

## CHAPTER 3

### Constrained Problems

In this chapter we propose to study two rectilinear distance constrained problems with the minimax objective. Sec. 3.1.1 discusses the equiweighted case while sec. 3.2.1 solves both the symmetric as well as asymmetric weighted problem.

**3.1.1 Geometric solution of a constrained rectilinear distance minimax location problem.\***

In a wide variety of situations persons interested in facility layout and location are confronted with the problem of locating a new facility in the midst of existing facilities (referred to as the one-centre problem in the literature) in such a way that the maximal distance from the new facility to the existing locations is minimised. The equal-weighted unconstrained single facility minimax location problem under the  $L_1$  metric has been well studied by Elzinga and Hearn [31], Francis and White [36], Wesolowsky [78] et al. The weighted and constrained minimax problem under the  $L_2$  metric in  $n$ -dimensional space has been solved by Scott et al. [69] using the concept of conjugate duality. Dutta and Chaudhuri [29] have given an elegant method of obtaining an exact solution to an equi-weighted planar constrained problem under the  $L_2$  or Euclidean norm. Hansen et al. [43], Francis et al. [37] Hansen et al. [44], Drezner and Shelah [22] and Drezner [21] have also discussed Facility Location Models at length. Morris [64] has also

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considered constrained multifacility minimax location problem under  $L_1$  metric. Our objective in this section is to present a geometric solution to the constrained version of the problem in the plane in which the distances are rectilinear. As a practical application of the problem addressed in this section one might consider locating an emergency service facility, for example - a health clinic in a rural area or a fire station in a large metropolitan city - where the facility is restricted to lie within a given region under the assumption that travel is allowed on a grid only.

### 3.1.2 Problem statement

Assume that a set  $D$  is defined by

$D = \{ P_i : i = 1, 2, \dots, n \}$  where  $P_i(a_i, b_i)$  are the  $n$  existing location points in the plane  $E_2$ . Also assume that a new facility is to be located at  $T(x, y)$  in such a way that the maximum rectilinear distance between the new facility and the  $n$  given locations is minimised subject to the restriction that  $T$  is constrained to lie in a convex polytope  $R$ . The rectilinear or rectangular distance between  $T$  and  $P_i$  is given by

$$d_1(T, P_i) = |x - a_i| + |y - b_i|, P_i \in D.$$

The minimax problem is

$$\min_{T \in R} \max_{1 \leq i \leq n} d_1(T, P_i) \quad (1)$$

where  $R \subset E_2$  is given by

$$R = \left\{ X : AX \leq b ; A = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \end{bmatrix}^t ; \right. \\ \left. b = [b_1, b_2, \dots, b_m]^t ; X = [x, y]^t \right\}$$

To solve problem (1) we make use of certain elementary properties of rectilinear distance function in a plane, the proofs of which being obvious will be omitted.

As may be seen in Figure 1, all the  $n$  given location points  $P_i$  may be covered by a rectangle  $A_1A_2A_3A_4$  by drawing a pair of parallel lines inclined at  $135^\circ$  with the  $x$ -axis through two of these points  $P_i$  farthest apart and another pair of parallel lines inclined at  $45^\circ$  through two similar points, i.e., points farthest from each other.

**The Unconstrained Case:** As long as point  $T$  lies to the right of the vertical  $A_3Q$  through  $A_3$  the rectilinear distance from  $T$  to any point on the side  $A_2A_3$  is greater than the same to any point on the side  $A_3A_4$ . On  $A_3Q$  itself these distances are equal. In the absence of any constraint this latter distance continually diminishes as one moves upwards along the vertical reaching its minimum value at  $Q$ , the point of intersection of the perpendicular bisector  $KQ$  of the longest side of the rectangle and  $A_3Q$  and maintains this value upto the point  $K$ , the other point of intersection of  $KQ$  and the vertical  $A_1K$  through  $A_1$ . Any point on  $KQ$  is a possible minimax location [31, 36].

We shall use the following definitions and notations in our

subsequent discussion.

**Dominating side:** The side or sides of the rectangle having the maximum contribution towards the rectilinear distance function at a point will be called the dominating side at the point.

**Cone of Descent Direction:** Let  $f$  be the objective function and  $x$  any point in  $R$ . The cone of descent direction of  $f$  at  $x$  is defined by

$$D_f(x) = \{ d : \exists \bar{\alpha} > 0 \text{ with } f(x + \alpha d) < f(x) \\ \forall 0 < \alpha \leq \bar{\alpha} \}.$$

Let a line  $l$  be drawn through  $x \in R$  parallel to the dominating side at  $x$ .

Also let  $H^+$  denote the closed half-space containing the dominating side defined by  $l$ . Then the cone of descent direction in the present case will be given by  $N_\delta(x) \cap H^+$  or  $R \cap N_\delta(x) \cap H^+$  according as  $x$  is an interior or a boundary point of  $R$ ,  $N_\delta(x)$  being a neighbourhood of  $x$ .

By  $\partial R$  we shall denote the boundary of the region  $R$ .

