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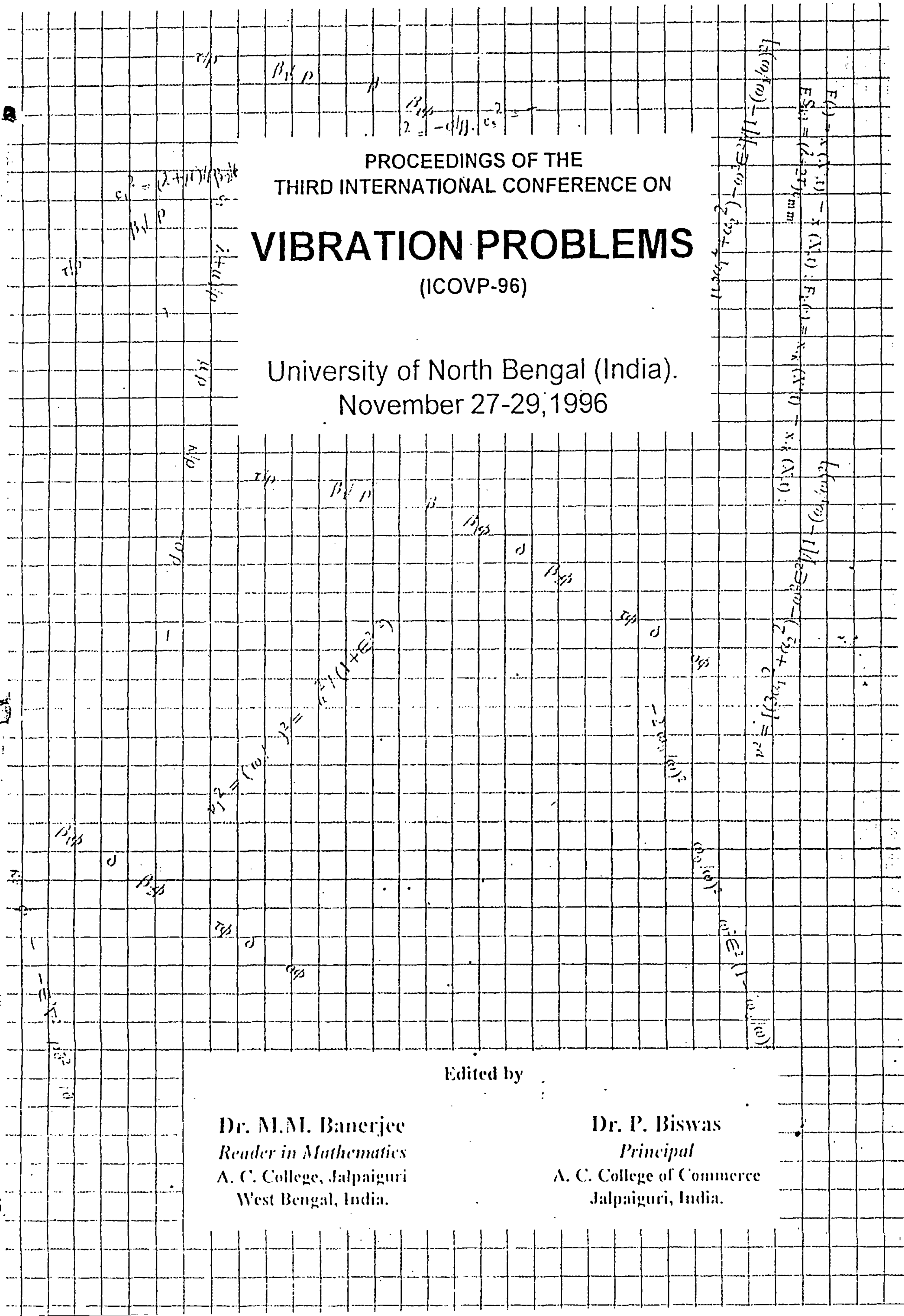

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MODIFIED SPHERICAL HARMONIC METHOD AND ITS APPLICATION

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1. Introduction :

The mathematical difficulties of working with the integro- differential equations have resulted in number of approximate methods of solving the equatiion of radiative transfer. Approximate solutiion of this equation can be obtained by means of Milne-Eddington Approximation and Schuster-Schwarzchild approximation. The Spherical Harmonic Method, the Moment Method and the Discrete ordinate Method are more elaborate schemes that may provide higher order approximation.

