CHAPTER - I.

PAPER - I.

LARGE DEFLECTION ANALYSIS OF A SQUARE PLATE OF NONHOMOGENEOUS MATERIAL SUBJECTED TO NORMAL PRESSURE AND HEATING*

I. Introduction.

Berger's [4] approximate plate theory for large deflection plate problems suggests, in deriving the differential equations from strain energy, to neglect the strain energy due to second strain invariant in the middle plane of the plate, so as to obtain a fourth order differential equation coupled with a non-linear second order equation. The maximum deflections obtained in this process tally with the known values whenever available from more exact theories or from experiments. This technique is also followed by various authors like Nash and Modeer [31], Nowinski and Ismail [32], Das [15],[17] and Basuli [3].

Strain Energy method and Berger's [4] technique are employed here to tackle the present large deflection problem of a square plate of uniform thickness made of nonhomogeneous material where the plate is under normal pressure and heating, and the case of an infinite strip of nonhomogeneous material with uniform thickness subjected to normal pressure and heating is also dealt with. The

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Young's modulus is supposed to vary directly as the distance in one direction which is evident in many non-homogeneous structures, particularly when they are made of concrete or when they are raw planks from the branch side toward the root side, used extensively on temporary bridges in war time. In both the cases temperature plays an important role to dehydrate the structure and thereby change the strength of it. Numerical results are obtained for special type temperature distributions. The effect of this nonhomogeneity for an infinite strip is shown in graph. It is interesting to note that the deflection for such a case is nearly one fourth of the deflection for the homogeneous strip.

2. Nomenclature.

The following nomenclature is used here. The x - y plane is the undeflected middle plane of the plate and z - axis is perpendicular to it in the downward direction.

q = normal load intensity,

w = lateral displacement,

u, v = displacement components in the middle plane of the plate,

h = uniform thickness of the plate,

E = $E_o x$ = Young's modulus, E_o a constant,

D = flexural rigidity of the plate $= \dfrac{Eh^3}{12(1-\nu^2)}$

$D_o = \dfrac{E_o h^3}{12(1-\nu^2)}$ = constant,

ν = Poisson's ratio,

α = Co-efficient of linear expansion,

e_{xx}, e_{yy} and e_{xy} are the middle surface strain components,

$e = e_{xx} + e_{yy}$, the first strain invariant,

$e_2 = e_{xx}\, e_{yy} - \frac{1}{4} e^2_{xy}$, the second strain invariant,

$\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

3. Analysis.

Combining the strain energy of a plate loaded normally without temperature undergoing large deflection and the strain energy due to heating only, one gets, Berger [4], Williams [43],[44], Boley and Weiner [7],

$$V = \iint_S \left[\frac{D}{2} \left\{ (\nabla^2 w)^2 + \frac{12}{h^2} e^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} - q w \right] dx\, dy - \iint_S \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E \alpha T(x,y,z)}{(1-\nu)} \left[e - z \nabla^2 w \right] dx\, dy\, dz. \qquad ..(1)$$

The strain energy due to second strain invariant is neglected here as done by Berger. The symbol $\iint_S$ indicates integration over the surface of the plate. Assuming that the temperature T(x,y,z) can be written as

$$T(x,y,z) = T_o(x,y) + g(z)\, T_1(x,y) \qquad ..(2)$$

and

$$\left. \begin{aligned} \int_{-\frac{h}{2}}^{\frac{h}{2}} g(z)\, dz &= F(h) \\ \int_{-\frac{h}{2}}^{\frac{h}{2}} z\, g(z)\, dz &= f(h) \end{aligned} \right\} \qquad ..(3)$$

where F = F(h) and f = f(h) are constants, the equation (1) with the help of equations (2) and (3) can be rewritten as

$$V=\iint_S\left[\frac{D}{2}\left\{(\nabla^2 w)^2+\frac{12e^2}{h^2}-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}-qw\right.$$
$$\left.-\frac{E\alpha}{1-\nu}\left\{e\left[hT_0(x,y)+FT_1(x,y)\right]-fT_1(x,y)\nabla^2 w\right\}\right]dx\,dy.$$

..(4)

For minimum value of the strain energy V Euler's variational equations give

$$\frac{\partial V}{\partial u}-\frac{\partial}{\partial x}\cdot\frac{\partial V}{\partial u_x}-\frac{\partial}{\partial y}\cdot\frac{\partial V}{\partial u_y}=0,$$
$$\frac{\partial V}{\partial v}-\frac{\partial}{\partial x}\cdot\frac{\partial V}{\partial v_x}-\frac{\partial}{\partial y}\cdot\frac{\partial V}{\partial v_y}=0,$$
$$\frac{\partial V}{\partial w}-\frac{\partial}{\partial x}\cdot\frac{\partial V}{\partial w_x}-\frac{\partial}{\partial y}\cdot\frac{\partial V}{\partial w_y}+\frac{\partial^2}{\partial x^2}\cdot\frac{\partial V}{\partial w_{xx}}+\frac{\partial^2}{\partial x\partial y}\cdot\frac{\partial V}{\partial w_{xy}}+\frac{\partial^2}{\partial y^2}\cdot\frac{\partial V}{\partial w_{yy}}=0,$$

where $u_x=\dfrac{\partial u}{\partial x}$, $w_{xx}=\dfrac{\partial^2 w}{\partial x^2}$ **etc.** ..(5)

We apply the equation (4) to all the equations of (5) and obtain the following equations

$$\frac{\partial}{\partial x}E\left[eh-(1+\nu)\alpha(T_0h+T_1F)\right]=0,$$

..(6)

$$\frac{\partial}{\partial y}E\left[eh-(1+\nu)\alpha(T_0h+T_1F)\right]=0,$$

..(7)

and

$$\frac{\partial^2}{\partial x^2}\left[D\nabla^2 w-D(1-\nu)w_{yy}+T_1f\frac{E\alpha}{1-\nu}\right]+\frac{\partial^2}{\partial y^2}\left[D\nabla^2 w-D(1-\nu)w_{xx}+T_1f\frac{E\alpha}{1-\nu}\right]$$
$$+\frac{\partial^2}{\partial x\partial y}\left[2w_{xy}D(1-\nu)\right]-\frac{\partial}{\partial x}\left\{\left[\frac{Eeh}{1-\nu^2}-\frac{E\alpha}{1-\nu}\left(T_0h+T_1F\right)\right]w_x\right\}$$
$$-\frac{\partial}{\partial y}\left\{\left[\frac{Eeh}{1-\nu^2}-\frac{E\alpha}{1-\nu}\left(T_0h+T_1F\right)\right]w_y\right\}-q=0.$$

..(8)

From equations (6) and (7)

$$E\left[e h - \alpha(1+\nu)(T_0 h + T_1 F)\right] = \frac{\beta^2 h^3}{12} \qquad ..(9)$$

where β^2 is a real normalised constant and $e = u_x + v_y + \frac{1}{2}(w_x^2 + w_y^2)$. ..(10)

Let the inhomogeneity of the plate be characterized by $E = E_0 x$, so that $D = D_0 x$. The differential equation for determining the normal deflection w now comes from the equations (8) and (9) for $D = D_0 x$, which is

$$\nabla^2\left(x\nabla^2 - \frac{\beta^2}{E_0}\right)w = \frac{1}{D_0}\left[q - \frac{\alpha f}{1-\nu}\nabla^2(T_1 E)\right] \qquad ..(11)$$

We shall assume $w = w_1 + w_2$ for the complementary part of this equation. And for the complementary function of equation (11) we get

$$\nabla^2\left(x\nabla^2 - \frac{\beta^2}{E_0}\right)w = 0 \quad \text{or} \quad \nabla^2(x\nabla^2 w_1) - \frac{\beta^2}{E_0}\nabla^2 w_1 + \nabla^2\left(x\nabla^2 w_2 - \frac{\beta^2}{E_0}w_2\right) = 0$$

which is clearly satisfied if we choose

$$\nabla^2 w_1 = 0 \qquad ..(12)$$

and

$$x\nabla^2 w_2 - \frac{\beta^2}{E_0}w_2 = 0 \qquad ..(13)$$

This method is also adopted by Gran Olsson or Reissner and quoted in Timoshenko and Woinosky - Krieger [40] for problems of rectangular plates of variable thickness.

Let the contour of the square plate be given by $a \leq x \leq 3a$ and $-a \leq y \leq a \, (a > 0)$. The equations that must be solved for this plate with variable loading and temperature are equations (9) and (11).

To solve these equations u, v and w can be expanded in appropriate series. Since the choice of the type of series that is best suited to the problem will depend on the boundary conditions, no general solution for completely arbitrary boundary conditions will be given for the plate.

As an illustration, we consider the solution when all the edges are simply supported and hence the boundary conditions are Timoshenko and Woinowsky-Krieger [40], Basuli [3], Berger [4],

$$\left.\begin{array}{ll} u = w = \dfrac{\partial^2 w}{\partial x^2} = 0, & x = a,\ x = 3a \\ v = w = \dfrac{\partial^2 w}{\partial y^2} = 0, & y = \pm a \end{array}\right\} \quad ..(14)$$

Now w_1 of equation (12) which will satisfy the boundary conditions at the edges $y = \pm a$ may be represented by the Fourier series

$$w_1 = \sum_{n=1,3\cdots}^{\infty} w_n(x) \sin \frac{d_n}{2} y \quad ..(15)$$

where $d_n = \dfrac{2n\pi}{a}$. Using this expression of w_1 in equation (12) one gets

$$w_n(x) = A_1 \exp\left(\frac{d_n x}{2}\right) + A_2 \exp\left(-\frac{d_n}{2} x\right).$$

Similarly for equation (13) and the same boundary conditions at $y = \pm a$ we choose

$$w_2 = \sum_{n=1,3,5\cdots}^{\infty} \bar{w}_n(x) \sin \frac{d_n}{2} y \quad ..(16)$$

We find on substitution of this w_2 in equation (13)

$$x\frac{d^2\bar{w}_n}{dx^2} - \bar{w}_n\left(\frac{d_n^2}{4}x + \frac{\beta^2}{E_o}\right) = 0.$$

..(17)

The solution of (17) may be found to be, Murphy [30]

$$\bar{w}_n(x) = x\exp\left(-\frac{d_n}{2}x\right)\Big[A_3\,{}_1F_1(\xi,2;d_nx) + A_4\Big\{{}_1F_1(\xi,2;d_nx)\,l_n(d_nx) + \sum_{m=1}^{\infty}\mu_m(d_nx)^m + \frac{\Gamma(\xi-1)}{\Gamma(\xi)}\cdot\frac{1}{d_nx}\Big\}\Big]$$

$$= A_3\,\eta_n(x) + A_4\,\zeta_n(x), \quad \text{say}$$

..(18)

in which $\xi = 1 + \dfrac{\beta^2}{d_nE_o}$ and

the confluent hypergeometric series

$${}_1F_1(\xi,2;d_nx) = 1 + \frac{\xi}{2}d_nx + \frac{\xi(\xi+1)}{2.3.\lfloor 2}(d_nx)^2 + \cdots\cdots + \lambda_m(d_nx)^m + \cdots$$

where $\lambda_m = \dfrac{\xi(\xi+1)\cdots\cdots(\xi+m-1)}{2.3\cdots\cdots(2+m-1)\lfloor m}$

..(19)

$$\mu_m = \frac{\Gamma(\xi+m)\,H_m}{\lfloor m\,.\,\lfloor m+1\,\Gamma(\xi)}, \quad H_m = \sum_{r=0}^{m-1}\left[\frac{1}{\xi+r} - \frac{1}{2+r} - \frac{1}{r+1}\right].$$

