B I B L I O G R A P H Y

1. ABRAMIAN, B.L., ARUTIUNIAN, N.Kh. and BABLOOIAN, A.A. (1964) :

   On two contact problems for the elastic sphere. Prikladnaia matematika i mekhanika. Vol. 21, p. 622.

2. ABRAMIAN, B.L. (1957) :

   On the plane problem of the theory of elasticity for a rectangle. Prikladnaia matematika i mekhanika. Vol. 21, p. 89.

3. ACHENBACH, J.D. (1976) :

   Wave propagation, elastodynamic stress intensity factors and fracture. Theoritical and Applied Mechanics, ed. W.T. Koiter, Proc. 14th IUTAM congress, North-Holland publishing co., Amsterdam.

4. ACHENBACH, J.D. (1975) :

   Wave propagation in elastic solids. North-Holland publishing company, Amsterdam-Oxford. American Elsevier Publishing Company, INC., New York.

5. ACHENBACH, J.D. (1972) :

   Dynamic effects in brittle fracture. Mechanics Today Vol. 1. ed. S. Nemat-Nasser, Pergamon Press, New York.

6. ADAMS, G.G. (1978) :

   A steadily moving load on an elastic strip resting on a rigid foundation. Int. J. Engng. Sci. Vol. 16, p. 659.

7. ADAMS, G.G. (1980) :

Crack interaction in an infinite elastic strip. Int.J. Engng. Sci. Vol. 18, p. 455.

8. ALBLAS, J.B. and KUIPERS, M. (1970) :

The contact problem of a rigid cylinder rolling on a thin viscoelastic layer. Int. J. Engng. Sci. vol. 8, p. 363.

9. ALBLAS, J.B. (1974) :

On the two dimensional contact problem of a rigid cylinder pressed between two elastic half-planes. Mech. Res. communications. Vol. 1, p. 15.

10. ALBLAS, J.B. and KUIPERS, M. (1971) :

The two dimensional contact problem of a rough stamp sliding slowly on an elastic layer-I. General considerations and thick layer asymptotics. Int. J. Solids Structures. Vol. 7, p. 99.

11. ALBLAS, J.B. and KUIPERS, M. (1971) :

The two dimensional contact problem of a rough stamp sliding solwly on an elastic layer-II. Thin layer asymptotics. Int. J. Solid Structures. Vol. 7, p. 225.

12. ALEXANDROV, A. P. and LAZURKIN I. S. (1940) :

Acta Phyic Chimica U.R.S.S. Vol. 12, p. 647.

13. ANG, D.D. (1960) :

Transient motion of a line load on the surface of an elastic half space. Quart. Appl. Math. Vol. 18, p. 251.

14. ANG, D.D. and KNOPOFF, L. (1964) :

Diffraction of scalar elastic waves by a finite crack.

Proc. Nat. Acd. Sci. Vol. 51, p. 593.

15. ANGEL, Y.C. and ACHENBACH, J.D. (1985) :
Reflection and transmission of elastic waves by a periodical array of cracks. J. Appl. Mech. vol. 52, p. 33.

16. ARIN, K. and ERDOGAN, F. (1971) :
Penny-shaped crack in an elastic layer bonded to dissimilar half spaces. Int. J. Engng. Sci. Vol. 9, p. 213.

17. ATKINSON, C. and LIST, R.D. (1972) :
A moving crack problem in viscoelastic solid. Int. J. Engng. Sci. Vol. 10, p. 309.

18. ATKINSON, C. (1977) :
Cracks in media with spatially varying elastic moduli. Mechanics of Fracture. Vol. 4, Editor G. C. Sih, Noordhoff International Publishing, Leyden.

19. ATKINSON, C. (1968) :
On axially symmetric expanding boundary value problems in classical elasticity. Int. J. Engng. Sci. Vol. 6, p. 27.

20. ATKINSON, C. and LIST, R.D. (1978) :
Steady state crack propagation into media with spatially varying elastic properties. Int. j. Engng. Sci. Vol. 16, p. 717.

21. AWOJOBI, O.A. and GROOTENHUIS, P. (1965) :
Vibration of rigid bodies on semi-infinite elastic

media. Proc. Roy. Soc. Vol. A223, p.521.

22. BAKER, B.R. (1962) :

Dynamic stress created by a moving crack. J. Appl. Mech. Vol 15, p. 449.

23. BORODACHEV, N.M. (1962) :

The plane contact problem for an elastic body of finite width. (in Russian). Akademiia nauk S.S.S.R. Izvestiia. Vol. 6, p. 629.

24. BORODACHEV, N.M. (1976) :

Contact problem for a stamp with a rectangular base. Prikladnaia matematika i mekhanika. Vol. 40, p. 554.

25. BREKHOVSKIKH, L.M. (1960) :

Waves in layered media. Academic Press, New York.

26. BOSE, S.K. (1968) :

The oscillatory indentation problem by a rigid circular disk. Bull. Cal. Math. Soc. Vol. 60, p. 87.

27. BOSTROM, A. (1987) :

Elastic wave scattering from an interface crack : Antiplane strain. J. Appl. Mech. Vol. 54, p. 503.

28. BOSTROM, A. and OLSSON, P. (1987) :

Scattering of elastic waves by non-planar cracks. Wave Motion. vol 9, p. 61.

29. BROCK, L.M. and DENG, Y.C. (1985) :

Dynamic studies of running half-plane and crusiform cracks. Int. J. Engng. Sci.. Vol 23, p. 163.

30. BROTHERS, P.W., SINCLAIR, G.B. and SEGEDIN, C.M. (1977) :

Uniform indentation of the elastic half-space by a rigid rectangular punch. Int. J. solids and structures. Vol. 13, p. 1059.

31. BULLEN, K.E. (1963) :

An introduction to the theory of seismology. University Press, Cambridge.

32. CAGNIARD, L. (1962) :

Rflection and refraction of progressive waves, translated and revised by E.A. Flinn and C.H.Dix. MeGraw-Hill Book Company, New York.

33. CEREPANOV, G.P. and AFANASEV, E.F. (1974) :

Some dynamic problems of the theory of elasticity-a review. Int. J. Engng. Sci. Vol. 12, p. 665.

34. CHANG, SHIH-JUNG. (1971) :

Diffraction of plane dilatational waves by a finite crack. Quart. J. Mech. Appl. Math.

35. CHEN, E.P. (1978) :

Sudden appearance of a crack in a stretched finite strip. J. Appl. Mech. vol. 45, p. 277.

36. COLE, J. and HUTH, J. (1958) :

Stress produced in a half-plane by miving loads. J. Alppl. Mech. Vol. 25, p. 43.

37. COOKE, J.C. (1963) :

Some further triple integral equation solutions. Proc. Camb. Phil. Soc. Vol. 13, p. 303.

38. COUSSY, O. (1987) :

A moving crack problem along the interface of two viscoelastic media. Int. J. Engng. Sci. Vol. 25, p. 609.

39. CRAGGS, J.W. (1966) :

The growth of a disk-shaped crack. Int. J. Engng. Sci. Vol. 4, p. 113.

40. DE, J. and PATRA, B. (1990) :

Mving Griffith crack in an orthotropic strip. Int. J. Engng. Sci. Vol. 28, p. 809.

41. DE SARKAR, P.K. (1983) :

On diffraction of SH-waves by cut in nonhomogeneous solids. Proc. Indian Acad. Science. (Math. Sci.). vol. 92, p. 41.

42. DHALIWAL, G. K. and DHALIWAL, R. S. (1978) :

On three coplanar cracks in an infinite transversely isotropic medium. Int. J. Engng. Sci. Vol. 16, p. 253.

43. DHALIWAL, R.S., SHING, B.M. and SNEDDON, I.N. (1979) :

A problem of Reissner-Sagoci type for an elastic cylinder embedded in an elastic half-space. Int. J. Engng. Sci. Vol. 16, p. 649.

44. DUNDURS, J., KIATTIKOMOL, K. and KEER, L. M. (1974) :

Contact between plates with sagged supports. American society of civil engineers. Engineering mechanics division, Journal. Vol. 100, p. 445.

45. EMBLEY, G.T. and SIH, G.C. (1971) :

Response of a penny-shaped crack to impact waves, Proceedings of the 12th Midwestern Mech. Conf. Vol. 6,

p. 473.

46. ENGLAND, A.H. (1961) :

A punch problem for a transversely isotropic layer. Proc. Camb. Phil. Soc.

47. ERDOGAN, F. (1965) :

Stress distribution in bonded dissimilar materials containing circular or ring-shaped cavities. J. Appl. Mech. Vol. 32, p. 829.

48. ERDOGAN, F. (1969) :

Approximate solutions of systems of singular integral equations. SIAM J. Appl. Mech. Vol. 17, p. 1041.

49. EWING, W.M., JARDETZKY, W.S. and PRESS, F. (1957) :

Elastic waves in layered media. McGraw-Hill Book company, New York.

50. FABRIKANT, V.I. (1991) :

Nonaxisymmetric annular punch problem. J. Appl. Mech. Vol. 58, p. 947.

51. FOX, L. and GOODWIN, E. T. (1953) :

The numerical solution of non-singular linear integral equations. Phil. Trans. Vol. A245, p.501.

52. FREUND, L.B. (1972) :

Wave motion in an elastic solid due to a non-uniformaly moving line load. Quart. Appl. Math. Vol. 30, p. 27.

53. FREUND, L.B. (1975) :

Dynamic crack propagation. The Mathematics of Fracture, ed. F. Erdogan, ASME AMD, Vol. 19, American Society of

Mechanical Engineers, New York.

54. FRUND, L.B. (1976) :

The analysis of elastodynamic crack tip stress fields. Mechanics Today, ed. S. Nemat- Nasser, Vol. III, Pergamon Press, New York.

55. GALIN, A.L. (1961) :

Contact problems in the theory of elasticity (ed. : I.N. Sneddon). North Carolina State University, Raleigh.

56. GARD, A.C. (1981) :

Stress distribution near periodic cracks at the interface of two bonded dissimilar orthotropic half-planes. Int. J. engng. Sci. Vol 19, p. 1101.

57. GEORGIADIS, H.G., THEOCARIS, P.S. and MOUSKOS, S.C. (1991) :

Plane impact of a cracked viscoelastic body. Int. J. Engng. Sci. Vol. 29, p. 171.

58. GHOSH, M. and GHOSH, M.L. (1985) :

Harmonic rocking of a rigid strip on a semi-infinite elastic medium. Indian J. Pure Appl. Math. Vol. 16, p. 938.

59. GLADWELL, G. M. L. (1968) :

Forced tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid. Int. J. Engng. Sci. Vol. 6, p. 591.

60. GLADWELL, G. M. L. (1980) :

Contact problems in the classical theory of elasticity. Sijthoff and Noordhoff, Alphen ann den Rijn.

61. GOLDEN, J.M. (1977) :

Hysteretic friction of a plane punch on a half-plane with arbitrary viscoelastic behaviour. Quart. J. Appl. Math. Vol. XXX, p. 23.

62. GOLDEN, J.M. (1979) :

The problem of a moving rigid punch on an unlubricated viscoelastic half-plane. Quart. J. Appl. Math. Vol. XXXII, p. 25.

63. GOLDEN, J. M. (1982) :

The problem of a rigid punch sliding on an elastic or a viscoelastic layer. Acta Mechanica. Vol. 43, p. 201.

64. GOLDEN, J. M. and GRAHAM, G. A. C. (1988) :

Boundary value problems in linear viscoelasticity. Spinger-Verlag, Berlin, Heidelberg, New York.

65. GOODMAN, L.E. and KEER, L.M. (1965) :

The contact stress problem for an elastic sphere indenting an elastic cavity. Int. J. of solid and Structures. Vol. 1, p. 407.

66. GRADSHTEYN, I. S. and RYZIK, I. M. (1965) :

Table of integrals, series and products. Academic Press, London, New York.

67. GRAFF, KARL F. (1973) :

Wave motion in elastic solids. Ohio State University Press.

68. GUBENKO, V.S. and MOSSAKOVSKII, V.I. (1960) :

Pressure of an axially symmetric circular die on an elastic half-space. Prikladnaia matematika i mekhanika. Vol. 24, p. 334.

69. HUDSON J. A. (1980) :

The excitation and propagation of elastic waves. Cambridge University Press, Cambridge.

70. HUNTER, S.C. (1961) :

The rolling contact of a rigid cylinder with a viscoelastic half-space. J. Appl. Mech. Vol. 4, p. 611.

71. ITOU, S. (1978) :

Dynamic stress concentration around two coplanar Griffith cracks in and infinite elastic medium. J. Appl. Mech. Vol. 4, p. 803.

72. ITOU, S. (1980) :

Diffraction of an antiplane shear wave by two coplanar Griffith cracks in an infinite elastic medium. Int. J. Solids Structures Vol. 16, p. 1142.

73. ITOU, S. (1980) :

Transient analysis of stress waves around two coplanar Griffith cracks under impact load. Engng. Fract. Mech. Vol. 13, p. 349.

74. JAIN, D.L. and KANWAL, R.P. (1972) :

Acustic diffraction of a Plane Wave by two coplanar parallel perfectly soft or rigid strips. Canadian Journal of Physics. Vol. 50, p. 928.

75. JAIN, D.L. and KANWAL, R.P. (1972) :

Dffraction of elastic waves by two coplanar and parallel rigid strips. Int. J. Engng. Sci. Vol. 10, p. 925.

76. JAIN, D.L. and KANWAL, R.P. (1972) :
Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium. Int. J. solids Structures. Vol. 8, p. 961.

77. JAIN, D.L. and KANWAL, R.P. (1970) :
Torsional oscillations of an elastic half-space due to an annuular disk and related problems. Int. J. Engng. Sci. Vol. 8, p. 687.

78. KALKER, J.J. (1972) :
On elastic line contact, J. Appl. Mech. Vol. 39, p. 1125.

79. KALKER, J.J. (1977) :
A survey of the mechanics of contact between solid bodies. ZAMM. Vol. 57.

80. KANNINEN, M.F. (1978) :
A critical appraisal of solution techniques in dynamic fracture mechanics. Numerical Methods in Fracture Mechanics, ed. A.R. Luxmoore and D.R.J. Owen, University College of Swansea, U.K.

81. KANWAL, R. P. (1971) :
Linear integral equations, theory and techniques. Academic Press, New York.

82. KARASUDHI, P., KEER, L.M. and LEE, S.L. (1968) :
Vibratory motion of a body on an elastic half-plane. J. Appl. Mech. Vol. 35, p. 697.

83. KASSIR, M.K. and TSE, S. (1983) :

Moving Griffith crack in an orthotropic material. Int. J. Engng. Sci. Vol. 21, p. 315.

84. KASSIR, M.K. and BANDYOPADHYA, K.K. (1983) :
Impact response of a cracked orthotropic medium. J. Appl. Mech. Vol. 50, p. 630.

85. KELLER, J.B. (1958) :
"A geometrical theory of diffraction" in calculus of variations and its applications. Vol. VIII McGraw-Hill. Publishing company, New York.

86. KNOPS, R. and PAYNE, L.E. (1971) :
Uniqueness theorems in linear elasticity. Springer-verlag, New York.

87. KOITER, W.T. (1954) :
Approximate solutions of Wiener-Hopf type equations with applications, Proc. Kon. Ned. Acad. Wet. Vol. 57, p. 2.

88. KOLSKY, H. (1963) :
Stress waves in solids. Dover Publications, INC. New York.

89. KOSTROV, B.V. (1964) :
The axisymmetric problem of propagation of a tensile crack. Appl. Math. Mech. Vol. 28, p. 793.

90. KRENK, S. and SCHMIDT, H. (1982) :
Elastic wave scattering by a circular crack. Phil Trans. R. Soc. Lond. Vol. A 308, p. 167.

91. KRYLOV, V.I. and KANTOROVICH, L.V. (1957) :
Approximate methods of higher analysis. Interscience, New York.

92. LEADERMANN, H. (1943) :

The elastic and creep properties of filamentous materials and other high polymers. Textile Foundation, National Bureau of standareds, Washington.

93. LOEBER, J.F. and SIH, G.C. (1968) :

Dffraction of antiplane shear waves by finite crack. J. Accous. Soc. Am. Vol. 44, p. 90.

94. LOVE, A.E.H. (1944) :

The mathematical theory of elasticity, 4th ed. dover Publications, Inc. New York.

95. LOWENGRUB, L. and SRIVASTAVA, K. N. (1968) :

On two coplanar cracks in an infinite elastic medium. Int. J. Engng. Sci. Vol.6, p.359

96. MAL, A.K., ANG, D.D. and KNOPOFF, L. (1968) :

Diffraction of elastic waves by a rigid circular disc. Proc. Camb. Phil. Soc. Vol. 64, p. 237.

97. MAL, A.K. (1970) :

Interaction of elastic waves with a griffith crack. Int. J. Engng. Sci. Vol. 8, p. 763.

98. MAL, A.K. (1970) :

Interaction of elastic waves with a penny-shaped crack. Int. J. Engng. Sci. Vol. 8, p. 381.

99. MAL, A.K. (1972) :

A note on the low frequency diffraction of elastic waves by a Griffith crack. Int. J. Engng. Sci. Vol 10, p. 609.

100. MATCZYNSKI, M. (1973) :

Motion of a crack in antiplane strain of an elastic strip. Archives of Mechanics. Vol. 25, p. 823.

101. MATYSIAK, S. and PASZT, A. (1986) :

On interaction between a crack and rigid inclusions in an elastic space. Bull. Polish Acad. Sci, Tech. Sci. Vol. 34, p. 495.

102. MAUE, A.W. (1954) :

Die entspannuangswelle bei plotzlichen einschnitt eines gespannten elastischen korpers. ZAMM. Vol. 34, p. 1.

103 MILLAR, G.F. and PURSEY, H. (1954) :

The field and radiation impedence of mechanical radiators on the surface of semi-infinite isotropic solid.

104. MILLER, M.K. and GUY, W.T. (1966) :

Numerical inversion of the Laplace Transform by use of Jacobi polynomials. STAM J. Numerical Analysis. Vol. 3, p. 624.

105. MUSKHELISHVILI, N.I. (1953) :

Some basic problems of the mathematical theory of elasticity. English translation, J.R.M. Radok. Noordhoff international publishing, Leyden.

106. NISHIDA, Y., SHINDO, Y. and ATSUMI, A. (1984) :

Diffraction of horizontal shear waves by a moving interface creack. Acta Mechanica. Vol. 54, p. 23.

107. NOBLE, B. (1958) :

Methods based on the Wiener-Hopf Technique. Pergamon Press, New York.

108. NOBLE, B. (1962) :

Electromagnetic waves. Ed. R.E. Langer, University of Wisconsin Press.

109. NOBLE, E.B. (1963) :

Proc. Camb. Philos. Vol. 59, p. 351.

110. NOLLE, A.W. (1949) :

J.Polymer Sci. Vol. 5, p. 1.

111. OKUBO, H. (1951) :

On the two dimensional problem of a semi-infinite elastic body compressed by an elastic plane. Quqrt. J. Mech. Appl. Math. Vol. 4, p. 260.

112. OLESIAK, Z. (1965) :

Annular punch on elastic semi-space. Archives of mechanics. Vol. 17, p. 633.

113 ONDER, Y., SUHUBI, E. S. and IDEMEN, M. (1975) :

Diffraction of scalar elastic waves by a rigid half-plane between two semi-infinite media. Lett. Appl. Engng. Sci. Vol. 3 ,p. 15.

114. ONG, M.K. and SRIVASTAVA R.P. (1985) :

A crusiform crack opened by internal pressure varying with time. Int. J. Engng. Sci. Vol. 23, p. 999.

115. PALAIYA, R. M. and MUJUMDER, P. (1981) :

Interaction of antiplane shear waves by rigid strips lying at the interface of two bonded dissinilar half spaces. ZAMM. Vol. 16, p. 120.

116. PANEK, C. and KALKER, J.J. (1977) :
A solution for a narrow rectangular punch. J. Elasticity. Vol. 7, p. 213.

117. PARIHAR, K.S. and LALITHA, S. (1987) :
Cracks located on a single line in an orthotropic elastic medium in which body forces are acting. Int. J. Engng. Sci. Vol. 25, p. 735.

118. PARTON, V. Z. and MOROZOV, E. M. (1978) :
Elastic plastic fracture mechanics. Mir Publishers, Moscow.

119. PILANT, WALTER L. (1979) :
Elastic waves in the earth. Elsevier Scientific Publishing Company, Amsterdam-Oxford- New York.

120. POULOS, H.G. and DAVIS, E.H. (1968) :
The settlement behaviour of single axially-loaded piles and piers. Geotechnique. Vol. 18, p. 351.

121. PRASAD, S.N. and CHATTERJEE, S.N. (1973) :
Some mixed boundary value problems of elasticity in a rectangular domain. Int. J. Solids Structures. Vol. 9, p. 1193.

122. PRASAD, S.N. and DASGUPTA, S. (1975) :
Effect of slidding friction on contact stresses in a rectangle compressed by rigid planes. J. Appl. Mech. Vol. 42, p. 656.

123. PROTSENKO, V.S. (1974) :

The pressure on an elastic half-space due to a strip-shaped punch. Prikladnaia mekhanika. (Soviet applied Mechanics). Vol. 8, p. 45.

124. RESSNER, R. E. (1937) :
Freie und erzwungene Torsionsschwingungen den elastischen Halbraumes. Ing. Arch. Vol. 8, p.229.

125. ROBERTSON, I. A. (1966) :
Forced vertical vibration of a rigid circular disc on a semi-infinite elastic solid. Proc. Camb. Phill. Soc. Vol. 62, p. 547.

126. ROY, ARABINDA and SHABINA, F.J. (1983) :
Low frequency acoustic diffraction by a soft elliptic disk. J. Acoust. Soc. Am. Vol. 73, p. 1494.

127. ROY, ARABINDA. (1986) :
Dynamic response of elliptical footings. Int. J. Solids Structures. Vol. 22, p. 293.

128. RVACHEV, V.L. (1967) :
Studies on the contact problems of elasticity theory in the Ukraine. Prikladnaia mekhanika (Soviet applied mechanics). Vol. 3, p. 109.

129. SARKAR, J., GHOSH, M.L. and MANDAL, S.C. (1991) :
Scattering of antiplane shear wave by a propagating crack at the interface of two dissimilar elastic media. Proc. Indian Acad. Sci. (Math. Sci.). Vol. 101, p. 183.

130. SHERBY, O.D. and DORN, J.E. (1958) :
J. Mech. Phys. Solids. Vol. 6, p. 145.

131. SHINDO, Y. (1979) :

Diffraction of torsional waves by a flat annular crack in an infinite elastic medium. J. Appl. Mech. Vol. 46, p. 827.