In Eddington Method, the basis of approach is an approximation to the angular distribution of radiation of intensity such that the integro-differential equation of radiative transfer is transformed into an ordinary differential equation. In case of Schuster-schwarzchild Method, the integro-differential equation for an isotropically scattering, plane-parallel medium can be transformed into a pair of ordinary differential equation.

The Spherical Harmonic Method provides a means to obtain a higher order approximate solution to the equation of radiative transfer at the expense of additional labour and calculation. The essential idea of the method, which is due to Eddington, lies in seeking a solution of the equation of transfer in the form of an expansion of intensity $I(\tau,\mu)$ in a series of Legendre polynimoals $P_l(\mu)$. These form a complete set of orthogonal functions in the interval (-1,1) which is just that through which μ varies.

2. The Spherical Harmonic Method :

Let us consider the equation of radiative transfer for plane parallel scattering atmosphere with constant net flux.

$$\mu(\partial/\partial\tau)I(\tau,\mu) = I(\tau,\mu) - {}^{1}/_{2}\int_{-1}^{1}p(\mu,\mu')I(\tau,\mu')d\mu' \tag{2.1}$$

where $p(\mu,\mu') = \int_0^{2\pi}p(\mu,\phi;\mu',\phi')d\phi'$ is the phase function through a ray scattered from the direction (μ,ϕ) into the direction (μ',ϕ').

$I(\tau,\mu)$ is the specific intensity of radiation at an optical depth τ and in direction ϕ with the outward drawn normal, $\mu = \cos(\phi)$ and τ is given by

$\tau = \int k_\nu \rho dz$ where ρ is the density of the meduim and k_ν is the absorption coefficient.

The phase function is assumed to be represented in a series of Legendre polynomials as

$$P(\mu,\mu') = \sum_{m=0}^{n}(2m+1)f_m P_m(\mu)P_m(\mu)P_m(\mu'), \quad f_0 = 1 \tag{2.2}$$

The intensity of radiation $I(\tau,\mu)$ can be expanded in a series of Legendre polynomials in the form,

$$I(\tau,\mu) = \sum_{m=0}^{n}\frac{2m+1}{4\pi}P_m(\mu)\psi_m(\tau) \tag{2.3}$$

If the functions $\psi_m(\tau)$ are known, the radiation intensity can be determined from(2.3). Therefore we shall be concerned with the determination of $\psi_m(\tau)$.

Using the expansions of (2.2) and (2.3) in (2.1) and considering orthogonal properties of Legendre polynomial as well as the recurrence relations one gets:

$$\sum_{m=0}^{n}[m(d/d\tau)\psi_m(\tau)P_{m-1}(\mu) + (m+1)(d/d\tau)\psi_m(\tau)P_{m+1}(\mu) + (2m+1)(f_m-1)P_m(\mu)\psi_m(\tau) = 0 \tag{2.4}$$

$$\sum_{m=0}^{\infty}[(m+1)(d/d\tau)\psi_{m+1}(\tau)+m(d/d\tau)\psi_{m-1}(\tau)+(2m+1)(f_m-1)\psi_m(\tau)]P_m(\mu)=0 \qquad (2.5)$$

If the equation (2.5) should be valid for all μ, the coefficients of $P_m(\mu)$ must vanish identically. This requirement results in the ordinary differential equation for the function $\psi_m(\tau)$, $m = 0,1,2,\ldots\ldots..$

$$(m+1)\psi'_{m+1} + m\psi'_{m-1} + (2m+1)(f_m - 1)\psi_m = 0, \quad m = 0,1,2,\ldots.. \qquad (2.6)$$

where $f_0 = 1$, the prime denotes differentiation w.r.t. τ.

For isotropic scattering all f_m are equal to zero except f_0 where $f_0 = 1$.