**The Constrained Case** The direction of movement is determined by the direction of descent which in turn is related to the cone of descent direction. The cone of descent direction as defined above is obtained from the dominating side of the rectangle.

Let  $C$  be the intersection of the cone of descent direction and the region  $R$ . The value of the objective function (i.e., the maximum rectilinear distance from  $T$ ) does not increase along any ray lying within  $C$  - rather, it has a

gradually diminishing value along  $\partial R$  constituting an extreme edge of  $C$  provided  $\partial R$  is not parallel to a dominating side or  $C$  is not degenerate.

Either of the following instances determines a stopping criterion.

I. The intersection of the two cones of descent direction reduces to their common vertex, which is the point of intersection of an active boundary, with either the vertical through  $A_1$  or  $A_3$ , or the horizontal through  $A_2$  or  $A_4$ .

II. The cone of descent direction degenerates into a point coinciding with an extreme point of an active boundary.

The algorithm of the present problem, which follows shortly, is justified by the lemma given below, the proof of which is given in sec. 3.1.7.

**Lemma:** For any point other than the one obtained by using either of the stopping criteria the rectilinear distance will be greater.

### 3.1.3. The Algorithm

Define the quantities  $c_1$  through  $c_4$  as follows:

$$c_1 = \min_{1 \leq i \leq n} (a_i + b_i); \quad c_2 = \max_{1 \leq i \leq n} (a_i + b_i);$$

$$c_3 = \min_{1 \leq i \leq n} (-a_i + b_i); \quad c_4 = \max_{1 \leq i \leq n} (-a_i + b_i).$$

The points  $P_i$  define a rectangle  $S$  the sides of which are given by

$$l_1: x+y = c_1; \quad l_2: x+y = c_2; \quad l_3: -x+y = c_3; \quad l_4: -x+y = c_4.$$

Label  $l_1 \cap l_3$ ,  $l_1 \cap l_4$ ,  $l_2 \cap l_4$ ,  $l_2 \cap l_3$  as  $A_1$ ,  $A_2$ ,  $A_3$  and  $A_4$ .

respectively.

Denote the segment of perpendicular bisector of one of the longer sides of  $S$  intercepted by lines through  $A_2$  and  $A_4$  drawn parallel to the  $x$ -axis (or through  $A_1$  and  $A_3$  drawn parallel to the  $y$ -axis) by  $KQ$ ,  $K$  having a greater ordinate than  $Q$ . Any point on  $KQ$  is a possible minimax location in the unconstrained case.

Without any loss of generality choose lines through  $A_2$  and  $A_3$  parallel to the axes as the coordinate axes and change the equations of constraints and the coordinates of the vertices of  $S$  accordingly.

#### Step 0 (Initialization Step)

If  $R$  contains the whole or any part of  $KQ$  then go to step 4. Starting from any point  $P(X,Y) \in \text{int } R$  move towards  $Oy$ , the  $y$ -axis along the line  $l: y = Y$ . If the point  $l \cap Oy = (0,Y) \in R$  then move towards the origin along the  $y$ -axis till the point  $M = \partial R \cap Oy$  is reached and go to step 1(a). Else denote the point  $l \cap \partial R$  by  $M$  and go to step 3.

**Step 1(a).** If  $M$  satisfies a stopping criterion then stop.  $M$  is the required point. Else go to step 1(b).

**Step 1(b).** Move along the extreme edge of the direction of descent until any one of the following four possibilities materializes:

- (i) an extreme point, say  $V$ , is reached : go to step 2(a)
- (ii) the point of intersection of the direction of descent and the line through  $A_1$  parallel to the  $y$ -axis is attained : call this point  $M$  and go to step 1(a).

(iii) the point, say  $N$ , of intersection of the direction of descent and the  $x$ -axis is arrived at: go to step 2(b), or

(iv) the point of intersection of the direction of descent and the line through  $A_4$  parallel to the  $x$ -axis is attained: call this point  $N$  and go to step 2(b).

**Step 2(a).** If  $V$  satisfies a stopping criterion then stop.  $V$  is the required location. Else obtain the adjacent edge through  $V$ , drop the current edge and go to step 1(b).

**Step 2(b).** If a stopping criterion holds good for the point  $N$  then stop.  $N$  is the sought after location point. Else go to step 1(b).

**Step 3.** Obtain the extreme edge of the direction of descent at  $M$  and move along it. If it meets the  $y$ -axis then call this point  $M$  and go to step 1(a). Else go to step 1(b).

**Step 4.** If  $R \cap KQ \neq \emptyset$  then any point  $T \in R \cap KQ$  is a minimax location.

**Note:** When an active constraint is parallel to a dominating side of the rectangle  $S$  then the required facility location will be any point belonging to the whole stretch of the active boundary included between the axes or between lines parallel to them.