132. SHINDO, Y. (1981) :

Sudden twisting of a flat annular crack. Int. J. Solids Structures. Vol. 17, p. 1103.

133. SHINDO, Y. (1981) :

Normal compression waves scattering at a flat annular crack in an infinite elastic solid. Quart. Appl. Math. Vol XXXIX, p. 305.

134. SHINDO, Y., NOZAKI, H. and HIGAKI, H. (1986) :

Impact response of a finite crack in an orthotropic strip. Acta Mechanica, Vol. 62, p. 87.

135. SIH, G.C. (1968) :

Some elastodynamic problems of cracks. Int. J. Fract. Mech. vol. 1, p. 51.

136. SIH, G.C. and LOEBER, J.F. (1969) :

Normal compression and radial shear waves scattering at a penny-shaped crack in an elastic solid. J. Acou. Soc. Am. vol. 46, p. 711.

137. SIH, G.C. and CHEN, E.P. (1980) :

Normal and shear impact of layered composite with a crack : Dynamic stress intensification. J. Appl. Mech. Vol. 47, p. 351.

138. SIH, G.C. and EMBLEY, G.T. (1971) :

Sudden twisting of a peny-shaped crack J. Appl. Mech. Vol. 39, p. 395.

139. SILLS, L.B. and BENVENSITE, Y. (1981) :
Steady state propagation of mode III interface crack between dissimilar viscoelastic media. Int. J. Engng. Sci. Vol. 19, p. 1255.

140. SINGH, B.M., MOODIE, T.B. and HADDOW, J.B. (1980) :
Torsion by an annular disk of an infinite cylinder embedded in an elastic half-space. Utilitas Mathematica. Vol. 18, p. 97.

141. SINGH, B. M. and DHALIWAL, R. S. (1977) :
Diffraction of SH-waves by Griffith cracks or rigid strips lying in a laterally and vertically nonhomogeneous infinite medium. Int. J. Engng. Sci. Vol. 15, p. 295.

142. SINGH, B. M., DHALIWAL, R. S. and ROKNE, J. G. (1980) :
Diffraction of antiplane shear waves by a Griffith crack or a rigid strip lying at the interface of homogeneous and nonhomogeneous elastic half-space. Acta Mechanica. Vol. 37, p. 137.

143. SLEEMAN, B.D. (1967) :
The low frequency scalar diffraction by an elliptic disc. Proc. Camb. Phil. Soc. Vol. 63, p. 1273.

144. SNEDDON, I.N. (1951) :
Fourier transform. McGraw-Hill Book company, New York.

145. SNEDDON, I.N. (1966) :

Mixed boundary value problems in potential theory. North Holland, Amsterdam.

146. SRIVASTAVA, K.N. and LOWENGRUB, M. (1968) :
Finite Hilbert transform technique for triple integral equations with trigonometrical kernels.

147. SRIVASTAVA, K.N., PALAIYA, R.M. and KARAULIA, D.S. (1980) :
Interaction of antiplane shear waves by a Griffith crack at the interface of two bonded dissimilar elastic half-spaces. Int. J. Fracture. Vol. 16, p. 349.

148. SRIVASTAVA, K.N., GUPTA, O.P. and PALAIYA, R.M. (1981) :
Interaction of elastic waves with a Griffith crack situated in an infinitely long strip. ZAMM. Vol. 61 p. 583.

149. SRIVATAVA, K.N., PALAIYA, R.M. and KARAULIA, D.S. (1983) :
Interaction of shear waves with a Griffith crack situated in an infinitely long elastic strip. Int. J. Fracture. Vol. 21, p. 39.

150. STALLYBRASS, M.P. and SCHERER, S.E. (1976) :
Forced vertical vibration of a rigid elliptic disc on elastic half-space. Int. J. Engng. Sci. Vol 14, p. 511.

151. SUHUBI, E.S. (1972) :
Moving rough punch on an elastic half-space. Bull. Tech. Univ. Istm. vol. 25, p. 71.

152. SVE, C. and KEER, L.M. (1969) :
Indentations of an elastic layer by moving punches. Int. J. Solids and structures. Vol. 5, p. 795.

153. TAIT, R.J. and MOODIE, T.B. (1981) :

Complex variable methods and closed from solutions to dynamic crack and punch problems in the classical theory of elasticity. Int. J. Engng. Sci. Vol. 19, p. 221.

154. TRANTER, C.J. (1960) :

Some triple integral equations. Proc. Glasgo Math. Asso. Vol. 4, p. 200.

155. TRANTER, C.J. (1968) :

Bessel functions with some physical applications. The English University Press Ltd., London.

156. TRICOMI, F.G. (1951) :

Quart. J. Math. Vol. 2, p. 199.

157. THOMAS, D.P. (1965) :

Acoustic diffraction by an annular disc. Int. J. Engng. Sci. Vol. 3, p. 405.

158. UEDA, S., SHINDO, Y. and ATSUMI, A. (1983) :

Torsional impact response of a penny-shaped crack lying on a bimaterial interface. Engng. Fract. Mech. vol. 18, p. 1059.

159. UFLIAND, IA. S. (1965) :

Survay of Articles on the applications of integral transforms in the theory of elasticity. English translation, Sneddon, I.N. (ed.). North Corolina State University, Raleigh.

160. VERMA, S.K. and JAIN, D.L. (1990) :

Diffraction of obliquely incident P-waves by two parallel and coplanar Griffith cracks. J. Appl. Phys. Vol. 67, p. 6787.

161. WHITTAKER, E.T. and WATSON, G.N. (1969) :

A course of modern analysis. Cambridge University Press, London.

162. WICKHAM, G.R. (1977) :

The forced two dimensional oscillations of a rigid strip in smooth contact with a semi-infinite elastic solid. Math Proc. Camb. Phil. Soc. Vol. 81, p. 291.

163. WILLIS, J.R. (1972) :

Crack propagation in a viscoelastic media. J. Mech. Phys. Solids, Vol. 15, p. 229.

164. WILLIAMS, W.E. (1963) :

Integral equation formulation of some three part boundary value problems. Proc. Edin. Math. Soc. Vol. 13, p. 317.

165. YOFFE, E.H. (1951) :

The moving Griffith crack. Philasophical Magazine. Vol. 42, p. 739.

166. ZENER, C. (1948) :

The Elasticity and Anelasticity of Metals. Chicago University Press.

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FORCED VERTICAL VIBRATION OF TWO RIGID STRIPS ON A SEMI-INFINITE ELASTIC SOLID

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The problem of two-dimensional oscillations of a pair of parallel rigid strips, situated on a homogeneous isotropic semi-infinite elastic solid and forced by a specified normal component of the displacement, is considered. The resulting mixed boundary value problem is solved by the application of an integral equation method. The normal stress just below the strips and the normal displacement away from the strips are derived. By using a similar procedure, the antiplane problem due to the motion of two strips on a semi-infinite elastic medium has also been solved. Finally, graphs are presented which illustrate the salient features of the displacement and stresses in both the cases.

1. INTRODUCTION

The study of the effect of a vibrating source of pressure in different forms on the surface of an elastic medium is almost classical. Reissner [1], and later Millar and Pursey [2], treated the case of a uniform vibrating pressure distribution applied to a circular region on the surface of an elastic half-space. The problem of most physical interest occurs when a displacement corresponding to indentation by a rigid body is prescribed over a given region and the surface tractions outside the region are zero. Analytical treatment of the dynamical response of footings and soil–structure interaction are usually available in the literature only for circular and elliptical footings and infinite strip loadings. Such results are important in view of their application in the design of foundations for machinery and buildings, and also in the study of the vibration of dams and large structures subjected to earthquakes. Awojobi and Grootenhuis [3], Robertson [4], Gladwell [5] and others have considered the problem of a circular footing in detail. Roy [6] considered the dynamic response of an elliptical footing in frictionless contact with a homogeneous elastic half-space. For low frequencies, both vertical and horizontal vibration were treated. A low frequency solution for the vertical, horizontal and rocking vibration of an infinite strip on a semi-infinite elastic medium has been obtained by Karasudhi, Keer and Lee [7] by reducing the governing dual integral equations into a single inhomogeneous Fredholm equation of the second kind. Wickham [8], however, removed the flaws occurring in the above paper and worked out in detail the problem of forced two-dimensional oscillation of a rigid strip in smooth contact with a semi-infinite elastic medium using a different technique.

To improve the dynamic models of buildings and other structures, it will be fruitful to have analytical results for foundations of more complicated nature. In what follows here the problem of vertical vibration of two rigid strips in smooth contact with a semi-infinite elastic medium is considered. The problem is also important in view of its application in the study of the vibration of an elastic medium caused by running wheels on a railway track. The resulting mixed boundary value problem is reduced to the solution of a triple

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integral equation, which is further reduced to the solution of an integro-differential equation. Finally, an iterative solution valid for low frequency is obtained. The integral equation was solved in a manner similar to that employed by Lowengrub and Srivastav [9] in solving static problems for two coplanar cracks in an infinite elastic medium. Jain and Kanwal [10, 11] also used the same technique to solve the problem of diffraction of elastic waves by two coplanar Griffith cracks and also by two coplanar rigid strips in an infinite elastic medium. In this connection, recently Itou [12] has also solved the problem of diffraction of SH-waves by two coplanar Griffith cracks in an infinite elastic medium using a different technique.

From the solution of the integral equation, the stresses just below the strips and also the vertical displacement at points outside the strip on the free surface are found. Finally, in the limit as the distance between the strips tends to zero, the results are found to become identical with the results given by Wickham [8] for the vertical vibration of a single strip on a semi-infinite elastic medium. A low frequency solution due to anti-plane motion of two strips on a semi-infinite elastic medium is also derived.

2. FORMULATION OF THE IN-PLANE PROBLEM

Consider the normal vibration of frequency ω of two rigid strips having smooth contact with a semi-infinite homogeneous isotropic elastic solid occupying the half-space $-\infty<X<\infty$, $Y\geqslant 0$, $-\infty<Z<\infty$ (see Figure 1). It is assumed that the motion is forced by prescribed displacement distribution $v_0\,e^{-i\omega t}$ normal to the two infinite strips located in the region $-a\leqslant X\leqslant -b$, $b\leqslant X\leqslant a$, $Y=0$, $|Z|<\infty$, where v_0 is constant. Normalizing all lengths with respect to a and putting $b/a=c$, one finds that the rigid strips are defined by $c\leqslant|x|\leqslant 1$, $y=0$, $|z|<\infty$.

With the time factor $e^{-i\omega t}$ suppressed throughout the analysis, the displacement components can be written as

$$u(x,y)=\partial\phi/\partial x-\partial\psi/\partial y,\qquad v(x,y)=\partial\phi/\partial y+\partial\psi/\partial x,\qquad w(x,y)=0, \tag{2.1}$$

where the displacement potentials $\phi(x,y)$ and $\psi(x,y)$ satisfy the Helmholtz equations

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+k_1^2\phi=0,\qquad \partial^2\psi/\partial x^2+\partial^2\psi/\partial y^2+k_2^2\psi=0, \tag{2.2}$$

in which $k_1^2=\omega^2a^2/c_1^2$ and $k_2^2=\omega^2a^2/c_2^2$. Consequently, the values of the stress components τ_{xy}, τ_{yy} and τ_{zy} are

$$\tau_{xy}=\mu[2\,\partial^2\phi/\partial x\,\partial y+\partial^2\psi/\partial x^2-\partial^2\psi/\partial y^2],$$

$$\tau_{yy}=-\mu[(k_2^2+2\,\partial^2/\partial x^2)\phi-2\,\partial^2\psi/\partial x\,\partial y],\qquad \tau_{zy}=0. \tag{2.3}$$

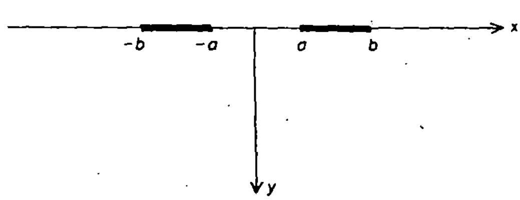

Figure 1. Geometry of the strips.

The boundary conditions are

$$v(x, 0) = v_0, \qquad c \leqslant |x| \leqslant 1,$$

$$\tau_{yy}(x, 0) = 0, \quad |x| < c, \ |x| > 1, \qquad \tau_{xy}(x, 0) = 0, \qquad -\infty < x < \infty. \tag{2.4}$$

The solution of the Helmholtz equation (2.2) can be written as

$$\phi = \int_{-\infty}^{\infty} A(\xi) \exp(\mathrm{i}\xi x - \gamma_1 y) \, \mathrm{d}\xi, \qquad \psi = \int_{-\infty}^{\infty} B(\xi) \exp(\mathrm{i}\xi x - \gamma_2 y) \, \mathrm{d}\xi, \tag{2.5}$$

where

$$\gamma_j = \begin{cases} (\xi^2 - k_j^2)^{1/2}, & |\xi| \geqslant k_j \\ -\mathrm{i}(k_j^2 - \xi^2)^{1/2}, & |\xi| \leqslant k_j \end{cases}, \qquad j = 1, 2, \tag{2.6}$$

and $A(\xi)$ and $B(\xi)$ are unknown functions, to be determined from the boundary conditions.

By using the last of the boundary conditions (2.4) it can be shown that

$$B(\xi) = -\{2\mathrm{i}\xi\gamma_1/(\xi^2 + \gamma_2^2)\} A(\xi).$$

Then the displacements and stresses given by expressions (2.1) and (2.3) become

$$u(x, y) = \int_{-\infty}^{\infty} \mathrm{i}\xi \left[\exp(-\gamma_1 y) - \frac{2\gamma_1\gamma_2}{\xi^2 + \gamma_2^2} \exp(-\gamma_2 y) \right] A(\xi) \exp(\mathrm{i}\xi x) \, \mathrm{d}\xi, \tag{2.7}$$

$$v(x, y) = \int_{-\infty}^{\infty} -\gamma_1 \left[\exp(-\gamma_1 y) - \frac{2\xi^2}{\xi^2 + \gamma_2^2} \exp(-\gamma_2 y) \right] A(\xi) \exp(\mathrm{i}\xi x) \, \mathrm{d}\xi, \tag{2.8}$$

$$\tau_{yy}(x, y) = -\mu \int_{-\infty}^{\infty} \left[(k_2^2 - 2\xi^2) \exp(-\gamma_1 y) + \frac{4\xi^2\gamma_1\gamma_2}{\xi^2 + \gamma_2^2} \exp(-\gamma_2 y) \right] \times A(\xi) \exp(\mathrm{i}\xi x) \, \mathrm{d}\xi, \tag{2.9}$$

$$\tau_{xy}(x, y) = \mu \int_{-\infty}^{\infty} 2\mathrm{i}\xi\gamma_1 [-\exp(-\gamma_1 y) + \exp(-\gamma_2 y)] A(\xi) \exp(\mathrm{i}\xi x) \, \mathrm{d}\xi. \tag{2.10}$$

Next, upon using the fact that $A(\xi)$ is an even function of ξ, and putting

$$P(\xi) = \{[(2\xi^2 - k_2^2)^2 - 4\xi^2\gamma_1\gamma_2]/[2\xi^2 - k_2^2]\} A(\xi),$$

the first and second of the boundary conditions (2.4) lead to the following dual integral equations in $P(\xi)$:

$$\int_0^{\infty} P(\xi) \cos \xi x \, \mathrm{d}\xi = 0, \qquad |x| < c, \qquad |x| > 1, \tag{2.11}$$

$$\int_0^{\infty} \frac{\gamma_1 k_2^2}{(2\xi^2 - k_2^2)^2 - 4\xi^2\gamma_1\gamma_2} P(\xi) \cos \xi x \, \mathrm{d}\xi = \tfrac{1}{2} v_0, \qquad c \leqslant |x| \leqslant 1. \tag{2.12}$$

3. SOLUTION OF THE IN-PLANE PROBLEM

Consider the solution of the integral equations (2.11) and (2.12) in the form

$$P(\xi)=\int_c^1 x_1 f(x_1^2)\cos\xi x_1\,\mathrm{d}x_1, \tag{3.1}$$

where $f(x_1^2)$ is an unknown function to be determined. The relation (2.11) is therefore satisfied automatically and equation (2.12) becomes

$$\int_c^1 x_1 f(x_1^2)\int_0^\infty \frac{\gamma_1 k_2^2}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}\cos\xi x\cos\xi x_1\,\mathrm{d}\xi\,\mathrm{d}x_1=\tfrac{1}{2}v_0,\qquad c\leqslant|x|\leqslant 1. \tag{3.2}$$

Using the relation

$$\frac{\sin\xi x\sin\xi x_1}{\xi^2}=\int_0^x\int_0^{x_1}\frac{wv\,\mathrm{J}_0(\xi w)\,\mathrm{J}_0(\xi v)\,\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}$$

converts equation (3.2) to the form

$$\frac{\mathrm{d}}{\mathrm{d}x}x_1 f(x_1^2)\frac{\partial}{\partial x_1}\int_0^x\int_0^{x_1}\frac{wvL_1(v,w)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}x_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}}=\tfrac{1}{2}v_0,\qquad c\leqslant|x|\leqslant 1, \tag{3.3}$$

where

$$L_1(v,w)=\int_0^\infty\frac{\gamma_1k_2^2}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}\mathrm{J}_0(\xi w)\,\mathrm{J}_0(\xi v)\,\mathrm{d}\xi. \tag{3.4}$$

By a simple contour integration technique [13], $L_1(v,w)$ can be written as

$$\begin{aligned}L_1(v,w)=&-\mathrm{i}\tau^2\int_0^1\frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\,\mathrm{H}_0^{(1)}(k_1\eta w)\,\mathrm{J}_0(k_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\mathrm{d}\eta\\&-4\mathrm{i}\tau^2\int_0^\tau\frac{\eta^2(\eta^2-1)(\tau^2-\eta^2)^{1/2}\,\mathrm{H}_0^{(1)}(k_1\eta w)\,\mathrm{J}_0(k_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\mathrm{d}\eta\\&+\pi\mathrm{i}\tau^2\left[\frac{(\eta^2-1)^{1/2}\,\mathrm{H}_0^{(1)}(k_1\eta w)\,\mathrm{J}_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0},\qquad w>v\\=&\frac{-\mathrm{i}\tau^2}{16(1-\tau^2)}\left[\sum_{j=0}^2 p_j\int_0^1\frac{(1-\eta^2)^{1/2}\,\mathrm{H}_0^{(1)}(k_1\eta w)\,\mathrm{J}_0(k_1\eta v)}{\eta^2-\tau_j^2}\mathrm{d}\eta\right.\\&\left.+\sum_{j=0}^2 s_j\int_0^\tau\frac{(\tau^2-\eta^2)^{1/2}\,\mathrm{H}_0^{(1)}(k_1\eta w)\,\mathrm{J}_0(k_1\eta v)}{\eta^2-\tau_j^2}\mathrm{d}\eta\right]\\&-\pi\mathrm{i}\tau^2\left[\frac{(\eta^2-1)^{1/2}\,\mathrm{H}_0^{(1)}(k_1\eta w)\,\mathrm{J}_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0},\qquad w>v,\end{aligned} \tag{3.5}$$

where $\tau=k_2/k_1=c_1/c_2$, $Q_0(\eta)=(2\eta^2-\tau^2)^2-4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}$ and τ_0 is the root of the Rayleigh wave equation $Q_0(\eta)=0$. τ_1, τ_2 are the roots of the equation

$(2\eta^2-\tau^2)^2+4\eta^2(\eta^2-1)^{1/2}(\eta^2-\tau^2)^{1/2}=0$. $Q_0'(\eta)$ denotes the derivative of $Q_0(\eta)$ with respect to η and

$$p_j=(2\tau_j^2-\tau^2)\Big/\prod_i(\tau_j^2-\tau_i^2),\qquad s_j=4\tau_j^2(\tau_j^2-1)\Big/\prod_i(\tau_j^2-\tau_i^2),\qquad i,j=0,1,2 \text{ and } i\neq j.$$

The corresponding expression for $L_1(v, w)$ for $w<v$ follows from equation (3.5) by interchanging w and v. For a Poisson ratio $\sigma=1/4$, the values of τ, τ_0, τ_1 and τ_2 are given by

$$\tau^2=\frac{2(1-\sigma)}{(1-2\sigma)}=3,\qquad \tau_0^2=\frac{3}{(0{\cdot}9194)^2},\qquad \tau_1^2=\frac{3}{(2+2/\sqrt{3})}\quad\text{and}\quad \tau_2^2=3/4.$$

Hence in this case $\tau_2<\tau_1<1<\tau<\tau_0$.

