

CHAPTER - II.

PAPER-I

THERMAL STRESSES IN AN ANISOTROPIC NONHOMOGENEOUS SPINNING DISK OF VARIABLE THICKNESS UNDER VARIABLE THERMAL LOADING*

1. Introduction

Rotating disks form an essential element of the rotating machinery, namely turbines, flywheels etc. For disks made of isotropic materials works have been done by various authors, e.g. Leopold [26], Sengupta [38], Mollah [29], with and without thermal loading both for constant and variable thickness.

But the present day trend is to lay special emphasis to use anisotropic and composite materials in designing the elements of rotating machinery. Therefore, it is worthy to develop the expressions of stresses taking into account the anisotropic character of the material. Gururaja and Srinath [21] obtain the elastic stresses in a homogeneous anisotropic rotating disk of variable thickness under thermal loading in which the density and co-efficients of thermal expansion do not vary with radius. In recent times, it has been observed that in a medium with definite type of temperature, the homogeneous character breaks down giving rise to nonhomogeneity to elastic moduli as well as co-efficients of thermal expansion. But because of the final equations being in general

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non-tractable in nature, analytic solution of the problem with all variable parameters is nonexistent.

The author's endeavour here is to get closed form solution of such problem where elastic moduli, thickness, density, coefficients of linear expansion and excess of temperature at any radius over the temperature at the origin vary as any power of the distance from the centre of the disk. The results of Gururaja and Srinath [21] for homogeneous anisotropic body and those of Mollah [29], Sengupta [38] and Leopold [26] for isotropic solids are all found here as particular cases of the general solution.

2. Fundamental Equations.

Let the Z-axis be the axis of the disk, u the mean radial displacement at a distance r from the centre due to stressing and heating. Due to rotational symmetry the strain displacement relations are obviously

$$e_{rr} = \frac{du}{dr}, \quad e_{\theta\theta} = \frac{u}{r} \quad \dots(1)$$

If $(\sigma_r h)$ and $(\sigma_\theta h)$ are stress resultants in r and θ - direction, $2h$ being the thickness at that point, then the equation of equilibrium is

$$\frac{d}{dr} (\sigma_r h r) - h \sigma_\theta + h \rho \omega^2 r^2 = 0 \quad \dots(2)$$

where $\rho = \rho(r)$ and ω are the density and constant angular velocity of the disk respectively.

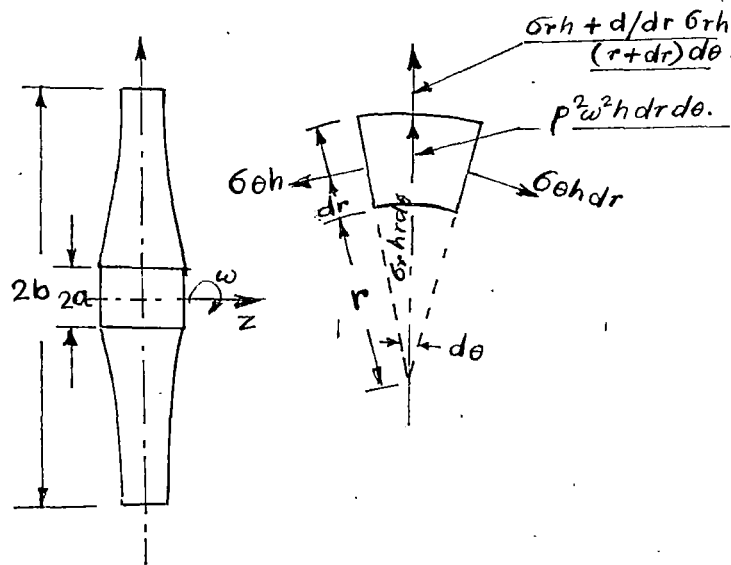


FIG. 1. ROTATING ANNULAR DISK AND AN ELEMENT OF THE DISK.