### 3.1.4. Analysis of Time Complexity

For constructing the rectangle  $S$  we are required to determine real numbers  $c_1$  through  $c_4$ . Again calculating any of these four quantities necessitates  $(n-1)$  comparisons. Hence  $S$  can be obtained in  $O(n)$  time. Then  $m$  given linear constraints defining the region  $R$  determine the extreme

points of a convex hull. To get the convex hull we have to find the slopes of all the constraints requiring  $m$  multiplications and sort the angles of inclination in  $O(m \log m)$  time. The extreme points are then obtained by solving each pair of consecutive constraints arranged in sorted angular order employing method of Gaussian elimination for which a total of  $5m$  multiplications is necessary. A few additional multiplications are needed for obtaining the point of intersection of the line  $y = Y$  with an active boundary or the  $y$ -axis as the case may be, as well as that of the direction of descent with either or both the axes of coordinates and / or lines parallel to them. Hence the worst case time complexity of the algorithm is  $\max\{O(n), O(m \log m)\}$ .

### 3.1.5. Sensitivity Analysis.

**Introduction or removal of a location point** If inserting an additional location point or deleting an existing one does not alter the configuration of the rectangle  $S$  or alters only any of its non-dominating sides then the current solution also remains unaltered. If, on the other hand, the above procedure affects a vertex of  $S$  with respect to which the current location point was obtained, then we regard the present solution as the starting solution, apply the algorithm and obtain the new optimal solution.

### **Introduction or removal of a constraint**

If we remove a non-binding constraint or introduce a constraint which, besides altering the region  $R$ , does not

have any impact on the currently active boundary with respect to which the optimum solution was obtained, modification of the solution is necessary. On the contrary, imposition of an additional constraint may result in either of the following cases.

I. The current optimal point  $T$  may now lie within the modified region  $R$ .  $P \leftarrow T$  and go to step 0.

II. The current optimal point  $T$  may lie outside the modified region  $R$  : denote the newly introduced constraint by  $L \leq 0$ . If the slope of  $L = 0$  is non-zero then move parallel to the  $x$ -axis until the point of intersection, say  $N$ , of  $L = 0$  and  $y = 0$  is reached. If  $N \in \partial R$  then go to step 2(b); else move towards an extreme point  $V$  along  $L = 0$  and go to step 2(a). If, on the contrary, the slope of  $L = 0$  is zero then move parallel to the  $y$ -axis and reach  $M$ , the point of intersection of  $L = 0$  and  $x = 0$ . If  $M \in \partial R$  then go to step 1(a); else move towards  $V$  along  $L = 0$  and go to step 2(a).

Again removal of a binding constraint renders the present facility point either an interior point or a boundary point. In the former case go to step 0; in the latter go to step 1(b).

### 3.1.6. Numerical Solutions

We demonstrate the algorithm given above by means of the following examples :

#### Problem 1.

Let the convex polytope be defined by  $R = \{ X : AX \leq b \}$

where

$$A = \begin{bmatrix} 2 & 5 & 2 & 0 & -1 & -1 & -5 \\ -3 & -1 & 1 & 1 & 1 & 0 & -4 \end{bmatrix}^t ;$$

$$X = [x, y]^t ; b = [-6, 4.5, 10, 11, 15, 7, 20]$$

and suppose that the set D consists of the following points, their respective coordinates being noted alongside each :

$$P_1 = (2.00, 10.00); P_2 = (0.00, 12.50); P_3 = (-0.25, 12.50);$$

$$P_4 = (7.00, 9.00); P_5 = (3.00, 13.00); P_6 = (3.60, 10.45);$$

$$P_7 = (4.50, 11.50); P_8 = (5.00, 12.25); P_9 = (7.00, 12.00);$$

$$P_{10} = (6.25, 8.75); P_{11} = (7.00, 10.65); P_{12} = (7.35, 9.80);$$

$$P_{13} = (8.30, 10.55); P_{14} = (3.25, 15.45); P_{15} = (3.80, 14.15);$$

$$P_{16} = (1.00, 14.00); P_{17} = (1.20, 13.85); P_{18} = (3.95, 14.60);$$

$$P_{19} = (5.15, 12.45); P_{20} = (6.30, 12.20).$$

Let P be the point  $(-3.00, 1.50) \in \text{int } R$ . Here KQ is given by the line segment joining  $K = (5.00, 12.50)$  and  $Q = (3.00, 10.50)$ . Since  $R \cap KQ = \emptyset$ , M is calculated to be  $(-0.75, 1.50)$  by step 0. Now move along the descent direction of the active constraint given by the equation  $2x - 3y = -6$  until the extreme point  $V = (1.50, 3.00)$  is reached. Drop this edge by step 2(a) and proceed along the next boundary given by the equation  $5x - y = 4.5$ . After successively executing three iterations attain the required facility point  $N = (-0.25, 10.50)$ , which, in this case, is the point of intersection of the binding constraint having equation  $2x + y = 10$  and the line through  $A_4 = (8.50, 10.50)$  parallel to the x-axis.

Problem 2.

Let us consider the effect of introducing an additional

constraint defined by  $L : -x + 0.5 \leq 0$  in Problem 1. The current facility point  $N$  now belongs to  $\text{int } R$ . Since the slope of  $L = 0$  is non-zero by section 5 move parallel to the  $x$ -axis and reach the point  $M = (0.50, 10.50)$  of intersection of  $NA_4$  and  $L = 0$ . Since this point  $M \notin \partial R$  move along  $L = 0$  towards the extreme point  $V = (0.50, 9.00)$  and following step 2(a) of the algorithm  $V$  is the required optimal location.