Equations (2.6) are infinite set of coupled ordinary differential equations for the function $\psi_m(\tau)$. In practice, only a finite number of equations $m = N$ are considered and the term ψ_{N+1} is neglected. Putting successively $m = 0,1,2,\ldots..,N-1,N$ one gets

$$\psi_1' = 0$$

$$2\psi_2' + \psi_0' + 3(f_1 - 1)\psi_1 = 0$$

$$3\psi_3' + 2\psi_1' + 5(f_1 - 1)\psi_2 = 0$$

....

....

$$N\psi_N' + (N-1)\psi_{N-2}' + (2N-1)(f_{N-1} - 1)\psi_N = 0$$

$$N\psi_N' + (2N+1)(f_N - 1)\psi_N = 0 \qquad (2.7)$$

Equations (2.7) provide (N+1) simultaneous linear differential equations for N+1 unknown functions $\psi_0, \psi_1, \ldots\ldots, \psi_N$ and is called the P_N approximation.

The desired solution of the system of equation (2.7) can be written as a linear sum of the solution.

We seek a solution of the form

$$\psi_m(\tau) = g_m e^{k\tau}, \quad m=0,1,2,\ldots\ldots..N \qquad (2.8)$$

where g_m are arbitrary constants and k is the exponent are to be determined.

Substituting (2.8) in (2.7) or (2.6) one gets,

$$k[(m+1)g_{m+1} + m g_{m-1}] + (2m+1)(f_m - 1)g_m = 0 \qquad (2.9)$$

for $m = 0,1,2, \quad ,N, \; f_0 = 1, \; g_{N+1} = 0$

For isotrpic scattering $f_m = \delta_{0m} = 1$ for m=0, 0 other wise

and equation (2.9) becomes

$$k[(m+1)g_{m+1} + m g_{m-1}] + (2m+1)(\delta_{0m} - 1)g_m = 0 \qquad (2.10)$$

Now if the system of homogeneous algebraic equations (2.12) should have a nontrivial solution then the determinant of the coefficients must vanish. In case of isotropic scattering, we have from (2.3) this requirement yields

$$\begin{vmatrix} 0 & k & 0 & 0 & 0 \ldots\ldots\ldots 0 & 0 & 0 & 0 \\ k & -3 & 2k & 0 & 0 \ldots\ldots\ldots 0 & 0 & 0 & 0 \\ 0 & 2k & -5 & 3k & 0 \ldots\ldots\ldots 0 & 0 & 0 & 0 \\ 0 & 0 & 3k & -7 & 4k \ldots\ldots\ldots 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \ldots\ldots (N-2)k & -(2n-3)k & (N-1)k & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 \dots\dots 0 & (N-1)k & -(2N-1) & Nk \\ 0 & 0 & 0 & 0 & 0 \dots\dots 0 & 0 & Nk & -(2N+1) \end{vmatrix} \quad (2.11)$$

Solving (2.11) one gets a different set of values of k, say $k_i, i = 0, 1, \dots N$, then from each k_i a set of $g_m(k_i)$s $m = 0, 1, 2 \dots N$ is determined from (2.9) and a solution is a linear combination of these, viz.

$$\psi_m(\tau) = \sum_{i=0}^{N} A_i g_m(k_i) e^{k_i \psi}, \quad m = 0, 1, \dots\dots\dots\dots N \qquad (2.12)$$

The unknown coefficients A_i associated with (2.12) are determined from the boundary condition of the problem. Once the function $\psi_m(\tau)$ is known the distribution of radiation intensity is determined from (2.10).

3. Works done so far :

The spherical harmonic method was used by for solving problems of radiative transfer and neutron transport in case of both plane and spherical geometry.and subsequently some other authors used several modifications of the intensity to solve the radiative transfer problems by double interval spherical harmonic method.

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A new Modification of spherical harmonic method in solving transport problems

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Abstract :

A new modification of the form of intensity in the double interval spherical harmonic method has been introduced. The equation of radiative transfer in isotropically scattering atmosphere has been solved with this modified spherical harmonic method.