By using the series expansions of J_0 and $H_0^{(1)}$ and evaluating the integrals arising in equation (3.5), one finds, after some algebraic manipulation,

$$L_1(v,w)=\begin{cases}(2/\pi)\tau^2[\{\gamma+\log(k_1w/2)-(\pi i/2)\}M+N-(P/4)(w^2+v^2)k_1^2\log k_1]+o(k_1^2), & w>v\\ (2/\pi)\tau^2[\{\gamma+\log(k_1v/2)-(\pi i/2)\}M+N-(P/4)(w^2+v^2)k_1^2\log k_1]+o(k_1^2), & w<v\end{cases},\tag{3.6}$$

where $\gamma=0{\cdot}5772157\ldots$ is Euler's constant,

$$M=-\pi/4(1-\tau^2),\tag{3.7}$$

$$N=\frac{\pi}{32(1-\tau^2)}\left[4\log\frac{4}{\tau}+\sum_{j=1}^{2}p_j\frac{\sqrt{(1-\tau_j^2)}}{\tau_j}\tan^{-1}\frac{\sqrt{(1-\tau_j^2)}}{\tau_j}-p_0\frac{\sqrt{(\tau_0^2-1)}}{\tau_0}\log\{\tau_0+\sqrt{(\tau_0^2-1)}\}\right.$$
$$\left.+\sum_{j=1}^{2}s_j\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j^2}\tan^{-1}\frac{\sqrt{(\tau^2-\tau_j^2)}}{\tau_j^2}-s_0\frac{\sqrt{(\tau_0^2-\tau^2)}}{\tau_0}\log\left\{\frac{\tau_0+\sqrt{(\tau_0^2-\tau^2)}}{\tau}\right\}\right],\tag{3.8}$$

$$P=\frac{\pi}{32(1-\tau^2)}\left[\sum_{j=0}^{2}p_j(\tfrac{1}{2}-\tau_j^2)+\sum_{j=0}^{2}s_j\left(\frac{\tau^2}{2}-\tau_j^2\right)\right].\tag{3.9}$$

Next, differentiating both sides of the relation (3.2) with respect to x, we obtain

$$\int_c^1 x_1 f(x_1^2)\int_0^\infty\frac{\gamma_1k_2^2}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}\,\xi\sin\xi x\cos\xi x_1\,d\xi\,dx_1=0,\qquad c\leqslant|x|\leqslant 1.$$

Following a procedure similar to that for deriving equation (3.3), one obtains

$$x\int_c^1\frac{x_1f(x_1^2)}{x^2-x_1^2}\,dx_1=\int_c^1 x_1f(x_1^2)\frac{\partial}{\partial x_1}\int_0^x\int_0^{x_1}\frac{wvL_2(v,w)\,dw\,dv\,dx_1}{(x^2-w^2)^{1/2}(x_1^2-v^2)^{1/2}},\ c\leqslant|x|\leqslant 1,\tag{3.10}$$

where

$$L_2(v,w)=\int_0^\infty\left[\xi-\frac{2\gamma_1\xi^2(k_1^2-k_2^2)}{(2\xi^2-k_2^2)^2-4\xi^2\gamma_1\gamma_2}\right]J_0(\xi w)\,J_0(\xi v)\,d\xi.\tag{3.11}$$

For small values of k_1 and k_2 such that $k_1 = o(k_2)$, one can use the contour integration technique mentioned above and obtain

$$L_2(v, w) = 2ik_1^2(1-\tau^2)\int_0^1 \frac{(1-\eta^2)^{1/2}(2\eta^2-\tau^2)^2\eta^2\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta$$
$$+4ik_1^2(1-\tau^2)\int_0^\tau \frac{2\eta^4(\eta^2-1)(\tau^2-\eta^2)^{1/2}\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{(2\eta^2-\tau^2)^4+16\eta^4(\eta^2-1)(\tau^2-\eta^2)}\,d\eta$$
$$-2\pi ik_1^2(1-\tau^2)\left[\frac{\eta^2(\eta^2-1)^{1/2}\,H_0^{(1)}(k_1\eta w)\,J_0(k_1\eta v)}{Q_0'(\eta)}\right]_{\eta=\tau_0}, \qquad w>v. \tag{3.12}$$

By a process similar to the one which led to equation (3.6), equation (3.12) can be written as

$$L_2(v, w) = -(4P/\pi)(1-\tau^2)k_1^2\log k_1 + o(k_1^2), \tag{3.13}$$

where P is given by equation (3.9).

Now consider

$$f(x_1^2) = f_0(x_1^2) + k_1^2\log k_1 f_1(x_1^2) + o(k_1^2). \tag{3.14}$$

Putting this expansion of $f(x_1^2)$ and the value of $L_2(v, w)$ given by equation (3.13) in equation (3.10) and equating the coefficients of equal powers of k_1 yields

$$\int_c^1 \frac{x_1 f_0(x_1^2)}{x^2-x_1^2}\,dx_1 = 0, \quad c\leqslant|x|\leqslant 1, \tag{3.15}$$

and

$$\int_c^1 \frac{x_1 f_1(x_1^2)}{x^2-x_1^2}\,dx_1 = -\frac{4P}{\pi}(1-\tau^2)\int_c^1 x_1 f_0(x_1^2)\,dx_1, \qquad c\leqslant|x|\leqslant 1. \tag{3.16}$$

Following Srivastava and Lowengrub [8] one finds the solutions of the integral equations (3.15) and (3.16) to be

$$f_0(x_1^2) = \frac{D}{(1-x_1^2)^{1/2}(x_1^2-c^2)^{1/2}}, \tag{3.17}$$

and

$$f_1(x_1^2) = \frac{4}{\pi}PD(1-\tau^2)\left[\frac{x_1^2-c^2}{1-x_1^2}\right]^{1/2} + \frac{B}{(x_1^2-c^2)^{1/2}(1-x_1^2)^{1/2}}, \tag{3.18}$$

where D and B are constants which can be calculated as follows. One substitutes the value of $L_1(v, w)$ from equation (3.6) as well as the expansion of $f(x_1^2)$ obtained from equations (3.14), (3.17) and (3.18) into equation (3.3). When the coefficients of like powers of k_1 on both sides of the resulting equation are equated the following results are obtained:

$$D = \frac{v_0}{2\tau^2[\{\gamma+\log(k_1/2)-(\pi i/2)+\log(1-c^2)^{1/2}\}M+N]}, \tag{3.19}$$

$$B = \frac{2\tau^2D^2P}{v_0}\left[\tfrac{1}{4}(2x^2+c^2+1) - \frac{M}{\pi}(1-\tau^2)(1-2x^2+c^2) - \frac{(1-c^2)(1-\tau^2)v_0}{\pi\tau^2 D}\right]. \tag{3.20}$$

One can now obtain the values of the vertical displacement in the plane $y=0$ from equations (2.8) and (3.1) as

$$v(x,0)=\begin{cases} v_0+2M\tau^2\left[D+k_1^2\log k_1\left\{B+\frac{2}{\pi}(1-\tau^2)(1-c^2)PD\right\}\right]\sinh^{-1}\left[\frac{c^2-x^2}{1-c^2}\right]^{1/2} \\ \quad -\frac{4\tau^2MPD(1-\tau^2)}{\pi}k_1^2\log k_1\{(1-x^2)(c^2-x^2)\}^{1/2}+o(k_1^2), \qquad |x|<c \\ v_0, \qquad c\leqslant|x|\leqslant 1 \\ v_0+2M^2\left[D+k_1^2\log k_1\left\{B+\frac{2}{\pi}(1-\tau^2)(1-c^2)PD\right\}\right]\sinh^{-1}\left[\frac{x^2-1}{1-c^2}\right]^{1/2} \\ \quad +\frac{4\tau^2MPD(1-\tau^2)}{\pi}k_1^2\log k_1\{(x^2-1)(x^2-c^2)\}^{1/2}+o(k_1^2), \qquad |x|>1 \end{cases} \tag{3.21}$$

The normal stress $\tau_{yy}(x, y)$ in the plane $y=0$ just below the strips can be found from the relation (2.9) as

$$\tau_{yy}(x,0)=\frac{\pi\mu|x|}{(1-x^2)^{1/2}(x^2-c^2)^{1/2}}(D+Bk_1^2\log k_1)$$
$$+4\mu xDP(1-\tau^2)\left[\frac{x^2-c^2}{1-x^2}\right]^{1/2}k_1^2\log k_1+o(k_1^2), \qquad c\leqslant|x|\leqslant 1. \tag{3.22}$$

Now putting $c=0$ in (3.20) one can obtain the normal stress for a single strip, $|x|\leqslant 1$, $y=0$, $-\infty<z<\infty$ as

$$\tau_{yy}(x,0)=\frac{\pi\mu D}{(1-x^2)^{1/2}}+\frac{\mu}{(1-x^2)^{1/2}}k_1^2\log k_1[4P(1-\tau^2)Dx^2+\pi B]+o(k_1^2),$$

where

$$D=\frac{v_0}{2\tau^2[(\gamma+\log(k_1/2)-(\pi i/2))M+N]},$$

$$B=\frac{2\tau^2D^2P}{v_0}\left[\tfrac{1}{4}(2x^2+1)-\frac{M}{\pi}(1-\tau^2)(1-2x^2)-\frac{(1-\tau^2)v_0}{\pi\tau^2D}\right].$$

Upon defining $\Delta_0=v_0/\pi^2D$, $\beta_0=-\tau^2/2\pi(1-\tau^2)$ and $\beta_2=-P/\pi^2$, as done by Wickham [8], one has

$$\tau_{yy}(x,0)=\frac{\mu v_0}{\pi\Delta_0(1-x^2)^{1/2}}\left\{1-\beta_2k_2^2\log k_2\left[\frac{1}{\Delta_0}+\frac{(1-2x^2)}{\beta_0}\right]\right\}+o(k_2^2),$$

which coincides with the result obtained by Wickham [8].

4. FORMULATION AND SOLUTION OF THE ANTI-PLANE PROBLEM

For an SH-wave the displacement and stress are $w(x, y, t) = w(x, y)\,e^{-i\omega t}$ and $\tau_{yz}(x, y) = \mu\,\partial w/\partial y$. As in the previous case one can write these expressions as

$$w(x, y) = \int_{-\infty}^{\infty} \frac{Q(\xi)}{\gamma_2} \exp(i\xi x - \gamma_2 y)\, d\xi, \tag{4.1}$$

$$\tau_{yz}(x, y) = -\mu \int_{-\infty}^{\infty} Q(\xi) \exp(i\xi x - \gamma_2 y)\, d\xi, \tag{4.2}$$

where $Q(\xi)$ is an unknown function to be determined from the boundary conditions, which are

$$w(x, 0) = w_0, \quad c \leqslant |x| \leqslant 1, \qquad \tau_{yz}(x, 0) = 0, \quad |x| < c,\ |x| > 1, \tag{4.3, 4.4}$$

where w_0 is a constant. By using a procedure similar to that followed for the solution of the in-plane problem, the values of stress $\tau_{yz}(x, y)$ and displacement $w(x, y)$ in the plane $y = 0$ can be found to be given by

$$\begin{aligned}\tau_{yz}(x, 0) = {} & \frac{-\pi\mu|x|}{(1-x^2)^{1/2}(x^2-c^2)^{1/2}}(D_1 + B_1 k_2^2 \log k_2) \\ & + \frac{\pi\mu|x|D_1}{2} k_2^2 \log k_2 \left[\frac{x^2-c^2}{1-x^2}\right]^{1/2} + o(k_2^2), \qquad c \leqslant |x| \leqslant 1,\end{aligned} \tag{4.5}$$

$$w(x, 0) = \begin{cases} w_0 - \pi[D_1 + k_2^2 \log k_2\{B_1 - D_1(1-c^2)/4\}]\sinh^{-1}\left[\dfrac{c^2-x^2}{1-c^2}\right]^{1/2} \\ \quad - (\pi D_1/4) k_2^2 \log k_2 [(1-x^2)(c^2-x^2)]^{1/2} + o(k_2^2), & |x| < c \\ w_0, \quad c \leqslant |x| \leqslant 1 \\ w_0 - \pi[D_1 + k_2^2 \log k_2\{B_1 - D_1(1-c^2)/4\}]\sinh^{-1}\left[\dfrac{x^2-1}{1-c^2}\right]^{1/2} \\ \quad + (\pi D_1/4) k_2^2 \log k_2 [(x^2-1)(x^2-c^2)]^{1/2} + o(k_2^2), & |x| > 1 \end{cases} \tag{4.6}$$

where

$$D_1 = \frac{w_0}{\pi[(\pi i/2) - \gamma - \log(k_2/4) - \log(1-c^2)^{1/2}]}, \tag{4.7}$$

$$B_1 = \frac{\pi D_1^2}{4w_0}\left[\frac{w_0(1-c^2)}{\pi D_1} - (1+c^2)\right]. \tag{4.8}$$

5. NUMERICAL RESULTS

The vertical and the transverse displacement fields for the in-plane and the anti-plane problems, respectively, for points near the rigid strips are illustrated graphically in Figures 2 and 3 for a Poisson solid ($\tau^2 = 3$). It is interesting to note from the graphs that the real parts of the displacements decrease with an increase in the value of k_2 in both cases.

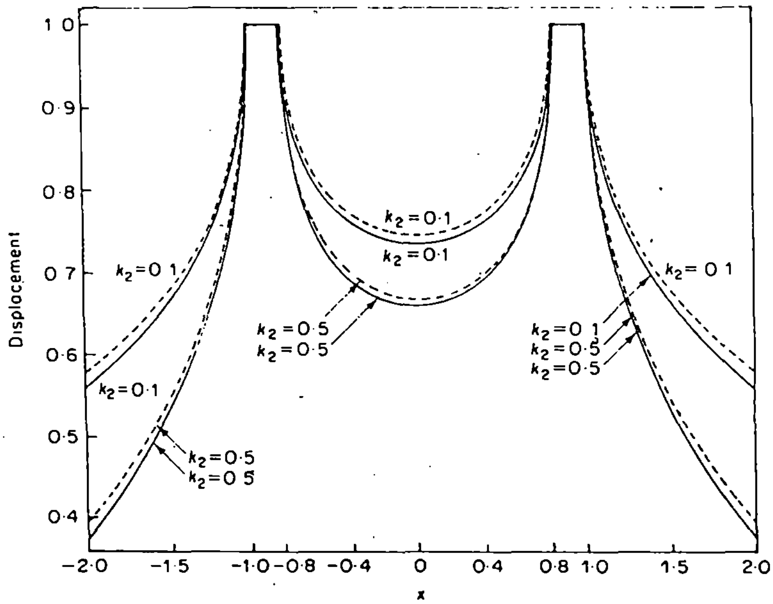

Figure 2. Displacement *vs.* distance. ——, Re $\{v(x,0)\}$ for in-plane problem; – – – –, Re $\{w(x,0)\}$ for anti-plane problem. $c=0\cdot8$.

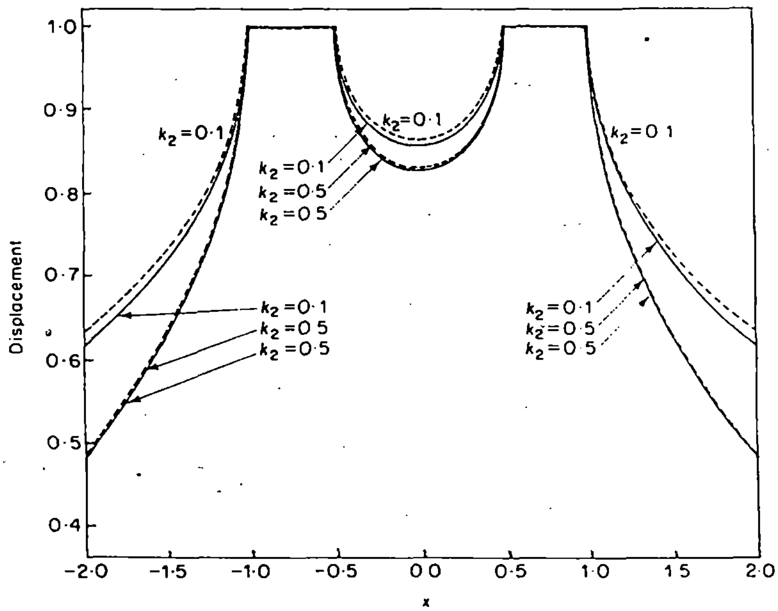

Figure 3. Displacement *vs.* distance. ——, Re $\{v(x,0)\}$ for in-plane problem; – – –, Re $\{w(x,0)\}$ for anti-plane problem. $c=0\cdot5$.

Graphs of the stress factors

$$\tau_1^* = \mathrm{Re}\left[\frac{\tau_{yy}\{(1-x^2)(x^2-c^2)\}^{1/2}}{\mu v_0}\right] \quad \text{and} \quad \tau_2^* = \mathrm{Re}\left[\frac{\tau_{yz}\{(1-x^2)(x^2-c^2)\}^{1/2}}{\mu w_0}\right]$$

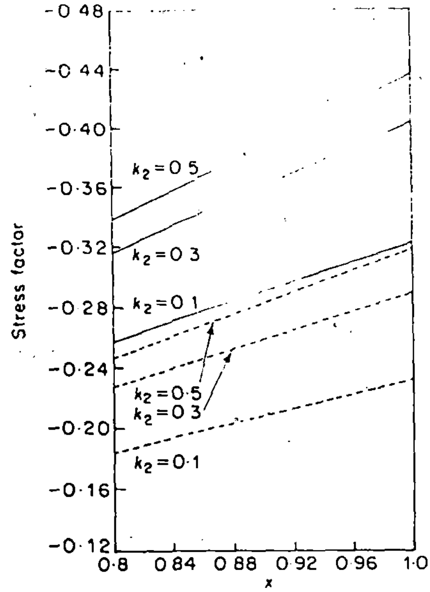

Figure 4. Stress factor *vs.* displacement. ——, τ_1^* for in-plane problem; – – –, τ_2^* for anti-plane problem. $c=0·8$.

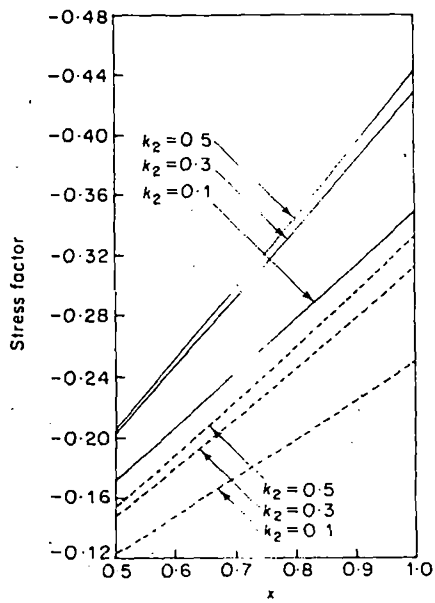

Figure 5. Stress factor *vs.* displacement. ——, τ_1^* for in-plane problem; – – –, τ_2^* for anti-plane problem. $c=0·5$.

vs. dimensionless distance x for the in-plane and the anti-plane problems, respectively, are shown in Figures 4 and 5, plotted for points just below the rigid strips. In both the cases the magnitude of the stress factor is found to increase as one proceeds from the inner to the outer edge of the strips.

ACKNOWLEDGMENT

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REFERENCES

1. E. Reissner 1937 *Ingenieur Archiv* **8**, 229–445. Freie und erzwungene Torsionsschwingungen den elastischen Halbraumes.
2. G. F. Miller and H. Pursey 1954 *Proceedings of the Royal Society, London* **A223**, 521–541. The field and radiation impedence of mechanical radiators on the surface of semi-infinite isotropic solid.
3. O. A. Awojobi and P. Grootenhuis 1965 *Proceedings of the Royal Society, London* **A287**, 27–63. Vibration of rigid bodies on semi-infinite elastic media.
4. I. A. Robertson 1966 *Proceedings of the Cambridge Philosophical Society* **62**, 547–553. Forced vertical vibration of a rigid circular disc on a semi-infinite elastic solid.
5. G. M. L. Gladwell 1968 *International Journal of Engineering Science* **6**, 591. Forced tangential and rotatory vibration of a rigid circular disc on a semi-infinite solid.
6. A. Ray 1986 *International Journal of Solids and Structures* **22**, 293–309. Dynamic response of elliptical footing.
7. P. Karasudhi, L. M. Keer and S. L. Lee 1968 *Journal of Applied Mechanics* **4**, 697–705. Vibratory motion of a body on an elastic half plane.
8. G. R. Wickham 1977 *Proceedings of the Cambridge Philosophical Society* **81**, 291–311. The forced two dimensional oscillations of a rigid strip in a smooth contact with a semi-infinite elastic solid.
9. M. Lowengrub and K. N. Srivastava 1968 *International Journal of Engineering Science* **6**, 359. On two coplanar cracks in an infinite elastic medium.
10. D. L. Jain and R. P. Kanwal 1972 *International Journal of Solids and Structures* **8**, 961–975. Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium.
11. D. L. Jain and R. P. Kanwal 1972 *International Journal of Engineering Science* **10**, 961–975. Diffraction of elastic waves by two coplanar and parallel rigid strips.
12. S. Itou 1980 *International Journal of Solids and Structures* **16**, 1147–1153. Diffraction of an antiplane shear wave by two coplanar Griffith cracks in an infinite elastic medium.
13. M. Ghosh and M. L. Ghosh 1985 *Indian Journal of Pure and Applied Mathematics* **16**(8), 938–955. Harmonic rocking of a rigid strip on a semi-infinite elastic medium.

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MOVING PUNCH ON A VISCOELASTIC SEMI-INFINITE MEDIUM

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We study the problem of a semi-infinite punch moving on the free surface of a semi-infinite viscoelastic medium and producing horizontal shear waves. The mixed boundary value problem has been solved by the use of integral transforms and the Wiener-Hopf technique for both the steady and non-steady cases. Two types of viscoelastic models, viz., the Maxwell solid and the standard linear solid have been considered. Solutions have been derived in close form in both the cases and graphs have been presented to bring out the salient features of the problem.

1. Introduction

Problems involving the motion of a punch on the surface of an elastic half-space or on the free boundaries of long strips are extremely important in view of their application in road construction technology and also in geophysical research. Punch problems within the classical theory of elasticity have been studied extensively by Galin[1] and by Gladwell[2] in their books. The motion of a rough punch on an elastic half-space has been treated in detail by Suhubi[3]. Recently problems involving antiplane motion due to punches moving along the surfaces of an elastic strip have been solved by complex variable methods by Tait and Moodie[4]. An analytical solution to the problem of a long rigid punch moving rapidly on a strip of a highly orthotropic elastic layer has been solved by Georgiadis[5] using integral transforms and the Wiener-Hopf techniques[6].

However, natural or artificial materials have generally disipative behaviour which often can be taken into account by viscoelastic models. Accordingly, problems involving the motion of a punch on a viscoelastic medium have drawn the attention of many scientists. The problem of a rigid cylinder rolling on the surface of a viscoelastic half space has been solved by Hunter[7]. The contact problem of rigid cylinder rolling slowly on a thin viscoelastic layer has been treated by Alblas and Kuipers[8] assuming that the layer thickness is small compared to the width of the contact region of the cylinder. The problem of a plane punch sliding without friction on a viscoelastic half space has been considered by Golden[9].

In the present paper, we have examined the stress and displacement field produced by a long punch moving on the boundary of a semi-infinte viscoelastic medium and producing Horizontal Shear waves. Two types of viscoelastic models viz. Maxwell solid and Standard Linear Solid have been considered and loading is assumed to be such that Mode III conditions prevail. The mathematical technique which is employed here consists of the application of integral transforms and the solution of the resulting Wiener-Hopf equations for the transformed unknown variables. Both the steady and nonsteady solutions of the problem have been derived. Displacement and stress on the free surface and at points below the punch have been derived analytically and the nature of their variations with the velocity of the moving punch has been shown by means of graphs.

2. Formulation of the Problem and Its Solution for Steady State Motion

Let us consider a semi-infinite viscoelastic medium which was set into motion by a semi-infinite rigid punch moving with a constant velocity v in the direction of the x-axis. The y-axis is taken vertically downwards into the medium (Fig. 1).