The particular integral of (11) can be taken as $\left(-\dfrac{q_oE_o}{2\beta^2D_o}x^2\right)$ where $\dfrac{1}{D_o}\left[q - \dfrac{\alpha f}{1-\nu}\nabla^2(T,E)\right]$ is assumed to be a function of y only, say $\dfrac{q_o(y)}{D_o}$ and it can be expanded in Fourier series in the form

$$\frac{q_o(y)}{D_o} = \frac{4q_o}{\pi D_o}\sum_{n=1,3,5\cdots}^{\infty}\frac{1}{n}\sin\frac{d_n}{2}y.$$

..(20)

Finally,

$$w(x,y) = \sum_{n=1,3,5\cdots}^{\infty}\Big[A_1 \exp\Big(\frac{d_n x}{2}\Big) + A_2 \exp\Big(-\frac{d_n x}{2}\Big) + A_3 \eta_n(x) + A_4 \zeta_n(x) - \frac{2 q_0 x^2 E_0}{\beta^2 D_0 n\pi}\Big] \sin\frac{d_n y}{2} \qquad ..(21)$$

Using the boundary conditions on w one can determine the constants A_1, A_2, A_3 and A_4 of the above equation which are as follows,

$A_1 = (-1)(a_2 a_6 - a_3 a_5)/(a_1 a_5 - a_2 a_4)$, $A_2 = (a_1 a_6 - a_3 a_4)/(a_1 a_5 - a_2 a_4)$,

$D_1 A_3 = \zeta_n(3a)\, b_2 - \zeta_n(a)\, b_1$, $D_1 A_4 = \eta_n(a)\, b_1 - \eta_n(3a)\, b_2$, where

$D_1 = \zeta_n(a)\eta_n(3a) - \eta_n(a)\zeta_n(3a)$,

$$b_1 = A_1 \exp(3n\pi) + A_2 \exp(-3n\pi) - \frac{18 a^2 q_0 E_0}{\beta^2 D_0 \pi n},$$

$$b_2 = A_1 \exp(n\pi) + A_2 \exp(-n\pi) - \frac{2 a^2 q_0 E_0}{\beta^2 D_0 \pi n},$$

$$a_1 = \Big[D_2\{\exp(3n\pi)\zeta_n(a) - \exp(n\pi)\zeta_n(3a)\} - D_1 \frac{d_n^2}{4}\{\exp(3n\pi)\zeta_n''(a) - \exp(n\pi)\zeta_n''(3a)\}\Big],$$

$$a_2 = \Big[D_2\{\exp(-3n\pi)\zeta_n(a) - \exp(-n\pi)\zeta_n(3a)\} - D_1 \frac{d_n^2}{4}\{\exp(-3n\pi)\zeta_n''(a) - \exp(-n\pi)\zeta_n''(3a)\}\Big],$$

$$a_3 = \frac{2 q_0 E_0}{\beta^2 D_0 n\pi}\Big[D_2\{9a^2\zeta_n(a) - a^2\zeta_n(3a)\} - 2D_1\{\zeta_n''(a) - \zeta_n''(3a)\}\Big],$$

$$a_4 = \Big[D_3 \exp(3n\pi)\{\zeta_n''(3a) - \zeta_n(3a)\frac{d_n^2}{4}\} - D_4 \exp(n\pi)\{\zeta_n''(a) - \zeta_n(a)\frac{d_n^2}{4}\}\Big],$$

$$a_5 = \Big[D_3 \exp(-3n\pi)\{\zeta_n''(3a) - \frac{d_n^2}{4}\zeta_n(3a)\} - D_4 \exp(-n\pi)\{\zeta_n''(a) - \frac{d_n^2}{4}\zeta_n(a)\}\Big],$$

$$a_6 = \frac{2 q_0 E_0}{\beta^2 D_0 n\pi}\Big[D_3\{9a^2\zeta_n''(3a) - 2\zeta_n(3a)\} - D_4\{a^2\zeta_n''(a) - 2\zeta_n(a)\}\Big],$$

$D_2 = \zeta_n''(a)\eta_n''(3a) - \eta_n''(a)\zeta_n''(3a)$, $D_3 = \eta_n(a)\zeta_n''(a) - \zeta_n(a)\eta_n''(a)$,

$$D_4 = \eta_n(3a)\zeta_n''(3a) - \zeta_n(3a)\eta_n''(3a). \qquad ..(22)$$

$(\ )''$ means double differentiation with respect to its argument.

To determine β we know from (9)

$$\frac{\beta^2 h^2}{12 E_0 x} + \frac{\alpha(1+\nu)}{h}(T_0 h + T_1 F) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad ..(23)$$

In order to find β let us assume that

$$\left.\begin{aligned} u &= \sum_{k=0}^{\infty} M_k(x) \cos\frac{k\pi y}{a} \\ \text{and} \quad v &= \sum_{k=1}^{\infty} N_k(x) \sin\frac{k\pi y}{a} \end{aligned}\right\} \qquad ..(24)$$

where this form has been chosen because we know that u is an even function of y and v is an odd function of y. Combining equations (21), (23) and (24) one can easily get

$$\sum_{k=0}^{\infty} M_k'(x) \cos\frac{k\pi y}{a} + \sum_{k=1}^{\infty} N_k(x) \frac{k\pi}{a} \cos\frac{k\pi y}{a}$$
$$+ \frac{1}{2}\left[\sum_{n=1,3,5\ldots}^{\infty}\left(\frac{d_n}{2} A_1 \exp\left(\frac{d_n x}{2}\right) - \frac{d_n}{2} A_2 \exp\left(-\frac{d_n x}{2}\right) + A_3 \eta_n'(x) + A_4 \zeta_n'(x) - \frac{4 q_0 E_0 x}{\beta^2 D_0 n \pi}\right) \sin\frac{d_n}{2} y\right]^2$$
$$+ \frac{1}{2}\left[\left(A_1 \exp\left(\frac{d_n}{2} x\right) + A_2 \exp\left(-\frac{d_n x}{2}\right) + A_3 \eta_n(x) + A_4 \zeta_n(x) - \frac{2 q_0 E_0 x^2}{\beta^2 D_0 n \pi}\right) \frac{d_n}{2} \cos\frac{d_n}{2} y\right]^2$$
$$= \frac{\beta^2 h^2}{12 E_0 x} + \frac{\alpha(1+\nu)(T_0 h + T_1 F)}{h} \qquad ..(25)$$

Integrating this equation over the plate we get the required equation to determine β which is given in Appendix I.

4. Plate strip of infinite length.

Nonhomogeneous case ($E = E_0 x$).

If the plate is infinite in the y - direction only, then the differential equation (11) will take the form

$$x\frac{d^4w}{dx^4} + 2\frac{d^3w}{dx^3} - \frac{\beta^2}{E_0}\frac{d^2w}{dx^2} = \left[\frac{1}{D_0}\left\{q - \frac{\alpha f(h)}{1-\nu}\frac{d^2}{dx^2}(T_1 E)\right\}\right]$$
$$= \frac{q_0}{D_0}, \text{ say} \qquad ..(26)$$

The solution of the above equation is

$$w = \frac{x^{\frac{1}{2}}}{\rho_0^2}\left[B_1 I_1(2\rho_0 x^{\frac{1}{2}}) + B_2 K_1(2\rho_0 x^{\frac{1}{2}})\right] + B_3 x + B_4 - \frac{q_0 x^2}{2\rho_0^2 D_0} \qquad ..(27)$$

where $\left(-\frac{q_0 x^2}{2\rho_0^2 D_0}\right)$ is the particular integral of the equation (26) and $\rho_0^2 = \frac{\beta^2}{E_0}$, I_1 and K_1 being the modified Bessel functions of the first and second kind.

The necessary boundary conditions for simply supported case are

$$\left.\begin{array}{ll} u = w = 0 & \text{at } x = a,\ x = b \\ & \qquad\qquad a,\ b > 0 \\ \frac{d^2w}{dx^2} = 0 & \text{at } x = a,\ x = b \end{array}\right\} \qquad ..(28)$$

Using the equations (27) and (28) we find that 'B's can be obtained from $\{B_1\ B_2\ B_3\ B_4\}$ of which the augmented matrix is

$$\begin{bmatrix} I_1(\rho_2) & K_1(\rho_2) & \rho_0^2 a^{\frac{1}{2}} & \rho_0^2 a^{-\frac{1}{2}} & \frac{q_0 a^{\frac{3}{2}}}{2D_0} \\ I_1(\rho_1) & K_1(\rho_1) & \rho_0^2 b^{\frac{1}{2}} & \rho_0^2 b^{-\frac{1}{2}} & \frac{q_0 b^{\frac{3}{2}}}{2D_0} \\ I_1(\rho_2) & K_1(\rho_2) & 0 & 0 & \frac{q_0 a^{\frac{1}{2}}}{\rho_0^2 D_0} \\ I_1(\rho_1) & K_1(\rho_1) & 0 & 0 & \frac{q_0 b^{\frac{1}{2}}}{\rho_0^2 D_0} \end{bmatrix}$$

..(29)

where $\rho_1 = 2\rho_0 b^{\frac{1}{2}}$ and $\rho_2 = 2\rho_0 a^{\frac{1}{2}}$ ..(30)

The equation for determining the value of β in this case reduces to

$$\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{\beta^2 h^2}{12E_0 x} + \frac{\alpha(1+\nu)}{h}(t_0 h + t_1 F)$$

where $T_0 = t_0(x)$ and $T_1 = t_1(x)$. ..(31)

If $u = U(x)$ simply, then

$$U'(x) + \frac{1}{2}\left[\frac{1}{\rho_0}\left\{B_1 I_0(2\rho_0 x^{\frac{1}{2}}) - B_2 K_0(2\rho_0 x^{\frac{1}{2}})\right\} + B_3 - \frac{q_0 x}{\rho_0^2 D_0}\right]^2$$
$$= \frac{\beta^2 h^2}{12E_0 x} + \frac{\alpha(1+\nu)}{h}(t_0 h + t_1 F)$$

..(32)

Integrating the above over the strip between the limits a and b we get the following equation

$$\frac{\rho_0^6}{3} \ln \frac{b}{a} + \frac{4\alpha(1+\nu)\rho_0^4(b-a)}{h^2}\left[t_0^{(1)} + \frac{t_1^{(1)} F}{h} \cdot \frac{b+a}{4} \right]$$

$$= \frac{1}{h^2}\Big[-\frac{B_1^2 \bar{z}^2}{2}\left\{ I_1^2(\bar{z}) - I_0^2(\bar{z}) \right\} - \frac{B_2^2 \bar{z}^2}{2}\left\{ K_1^2(\bar{z}) - K_0^2(\bar{z}) \right\} + B_3^2 \rho_0^2 \frac{\bar{z}^2}{2}$$

$$+ \frac{q_0^2}{96\rho_0^6 D_0^2} \cdot \bar{z}^6 + 2 B_1 B_3 \rho_0 \bar{z} I_1(\bar{z})$$

$$- \frac{B_1 q_0}{2\rho_0^3 D_0}\left\{ \bar{z}^3 I_1(\bar{z}) - 2\bar{z}^2 I_2(\bar{z}) \right\} + 2 B_2 B_3 \rho_0 \bar{z} K_1(\bar{z})$$

$$- \frac{B_2 q_0}{2\rho_0^3 D_0}\left\{ \bar{z}^3 K_1(\bar{z}) - 2\bar{z}^2 K_2(\bar{z}) \right\} - \frac{B_3 q_0}{8\rho_0^2 D_0} \bar{z}^4 \Big\} \Big]_{2\rho_0\sqrt{a}}^{2\rho_0\sqrt{b}}$$

$$- 2\frac{B_1 B_2}{h^2}\left\{ \phi(b) - \phi(a) \right\} \qquad ..(33)$$

where we have chosen

$$t_0(x) = t_0^{(1)} = \text{constant}, \quad t_1(x) = t_1^{(1)} \frac{x}{2}. \qquad ..(34)$$

