CHAPTER II

SECTION ONE : PROPAGATION OF ELECTROMAGNETIC WAVES IN PLASMA FILLED ELLIPTICAL WAVEGUIDES.

SECTION TWO : SPACE-CHARGE WAVES IN PLASMA FILLED ELLIPTIC WAVEGUIDES IN PRESENCE OF MAGNETIC FIELD.

SECTION THREE: PROPAGATION OF ELECTROMAGNETIC WAVES IN WARM PLASMA FILLED ELLIPTIC WAVEGUIDES.

Propagation of electromagnetic waves in hollow conducting cylinder of elliptical cross-section was presented by Chu (1938), Kinzer and Wilson (1947), Velenzuela (1960), Piefke (1964) and others. Problems relating to plasma filled cylindrical waveguide of elliptical cross-section was discussed by Gajewski (1972), Van Den Berg et. al (1973), Strauss (1974), Dwar et. al (1974) and others.

The main advantages of elliptical waveguides are that long continuous lines are easily manufactured and transported. Furthermore, there is no mode splitting or rotation of the polarization plane for slight deformations of the cross-section while simple matched connecting parts of rectangular and circular waveguides are possible. Although, elliptical waveguides are commercially available and have already been used in several applications such as multichannel communication and rader feed lines there still remain a lot of unresolved problems in this domain.

In this chapter we consider the propagation of waves in plasma filled perfectly conducting cylinder of elliptical cross-section. This chapter consists of three section. In the first section we discuss the propagation of electromagnetic waves in cold plasma filled cylindrical waveguide of elliptical cross-section. Mathieu functions and modified Mathieu functions are used to derive the dispersion relation. The time rate of energy flow through the waveguide is calculated as integral of Mathieu functions. Lastly, wave propagation through cold plasma filled cylinder of circular cross-section is derived as a limiting process. In the second section attemps are made to discuss the nature of plasma oscillation or space charge waves in an elliptic plasma column in the presence of finite and infinite axial magnetic field. The dispersion relations for finite and infinite magnetic field are derived and shown graphically. Lastly, the dispersion relation for circular waveguide is derived as a limiting process and it is interesting to note that these dispersion relation are same as was obtained by Trivelpiece and Gould (1959). The third section comprises the propagation of electromagnetic waves in warm plasma filled cylindrical waveguide of elliptical cross-section. To consider the temperature effect we use the compressible plasma theory. The dispersion relations for TE- modes and TM- modes are derived and shown graphically. The time rate of energy flow through the warm plasma is derived as integral of Mathieu functions. With the usual limiting process the dispersion relations for propagation of TE- and TM- modes in circular waveguide have been deduced and these are found to agree with those obtained by Azakami et. al, (1972).

SECTION ONE

*PROPAGATION OF ELECTROMAGNETIC WAVES IN PLASMA FILLED ELLIPTIC WAVEGUIDES.

2.1.1. INTRODUCTION:

Wave propagation in plasma-filled rectanguler and circular waveguides was investigated by many authors such as Stix (1957), Korper (1957), Dawson and Oberman (1959), Willett (1963), Trivelpieceand Gould (1959) etc. In this section we discuss the problem of wave propagation in a plasma-filled perfectly conducting elliptical waveguide. To solve the above problem we use cold plasma theory. Mathieu functions and modified Mathieu functions are used to calculate the dispersion relations.

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2.1.2. BASIC EQUATIONS:

To study the problem we assume that the density of the plasma is finite, zero pressure corresponding to zero temperature and immersed in a uniform axial constant external magnetic field $\vec{B}_o$. We also assume that the effects of resistivity, viscosity, gravity and electron inertia are negligible. On the above assumptions the Magnetohydrodynamic equations takes the following form:

$$\rho \frac{\partial \vec{v}}{\partial t} = \vec{j} \times \vec{B}_{o/c} , \tag{2.1}$$

$$\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}_o = 0 , \tag{2.2}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} , \tag{2.3}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} , \tag{2.4}$$

$$\nabla \cdot \vec{B} = 0 , \tag{2.5}$$

$$\nabla \cdot \vec{E} = 0 , \tag{2.6}$$

where ρ, $\vec{v}$, $\vec{j}$, c, $\vec{E}$, $\vec{B}$ are respectively the unperturbed plasma density, plasma velocity, current density, velocity of light in vacuum, electric field and magnetic field.

Equation (2.3) can be put in the following form by using equations (2.1) and (2.2)

$$[(\nabla \times \vec{B}) \times \vec{B}_o] \times \vec{B}_o = -4\pi\rho c \frac{\partial \vec{E}}{\partial t} + \frac{1}{c}\left(\frac{\partial \vec{E}}{\partial t} \times \vec{B}_o\right) \times \vec{B}_o \qquad (2.7)$$

Let us take the time dependence of the field quantities in the form $e^{-i\omega t}$ and $\vec{B}_o$ in the direction of z- axis i.e, $\vec{B}_o = \hat{z} B_o$, where $\hat{z}$ is a unit vector along positive z-axis. Then equation (2.4) and (2.7) takes the form:

$$\nabla \times \vec{E} = \frac{i\omega}{c} \vec{B}, \qquad (2.8)$$

and

$$[(\nabla \times \vec{B}) \times \hat{z}] \times \hat{z} = \frac{4\pi i\omega\rho c}{B_o^2} \vec{E} - \frac{i\omega}{c} (\vec{E} \times \hat{z}) \times \hat{z} . \qquad (2.9)$$

We introduce elliptic coordinates (μ, θ, z) defined by $x = a \cosh \mu \cos\theta$, $y = a \sinh \mu \sin\theta$, $z = z$, where $a = (A^2 - B^2)^{1/2}$ and A, B are the semi-major and semi-minor axes of the boundary of the elliptic cylinder Fig. (2.1.). Let us take the z-dependence of the field quantities in the form e^{ikz}. Then, the components of the equations (2.8) and (2.9) are as follows:

$$-\frac{i\omega}{c}\left(1 + \frac{4\pi\rho c^2}{B_o^2} - \frac{k^2c^2}{\omega^2}\right) E_\mu = \frac{1}{a(\sinh^2\mu + \sin^2\theta)^{1/2}} \frac{\partial B_z}{\partial \theta}, \qquad (2.10)$$

$$\frac{i\omega}{c}\left(1 + \frac{4\pi\rho c^2}{B_o^2} - \frac{k^2c^2}{\omega^2}\right) E_\theta = \frac{1}{a(\sinh^2\mu + \sin^2\theta)^{1/2}} \frac{\partial B_z}{\partial \mu}, \qquad (2.11)$$

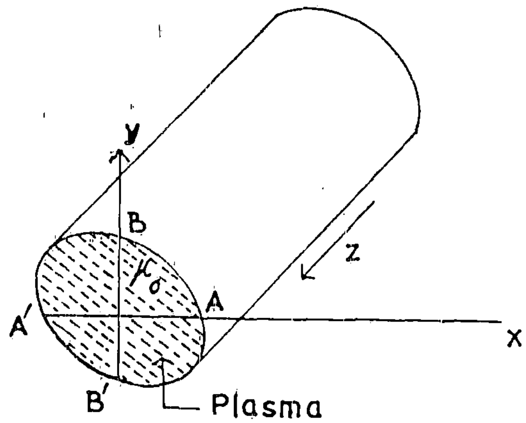

Figure 2.1. Plasma filled elliptical waveguide.

$$E_z = 0 , \tag{2.12}$$

$$B_\mu = -\frac{kc}{\omega} E_\theta , \tag{2.13}$$

$$B_\theta = \frac{kc}{\omega} E_\mu , \tag{2.14}$$

$$B_z = \frac{c}{i\omega} \frac{1}{a^2(\sinh^2\mu+\sin^2\theta)} \left[\frac{\partial}{\partial\mu} \left(a\sqrt{\sinh^2\mu+\sin^2\theta}\; E_\theta\right) - \frac{\partial}{\partial\theta} \left(a\sqrt{\sinh^2\mu+\sin^2\theta}\; E_\mu\right)\right] \tag{2.15}$$

Now, from equations (2.10), (2.11) and (2.15), we have

$$\frac{\partial^2 B_z}{\partial\mu^2} + \frac{\partial^2 B_z}{\partial\theta^2} + 2p(\cosh 2\mu - \cos 2\theta)\, B_z = 0, \tag{2.16}$$

where

$$4p = a^2 \left(\frac{\omega^2}{c^2} - k^2 + \frac{4\pi\rho\omega^2}{B_o^2} \right). \tag{2.17}$$

2.1.3 THE SOLUTION AND DISPERSION RELATION:

The general solution of equation (2.16) is (MacLachlan, p.347)

$$B_z = \sum_{m=o}^{\infty} C_m \, Ce_m(\mu, p)\, ce_m(\theta, p) + \sum_{m=1}^{\infty} S_m \, Se_m(\mu, p)\, se_m(\theta, p) \tag{2.18}$$

where C_m, S_m are constants, $ce_m(\theta, p)$, $se_m(\theta, p)$ are even and odd Mathieu functions while $Ce_m(\mu, p)$ and $Se_m(\mu, p)$ are the corresponding modified Mathieu functions. The other field components are easily obtained by applying equations (2.10) - (2.14). Each individual solution in (2.18), for $m = 0, 1, 2 \ldots$ corresponds to a different mode of propagation.

For any m we consider two types of solutions,

$$B_{zmc} = C_m \, Ce_m(\mu, p) \, ce_m(\theta, p), \quad \text{(even solution)} \qquad (2.19)$$

and

$$B_{zms} = S_m \, Se_m(\mu, p) \, se_m(\theta, p), \quad \text{(odd solution)} \qquad (2.20)$$

a) Firstly, we determine all the field components and the dispersion relation for the even solution given in (2.19).

