use. Our method consists in moving along an equipolygon maintaining primal feasibility. In so moving we shall either obtain the optimal solution or attain a point G such that $\rho(G, A) = \rho(G, B) = \rho(G,C)$, $C \in S\setminus\{A, B\}$. From theorem 1 we know that a path different from the current one is to be chosen at G. Moreover, an edge of an equipolygon can not intersect that of another more than once in a particular zone and there remain (n-2) other equipolygons. Thus if a point is excluded at a particular iteration it will cease to be required if the iteration is restricted to the same zone. Thus at most (n-2) pairs of linear equations need to be solved for a particular iteration in a given zone. Again, inasmuch as the number of sides of an equipolygon is at most six the number of operations is $O(n^2)$ in the worst case.

2.2.7 Appendix

Choosing the direction of movement: Let E_{AB} be an equipolygon enclosing A where $x_A < x_B$. We divide the whole xy-plane into nine zones - I through IX - by drawing lines parallel to the coordinate axes through A and B. Let us assume for the preseent $y_A < y_B$. E_{AB} has generally six edges lying in zones I to VI as shown in figure 3. In II, IV and VI the directions of movement are from D to E, C to H and F to G respectively. In zone I the movement is along CD or DC according as $u_1^- / v_1^- >$ or $< u_2^- / v_2^-$. For zone III the direction is from E to F or the other way round depending on whether $u_1^- / v_1^+ <$ or $> u_2^- / v_2^+$. If it is along FE no movement along E_{AB} is possible without violating primal feasibility. Hence, in order to reduce the objective further we make a detour via the isoline having a smaller gradient until the point $U \in FG$ in VI is encountered whence movement will be along UG towards G. In zone V the direction of movement is governed by the **Stopping Criteria**. If $y_A > y_B$ the direction of movement in II is from E to D and the roles of I and III discussed above will simply be interchanged. Similar remarks hold when E_{AB} encloses B instead.