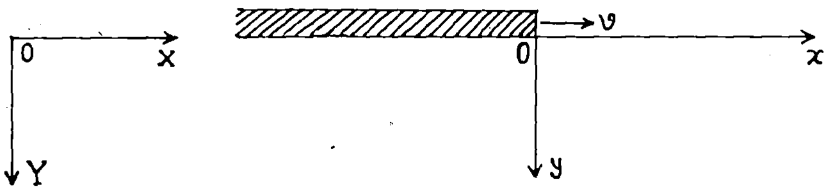

Fig. 1

For horizontal shear waves, the displacements along X and Y directions are zero and only the displacement $W = W(X, Y, t)$ along z-direction exists. The stresses under the punch are

$$\sigma_{13} = \sigma_{13}(X, Y, t) \text{ and } \sigma_{23} = \sigma_{23}(X, Y, t). \qquad \ldots(1)$$

The non-vanishing strains are

$$e_{13} = \frac{1}{2}\frac{\partial W}{\partial X} \text{ and } e_{23} = \frac{1}{2}\frac{\partial W}{\partial Y}. \qquad \ldots(2)$$

Considering a 'standard linear solid' as the viscoelastic model, the stress strain relations are

$$\frac{\partial \sigma_{i3}}{\partial t} + \beta\sigma_{i3} = 2\mu\left(\frac{\partial e_{i3}}{\partial t} + \alpha\, e_{i3}\right),\ i = 1, 2 \qquad \ldots(3)$$

where α, β are positive constants and μ is the instantaneous elastic modulus of rigidity of the material.

The equation of motion is

$$\frac{\partial \sigma_{13}}{\partial X} + \frac{\partial \sigma_{23}}{\partial Y} = \rho\,\frac{\partial^2 W}{\partial t^2} \qquad \ldots(4)$$

where ρ is the density of the material.

The boundary conditions of the problem are

$$\left.\begin{aligned} W(X, 0, t) &= w_0, \quad X - vt < 0 \\ W(X, \infty, t) &= 0, \quad -\infty < X < \infty \\ \sigma_{23}(X, 0, t) &= 0, \quad X - vt > 0 \end{aligned}\right\} \quad \ldots(5)$$

Since we are going to investigate the steady state propagation of a punch, it is convenient to define a moving co-ordinate system (x, y) whose origin coincides with the tip of the punch and whose axes are parallel to the fixed (X, Y) axes, respectively (Fig. 1)

Hence putting $x = X - vt$, $y = Y$ eqns. (1) to (4) become respectively

$$\sigma_{13} = \sigma_{13}(x, y), \quad \sigma_{23} = \sigma_{23}(x, y) \quad \ldots(6)$$

$$e_{13} = \frac{1}{2}\frac{\partial W}{\partial x}(x, y), \quad e_{23} = \frac{1}{2}\frac{\partial W}{\partial y}(x, y) \quad \ldots(7)$$

$$\left.\begin{aligned} -v\frac{\partial \sigma_{13}}{\partial x} + \beta\sigma_{13} &= \mu\left(-v\frac{\partial^2 W}{\partial x^2} + \alpha\frac{\partial W}{\partial x}\right) \\ -v\frac{\partial \sigma_{23}}{\partial x} + \beta\sigma_{23} &= \mu\left(-v\frac{\partial^2 W}{\partial x \partial y} + \alpha\frac{\partial W}{\partial y}\right) \end{aligned}\right\} \quad \ldots(8)$$

and

$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial y} = \rho v^2 \frac{\partial^2 W}{\partial x^2}. \quad \ldots(9)$$

The boundary conditions (5), now become

$$\left.\begin{aligned} W(x, 0) &= w_0, \quad x < 0 \\ W(x, \infty) &= 0, \quad -\infty < x < \infty \\ \sigma_{23}(x, 0) &= 0, \quad x > 0. \end{aligned}\right\} \quad \ldots(10)$$

Now introduce Fourier transform

$$\bar{f}(\xi, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x, y)\, e^{i\xi x}\, dx$$

so that $$f(x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(\xi, y)\, e^{-i\xi x}\, dx.$$

Taking Fourier transform of (8) and (9) we get

$$(i\xi v + \beta)\,\bar{\sigma}_{13} = \mu(\xi^2 v - i\xi\alpha)\,\bar{W} \quad \ldots(12)$$

$$(i\xi v + \beta)\,\bar{\sigma}_{23} = \mu(i\xi v + \alpha)\frac{d\bar{W}}{dy} \quad \ldots(13)$$

and

$$i\xi\bar{\sigma}_{13} + \frac{d\,\bar{\sigma}_{23}}{dy} = -\rho v^2\xi^2\,\bar{W}. \qquad \ldots(14)$$

Eliminating $\bar{\sigma}_{13}$, $\bar{\sigma}_{23}$ from (12), (13) and (14) we obtain,

$$\frac{d^2\bar{W}}{dy^2} - \gamma^2\bar{W} = 0 \qquad \ldots(15)$$

where

$$\gamma^2 = \frac{\xi^2}{\left(\xi - \frac{i\alpha}{v}\right)}\left[\left(1 - \frac{v^2}{c^2}\right)\xi + i\left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right)\right],\ c^2 = \frac{\mu}{\rho}. \qquad \ldots(16)$$

The branches of γ are so chosen that

$$\text{Re}\,(\gamma) > 0 \quad \text{for} \quad -a < \text{Im}\,(\xi) < 0$$

where $$a = \left(\frac{v\beta}{c^2} - \frac{\alpha}{v}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right). \qquad \ldots(17)$$

Now the solution of eqn. (15) bounded as $y \to \infty$ is

$$\bar{W}\,(\xi, y) = B\,(\xi)\,e^{-\gamma y} \qquad \ldots(18)$$

Let us consider

$$W\,(x, 0) = w_0 = W_0\,e^{\epsilon x},$$

$x < 0$, $\epsilon > 0$ and ϵ will be made to tend to zero finally ...(19)

$$= W_0\,p\,(x) \quad \text{(say)}\ ,\ x > 0 \qquad \ldots(19)$$

$$\sigma_{23}\,(x, 0) = 0\ ,\ x > 0$$

$$= W_0\,t\,(x) \quad \text{(say)}\ ,\ x < 0 \qquad \ldots(20)$$

where $p\,(x)$ and $t\,(x)$ are unknown functions such that

$$p\,(x) \sim O\,(e^{-k_1 x}) \quad \text{as} \quad x \to \infty,\ k_1 > 0$$

$$t\,(x) \sim O\,(e^{+k_2 x}) \quad \text{as} \quad x \to -\infty,\ k_2 > 0.$$

Taking Fourier transform of (19)

$$\bar{W}\,(\xi, 0) = \frac{W_0}{\sqrt{2\pi}\,(\epsilon + i\xi)} + \frac{W_0}{\sqrt{2\pi}}\,P_+\,(\xi) \qquad \ldots(21)$$

where

$$P_+(\xi) = \int_0^\infty p\,(x)\,e^{i\xi x}\,dx,\ (\xi = \sigma + i\,\tau). \qquad \ldots(22)$$

In (21) the first term on the right-hand side is analytic in the lower half plane $\text{Im}\,(\xi) = \tau < \epsilon$ and $P_+(\xi)$ is analytic in the upper half plane $\tau > -k_1$ ($k_1 < a$, say). Again taking Fourier transforms of (20)

$$\bar{\sigma}_{23}(\xi, 0) = \frac{W_0}{\sqrt{2\pi}} T_-(\xi) \qquad ...(23)$$

where $$T_-(\xi) = \int_{-\infty}^{0} t(x)\, e^{i\xi x}\, dx. \qquad ...(24)$$

$T_-(\xi)$ is analytic in the lower half plane $\tau < k_2$. Therefore, $\bar{W}(\xi, 0)$ is analytic for $-k_1 < \tau < \epsilon$ and $\bar{\sigma}_{23}(\xi, 0)$ is analytic in the lower half plane $\tau < k_2$.

From (13), $$\left[(i\xi\nu + \beta)\,\bar{\sigma}_{23}\right]_{y=0} = \left[\mu\,(i\xi\nu + \alpha)\,\frac{d\bar{W}}{dy}\right]_{y=0}$$

Using (18), (21) and (23) this becomes

$$T_-(\xi) = -H(\xi)\left[P_+(\xi) - \frac{i}{\xi - i\epsilon}\right] \qquad ...(25)$$

where $$H(\xi) = \frac{\mu\left(\xi - \frac{i\alpha}{\nu}\right)^{1/2}}{\left(\xi - \frac{i\beta}{\nu}\right)}\,\xi.\left[\left(1 - \frac{\nu^2}{c^2}\right)\xi + i\left(\frac{\nu\beta}{c^2} - \frac{\alpha}{\nu}\right)\right]^{1/2}. \qquad ...(26)$$

It may be noted that the problem has been reduced to a form suitable for the application of the Wiener-Hopf technique. Now $H(\xi)$ can be written as

$$H(\xi) = H_+(\xi)\, H_-(\xi) \qquad ...(27)$$

where

$$H_+(\xi) = \mu\left[\left(1 - \frac{\nu^2}{c^2}\right)\xi + i\left(\frac{\nu\beta}{c^2} - \frac{\alpha}{\nu}\right)\right]^{1/2} \qquad ...(28)$$

and

$$H_-(\xi) = \frac{\left(\xi - \frac{i\alpha}{\nu}\right)^{1/2}\xi}{\left(\xi - \frac{i\beta}{\nu}\right)}. \qquad ...(29)$$

$H_+(\xi)$ is analytic in the upper half plane $\tau > -a$ and

$H_-(\xi)$ is analytic in the lower half plane $\tau < 0$.

Applying well known Wiener-Hopf technique we get,

$$T_-(\xi) = i\mu\left[i\,\frac{\nu\beta}{c^2} - \frac{\alpha}{\nu}\right]^{1/2} \frac{\left(\xi - \frac{i\alpha}{\nu}\right)^{1/2}}{\xi - \frac{i\beta}{\nu}} \quad \text{as } \epsilon \to 0.$$

So from (23)

$$\bar{\sigma}_{23}(\xi, 0) = \frac{W_0}{\sqrt{2\pi}} \cdot i\mu \left[i\left(\frac{\beta\nu}{c^2} - \frac{\alpha}{\nu}\right)\right]^{1/2} \cdot \frac{\left(\xi - \frac{i\alpha}{\nu}\right)^{1/2}}{\xi - \frac{i\beta}{\nu}} .$$

Therefore for $x < 0$

$$\sigma_{23}(x, 0) = \frac{iW_0\mu}{\sqrt{2\pi}} \left[i\left(\frac{\beta\nu}{c^2} - \frac{\alpha}{\nu}\right)\right]^{1/2} \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{\nu}\right)^{1/2}}{\left(\xi - \frac{i\beta}{\nu}\right)} e^{-i\xi x}\, d\xi, \quad \text{...(30)}$$

Considering a branch cut along the positive imaginary axis starting from $\xi = i\alpha/\nu$ and changing the path of integration from real ξ-axis to the path around the branch cut it can easily be shown that the integral

$$I = \int_{-\infty}^{\infty} \frac{\left(\xi - \frac{i\alpha}{\nu}\right)}{\left(\xi - \frac{i\beta}{\nu}\right)} e^{-i\xi x}\, d\xi \qquad \text{(assuming } \beta > \alpha)$$

can be converted to the following integral

$$I = 2e^{\frac{\pi i}{4} - \frac{\alpha}{\nu}x_1} \fint_0^{\infty} \frac{u^{1/2} e^{-ux_1}}{u - \left(\frac{\beta}{\nu} - \frac{\alpha}{\nu}\right)} du \quad \text{...(31)}$$

where x has been replaced by $-x_1$, $\fint_0^{\infty}$ denotes the principal value of the integral. For large values of $(\beta - \alpha)\, x_1/\nu = mx_1$, where $m = (\beta - \alpha)/\nu$ the integral (36) can be evaluated in the form

$$I = \frac{-2e^{\frac{\pi i}{4}} e^{-\frac{\alpha}{\nu}x_1}}{x_1^{1/2}} \left[\frac{\Gamma(3/2)}{mx_1} + \frac{\Gamma(5/2)}{m^2x_1^2} + \frac{\Gamma(7/2)}{m^3x_1^3} + \dots\right] \quad \text{...(32)}$$

and for small values of mx_1 it can be shown that

$$I = 2e^{\pi i/4}\, e^{-\alpha x_1/\nu} \sqrt{\frac{\pi}{x_1}}. \quad \text{...(33)}$$

Therefore using (30) and (31) we obtain for $x < 0$,

$$\sigma_{23}(x, 0) = -\frac{W_0\,\mu}{\pi}\left(\frac{\beta\nu}{c^2} - \frac{\alpha}{\nu}\right)^{1/2} e^{-\alpha x_1/\nu} \int_{\infty}^{\infty} \frac{u^{1/2} e^{-ux_1}}{\left|u - \left(\frac{\beta}{\nu} - \frac{\alpha}{\nu}\right)\right|} du, \; x < 0. \quad \text{...(34)}$$

Using the value of the integral arising in (34) by (33), we get for small values of mx_1,

$$\sigma_{23}(x, 0) = \frac{-W_0 \mu}{\sqrt{\pi m x_1}} \left[m \left(\frac{\beta v}{c^2} - \frac{\alpha}{v} \right) \right]^{1/2} e^{-\alpha x_1 / v}, \; x_1 \to 0^+. \qquad \ldots(35)$$

Also with the help of (34) and (32), for large values of mx_1 $(x < 0)$

$$\sigma_{23}(x, 0) = \frac{W_0 \mu e^{-\alpha x_1 / v}}{\pi \sqrt{m x_1}} \left[m \left(\frac{\beta v}{c^2} - \frac{\alpha}{v} \right) \right]^{1/2}$$

$$\left[\frac{\Gamma(3/2)}{m x_1} + \frac{\Gamma(5/2)}{m^2 x_1^2} + \frac{\Gamma(7/2)}{m^3 x_1^3} + \ldots \right]. \qquad \ldots(36)$$

Now from (21) we get as $\epsilon \to 0$

$$\overline{W}(\xi, 0) = \frac{-iW_0}{2\pi} \sqrt{ia} \frac{1}{\xi (\xi + ia)^{1/2}}, \; a = \left(\frac{v\beta}{c^2} - \frac{\alpha}{v} \right) / \left(1 - \frac{v^2}{c^2} \right).$$

Taking inverse Fourier transform

$$W(x, 0) = -\frac{iW_0}{2\pi} \sqrt{ia} \int_{-\infty - id}^{\infty - id} \frac{e^{-i\xi x}}{\xi (\xi + ia)^{1/2}} d\xi, \; x > 0 \; (0 < d < a). \qquad \ldots(37)$$

Transforming the integral in (37) to an integral along the contour around the branch cut from $-ia$ to $-i\infty$, it can be shown that

$$W(x, 0) = \frac{-iW_0}{\pi} \sqrt{ia}\, e^{-ax} e^{\pi i/4} x^{1/2} \int_0^\infty \frac{e^{-U} U^{-1/2}}{U + ax} dU \; (x > 0)$$

which can be written as

$$W(x, 0) = \frac{W_0}{\sqrt{\pi}} e^{-\frac{1}{2} ax} (ax)^{-1/4} W_{-1/4, -1/4}(ax), \; (x > 0) \qquad \ldots(38)$$

where $W_{k,m}$ is the Whittaker function[10].

Therefore, for small values of ax $(x > 0)$ we obtain from (38)

$$W(x, 0) = W_0 e^{-ax} - \frac{2W_0}{\sqrt{\pi}} e^{-ax} (ax)^{1/2}, \; x \to 0^+ \qquad \ldots(39)$$

and for large values of ax $(x > 0)$

$$W(x, 0) = \frac{W_0}{\sqrt{\pi}} \frac{e^{-ax}}{\sqrt{ax}}, \; x \to \infty \; (x > 0). \qquad \ldots(40)$$

3. Steady State Solution for Maxwell Solid

For 'Maxwell solid' the stress-strain relations obtained from (3) putting $\alpha = 0$ are

$$\frac{\partial \sigma_{i3}}{\partial t} + \beta \sigma_{i3} = 2\mu \frac{\partial e_{i3}}{\partial t}, \; i = 1, 2. \qquad \ldots(41)$$

The stress can be found by putting $\alpha = 0$ in (34) as (for $x < 0$, $y = 0$)

$$\sigma_{23}(x, 0) = -\frac{W_0 \mu}{\pi} \left(\frac{\beta}{c} \cdot \frac{v}{c}\right)^{1/2} \int_0^\infty \frac{e^{-ux_1} u^{1/2}}{(u - \beta/v)} du. \qquad \ldots(42)$$

For small values of $\frac{\beta}{v} x$, $x < 0$, putting $\alpha = 0$ in (35) we get

$$\sigma_{23}(x, 0) = \frac{-W_0 \mu}{\sqrt{\pi}} \left(\frac{\beta}{c} \cdot \frac{v}{c}\right)^{1/2} \frac{(\beta/v)^{1/2}}{\sqrt{x_1 \beta/v}} \qquad \ldots(43)$$

$$x_1 \to 0^+, \; (x_1 = -x).$$

Again for large values of $\beta x/v$, $(x < 0)$, from (36)

$$\sigma_{23}(x, 0) = \frac{W_0 \mu}{\sqrt{\pi}} \left(\frac{\beta}{c} \cdot \frac{v}{c}\right)^{1/2} \frac{(\beta/v)^{1/2}}{\sqrt{x_1 \beta/v}}$$

$$\times \left[\frac{v}{\beta x_1} \Gamma\left(\frac{3}{2}\right) + \frac{v^2}{\beta^2 x_1^2} \Gamma\left(\frac{5}{2}\right) + \frac{v^3}{\beta^3 x_1^3} \Gamma\left(\frac{7}{2}\right) + \ldots\right]. \qquad \ldots(44)$$

Putting $\alpha = 0$ the displacement on the free surface $(y = 0, x > 0)$ is obtained from (38) as

$$W(x, 0) = \frac{W_0}{\sqrt{\pi}} e^{-\frac{1}{2}kx} (kx)^{-\frac{1}{4}} \cdot W_{-\frac{1}{4}, -\frac{1}{4}}(kx), \; x > 0 \qquad \ldots(45)$$

where $k = \left(\frac{\beta}{c} \cdot \frac{v}{c}\right) / \left(1 - \frac{v^2}{c^2}\right)$

which for small values of $kx > 0$ becomes by help of (39)

$$W(x, 0) = W_0 e^{-kx} - \frac{2W_0}{\sqrt{\pi}} e^{-kx} (kx)^{\frac{1}{2}}, \; x \to 0^+ \qquad \ldots(46)$$

and for large values of $kx > 0$, using (40) we obtain

$$W(x, 0) = \frac{W_0}{\sqrt{\pi}} \frac{e^{-kx}}{\sqrt{kx}}, \; x \to \infty, \; (x > 0). \qquad \ldots(47)$$

4. Solution of the Problem for Non Steady State Motion

In this case it is assumed that at time $t = 0$ a semi-infinite punch starts to move with a constant velocity v at $X = Y = 0$ on the surface of the semi-infinite viscoelastic medium.

The 'standard linear solid' is taken as the viscoelastic model. Shifting the origin at $X=vt$ and putting $X - vt=x$ and $Y=y$ so that $\frac{\partial}{\partial X} = \frac{\partial}{\partial x}$, $\frac{\partial}{\partial Y} = \frac{\partial}{\partial y}$ and time derivative equal to $- v \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ the stress-displacement relations given by (8) become in this case

$$\left.\begin{aligned} - v \frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{13}}{\partial t} + \beta \sigma_{13} &= \mu \left(- v \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial t\, \partial x} + \alpha \frac{\partial W}{\partial x} \right) \\ - v \frac{\partial \sigma_{23}}{\partial x} + \frac{\partial \sigma_{23}}{\partial t} + \beta \sigma_{23} &= \mu \left(- v \frac{\partial^2 W}{\partial x\, \partial y} + \frac{\partial^2 W}{\partial t\, \partial y} + \alpha \frac{\partial W}{\partial y} \right) \end{aligned}\right\} \quad \ldots(48)$$

Both these equations can be reduced to ordinary differential equations by the application of the Laplace transform over t and the Fourier transform over x.

Let us denote the Laplace transform by a single bar

$$\bar{f} \equiv \bar{f}\,(x, y, p) = \int_0^\infty e^{-pt} f\,(x, y, t)\, dt \ldots(49)$$

and Fourier transform by two bars

$$\bar{\bar{f}} \equiv \bar{\bar{f}}\,(\xi, y, p) = \int_{-\infty}^{\infty} e^{i\xi t}\, \bar{f}\,(x, y, p)\, dx \ldots(50)$$

Applying these transforms to (48) we get

$$(i\xi v + p + \beta)\, \bar{\bar{\sigma}}_{13} = \mu\, (v\xi^2 - i\xi p - i\xi\alpha)\, \bar{\bar{W}} \quad \ldots(51)$$

$$(i\xi v + p + \beta)\, \bar{\bar{\sigma}}_{23} = \mu\, (vi\xi + p + \alpha)\, \frac{d\bar{\bar{W}}}{dy} \quad \ldots(52)$$

Now the equation of motion given by (4) becomes

$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{23}}{\partial y} = \rho \left(v^2 \frac{\partial^2 W}{\partial x^2} - 2v \frac{\partial^2 W}{\partial x \partial t} + \frac{\partial^2 W}{\partial t^2} \right)$$

which after taking Laplace and Fourier transforms takes the form

$$- i\xi\, \bar{\bar{\sigma}}_{13} + \frac{d\bar{\bar{\sigma}}_{23}}{dy} = \rho\, (- v^2\xi^2 + 2\, vi\xi p + p^2)\, \bar{\bar{W}}. \quad \ldots(53)$$

Substituting for $\bar{\bar{\sigma}}_{13}$ and $\bar{\bar{\sigma}}_{23}$ from (51) and (52) in (53) we have

$$\frac{d^2\bar{\bar{W}}}{dy^2} - \gamma^2\, \bar{\bar{W}} = 0 \quad \ldots(54)$$

where $\gamma^2 = \dfrac{1}{(Vi\xi+p+\alpha)} \left\{ \xi^2(Vi\xi+p+\alpha) + \dfrac{\rho}{\mu} (Vi\xi+p)^2 (Vi\xi+p+\beta) \right\}$. ...(55)

The branches of γ are defined by Re $(\gamma) > 0$.
Since the stresses are bounded as $y \to \infty$, $W(x, y, t)$ and hence also $\bar{\bar{W}}(\xi, y, p)$ must remain bounded as $y \to \infty$. Hence, the solution of eqn. (54) is given by

$$\bar{\bar{W}}(\xi, y, p) = A(\xi, p)\, e^{-\gamma y}.$$

Now the boundary conditions are

$$\left.\begin{aligned} W(x, 0, t) &= W_0 H(t), & x &< 0 \\ W(x, \infty, t) &= 0, & -\infty &< x < \infty \\ \sigma_{23}(x, 0, t) &= 0, & x &> 0. \end{aligned}\right\} \quad ...(56)$$

Taking Laplace transforms with respect to t, these conditions become

$$\left.\begin{aligned} \bar{W}(x, 0, p) &= \frac{W_0}{p}, & x &< 0 \\ \bar{\sigma}_{23}(x, 0, p) &= 0, & x &> 0. \end{aligned}\right\} \quad ...(57)$$

Let us consider

$$\bar{W}(x, 0, p) = W_0\, p(x) \quad \text{(say)}, \; x > 0$$

and $$\left.\bar{\sigma}_{23}(x, 0, p) = \mu W_0\, t(x) \; \text{(say)}, \; x < 0. \right\} \quad ...(58)$$

The functions $p(x)$ and $t(x)$ are such that

$$p(x) \sim O(e^{-k_1 x}) \text{ as } x \to \infty, \quad k_1 > 0$$

and $$p(x) \sim O(e^{-k_2 x}) \text{ as } x \to -\infty, \quad k_2 > 0.$$

Taking Fourier transform of (58) and (59) we obtain

$$\bar{\bar{W}}(\xi, 0, p) = \frac{W_0}{ip\xi} + W_0 P_+(\xi) \quad ...(59)$$

where $$P_+(\xi) = \int_0^\infty e^{i\xi x} p(x)\, dx, \quad (\xi = \sigma + i\tau)$$

and $$\bar{\sigma}_{23}(\xi, 0, p) = \mu W_0 T_-(\xi) \quad ...(60)$$

where $$T_-(\xi) = \int_{-\infty}^0 e^{i\xi x} t(x)\, dx.$$

The integral of $\bar{\bar{W}}(\xi, 0, p)$ over $(-\infty, 0)$ converges if and only if Im $(\xi) = \tau < 0$ and integral over $(0, \infty)$ converages if $\tau > -k_1$. $\bar{\sigma}_{23}$ is analytic over $(-\infty, 0)$ if $\tau < k_2$.