### 3.1.7. Appendix

**Proof of Lemma 1.** Let us suppose that the point  $N$  of intersection of an active boundary with the horizontal through  $A_2$  is the sought for minimax location. Then the maximum rectilinear distance of  $N$  from the set  $D$  is given by  $NA_2$ . For any other point  $U \in R$  or  $\partial R$  lying above or on  $NA_2$  the maximum rectilinear distance being given by the horizontal line segment included between  $U$  and the dominating side  $A_1A_2$ , produced if necessary, is clearly greater than or equal to  $NA_2$ . Suppose  $U$  lies below  $NA_2$  and on the same side of the vertical through  $A_3$  as  $N$ . Then the maximum rectilinear distance of  $U$  from  $A_3A_2$ , the dominating side, measured horizontally is also easily seen to be greater than  $NA_2$ . Furthermore, when  $U$  and  $N$  are on opposite sides of the vertical through  $A_3$  the maximum rectilinear distance of  $U$  from the set  $D$  (being given by the horizontal distance from the dominating side  $A_3A_4$ ) is no less than the maximum rectilinear distance of  $U$  from  $A_3A_2$  which is greater than or equal to  $NA_2$ . Hence for all positions of the point  $U$  the

rectilinear distance of  $U$  is no less than  $NA_2$ . Thus  $NA_2$  is the minimum rectilinear distance. The same holds when the facility point is the point of intersection of a boundary and the horizontal through  $A_4$ .

It can be shown in a similar way that the above conclusion is valid when the optimal location is either the point  $M$  of intersection of an active boundary of  $R$  with the vertical through  $A_3$  or  $A_1$ , or is an extreme point  $V$ .

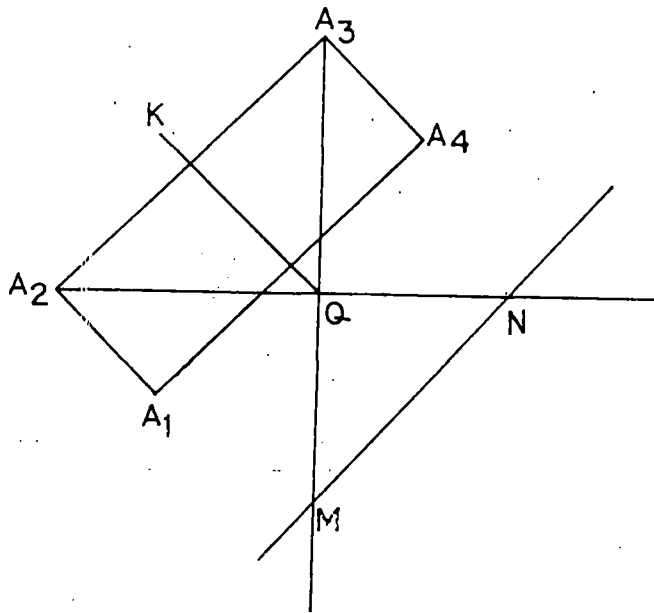


Fig 1

### 3.2.1 Geometric solution to some planar constrained minimax problems involving the weighted rectilinear metric\*

In most practical situations we have to locate a facility point within a constrained, rather than an unconstrained, region owing to the nature of space available for the purpose of locating a facility. By incorporating constraints the facility location problem more closely resembles a real life situation. The rectilinear metric is ideally suited to urban applications in view of the assumption that travel is usually restricted to a rectangular grid. In this section we have studied the one-centre problem, also known as the single facility minimax location problem, in the presence of linear constraints forming a convex polytope in  $E_2$  wherein we assume that besides being rectilinear the underlying distance metric is weighted by some importance factor.

When the distance between two points is a symmetric function of their positions we assign a positive weight to each demand point commensurate with the intensity of demand at that point. But for peak-hour traffic and many similar situations, when the distance is not a symmetric function, we associate four weights along the four principal directions - horizontally left or right and vertically up or down - with respect to the demand point.

Our method of approach in this problem consists first

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in finding the maximum weighted rectilinear distance of the set of given locations from a point, arbitrarily selected within the constrained region, and subsequently reducing this distance gradually so that primal feasibility is never violated. With a view to achieving this we have appealed to methods of plane analytic geometry.

Although the solution technique has been developed for polyhedral sets, it is applicable to any convex set.