Here $$\phi(b) - \phi(a) = \int_{2\rho_0\sqrt{a}}^{2\rho_0\sqrt{b}} \bar{z}\, I_0(\bar{z})\, K_0(\bar{z})\, d\bar{z} \qquad ..(35)$$

Homogeneous case ($E = E_0$)

The governing equations for determining the normal deflection w come from the equations (8) and (9). When the plate is infinite in the y - direction only, the equation (8) becomes

$$\frac{d^4 w}{dx^4} - \frac{\beta^2}{E_0}\frac{d^2 w}{dx^2} = \frac{q_0}{D_0} \qquad ..(36)$$

where

$$q_0 = q - \frac{\alpha E_0 f}{1-\nu} \cdot \frac{d^2 T_1}{dx^2}$$

..(37)

The solution of the above differential equation (36) is evidently

$$w = c_1 \exp(\rho_0 x) + c_2 \exp(-\rho_0 x) + c_3 x + c_4 - \frac{q_0}{2\rho_0^2 D_0} x^2$$

..(38)

in which $\left(-\frac{q_0 x^2}{2\rho_0^2 D_0}\right)$ is the particular integral of (36). The arbitrary constants in (38) can be determined for the same boundary conditions stated in equations (28).

The equation corresponding to the equation (32) can be derived from the equation (9) with the help of (38). This equation, when integrated between the limits a and b, comes out to be

$$\frac{\rho_0^2}{6}(b-a) + \frac{2\alpha(1+\nu)(b-a)}{h^2}\left\{ t_0^{(1)} + \frac{t_1^{(1)} F}{h} \cdot \frac{(b+a)}{4} \right\}$$

$$= \frac{1}{h^2}\Big[\frac{\rho_0 c_1^2}{2} \exp(2\rho_0 x) - \frac{\rho_0 c_2^2}{2} \exp(-2\rho_0 x) + c_3^2 x + \frac{q_0^2 x^3}{3\rho_0^2 D_0^2} - 2\rho_0^2 c_1 c_2 x$$

$$+ 2c_1 c_3 \exp(\rho_0 x) - \frac{2c_1 q_0}{\rho_0 D_0}\left[x \exp(\rho_0 x) - \frac{\exp(\rho_0 x)}{\rho_0}\right] + 2c_2 c_3 \exp(-\rho_0 x)$$

$$- \frac{2c_2 q_0}{\rho_0 D_0}\left[x \exp(-\rho_0 x) + \frac{1}{\rho_0}\exp(-\rho_0 x)\right] - \frac{c_3 q_0}{\rho_0 D_0} x^2 \Big]_a^b$$

..(39)

5. Numerical Calculations.

1) Square plate :

As a test case we choose here a plate of thickness 2h having 400 square-inch-area i.e., $a = 10$ inches of which $E_0 = 40{,}000$ psi and assume that the value of β is such that $\beta^2 = 10$. As an example, we treat here a special type temperature distribution in the plate to show that the variability of temperature function may also be included and assume

$$T_0(x,y) = T_0^{(1)} = \text{constant}, \quad T_1(x,y) = T_1^{(1)} \frac{y^2}{x}$$

and $\frac{40\alpha(1+\nu)}{h^3}\left\{2T_0^{(1)} + \frac{\alpha F(h) T_1^{(1)} \ln 3}{6h}\right\} = \cdot 91550 \times 10^{-4}$, Williams [43],

and using these values in the equation obtainable from the equation (25) after integrating the same over the plate we get the corresponding load factor for that assumed value of β in the form

$$\frac{q_0}{D_0 h} = 106\cdot 05 \times 10^{-7}$$

The equation (21) with the above load factor helps us determine the deflection of the plate at ($x = 20$, $y = 5$, middle point of the portion of the plate in the first quadrant) as

$$\frac{w}{h} = 5.42$$

2) Plate of infinite strip :

We choose here an infinite strip such that $b = 16\ a$ and a = unit length. We further assume that $\beta = 100$ and $E_0 = 40{,}000$ psi both for homogeneous and nonhomogeneous cases so that $\rho_0 = 0.5$. For both the cases, the strips are under the same temperature distribution mentioned in (34) along with

$$\frac{30\alpha(1+\nu)}{h^2}\left[t_0^{(1)} + 4.25\,\frac{t_1^{(1)}F}{h}\right] = 0.375\,, \qquad \text{Williams [43].}$$

<u>Nonhomogeneous strip.</u>

With the above mentioned values one can easily find out the arbitrary constants 'B's of the equation (27) from the augmented matrix (29) such as

$$\left\{ B_1 \ B_2 \ B_3 \ B_4 \right\} = \frac{q_0}{D_0}\left\{ 1.638 \quad 5.109 \quad 18.000 \quad -32.000 \right\} \qquad ..(40)$$

Equation (40) when applied on (33) along with the chosen values of the parameters the appropriate load factor comes out to be

$$\frac{q_0}{D_0 h} = 65.078 \times 10^{-4} \qquad ..(41)$$

When relations (40) and (41) are inserted in (27), the nature of normal deflection becomes known to us in terms of h and it is shown in the graph.

<u>Homogeneous Strip.</u>

For the same chosen values of the parameters the arbitrary constants 'c's of (38) for the boundary conditions (28) are found to be $C_1 = .005\ \frac{q_0}{D_0}$, $C_2 = 26.363\ \frac{q_0}{D_0}$, $C_3 = 34.000\ \frac{q_0}{D_0}$ and $C_4 = -48.00\ \frac{q_0}{D_0}$.

These values are used in (39) to pick up the appropriate load factor as

$$\frac{q_0}{D_0 h} = 136.23 \times 10^{-4}$$

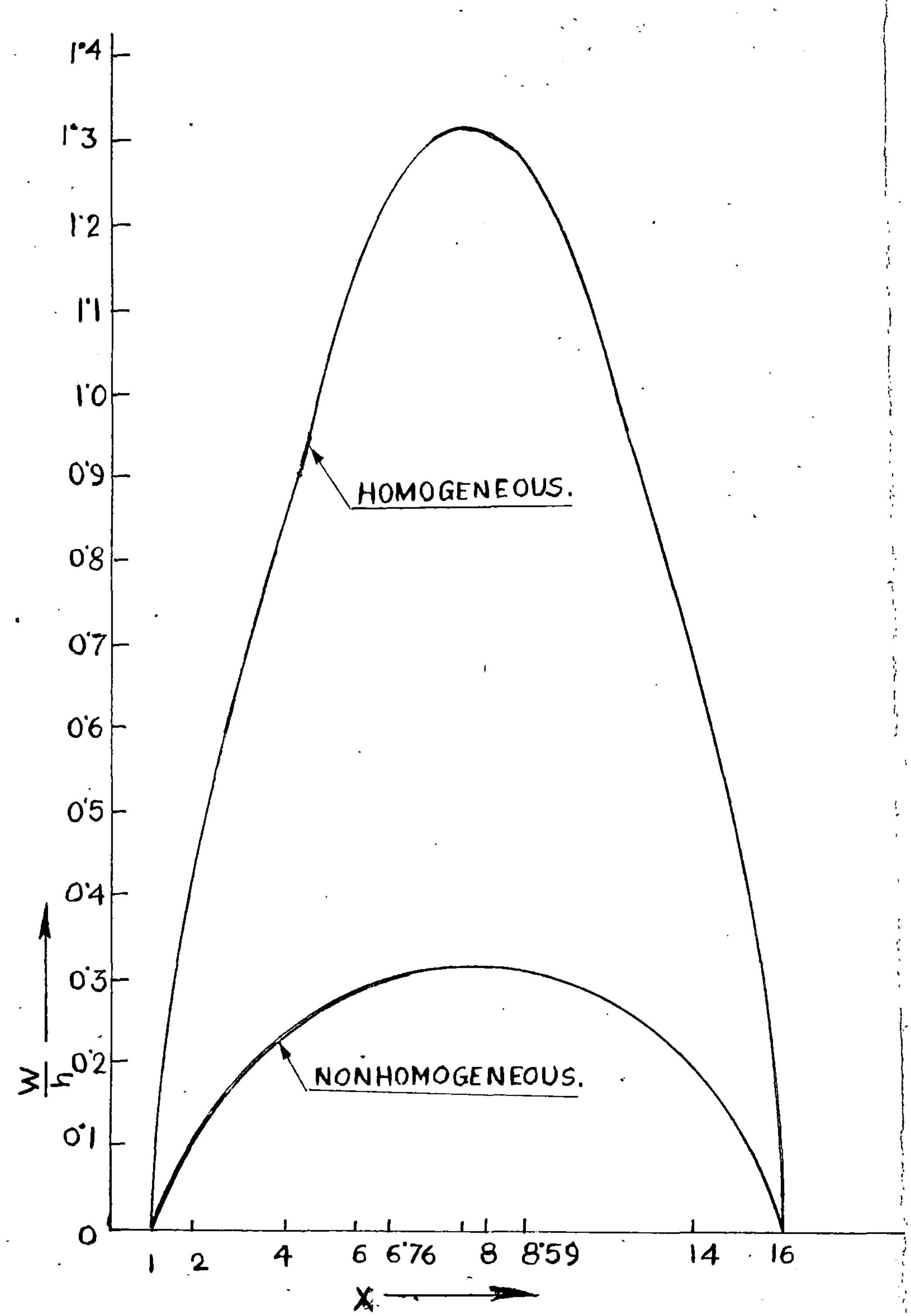
1·4
1·3
1·2
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
W/h
HOMOGENEOUS.
NONHOMOGENEOUS.
1
2
4
6
6·76
8
8·59
14
16
X

FIG.-

The relation (38) with these values of 'C's and the load factor clearly projects the nature of normal deflection in terms of h and it is depicted in the same graph.

6. Discussion.

The graph tells the fact that the normal deflection for the nonhomogeneous strip is always smaller than that for the homogeneous strip and roughly, we can say, normal deflection in homogeneous case=4 times that in nonhomogeneous case. The maximum deflection is, as usual, noticed at the middle of the strip in both the cases.

It is noteworthy to mention that the most practical variation of temperature in Z - direction is linear in Z so that $g(Z) = Z$ and therefore $f(h) = \frac{h^3}{12}$ and $F(h) = 0$ and $f(h)$ vanishes when $g(Z)$ is a constant where as $F(h) = h \times$ (that constant). For large deflection plate problems the deflection surface is vitally important and it is known only when the lateral displacement w is known. Stress distribution can then be revealed with the help of w from well established formulas given in any classical literature.

Appendix - I.