1. Introduction

Kourganoff (1952) analyzed the method of single interval spherical harmonics for solving the equation of transfer and suggested a possible modification. Wilson and Sen (1963, 1964, 1964a, 1964b, 1965a, 1965b, 1965c) solved several radiative transfer problems in plane and spherical geometry and also a nuetron transport problem using some approximations to the intensity with the spherical harmonic method.

Bishnu (1968) solved the equation of transfer for plane parallel isotropic scattering using a different approximate form of the intensity.

Karanjai and Talukdar (1992) solved the equation of transfer with general phase function using the form of intensity given by Bishnu (1964) and deduced the results with phase functions like (i) Planetary, (ii) Rayleigh, (iii) Henyey Greenstein from the general solution.

Wan et el (1977, 1986) used another form of intensity function to get the solution of the equation of transfer.

Karanjai and Biswas (1992, 1993) applied the same method with the form of intensity given by Wan et al (1986) to solve transfer equation with Rayleigh phase function and with

$$p(\mu,\mu') = 1 + \omega_1 P_1(\mu) P_1(\mu') + \omega_2 P_2(\mu) P_2(\mu').$$

Here in this paper we like to introduce a new form of the intensity viz,

$$I^{+}(\tau,\mu)=I(0,0)\,[\phi(\tau)+\psi(\mu)+\sum_{1=0}^{l_0}(2l+1)I_1(\mu)\mu\,P_1(2\mu-1)]\quad 0\le\mu\le 1 \tag{1.1}$$

$$I^{-}(\tau,\mu)=I(0,0)\,[\phi(\tau)+\psi(\mu)+\sum_{l=0}^{l_0}(2l+1)I_1(\mu)\mu\,P_1(2\mu+1)]\quad -1\le\mu<0 \tag{1.2}$$

where $\phi(\tau)$ a function of τ only and $\psi(\mu)$ is given by,

$$\psi(\mu) = 1 \text{ if } 0 \le \mu \le 1 \tag{1.3a}$$

$$\psi(\mu) = 0 \text{ if } -1 \le \mu < 0 \tag{1.3a}$$

2. The Equation of Transfer and boundary conditions

The equation of transfer for plane parallel isotropically scattering atmosphere is given by

$$\frac{\partial}{\partial\tau} I(\tau,\mu) = I(\tau,\mu) - \tfrac{1}{2}\int_{-1}^{+1} I(\tau,\mu')d\mu' \qquad (2.1)$$

Where I (τ,μ) is the specific intensity of radiation at an optical depth τ and in a direction θ with the outward drawn normal and $\mu = \cos(\theta)$. The optical thicness is given by

$$\tau = \int_z^{\alpha} k\rho dz \qquad (2.2)$$

where k is the absorption coefficient.

The Equation of Trasfer (2.1) is to be solved subject to the boundary conditions,

(a) Absence of incident radiation from outside at the free surface $\tau = 0$, i.e.,

$$I(\tau, \mu) \equiv 0 \text{ for } -1 \le \mu \le 0 \qquad (2.3)$$

(b)The convergence of intensity as τ tends to α i.e,

$$I(\tau, \mu)e^{-\tau} \rightarrow 0 \text{ as } \tau \rightarrow \alpha \qquad (2.4)$$

We shall seek a solution to equation (2.1) so that $I(\tau,\mu)$ can be represented by two different expressions $I^+(\tau,\mu)$ and $I^-(\tau,\mu)$ for μ in the intervals (0,1) and (–1,0) respectively in the form given by equations (1.1) – (1.3).

With these two representations the equation of transfer (2.1) takes the following forms,

$$\mu\frac{\partial I^+}{\partial\tau}(\tau,\mu) = I^+(\tau,\mu) - \tfrac{1}{2}\left[\int_0^{+1} I^+(\tau,\mu)d\mu + \int_{-1}^{0} I^-(\tau,\mu)d\mu\right] \qquad (2.5)$$

$$\mu\frac{\partial I^-}{\partial\tau}(\tau,\mu) = I^-(\tau,\mu) - \tfrac{1}{2}\left[\int_0^{+1} I^+(\tau,\mu)d\mu + \int_{-1}^{0} I^-(\tau,\mu)d\mu\right] \qquad (2.6)$$

We shall use the recurrance formulae,

$$\mu P_1(2\mu\pm1) = \frac{1}{2(2l+1)}\left[(l+1)P_{l+1}(2\mu\pm1) \mp (2l+1)P_l(2\mu\pm1) + lP_{l-1}(2\mu\pm1)\right] \qquad (2.7)$$

We shall take advantage of orthogonal properties of $P_l(2\mu-1)$ and $P_l(2\mu+1)$ in (0,1) and (–1,0) respectively.