### 3.2.2 Formulation of the problem

Suppose that  $P_i(x_i, y_i)$ ,  $i \in I = \{1, 2, \dots, n\}$  are the given demand points comprising the set  $S$ ,  $(x, y)$  is the location of the facility (to be determined),  $W_i = \begin{bmatrix} U_i \\ V_i \end{bmatrix}$  is the weight associated with the  $i$ th demand point, where

$$U_i = \begin{cases} U_i^- & \text{for } x < x_i \\ U_i^+ & \text{for } x \geq x_i \end{cases} \quad \text{and} \quad V_i = \begin{cases} V_i^- & \text{for } y < y_i \\ V_i^+ & \text{for } y \geq y_i \end{cases}$$

Let  $F(x, y) = \max_i \{ U_i |x - x_i| + V_i |y - y_i| \}$ . Our problem may then be stated as follows.

$$\begin{aligned} & \text{minimize} && F(x, y) && (1) \\ & (x, y) \in E_2 \end{aligned}$$

subject to  $a_j x + b_j y \leq c_j$ ,  $j \in \{1, 2, \dots, m\}$ ,  $a_j$ ,  $b_j$  and  $c_j$  being constants defining the linear constraints, forming the convex polyhedron  $R$ . For the symmetric distance case,  $U_i^- = U_i^+ = V_i^- = V_i^+$  for all  $i$ .

The unweighted 1-centre problem in the absence of const-

rains has been elegantly dealt with by Wesolowsky [78] and Elzinga and Hearn [31] using the rectilinear norm. Megiddo [57, 58] has proposed a linear time algorithm that solves the weighted but unconstrained problem. Francis and White [36] have given an extensive treatment to, inter alia, single and multiple facility unconstrained minimax location problems with or without weights. The equiweighted problem corresponding to (1) has been solved by Chakrabarty and Chaudhuri [7]. The method of solution required four pairs of lines each pair being drawn parallel to the axes of coordinates through a vertex of the smallest rectangle enclosing all the demand points by two lines inclined at angles of  $45^\circ$  and  $135^\circ$  respectively with the x-axis. These four lines are actually the equipolygon with respect to a pair of dominating points associated with the objective function. These lines together with the boundary were sufficient to determine the optimal solution whereas in the present case  ${}^nC_2$  equipolygons in addition to the boundary need to be considered. The unconstrained version of the present problem has been studied using the symmetric distance rectilinear metric by Chakrabarty and Chaudhuri [8]. Batta et al. [4] presented an algorithm to solve the locational problem involving weights using  $L_1$ -norm with barriers of arbitrary shape and forbidden convex regions. The solution to the asymmetric distance minimax problem has been obtained by Dykstra et al. [30], Hodgson et al. [46],

Drezner and Wesolowsky [25] and Chakrabarty and Chaudhuri [9]. Furthermore, Drezner [19] has presented an algorithm for solving, among other cases, the weighted minimax problem subject to planar constraints.

### 3.2.3 Preliminaries

We now provide requisite definitions and notations for developing our algorithm.

1)  $R_{ij}$  denotes the rectangle having sides parallel to the coordinate axes through diagonally opposite vertices  $P_i$  and  $P_j$ .

2)  $P_{ij}$  denotes the locus of points at which the weighted rectilinear distances of  $P_i$  and  $P_j$  are equal. This locus will be hereafter called the *equipolygon* of  $P_i$  and  $P_j$ . Refer to figure 1 for the diagram of the equipolygon  $P_{ij}$ .

3)  $L(P_i, P)$  represents an "L-shaped" curve having a straight line segment through  $P_i$  parallel to y-axis joined end-to-end to another segment through  $P$  parallel to the x-axis.

4) Let  $P_iX$  and  $P_iY$  be lines drawn through  $P_i$  parallel to the axes of coordinates. The *cone of descent direction* at  $(\alpha, \beta)$  for the objective function at  $(x_i, y_i)$  is then defined by the set

$$H = \left\{ (x, y) : U_i |x - x_i| + V_i |y - y_i| \leq U_i |\alpha - x_i| \right.$$

$\left. + V_i |\beta - y_i| \text{ and bounded by } P_iX \text{ and } P_iY \text{ with respect to which the points } (x, y) \text{ and } (\alpha, \beta) \text{ lie in the same quadrant} \right\}$ .

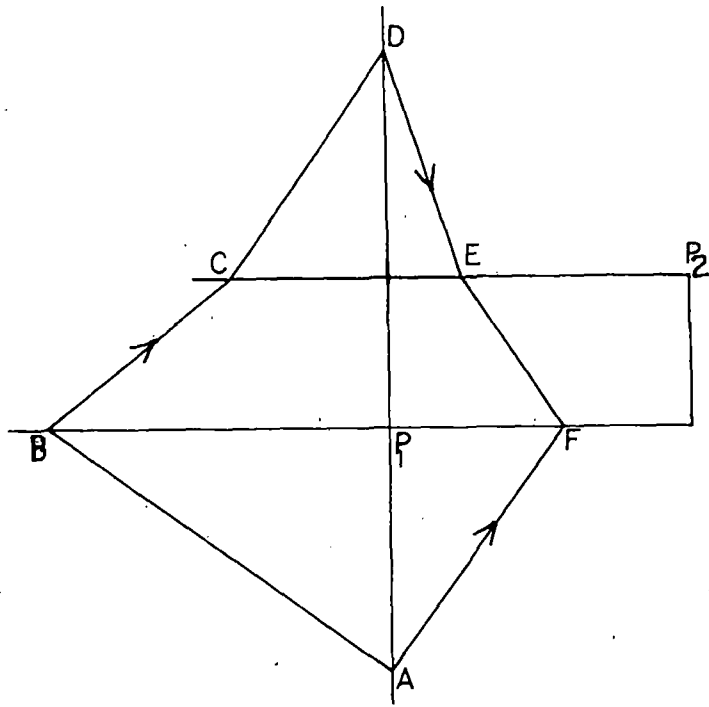


Fig 1.