From the equation (25) the required equation for β may be obtained as

$$\left\{ \frac{\beta^2 h^2}{3E_0} \ln 3 \right\} + \frac{2}{a} \frac{\alpha(1+\nu)}{h} \int_a^{3a}\int_{-a}^{a} \frac{T_0 h + T_1 F}{h}\, dx\,dy$$

$$= A_1^2 \frac{d_n}{2}\left\{-\exp(6n\pi) - \exp(2n\pi)\right\} - A_2^2 \frac{d_n}{2}\left\{\exp(-6n\pi) - \exp(-2n\pi)\right\}$$

$$+ A_3^2 \frac{d_n^2}{4} \sum_{m,s=0}^{\infty} \lambda_m \lambda_s (d_n)^{m+s} \left[I^{(x)}_{m+s+2,\,n} \right]_a^{3a}$$

contd.

$$+ A_3^2 \sum_{m,s=0}^{\infty} \lambda_m \lambda_s (d_n)^{m+s} \Big[(m+1)(s+1)\, I_{m+s,n}^{(x)} - \frac{d_n}{2}(m+s+2)\, I_{m+s+1,n}^{(x)} + \frac{d_n^2}{2} I_{m+s+2,n}^{(x)} \Big]_a^{3a}$$

$$+ \frac{16 q_0^2 E_0^2}{3\beta^4 D_0^2 n^2 \pi^2} \times 26a^3 + \frac{4 q_0^2 E_0^2 n\pi}{5\beta^4 D_0^2} \times 242 a^3 + d_n A_1 A_3 \Big[\exp\Big(\frac{d_n x}{2}\Big) \eta_n(x) \Big]_a^{3a}$$

$$+ d_n A_1 A_4 \Big[\exp\Big(\frac{d_n x}{2}\Big) \xi_n(x) \Big]_a^{3a} - \frac{4 q_0 E_0}{a\beta^2 D_0} A_1 \Big[x^2 \exp\Big(\frac{d_n x}{2}\Big) \Big]_a^{3a}$$

$$+ d_n A_2 A_3 \Big[-\exp\Big(-\frac{d_n x}{2}\Big) \eta_n(x) \Big]_a^{3a} + d_n A_2 A_4 \Big[-\exp\Big(-\frac{d_n x}{2}\Big) \xi_n(x) \Big]_a^{3a}$$

$$+ \frac{4 q_0 E_0}{a\beta^2 D_0} A_2 \Big[x^2 \exp\Big(-\frac{d_n x}{2}\Big) \Big]_a^{3a} - \frac{8 q_0 E_0}{\beta^2 D_0 n\pi} A_4 \Big[\sum_{m=0}^{\infty} \lambda_m (d_n)^m \Big[\Big\{ (m+1) \ln(d_n x)\, I_{m+1,n}^{(x)}$$

$$+ \frac{I_{m,n}^{(x)}}{\frac{d_n}{2}} + \frac{(m+1) I_{m-1,n}^{(x)}}{\left(\frac{d_n}{2}\right)^2} + \cdots\cdots + \frac{\lfloor m+1}{(d_n)^{m+2}} \chi_n(x) \Big\} - \Big\{ \frac{\exp\left(-\frac{d_n x}{2}\right)}{\frac{d_n}{2}} J_{m+1,n}^{(x)} \Big\}$$

$$+ \Big\{ \exp\Big(-\frac{d_n x}{2}\Big) J_{m+2,n}^{(x)} \Big\} \Big] + \sum_{m'=1}^{\infty} \mu_{m'} (d_n)^{m'} \Big[- \Big\{ (m'+1) \frac{\exp\left(-\frac{d_n x}{2}\right)}{\frac{d_n}{2}} J_{m'+1,n}^{(x)} \Big\}$$

$$+ \exp\Big(-\frac{d_n x}{2}\Big) J_{m'+2,n}^{(x)} + \frac{1}{2} \frac{\Gamma(\xi-1)}{\Gamma(\xi)} \frac{\exp\left(-\frac{d_n x}{2}\right)}{d_n} \Big(x + \frac{1}{\frac{d_n}{2}} \Big) \Big] \Big]_a^{3a} - \frac{4 q_0 E_0 n\pi}{a^2 \beta^2 D_0} A_4 \Big[$$

$$\sum_{m=0}^{\infty} \lambda_m (d_n)^m \Big\{ \ln(d_n x)\, I_{m+3,n}^{(x)} + \frac{I_{m+2,n}^{(x)}}{\frac{d_n}{2}} + \frac{(m+3) I_{m+1,n}^{(x)}}{\left(\frac{d_n}{2}\right)^2} + \cdots + \frac{\lfloor m+3}{\left(\frac{d_n}{2}\right)^{m+4}} \chi_n(x) \Big\}$$

$$- \sum_{m=1}^{\infty} \mu_m (d_n)^m \Big\{ \frac{\exp\left(-\frac{d_n x}{2}\right)}{\frac{d_n}{2}} \Big\} J_{m+3,n}^{(x)} - \frac{\Gamma(\xi-1)}{\Gamma(\xi)} \frac{1}{d_n} \frac{\exp\left(-\frac{d_n x}{2}\right)}{\frac{d_n}{2}} \Big\{ x^2 + \frac{2x}{\frac{d_n}{2}} + \frac{2}{\left(\frac{d_n}{2}\right)^2} \Big\} \Big]_a^{3a}$$

$$+ \frac{8 q_0 E_0}{\beta^2 D_0 n\pi} A_3 \Big[\sum_{m=0}^{\infty} \lambda_m (d_n)^m \Big[-(m+1) J_{m+1,n}^{(x)} \frac{\exp\left(-\frac{d_n x}{2}\right)}{\frac{d_n}{2}} + \exp\Big(-\frac{d_n x}{2}\Big) J_{m+2,n}^{(x)} \Big] \Big]_a^{3a}$$

$$+ \frac{4 q_0 E_0 n\pi}{a^2 \beta^2 D_0} A_3 \Big[\sum_{m=0}^{\infty} \lambda_m (d_n)^m \Big[\frac{\exp\left(-\frac{d_n x}{2}\right)}{\frac{d_n}{2}} J_{m+3,n}^{(x)} \Big] \Big]_a^{3a} + A_4^2 \Big[\sum_{m,s=0}^{\infty} \lambda_m \lambda_s (d_n)^{m+s} \Big[$$

$$(m+1)(s+1) \Big\{ (\ln d_n x)^2 I_{m+s,n}^{(x)} + \frac{2}{d_n} \Big[\Big(\ln(d_n x)\, I_{m+s-1,n}^{(x)} + \frac{I_{m+s-2,n}^{(x)}}{d_n}$$

contd.

$$+\frac{(m+s-1)I^{(x)}_{m+s-3,n}}{d_n^2}+\cdots\cdots+\frac{\lfloor m+s-1}{(d_n)^{m+s}}\mathcal{X}_n(x)\Big)+\cdots+\frac{\lfloor m+s}{(d_n)^{m+s}}\Big(\ln(d_n x)$$

$$\mathcal{X}_n(x)-\Big\{\frac{(l_n x)^2}{2}-d_n x+\frac{d_n^2}{\lfloor 2}\cdot\frac{x^2}{4}-\cdots\cdots\Big\}\Big)\Big]\Big]\Big\}$$

$$+(m+s+2)\Big\{(\ln d_n x)I^{(x)}_{m+s,n}+\frac{I^{(x)}_{m+s-1,n}}{d_n}+\frac{(m+s)I^{(x)}_{m+s-2,n}}{d_n^2}+\cdots+\frac{\lfloor m+s}{(d_n)^{m+s+1}}\mathcal{X}_n(x)\Big\}$$

$$-(m+s+2)\frac{d_n}{2}\Big\{(\ln d_n x)^2 I^{(x)}_{m+s+1,n}+\frac{2}{d_n}\Big[\ln(d_n x)I^{(x)}_{m+s,n}+\frac{I^{(x)}_{m+s-1,n}}{d_n}$$

$$+\frac{(m+s)I^{(x)}_{m+s-2,n}}{d_n^2}+\cdots+\frac{\lfloor m+s}{(d_n)^{m+s+1}}\mathcal{X}_n(x)\Big)+\cdots+\frac{\lfloor m+s+1}{(d_n)^{m+s+1}}\Big(\ln(d_n x)\mathcal{X}_n(x)$$

$$-\Big\{\frac{(l_n x)^2}{2}-d_n x+\frac{d_n^2}{\lfloor 2}\cdot\frac{x^4}{4}-\cdots\Big\}\Big)\Big]\Big]-d_n\Big\{\ln(d_n x)I^{(x)}_{m+s+1,n}+\frac{I^{(x)}_{m+s,n}}{d_n}$$

$$+\frac{(m+s+1)I^{(x)}_{m+s-1,n}}{d_n^2}+\cdots+\frac{\lfloor m+s+1}{(d_n)^{m+s+2}}\mathcal{X}_n(x)\Big\}+I^{(x)}_{m+s,n}+\frac{d_n^2}{4}\Big\{(\ln d_n x)^2 I^{(x)}_{m+s+2,n}$$

$$+\frac{2}{d_n}\Big[\ln(d_n x)I^{(x)}_{m+s+1,n}+\frac{I^{(x)}_{m+s,n}}{d_n}+\frac{(m+s+1)I^{(x)}_{m+s-1,n}}{d_n^2}+\cdots\cdots+\frac{\lfloor m+s+1}{(d_n)^{m+s+2}}\mathcal{X}_n(x)+$$

$$\cdots\cdots+\frac{\lfloor m+s+2}{(d_n)^{m+s+2}}\Big((\ln d_n x)\mathcal{X}_n(x)-\Big\{\frac{(l_n x)^2}{2}-d_n x+\frac{d_n^2}{\lfloor 2}\cdot\frac{x^4}{4}-\cdots\cdots\Big\}\Big)\Big]\Big]\Big\}\Big]$$

$$+\sum_{m',s'=1}^{\infty}\mu_{m'}\mu_{s'}(d_n)^{m'+s'}\Big\{(m'+1)(s'+1)I^{(x)}_{m'+s',n}-(m'+s'+2)\frac{d_n}{2}I^{(x)}_{m'+s'+1,n}$$

$$-\Big(\frac{d_n}{2}\Big)^2 I^{(x)}_{m'+s'+2,n}-\frac{1}{4}\Big[\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\Big]^2\frac{\exp(-d_n x)}{d_n}+2\sum_{\substack{m=0\\m'=1}}(d_n)^{m+m'}\lambda_m\mu_{m'}\Big[$$

$$(m+1)(m'+1)\Big\{\ln(d_n x)I^{(x)}_{m+m',n}+\frac{I^{(x)}_{m+m'-1,n}}{d_n}+\frac{(m+m')I^{(x)}_{m+m'-2,n}}{d_n^2}+\cdots\cdots$$

$$+\frac{\lfloor m+m'}{(d_n)^{m+m'+1}}\mathcal{X}_n(x)\Big\}+(m'+1)I^{(x)}_{m+m',n}-\frac{d_n}{2}(m+m'+2)\Big\{\ln(d_n x)I^{(x)}_{m+m'+1,n}$$

contd.