Now (52) becomes with the help of (56), (59) and (60)

$$\frac{(Vi\xi+p+\beta)\ T_-(\xi)}{(Vi\xi+p+\alpha)\ \gamma} = \frac{1}{ip\xi} + P_+(\xi). \qquad \ldots(61)$$

In this form of equation Wiener-Hopf technique can easily be applied.

5. Non Steady State Solution for Maxwell Solid

For general α and β, γ does not readily factorize. Expressions for the roots of $\gamma = 0$ can be obtained but these are difficult to handle. We discuss here the case of the Maxwell solid, where $\alpha = 0$.

In this case γ^2 reduces to

$$\gamma^2 = \left(1 - \frac{v^2}{c^2}\right)\left\{\xi^2 + \frac{vi\xi}{c^2}\cdot\frac{2p+\beta}{\left(1-\frac{v^2}{c^2}\right)} + \frac{p\,(p+\beta)}{c^2\left(1-\frac{v^2}{c^2}\right)}\right\},\ c^2 = \frac{\mu}{\rho}. \qquad \ldots(62)$$

$$\text{Hence}\quad \gamma = \sqrt{\left(1-\frac{v^2}{c^2}\right)}\,(\xi+iX_1)^{1/2}\,(\xi-iX_2)^{1/2},\ \text{Re}\ X_1, X_2 > 0. \qquad \ldots(63)$$

$$\text{where}\ X_1 = \frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[\frac{v\,(2p+\beta)}{c^2} + \frac{2}{c}\sqrt{p\,(p+\beta)+\frac{v^2\beta^2}{4c^2}}\right]$$

$$X_2 = \frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[\frac{-v\,(2p+\beta)}{c^2} + \frac{2}{c}\sqrt{p\,(p+\beta)+\frac{v^2\beta^2}{4c^2}}\right]. \qquad \ldots(64)$$

Branches are chosen so that $\gamma \to +\infty$ as $\xi \to \pm\infty$.

Thus for a Maxwell solid, (61) can be written after simplification as

$$-\left(1-\frac{v^2}{c^2}\right)^{-1/2}\frac{\left(\xi - i\left(\frac{p+\beta}{v}\right)\right)}{\left(\xi-\frac{ip}{v}\right)}\cdot\frac{T_-(\xi)}{(\xi-iX_2)^{1/2}} - \frac{(iX_1)^{1/2}}{i\xi p}$$

$$= \frac{(\xi+iX_1)^{1/2}-(iX_1)^{1/2}}{ip\xi} + (\xi+iX_1)^{1/2}\,P_+(\xi). \qquad \ldots(65)$$

Applying the Wiener-Hopf technique we get

$$P_+(\xi) = \frac{(iX_1)^{1/2}}{ip\xi\ (\xi+iX_1)^{1/2}} - \frac{1}{ip\xi} \qquad \ldots(66)$$

and $$T_{-}(\xi) = -\left(1-\frac{v^2}{c^2}\right)^{1/2} \frac{(iX_1)^{1/2}}{ip\xi} \cdot \frac{\left(\xi-\frac{ip}{v}\right)(\xi-iX_2)^{1/2}}{\left(\xi-\frac{i(p+\beta)}{v}\right)}. \qquad \text{...(67)}$$

Therefore, $\bar{\bar{W}}(\xi, 0, p)$ given in (59), with the help of (66), takes the form

$$\bar{\bar{W}}(\xi, 0, p) = \frac{W_0 (iX_1)^{1/2}}{ip\xi\,(\xi+iX_1)^{1/2}}. \qquad \text{...(68)}$$

Taking inverse transforms, one gets

$$W(x, 0, t) = \frac{1}{2\pi i}\frac{W_0}{i}\int_{c'-i\infty}^{c'+i\infty} \frac{(iX_1)^{1/2} e^{pt}}{p}\,dp \times$$

$$\times \frac{1}{2\pi}\int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}}{\xi\,(\xi+iX_1)^{1/2}}\,d\xi,$$

$$0 < d < k_1,\ x > 0. \qquad \text{...(69)}$$

Taking the path of integration around the branch cut along negative imaginary axis from $-iX_1$ to $-i\infty$ the integral

$$I = \int_{-\infty-id}^{\infty-id} \frac{e^{-i\xi x}}{\xi\,(\xi+iX_1)^{1/2}}\,d\xi$$

can be converted to the integral

$$I = 2e^{\pi i/4} e^{-X_1 x} (x)^{1/2} \int_0^{\infty} \frac{e^{-U} U^{-1/2}}{U+xX_1}\,dU$$

which is finally evaluated as

$$I = 2\sqrt{\pi}\, e^{\pi i/4} e^{-\frac{1}{2}xX_1}.(x)^{1/2} (xX_1)^{-3/4} W_{-1/4,-1/4}(xX_1).$$

Putting this value of the integral in (69) we obtain

$$W(x, 0, t) = \frac{W_0}{\sqrt{\pi}} \cdot \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt} e^{-\frac{1}{2}xX_1}}{p} (xX_1)^{-1/4} W_{-1/4,-1/4}(xX_1)\,dp,\ x>0. \qquad \text{...(70)}$$

Now for small p,

$$X_1 = \left(\frac{v}{c}\cdot\frac{\beta}{c}\right) / \left(1-\frac{v^2}{c^2}\right) \doteq k \text{ (say)} \qquad \text{...(71)}$$

and for large p, $X_1 = \dfrac{p}{c-V}$

therefore for large $\frac{px}{c-V}$, $W_{-\frac{1}{4},-\frac{1}{4}}\left(p\frac{x}{c-V}\right) \sim \exp\left(-\frac{1}{2}\frac{px}{c-V}\right)\left(\frac{px}{c-V}\right)^{-\frac{1}{4}}$. ...(72)

So in eqn. (70), putting the value of X_1 for small p given by (71), we obtain for large time t,

$$W(x, 0, t) = \frac{W_0}{\sqrt{\pi}} e^{-\frac{1}{2}kx} (kx)^{-\frac{1}{4}} W_{-\frac{1}{4},-\frac{1}{4}}(kx) \quad ...(73)$$

which is same as the result for the steady state case for all $x > 0$ given by (45).

For large p, i.e. for small time t and for all finite x such that $px/(c-V)$ is large, using (72) we obtain from (70)

$$W(x, 0, t) = \frac{2W_0}{\pi}\sqrt{\frac{c-V}{x}}\left(t-\frac{x}{c-V}\right)^{\frac{1}{2}} H\left(t-\frac{x}{c-V}\right). \quad ...(74)$$

Now, using (67), (60) becomes

$$\bar{\bar{\sigma}}_{23}(\xi, 0, p) = \frac{-\mu W_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} (iX_1)^{\frac{1}{2}}\left(\xi-\frac{ip}{v}\right)(\xi-iX_2)^{\frac{1}{2}}}{i\xi p\left(\xi-\frac{i(p+\beta)}{v}\right)}.$$

After taking inverse transforms it converts to

$$\sigma_{23}(x, 0, t) = \frac{-\mu W_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{pt}}{p}(iX_1)^{\frac{1}{2}}\,dp.\,\frac{1}{2\pi i}$$

$$\times\int_{-\infty-id}^{\infty-id}\frac{e^{-i\xi x}\left(\xi-\frac{ip}{v}\right)(\xi-iX_2)^{\frac{1}{2}}}{\xi\left(\xi-\frac{i(p+\beta)}{v}\right)}\,d\xi. \quad ...(75)$$

Evaluating the integral with respect to ξ for small x, it can be shown that

$$\sigma_{23}(x, 0, t) = -\frac{\mu W_0(1-v^2/c^2)^{\frac{1}{2}}}{\sqrt{\pi x_1}}\cdot B, \text{ as } -x = x_1 \to 0^+ \quad ...(76)$$

(for all $t > 0$)

where

$$B = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{(X_1)^{\frac{1}{2}} e^{pt}}{p}\,dp. \quad ...(77)$$

The evaluation of B for all t ($t > 0$) has been done in the appendix.

For small p i.e., for large t, using from (71) the result that $X_1 \sim k$,

we have $$B = \sqrt{k} = \sqrt{\left(\frac{\beta}{c} \cdot \frac{v}{c}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right)}. \qquad \ldots(78)$$

Substituting the value of B given by (78) in (77) we obtain

$$\sigma_{23}(x, 0) = -\frac{\mu W_0 \left(\frac{\beta}{c} \cdot \frac{v}{c}\right)^{1/2}}{\sqrt{\pi x_1}}, \quad x_1 \to 0^+$$

which is same as the result for the steady state case given by (43). The variation of the nondimensional values of B given by $B^* = (1 - v^2/c^2)^{1/2} \times B\sqrt{c/\beta}$ has been plotted against nondimensional time $t_1 = \beta t$ for various values of $v/c = 0.5, 0.7$ and 0.9 and has been shown by means of the following graphs (Fig. 2)

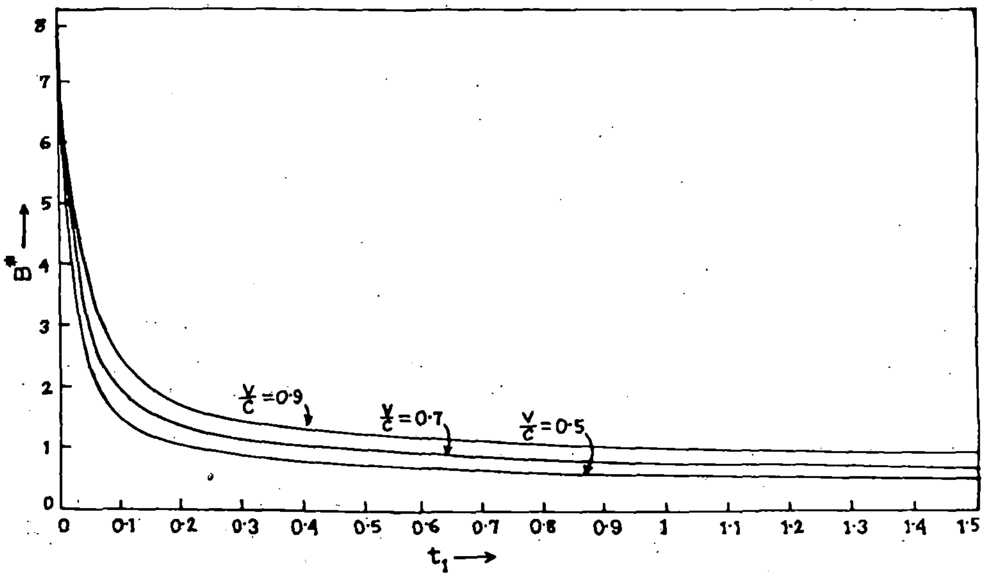

FIG. 2

Now for all values of x i.e., for general value of ξ the integral

$$\frac{1}{2\pi i} \int_{-\infty - id}^{\infty - id} \frac{e^{-i\xi x}\left(\xi - \frac{ip}{v}\right)(\xi - iX_2)^{1/2}}{\xi\left[\xi - \frac{i(p+\beta)}{v}\right]} d\xi$$

appearing in eqn. (75) can be converted to the integral

$$I_1 = e^{\frac{3\pi i}{4}} \cdot \frac{p}{p+\beta}(X_2)^{1/2} + \frac{e^{\pi i/4} e^{X_2 x}}{\pi i} \int_0^\infty \frac{e^{ux} u^{1/2}\left(u + X_2 - \frac{p}{v}\right)}{(u + X_2)\left(u + X_2 - \frac{p+\beta}{v}\right)} du,$$

$$\left(\because \frac{p+\beta}{v} > X_2\right) \qquad \ldots(79)$$

considering the path of integration around the branch cut along positive imaginary axis from iX_2 to $i\infty$.

So using (79), (75) becomes

$$\sigma_{23}(x, 0, t) = -\frac{\mu W_0}{\pi}\left(1-\frac{v^2}{c^2}\right)^{1/2} \cdot \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p}\,(X_1)^{1/2}\, e^{X_2 x}$$

$$\times \int_0^{\infty} \frac{e^{ux}\, u^{1/2}\left(u+X_2-\dfrac{p}{v}\right)}{(u+X_2)\left(u+X_2-\dfrac{p+\beta}{v}\right)}\, du + \mu W_0\left(1-\frac{v^2}{c^2}\right)^{1/2} \cdot \frac{1}{2\pi i}$$

$$\int_{c'-i\infty}^{c'+i\infty} \frac{e^{pt}}{p+\beta}\,(X_1X_2)^{1/2}\, dp.$$

For large t (i.e., for small p using $X_1 = k$, $X_2 = 0$) and for all x ($x < 0$) we obtain from (80)

$$\sigma_{23}(x, 0, t) = -\frac{\mu W_0}{\pi}\left(\frac{\beta}{c}\cdot\frac{v}{c}\right)^{1/2}\int_0^{\infty} \frac{e^{ux}\, u^{1/2}}{\left(u-\dfrac{\beta}{v}\right)}\, du$$

$$k = \left(\frac{v}{c}\cdot\frac{\beta}{c}\right) \Big/ \left(1-\frac{v^2}{c^2}\right)$$

which is same as the solution for the steady case for all values of $x > 0$ given by (42).

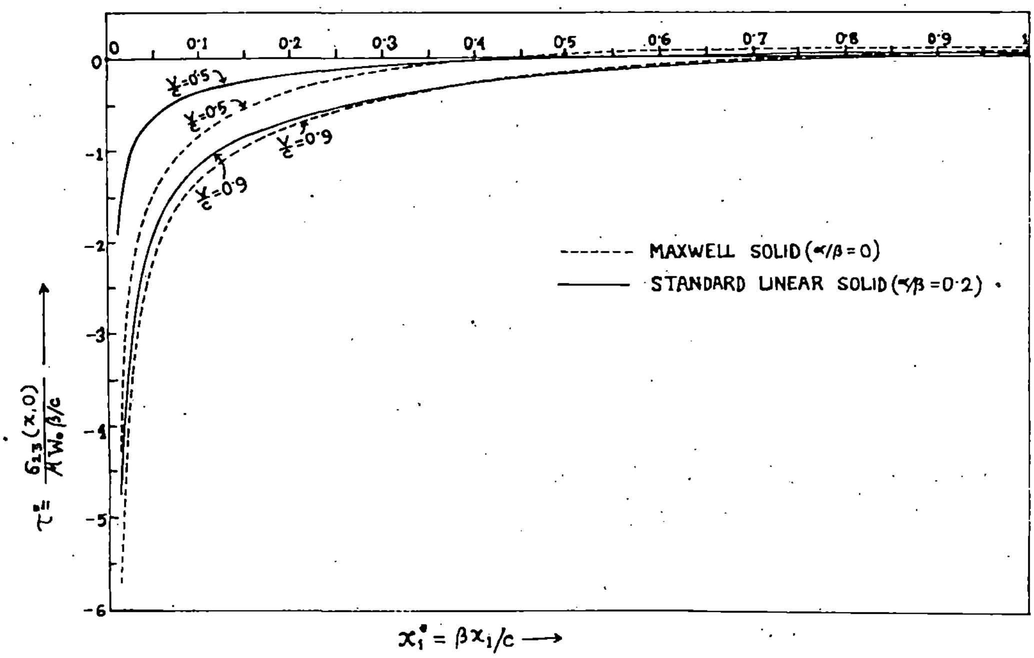

FIG. 3

6. Results and Discussion

The stress σ_{23} $(x, 0)$ just below the punch $(x < 0)$ and the displacement W $(x, 0)$ on the free surface $(y=0, x > 0)$ have been computed numerically from eqns. (39) and (43) for different values of parameters v/c and α/β. The case $\alpha/\beta = 0$ corresponds to Maxwell-Solid. In Fig. 3 non dimensional stress $\tau^* = \dfrac{\sigma_{23}\,(x,\ 0)}{\mu W_0\ \beta/c}$ has been plotted against non-dimensional distance $x_1^* = \beta x_1\ /c$ for values of the paramter $v/c=0.5$, 0.9 and for values of the parameter $\alpha/\beta=0$ and 0.2.

For the same sets of the parameter values non dimensional displacement $W^* = W/W_0$ has been plotted versus non-dimensional distance $x^* = \beta x/c$ in Fig. 4. W^* varies from 1 to zero as x^* changes gradually from $x^* = 0$ to ∞.

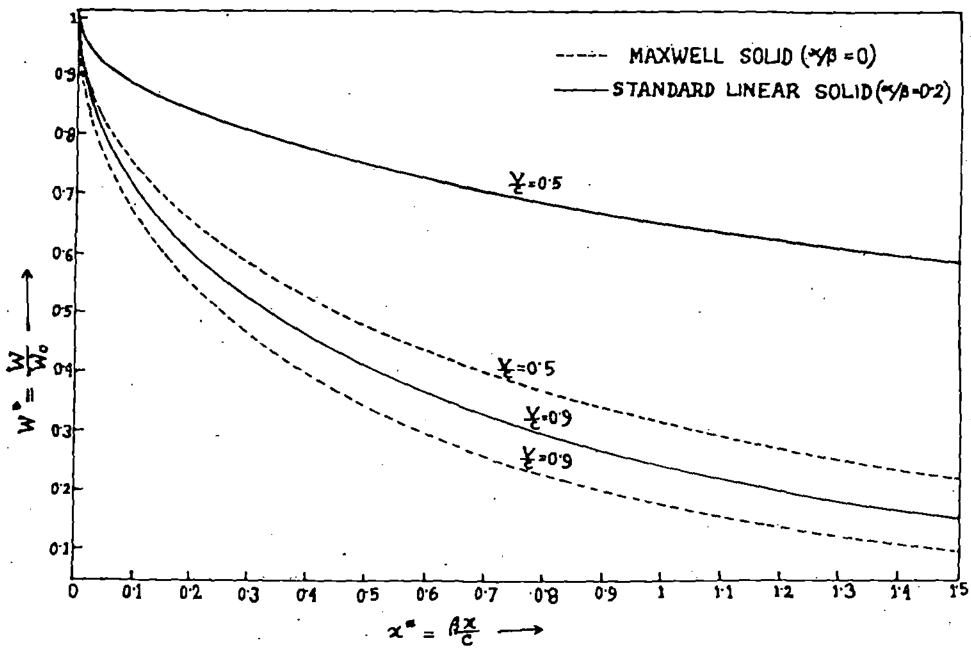

Fig. 4

It may be noted from the graphs that variation of the values of W^* with x^* is rapid with the increase in the values of the paramter v/c. Further it is found that the graphs become steeper with the decrease in the values of the paramter α/β. From Fig. 3 it is found that nondimensional stress τ^* changes rapidly with the decrease in the values of v/c where as for a fixed value of v/c graphs become flat with the increase in the values of α/β.

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References

1. A. L. Galin, *Contact Problems in the Theory of Elasticity* (ed.: I. N. Sneddon), North Carolina State University, Raleigh, 1961.
2. G. M. L. Gladwell, *Contact Problems in the Classical Theory of Elasticity*, Sijthoff and Noordhoff, Alphen ann den Rijn, 1980.
3. E. S. Suhibi, *Bull. Tech. Univ. of Istambul* **25** (1972), 72.
4. R. J. Tait and T. B. Moodie, *Int. J. Engeng. Sci.* **19** (1981), 221.
5. H. G. Georgiadis, *Acta Mechanica* **68** (1987), 193.
6. B. Noble, *Methods based on the Wiener-Hopf Technique*, Pergamon Press, New York, 1958, p. 13.
7. S. C. Hunter, *J. Appl. Mech.* **4** (1961), 611.
8. J. B. Alblas and M. Kuipers, *Int. J. Engng. Sci.* **8** (1970), 363
9. J. M. Golden, *Q. J. Mech. appl. Math.* **30** (1977), 23.
10. E. T. Whittacker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1969, p. 337.

APPENDIX

I. Evaluation of the Integral B

The integral in (77)

$$B = \frac{1}{2\pi i} \frac{1}{\sqrt{c\left(1-\frac{v^2}{c^2}\right)}} \int_{c'-i\infty}^{c'+i\infty} e^{pt} \frac{\sqrt{\frac{v}{c}\left(p+\frac{\beta}{2}\right)+\sqrt{p(p+\beta)+\frac{v^2\beta^2}{4c^2}}}}{p} dp$$

has a simple pole at $p = 0$ and branch points at $p = -\beta$

$$\alpha_1 = \frac{\beta}{2}\left(-1+\sqrt{1-\frac{v^2}{c^2}}\right)$$

$$\alpha_2 = \frac{\beta}{2}\left(-1-\sqrt{1-\frac{v^2}{c^2}}\right).$$

Taking the branch cut along the negative real axis from α_1 to $-\infty$ the integral can be considered as a contour integral around the path as shown in Fig. 5.