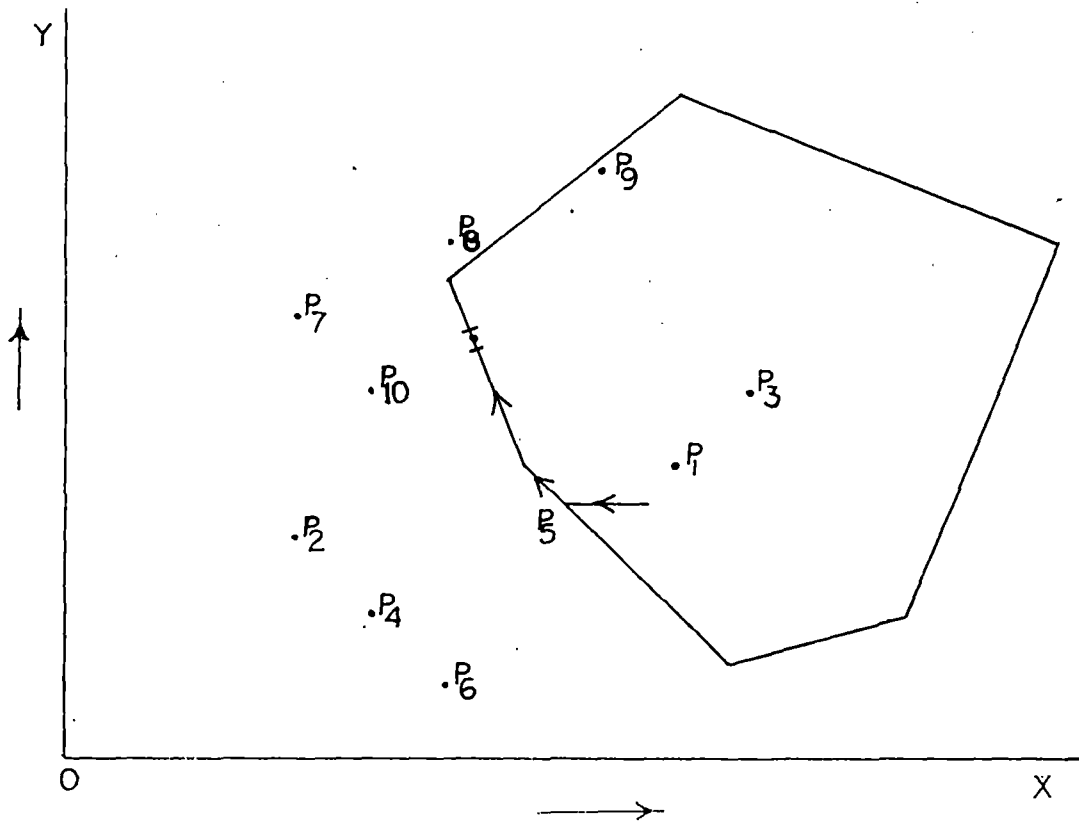


Fig 2

5)  $D_{ij} = U_i^+ V_j^k - U_j^- V_i^l$ , where  $k = -1, l = 1$

or  $k = 1, l = -1$  according as  $y_i < y_j$  or  $y_i > y_j$ ,

with  $x_i < x_j$ .

6)  $\Gamma_{ij}$  is the portion of  $P_{ij}$  lying within  $R_{ij}$ .

7)  $\partial R$  defines the boundary of  $R$ .

8)  $d(A, B)$  indicates the rectilinear distance between  $A$  and  $B$  and  $\rho(A, P_i)$  denotes the weighted rectilinear distance between  $A$  and  $P_i$ .

9) Let the index set  $J \subset I$  be such that  $\rho(Q, P_i) = \rho(Q, P_j) > \rho(Q, P_k)$  where  $i, j \in J$  and  $k \in I \setminus J$ . If  $Q$  is not an optimal solution and moving along the direction of descent of  $P_{ij}$  does not violate the condition of primal feasibility then the points  $P_i, P_j$  are to be called the *Dominating points* at  $Q$ . For its determination we shall make use of a criterion given in sec. 3.2.6.

### Stopping Criteria (SC)

I Let  $Q$  be a point such that  $\rho(Q, P_i) = \rho(Q, P_j) = \rho(Q, P_k) \geq \rho(Q, P_l)$  where  $i, j, k \in I$  and  $l \in I - \{i, j, k\}$ . If a movement away from  $Q$  in the direction of descent of any equipolygon violates primal feasibility then  $Q$  is optimal.