Using these we find that

$$\int_0^{+1} I^+(\tau,\mu)d\mu = I(0,0)\left[\phi(\tau) + 1 + \tfrac{1}{2}(I^+_0 + I^+_1)\right] \qquad (2.8a)$$

and

$$\int_{-1}^{0} I^-(\tau,\mu)d\mu = I(0,0)\left[\phi(\tau) + \tfrac{1}{2}(-I^-_0 + I^-_1)\right] \qquad (2.8b)$$

using (2.8a) and (2.8b) in the equation of transfer (2.5) we find that,

$$\mu\frac{\partial I^+}{\partial\tau}(\tau,\mu) = I^+(\tau,\mu) - \tfrac{1}{2}I(0,0)\left[2\phi(\tau) + 1 + \tfrac{1}{2}(I^+_0 - I^-_0 + I^+_0 + I^-_0)\right] \qquad (2.9)$$

Similarly using (2.8a) and (2.8b) in the equation of transfer (2.6) we find that

$$\frac{\mu\partial I^-}{\partial\tau}(\tau,\mu) = I^-(\tau,\mu) - \tfrac{1}{2} I(0,0)\left[2\phi(\tau) + 1 + \tfrac{1}{2}(I^+_0 - I^-_0 + I^+_1 + I^-_1)\right] \tag{2.10}$$

Using (1.1) in (2.9) and then multiplying by $P_l(2\mu-1)$, we find after integrating over [0,1],

$$\phi'(\tau)\int_0^1 \mu P_l(2\mu-1)d\mu + \frac{1}{4(2l+1)}\left[\frac{l^2-1}{2l-1} I^{+'}_{l-2} + 2l I^{+'}_{l-1} + \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)} I^{-'}_{l}\right.$$
$$\left. + 2(l+1) I^{+'}_{l+1} + \frac{l^2+3l+2}{2l+3} I^{+'}_{l+2}\right] = \int_0^1 \psi(\mu)P_l(2\mu-1)d\mu + \frac{1}{2(2l+1)}\left[l I^{+}_{l-1}\right.$$
$$+ (2l+1) I^+_l + (l+1) I^+_{l+1} - \tfrac{1}{2}\delta_{0l} - \tfrac{1}{4}\delta_{0l}\left[I^+_0 - I^-_0 + I^+_1 + I^-_1\right] \tag{2.11}$$

Similarly using (1.2) in (2.10) and then multiplying (2.10) by $P_l(2\mu+1)$ and integrating over [–1,0] we find that,

$$\phi'(\tau)\int_{-1}^{0} \mu P_l(2\mu+1)d\mu + \frac{1}{4(2l+1)}\left[\frac{l^2-1}{2l-1} I^{-'}_{l-2} - 2l I^{-'}_{l-1} \frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)} I^{-'}_{l}\right.$$
$$\left. - 2(l+1) I^{-'}_{l+1} + \frac{l^2+3l+2}{2l+3} I^{-'}_{l+2}\right] = \int_{-1}^{0} \psi(\mu)P_l(2\mu+1)d\mu + \frac{1}{2(2l+1)}\left] l I^{-}_{l-1} - \right.$$
$$(2l+1) I^-_l + (l+1) I^-_{l+1}\Big] - \tfrac{1}{2}\delta_{0l} - \tfrac{1}{4}\delta_{0l}\left[I^+_0 - I^-_0 + I^+_1 + I^-_1\right] \tag{2.12}$$

where I'_l are derivatives of I_l with respect to the optical thickness τ.