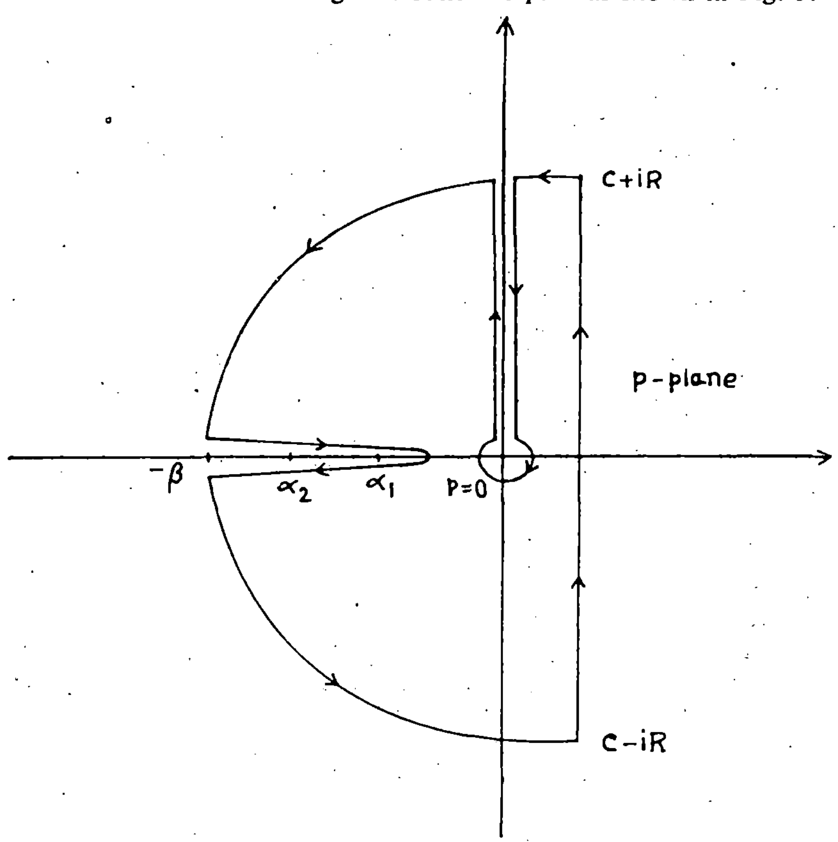

FIG. 5

Let $I = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \epsilon^{pt} \frac{\sqrt{\frac{v}{c}\left(p+\frac{\beta}{2}\right) + \sqrt{(p-\alpha_1)(p-\alpha_2)}}}{p} dp$

which can be written as $I = \sqrt{\frac{v\beta}{c}} + I_1$ where $\sqrt{\frac{v\beta}{c}}$ is the contribution to the integral from pole at $p=0$.

Let $I_1 = I_2 + I_3$ where I_2 is the value of the integral I_1 around the branch cut from α_1 to α_2 and I_3 is its value round the branch cut from α_2 to $-\infty$.
Now it can be shown that

$$I_2 = \frac{\sqrt{\beta}}{\pi} \int_0^{\sqrt{1-v^2/c^2}} \frac{e^{t_1(\alpha_1^*-r)}\sqrt{\frac{R^*-x^*}{2}}}{(\alpha_1^*-r)} dr.$$

In the interval $(\alpha_2, -\infty)$

$$I_3 = \frac{\sqrt{\beta}}{\pi} \int_0^{\sqrt{1-v^2/c^2}} \frac{e^{t_1(\alpha_1^*-r)}\sqrt{-x^{**}}}{(\alpha_1^*-r)} dr.$$

where $\alpha_1 = \beta\alpha_1^*$, $\alpha_2 = \beta\alpha_2^*$, $t_1 = \beta t$,

$$x^* = \frac{v}{c}\left(\frac{1}{2}\sqrt{1-\frac{v^2}{c^2}} - r\right), \quad y^* = \sqrt{r}\sqrt{\sqrt{1-\frac{v^2}{c^2}} - r}$$

$$R^* = \sqrt{(x^*)^2 + (y^*)^2}$$

$$x^{**} = -\frac{v}{c}\left(r - \frac{1}{2}\sqrt{1-\frac{v^2}{c^2}}\right) - \sqrt{r}\sqrt{r - \sqrt{\left(1-\frac{v^2}{c^2}\right)}}.$$

Finally, we obtain

$$B = \frac{1}{\sqrt{c\left(1-\frac{v^2}{c^2}\right)}}\left[\sqrt{\frac{v\beta}{c}} + \frac{\sqrt{\beta}}{\pi}\int_0^{\sqrt{1-v^2/c^2}} \frac{e^{t_1(\alpha_1^*-r)}\sqrt{\frac{R^*-x^*}{2}}}{(\alpha_1^*-r)} dr + \int_{\sqrt{1-v^2/c^2}}^{\infty} \frac{e^{t_1(\alpha_1^*-r)}\sqrt{-x^{**}}}{(\alpha_1^*-r)} dr\right].$$

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ANTIPLANE DYNAMIC CRACK PROPAGATION IN AN INHOMOGENEOUS VISCOELASTIC SOLID

S. C. MANDAL and M. L. GHOSH (DARJEELING)

1. Introduction

Until now many authors, BAKER [1], CHEREPANOV and AFANASEV [2] and others have investigated the dynamic crack propagation in a homogeneous elastic medium. This problem presents an interest for a better understanding of the brittle behaviour of the material. However, natural or atificial materials are usually inhomogeneous. There exist very few solutions to the problem of dynamic crack propagation in inhomogeneous elastic media. ARKINSON and LIST [3] and ATKINSON [4] considered steady-state crack propagation in different types of inhomogeneous elastic media. In addition, if the materials are dissipative, that effect can be taken into account by considering the material to be viscoelastic. Crack propagation in viscoelastic medium has been studied by WILLIS [5], ATKINSON and LIST [6], COUSSY [7] and others. WILLIS [5] considered steady-state Mode III crack propagation for a standard linear solid under general type of loading on the crack surfaces. ATKINSON and LIST [6] studied nonsteady SH-wave type crack propagation starting at $t = 0$ and moving with a constant velocity in the "Maxwell Solid" or using the viscoelastic model suggested by Achenbach and Chao. Finally, SILLS and BENVENISTE [9] and COUSSY [7] studied steady state crack propagation of SH-type at the interface between two viscoelastic media.

In our case we have considered steady and nonsteady cases of Mode III crack propagation in an inhomogeneous viscoelastic medium. Two types of viscoelastic models, namely Maxwell Solid and Standard Linear Solid have been considered. Material properties have been assumed to vary exponentially in the direction perpendicular to the direction of crack propagation. We have studied how the material inhomogeneity affects the stress intensity factor and also the crack opening displacement when a Mode III type crack propagates through the inhomogeneous viscoelastic medium.

2. Formulation of the Problem and its Solution for Nonsteady Case in Maxwell Solid

Let us consider an inhomogeneous viscoelastic medium which was set in motion by a semi-infinite crack suddenly appearing at $t = 0$ and moving with a constant velocity V in the direction of the X-axis. The Y-axis is taken perpendicular to the X-axis (Fig. 1). For SH-waves, the displacements along X and Y directions are zero and only the displacement $W = W(X, Y, t)$ along the Z-direction exists.

The shear modulus is $\mu(Y) = \mu_0 \exp(2\beta Y)$ and density $\varrho(Y) = \varrho_0 \exp(2\beta Y)$, where β, μ_0, and ϱ_0 are constants.

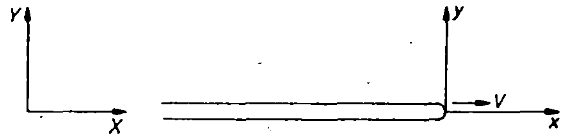

FIG. 1. The crack geometry.

The non-zero stresses are

(2.1) $$\sigma_{YZ} = \sigma_{YZ}(X, Y, t) \quad \text{and} \quad \sigma_{XZ} = \sigma_{XZ}(X, Y, t),$$

and the nonvanishing strains are

(2.2) $$e_{XZ} = {}^1/_2(\partial W/\partial X), \quad e_{YZ} = {}^1/_2(\partial W/\partial Y).$$

Considering a Maxwell Solid as the viscoelastic model, the stress-strain relations are

(2.3) $$\begin{aligned}(\partial\sigma_{YZ}/\partial t)+\beta_1\sigma_{YZ} &= 2\mu(Y)(\partial e_{YZ}/\partial t),\\ (\partial\sigma_{XZ}/\partial t)+\beta_1\sigma_{XZ} &= 2\mu(Y)(\partial e_{XZ}/\partial t),\end{aligned}$$

where β_1 is a positive constant.

The equation of motion has the form

(2.4) $$(\partial\sigma_{XZ}/\partial X)+(\partial\sigma/\partial Y_{YZ}) = \varrho(Y)(\partial^2 W/\partial t^2),$$

and the boundary conditions of the problem are

(2.5) $$\begin{aligned}W(X, 0, t) &= 0, \quad X-Vt > 0, \quad t > 0,\\ \sigma_{YZ}(X, 0, t) &= -\sigma H(t), \quad X-Vt < 0, \quad t > 0,\\ \sigma_{YZ}(X, Y, t) &\to 0 \quad \text{as} \quad X^2+Y^2 \to \infty.\end{aligned}$$

It is convenient to shift the origin of coordinate to the tip of the crack at $X = Vt$. New coordinate axes (x, y) are parallel to the respective fixed ones (X, Y).

Hence, putting $x = X-Vt$, $y = Y$, we obtain $(\partial/\partial X) = (\partial/\partial x)$, $(\partial/\partial Y) = (\partial/\partial y)$ and the time derivative transforms to $-V(\partial/\partial x)+(\partial/\partial t)$. Equations (2.1), (2.2), (2.3) and (2.4) become

(2.6) $$\sigma_{xz} = \sigma_{xz}(x, y, t), \quad \sigma_{yz} = \sigma_{yz}(x, y, t),$$

(2.7) $$e_{xz} = {}^1/_2[\partial W(x, y, t)/\partial x], \quad e_{yz} = {}^1/_2[\partial W(x, y, t)/\partial y],$$

(2.8) $$\begin{aligned}-V(\partial\sigma_{yz}/\partial x)+(\partial\sigma_{yz}/\partial t)+\beta_1\sigma_{yz} &= \mu(y)[-V(\partial^2 W/\partial x\,\partial y)+(\partial^2 W/\partial t\,\partial y)],\\ -V(\partial\sigma_{xz}/\partial x)+(\partial\sigma_{xz}/\partial t)+\beta_1\sigma_{xz} &= \mu(y)[-V(\partial^2 W/\partial x^2)+(\partial^2 W/\partial t\,\partial x)]\end{aligned}$$

and

(2.9) $$(\partial\sigma_{xz}/\partial x)+(\partial\sigma_{xz}/\partial y) = \varrho(y)[V^2(\partial^2 W/\partial x^2)-2V(\partial^2 W/\partial x\,\partial t)+(\partial^2 W/\partial t^2)].$$

The boundary conditions (2.5) now assume the form

(2.10) $$\begin{aligned}W(x, 0, t) &= 0, \quad x > 0,\\ \sigma_{yz}(X, 0, t) &= -\sigma H(t), \quad x < 0,\\ \sigma_{yz}(x, y, t) &\to 0 \quad \text{as} \quad x^2+y^2 \to \infty.\end{aligned}$$

Let us denote the Laplace transform by a single bar

$$\bar{f} \equiv \bar{f}(x, y, p) = \int_0^{\infty} f(x, y, t)\exp(-pt)dt,$$

and the Fourier transform by two bars

$$\bar{\bar{f}} \equiv \bar{\bar{f}}(\xi, y, p) = 1/\sqrt{(2\pi)} \int_{-\infty}^{\infty} \bar{f}(x, y, p)\exp(i\xi x)dx.$$

Applying these transforms to Eqs. (2.8) and (2.9), we get

$$(i\xi V + p + \beta_1)\bar{\bar{\sigma}}_{yz} = \mu(y)(Vi\xi + p)(d\bar{\bar{W}}/dy), \tag{2.11}$$

$$(i\xi V + p + \beta_1)\bar{\bar{\sigma}}_{xz} = \mu(y)(V\xi^2 - i\xi p)\bar{\bar{W}} \tag{2.12}$$

and

$$-i\xi\bar{\bar{\sigma}}_{xz} + (d\bar{\bar{\sigma}}_{yz}/dy) = \varrho(y)(-V^2\xi^2 + 2Vi\xi p + p^2)\bar{\bar{W}}. \tag{2.13}$$

Eliminating $\bar{\bar{\sigma}}_{xz}, \bar{\bar{\sigma}}_{yz}$ from Eqs. (2.11), (2.12) and (2.13), we obtain

$$(d^2\bar{\bar{W}}/dy^2) + 2\beta(d\bar{\bar{W}}/dy) - \gamma^2\bar{\bar{W}} = 0, \tag{2.14}$$

where

$$\gamma^2 = \xi^2 + (1/c^2)(Vi\xi + p)(Vi\xi + p + \beta_1), \quad c^2 = \mu_0/\varrho_0. \tag{2.15}$$

The branches of γ are chosen so that $\mathrm{Re}(\gamma) > 0$.

Since $\bar{\bar{W}}$ must remain bounded as $y \to \pm\infty$, so solutions of (2.14) are

$$\bar{\bar{W}}^{(1)} = B_1\exp\left[-(\beta + \sqrt{(\beta^2 + \gamma^2)})y\right], \quad y > 0, \tag{2.16}$$

and

$$\bar{\bar{W}}^{(2)} = A_2\exp\left[(-\beta + \sqrt{(\beta^2 + \gamma^2)})y\right], \quad y < 0, \tag{2.17}$$

where $W^{(1)}$, and $W^{(2)}$ denote the displacement in the upper and lower half-plane, respectively.

Let us consider the case when for $y = 0$

$$\bar{W}^{(1)} - \bar{W}^{(2)} = h(x, p), \quad x < 0, \qquad = 0, \quad x > 0, \tag{2.18}$$

where $h(x, p)$ is an unknown function such that

$$h(x, p) \sim 0\{\exp(k_1 x)\} \quad \text{as} \quad x \to -\infty, \quad k_1 > 0,$$

Applying the Fourier transform to Eq. (2.18), we get

$$\bar{\bar{W}}^{(1)} - \bar{\bar{W}}^{(2)} = B_1 - A_2 = 1/\sqrt{(2\pi)} \int_{-\infty}^{\infty} h(x, p)\exp(i\xi x)dx, \tag{2.19}$$

$$= H_-(\xi, p),$$

where $H_-(\xi, p)$ is an analytic function in the lower half-plane $\tau < k_1$ and $\xi = \sigma + i\tau$.

Now from Eqs. (2.11), (2.16), and (2.17), we obtain

$$(2.20)\quad \bar{\sigma}_{yz}^{(1)} = \frac{(V_i\xi+p)}{(V_i\xi+p+\beta_1)}\mu(y)(\partial\bar{\bar{W}}^{(1)}/\partial y) = -\mu(y)B_1\left(\beta+\sqrt{(\beta^2+\gamma^2)}\right)\times$$

$$\times\exp\left[-\left(\beta+\sqrt{(\beta^2+\gamma^2)}\right)y\right]\frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)},\quad y>0,$$

$$\bar{\bar{\sigma}}_y^{(2)} = \frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\cdot\mu(y)A_2\left(-\beta+\sqrt{(\beta^2+\gamma^2)}\right)\exp\left[\left(-\beta+\sqrt{(\beta^2+\gamma^2)}\right)y\right],\ y<0,$$

where $\sigma_{yz}^{(1)}$ and $\sigma_{yz}^{(2)}$ are the stresses on the upper and lower surfaces of the crack.

Since the stresses are continuous for $y = 0$,

$$\bar{\bar{\sigma}}_{yz}^{(1)} = \bar{\bar{\sigma}}_{yz}^{(2)}.$$

Using Eqs. (2.20), we obtain

$$(2.21)\quad B_1 = -\frac{\left[-\beta+\sqrt{(\beta^2+\gamma^2)}\right]}{\left[\beta+\sqrt{(\beta^2+\gamma^2)}\right]}A_2.$$

Using Eq. (2.21), (2.19) becomes

$$(2.22)\quad H_-(\xi,p) = -\frac{2\sqrt{(\beta^2+\gamma^2)}}{\left[\beta+\sqrt{(\beta^2+\gamma^2)}\right]}A_2.$$

Again let us assume that for $y = 0$

$$(2.23)\quad \begin{aligned}\bar{\sigma}_{yz} = \bar{\sigma}_{yz}^{(1)} = \bar{\sigma}_{yz}^{(2)} &= -[\sigma_0\exp(\lambda x)]/p, & x<0,\\ &= e(x), & x>0.\end{aligned}$$

Here $e(x)$ is an unknown function such that

$$e(x)\sim 0\{\exp(-k_2x)\},\quad x\to\infty,\quad k_2>0.$$

Taking Fourier transforms of Eq. (2.23) we get

$$(2.24)\quad \bar{\bar{\sigma}}_{yz}^{(2)} = \frac{(Vi\xi+p)}{(Vi\xi+p+\beta_1)}\mu_0A_2\left[-\beta+\sqrt{(\beta^2+\gamma^2)}\right]$$

$$= 1/\sqrt{(2\pi)}\int_0^\infty \bar{\sigma}_{yz}^{(2)}\exp(i\xi x)dx + 1/\sqrt{(2\pi)}\int_{-\infty}^0 \bar{\sigma}_{yz}^{(2)}\exp(i\xi x)dx =$$

$$= E_+(\xi,p) - \sigma_0/\left[\sqrt{(2\pi)}\,(\lambda+i\xi)p\right],$$

where

$$(2.25)\quad E_+(\xi,p) = 1/\sqrt{(2\pi)}\int_0^\infty \bar{\sigma}_{yz}^{(2)}\exp(i\xi x)dx$$

and is an analytic function in the upper half-plane $\tau > -k_2$ and $\sigma_0/[i\sqrt{(2\pi)}(\xi-i\lambda)p]$ is analytic in the lower half-plane $\tau < \lambda$.

From Eqs. (2.22) and (2.24), we get

$$(2.26)\qquad -\frac{\mu_0(Vi\xi+p)\gamma^2 H^-(\xi,p)}{2(Vi\xi+p+\beta_1)\sqrt{(\beta^2+\gamma^2)}} = E_+(\xi,p)-\sigma_0/\left[\sqrt{(2\pi)}\,(\lambda+i\xi)p\right].$$

It may be noted that the problem has been reduced to a form suitable for application of the Wiener–Hopf technique.

Now

$$(2.27)\qquad \gamma^2 = (1-V^2/c^2)(\xi+iX_1)(\xi-iX_2),$$

where

$$(2.28)\qquad X_1 = {}^1/_2(1-V^2/c^2)^{-1}\left[(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2 V^2/c^4+} \right.$$
$$\left. +4p(p+\beta_1)(1-V^2/c^2)/c^2\}\right],$$

$$(2.29)\qquad X_2 = {}^1/_2(1-V^2/c^2)^{-1}\left[-(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2 V^2/c^4+}\right.$$
$$\left. +4p(p+\beta_1)(1-V^2/c^2)/c^2\}\right]$$

and

$$(2.30)\qquad \sqrt{(\beta^2+\gamma^2)} = (\xi+iY_1)^{1/2}(\xi-iY_2)^{1/2}(1-V^2/c^2)^{1/2},$$

where

$$(2.31)\qquad Y_1 = {}^1/_2(1-V^2/c^2)^{-1}\left[(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2 V^2/c^4+}\right.$$
$$\left. +4\{p(p+\beta_1)/c^2+\beta^2\}(1-V^2/c^2)\}\right],$$

$$(2.32)\qquad Y_2 = {}^1/_2(1-V^2/c^2)^{-1}\left[-(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2 V^2/c^4+}\right.$$
$$\left. +4\{p(p+\beta_1)/c^2+\beta^2\}(1-V^2/(c^2))\}\right].$$

Using Eqs. (2.27) and (2.30), (2.26) becomes

$$(2.33)\qquad -\frac{\mu_0(1-V^2/c^2)^{1/2}(\xi-ip/V)(\xi-iX_2)H_-(\xi,p)}{2[\xi-i(p+\beta_1)/V](\xi-iY_2)^{1/2}}+\frac{\sigma_0(i\lambda+iY_1)^{1/2}}{\sqrt{(2\pi)}\,i(\xi-i\lambda)(i\lambda+iX_1)p} =$$
$$= \frac{(\xi+iY_1)^{1/2}E_+(\xi,p)}{(\xi+iX_1)}-\frac{\sigma_0}{\sqrt{(2\pi)}\,ip(\xi-i\lambda)}\left[\frac{(\xi+iY_1)^{1/2}}{(\xi+iX_1)}-\frac{(i\lambda+iY_1)^{1/2}}{(i\lambda+iX_1)}\right].$$

The functions on the R.H.S. of Eq. (2.33) are analytic and non-zero in the upper half-plane $\tau > -k_2$, and functions on the L.H.S. are analytic and non-zero in the lower half-plane $\tau < \lambda$ ($\lambda < k_1$). Since both the functions are analytic in the strip $-k_2 < \tau < \lambda$, the principle of an analytic continuation states that each of them represents an entire function $M(\xi)$ in the whole ξ-plane.

Now the L.H.S. of (3.33) approaches zero as $|\xi| \to \infty$. It may then be concluded by Liouville's theorem that $M(\xi) = 0$, and therefore

$$(2.34)\qquad H_-(\xi,p) = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(\xi-iY_2)^{1/2}(1-V^2/c^2)^{-1/2}(\xi-i(p+\beta_1)/V)}{\mu_0\sqrt{(2\pi)}\,ip(\xi-i\lambda)(i\lambda+iX_1)(\xi-iX_2)(\xi-ip/V)}$$

and

$$E_+(\xi, p) = \frac{\sigma_0}{\sqrt{(2\pi)}\; ip(\xi - i\lambda)} - \frac{\sigma_0(i\lambda + iYy)^{1/2}(\xi + iX_1)}{\sqrt{(2\pi)}\; ip(\xi - i\lambda)(i\lambda + iX_1)(\xi + iY_1)^{1/2}} \tag{2.35}$$

From Eq. (2.34) it follows that

$$H_-(\xi, p) = \frac{2\sigma_0(1 - V^2/(c^2)^{-1/2}(i\lambda + iY_1)^{1/2}}{\mu_0 \sqrt{(2\pi)}\; ip(i\lambda + iX_1)} \xi^{-3/2}, \quad \xi \to \infty.$$

Application of the inverse Fourier transform yields

$$h(x, p) = \frac{4\sigma_0}{\mu_0}(1 - V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2}\frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p}, \quad x \to 0^-.$$

Again, taking the inverse Laplace transform, the displacement jump across the surface of the crack near the crack tip is

$$W^{(1)} - W^{(2)} = (4\sigma_0/\mu_0)(1 - V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2} \times \tag{2.36}$$

$$\times (1/2\pi i) \int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p} \exp(pt)\, dp.$$

From Eq. (2.35)

$$E_+(\xi, p) = -\frac{\sigma_0(i\lambda + iY_1)^{1/2}}{\sqrt{(2\pi)}\; ip(i\lambda + iX_1)} \xi^{-1/2}, \quad \xi \to \infty.$$

Taking the inverse Fourier transform we obtain

$$e(x, p) = \sigma_0 \pi^{-1/2}(x)^{-1/2}\frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p}, \quad x \to 0^+.$$

Again, taking the inverse Laplace transform

$$\sigma_{yz} = \sigma_0 \pi^{-1/2}(x)^{-1/2}(1/2\pi i) \int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p} \exp(pt)\, dp. \tag{2.37}$$

If ΔW is the displacement jump, then the crack opening displacement near the crack tip is given by

$$\mu_0 \Delta W = 4\sigma_0(1 - V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2} \cdot A, \quad (-1 \ll x < 0), \tag{2.38}$$

and the stress near the crack tip is

$$\sigma_{yz} = \sigma_0 \pi^{-1/2}(x)^{-1/2} \cdot A, \quad (0 < x \ll 1), \tag{2.39}$$

where

$$A = (1/2\pi i) \int_{c'-i\infty}^{c'+i\infty} \frac{(\lambda + Y_1)^{1/2}}{(\lambda + X_1)p} \exp(pt)\, dp = \tag{2.40}$$

$$= (1/2\pi i) \int_{c'-i\infty}^{c'+i\infty} \frac{(Y_1)^{1/2}}{X_1 p} \exp(pt)\, dp, \quad \lambda \to 0.$$

Evaluation of the integral A given by Eq. (2.40) corresponding to constant stress $-\sigma_0$ on the crack surfaces is presented in the Appendix.