From equations (2.10) - (2.14), the field components in the elliptic coordinate system are

$$E_\mu = \frac{\omega}{kc} B_\theta = \frac{ia\omega}{4pc} \cdot \frac{C_m \, Ce_m(\mu, p) \, ce'_m(\theta, p)}{(\sinh^2\mu + \sin^2\theta)^{1/2}}, \qquad (2.21)$$

$$E_\theta = -\frac{\omega}{kc} B_\mu = -\frac{ia\omega}{4pc} \cdot \frac{C_m \, Ce'_m(\mu, p) \, ce_m(\theta, p)}{(\sinh^2\mu + \sin^2\theta)^{1/2}} \qquad (2.22)$$

$$E_z = 0 \qquad (2.23)$$

$$B_z = C_m \, Ce_m(\mu, p) \, ce_m(\theta, p). \qquad (2.24)$$

The prime denotes the derivatives with respect to μ and θ as the case may be. The boundary condition at the perfectly conducting waveguide wall requires that the tangential components of the electric field vanishes at the wall i.e., $E_\theta(\mu = \mu_o) = 0$. From equation (2.22) we have

$$Ce_m'(\mu_o, p) = 0 . \tag{2.25}$$

Now μ_o is fixed, so we need those positive values of p, say $p_{m,\nu}$ for which $Ce_m'(\mu_o, p)$ vanishes. These may be regarded as the parametric zeros of the function. These roots are used in equation (2.17) to get the dispersion relation

$$(ka)^2 = \frac{a^2\omega^2}{c^2}\left(1 + \frac{4\pi\rho c^2}{B_o^2}\right) - 4p_{m,\nu} \tag{2.26}$$

The phase characteristic frequency (ω) vs. wavenumber (k) are shown graphically in curve I (Fig.No.2.2) for the m=1 mode. The lines of force of electric and magnetic vectors of a cross-section of an elliptical waveguide are found to be same as those obtained by Chu (1938).

b) In case of the odd solution given in (2.20) the field components for the mth mode in elliptic coordinates are

$$E_{\mu} = \frac{\omega}{kc} B_{\theta} = \frac{ia\omega}{4pc} \cdot \frac{S_m \, Se_m(\mu, p) \, se_m'(\theta, p)}{(\sinh^2\mu + \sin^2\theta)^{1/2}} \tag{2.27}$$

$$E_{\theta} = -\frac{\omega}{kc} B_{\mu} = -\frac{ia\omega}{4pc} \cdot \frac{S_m \, Se_m'(\mu, p) \, se_m(\theta, p)}{(\sinh^2\mu + \sin^2\theta)^{1/2}} \tag{2.28}$$

$$E_z = 0 \tag{2.29}$$

$$B_z = S_m \, Se_m(\mu, p) \, se_m(\theta, p) \tag{2.30}$$

The boundary condition of the perfectly conducting waveguide wall requires that $E_{\theta}(\mu = \mu_o) = 0$. Thus, from equation (2.28) we have

$$Se_m'(\mu_o, p) = 0 \tag{2.31}$$

The positive parametric zeros of equation (2.31) (say $\bar{p}_{m,\nu}$) together with equation (2.17) gives the dispersion relation

$$(ka)^2 = \frac{a^2\omega^2}{c^2}\left(1 + \frac{4\pi\rho c^2}{B_o^2}\right) - 4\,\bar{p}_{m,\nu} \tag{2.32}$$

The phase characteristic frequency (ω) Vs. wavenumber (k) are shown graphically in curve II (Fig. No. 2.2.) for the m=1 mode. In this case also the lines of force of the electric and magnetic vectors of a cross-section of an elliptical waveguide are same as those of Chu (1938).

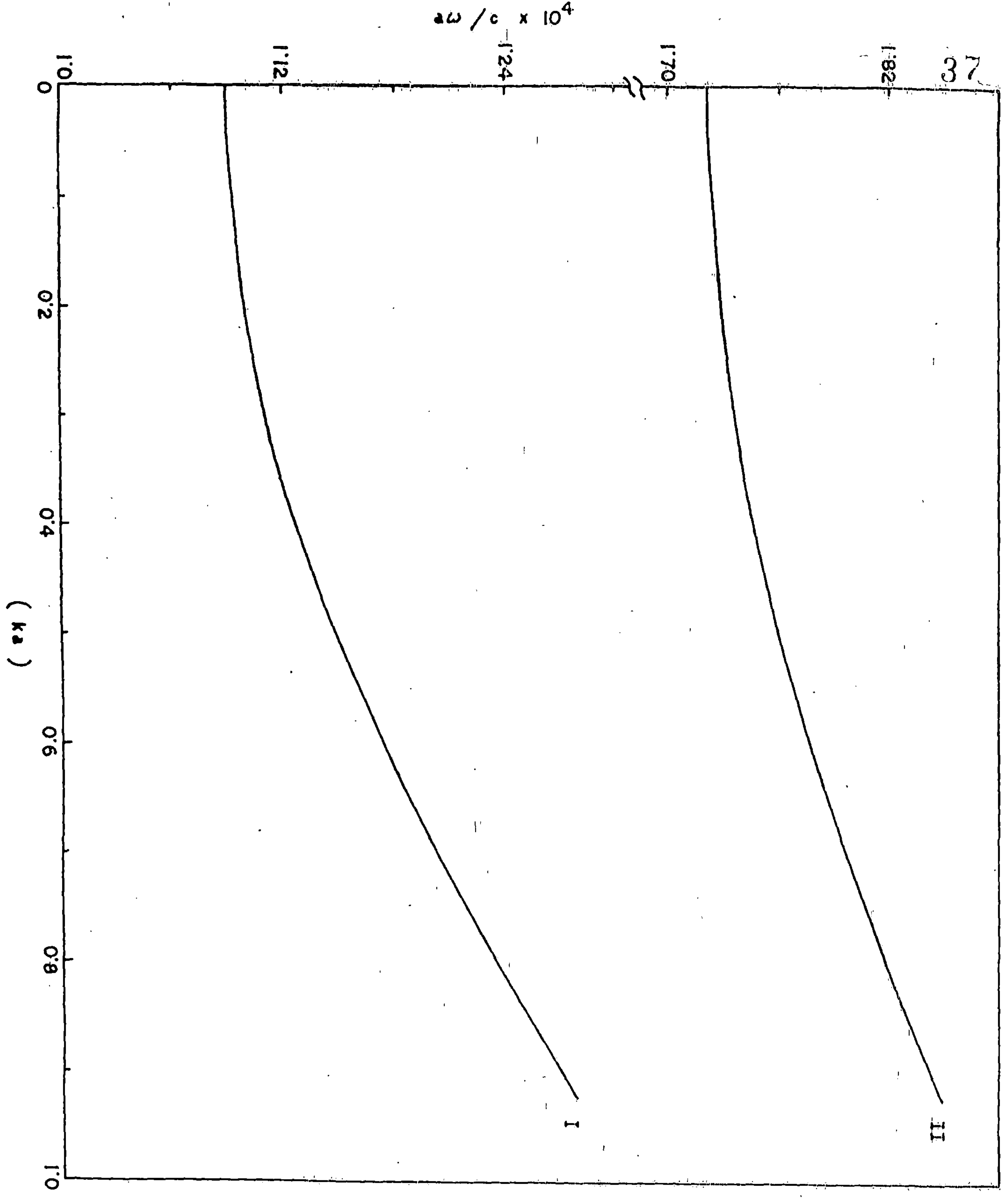

Figure 2.2. Frequency Vs. wavenumber diagram for $m = 1$ even modes.

(Curve I, $p_{1,1} = 0.56$, $\rho = 1.67 \times 10^{-12}$ gm/c.c., $B_o = 10$ Gauss.)

(Curve II, $p_{1,1} = 1.4$, $\rho = 1.67 \times 10^{-12}$ gm/c.c., $B_o = 10$ Gauss.)

The propagation of waves without the components of E_μ and B_θ are generally impossible in elliptical waveguides. Because their non existance demands the independence of B_z from θ, as may be seen by putting $E_\mu = 0$ in equation (2.10) and $B_\theta = 0$ in equation (2.14). But B_z cannot be independent of θ, since equation (2.16) contains a term in $\cos 2\theta$. In case of circular cylindrical waveguide the waves can propagate in absence of the components E_μ and B_θ, because in this case $\cos 2\theta$ is negligible compaired to $\cosh 2\mu$ in equation (2.16).

2.1.4. POWER TRANSMITTED IN THE DIRECTION OF WAVE PROPAGATION.

The power passing through each unit area of the cross-section in the direction of wave propagation, i.e., the time rate of energy flow per unit area, may be calculated with the aid of the complex poynting vector $\vec{S}$, Ferraro (1970).

The total mean power for the mth mode can be obtained by integrating

$$S_z = \frac{C}{8\pi} (E_\mu \bar{H}_\theta - E_\theta \bar{H}_\mu) \qquad (2.33)$$

over the cross-section of the cylinder, where $\bar{H}_\theta$ and $\bar{H}_\mu$ denotes the complex canjugate of the components H_θ and H_μ respectively.