$$+\frac{I^{(x)}_{m+m',n}}{d_n}+\frac{(m+m'+1)\,I^{(x)}_{m+m'-1,n}}{d_n^2}+\cdots\cdots+\frac{\lfloor m+m'+1}{(d_n)^{m+m'+2}}\mathcal{X}_n(x)\Big\}-\left(\frac{d_n}{2}\right)I^{(x)}_{m+m'+1,n}$$

$$+\left(\frac{d_n}{2}\right)^2\Big\{\ln(d_nx)\,I^{(x)}_{m+m'+2,n}+\frac{I^{(x)}_{m+m'+1,n}}{d_n}+\frac{(m+m'+2)\;I^{(x)}_{m+m',n}}{d_n^2}+\cdots\cdots$$

$$+\frac{\lfloor m+m'+2}{(d_n)^{m+m'+3}}\mathcal{X}_n(x)\Big\}\Big]-\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\sum_{m=0}^{\infty}\lambda_m\,(d_n)^m\Big[(m+1)\Big\{\ln(d_nx)\,I_{m,n}^{(x)}$$

$$+\frac{I^{(x)}_{m-1,n}}{d_n}+\frac{m\,I^{(x)}_{m-2,n}}{d_n^2}+\cdots\cdots+\frac{\lfloor m}{(d_n)^{m+1}}\mathcal{X}_n(x)\Big\}+I^{(x)}_{m,n}-\frac{d_n}{2}\Big\{\ln(d_nx)\,I^{(x)}_{m+1,n}$$

$$+\frac{I^{(x)}_{m,n}}{d_n}+\frac{(m+1)\,I^{(x)}_{m-1,n}}{d_n^2}+\cdots\cdots+\frac{\lfloor m+1}{(d_n)^{m+2}}\mathcal{X}_n(x)\Big\}\Big]-\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\sum_{m'=1}^{\infty}\mu_{m'}\,(d_n)^{m'}$$

$$\Big[(m'+1)\,I^{(x)}_{m',n}-\frac{d_n}{2}\;I^{(x)}_{m'+1,n}\Big]+\frac{d_n^2}{4}\Big[\sum_{m,s=0}^{\infty}\lambda_m\lambda_s\,(d_n)^{m+s}\Big\{(\ln d_nx)^2\,I^{(x)}_{m+s+2,n}$$

$$+\frac{2}{d_n}\Big[\ln(d_nx)\,I^{(x)}_{m+s+1,n}+\frac{I^{(x)}_{m+s,n}}{d_n}+\frac{(m+s+1)\,I^{(x)}_{m+s-1,n}}{d_n^2}+\cdots\cdots+\frac{\lfloor m+s+1}{(d_n)^{m+s+2}}\mathcal{X}_n(x)+\cdots$$

$$+\frac{\lfloor m+s+2}{(d_n)^{m+s+2}}\Big\{\ln(d_nx)\,\mathcal{X}_n(x)-\Big(\frac{(\ln x)^2}{2}-d_nx+\frac{d_n^2}{\lfloor 2}\cdot\frac{x^2}{4}-\cdots\cdots\Big)\Big\}\Big]\Big\}$$

$$+\sum_{m',s'=1}^{\infty}\mu_{m'}\mu_{s'}(d_n)^{m'+s'}\,I^{(x)}_{m'+s'+2,n}-\left(\frac{1}{d_n}\right)^3\left\{\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\right\}^2\exp(-d_nx)$$

$$+2\sum_{\substack{m=0\\m'=1}}^{\infty}(d_n)^{m+m'}\mu_{m'}\lambda_m\Big\{\ln(d_nx)\,I^{(x)}_{m+m'+2,n}+\frac{I^{(x)}_{m+m'+1,n}}{d_n}+\frac{(m+m'+2)\,I^{(x)}_{m+m',n}}{d_n^2}$$

$$+\cdots\cdots+\frac{\lfloor m+m'+2}{(d_n)^{m+m'+3}}\mathcal{X}_n(x)\Big\}+2\sum_{m=0}^{\infty}\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\left(\frac{1}{d_n}\right)\lambda_m(d_n)^m\Big\{\ln(d_nx)\,I^{(x)}_{m+1,n}$$

$$+\frac{I^{(x)}_{m,n}}{d_n}+\frac{(m+1)\,I^{(x)}_{m-1,n}}{d_n^2}+\cdots\cdots+\frac{\lfloor m+1}{(d_n)^{m+2}}\mathcal{X}_n(x)\Big\}+2\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\left(\frac{1}{d_n}\right)\sum_{m'=1}^{\infty}\mu_{m'}(d_n)^{m'}I^{(x)}_{m'+1,n}\Big]\Big]_a^{3a}$$

$$+2A_4A_3\Big[\sum_{m,s=0}^{\infty}\lambda_m\lambda_s\,(d_n)^{m+s}\Big[\Big\{(m+1)(s+1)\Big[\ln(d_nx)\,I^{(x)}_{m+s,n}+\frac{I^{(x)}_{m+s-1,n}}{d_n}$$

contd.

$$+\frac{(m+s)I^{(x)}_{m+s-2,n}}{d_n^2}+\cdots+\frac{\lfloor m+s}{(d_n)^{m+s+1}}\chi_n(x)\Big]\Big\}+(m+1)I^{(x)}_{m+s,n}-\frac{d_n}{2}(m+2)I^{(x)}_{m+s+1,n}$$

$$-\frac{d_n}{2}(s+1)\Big\{l_n(d_n x)I^{(x)}_{m+s+1,n}+\frac{I^{(x)}_{m+s,n}}{d_n}+\frac{(m+s+1)I^{(x)}_{m+s-1,n}}{d_n^2}+\cdots\cdots$$

$$+\frac{\lfloor m+s+1}{(d_n)^{m+s+2}}\chi_n(x)\Big\}+\frac{d_n^2}{4}I^{(x)}_{m+s+2,n}\Big]+\sum_{\substack{m=0\\m'=1}}^{\infty}\lambda_m\mu_{m'}(d_n)^{m+m'}\Big[(m+1)(m'+1)I^{(x)}_{m+m',n}$$

$$-\frac{d_n}{2}(m+m'+2)I^{(x)}_{m+m'+2,n}+\frac{d_n^2}{4}I^{(x)}_{m+m'+2,n}\Big]-\frac{1}{2}\sum_{m=0}^{\infty}\lambda_m(d_n)^m\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\times$$

$$\Big[(m+1)I^{(x)}_{m,n}-\frac{d_n}{2}I^{(x)}_{m+1,n}\Big]+\frac{d_n^2}{4}\Big[\sum_{m,s=0}^{\infty}\lambda_m\lambda_s(d_n)^{m+s}\Big\{l_n(d_n x)I^{(x)}_{m+s+2,n}$$

$$+\frac{I^{(x)}_{m+s+1,n}}{d_n}+\frac{(m+s+2)I^{(x)}_{m+s,n}}{d_n^2}+\cdots+\frac{\lfloor m+s+2}{(d_n)^{m+s+3}}\chi_n(x)\Big\}$$

$$+\sum_{\substack{m=0\\m'=1}}^{\infty}\lambda_m\mu_{m'}(d_n)^{m+m'}I^{(x)}_{m+m'+2,n}+\frac{\Gamma(\xi-1)}{\Gamma(\xi)}\sum_{m=0}^{\infty}(d_n)^{m-1}I^{(x)}_{m+1,n}\Big]\Big]_a^{3a}$$

where $$I^{(x)}_{m,n}=-\frac{\exp(-d_n x)}{d_n}\Big\{x^m+\frac{m x^{m-1}}{d_n}+\cdots+\frac{\lfloor m}{(d_n)^m}\Big\}$$

and $$\chi_n(x)=L_n(x)-d_n\cdot x+\frac{d_n^2}{\lfloor 4}\cdot\frac{x^2}{2}-\cdots\cdots$$

$$l_n(x)=\log_e x$$

& $$J_{m,n}(x)=x^m+\frac{m x^{m-1}}{\left(\frac{d_n}{2}\right)}+\cdots\cdots+\frac{\lfloor m}{\left(\frac{d_n}{2}\right)^m}.$$

PAPER - II.

LARGE DEFLECTION ANALYSIS OF ANNULAR PLATE OF NONHOMOGENEOUS MATERIAL SUBJECTED TO VARIABLE NORMAL PRESSURE AND HEATING*

1. Introduction

Strain energy method and Berger's [4] technique are used here also for the large deflection problem of an annular circular plate of nonhomogeneous material where the plate is under variable normal pressure and is subjected to two dimensional temperature distribution i.e. the temperature varies along the radius and the width of the plate. The Young's modulus is supposed to vary as any power of the radius and it characterizes the nonhomogeneity of the plate. Calculations for the specific cases of both clamped and simply supported plates subjected to linearly varying pressure and a general type temperature distribution for two different types of nonhomogeneity are presented.

2. Nomenclature

a, b	=	the outer and inner radii of the plate,
h	=	thickness of the plate,
$q = q(r)$	=	normal load intensity,
$T = T(r, z)$	=	the temperature distribution in the plate,
α	=	co-efficient of linear expansion of the material of the plate,

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$E = E_o r^{2m}$ = Young's modulus of the material of the plate at a distance r from the centre,

ν = Poisson's ratio of the material of the plate,

$D = \frac{Eh^3}{12(1-\nu^2)} = \frac{E_o h^3}{12(1-\nu^2)} \cdot r^{2m} = D_o r^{2m}$ = the flexural rigidity of the plate,

u, w = radial and lateral displacements,

$e_{rr} = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2$, $e_{\theta\theta} = \frac{u}{r}$ radial and tangential strains,

$e = e_{rr} + e_{\theta\theta}$ = first strain invariant,

$e_2 = e_{rr} \cdot e_{\theta\theta}$ = second strain invariant,

$\bar{e}_{rr} = e_{rr} - z\frac{d^2w}{dr^2}$, $\bar{e}_{\theta\theta} = (e_{\theta\theta}) - \left(\frac{z}{r}\frac{dw}{dr}\right)$, e_{zz}, three dimensional thermal strains,

λ', μ' = Lamé's constants,

σ_z = lateral stress,

$\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$

V = Potential energy of deformation,

U_m, U_b, U_q = membrane strain energy, bending strain energy, and energy contribution from pressure loading, respectively, in absence of heating or cooling,

W_T = energy contribution from heating.

3. Theory.

The undeflected middle plane of the plate is chosen to be the plane of reference and its centre is taken as origin. The Z - axis is perpendicular to the reference plane in the downward direction. The potential energy of deformation may be written as

$$V = U_b + U_m - U_q - W_T \qquad ..(1)$$

in which $$U_b = \pi \int_b^a D\left[\nabla^2 w - 2(1-\nu)\frac{1}{r}\frac{dw}{dr}\cdot\frac{d^2w}{dr^2}\right] r\,dr, \qquad ..(2)$$

$$U_m = \pi \int_b^a \frac{12D}{h^2}\left[e^2 - 2(1-\nu)e_2\right] r\,dr, \qquad ..(3)$$

$$U_q = 2\pi \int_b^a q.w.r\;dr \qquad ..(4)$$

and lastly the expression for W_T is given by, Hemp [22], Williams [43], [44],

$$W_T = \int_0^{2\pi}\int_b^a\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\alpha T(r,z)}{1-2\nu}\left[\bar{e}_{rr} + \bar{e}_{\theta\theta} + \bar{e}_{zz}\right] r\,d\theta\,dr\,dz, \qquad ..(5)$$

Now the plane-stress assumption of an isotropic non-homogeneous thin plate leads to the relation, Williams [43], [44],

$$\sigma_z = \lambda'(\bar{e}_{rr} + \bar{e}_{\theta\theta} + \bar{e}_{zz}) + 2\mu'\bar{e}_{zz} = 0,$$

or,

$$\bar{e}_{zz} = -\left(\frac{\lambda'}{\lambda'+2\mu'}\right)(\bar{e}_{rr} + \bar{e}_{\theta\theta}) \qquad ..(6)$$

Hence $$\bar{e}_{rr} + \bar{e}_{\theta\theta} + \bar{e}_{zz} = \frac{1-2\nu}{1-\nu}(e - z\nabla^2 w) \qquad ..(7)$$

From (5) and (7) W_T becomes

$$W_T = 2\pi \int_b^a\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\alpha T}{1-\nu}\left[e - z\nabla^2 w\right] r\,dr\,dz \qquad ..(8)$$