The particular type of material considered is an anisotropic solid possessing three orthogonal planes of symmetry in anisotropy at every point. The intersections of these planes, are called the principal axes of anisotropy. No shear deformation is produced by the normal stress applied in the directions of the principal axes of anisotropy. It is assumed for this analysis that the principal axes of anisotropy coincide with those of the stress.

For anisotropic materials the stress-strain relations are Lekhnitskii [25] or Gururaja and Srinath [21].

$$\left. \begin{aligned} e_{rr} &= \frac{\sigma_r}{E_1} - \nu_{21} \frac{\sigma_\theta}{E_2} + \alpha_1 T, \\ e_{\theta\theta} &= \frac{\sigma_\theta}{E_2} - \nu_{12} \frac{\sigma_r}{E_1} + \alpha_2 T, \end{aligned} \right\} \dots(3)$$

with $\nu_{12} E_2 = \nu_{21} E_1$

where E_1, E_2 are elastic moduli in r and θ directions, ν_{21} elastic constant characterizing compression in r -direction due to tension in θ - direction and α_1, α_2 are coefficients of thermal expansion in the r and θ directions respectively. $T = T(r)$ is the temperature at any radius r .

The condition of compatibility in the case of deformation symmetrical about central axis is

$$\frac{d}{dr} (r e_{\theta\theta}) - e_{rr} = 0$$

..(4)

3. Method of Solution.

We substitute the expressions of e_{rr} , $e_{\theta\theta}$ from equations (3) and eliminate σ_{θ} with the help of equation (2). The compatibility equation (4) now stands as

$$\begin{aligned} & r^2 \frac{d^2 \sigma_r}{dr^2} + \left(3 + \frac{r}{h} \frac{dh}{dr} - \frac{r}{E_2} \frac{dE_2}{dr} \right) r \frac{d\sigma_r}{dr} + \left\{ \frac{r}{h} \frac{dh}{dr} \left(2 - \frac{r}{E_2} \frac{dE_2}{dr} + \nu_{21} \right) \right. \\ & \left. + r^2 \frac{d}{dr} \left(\frac{1}{h} \frac{dh}{dr} \right) - r \left(\frac{1}{E_2} \frac{dE_2}{dr} - \frac{\nu_{21} E_2}{E_1^2} \cdot \frac{dE_1}{dr} \right) + \left(1 - \frac{E_2}{E_1} \right) \right\} \sigma_r \\ & = \left[\left\{ -(3 + \nu_{21}) + \frac{r}{E_2} \frac{dE_2}{dr} \right\} \rho - r \frac{d\rho}{dr} \right] \omega^2 r^2 + (\alpha_1 - \alpha_2) E_2 T - r E_2 \frac{d}{dr} (\alpha_2 T). \end{aligned} \quad \dots(5)$$

The material of the structure is anisotropic (anisotropy defined earlier) and, in addition, is continuously inhomogeneous, Greif and Chou [20], so that the Young's moduli are of the form

$$E_i = C_i \left(\frac{r}{b} \right)^{2m} \quad (i = 1, 2) \quad \dots(6)$$

' C_i 's are constants and m may take up any value, b is the outer radius of the disk.

Substituting E_1 and E_2 from the equation (6) to the equation (5) we get

$$\begin{aligned} & r^2 \frac{d^2 \sigma_r}{dr^2} + \left(3 + \frac{r}{h} \frac{dh}{dr} - 2m \right) r \frac{d\sigma_r}{dr} + \left\{ \frac{r}{h} \frac{dh}{dr} (2 - 2m + \nu_{21}) + r^2 \frac{d}{dr} \left(\frac{1}{h} \frac{dh}{dr} \right) \right. \\ & \left. - 2m \left(1 - \nu_{21} \frac{C_2}{C_1} \right) + \left(1 - \frac{C_2}{C_1} \right) \right\} \sigma_r = \left[\left\{ -(3 + \nu_{21}) + 2m \right\} \rho - r \frac{d\rho}{dr} \right] \omega^2 r^2 \\ & + (\alpha_1 - \alpha_2) C_2 \left(\frac{r}{b} \right)^{2m} T - r C_2 \left(\frac{r}{b} \right)^{2m} \frac{d}{dr} (\alpha_2 T) \end{aligned} \quad \dots(7)$$