The total mean power for the even solution can be obtained by substituting the values of E_μ, E_θ, $\bar{H}_\mu$, $\bar{H}_\theta$ from equations (2.21) and (2.22) into equation (2.33):

$$W_e = \frac{k\omega}{8\pi\mu_o} \frac{|C_m|^2}{\left(\frac{\omega^2}{c^2} - k^2 + \frac{4\pi\rho\omega^2}{B_o^2}\right)^2} \int_0^{\mu_o} \int_0^{2\pi} [\, Ce_m^2(\mu,p)\; ce_m'^2(\theta,p) + Ce_m'^2(\mu,p)\; ce_m^2(\theta,p)]\,d\mu\, d\theta\,. \tag{2.34}$$

On using the relations (McLachlun , p. 24, 264)

$$\int_0^{2\pi} ce_m^2(\theta,p)\, d\theta = \pi$$

$$\int_0^{2\pi} ce_m'^2(\theta,p)\, d\theta = \pi\, \vartheta_m$$

the above equation finally takes the form:

$$W_e = \frac{k\omega|C_m|^2}{8\mu_o\left(\frac{\omega^2}{c^2} - k^2 + \frac{4\pi\rho\omega^2}{B_o^2}\right)^2} \int_0^{\mu_o} [Ce_m'^2(\mu,p) + \vartheta_m\, Ce_m^2(\mu,p)]\,d\mu\,. \tag{2.35}$$

Similarly, the total mean power for odd solutions can be obtained by substituting the values of E_μ, E_θ, $\bar{H}_\mu$, $\bar{H}_\theta$ from equations (2.27) and (2.28) into equation (2.33) and using relations (McLachlun p.24, 264)

$$\int_0^{2\pi} se_m^2(\theta,p)\,d\theta = \pi$$

$$\int_0^{2\pi} se_m'^2(\theta,p)\,d\theta = \pi\,\bar{\omega}_m$$

in the following form:

$$W_o = \frac{k\omega\,|S_m|^2}{8\mu_o\left(\frac{\omega^2}{c^2} - k^2 + \frac{4\pi\rho\omega^2}{B_o^2}\right)^2} \int_0^{\mu_o} [Se_m'^2(\mu,p) + \bar{\omega}_m\, Se_m^2(\mu,p)]\,d\mu. \tag{2.36}$$

2.1.5. CIRCULAR WAVEGUIDE (AS A LIMITING CASE)

When a confocal ellipse of semi-major axis r tends to a circle with radius r , $\mu \to \infty$, $a \to o$ such that $a \cosh\mu \to r$. For this limiting case equations (2.25) and (2.31) take the form (McLachlan , p. 368)

$$J_m'(k_1 r) = 0\,, \tag{2.37}$$

with

$$k_1 a = (2p)^{1/2}. \tag{2.38}$$

From equations (2.17), (2.37) and (2.38) the dispersion for a circular waveguide can be written as

$$(kr)^2 = \frac{\omega^2 r^2}{c^2}\left(1 + \frac{4\pi\rho c^2}{B_o^2}\right) - 2q_{m,\nu}^2 \tag{2.39}$$

where $q_{m,\nu}$ are the νth root of equation (2.37).

2.1.6. DISCUSSIONS:

The basic equations to determine the electromagnetic field components in elliptic coordinates are derived and then the components of field vectors for even and odd types of solution are obtained in terms of Mathieu functions. Here we see that the components of field vectors in plasma filled elliptical waveguide are different from those of empty elliptical waveguide Chu (1938) but the electric and magnetic lines of force of or cross-section of the plasma-filled elliptical waveguide are the same as those of Chu (1938). The propagation of waves without the components of E_μ and B_θ are generaly impossible in elliptical waveguides but the vanishing of E_μ and B_θ does not effects the propagation in circular waveguide. Then the dispersion relations are derived for even and odd solutions to calculate the pulsatances of the modes. The phase characteristics for the two types of solution are shown graphically. The time rate of energy flow through the ellipsical waveguide in the direction of wave propagation is obtained as integrals of Mathieu functions with the usual limiting process the dispersion relations for wave propagation in circular waveguides have been deduced. It is interesting to note that this didpersion relation represents both even and odd solutions of elliptical waveguide.

SECTION TWO

*SPACE CHARGE WAVES IN PLASMA-FILLED ELLIPTIC WAVEGUIDES IN PRESENCE OF MAGNETIC FIELD.

2.2.1 INTRODUCTION:

Trivelpiece and Gould (1959) have discussed the nature of plasma oscillation of a circular cylindrical plasma column in the presence of a constant axial magnetic field. In this section, we discuss the nature of plasma oscillation or space charge wave of an elliptic plasma column in the presence of finite and infinite constant axial magnetic field. Mathieu functions are used to derive the dispersion relations. First, we derive the dispersion relation for infinite axial magnetic field. The phase characteristic for a particular mode are shown graphically. Then, we derive the dispersion relation for finite axial magnetic field. The phase characteristic for a particular mode are also shown graphically. The phase characteristic in both the cases are of same nature as that of Trivelpiece and Gould. Lastly, the dispersion relation for circular waveguides is derived as a limiting case.

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2.2.2 BASIC EQUATIONS:

We assume time dependence of the perturbed field quantities and currents in the form $e^{-i\omega t}$. Then the linearized fluid equations relate the charge density and current in the plasma to the electric and magnetic fields in the plasma. The Maxwell's equations for the perturbed variables in terms of dielectric tensor ϵ are taken in this form:

$$\nabla \times \vec{E}_1 = i\omega \vec{B}_1 \tag{2.40}$$

$$\nabla \times \vec{B}_1 = -i\omega \epsilon \cdot \vec{E}_1 \tag{2.41}$$

$$\nabla \cdot \vec{B}_1 = 0, \tag{2.42}$$

$$\nabla \cdot (\epsilon \cdot \vec{E}_1) = 0 \tag{2.43}$$

where the dielectric tensor ϵ is given by

$$\epsilon = \begin{bmatrix} \epsilon_1 & i\epsilon_2 & 0 \\ -i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \tag{2.44}$$

when the ion motion and electron-collisions are neglected, ϵ_1, ϵ_2 and ϵ_3 are given by

$$\epsilon_1 = \epsilon_o \{ 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \}$$

$$\epsilon_2 = -\epsilon_o \frac{\omega_p^2 \omega_c}{\omega(\omega^2 - \omega_c^2)} \qquad (2.45)$$

$$\epsilon_3 = \epsilon_o \{ 1 - \omega_p^2/\omega^2 \}$$

where, $\omega_c = (eB_o/m)$ is the electron cyclotron frequency and $\omega_p = (ne^2/m\epsilon_o)^{1/2}$ is the electron plasma frequency.

2.2.3 SPACE-CHARGE WAVES IN A PLASMA-FILLED ELLIPTIC CYLINDER IN AN INFINITE AXIAL MAGNETIC FIELD.

Consider a perfectly conducting elliptic cylinder filled with cold plasma and immersed in an infinite axial magnetic field. Due to infinite axial magnetic field, the electrons are constrained to move only in the z-direction (i.e., along the axis of cylinder) and hence cannot interact with modes that have only transverse electric field components. This means that, for this case, transverse electric (TE)-modes are unaffected by the presence of plasma. The transverse magnetic (TM)-modes have an electric component in the direction of the steady magnetic field and they are affected by the presence of plasma.

For infinite axial magnetic field the electron cyclotron frequency tends to zero, so the dielectric tensor ϵ in equation (2.44) takes the form:

$$\epsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\frac{\omega_p^2}{\omega^2} \end{bmatrix} \qquad (2.46)$$

From equations (2.40) and (2.41), we have

$$\nabla(\nabla.\vec{E}_1) - \nabla^2\vec{E}_1 = \omega^2\epsilon.\vec{E}_1 \,.$$

The z-component of the above equation gives

$$\nabla^2 E_z + \omega^2\epsilon_o(1 - \frac{\omega_p^2}{\omega^2})E_z = [\nabla(\nabla.\vec{E}_1)]_z \qquad (2.47)$$

From equations (2.43) and (2.44), we have

$$[\nabla(\nabla.\vec{E}_1)]_z = \frac{\omega_p^2}{\omega^2} \quad \frac{\partial^2 E_z}{\partial z^2} \qquad (2.48)$$

Therefore, from equations (2.47) and (2.48)

$$\nabla^2 E_z + \omega^2\epsilon_o(1 - \frac{\omega_p^2}{\omega^2})E_z - \frac{\omega_p^2}{\omega^2}\ \frac{\partial^2 E_z}{\partial z^2} = 0 \qquad (2.49)$$

Let us assume the z-dependence in the form exp(ikz), then equation (2.49) takes the form:

$$(\nabla_\perp^2 + T^2)\, E_z = 0 \qquad (2.50)$$

where,
$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} ,$$

$$T^2 = (1 - \frac{\omega_p^2}{\omega^2})\,(\omega^2\epsilon_o - k^2). \qquad (2.51)$$

In elliptical coordinate system i.e., using

$x = a \cosh\mu \cos\theta$, $y = a \sinh\mu \sin\theta$, we have

$$\nabla_\perp^2 = \frac{2}{a^2(\cosh 2\mu - \cos 2\theta)} \left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \theta^2} \right).$$

Then, in this coordinate system, equation (2.50) takes the form:

$$\frac{\partial^2 E_z}{\partial \mu^2} + \frac{\partial^2 E_z}{\partial \theta^2} + 2p^2(\cosh 2\mu - \cos 2\theta)\, E_z = 0 \qquad (2.52)$$

where
$$2p^2 = a^2T^2/2 \qquad (2.53)$$

Let us take the general solution of equation (2.52) in the form (McLachlan, P. 174)

$$E_z = \sum_{m=o}^{\infty} C_{2m}\, Ce_{2m}\,(\mu, p)\, ce_{2m}\,(\theta, p) \qquad (2.54)$$

where

$$Ce_{2m}\,(\mu, p) = \sum_{r=o}^{\infty} A_{2r}^{(2m)} \cosh 2\mu r$$

$$ce_{2m}\,(\theta, p) = \sum_{r=o}^{\infty} A_{2r}^{(2m)} \cos 2r\theta$$

C_{2m} are constants and $A_{2r}^{(2m)}$ are real function of p.