The temperature $T(r,z)$ is assumed to take the following form

$$T(r,z) = T_0(r) + g(z)\, T_1(r) \qquad ..(9)$$

and we suppose that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} z\, g(z)\, dz = f(h) \qquad ..(10)$$

and $$\int_{-\frac{h}{2}}^{\frac{h}{2}} g(z)\, dz = F(h) \qquad ..(11)$$

with the aid of equations (2), (3), (4), (8), (9), (10) and (11) the equation (1) becomes, on neglecting the strain energy due to second strain invariant,

$$V = \pi \int_b^a \Big[D\Big\{ \Big(\frac{d^2w}{dr^2}\Big)^2 + \frac{1}{r^2}\Big(\frac{dw}{dr}\Big)^2 + \frac{12e^2}{h^2} + \frac{2\nu}{r}\frac{dw}{dr}\cdot\frac{d^2w}{dr^2} \Big\}$$

$$- 2qw - \frac{2E\alpha}{1-\nu}\Big\{ e(T_0 h + T_1 F) - f(h)\, T_1(r)\, \nabla^2 w \Big\} \Big] r\, dr. \qquad ..(12)$$

For minimum of V Euler's variational equations are

$$\frac{\partial V}{\partial u} - \frac{\partial}{\partial r}\frac{\partial V}{\partial u_r} = 0, \qquad ..(13)$$

$$\frac{\partial V}{\partial w} - \frac{\partial}{\partial r}\frac{\partial V}{\partial w_r} + \frac{\partial^2}{\partial r^2}\frac{\partial V}{\partial w_{rr}} = 0 \qquad ..(14)$$

The equations (12) and (13) lead to

$$\frac{\partial}{\partial r}\Big[\frac{12De}{h^2} - \frac{E\alpha}{1-\nu}(T_0 h + T_1 F) \Big] = 0, \qquad ..(15)$$

from which $$\frac{12De}{h^2} - \frac{E\alpha}{1-\nu}\left[T_0(r).h + T_1(r).F(h)\right] = \beta^2 D_0$$

..(16)

where β is a normalized constant of integration.

Equation (12) and (14) yield with the aid of (16)

$$\frac{d}{dr}\left[\frac{d}{dr}\left(D.r\frac{d^2w}{dr^2}\right) - \frac{D}{r}\frac{dw}{dr} + \nu\frac{dD}{dr}\frac{dw}{dr} - D_0\beta^2\left(r\frac{dw}{dr}\right)\right]$$
$$= r\left[q(r) - \frac{\alpha f(h)}{1-\nu}\nabla^2\left\{E(r)T_1(r)\right\}\right]$$

..(17)

For $E(r) = E_o\, r^{2m}$

..(18)

in which E_o is a constant and m may take up any value and for

$$q(r) - \frac{\alpha f(h)}{1-\nu}.E_o.\nabla^2\left\{r^{2m}T_1(r)\right\} = r^{n-1}\left(\sum a_p r^p\right)$$

..(19)

n being any number greater than -1 and ($p = 0, 1, 2 \ldots \infty$), the equation (17) stands as

$$\frac{d}{dr}\left[r^{2m+1}\left\{\frac{d^3w}{dr^3} + \frac{2m+1}{r}\frac{d^2w}{dr^2} + \left(\frac{2m\nu-1}{r^2} - \frac{\beta^2}{r^{2m}}\right)\frac{dw}{dr}\right\}\right]$$
$$= \frac{1}{D_0}\sum_{p=0}^{\infty} a_p r^{n+p}$$

..(20)

The general expression of the lateral displacement w may be found from (20) to be

$$W = \frac{A_1}{2(i\beta)^2 D_0(1-m)} \sum_{s=0}^{\infty} \frac{(-1)^s \rho^{2(1+s)}}{(1+s)\left[(1^2-\mu^2)\cdots\left\{(2s+1)^2-\mu^2\right\}\right]} + \frac{\rho}{i\beta}\left[A_2\left\{\right.\right.$$

$$(\mu-1)J_{\mu}(\rho)S_{-1,\mu-1}(\rho) - J_{\mu-1}(\rho)S_{0,\mu}(\rho)\Big\} + A_3\left\{(\mu-1)Y_{\mu}(\rho)S_{-1,\mu-1}(\rho) - Y_{\mu-1}(\rho)S_{0,\mu}(\rho)\right\}\Big]$$

$$+ \frac{1}{D_0(1-m)^3}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\frac{a_p}{n+p+1}\left(\frac{1-m}{i\beta}\right)^{\eta+2}\frac{(-1)^s \rho^{\eta+2+2s}}{(\eta+2+2s)\left[\left\{(\eta+1)^2-\mu^2\right\}\cdots\cdots\left\{(\eta+2s+1)^2-\mu^2\right\}\right]}$$

..(21)

for m ≠ 1; where $\mu^2 = 1 + \dfrac{2m(1-\nu)}{(1-m)^2}$

$$\left.\begin{aligned} \rho &= \frac{i\beta}{1-m}\, r^{1-m} \\ \text{and}\quad \eta &= \frac{n+p+1}{1-m} \end{aligned}\right\}$$

..(22)

Here $J_{\mu}(\rho)$, $Y_{\mu}(\rho)$ are the Bessel functions of the first and the second kind, $S_{\eta,\mu}(\rho)$ is the Lommel's function and A_1, A_2, A_3, A_4 are integration constants. On assuming u to be zero on both the boundaries equations (16) and (21) lead to the result,

$$\frac{\beta^2 h^2}{24(1-m)}\left[a^{2(1-m)} - b^{2(1-m)}\right] + \frac{\alpha(1+\nu)}{h}\left[\tau(a) - \tau(b)\right]$$

$$= \frac{1-m}{2(i\beta)^2}\left[\frac{A_2^2}{2}\left[\rho^2\left\{J_\mu^2(\rho) - J_{\mu-1}(\rho)\,J_{\mu+1}(\rho)\right\}\right]_{\rho_b}^{\rho_a} + \frac{A_3^2}{2}\left[\rho^2\left\{Y_\mu^2(\rho) - Y_{\mu-1}(\rho)\;Y_{\mu+1}(\rho)\right\}\right]_{\rho_b}^{\rho_a}\right.$$

$$+ \frac{A_2 A_3}{2}\left[\rho^2\left\{2J_\mu(\rho)Y_\mu(\rho) - J_{\mu-1}(\rho)\,Y_{\mu+1}(\rho) - J_{\mu+1}(\rho)\,Y_{\mu-1}(\rho)\right\}\right]_{\rho_b}^{\rho_a}$$

$$+ \frac{2A_1A_3}{D_0(i\beta)(1-m)}\sum_{s=0}^{\infty}\frac{(-1)^s\left[\rho\left\{(2s+\mu+1)Y_\mu(\rho)\,S_{2s+1,\mu-1}(\rho) - Y_{\mu-1}(\rho)\,S_{2+2s,\mu}(\rho)\right\}\right]_{\rho_b}^{\rho_a}}{(1^2-\mu^2)\cdots\cdots\cdots\{(2s+1)^2-\mu^2\}}$$

$$+ \frac{2A_1A_2}{D_0(i\beta)(1-m)}\sum_{s=0}^{\infty}\frac{(-1)^s\left[(2s+\mu+1)\rho J_\mu(\rho)\,S_{2s+1,\mu-1}(\rho) - \rho J_{\mu-1}(\rho)\,S_{2+2s,\mu}(\rho)\right]_{\rho_b}^{\rho_a}}{(1^2-\mu^2)\cdots\cdots\{(2s+1)^2-\mu^2\}}$$

$$+ \frac{2i\beta A_2}{D_0(1-m)^3}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\frac{a_p}{n+p+1}\left(\frac{1-m}{i\beta}\right)^{\eta+2}\frac{(-1)^s\left[\rho\left\{(\eta+1+2s+\mu)J_\mu(\rho)S_{\eta+1+2s,\mu-1}(\rho) - J_{\mu-1}(\rho)\,S_{\eta+2+2s,\mu}(\rho)\right\}\right]_{\rho_b}^{\rho_a}}{\{(\eta+1)^2-\mu^2\}\cdots\cdots\{(\eta+2s+1)^2-\mu^2\}}$$

$$+ \frac{2i\beta A_3}{D_0(1-m)^3}\sum_{p=0}^{\infty}\sum_{s=0}^{\infty}\frac{a_p}{n+p+1}\left(\frac{1-m}{i\beta}\right)^{\eta+2}\frac{(-1)^s\left[\rho\left\{(\eta+1+2s+\mu)Y_\mu(\rho)S_{\eta+2s+1,\mu-1}(\rho) - Y_{\mu-1}(\rho)\,S_{\eta+2+2s,\mu}(\rho)\right\}\right]_{\rho_b}^{\rho_a}}{[(\eta+1)^2-\mu^2]\cdots\cdots[(\eta+2s+1)^2-\mu^2]}$$

$$+ \frac{A_1^2}{2D_0^2(1-m)^2(i\beta)^2}\sum_{s=0}^{\infty}\sum_{s_1=0}^{\infty}\frac{(-1)^s(-1)^{s_1}\left[\rho_a^{2(2+s+s_1)} - \rho_b^{2(2+s+s_1)}\right]}{(2+s+s_1)\left[(1^2-\mu^2)\cdots\cdots\{(1+2s)^2-\mu^2\}\right].\left[(1^2-\mu^2)\cdots\cdots\{(1+2s_1)^2-\mu^2\}\right]}$$

$$+ \frac{2A_1}{D_0^2(1-m)^4}\sum_{p=0}^{\infty}\frac{a_p}{n+p+1}\left(\frac{1-m}{i\beta}\right)^{\eta+2}\sum_{s=0}^{\infty}\sum_{s_1=0}^{\infty}\frac{(-1)^s(-1)^{s_1}\left[\rho_a^{4+\eta+2s+2s_1} - \rho_b^{4+\eta+2s+2s_1}\right]}{(4+\eta+2s+2s_1)\left[(1^2-\mu^2)\cdots\cdots\{(1+2s)^2-\mu^2\}\right].\left[\{(\eta+1)^2-\mu^2\}\cdots\cdots\{(\eta+1+2s_1)^2-\mu^2\}\right]}$$

$$\left.+ \frac{(i\beta)^2}{D_0^2(1-m)^6}\sum_{p=0}^{\infty}\sum_{p_1=0}^{\infty}\frac{a_p . a_{p_1}}{(n+p+1)(n+p_1+1)}\left(\frac{1-m}{i\beta}\right)^{\eta+\eta_1+4}\sum_{s=0}^{\infty}\sum_{s_1=0}^{\infty}\frac{(-1)^s(-1)^{s_1}\left[\rho_a^{4+\eta+\eta_1+2s+2s_1} - \rho_b^{4+\eta+\eta_1+2s+2s_1}\right]}{(4+\eta+\eta_1+2s+2s_1)\left[\{(\eta+1)^2-\mu^2\}\cdots\cdots\{(\eta+2s+1)^2-\mu^2\}\right]\times\left[\{(\eta_1+1)^2-\mu^2\}\cdots\cdots\{(\eta_1+2s_1+1)^2-\mu^2\}\right]}\right] \quad ..(23)$$

in which we have taken

$$\left.\begin{aligned} r\{h T_0(r) + F(h) T_1(r)\} &= \frac{d\tau}{dr} \\ \frac{i\beta}{1-m} a^{1-m} &= \rho_a \\ \frac{i\beta}{1-m} b^{1-m} &= \rho_b \\ \frac{n+p_1+1}{1-m} &= \eta_1 \end{aligned}\right\} \quad ..(24)$$

4. Relevant Equations For m = 1.

The solution of the equation (20) for m = 1 gives the lateral displacement w in the following form

$$w = B_1 \log r + \frac{B_2}{k} r^k - \frac{B_3}{k} r^{-k} + B_4 + \frac{1}{D_0} \sum_{p=0}^{\infty} \frac{a_p r^{n+p+1}}{(n+p+1)^2\{(n+p+1)^2 - k^2\}} \quad ..(25)$$

in which $k^2 = 2(1-\nu) + \beta^2$ and $k \neq \pm(n+p+1)$ ..(26)

Even if $k = (n+p+1)$ or $-(n+p+1)$ or both the last term in (25) would be slightly changed through indefinite integrations like $\int r^c \log r\,dr$ or $\int r^c (\log r)^2\,dr$ as the case arises.