Equation (7) can be solved in closed form by considering the hyperbolic thickness function

$$h = h_0 x^{-\beta} \quad \text{where } x = \frac{r}{b} \quad \dots(8)$$

The thickness $h = h(r)$ is characterized by the order of the hyperbola namely β . The profile is divergent when β is less than zero, convergent when β is greater than zero and it becomes uniformly thick disk, when β is equal to zero, which forms a special case of this analysis.

As the analysis concerns an annular disk neither zero nor infinite thickness at the centre is encountered.

Introducing the equation (8) into equation (7) the governing differential equation is

$$x^2 \frac{d^2 \sigma_r}{dx^2} + (3 - \beta - 2m)x \frac{d\sigma_r}{dx} + M \sigma_r = - (3 + \nu_{21} - 2m + q_1) \rho \omega^2 b^2 x^{\nu+2} + C_2 T_0 \{ A_1 - (1 + 2\beta) A_2 \} x^{2\beta+2m} + C_2 T_0 k \{ A_1 - (1 + 2\beta + n) A_2 \} x^{2\beta+2m+n} \quad \dots(9)$$

where M is a constant given by

$$M = \left\{ -\beta (1 - 2m + \nu_{21}) - 2m \left(1 - \nu_{21} \frac{C_2}{c_1} \right) + 1 - \frac{C_2}{c_1} \right\} \quad \dots(10)$$

In equation (9) it is assumed that the temperature distribution, density and co-efficients of thermal expansion of the disk under consideration obey the following laws :-

for temperature $T = T_0 (1 + k x^n)$..(11)

for density $\rho = \rho_0 x^q$..(12)

for coefficients of thermal expansion

$$\alpha_i = A_i x^{2p} \quad (i=1,2) \quad \text{..(13)}$$

n, q, p may take up any value. T_0, k, ρ_0, A_i are constants. For $n = 2, k = 1.5$ and $T_0 = 220$ the temperature distribution becomes typical of that of a high temperature gas turbine disk and temperature distribution profile for such a case is given in Fig. 2.

The general solution of the differential equation (9) is

$$\sigma_r = Ax^{\alpha_1} + Bx^{\beta_1} + F(x) \quad \text{..(14)}$$

where $\alpha_1, \beta_1 = \frac{-(2-\beta-2m) \pm \sqrt{(2-\beta-2m)^2 - 4M}}{2}$

$$F(x) = -\frac{(3+\nu_1-2m+q)\rho_0 \omega^2 b^2}{(q+2)(q+4-\beta-2m)+M} x^{\nu+2} + \frac{C_2 T_0 \{A_1 - (1+2p)A_2\}}{2(p+m)(2p+2-\beta)+M} x^{2p+2m}$$

$$+ \frac{C_2 T_0 k \{A_1 - (1+2p+\eta)A_2\}}{(2p+2m+\eta)(2p+\eta-\beta+2)+M} x^{2p+2m+\eta}$$

..(15)

A and B are integration constants.

4. Stress-Free Annular Disk

We consider here a circular disk $a \leq r \leq b$. The structure is made of nonhomogeneous anisotropic material. The body is under no influence of pressure on the boundaries.