II(a) When moving away from  $Q$  defined in I in the direction of descent of  $P_{ij}$  and maintaining primal feasibility, if no  $Q_1$  exists such that  $\rho(Q_1, P_i) = \rho(Q_1, P_j) = \rho(Q_1, P_k)$ ,  $k \in I - \{i, j\}$  then an extreme point of  $\Gamma_{ij}$  is a unique optimum if

$D_{ij} \neq 0$  (sec. 3.2.6). If, however,  $D_{ij} = 0$  then the whole of  $\Gamma_{ij}$  or the portion extending from  $Q$  to the extreme point beyond which primal feasibility ceases to be valid, forms the set of optimal points according as  $Q \notin$  or  $\in R_{ij}$ .

(b) If  $D_{ij} = 0$  and the objectives at  $Q, Q_1 \in \Gamma_{ij}$  are equal then any point on the line segment  $QQ_1$  is optimal; else, if  $Q_1 \in \Gamma_{ij}$  and  $Q \notin \Gamma_{ij}$  then the part of  $\Gamma_{ij}$  from its extreme point, encountered first, along the descent direction up to  $Q_1$ , forms the set of optimal points (sec. 3.2.6).

### III (For a Boundary Point)

(a) Let  $(x_1, y_1)$  be the weighted farthest point from  $(\alpha, \beta) \in \partial R$  with respect to the chosen norm. If  $\partial R \cap H = \{(\alpha, \beta)\}$  then we have a unique optimum thereat. But if  $\partial R \cap H$  consists of a segment of  $\partial R$  then any point contained in this segment is optimal.

(b) If the direction of descent of  $P_{ij}$  at  $Q = P_{ij} \cap \partial R$  is outside  $R$  and, moreover, if any movement along a direction defined by  $Q$  and  $\partial R$  destroys primal feasibility, then  $Q$  is the unique optimum. We shall be making use of the above criteria while developing our algorithm which now follows.

### 3.2.4 Algorithm

**Step 0** Take any  $P \in \text{Int } R$  as the starting point. Find  $P_i \in S$  so that  $\rho(P, P_i)$  is a maximum. Let  $1 \leftarrow i$ . Determine a point  $Q \in L(P_1, P)$  such that  $\rho(Q, P_1) = \rho(Q, P)$ ,  $j \in I$  -

(1) and  $d(Q, P)$  is minimum. If  $Q \notin R$  then  $Q \leftarrow \partial R \cap L(P_1, P)$  and go to step 1; else  $P_2 \leftarrow P_j$ . If  $Q \in \partial R$  then go to step 2(a); else go to step 3.

**Step 1** If  $Q$  satisfies SC-III(a) then stop.  $Q$  is the required location; else move along the direction of descent of  $\partial R$  until the first occurrence of any one of the following:

(i) an extreme point  $B$  of the currently active boundary is attained: replace the boundary with the succeeding one,  $Q \leftarrow B$  and go to step 1;

(ii) a point  $B \in \partial R$ , from which the weighted distances of three or more points are equal, is reached:  $Q \leftarrow B$  and go to step 2(a).

(iii) a point  $B \in \partial R$ , from which the weighted distances of two location points - call them  $P_1$  and  $P_2$  - are equal, is obtained:  $Q \leftarrow B$  and go to step 2(b).

**Step 2(a)** If a single equipolygon passes through  $Q$  then call it  $P_{12}$  and go to step 2(b). For three or more equipolygons meeting at  $Q$  apply SC-I. If  $Q$  satisfies it then stop;  $Q$  is optimal; else, choose the two dominating points from among them (sec. 3.2.6) required for the next iteration (call them  $P_1$  and  $P_2$ ) and go to step 2(b).

**Step 2(b)** If SC-III(b) applies at  $Q$  then it is the optimal point; else, if one of the points is non-dominating, then drop it and go to step 1; else go to step 3.

**Step 3** Moving in the direction of descent of  $P_{12}$  at  $Q$  and

maintaining primal feasibility, if a point  $Q_1 \in R$  can be found such that  $\rho(Q_1, P_1) = \rho(Q_1, P_2) = \rho(Q_1, P_k)$ ,  $k \in I - \{1, 2\}$  then  $Q \leftarrow \partial R \cap P_{12}$  and go to step 2(b); else, if  $Q_1 \in R$  then apply SC-I and, if necessary, SC-II(b) (when  $D_{12} = 0$ ), to  $Q_1$ . If  $Q_1$  satisfies the former or both depending upon the value of  $D_{12}$  then stop, with the optimal solution; else, if  $Q_1$  satisfies neither, then  $Q \leftarrow Q_1$ , choose an equipolygon by dropping a non-dominating point from among  $P_1, P_2$  and  $P_k$ , call it  $P_{12}$ , and repeat step 3; else, if no such  $Q_1$  exists, then go to step 4.

**Step 4** Apply SC-II(a) to obtain the optimal solution or solutions as the case may be.

**N.B.** 1. For the symmetric distance rectilinear metric  $D_{ij} = 0$  always. 2. The algorithm has been designed in such a way that the movement always takes place along the direction of descent of an equipolygon having a finite number of sides (sec. 3.2.6) or an edge of  $\partial R$  - the number of edges that make  $R$  is also finite - maintaining all the while primal feasibility. This, therefore, ensures that cycling never occurs.