Separating the equations for $l=0$ and $l=1$ from rest of the equations in (2.11) we have

$l=0$:

$$\tfrac{1}{2}\phi'(\tau) + \tfrac{1}{4}\left[\tfrac{4}{3} I^{+'}_0 + 2I^{'}_1 + \tfrac{2}{3} I^{+'}_2\right] = \tfrac{1}{2} + \tfrac{1}{2}(I^+_0 + I^+_1) - \tfrac{1}{4}(I^+_0 - I^-_0 + I^+_0 + I^-_1) \tag{2.13}$$

$l=1$:

$$(2I^{+'}_0 + \tfrac{24}{5} I^{+'}_1 + 4I^{+'}_2 + \tfrac{6}{5} I^{+'}_3) - 2[I^{+}_0 + 3I^{+}_1 + 2I^{+'}_2] = -2\phi'(\tau) \tag{2.14}$$

$l \neq 0, 1$:

$$\phi'(\tau)\int_0^1 \mu P_l(2\mu-1)d\mu + \frac{1}{4(2l+1)}\left[\frac{l^2-1}{2l-1} I^{+'}_{l-2} + 2l I^{+'}_{l-1}\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)} I^{+'}_{l}\right.$$
$$\left. + 2(l+1) I^{+'}_{l+1} + \frac{l^2+3l+2}{2l+3} I^{+'}_{l+2}\right] = \int_0^1 \psi(\mu)P_l(2\mu-1)d\mu + \frac{1}{2(2l+1)}\Big] l I^{+}_{1-1}$$
$$(2l+1) I^+_l + (l+1) I^+_{1+1}\Big] \tag{2.15}$$

Similarly separating the equations for l=0 and l=1 from rest of the Equations in (2.12) we have,

l=0:

$$-\frac{1}{2}\phi'(\tau)+\frac{1}{4}\left[\frac{4}{3}I^{-'}_{0}-2I^{-'}_{1}+\frac{2}{3}I^{-'}_{2}\right]=-\frac{1}{2}-\frac{1}{2}\left(-I^{-}_{0}+I^{-}_{1}\right)-\frac{1}{4}\left(I^{+}_{0}-I^{-}_{0}+I^{+}_{1}+I^{-}_{1}\right) \qquad (2.16)$$

l=1:

$$\left(-2I^{-'}_{0}+\frac{24}{5}I^{-'}_{1}-4I^{-'}_{2}+\frac{6}{5}I^{-'}_{3}\right)-2\left[I^{-}_{0}-3I^{-}_{1}+2I^{-}_{2}\right]=-2\phi'(\tau) \qquad (2.17)$$

l≠0, 1:

$$\phi'(\tau)\int_{-1}^{0}\mu P_l(2\mu+1)d\mu+\frac{1}{4(2l+1)}\left[\frac{l^2-1}{21-1}I^{-'}_{l-2}+21I^{-'}_{l-1}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I^{-'}_{l}-2(1+1)I^{-'}_{1+1}+\frac{l^2+3l+2}{2l+3}I^{-'}_{l+2}\right]=\int_{-1}^{0}\psi(\mu)P_l(2\mu+1)d\mu+\frac{1}{2(2l+1)}\left[lI^{-}_{l-1}-(2l+1)I^{-}_{l}+(l+1)I^{-}_{l+1}\right] \qquad (2.18)$$

Using the condition given in (1.3) we find that equations (2.15) and 2.18) reduce for 1≠0, 1:

$$\phi'(\tau)\int_{0}^{1}\mu P_l(2\mu-1)d\mu+\frac{1}{4(2l+1)}\left[\frac{l^2-1}{2l-1}I^{+'}_{l-2}+2lI^{+'}_{1-1}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I^{+'}_{l}+2(l+1)I^{+'}_{1+1}+\frac{l^2+3l+2}{2l+3}I^{+'}_{l+2}\right]=\int_{0}^{1}\psi(\mu)P_l(2\mu-1)d\mu+\frac{1}{2(2l+1)}\left[lI^{+}_{l-1}+(2l+1)I^{+}_{l}+(l+1)I^{+}_{l+1}\right] \qquad (2.19)$$

and

$$\phi'(\tau)\int_{-1}^{0}\mu P_l(2\mu+1)d\mu+\frac{1}{4(2l+1)}\left[\frac{l^2-1}{2l-1}I^{-'}_{l-2}-2lI^{-'}_{1-1}+\frac{12l^3+18l^2-2l-4}{(2l+3)(2l-1)}I^{-'}_{l}-2(l+1)I^{-'}_{l+1}+\frac{l^2+3l+2}{2l+3}I^{-'}_{l+2}\right]=\frac{1}{2(2l+1)}\left[lI^{-}_{l-1}-(2l+1)I^{-}_{l}+(l+1)I^{-}_{l}\right] \qquad (2.20)$$