In the fracture mechanics, it is customary to write $\sigma_{yz} = (0^+, 0, t)$ in the form $K/\sqrt{(2\pi x)}$, where K is the stress intensity factor. In our case

$$K = \sqrt{2\sigma_0 A}. \tag{2.41}$$

Putting $\beta = 0$ in the expression for A, we obtain the stress intensity factor in a homogeneous viscoelastic medium as

$$K = \sqrt{2\sigma_0 A_1},$$

where

$$A_1 = (1/2\pi i)\sqrt{2(1-V^2/c^2)} \times$$
$$\times \int_{c'-i\infty}^{c'+i\infty} \frac{\exp(pt)\,dp}{p[(2p+\beta_1)V/c^2+\sqrt{\{(2p+\beta_1)^2 V^2/c^4+4p(p+\beta_1)(1-V^2/c^2)/c^2\}}]},$$

which agrees with the results of ATKINSON and LIST [6]

3. Steady State Case for Maxwell Solid

Steady state solutions are the results of Sect. 2 corresponding to the case of t approaching infinity. So for the steady state case, passing to the limit $p \to 0$ and using the Tauberian theorem we obtain from Eq. (2.34)

$$H_-(\xi, p) = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(\xi-iY_2)^{1/2}(1-V^2/c^2)^{-1/2}(\xi-i\beta_1/V)}{\mu_0\sqrt{(2\pi)}\; i(\xi-i\lambda)(i\lambda+iX_1)\xi^2}.$$

Applying the inverse Fourier transform we obtain

$$W^{(1)}-W^{(2)} = \frac{2\sigma_0(i\lambda+iY_1)^{1/2}(1-V^2/c^2)^{-1/2}}{\mu_0 2\pi i(i\lambda+iX_1)} \times \tag{3.1}$$
$$\times \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{(\xi-iY_2)^{1/2}(\xi-i\beta_1/V)}{(\xi-i\lambda)\xi^2}\exp(-i\xi x)\,d\xi =$$
$$= \frac{2\sigma_0(i\lambda+iY_1)^{1/2}}{\mu_0 2\pi i(i\lambda+iX_1)}(1-V^2/c^2)^{-1/2} I,$$

where

$$I = \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{(\xi-iY_2)^{1/2}(\xi-i\beta_1/V)}{(\xi-i\lambda)\xi^2}\exp(-i\xi x)\,d\xi.$$

For $x < 0$, the above integral can be replaced with the integral taken along the positive imaginary ξ-axis and round the branch point at $\xi = iY_2$, together with the contribution from the poles at $\xi = 0$ and $\xi = i\lambda$ as shown in Fig. 2.

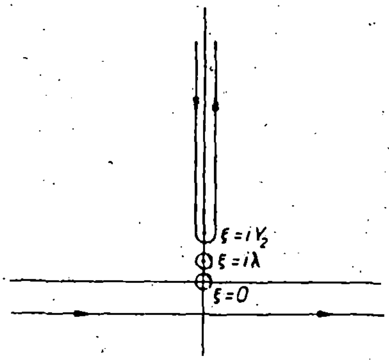

FIG. 2. The path of integration of the integral I.

Thus it can be shown that

$$(3.2)\quad I = \exp\{-(\pi i/4+x_1 Y_2)\}\Big[(2/\lambda)(\beta_1/\lambda V-1)\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{u+Y_2}\,du+ +(2\beta_1/\lambda V)\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{(u+Y_2)^2}\,du+ -(2/\lambda)(\beta_1/\lambda V-1)\int_0^\infty \frac{u^{1/2}\exp(-ux_1)}{u+(Y_2-\lambda)}\,du\Big] +$$

$$+(2\pi/\lambda)[(\beta_1/\lambda V-1)(-iY_2)^{1/2}+(\beta_1/V)\{x(-iY_2)^{1/2}+(i/2)(-iY_2)^{-1/2}\}+ -(\beta_1/\lambda V-1)(i\lambda-iY_2)^{1/2}\exp(-\lambda x_1)] =$$

$$=(1/\lambda)\exp(-\pi i/4)[(\beta_1/\lambda V-1)(x_1)^{-1/2}(x_1 Y_2)^{-1/4}\sqrt{\pi}\,\exp(-{}^1/_2 x_1 Y_2)\times \times W_{-3/4,\,1/4}(x_1 Y_2)+(\beta_1/V)\sqrt{(\pi x_1)}\,(x_1 Y_3)^{-3/4}\exp(-{}^1/_2 x_1 Y_2)\times \times W_{-5/4,\,-1/4}(x_1 Y_2)+(1-\beta_1/\lambda V)(x_1)^{-1/2}(x_1(Y_2-\lambda))^{-1/4}\sqrt{\pi}\,\exp\{-{}^1/_2 x_1(Y_2+\lambda)\}\times \times W_{-3/4,\,1/4}(x_1(Y_2-\lambda))]+(2\pi/\lambda)[(\beta_1/\lambda V-1)(-iY_2)^{1/2}+ +(i\beta_1/V)\{{}^1/_2(-iY_2)^{-1/2}+ix_1(-iY_2)^{1/2}\}+ +(1-\beta_1/\lambda V)(i\lambda-iY_2)^{1/2}\exp(-\lambda x_1)],$$

where $W_{k,m}$ is the Whittaker function [9].

Therefore the displacement jump ΔW across the surface of the crack ($x<0$) is given by

$$(3.3)\qquad \mu_0\Delta W = -\frac{\sigma_0(\lambda+Y_1)^{1/2}(1-V^2/c^2)^{-1/2}}{\pi(\lambda+X_1)}\exp(\pi i/4)\cdot I,$$

where I is given by Eq. (3.2).

Using the result that

$$W_{k,m}(z)=\frac{\Gamma(-2m)}{\Gamma({}^1/_2-m-k)}(z)^{1/2+m}\exp(-z/2)+\frac{\Gamma(2m)}{\Gamma({}^1/^2+m+k)}(z)^{1/2-m}\exp(-z/2)$$

for small z,

we find that for small $x_1 Y_2$, Eq. (3.2) yields

$$I = -4\sqrt{(\pi x_1)}\exp(-\pi i/4), \tag{3.4}$$

Substituting the value of I from Eq. (3.4) into Eq. (3.3) we get

$$\mu_0 \Delta W = 4\sigma_0(1-V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2}\frac{(\lambda+\alpha_1)^{1/2}}{(\lambda+\alpha^2)}, \quad -1 \leqslant x < 0, \tag{3.5}$$

where

$$\alpha_1 = {}^1/_2(1-V^2/c^2)^{-1}\left[(\beta_1 V/c^2)+\sqrt{\{(\beta_1^2 V^2/c^4)+4\beta^2(1-V^2/c^2)\}}\right] \tag{3.6}$$

and

$$\alpha_2 = \beta_1 V(1-V^2/c^2)^{-1}/c^2. \tag{3.7}$$

Again, letting $p \to 0$ and using the Tauberian theorem we find from Eqs. (2.35) and (2.24) that in the steady-state case

$$\bar{\sigma}_{yz} = -\frac{\sigma_0(i\lambda+iY_1)^{1/2}(\xi+iX_1)}{\sqrt{(2\pi)}\, i(\xi-i\lambda)(i\lambda+iX_1)(\xi+iY_1)^{1/2}}. \tag{3.8}$$

Taking the inverse Fourier transform we obtain

$$\sigma_{yz} = -\frac{\sigma_0(i\lambda+iY_1)^{1/2}}{2\pi i(i\lambda+iX_1)}\int_{-\infty-is}^{\infty-is}\frac{(\xi+iX_1)\exp(-i\xi x)}{(\xi-i\lambda)(\xi+iY_1)^{2/1}}d\xi = -\frac{\sigma_0(i\lambda+iY_1)^{1/2}}{2\pi i(i\lambda+iX_1)}I_1, \tag{3.9}$$

where

$$I_1 = \int_{-\infty-is}^{\infty-is}\frac{(\xi+iX_1)\exp(-i\xi x)}{(\xi-i\lambda)(\xi+iY_1)^{1/2}}d\xi = \tag{3.10}$$

$$= 2\sqrt{\pi}\exp(-\pi i/4)\left[(x)^{-1/2}\exp(-Y_1 x)-(\lambda+X_1)(x)^{1/2}(x(\lambda+Y_1))^{-3/4}\right.$$
$$\left.\times\exp\{{}^1/_2 x(\lambda-Y_1)\}W_{-1/4,-1/4}(x(\lambda+Y_1))\right].$$

Thus the stress at $y = 0$ for all x ($x > 0$) is given by Eq. (3.9).

Now for small $(\lambda+Y_1)x$

$$I_1 = 2\sqrt{(\pi/x)}\exp(-\pi i/4), \tag{3.11}$$

so from Eq. (3.9) it follows that

$$\sigma_{yz} = \sigma_0(\pi x)^{-1/2}\frac{(\lambda+\alpha_1)^{1/2}}{(\lambda+\alpha_2)}, \quad (0 < x \leqslant 1), \quad y = 0. \tag{3.12}$$

Stress intensity factor K is given by

$$K = \sqrt{2}\sigma_0 B, \tag{3.13}$$

where

$$B = \frac{(\lambda+\alpha_1)^{1/2}}{(\lambda+\alpha_2)}.$$

Now putting $\beta_1 = 0$ in the expression for α_1 and α_2 we get from Eqs. (3.5) and (3.12) the displacement jump and stress intensity factor in an inhomogeneous elastic medium,

(3.14) $$\mu_0 \Delta W = 4\sigma_0(1-V^2/c^2)^{-1/2}(-x)^{1/2}\pi^{-1/2}\lambda^{-1}\left[\lambda+\beta/\sqrt{(1-V^2/c^2)}\right]^{1/2}$$

and

(3.15) $$\sigma_{yz} = \sigma_0(\pi x)^{-1/2}\lambda^{-1}\left[\lambda+\beta/\sqrt{(1-V^2/c^2)}\right]^{1/2},$$

which agree with the results derived by Atkinson [4]

4. Steady State Solution for Standard Linear Solid

In this case the stress-strain relations are

(4.1) $$\begin{aligned}(\partial\sigma_{YZ}/\partial t)+\beta_1\sigma_{YZ} &= 2\mu(Y)\left[\partial e_{YZ}/\partial t)+\alpha e_{YZ}\right],\\ (\partial\sigma_{XZ}/\partial t)+\beta_1\sigma_{XZ} &= 2\mu(Y)\left[(\partial e_{XZ}/\partial t)+\alpha e_{XZ}\right],\end{aligned}$$

where β_1 and α are constants.

Equation of motion has the form

(4.2) $$(\partial\sigma_{XZ}/\partial X)+(\partial\sigma_{YZ}/\partial Y) = \varrho(Y)(\partial^2 W/\partial t^2).$$

Now, putting $x = X-Vt$ and $y = Y$ so that

$$(\partial/\partial X) = (\partial/\partial x), \quad (\partial/\partial Y) = (\partial/\partial y), \quad \text{and} \quad (\partial/\partial t) = -V(\partial/\partial x).$$

Equations (4.1) and (4.2) become

(4.3) $$\begin{aligned}-V(\partial\sigma_{yz}/\partial x)+\beta_1\sigma_{yz} &= \mu(y)\left[-V(\partial^2 W/\partial x\,\partial y)+\alpha(\partial W/\partial y)\right],\\ -V(\partial\sigma_{xz}/\partial x)+\beta_1\sigma_{xz} &= \mu(y)\left[-V(\partial^2 W/\partial x^2)+\alpha(\partial W/\partial x)\right]\end{aligned}$$

and

(4.4) $$(\partial\sigma_{xz}/\partial x)+(\partial\sigma_{yz}/\partial y) = \varrho(y)V^2(\partial^2 W/\partial x^2).$$

Introducing the Fourier transform denoted by

(4.5) $$\overline{f}(\xi, y) = 1/\sqrt{(2\pi)}\int_{-\infty}^{\infty} f(x,y)\exp(i\xi x)\,dx.$$

Equations (4.3) and (4.4) can be transformed to

(4.6) $$(i\xi V+\beta_1)\overline{\sigma}_{yz} = \mu(y)(i\xi V+\alpha)(d\overline{W}/dy),$$

(4.7) $$(i\xi V+\beta_1)\overline{\sigma}_{xz} = \mu(y)(\xi^2 V-i\xi\alpha)\overline{W}$$

and

(4.8) $$i\xi\overline{\sigma}_{xz} = (d\overline{\sigma}_{yz}/dy) = -\varrho(y)V^2\xi^2\overline{W}.$$

Eliminating $\overline{\sigma}_{yz}$ and $\overline{\sigma}_{xz}$ from Eqs. (4.6), (4.7) and (4.8) we get

(4.9) $$(d^2\overline{W}/dy^2)+2\beta(d\overline{W}/dy)-\gamma^2\overline{W} = 0,$$

where

(4.10) $$\gamma^2 = \xi^2\left[(1-V^2/c^2)\xi+i(V\beta_1/c^2-\alpha/V)\right]/(\xi-i\alpha/V).$$

Branches of γ are chosen so that $\mathrm{Re}(\gamma) > 0$.

Since $\overline{W}$ must remain bounded as $y \to \pm\infty$, solutions of Eq. (4.9) are

$$\text{(4.11)} \qquad \begin{aligned} \overline{W}^{(1)} &= B_1 \exp\left[-\{\beta+\sqrt{(\beta^2+\gamma^2)}\}y\right], \quad y > 0 \\ \overline{W}^{(2)} &= A_2 \exp\left[\{-\beta+\sqrt{(\beta^2+\gamma^2)}\}y\right], \quad y < 0 \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ denote the displacements in the upper and lower half-planes.

Let us consider the case when for $y = 0$

$$\text{(4.12)} \qquad \begin{aligned} W^{(1)} - W^{(2)} &= h(x), \quad x < 0, \\ &= 0, \quad x > 0, \end{aligned}$$

where $h(x)$ is an unknown function such that

$$h(x) \sim 0[\exp(k_1 x)] \quad \text{as} \quad x \to -\infty \quad \text{and} \quad k_1 > 0$$

and

$$\text{(4.13)} \qquad \begin{aligned} \sigma_{yz} &= -\sigma_0 \exp(\lambda x), \quad x < 0, \\ &= e(x), \quad x > 0, \end{aligned}$$

where $e(x)$ is an unknown function satisfying the condition

$$e(x) \sim 0[\exp(-k_2 x)] \quad \text{as} \quad x \to \infty \quad \text{and} \quad k_2 > 0.$$

In this case Eq. (2.26) becomes

$$\text{(4.14)} \qquad -\frac{\mu_0(Vi\xi+\alpha)\gamma^2 H_-(\xi)}{2(Vi\xi+\beta_1)\sqrt{(\beta^2+\gamma^2)}} = E_+(\xi) - \frac{\sigma_0}{\sqrt{(2\pi)}\, i(\xi - i\lambda)}.$$

This equation holds in the region of regularity of the functions appearing in Eq. (4.14). Owing to our former assumptions regarding the behaviour of $e(x)$ and $h(x)$ at infinity, this region is represented by the inequality $-k_2 < \tau < \lambda < k_1$ where $\xi = \sigma + i\tau$.

Now Eq. (4.14) is suitable for the application of the Wiener-Hopf technique. Again,

$$\text{(4.15)} \qquad \gamma^2 = \xi^2(1 - V^2/c^2)(\xi + ia)/(\xi - i\alpha/V),$$

where

$$a = (V\beta_1/c^2 - \alpha/V)/(1 - V^2/c^2)$$

and

$$\text{(4.16)} \qquad \tau^2 + \beta^2 = [\xi^3(1-V^2/c^2) + i(V\beta_1/c_2 - \alpha/V)\xi^2 + \beta^2(\xi - i\alpha/V)]/(\xi - i\alpha/V).$$

Since it is difficult tofactorize $\sqrt{\gamma^2+\beta^2}$, i.e. to represent it as a product of two functions, one analytic in the upper half-plane and the other analytic in the lower half-plane, we follow the approximate method of KOITER [10] of solving Wiener–Hopf type equations. Accordingly, we write $P(\xi) = \sqrt{(\gamma^2+\beta^2)}$ in the form $P(\xi) = \overline{P}(\xi)P_1(\xi)$, where the function $\overline{P}(\xi)$ is required to behave at $|\xi| \to \infty$ and at $|\xi| \to 0$ in the same manner as $P(\xi)$. The auxiliary function $P_1(\xi)$ should be non-zero and should have no singularity within the strip $-k_2 < -\tau_1 < \tau < \tau_2 < \lambda$; it has to be suitably chosen such that $P(\xi)$ is non-zero and possesses no singularity within the strip $-\tau_1 < \tau < \tau_2$. Now we note that

$$P(\xi) = \sqrt{(\tau^2+\beta^2)} \approx [(1 - V^2/c^2)\xi^2 + i(V\beta_1/c^2 - \alpha/V)\xi]^{1/2} \quad \text{as} \quad |\xi| \to \infty$$

and

$$\sqrt{(\tau^2+\beta^2)} \approx \beta \quad \text{as} \quad |\xi| \to 0.$$

Therefore we choose P in the form

$$\bar{P}(\xi) = [(1-V^2/c^2)\xi^2+i(V\beta_1/c^2-\alpha/V)\xi+\beta^2]^{1/2}, \tag{4.17}$$

which behaves in the same manner as $P(\xi)$ for $|\xi| \to \infty$ and $|\xi| \to 0$.

Now $\bar{P}(\xi)$ can be written as

$$\bar{P}(\xi) = (1-V^2/c^2)^{1/2}(\xi-ia_2)^{1/2}(\xi+ia_1)^{1/2}, \tag{4.18}$$

where

$$a_1 = {}^1/_2\left[(V\beta_1/c^2-\alpha/V)+\sqrt{\{(V\beta_1/c^2-\alpha/V)^2+4\alpha\beta^2(1-V^2/c^2)\}}\right]/(1-V^2/c^2) \tag{4.19}$$

and

$$a_2 = {}^1/_2\left[-(V\beta_1/c_2-\alpha/V)+\sqrt{\{/V\beta_1/c^2-\alpha/V)^2+4\beta^2(1-V^2/c^2)\}}\right]/(1-V^2/c^2). \tag{4.20}$$

It consequently follows that the assumptions concerning $P_1(\xi)$ are satisfied, and in view of the fact that $P_1(\xi) \to 1$ in the strip $-\tau_1 < \tau < \tau_2$ for $\xi \to \infty$, the function may be represented in the form

$$P_1(\xi) = P_1^+(\xi)P_1^-(\xi) \tag{4.21}$$

where

$$P_1^+(\xi) = \exp\left[\frac{1}{2\pi i}\int_{-\infty+id_2}^{\infty+id_2}\frac{\ln P_1(\eta)}{\eta-\xi}\,d\eta\right],$$
$$P_1^-(\xi) = \exp\left[-\frac{1}{2\pi i}\int_{-\infty+id_1}^{\infty+id_1}\frac{\ln P_1(\eta)}{\eta-\xi}\,d\eta\right], \tag{4.22}$$

where $-\tau_1 < d_1 < d_2 < \tau_2$ and the functions $P_1^\pm(\xi)$ are regular in the respective half-planes $\tau > -\tau_1$ and $\tau < \tau_2$.