The integration constants can then be found from stress-free inner and outer boundaries, namely

$$\left(\sigma_r\right)_{r=a} = 0, \quad \left(\sigma_r\right)_{r=b} = 0 \quad \dots(16)$$

$$\text{as } A = \frac{K^{\beta_1} F(1) - F(K)}{N}$$

$$B = - \left\{ \frac{K^{\alpha_1} F(1) - F(K)}{N} \right\}$$

$$\dots(17)$$

$$\text{where } N = K^{\alpha_1} - K^{\beta_1}$$

$$\text{and } K = \frac{a}{b}$$

$$\dots(18)$$

Using the values of A and B in equation (14) the radial stress is found to be

$$\sigma_r = \frac{1}{N} \left[\left\{ K^{\beta_1} F(1) - F(K) \right\} x^{\alpha_1} - \left\{ K^{\alpha_1} F(1) - F(K) \right\} x^{\beta_1} \right] + F(x)$$

$$\dots(19)$$

Applying the above expression of σ_r in equation (2) the stress in the circumferential direction becomes

$$\sigma_\theta = \frac{(\alpha_1 - \beta_1 + 1)}{N} \left\{ K^{\beta_1} F(1) - F(K) \right\} x^{\alpha_1} - \frac{(\beta_1 - \beta_1 + 1)}{N} \left\{ K^{\alpha_1} F(1) - F(K) \right\} x^{\beta_1}$$

$$+ (-\beta) F(x) + x F'(x) + \rho \omega^2 b^2 x^{\beta+2}$$

$$\dots(20)$$

The prime indicates derivative with respect to its argument.

4.1. Anisotropic Homogeneous Body and Gururaja & Srinath's Results.

For anisotropic homogeneous case $m = 0$ and if $q = 0$, $p = 0$, $n = 2$ i.e., where mechanical properties of the solid are invariant, density is uniform, co-efficients of thermal expansion are not varying with radius and the temperature profile is of Leopold's type, σ_r , σ_θ of equations (19), (20) come out to be

$$\sigma_r = \frac{\kappa_1^{\beta^*} F_0(1) - F_0(\kappa)}{\kappa^{\alpha_1^*} - \kappa^{\beta_1^*}} \cdot x^{\alpha_1^*} - \frac{\kappa^{\alpha_1^*} F_0(1) - F_0(\kappa)}{\kappa^{\alpha_1^*} - \kappa^{\beta_1^*}} x^{\beta_1^*} + F_0(x)$$

$$\sigma_\theta = \frac{\alpha_1^* - \beta + 1}{\kappa^{\alpha_1^*} - \kappa^{\beta_1^*}} \left[\kappa^{\beta_1^*} F_0(1) - F_0(\kappa) \right] x^{\alpha_1^*} - \frac{\beta_1^* - \beta + 1}{\kappa^{\alpha_1^*} - \kappa^{\beta_1^*}} \left[\kappa^{\alpha_1^*} F_0(1) - F_0(\kappa) \right] x^{\beta_1^*}$$

$$+ (1 - \beta) F_0(x) + x F_0'(x) + \rho_0 \omega^2 b^2 x^2$$

..(21)

in which $\alpha_1^*, \beta_1^* = \frac{-(2 - \beta) \pm \sqrt{\beta^2 + 4\beta\gamma_{21} + 4\frac{C_2}{c_1}}}{2}$

$$F_0(x) = \frac{C_0 T_0 k (A_1 - 3A_2) - (3 + \gamma_{21}) \rho_0 \omega^2 b^2}{9 - \frac{C_2}{c_1} - \beta(3 + \gamma_{21})} x + \frac{C_2 T_0 (A_1 - A_2)}{1 - \frac{C_2}{c_1} - \beta(1 + \gamma_{21})}$$

The above results for σ_r and σ_θ as in equations (21) when written in full on inserting $F_0(x)$, α_1^* and β_1^* tally with those obtained by Gururaja and Srinath [21] but for one mistake there in the expression of σ_r . They preferred symbols different from those used here, but if the symbols of this paper are converted to their notations one gets the results from (21) in their forms.