The equation corresponding to the equation (23) happens to be

$$\frac{\beta^2 h^2}{12} \log\left(\frac{a}{b}\right) + \frac{\alpha(1+\nu)}{h}\left[\tau(a) - \tau(b)\right]$$

$$= \frac{1}{2}\left[(B_1^2 + 2B_2B_3)\left(\log\frac{a}{b}\right) + \frac{B_2^2}{2k}\left(a^{2k} - b^{2k}\right) - \frac{B_3^2}{2k}\left(a^{-2k} - b^{-2k}\right)\right.$$

contd.

$$+ \frac{2B_1B_2}{k}\left(a^{k}-b^{k}\right) - \frac{2B_1B_3}{k}\left(a^{-k}-b^{-k}\right)$$

$$+ \frac{2B_1}{D_0}\sum_{p=0}^{\infty}\frac{a_p\left(a^{n+p+1}-b^{n+p+1}\right)}{(n+p+1)^2\{(n+p+1)^2-k^2\}} + \frac{2B_2}{D_0}\sum_{p=0}^{\infty}\frac{a_p\left(a^{n+p+1+k}-b^{n+p+1+k}\right)}{(n+p+1)(n+p+1+k)^2(n+p+1-k)}$$

$$+ \frac{2B_3}{D_0}\sum_{p=0}^{\infty}\frac{a_p\left(a^{n+p+1-k}-b^{n+p+1-k}\right)}{(n+p+1)(n+p+1-k)^2(n+p+1+k)}$$

$$+ \frac{1}{D_0^2}\sum_{p=0}^{\infty}\sum_{p_1=0}^{\infty}\frac{a_pa_{p_1}\left(a^{2n+p+p_1+2}-b^{2n+p+p_1+2}\right)}{(n+p+1)(n+p_1+1)\{(n+p+1)^2-k^2\}\{(n+p_1+1)^2-k^2\}(2n+p+p_1+2)}\Bigg]$$

..(27)

5.1 Numerical Results.

Clamped Boundaries : .

On a clamped edge we shall have

$$u = w = \frac{dw}{dr} = 0$$

..(28)

using the boundary conditions (28) for the edges $r = a$ and $r = b$ we get the constants 'A' S from (21) in a matrix form $\{A_1\ A_2\ A_3\ A_4\}$ from its augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & b_{15} \\ a_{21} & a_{22} & a_{23} & 1 & b_{25} \\ \dfrac{S_{0,\mu}(\rho_a)}{(i\beta)D_0(1-m)} & J_\mu(\rho_a) & Y_\mu(\rho_a) & 0 & b_{35} \\ \dfrac{S_{0,\mu}(\rho_b)'}{(i\beta)D_0(1-m)} & J_\mu(\rho_b) & Y_\mu(\rho_b) & 0 & b_{45} \end{bmatrix}$$

..(29)

where

$$a_{11} = \frac{1}{2(i\beta)^2 D_0 (1-m)} \sum_{s=0}^{\infty} \frac{(-1)^s \rho_a^{2(1+s)}}{\left[(1^2-\mu^2).(3^2-\mu^2)\cdots\cdots\left\{(2s+1)^2-\mu^2\right\}\right](1+s)}$$

$$a_{12} = \frac{\rho a}{(i\beta)} \left\{(\mu-1) J_{\mu}(\rho a) \cdot S_{-1,\mu-1}(\rho a) - J_{\mu-1}(\rho a)\, S_{0,\mu}(\rho a)\right\}$$

$$a_{13} = \frac{\rho a}{(i\beta)} \left\{(\mu-1) Y_{\mu}(\rho a)\, S_{-1,\mu-1}(\rho a) - Y_{\mu-1}(\rho a)\, S_{0,\mu}(\rho a)\right\}$$

$$b_{15} = \frac{1}{(m-1)^3 D_0} \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{1-m}{i\beta}\right)^{\eta+2} \frac{a_p}{\eta+p+1} \frac{(-1)^s \rho_a^{\eta+2+2s}}{\left\{\left[(\eta+1)^2-\mu^2\right]\cdots\cdots\left[(\eta+2s+1)^2-\mu^2\right]\right\}(\eta+2+2s)}$$

$$b_{35} = \frac{i\beta}{(m-1)^3 D_0} \sum_{p=0}^{\infty} \left(\frac{1-m}{i\beta}\right)^{\eta+2} \frac{a_p}{\eta+p+1}\, S_{\eta,\mu}(\rho a) \qquad \ldots(30)$$

Replacing ρ_a by ρ_b in the expressions of a_{11}, a_{12}, a_{13}, b_{15}, b_{35} we get a_{21}, a_{22}, a_{23}, b_{25}, b_{45} respectively.

For a particular case in which

$$T_0(r) = \frac{T_0^{(1)}}{r} , \quad T_1(r) = T_1^{(1)} \left\{ r^{-2m} \left(1 + \log r + \frac{1}{9} r^3 \right) \right\}$$

$$\vartheta_0 = \frac{\alpha T_0^{(1)}}{h^2} = 0.01 \quad \text{and} \quad \vartheta_1 = \frac{\alpha T_1^{(1)} F(h)}{h^3} = -0.20$$

..(31)

and where $a = 1$, $b = .01$, $p = 1 = n$, $a_0 = 0$, $\nu = 0.25$, $m = 0.5$ ..(32)

and for an assumed value of $i\beta = 0.5$, the augmented matrix (29) gives

$$\left\{ A_1 \; A_2 \; A_3 \; A_4 \right\} = -\frac{a_1}{D_0} \left\{ .029930\, D_0 \quad .535250 \quad -.000026 \quad .0010137 \right\}$$

..(33)

Now (31), (32), (33) and $i\beta = 0.5$ help us determine the appropriate load factor from the equation (23) to be

$$\frac{a_1}{D_0 h} = 8.6631$$

..(34)

The equations (33) and (34) clearly give the value of the arbitrary constants A_1, A_2, A_3, A_4 of the equation (21) as

$$\left\{ A_1 \; A_2 \; A_3 \; A_4 \right\} = -8.6631\, h \left\{ .029930\, D_0 \quad .535250 \quad -.000026 \quad .0010137 \right\}$$

..(35)

For such a case we have plotted $\frac{10^3 w}{8.6631 h}$ against r in Fig. 1.

Simply Supported Boundaries :

For the simply supported edges the required boundary conditions are

$$\left[u \right]_{r=a,b} = 0 , \left[w \right]_{r=a,b} = 0 \quad \text{and} \quad \left[\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right]_{r=a,b} = 0 .$$

..(36)

The corresponding augmented matrix for the evalution of 'A' in this case is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & b_{15} \\ a_{21} & a_{22} & a_{23} & 1 & b_{25} \\ a_{31} & a_{32} & a_{33} & 0 & M_1 \\ a_{41} & a_{42} & a_{43} & 0 & M_2 \end{bmatrix} \qquad ..(37)$$

where

$$a_{31} = \frac{1}{i\beta(D_0)}\left[\frac{\nu-m}{1-m} S_{0,\mu}(\rho_a) + \sum_{s=0}^{\infty} \frac{(-1)^s(1+2s)\,\rho_a^{2s+1}}{(1^2-\mu^2)(3^2-\mu^2)\cdots\left[(2s+1)^2-\mu^2\right]}\right]$$

$$a_{32} = \left[(\nu-m) J_\mu(\rho_a) + \frac{1-m}{2}\rho_a\left\{J_{\mu-1}(\rho_a) - J_{\mu+1}(\rho_a)\right\}\right]$$

$$a_{33} = \left[(\nu-m) Y_\mu(\rho_a) + \frac{1-m}{2}\rho_a\left\{Y_{\mu-1}(\rho_a) - Y_{\mu+1}(\rho_a)\right\}\right]$$

$$M_1 = \frac{i\beta}{(m-1)^3}\sum_{p=0}^{\infty}\left(\frac{1-m}{i\beta}\right)^{\eta+2}\frac{a_p}{n+p+1}\left[(\nu-m) S_{\eta,\mu}(\rho_a) + (1-m)\sum_{s=0}^{\infty}\frac{(-1)^s(\eta+1+2s)\,\rho_a^{\eta+1+2s}}{\{(\eta+1)^2-\mu^2\}\cdots\{(\eta+2s+1)^2-\mu^2\}}\right] \qquad ..(38)$$

Replacing ρ_a by ρ_b in the expressions of a_{31}, a_{32}, a_{33}; M_1 we get a_{41}, a_{42}, a_{43}; M_2 respectively.

For a particular case mentioned in (31) and (32) and for the same assumed value of $i\beta = 0.5$, the augmented matrix (37) gives

$$\left\{ A_1 \; A_2 \; A_3 \; A_4 \right\} = - \frac{a_1}{D_0} \left\{ 0.078773 \, D_0 \quad 1.2803 \quad 0.000008 \quad 0.000729 \right\} \quad ..(39)$$

The appropriate load factor is obtained from (23) with the help of (31), (32), (33) and (39) in the form

$$\frac{a_1}{D_0 \, h} = 10.426 \quad ..(40)$$

The arbitrary constants A_1, A_2, A_3, A_4 of the equation (21) may now be obtained from (39) and (40) in completely known terms as

$$\left\{ A_1 \; A_2 \; A_3 \; A_4 \right\} = - 10.426 \, h \left\{ 0.078773 D_0 \quad 1.2803 \quad 0.000008 \quad 0.000729 \right\} \quad ..(41)$$