The boundary condition at the perfectly conducting wall requires that the tangential component of the electric field vanishes at the wall i.e., $E_z(\mu = \mu_o) = 0$.

Thus,

$$\sum_{m=o}^{\infty} C_{2m}\, Ce_{2m}(\mu_o, p)\, ce_{2m}(\theta, p) = 0.$$

for the mth mode,

$$Ce_{2m}(\mu_o, p) = 0. \tag{2.55}$$

As μ_o is fixed, we need those positive values of p, say $p_{2m,\nu}$ for which $Ce_{2m}(\mu_o, p)$ vanish. These may be regarded as the positive parametric zeros of the function. The roots of equation (2.55) are used to obtain the dispersion relation.

Therefore, with the help of equation (2.53), We get

$$(ak)^2 = a^2 \epsilon_o \omega^2 - \frac{4p_{2m,\nu}^2}{(1 - \omega_p^2/\omega^2)} . \tag{2.56}$$

This is the dispersion relation which gives the nature of wave propagation. From the dispersion relation, it is clear that the empty waveguide cut-off frequency $\omega_o = 2p_{2m,\nu}/a\sqrt{\epsilon_o}$ and the plasma filled waveguide frequency is

$$\omega = [\ \omega_p^{\ 2} + 4\ p_{2m,\nu}^{2}/(a^2\ \epsilon_o)\ ]^{1/2}\ .$$

Thus we see that the plasma-filled waveguide mode cut-off frequency is shifted upwards as in case of circular waveguide. The graphical representation of the dispersion relation and the position of the cut-off frequency are shown in (Fig.2.3) for m= 1 mode and for the first parametric zero of the equation $Ce_2(\mu_o, p) = 0$ (i.e., $p_{2,1} = 2.416$). The upper passband in the figure (2.3) represents the waveguide modes which in absence of plasma, would still propagate. The lower pass-band between $\omega = 0$ and $\omega = \omega_p$ depends on the presence of plasma, and it represents the plasma oscillation in finite plasma.

2.2.4. CIRCULAR WAVEGUIDE (AS A LIMITING CASE).

In this case equation (2.55) takes the form (McLachlan, p. 367)

$$J_{2m}(k_1 r) = 0 \tag{2.57}$$

with $2p = k_1 a$.

Then from equation (2.57) , using equation (2.53)

$$(Tr)^2 = \alpha_{2m,\nu}^2 \tag{2.58}$$

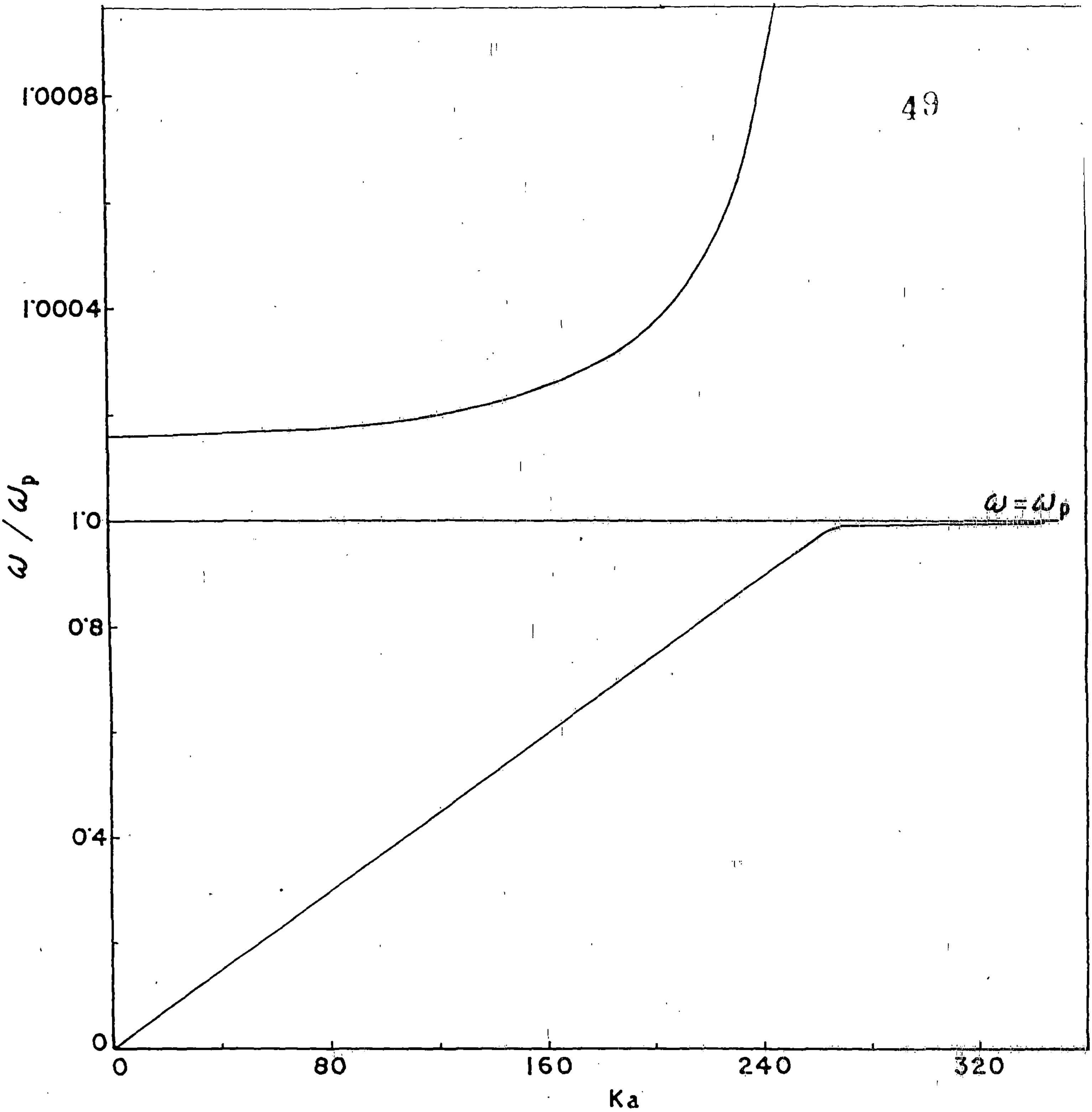

Figure 2.3. Frequency Vs. wavenumber diagram for the transverse magnetic modes in a plasma filled elliptical waveguide in an infinite axial magnetic field.
($p_{2,1} = 2.416$, $\omega_p = 9 \times 10^9$ Hz, $a = 10^{-2}$ m, $\varepsilon_o = \frac{1}{36\pi} \times 10^{-9}$ F/m.)

where $\alpha_{2m,\nu}$ is the νth zero of the 2mth-order Bessel function. On simplification, equation (2.58) can be written in the form

$$(kr)^2 = \epsilon_0 \omega^2 r^2 - \frac{\alpha^2_{2m,\nu}}{(1 - \omega_p^2/\omega^2)} \quad . \tag{2.59}$$

The above dispersion relation is identical with that of Trivelpiece and Gould (1959).

2.2.5. SPACE-CHARGE WAVES IN COLD FINITE PLASMA IN A FINITE MAGNETIC FIELD.

In the previous sub-sections, propagation of space-charge waves in the presence of an infinite axial magnetic field has been discussed. In this sub-section, the dispersion relation for propagation of space-charge waves in the presence of a finite axial magnetic field is derived. For the purpose of simplification, we utilize the fact that the space-charge waves are slow waves. Assuming that the phase velocity of the wave solution of Maxwell's equation is much less than the velocity of light, we can neglect the magnetic field associated with the wave.

Therefore, equation (2.40) gives

$$\vec{E}_1 = -\nabla \phi . \tag{2.60}$$

Substitution of the value of $\vec{E}_1$ from equation (2.60) into equation (2.43) gives

$$\epsilon_1 \nabla_\perp^2 \phi + \epsilon_3 \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad (2.61)$$

Let us assume the z-dependence in the form exp(ikz), then equation (2.61) takes the form

$$(\nabla_\perp^2 - k^2 \epsilon_3/\epsilon_1)\, \phi = 0 \qquad (2.62)$$

In elliptic coordinate system, (2.62) takes the form

$$\frac{\partial^2 \phi}{\partial \mu^2} + \frac{\partial^2 \phi}{\partial \theta^2} + 2q^2(\cosh 2\mu - \cos 2\theta) = 0, \qquad (2.63)$$

where, $2q^2 = 0.5\, a^2 k^2 (- \epsilon_3/\epsilon_1)$. (2.64)

Let us take the general solution of equation (2.63), as before, in the form

$$\phi = \sum_{m=0}^{\infty} D_{2m}\, Ce_{2m}(\mu, q)\, ce_{2m}(\theta, q) \qquad (2.65)$$

where $Ce_{2m}(\mu, q) = \sum_{r=0}^{\infty} B_{2r}^{(2m)} \cosh 2\mu r$

$$ce_{2m}(\theta, q) = \sum_{r=0}^{\infty} B_{2r}^{(2m)} \cos 2r\theta,$$

D_{2m} are constants and $B_{2r}^{(2m)}$ are real functions of q.