4.2. Isotropic Nonhomogeneous Body and Mollah's Results.

For isotropic nonhomogeneous case with uniform density one has to take $\nu_{21} = \nu_{12} = \nu$ say, $E_1 = E_2$ (so $C_1 = C_2 = C_0$ say,) and $\alpha_1 = \alpha_2$, (so $A_1 = A_2 = A_0$ say), also $q = 0$ (uniform density). Applying them in σ_r and σ_θ of equations (19) and (20) respectively one can have

$$\begin{aligned}
 \sigma_r = & - \frac{(3+\nu-2m)\rho_0\omega^2b^2}{8-\beta(3+\nu-2m)-2m(3-\nu)} \left[x^2 + \frac{\kappa_{\bar{\alpha}_1} - \kappa^2}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\beta}_1} - \frac{\kappa_{\bar{\beta}_1} - \kappa^2}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\alpha}_1} \right] \\
 & - \frac{C_0 A_0 (2p) T_0}{4p(p+m) + 4p + 2m(1+\nu) - \beta(2p+1+\nu)} \left[x^{2p+2m} + \frac{\kappa_{\bar{\alpha}_1} - \kappa^{2p+2m}}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\beta}_1} - \frac{\kappa_{\bar{\beta}_1} - \kappa^{2p+2m}}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\alpha}_1} \right] \\
 & - \frac{C_0 A_0 (2p+n) T_0 k}{4p(p+m+1) + 2m(1+\nu) + n(4p+2m) - \beta(2p+1+n+\nu) + n(2+n)} \left[x^{2p+2m+n} + \frac{\kappa_{\bar{\alpha}_1} - \kappa^{2p+2m+n}}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\beta}_1} \right. \\
 & \left. - \frac{\kappa_{\bar{\beta}_1} - \kappa^{2p+2m+n}}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\alpha}_1} \right] \\
 \sigma_\theta = & - \frac{(3+\nu-2m)\rho_0\omega^2b^2}{8-\beta(3+\nu-2m)-2m(3-\nu)} \left[\frac{1+3\nu-2m\nu}{3+\nu-2m} x^2 + (\bar{\beta}_1 - \beta + 1) \frac{\kappa_{\bar{\alpha}_1} - \kappa^2}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\beta}_1} \right. \\
 & \left. - (\bar{\alpha}_1 - \beta + 1) \frac{\kappa_{\bar{\beta}_1} - \kappa^2}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\alpha}_1} \right] - \frac{C_0 A_0 (2p) T_0}{4p(p+m) + 4p + 2m(1+\nu) - \beta(2p+1+\nu)} \left[(2p+2m-\beta+1) x^{2p+2m} \right. \\
 & \left. + (\bar{\beta}_1 - \beta + 1) \frac{\kappa_{\bar{\alpha}_1} - \kappa^{2p+2m}}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\beta}_1} - (\bar{\alpha}_1 - \beta + 1) \frac{\kappa_{\bar{\beta}_1} - \kappa^{2p+2m}}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\alpha}_1} \right] \\
 & - \frac{C_0 A_0 T_0 (2p+n) (T_0 k)}{4p(p+m+1) + 2m(1+\nu) + n(4p+2m) - \beta(2p+1+n+\nu) + n(2+n)} \left[(2p+2m+n-\beta+1) x^{2p+2m+n} \right. \\
 & \left. + (\bar{\beta}_1 - \beta + 1) \frac{\kappa_{\bar{\alpha}_1} - \kappa^{2p+2m+n}}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\beta}_1} - (\bar{\alpha}_1 - \beta + 1) \frac{\kappa_{\bar{\beta}_1} - \kappa^{2p+2m+n}}{\kappa_{\bar{\beta}_1} - \kappa_{\bar{\alpha}_1}} x^{\bar{\alpha}_1} \right]
 \end{aligned}$$

where $\bar{\alpha}_1, \bar{\beta}_1 = \frac{-(2-\beta-2m) \pm \sqrt{(2-\beta-2m)^2 + 4\beta(1+\nu)-2m) + 8m(1-\nu)}}{2}$.. (22)

Let $2m$ as well as $2p$ be replaced by m , $E_0 b^m$ and $\alpha_0 b^m$ be substituted instead of C_0 and A_0 respectively, $T_0 k$ be changed by kb^n and (β) be replaced by β . These changes are necessary to bring about the variations of parameters as taken by Mollah [29]. In this particular case, the above equations (22) would be same as found by Mollah with $\bar{\alpha}_1 = \alpha_1$ and $\bar{\beta}_1 = \beta_1$ in his paper.