We take the form of $\phi(\tau)$ as

$\phi(\tau)=Ae^{-\tau}+Be^{\tau}$ where τ is small and A and B are constants. Equations (2.13), (2.14), (2.16), (2.17) are to be solved subject to the boundary conditions (2.3) and (2.4).

It is assumed that when we are working in the l^{th} approximation,

$$I^{+}_{l+1}=I^{-}_{l+1}=0 \qquad (2.21)$$

3. First Approximation

We try a solution of the form

$$I^{+}_{1}(\tau) = g_{1,\alpha}\, e^{-k\tau} + g_{1,\beta} e^{-\tau} + g_{1,\gamma}\, e^{\tau} \tag{3.1a}$$

$$I^{-}_{1}(\tau) = h_{1,\alpha}\, e^{-k\tau} + h_{1,\beta} e^{-\tau} + h_{1,\gamma}\, e^{\tau} \tag{3.1b}$$

For first approximation solution we have $I_o = 1$

Hence (1.1) and (1.2) give us

$$I^{+}(\tau,\mu) = I\,(0,0)\,[\phi(\tau) + \psi(\mu) + I^{+}_{0}(\tau)\mu + 3I^{+}_{1}(\tau)\mu P_1(2\mu-1)]\ 0 \le \mu \ge 1 \tag{3.2a}$$

$$I^{-}(\tau,\mu) = I\,(0,0)\,[\phi(\tau) + \psi(\mu) + I^{-}_{0}(\tau)\mu + 3I^{-}_{1}(\tau)\mu P_1(2\mu+1)] - 1 \le \mu \ge 0 \tag{3.2b}$$

Taking $I^{+\prime}_{2} = 0$ we get from (2.13)

$$\left(\tfrac{4}{3} I^{+\prime}_{0} + 2I^{+\prime}_{1}\right) - \left(I^{+}_{0} + I^{-}_{0} + I^{+}_{1} - I^{-}_{1}\right) = 2\,[\,1 - \phi'(\tau)] \tag{3.3}$$

Similarly from (2.16) we get

$$\left(\tfrac{4}{3} I^{-\prime}_{0} - 2I^{-\prime}_{1}\right) - \left(I^{+}_{0} + I^{-}_{0} + I^{+}_{1} - I^{-}_{1}\right) = 2\,[\phi'(\tau) - 1] \tag{3.4}$$

Likewise (2.14) and (2.17) give us

$$\left(2I^{+\prime}_{0} + \tfrac{24}{5} I^{+\prime}_{1}\right) - 2\left(I^{+}_{0} + 3I^{+}_{1}\right) = -\,2\phi'(\tau) \tag{3.5}$$

$$\left(-2I^{-\prime}_{0} + \tfrac{24}{5} I^{-\prime}_{1}\right) - 2\left(I^{-}_{0} + 3I^{-}_{1}\right) = -\,2\phi'(\tau) \tag{3.6}$$

Substituting (3.1a) and (3.1b) in (3.3), (3.4), (3.5) and (3.6) and then equating the coefficients of $e^{-k\tau}$ on each side we obtain the following equations,

$$\left.\begin{aligned}
&(1 + \tfrac{4K}{3})g_{0,\alpha} + (1 + 2K)g_{1,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0\\
&(2 + 2K)g_{0,\alpha} + 6(1 + \tfrac{4K}{5})g_{1,\alpha} = 0\\
&g_{0,\alpha} + g_{1,\alpha} + (1 - \tfrac{4K}{3})\, h_{0,\alpha} - (1 - 2K)h_{1,\alpha} = 0\\
&-(2 - 2K)h_{0,\alpha} + 6(1 - \tfrac{4K}{5})h_{1,\alpha} = 0
\end{aligned}\right] \tag{3.7}$$