We have at the boundary $(\mu = \mu_0)$, $\phi = 0$ i.e.,

$$\sum_{m=o}^{\infty} D_{2m}\, Ce_{2m}(\mu_o, q)\, ce_{2m}(\theta, q) = 0$$

For the mth mode

$$Ce_{2m}(\mu_o, q) = 0 \tag{2.66}$$

As before,

$$(ak)^2 = 4\, q^2_{2m,\nu}(-\epsilon_3/\epsilon_1),$$

where $q_{2m,\nu}$ are those positive values of q for which $Ce_{2m}(\mu_o, q)$ vanish. On using equation (2.45) the above equation can be written in the form

$$(ak)^2 = 4\, q^2_{2m,\nu}\left[\frac{-\omega^2(\omega^2 - \omega_c^2 - \omega_p^2)}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_p^2)}\right] \tag{2.67}$$

This is the dispersion relation which gives the nature of wave propagation. The phase characteristics (ω Vs. k) obtained from equation (2.67) for m=1 mode and first parametric zero of the equation $Ce_2(\mu_o, p) = 0$ are shown in figure (2.4) for a strong magnetic field $\omega_c > \omega_p$, and for a weak magnetic field $\omega_c < \omega_p$ are shown in figure (2.5). For the case $\omega_c > \omega_p$, in addition to the mode $(\omega < \omega_p)$ predicted from the $B_o = \infty$ analysis, the upper hybrid mode $\omega_c < \omega < \sqrt{\omega_p^2 + \omega_c^2}$, appears

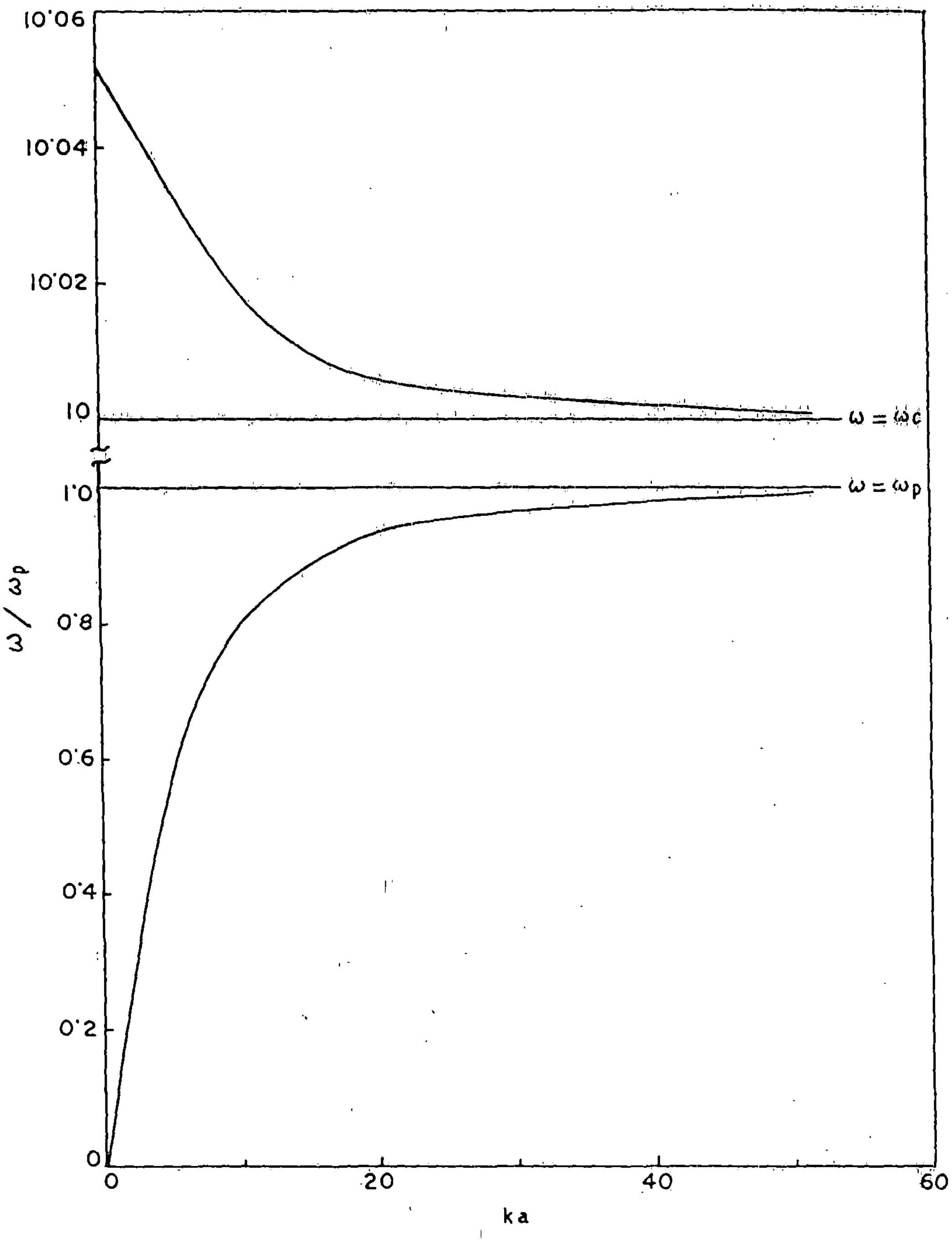

Figure 2.4. Frequency Vs. wavenumber diagram for a plasma filled waveguide in a finite axial magnetic field.

($p_{2,1} = 2.416$, $\omega_p = 9 \times 10^{9}$ Hz, $a = 10^{-2}$ m, $\omega_c = 9 \times 10^{10}$ Hz.)

as a characteristic frequency in the plasma. This mode has the interesting feature that it is a backward wave. As the magnetic field is further reduced ($\omega_p > \omega_c$), it is seen that waves propagate for frequencies less than the cyclotron frequency. The backward wave mode now propagates in the frequency range $\omega_p < \omega < \sqrt{\omega_p^2 + \omega_c^2}$. If the steady magnetic field strength is reduced to zero, the upper pass-band reduce to the plasma resonance at ω_p and the lower passband reduces towards $\omega = 0$. In both cases, the waves cease to propagate. That is, plasma filled waveguide without an external magnetic field will not propagate space-charge waves. Thus from the above asymptotic behaviour of the solution at $B_o = \infty$ and $B_o = 0$, we see that the nature of wave propagation in plasma filled elliptical waveguide is of same nature as was obtained by Trivelpiece and Gould in the case of circular wave guide (1959).

2.2.6. CIRCULAR WAVEGUIDE (AS A LIMITING CASE)

In this case, equation (2.66) gives

$$J_{2m}(k_2 r) = 0 \tag{2.68}$$

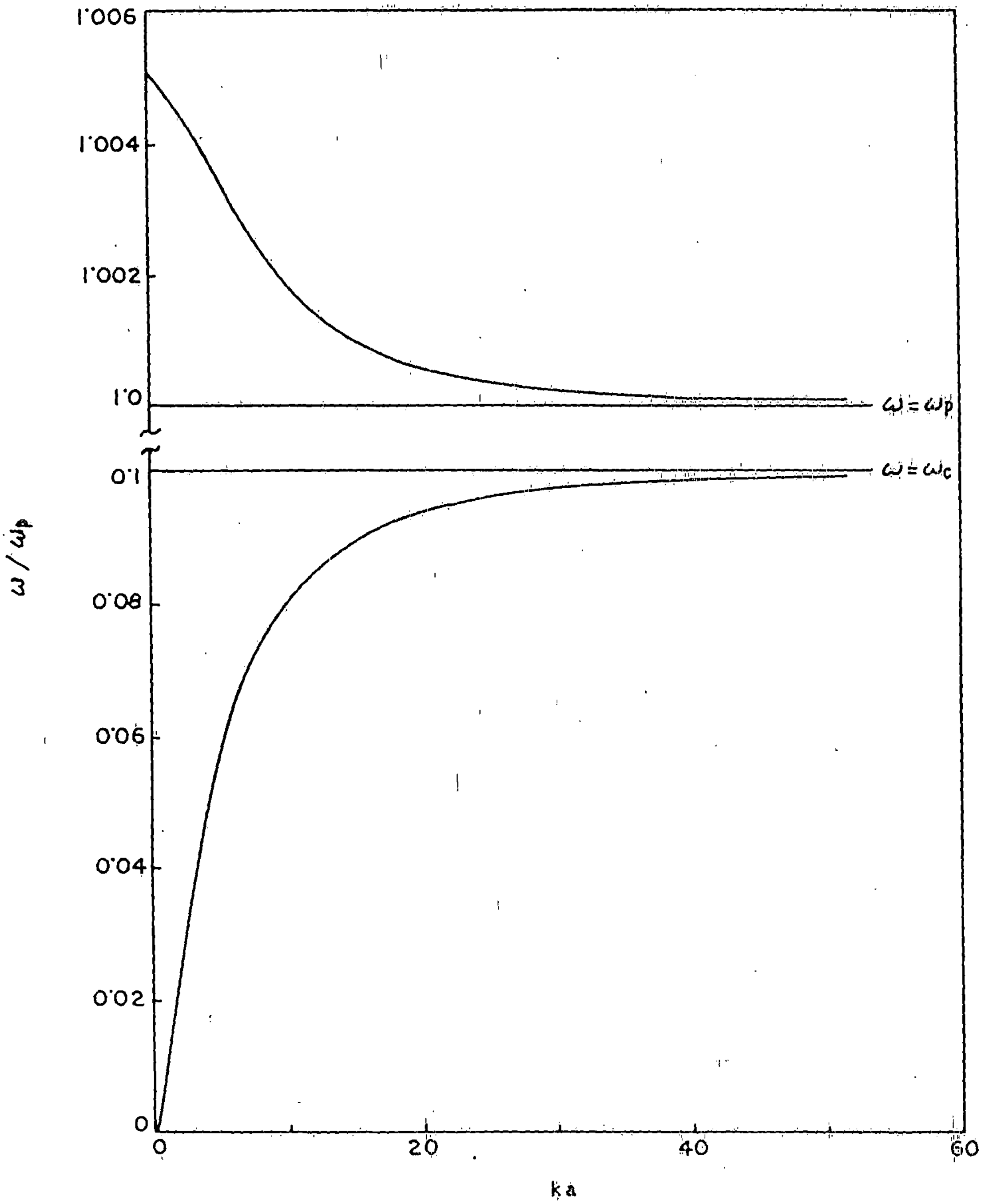

Figure 2.5. Frequency Vs. wavenumber diagram for a plasma filled waveguide in a finite axial magnetic field.