It follows from (4.22) and from the fact $P_1(0) = P_1(\infty) = 1$ that these functions satisfy the additional condition $P_1^\pm(0) = P_1^\pm(\infty) = 1$ with the help of (4.15), (4.18), (4.21) and the relation $P(\xi) = \bar{P}(\xi)P_1(\xi)$, Eq. (4.14) becomes

$$-\frac{\mu_0(1-V^2/c^2)^{1/2}H_-(\xi)\xi^2}{2(\xi-i\beta_1/V)(\xi-ia_2)^{1/2}P_1^-(\xi)}+\frac{\sigma_0P_2^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\;i(\xi-i\lambda)(i\lambda+ia)} = \tag{4.23}$$
$$= \frac{P_1^+(\xi)(\xi+ia_1)^{1/2}E_+(\xi)}{(\xi+ia)}-\frac{\sigma_0}{\sqrt{(2\pi)}\;i(\xi-i\lambda)}\times$$
$$\times\left[\frac{P_1^+(\xi)(\xi+ia_1)^{1/2}}{(\xi+ia)}-\frac{P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{(i\lambda+ia)}\right].$$

Using the same arguments as in Eq. (2.33) we get

$$(4.24)\quad H_-(\xi) = -\frac{2\sigma_0(1-V^2/c^2)^{-1/2}P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}(\xi-ia_2)^{1/2}}{\sqrt{(2\pi)}\ \mu_0(\xi-i\lambda)(\lambda+a)\xi}P_1^-(\xi)(\xi-i\beta_1/V) =$$

$$= -\frac{2\sigma_0(1-V^2/c^2)^{-1/2}P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\ \mu_0(\lambda+a)}\xi^{-3/2} \quad \text{as} \quad \xi\to\infty$$

and

$$(4.25)\quad E_+(\xi) = \frac{\sigma_0}{\sqrt{(2\pi)}\ i(\xi-i\lambda)} - \frac{\sigma_0 P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}(\xi+ia)}{\sqrt{(2\pi)}\ i(\xi-i\lambda)(i\lambda+ia)P_1^+(\xi)(\xi+ia_1)^{1/2}} =$$

$$= -\frac{\sigma_0 P_1^+(i\lambda)(i\lambda+ia_1)^{1/2}}{\sqrt{(2\pi)}\ i(i\lambda+ia)}\xi^{-1/2} \quad \text{as} \quad \xi\to\infty.$$

Now taking the inverse Fourier transform we get from Eqs. (4.24) and (4.25)

$$(4.26)\quad h(x) = \frac{4\sigma_0(1-V^2/c^2)^{-1/2}P_1^+(i\lambda)(\lambda+a_1)^{1/2}}{\mu_0(\lambda+a)\sqrt{\pi}}(-x)^{1/2}, \quad -1\leqslant x<0$$

and

$$(4.27)\quad e(x) = \frac{\sigma_0\pi^{-1/2}P_1^+(i\lambda)(\lambda+a_1)^{1/2}}{(\lambda+a)}(x)^{-1/2}, \quad 0<x\leqslant 1.$$

The corresponding results for the case of constant loading $\sigma_{yz} = -\sigma_0$ $(x<0)$ on the crack surface are obtained by putting $\lambda = 0$ in the above equation. If ΔW is the displacement jump then the crack opening displacement in this case is given by

$$(4.28)\quad \mu_0\Delta W = [4\sigma_0(1-V^2/c^2)^{-1/2}\pi^{-1/2}(a_1)^{1/2}(-x)^{1/2}]/a, \quad -1\leqslant x<0$$

and also the stress near the crack tip is

$$(4.29)\quad \sigma_{yz} = [\sigma_0\pi^{-1/2}(a_1)^{1/2}(x)^{-1/2}]/a, \quad 0<x\leqslant 1 \quad (\text{since} \quad P_1^+(0)=1).$$

Therefore the stress intensity factor is equal to

$$(4.30)\quad K = \sqrt{2}\sigma_0 B_1 \quad \text{where} \quad B_1 = (a_1)^{1/2}/a.$$

Now putting $\alpha = 0$ in Eqs. (4.28) and (4.30) we get the crack opening displacement and stress intensity factor for the Maxwell Solid

$$(4.31)\quad \mu_0\Delta W = [4\sigma_0(1-V^2/c^2)^{-1/2}\pi^{-1/2}(\alpha_1)^{1/2}(-x)^{1/2}]/\alpha_2$$

and

$$(4.32)\quad K = \sqrt{2}\sigma_0 B \quad \text{where} \quad B = (\alpha_1)^{1/2}/\alpha_2$$

which agree with the results given by (3.12) and (3.13) in the Maxwell Solid corresponding to $\lambda = 0$.

5. Results and Discussion

5.1. The Maxwell solid. In this case time variation of the stress intensity factor is given by $K = \sqrt{2}\sigma_0 A$ where A is given by Eq. (2.40) and has been evaluated in the Appendix.

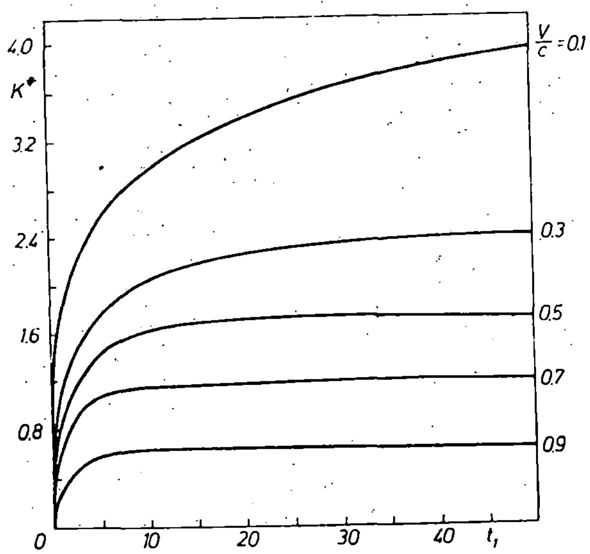

FIG. 3. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0$ (homogeneous medium).

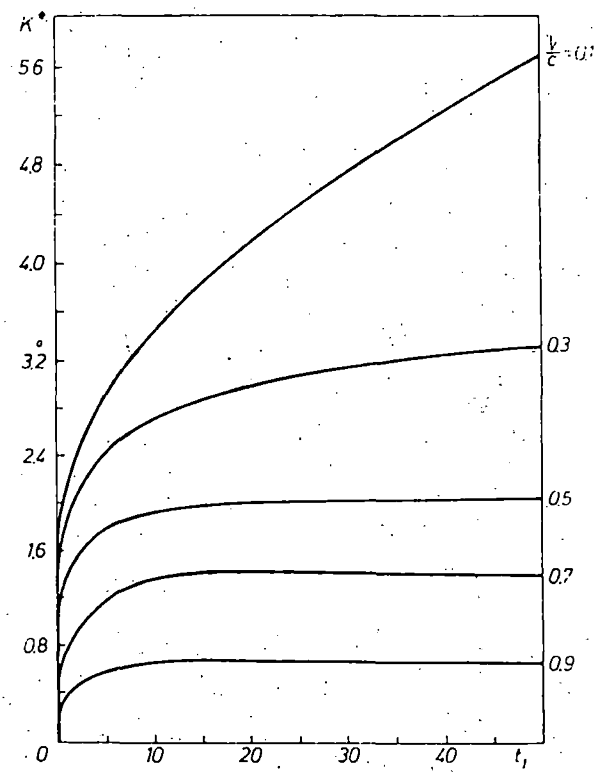

FIG. 4. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0.1$.

The dimensionless stress intensity factor $K^* = (K/\sigma_0)(\beta_1/c)^{1/2}$ has been plotted against $t_1 = \beta_1 t$ for the range of values of $V/c = 0.1, 0.3, 0.5, 0.7$ and 0.9 for different values of the inhomogeneity factor $\beta^* = 4\beta^2c^2/\beta_1^2$.

It is interesting to note by inspecting the graphs given in Fig. 3, Fig. 4 and Fig. 5 that

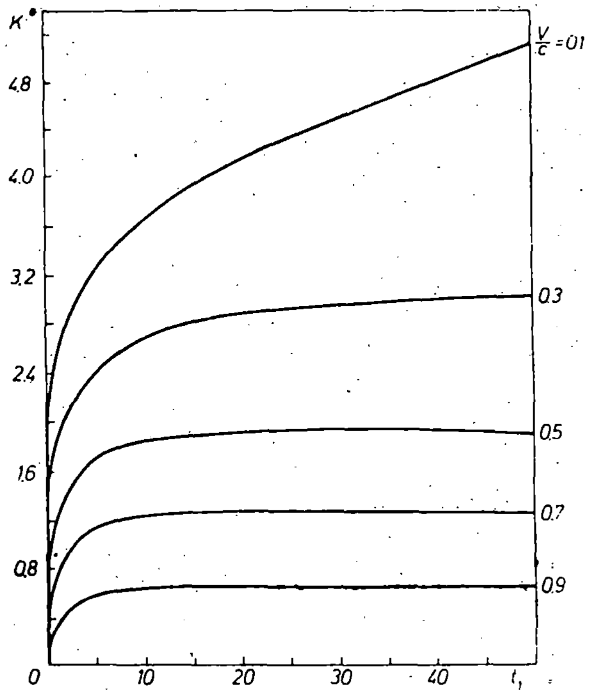

FIG. 5. K^* vs. t_1 for the Maxwell solid in non-steady state case. $\beta^* = 0.2$.

the effect of inhomogeneity of the medium introduced through the factor β^* in the stress intensity factor K^* becomes more significant for small values of V/c, whereas for values of V/c differing slightly from unity, the effect of inhomogeneity of the medium on the stress intensity factor is negligible

5.2. Standard linear solid. In this case the stress intensity factor for the steadily propagating crack is given by $K = \sqrt{2}\,\sigma_0 B_1$, where B_1 is given by Eq. (4.30).

We have plotted also the stress intensity factor $K^* = (K/\sigma_0)(\beta_1/c)^{1/2}$ against β^* for various values of V/c, $V/c = 0.5, 0.6, 0.7, 0.8$ and 0.9, and for different values of $\alpha/\beta_1 = 0$, 0.1, 0.2. The case $\alpha/\beta_1 = 0$ corresponds to the steady-state values of K^* for the Maxwell solid. It is evident from the graphs given in Fig. 6. Fig. 7 and Fig. 8 that at large values of α/β_1, values of K^* increase rapidly with the increase in values of β^* if V_0 V/c is very small. But for values of V/c close to unity the variation of K^* with the change in the value of β^* is small showing that the inhomogeneity effect is negligibly in this case. This is also evident from the expressions (4.31) and (4.19).

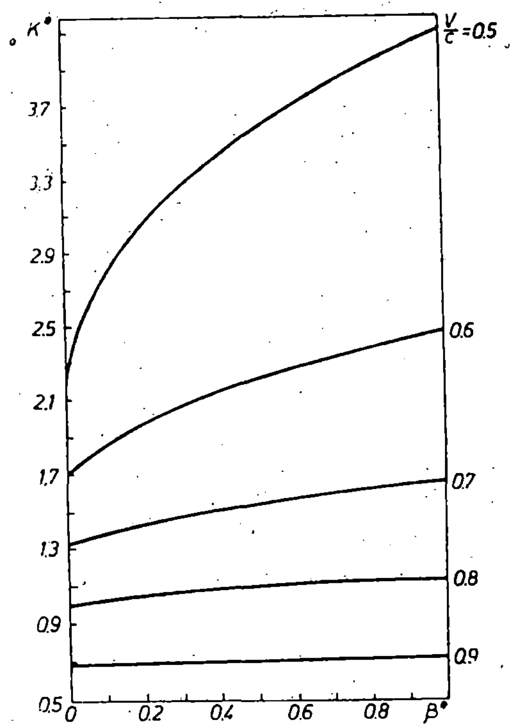

FIG. 6. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0$ (Maxwell solid).

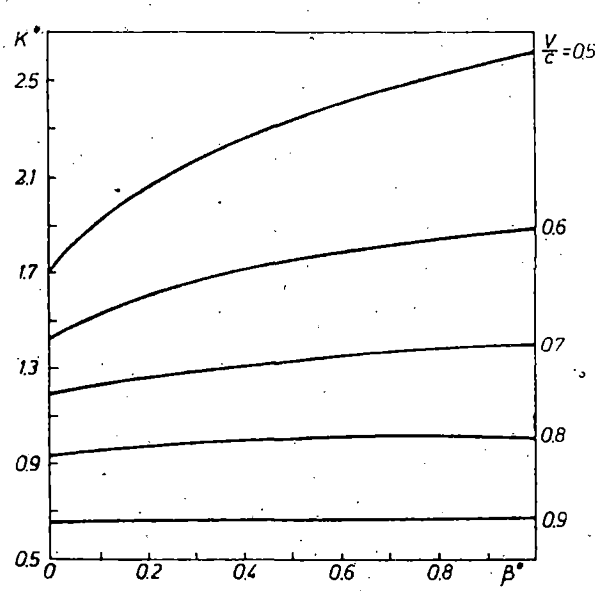

FIG. 7. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0.1$.

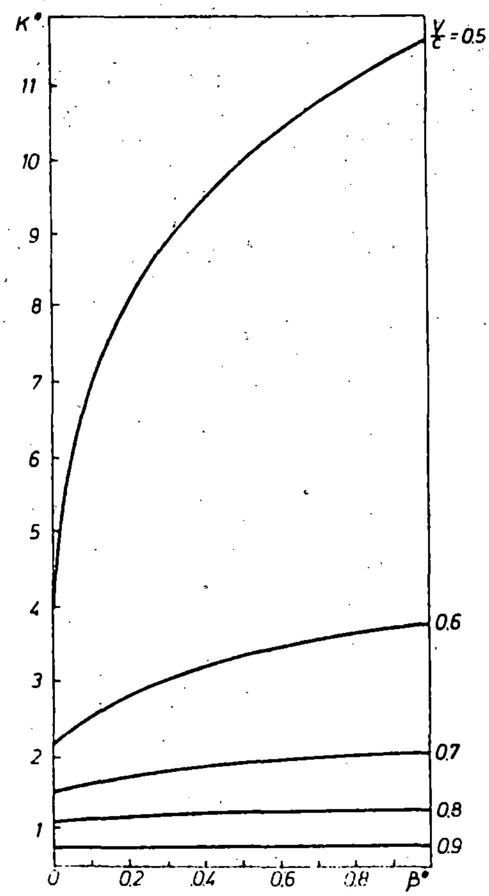

FIG. 8. K^* vs. β^* for the standard linear solid in steady state case. $\alpha/\beta_1 = 0.2$.

Appendix. Evaluation of the Integral A in Eq. (2.40).

The integral

$$A = (1/2\pi i) \int_{c'-i\infty}^{c'+i\infty} \frac{(Y_1)^{1/2}}{pX_1} \exp(pt)dp.$$

The integrand has poles at $p = 0$ and also at $p = -\beta_1$ which correspond to the zero of X_1. Further the integrand has branch points at

$$\delta_1 = (\beta_1/2)\left[-1+\sqrt{\{(1-V^2/c^2)(1-4z)\}}\right],$$
$$\delta_2 = (\beta_1/2)\left[-1-\sqrt{\{(1-V^2/c^2)(1-4z)\}}\right],$$
$$\delta_3 = (\beta_1/2)\left[-1-\sqrt{(1-4z)}\right],$$
$$\delta_4 = (\beta_1/2)\left[-1+\sqrt{(1-V^2/c^2)}\right],$$
$$\delta_5 = (\beta_1/2)\left[-1-\sqrt{(1-V^2(c^2)}\right],$$

where $z = \beta^2c^2/\beta_1^2$ which is assumed to be less than 1/4.

Evidently, $\delta_4 > \delta_1 > \delta_2 > \delta_5 > \delta_3$.

Now taking the branch cut along the negative real axis from δ_4 to $-\infty$, the integral can be considered as a contour integral around the path shown in Fig. 9.

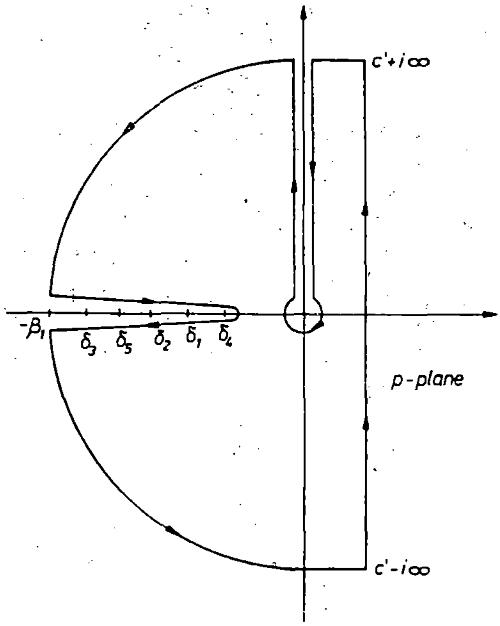

FIG. 9. The integration contour to evaluate A for the Maxwell solid.

Now

$$A = (1/2\pi i)\sqrt{2}\,(1-V^2/c^2)^{1/2} \times$$

$$\times \int_{c'-i\infty}^{c'+i\infty} \frac{\left[(2p+\beta_1)V/c^2+(2/c)\sqrt{\{(p-\delta_1)(p-\delta_2)\}}\right]^{1/2}}{p\left[(2p+\beta_1)V/c^2+(2/c)\sqrt{\{(p-\delta_4)(p-\delta_5)\}}\right]}\exp(pt)\,dp.$$

It can be shown that

$$A = \sqrt{2}\,(1-V^2/c^2)^{1/2}\left[{}^1/_2(c/V)\sqrt{(c/\beta_1)}\,\sqrt{\{V/c+\sqrt{(V^2/c^2+4z(1-V^2/c^2))}\}}\right]+$$

$$+\sqrt{2}\,(1-V^2/c^2)^{1/2}\left\{\sqrt{(c/\beta_1)}/\pi\right\}[I_1+I_2-I_3-I_4],$$

where

$$I_1 = \int_0^{b_1} \frac{\sqrt{(x_1^{**})}\;y_1^*}{(\delta_4^*-r)R_1^*}\exp[(\delta_4^*-r)t_1]\,dr,$$

$$I_2 = \int_0^{b_2} \frac{\sqrt{[\{R_2^{**}-(x_2^*)^3+(y_2^{**})^2x_2^*-2x_2^*y_2^*y_2^{**}\}/2]}}{(\delta_1^*-r)R_2^*}\exp[(\delta_1^*-r)t_1]\,dr,$$

$$I_3 = \int_{b_2}^{b_3} \frac{\sqrt{(x_3^{**})}\; x^*}{(\delta_1^* - r) R_3^*} \exp[(\delta_1^* - r) t_1)] dr,$$

$$I_4 = \int_{b_3}^{\infty} \frac{\sqrt{(x_3^{**})}}{(\delta_1^* - r) x_4^*} \exp[(\delta_1^* - r) t_1] dr,$$

where

$$\delta_1 = \beta_1 \delta_1^*, \quad \delta_2 = \beta_1 \delta_2^*, \quad \delta_3 = \beta_1 \delta_3^*, \quad \delta_4 = \beta_1 \delta_1^*, \quad \delta_5 = \beta_1 \delta_5^*, \quad t_1 = \beta_1 t,$$

$$b_1 = {}^1/_2 \sqrt{(1 - V^2/c^2)}\, [1 - \sqrt{(1-4z)}], \quad b_2 = \sqrt{[(1 - V^2/c^2)(1-4z)]},$$

$$b_3 = {}^1/_2 \sqrt{(1 - V^2/c^2)}\, [1 + \sqrt{(1-4z)}],$$

$$x_1^{**} = [\sqrt{(1 - V^2/c^2)} - 2r](V/c) + 2\sqrt{[r^2 - r\sqrt{(1 - V^2/c^2)} + (1 - V^2/c^2) z]},$$

$$x_1^* = [\sqrt{[(1 - V^2/c^2)} - 2r](V/c),$$

$$y_1^* = 2\sqrt{[r\sqrt{(1 - V^2/c^2)} - r^2]},$$

$$R_3^* = (x_1^*)^2 + (y_1^*)^2,$$

$$x_2^* = [\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - 2r](V/c),$$

$$y_2^* = 2\sqrt{[r\sqrt{(1 - V^2/c^2)(1-4z)} - r^2]},$$

$$y_2^{**} = 2\sqrt{[-r^2 + r\sqrt{\{(1 - V^2/c^2)(1-4z)\}} + z(1 - V^2/c^2)]},$$

$$R_2^{**} = [(x_2^*)^2 + (y_2^{**})^2][(x_2^*)^2 + (y_2^*)^2]^{1/2},$$

$$R_2^* = (x_2^*)^2 + (y_2^{**})^2,$$

$$x_3^{**} = -[\sqrt{[\{(1 - V^2/c^2)(1-4z)\}} - 2r](V/c) + 2\sqrt{[r^2 - r\sqrt{\{(1 - V^2/c^2)(1-4z)\}}]},$$

$$x_3^* = [\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - 2r](V/c),$$

$$y_3^* = 2\sqrt{[z(1 - V^2(c^2) + r\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - r^2]},$$

$$R_3^* = (x_3^*)^2 + (y_3^*)^2,$$

$$x_4^* = [\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - 2r](V/c) - 2\sqrt{[r^2 - r\sqrt{\{(1 - V^2/c^2)(1-4z)\}} - z(1 - V^2/c^2)]}.$$

References

1. B. R. BAKER, *Dynamic stress created by a moving crack*, J. Appl. Mech., **15**, 449–459, 1962.
2. G. P. CEREPANOV and E. F. AFANASEV, *Some dynamic problems of the theory of elasticity—a review*, Int. J. Engng. Sci., **12**, 665 - 690, 1974.
3. C. ATKINSON and R. D. LIST, *Steady state crack propagation into media with spatially varying elastic properties*, Int. J. Engng. Sci., **16**, 717 - 730, 1978.
4. C. ATKINSON, *Cracks in media with spatially varying elastic moduli*, Mechanics of Fracture, Vol-4, [Ed.] G. C. SIH, Noordhoff International Publishing, Leyden 1977.
5. J. R. WILLIS, *Crack propagation in a viscoelastic media*, J. Mech. Phys. Solids, **15**, 229–240, 1972.

6. C. Atkinson and R. D. List, *A moving crack problem in a viscoelastic solid*, Int. J. Engng. Sci., **10**, 309-322, 1972.
7. O. Coussy, *A moving crack problem along the interface of two viscoelastic media*, Int. J. Engng. Sci., **25**, 609–620, 1987.
8. L. B. Sills and Y. Benveniste, *Steady state propagation of mode III interface crack between dissimilar viscoelastic media*, Int. J. Engng. Sci., 19, 1255-1268, 1981.
9. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge University Press, London 1969.
10. W. T. Koiter, *Approximate solutions of Wiener-Hopf type equations with applications*, Proc. Kon. Ned. Acad. Wet., **57**, 2, 1954.

Streszczenie

DYNAMICZNA PROPAGACJA ANTYPŁASKIEJ SZCZELINY W NIEJEDNORODNYM OŚRODKU LEPKOSPRĘŻYSTYM

Rozważono model półnieskończonej szczeliny w antypłaskim stanie odkształcenia, która w chwili $t = 0$ zaczyna propagować się ze stałą prędkością w niejednorodnym, liniowo-lepkosprężystym ośrodku. Rozważono makswellowski i standardowy model ciała lepkosprężystego oraz przyjęto, że stałe materiałowe zmieniają się wykładniczo w kierunku prostopadłym do kierunku propagacji. Zastosowanie transformacji całkowych pozwala zredukować zagadnienie do rozwiązania problemu Wienera-Hopfa. Zmienność czasowa współczynnika intensywności naprężenia jest funkcją prędkości propagacji szczeliny i stałych materiałowych ośrodka. Zmienność tę zilustrowano wykresami.

Резюме

ДИНАМИЧЕСКОЕ РАСПРОСТРАНЕНИЕ АНТИПЛОСКОЙ ТРЕЩИНЫ В НЕОДНОРОДНОЙ ВЯЗКОУПРУГОЙ СРЕДЕ

Рассмотрена модель полубесконечной трещины в антиплоском деформационном состоянии, которая в момент $t = 0$ начинает распросраняться с постоянной скоростью в неоднородной, линейно-вязкоупругой среде. Рассмотрены максвелловская и стендартная модели вязкоупругого тела и принято, что материальные постоянные изменяются экспоненциальным образом в перпендикулярном направлении к направлении распространения. Применение интегральных преобразований позволяет свести задачу к решению задачи Випера-Хопфа. Временная переменность коэффициента интенсивности напряжения является функцией скорости распространения трещины и материальных постоянных среды. Эта переменность иллюстрируется диаграммами.

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