4.3. Isotropic Homogeneous Body and Sengupta's Results.

Again for isotropic homogeneous disk of variable thickness ($\nu_{21} = \nu_{12} = \nu$, $C_0 = E_0$ and $A_0 = \alpha_0$ since $m = 0$, $p = 0$) with uniform density ($q = 0$), the equation (19) and (20) coincide with the results obtained in (2.8) and (2.9) respectively of Sengupta's paper [38]. Sengupta used λ instead of β and m_1, m_2 for α_1, β_1 of this paper.

4.4. Isotropic Homogeneous Body with Uniform Thickness and Leopold's Results.

For homogeneous uniform disk with uniform density ($m=0, p=0, \beta=0, q=0$) made of isotropic material ($\nu_{21} = \nu_{12} = \nu, C_0 = E_0, A_0 = \alpha_0$), the equations (19) and (20) coincide with the equations (7 a) and (7 b) of Leopold's paper [26].

5. Numerical Results.

All the results are for disks with finite outer radius b which is twice the inner radius a . It should be noted that the results previously derived are quite general. The problem investigated involve inhomogeneous materials with properties varying as any power of the radius.

We choose here $m = 0.5$, $2\beta = 1$, $p = .05$, $q = .25$,

$\nu_{21} = .3$, $C_2 = 2C_1$, $A_1 = 4 A_2$ and for temperature $n = 2$, $T_0 = 220$,
 $k = 1.5$ relevant to gas turbine temperature profile.

We make use of the above values in equation (10) to obtain $M = -1.5$ and
 as such equations (14) and (17) give ($\alpha_1 = 1$, $\beta_1 = -1.5$) and

$$A = 2.4783 \rho_0 \omega^2 a^2 - 1262.5 A_1 C_1$$

$$B = -.27435 \rho_0 \omega^2 a^2 + 20.18 A_1 C_1,$$

under the supposition that the edges are stress-free. We can now compute
 the different values of σ_r and σ_θ from (19) and (20) respectively
 for different values of r and they are presented here in tabular form.

$\frac{r}{b}$	σ_r	σ_θ
0.5	0	$2.20 \rho_0 \omega^2 a^2 - 81.64 A_1 C_1$
0.6	$.20 \rho_0 \omega^2 a^2 - 11.39 A_1 C_1$	$2.17 \rho_0 \omega^2 a^2 - 49.12 A_1 C_1$
0.7	$.28 \rho_0 \omega^2 a^2 - 15.43 A_1 C_1$	$2.15 \rho_0 \omega^2 a^2 - 15.65 A_1 C_1$
0.8	$.26 \rho_0 \omega^2 a^2 - 14.23 A_1 C_1$	$2.11 \rho_0 \omega^2 a^2 + 20.34 A_1 C_1$
0.9	$.17 \rho_0 \omega^2 a^2 - 8.99 A_1 C_1$	$2.04 \rho_0 \omega^2 a^2 + 60.08 A_1 C_1$
1.0	0	$1.93 \rho_0 \omega^2 a^2 + 104.61 A_1 C_1$

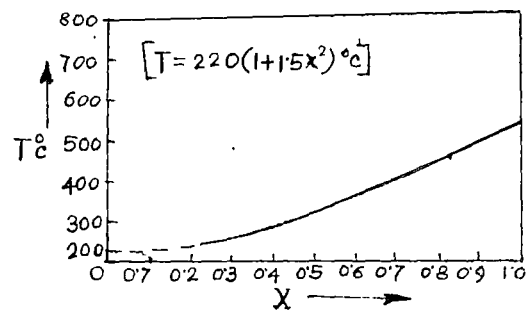


FIG.2 TEMPERATURE DISTRIBUTION PROFILE.