($P_{2,1} = 2.416$, $\omega_p = 9 \times 10^{9}$ Hz, $a = 10^{-2}$ m, $\omega_c = 9 \times 10^{8}$ Hz.)

with $k_2 a = 2q$

i.e., $$k_2^{\,2} = (- \epsilon_3/\epsilon_1)\, k^2 \qquad (2.69)$$

Then, from equations (2.68) and (2.69), we can write

$$(kr)^2 = \beta^2_{2m,\nu} \left[\frac{-\omega^2(\omega^2-\omega_p{}^2-\omega_c{}^2)}{(\omega^2-\omega_c{}^2)(\omega^2-\omega_p{}^2)} \right] \qquad (2.70)$$

where, $\beta_{2m,\nu}$ is the νth zero of the 2mth order Bessel function. Equation (2.70) is identical to that of Trivelpiece and Gould (1959).

SECTION THREE

PROPAGATION OF ELECTROMAGNETIC WAVES IN WARM PLASMA FILLED ELLIPTIC WAVEGUIDES.

2.3.1. INTRODUCTION.

In this section we consider the wave propagation in warm plasma filled elliptical waveguide. A complete analysis of this problem is not possible within the framework of fluid theory, unless an equation of state is considered. Even if an equation of state is chosen, phenomena which depend on the details of the velocity distribution, such as Landau damping, will of course, not appear in a fluid description. Nevertheless, some of the general features of these waves can be studied with the fluid equations. Here we consider a homogeneous field free plasma inside elliptical waveguide allowing the electrons to have a finite temperature. To study the problem Mathieu functions are used. Dispersion relations for transverse electric (TE) and transverse magnetic (TM) modes are derived and discussed graphically. Power transmission in the direction of wave propagation for TE- and TM- modes are calculated. Lastly, with the usual limiting process dispersion relation in circular waveguide have been derived.

2.3.2. BASIC EQUATIONS:

The equations of continuity, momentum along with Maxwell's equations for warm plasma in absence of magnetic and electric sources given by Waite (1968) are

$$U^2 m N \; \nabla . \vec{V} = - j\omega p , \tag{2.71}$$

$$mN(j\omega + \nu) \vec{V} = - Ne \vec{E} - \nabla p , \tag{2.72}$$

$$\nabla X \vec{E} = - j\omega\mu_o \vec{H} , \tag{2.73}$$

$$\nabla X \vec{H} = j\omega\varepsilon_o \vec{E} - Ne\vec{V} , \tag{2.74}$$

where $U(= (\frac{\gamma \kappa T}{m})^{1/2}$, m, N, p, γ, κ, ν, T, e, $\vec{V}$, $\vec{E}$ and $\vec{H}$ are respectively the velocity of sound wave propagation in warm plasma, electron mass, electron density, plasma pressure, ratio of specific heats, Boltzmann's constant, electron collision frequency, temperature of the warm plasma, electron charge, electron velocity, electric field and magnetic field.

Eliminating p and $\vec{V}$ from equation (2.72) with the help of equations (2.71) and (2.74) we have

$$\nabla X \vec{H} = j\omega\varepsilon_o\varepsilon_p\vec{E} + j \frac{\varepsilon_o U^2}{(\omega - j\nu)} \nabla (\nabla . \vec{E}) \tag{2.75}$$

where

$$\varepsilon_p = 1 - \omega_p^2/\omega(\omega - j\nu)$$
$$\omega_p = (Ne^2/m\varepsilon_o)^{1/2} \tag{2.76}$$

and ω_p is the electron plasma frequency. The equations (2.73) and (2.75) are the expanded Maxwell's equation for warm plasma.

To find the solution of above set of equations, we assume that the waves propagate in the z-direction in the form of exp(-jkz). We also assume that the electron collision frequency ν is much smaller than the wave frequency ω .

Let us now express the electric, magnetic and electron velocity vectors in terms of perpendicular and parallel components:

$$\vec{E} = \vec{E}_\perp + \hat{z} E_z, \qquad \vec{H} = \vec{H}_\perp + \hat{z} H_z ,$$
$$\vec{V} = \vec{V}_\perp + \hat{z} V_z; \qquad \nabla = \nabla_\perp - jk\hat{z}. \tag{2.77}$$

where $\hat{z}$ is the unit vector along positive z-direction.

Substitution of (2.77) into equations (2.73) and (2.75) yields,

$$\nabla_\perp X \vec{E}_\perp = - j\omega\mu_o \hat{z} H_z , \tag{2.78}$$

$$(\nabla_\perp X \hat{z})E_z - j k (\hat{z} X \vec{E}_\perp) = - j\omega\mu_o \vec{H}_\perp , \tag{2.79}$$

$$\nabla_\perp \times \vec{H}_\perp = j\omega\varepsilon_o\varepsilon_p \hat{z}E_z + \hat{z}(\varepsilon_o kU^2/\omega)\,(\nabla_\perp \cdot \vec{E}_\perp - jkE_z)\ , \tag{2.80}$$

$$(\nabla_\perp \times \hat{z})\, H_z - j\,k(\hat{z} \times \vec{H}_\perp) = j\omega\varepsilon_o\,\varepsilon_p\,\vec{E}_\perp + j(\varepsilon_o U^2/\omega)\nabla_\perp(\nabla_\perp \cdot \vec{E}_\perp - jkE_z) \tag{2.81}$$

From equations (2.78) - (2.81) using simplified assumptions, we get the following differential equations for E_z and H_z:

$$(\nabla_\perp^2 + k_s^2)\,(\nabla_\perp^2 + k_e^2)\,E_z = 0 \tag{2.82}$$

$$(\nabla_\perp^2 + k_e^2)\,H_z = 0 \tag{2.83}$$

and we obtain equations for $\vec{E}_\perp$ and $\vec{H}_\perp$ in terms of E_z and H_z:

$$k_e^2\vec{E}_\perp = -k\nabla_\perp E_z + j\,\omega\mu_o\,(\hat{z} \times \nabla_\perp)H_z - j\,\frac{(U/c)^2}{k\gamma}\nabla_\perp \cdot (\nabla_\perp^2 E_z + k_e^2 E_z)\ , \tag{2.84}$$

$$k_e^2\vec{H}_\perp = -jk\nabla_\perp H_z - j\omega\varepsilon_o\varepsilon_p(\hat{z} \times \nabla_\perp)E_z - j\,\frac{(U^2\varepsilon_o/w)^2}{\gamma}(\hat{z} \times \nabla_\perp)\cdot(\nabla_\perp^2 E_z + k_e^2 E_z)\ , \tag{2.85}$$

where

$$k_e^2 = \frac{\omega^2}{c^2}\varepsilon_p - k^2, \qquad k_s^2 = \frac{\omega^2}{U^2}\varepsilon_p - k^2\ ,$$
$$\gamma = 1 - U^2/c^2 \qquad \varepsilon_p = 1 - \omega_p^2/\omega^2\ . \tag{2.86}$$

Again, from equations (2.71), (2.72) and (2.74) we can get $\vec{V}_\perp$ interms of $\vec{E}_\perp$ and E_z and V_z interms of E_z.

$$\vec{V}_\perp = j\,\frac{e}{m\omega}\,[\vec{E}_\perp - \frac{U^2}{\omega_p^2}\nabla_\perp(\nabla_\perp \cdot \vec{E}_\perp - j\,k\,E_z)\,] \tag{2.87}$$

$$V_z = j \frac{e}{m\omega} [E_z + jk \frac{U^2}{\omega_p^2} (\nabla_\perp \cdot \vec{E}_\perp - jkE_z)] \quad . \tag{2.88}$$

Now let us put

$$E_z = E_{z1} + E_{z2} \quad , \tag{2.89}$$

then from equation (2.82), we have

$$\nabla_\perp^2 E_{z1} + k_s^2 E_{z1} = 0 \quad , \tag{2.90}$$

$$\nabla_\perp^2 E_{z2} + k_e^2 E_{z2} = 0 \quad , \tag{2.91}$$

and the other components of the field vectors and electron velocity from equations (2.84) - (2.88) are

$$k_e^2 \vec{E}_\perp = - jk \nabla_\perp E_{z2} + j\omega\mu_o (\hat{z} \times \nabla_\perp) H_z + j \frac{k_e^2}{k} \nabla_\perp E_{z1} \quad , \tag{2.92}$$

$$k_e^2 \vec{H}_\perp = - jk \nabla_\perp H_z - j \omega \varepsilon_o \varepsilon_p (\hat{z} \times \nabla_\perp) E_{z2} \quad , \tag{2.93}$$

$$\vec{V}_\perp = \frac{e}{m \omega k_e^2} [k \nabla_\perp E_{z2} - \omega \mu_o (\hat{z} \times \nabla_\perp) H_z - \frac{k_e^2}{k} (\omega/\omega_p)^2 \nabla_\perp E_{z1}] \quad , \tag{2.94}$$

$$V_z = j \frac{e}{m\omega} [E_{z2} + (\omega/\omega_p)^2 E_{z1}] \quad . \tag{2.95}$$

Thus, we have obtained all the field components and the electron velocity interms of E_z and H_z.