### 3.2.5 Numerical Examples

We shall consider two examples - the first one involving asymmetric distance and the other employing symmetric distance- to illustrate how the algorithm works. In the computation of the optimal solution in both these cases we

shall retain figures correct to 3 decimal places.

1. **Asymmetric distance case:** Let us assume that the location points and the associated weights in the four principal directions are as shown in Table 1 below. Also, let the convex polyhedral region  $R$ , within or on the boundary of which the required facility point is to be located, be given by

$$-x - y \leq -10 \quad (1) \qquad -5x - 2y \leq -38 \quad (2)$$

$$-5x + 6y \leq 14 \quad (3) \qquad 2x + 5y \leq 61 \quad (4)$$

$$5x - 2y \leq 51 \quad (5) \qquad 2x - 7y \leq 8 \quad (6)$$

$P_i$	$x_i$	$y_i$	$U_i^-$	$U_i^+$	$V_i^-$	$V_i^+$
$P_1$	8	4	0.8	1.2	0.12	0.48
$P_2$	3	3	1.6	2.4	0.24	0.96
$P_3$	9	5	0.6	0.9	0.09	0.36
$P_4$	4	2	1.2	1.8	0.18	0.72
$P_5$	6	3	3.2	4.8	0.48	1.92
$P_6$	5	1	0.4	0.6	0.06	0.24
$P_7$	3	6	2.4	3.6	0.36	1.44
$P_8$	5	7	2.8	4.2	0.42	1.68
$P_9$	7	8	1.6	2.4	0.24	0.96
$P_{10}$	4	5	2.0	3.0	0.30	1.20

Table 1

Let  $P = (8, 3.5) \in \text{Int } R$  be the initial point. Following step 0 of the algorithm the weighted farthest point  $P_1 = (3,$

$\delta$ ),  $P_2 = (6, 3)$ , and  $Q = (4.421, 3.500)$ . As  $Q \notin R$ ,  $Q \leftarrow \partial R \cap L(P_1, P) = (6.500, 3.500)$ . By step 1, we move along the descent direction of  $\partial R$  until the extreme point  $B = (6, 4)$  is obtained. We next replace the currently active boundary of (1) with the succeeding one corresponding to (2) and  $Q \leftarrow B$ . After two consecutive iterations of step 1 and another of step 2(b), we finally obtain the optimal location to be any point  $\in$  the segment of  $\partial R$  from  $(5.253, 5.867)$  to  $(5.220, 5.950)$  with the corresponding optimum objective = 8.160. Refer to figure 2, which clearly explains how the optimal solution of problem 1 converges relative to the constrained set.

**2. Symmetric distance case:** We next consider a problem in which the coordinates of the location points and the constraints defining the polygonal region  $R$  are the same as in problem 1, but the weights associated with the respective location points are fixed constants (shown alongside each entry in a separate column in Table 2 below) independent of the directions in order to demonstrate how symmetry in the distance norm affects the optimal location.

Let  $P = (8, 3.5) \in \text{int } R$  as before. By step 0 of the algorithm,  $P_1 = (5, 7)$ ,  $Q = (7.500, 3.500)$  and  $P_2 = (3, 6)$ . As  $Q \in \text{int } R$  we obtain following step 3,  $Q_1 = (4.833, 3.167) \notin R$ .  $Q \leftarrow \partial R \cap P_{12} = (7.000, 3.000)$ . After two more iterations of steps 1 and 2(b) the optimal is found to be any point  $\in$  the portion of  $\Gamma_{12}$  from  $(5.551, 5.122)$  to  $(5.786, 5.357)$

and the corresponding optimal objective = 10.286.

$P_i$	$x_i$	$y_i$	$W_i$
$P_1$	8	4	1.00
$P_2$	3	3	2.00
$P_3$	9	5	0.75
$P_4$	4	2	1.50
$P_5$	6	3	4.00
$P_6$	5	1	0.50
$P_7$	3	6	3.00
$P_8$	5	7	3.50
$P_9$	7	8	2.00
$P_{10}$	4	5	2.50

Table 2

The above example clearly validates the observation made regarding the simplifications inherent in the second problem.

### 3.2.6 Appendix

In order to develop our algorithm we have made use of the following results given in [9].

1. The optimal objective value with respect to any two location points  $P_i$  and  $P_j$  occurs within  $R_{ij}$  in the absence of constraints.
2. The equipolygon of two points  $P_i$  and  $P_j$  with unequal

weights is a closed figure having 4 or 6 vertices enclosing the point associated with the greater weight, as shown in figure 1. In case the weights are equal it consists of a straight line segment lying within  $R_{ij}$  flanked by two semi-infinite straight lines perpendicular to the sides of  $R_{ij}$ .

3. The weighted rectilinear distance from any point outside the equipolygon to the location point with greater weight is greater than that to the other point and vice versa.