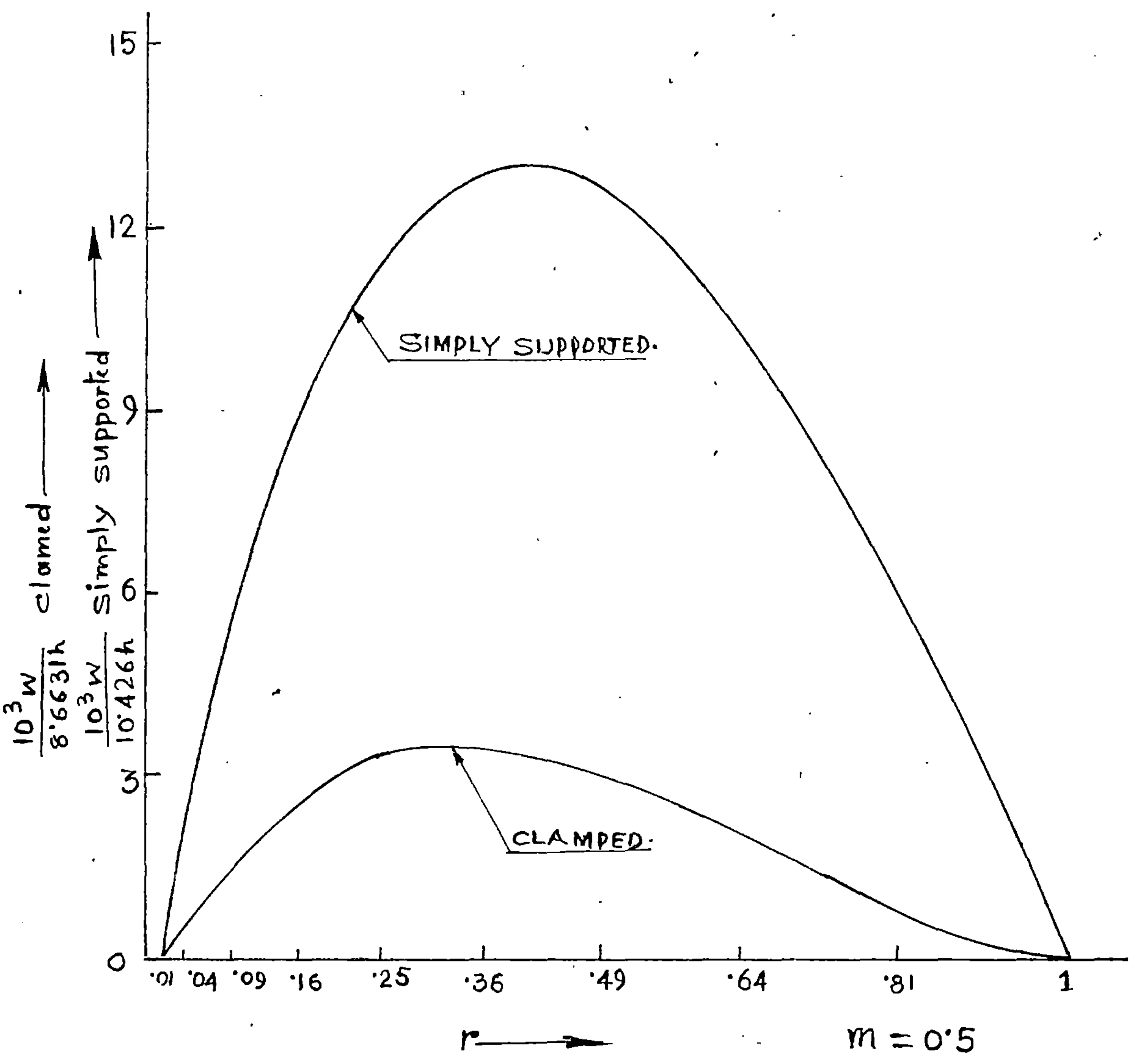
In this case also a graph of $\dfrac{10^3 w}{10.426 \, h}$ against r is drawn in Fig.1.

5.2 Numerical Results For m = 1.

Clamped Boundaries :

Using the boundary conditions (28) for the edges r = a and r = b we get the constants 'B' S from (25) in a matrix form $\left\{ B_1 \; B_2 \; B_3 \; B_4 \right\}$ from its augmented matrix

$$\begin{bmatrix} \log a & \dfrac{a^k}{k} & \dfrac{a^{-k}}{(-k)} & 1 & C_{15} \\ \log b & \dfrac{b^k}{k} & \dfrac{b^{-k}}{(-k)} & 1 & C_{25} \\ 1 & a^k & a^{-k} & 0 & C_{35} \\ 1 & b^k & b^{-k} & 0 & C_{45} \end{bmatrix} \quad ..(42)$$



15
12
9
6
3
0
$\frac{10^3 w}{8.6631h}$ clamed
$\frac{10^3 w}{10.426h}$ simply supported
SIMPLY SUPPORTED.
CLAMPED.
·01 ·04 ·09 ·16 ·25 ·36 ·49 ·64 ·81 1
r
m = 0·5

FIG. 1

where

$$C_{15} = \frac{1}{D_0} \sum_{p=0}^{\infty} a_p \frac{a^{n+p+1}}{(n+p+1)^2\{k^2-(n+p+1)^2\}}$$

$$C_{35} = \frac{1}{D_0} \sum_{p=0}^{\infty} a_p \frac{a^{n+p+1}}{(n+p+1)\{k^2-(n+p+1)^2\}}$$

..(43)

also C_{25}, C_{45} can be written from the expressions of C_{15} and C_{35} respectively on replacing a^{n+p+1} by b^{n+p+1} in each case.

For a particular case in which

$$a = 10, \quad b = 1, \quad p = 1 = n, \quad a_0 = 0, \quad \nu = 0.25, \quad \underline{m = 1} \qquad ..(44)$$

and for an assumed value of $\beta^2 = 2.5$, the augmented matrix (42) gives

$$\{B_1\ B_2\ B_3\ B_4\} = -\frac{a_1}{D_0}(-7.49020 \quad .728718 \quad 6.83065 \quad 3.07341) \qquad ..(45)$$

Now equations (31), (44), (45) and $\beta^2 = 2.5$ are utilised to get the appropriate load factor from the equation (27) in the form

$$\frac{a_1}{D_0\,h} = 0.15863 \qquad ..(46)$$

The equations (45) and (46) give clearly the value of the arbitrary constants of equation (25) as

$$\{B_1\ B_2\ B_3\ B_4\} = -0.15863\,h\,\{-7.49020 \quad 0.728718 \quad 6.83065 \quad 3.07341\} \qquad ..(47)$$

For such a case we have plotted $\dfrac{w}{0.15863\ h}$ against r in Fig. 2.

Simply Supported Boundaries :

Using the boundary conditions mentioned in (36) we get the constants 'B' S from (25) in a matrix form $\{B_1\ B_2\ B_3\ B_4\}$ from its angmented matrix

$$\begin{bmatrix} \log a & \dfrac{a^{k}}{k} & \dfrac{a^{-k}}{(-k)} & 1 & C_{15} \\ \log b & \dfrac{b^{k}}{k} & \dfrac{b^{-k}}{(-k)} & 1 & C_{25} \\ 1-\nu & (1-\nu-k)a^{k} & (1-\nu+k)a^{-k} & 0 & N_1 \\ 1-\nu & (1-\nu-k)b^{k} & (1-\nu+k)b^{-k} & 0 & N_2 \end{bmatrix} \qquad ..(48)$$

in which $N_1 = \dfrac{1}{D_o}\displaystyle\sum_{p=0}^{\infty} \dfrac{a_p\ (\nu+n+p)\ a^{n+p+1}}{(n+p+1)\left\{(n+p+1)^2-k^2\right\}}$ and N_2 may be obtained from the expression of N_1 just by replacing a^{n+p+1} by b^{n+p+1}

For the particular case mentioned by (31) and (44) and for the same assumed value of $\beta^2 = 2.5$, the augmented matrix (48) gives

$$\{B_1\ B_2\ B_3\ B_4\} = -\ \frac{a_1}{D_o}\ \{-15.2107 \quad 1.10972 \quad 4.64860 \quad 1.79170\} \qquad ..(49)$$

The appropriate load factor is obtained from (27) with the help of (31), (44), (49) and $\beta^2 = 2.5$ in the form

$$\frac{a_1}{D_o\ h} = 0.28673 \qquad ..(50)$$

The arbitrary constants B_1, B_2, B_3, B_4 of equation (25) may be obtained from (49) and (50) as

$$\left\{ B_1 \; B_2 \; B_3 \; B_4 \right\} = -0.28673 \, h \left\{ -15.2107 \quad 1.10972 \quad 4.64860 \quad 1.79170 \right\} \quad ..(51)$$

In this case also we have plotted $\dfrac{w}{0.28673 \, h}$ against r in Fig.2.

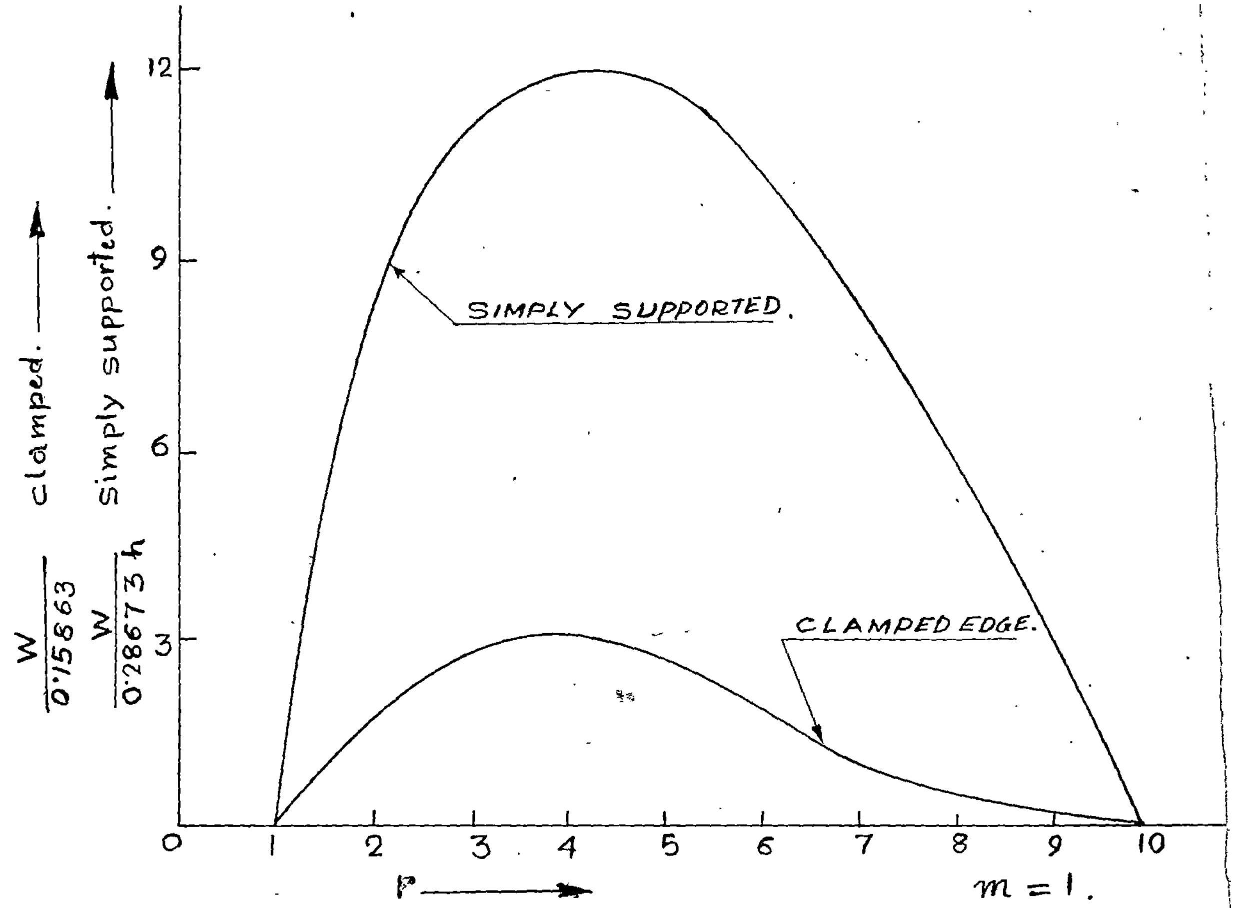

12
9
6
3
0
1
2
3
4
5
6
7
8
9
10
SIMPLY SUPPORTED.
CLAMPED EDGE.
W / 0·15863 clamped.
W / 0·28673 h Simply supported.
r
m = 1.

FIG. 2.