Let us introduce the elliptic coordinates (μ, θ, z) defined by $x = a \cosh\mu \cos\theta$, $y = a \sinh\mu \sin\theta$, $z = z$, where $a = (A^2 - B^2)^{1/2}$ and A, B are the semi-major and semi-minor axes of the boundary of the elliptic cylinder.

In this coordinate system equation (2.83) and equations (2.90) - (2.95) respectively take the following form:

$$\frac{\partial^2 H_z}{\partial \mu^2} + \frac{\partial^2 H_z}{\partial \theta^2} + 2q\,(\cosh^2\mu - \cos^2\theta)\,H_z = 0\,, \qquad (2.96)$$

$$\frac{\partial^2 E_{z1}}{\partial \mu^2} + \frac{\partial^2 E_{z1}}{\partial \theta^2} + 2p\,(\cosh^2\mu - \cos^2\theta)\,E_{z1} = 0\,, \qquad (2.97)$$

$$\frac{\partial^2 E_{z2}}{\partial \mu^2} + \frac{\partial^2 E_{z2}}{\partial \theta^2} + 2p\,(\cosh^2\mu - \cos^2\theta)\,E_{z2} = 0\,, \qquad (2.98)$$

$$ak_e^2(\sinh^2\mu + \sin^2\theta)^{1/2} E_\mu = -jk\frac{\partial E_{z2}}{\partial \mu} - j\omega\mu_o\frac{\partial H_z}{\partial \theta} + j\frac{k_e^2}{k}\frac{\partial E_{z1}}{\partial \mu}\,, \qquad (2.99)$$

$$ak_e^2(\sinh^2\mu + \sin^2\theta)^{1/2} E_\theta = -jk\frac{\partial E_{z2}}{\partial \theta} + j\omega\mu_o\frac{\partial H_z}{\partial \mu} + j\frac{k_e^2}{k}\frac{\partial E_{z1}}{\partial \theta}\,, \qquad (2.100)$$

$$ak_e^2(\sinh^2\mu + \sin^2\theta)^{1/2} H_\mu = -jk\frac{\partial H_z}{\partial \mu} + j\omega\varepsilon_o\varepsilon_p\frac{\partial E_{z2}}{\partial \theta}\,, \qquad (2.101)$$

$$ak_e^2(\sinh^2\mu + \sin\theta)^{1/2} H_\theta = -jk\frac{\partial H_z}{\partial \theta} - j\omega\varepsilon_o\varepsilon_p\frac{\partial E_{z2}}{\partial \mu}\,, \qquad (2.102)$$

$$a(\sinh^2\mu + \sin^2\theta)^{1/2}\frac{\omega m k_e^2}{e} V_\mu = k\frac{\partial E_{z2}}{\partial \mu} + \omega\mu_o\frac{\partial H_z}{\partial \theta} - \frac{k_e^2}{k}(\omega/\omega_p)^2\frac{\partial E_{z1}}{\partial \mu}\,, \qquad (2.103)$$

$$a(\sinh^2\mu + \sin^2\theta)^{1/2}\frac{\omega m k_e^2}{e} V_\theta = k\frac{\partial E_{z2}}{\partial \theta} - \omega\mu_o\frac{\partial H_z}{\partial \mu} - \frac{k_e^2}{k}(\omega/\omega_p)^2\frac{\partial E_{z1}}{\partial \theta}\,, \qquad (2.104)$$

$$V_z = j \frac{e}{m\omega}[\ E_{z2} + (\omega/\omega_p)^2\ E_{z1}\ ] \quad , \tag{2.105}$$

where

$$2p = a^2 k_s^{\ 2} \quad \text{and} \quad 2q = a^2 k_e^{\ 2} \ . \tag{2.106}$$

2.3.3. DISPERSION RELATION

We take a solution of equations (2.96) - (2.98) in the form

$$E_{z1} = \sum_{m=o}^{\infty} C_{2m}\ Ce_{2m}\ (\mu,\ p)\ ce_{2m}\ (\theta,\ p) \tag{2.107}$$

$$E_{z2} = \sum_{m=o}^{\infty} \bar{C}_{2m}\ Ce_{2m}\ (\mu,\ q)\ ce_{2m}\ (\theta, q) \tag{2.108}$$

$$H_z = \sum_{m=o}^{\infty} D_{2m}\ Ce_{2m}\ (\mu,\ q)\ ce_{2m}\ (\theta, q) \tag{2.109}$$

where C_{2m} , $\bar{C}_{2m}$, D_{2m} are constants and

$$\begin{aligned} Ce_{2m}\ (\mu,\ p) &= \sum_{r=o}^{\infty} A_{2r}^{2m} \cosh 2\mu r \\ Ce_{2m}\ (\mu,\ q) &= \sum_{r=o}^{\infty} B_{2r}^{2m} \cosh 2\mu r \\ ce_{2m}\ (\theta\ ,\ p) &= \sum_{r=o}^{\infty} A_{2r}^{2m} \cos 2r\theta \\ ce_{2m}\ (\theta\ ,\ q) &= \sum_{r=o}^{\infty} B_{2r}^{2m} \cos 2r\theta \end{aligned} \tag{2.110}$$

A_{2r}^{2m} , B_{2r}^{2m} are functions of p and q respectively.

At the boundary $\mu = \mu_o$, we have

$$E_z = 0, \; E_\theta = 0 \text{ and } V_\mu = 0 \qquad (2.111)$$

For a particular m, the equations (2.89), (2.100), (2.103) together with the boundary conditions (2.111) gives the following dispersion relation:

$$Ce'_{2m}(\mu_o, q)[k^2 Ce_{2m}(\mu_o, p)Ce'_{2m}(\mu_o, q) + (\omega/\omega_p)^2 k_e^{\,2} Ce_{2m}(\mu_o, q)Ce'_{2m}(\mu_o, p$$

$$= [(\frac{B_1}{B})^2 k^2 + \frac{A_1 B_1}{AB} k_e^{\,2}]Ce_{2m}(\mu_o, p)Ce^2_{2m}(\mu_o, q) \qquad (2.112)$$

where

$$A = \sum_{r=o}^{\infty} A^{2m}_{2r} \cos 2r\,\theta, \qquad B = \sum_{r=o}^{\infty} B^{2m}_{2r} \cos 2r\,\theta$$

$$A_1 = \sum_{r=o}^{\infty} A^{2m}_{2r} (2r \sin 2r\,\theta), \quad B_1 = \sum_{r=o}^{\infty} B^{2m}_{2r} (2r \sin 2r\,\theta).$$

This equation shows that we can easily classify these waves into two modes of groups TE and TM modes, only when r = o and we cannot do when r ≠ 0, so they are hybrid modes. In case of r = 0, the dispersion relation (2.112) take the form

$$Ce'_{2m}(\mu_o, q) = 0 \qquad (2.113)$$

$$\frac{Ce'_{2m}(\mu_o,p)}{Ce_{2m}(\mu_o,p)} + \frac{k^2}{k_e^2}\left(\omega_p/\omega\right)^2 \frac{Ce'_{2m}(\mu_o,q)}{Ce_{2m}(\mu_o,q)} = 0 \qquad (2.114)$$

The equation (2.113) gives the wave propagation of TE modes and equation (2.114) gives the wave propagation of TM modes.

As μ_o is fixed, We need those positive values of q, say $q_{2m,\nu}$ for which $Ce'_{2m}(\mu_o,q)$ vanish. These may be regarded as the positive parametric zeros of the function. These roots of equation (2.113) are used to obtain the dispersion relation. Therefore, with the help of equation (2.106) we get

$$(ak)^2 = \left(\frac{a\omega_p}{c}\right)^2 \left\{\left(\frac{\omega}{\omega_p}\right)^2 - 1\right\} - 2\, q_{2m,\nu} \qquad (2.115)$$

This is the dispersion relation which gives the nature of wave propagation for the TE- modes. The graphical representation of the dispersion relation for m = 1 mode and for the first parametric zero of the equation $Ce'_2(\mu_o, q) = 0$ are shown in fig. (2.6).

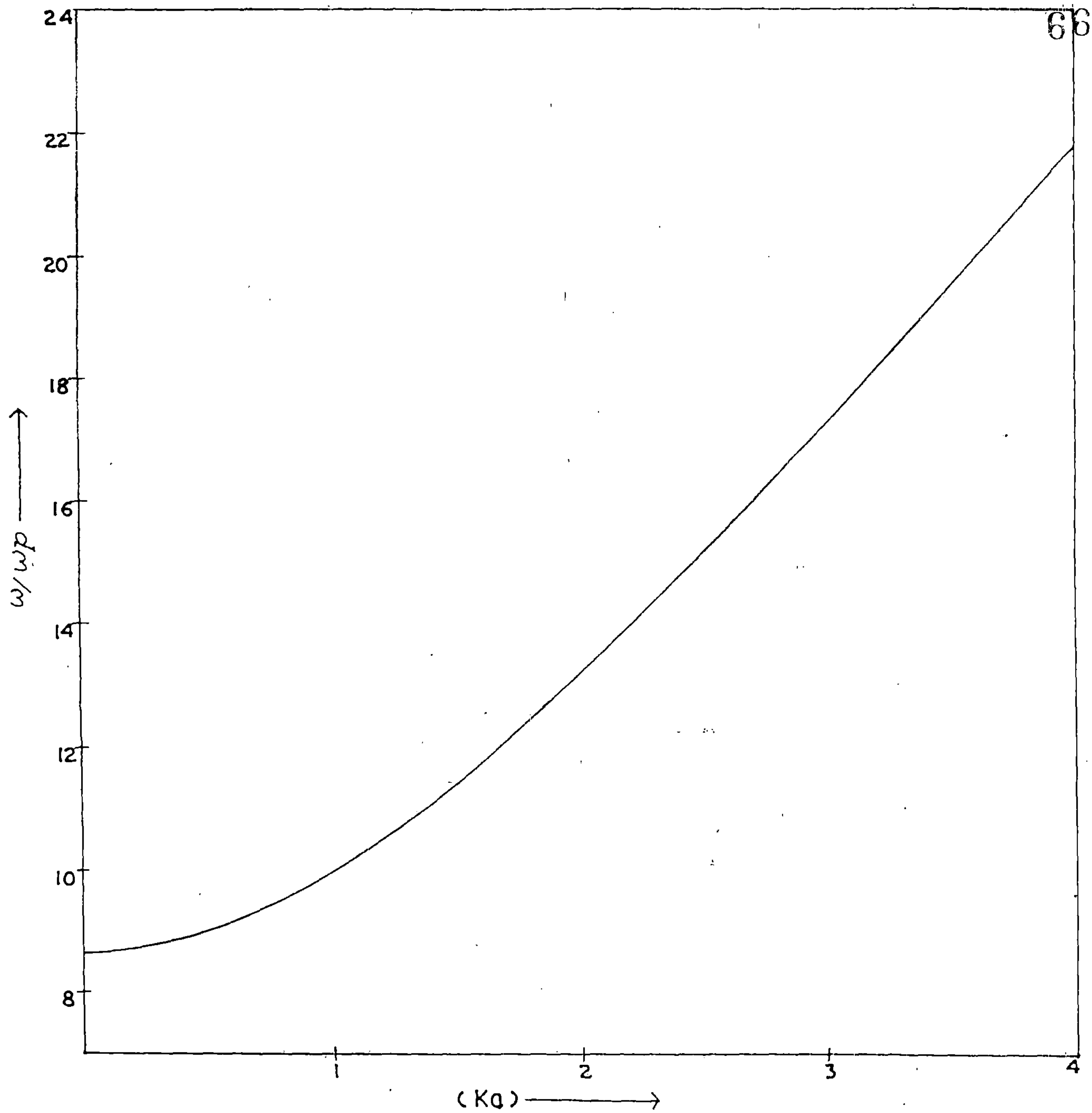

Figure 2.6. Frequency Vs. wavenumber diagram for a warm plasma filled waveguide for m = 1 mode.

($q_{2,1} = 1.459$, $\omega_p = 6 \times 10^{11}$/S, $a = 10^{-2}$ m ,

$c = 3 \times 10^{10}$ cm/S)

2.3.4. POWER TRANSMITTED IN THE DIRECTION OF WAVE PROPAGATION.

The total mean power for TM and TE modes can be obtained by substituting the values of E_μ, E_θ, $\bar{H}_\mu$, $\bar{H}_\theta$ from equations (2.99) - (2.102) into equation (2.33).

$$W_{TM} = \frac{|\bar{C}_{2m}|^2 \omega c \varepsilon_o \varepsilon_p k}{8\pi k_e^4} \int_0^{\mu_o} \int_0^{2\pi} [Ce_{2m}^2(\mu,q) ce_{2m}^2(\theta,q) + Ce_{2m}'^2(\mu,q)\, ce_{2m}^2(\theta,q)] d\mu d\theta - \frac{C_{2m}\,\bar{C}_{2m}\omega c \varepsilon_o \varepsilon_p}{8\pi k\, k_e^2} \int_0^{\mu_o} \int_0^{2\pi} [Ce_{2m}'(\mu,p) \cdot Ce_{2m}'(\mu,q) ce_{2m}(\theta,p) ce_{2m}(\theta,q) + Ce_{2m}(\mu,q)\, Ce_{2m}(\mu,p)\, ce_{2m}'(\theta,q)\, ce_{2m}'(\theta,p)]\, d\mu d\theta \,. \quad (2.116)$$

$$W_{TE} = \frac{|D_{2m}|^2 \omega c \mu_o k}{8\pi k_e^4} \int_0^{\mu_o} \int_0^{2\pi} [\, Ce_{2m}^2(\mu,q)\, ce_{2m}'^2(\theta,q) + Ce_{2m}'^2(\mu,q)\, ce_{2m}^2(\theta,q)\,]\, d\mu d\theta \quad (2.117)$$

On using relations (McLachlan, p. 23, 1964)

$$\int_0^{2\pi} ce_{2m}^2(\theta,q)\, d\theta = 2\pi [\, B_o^{(2m)}\,]^2 + \pi \sum_{r=1}^{\infty} [\, B_{2r}^{(2m)}\,]^2 ,$$

$$\int_0^{2\pi} ce_{2m}(\theta,p)\, ce_{2m}(\theta,q)\, d\theta = \pi \sum_{r=1}^{\infty} [\, A_{2r}^{(2m)} \; B_{2r}^{(2m)}],$$

$$\int_0^{2\pi} ce'^{\,2}_{2m}(\theta,q)\, d\theta = 4\pi \sum_{r=1}^{\infty} [\, r\, A_{2r}^{(2m)}]^2 ,$$

$$\int_0^{2\pi} ce'_{2m}(\theta,q)\, ce'_{2m}(\theta,p)\, d\theta = 4\pi \sum_{r=1}^{\infty} [\, r^2 A_{2r}^{(2m)} \; B_{2r}^{(2m)}],$$

the equations (2.116) and (2.117) finally take the form

$$W_{TM} = \frac{|\bar{C}_{2m}|^2 \omega\, c\, \varepsilon_o \varepsilon_p k}{8\, k_e^4} \int_0^{\mu_o} [(\sum_{r=1}^{\infty} [2r\, A_{2r}^{(2m)}]^2)\, Ce^2_{2m}(\mu,q) +$$

$$+ (2\,[\, B_o^{(2m)}]^2 + \sum_{r=1}^{\infty} [\, B_{2r}^{(2m)}])\, Ce'^{\,2}_{2m}(\mu,q)\,]\, d\mu \; -$$

$$- \frac{C_{2m}\bar{C}_{2m}\omega\, c\, \varepsilon_o \varepsilon_p}{8k\, k_e^2} \int_0^{\mu_o} [\, (\sum_{r=1}^{\infty} [\, A_{2r}^{(2m)} \; B_{2r}^{(2m)}]\,)\, Ce'_{2m}(\mu,q) Ce'_{2m}(\mu,$$

$$+ (\sum_{r=1}^{\infty} [\, 4r^2 A_{2r}^{(2m)} B_{2r}^{(2m)}]) Ce_{2m}(\mu,q) Ce_{2m}(\mu,p)]\, d\mu . \qquad (2.118)$$

and

$$W_{TE} = \frac{|D_{2m}|^2 \omega\, c\, \mu_o\, k}{8k_e^4} \int_0^{\mu_o} [(\sum_{r=1}^{\infty} [2r\, A_{2r}^{(2m)}]^2)\, Ce^2_{2m}(\mu,q) +$$

$$+ (\, 2\,[\, B_o^{(2m)}]^2 + \sum_{r=1}^{\infty} [\, B_{2r}^{(2m)}]^2\, Ce'^{\,2}_{2m}(\mu, q)\,]\, d\mu . \qquad (2.119)$$

2.3.5. CIRCULAR WAVEGUIDE [AS A LIMITING CASE]

When a confocal ellipse of semi-major axes r tends to a circle with radius r, $\mu \to \infty$, $a \to 0$ such that $a \cosh \mu \to r$. For this limiting case equations (2.113) and (2.114) take the form (McLachlan, p.368)

$$J'_{2m}(k_1 r) = 0 \qquad (2.120)$$

$$\text{with} \quad k_1 r = (2q)^{1/2}$$

and

$$\frac{J'_{2m}(k_2 r)}{J_{2m}(k_2 r)} + \frac{k^2}{k_e k_s} \left({}^{\omega_p}/_{\omega} \right)^2 \frac{J'_{2m}(k_1 r)}{J_{2m}(k_1 r)} = 0 \qquad (2.121)$$

$$\text{with} \quad k_1 r = (2q)^{1/2}, \quad k_2 r = (2p)^{1/2} .$$

For m = o modes the dispersion relations (2.120) and (2.121) coincides with the dispersion relation for circular waveguide obtained by Azakami et al. (1972).

2.3.6. DISCUSSION.

In this section the basic equations to determine the electromagnetic field components and electron velocity in warm plasma filled elliptical waveguide in elliptical coordinates are derived and then the components of field vectors and electron velocity are obtained in terms of Mathieu functions. Here we see that the waves combining the electron sound wave mode with the electromagnetic wave mode can propagate in elliptical waveguide containing the warm plasma. The waves can be separated into TM-modes and TE- modes only at $r = 0$ mode, and the other modes become hybrid modes. These results cannot be explained upto the present by the cold plasma approximation. The phase characteristics of the equation (2.113) are illustrated graphically for a particular case. Here we see waves propagate above the plasma frequency. Lastly, with the usual limiting process the dispersion relation for TM- and TE- modes in circular waveguide has been deduced and these are found to agree with those obtained by Azakami et al. (1972).

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