The determinant corresponding to (3.7) will be

$$D_1(K) = \begin{vmatrix} 1 + \frac{4K}{3} & 1+2K & 1 & -1 \\ 2+2K & 6(1 + \frac{4K}{5}) & 0 & 0 \\ 1 & 1 & 1 - \frac{4K}{3} & -1+2K \\ 0 & 0 & 2-2K & 6(1 - \frac{4K}{5}) \end{vmatrix}$$

where $D_1(K) = 0$ gives us

$(16K^2/225)\,(81K^2 - 270) = 0$

$\therefore\ K = 0,\, 0,\, \pm 1.8257$

Retaining the positive root we have K=1. 8257.
We have a set of boundary conditions,

$$I^{+}_{0}(0) = I^{+}_{1}(0) = 0,\ I^{-}_{0}(\tau_0) = I^{-}_{1}(\tau_0) = 1 \tag{3.8}$$

Using the first equation in (3.7) on $\phi(\tau) = Ae^{-\tau} + Be^{\tau'}$ we obtain A= 5.5166547, B=4.5166547

Again equating the coefficients of e^{τ} and $e^{-\tau}$ on each side we obtain,

$$\tfrac{7}{3} g_{0,\beta} + 3g_{1,\beta} + h_{0,\beta} - h_{1,\beta} = -2A \tag{3.9a}$$

$$\tfrac{7}{3} g_{0,\gamma} + 3g_{1,\gamma} + h_{0,\gamma} - h_{1,\gamma} = -2B \tag{3.9b}$$

$$2g_{0,\beta} + \tfrac{27}{5} g_{1,\beta} = -A \tag{3.9c}$$

$$\tfrac{1}{5} g_{1,\gamma} = B \tag{3.9d}$$

$$g_{0,\beta} + g_{1,\beta} - \tfrac{1}{3} h_{0,\beta} + h_{1,\beta} = -2A \tag{3.9e}$$

$$g_{0,\gamma} + g_{1,\gamma} + \tfrac{7}{3} h_{0,\gamma} - h_{1,\gamma} = 2B \tag{3.9f}$$

$$\tfrac{4}{5} h_{1,\beta} = 2B \tag{3.9g}$$

$$-2h_{0,\gamma} + \tfrac{27}{5} h_{1,\gamma} = -B \tag{3.9h}$$

On solving the set of equations in (3.9) we obtain

$g_{0,\beta} = 21.1952$ $\qquad$ $h_{0,\beta} = -46.6974$

$g_{1,\beta} = -8.8717$ $\qquad$ $h_{1,\beta} = 13.7915$

$g_{0,\gamma} = 18.6307$ $\qquad$ $h_{0,\gamma} = 48.0470$

$g_{1,\gamma} = -7.5276$ $\qquad$ $h_{1,\gamma} = -5.6467$

Now solving the set of equations in (3.7) and using the values of $g_{1,\alpha}$'s and $h_{1,\alpha}$'s obtained from (3.9) and using the boundary conditions (3.8) we find that,

$g_{0,\alpha} = -39.9259$ $\qquad$ $h_{0,\alpha} = -13.0211$
$g_{1,\alpha} = 16.3993$ $\qquad$ $h_{1,\alpha} = -7.4875$

Finally from (3.1a) and (3.1b) we have
$I^{+}_{0} = 6.508$, $I^{+}_{1} = -2.6912$, $I^{-}_{0} = 1.945496 10^{-3}$, $I^{-}_{1} = -2.0504 10^{-5}$

4.Second Approxiamtion

In this case we will take $l_0 = 2$ and as before we will consider (2.11) and (2.12). We shall take $\phi(\tau) = -Ae^{-\tau} + Be^{\tau}, \tau$ is small. For $l_0=2$, the equations (1.1) and (1.2) respectively become

$$I^{+}(\tau,\mu) = I(0,0)\,[\,\phi(\tau) + \psi(\mu) + I^{+}_{0}(\tau)\mu + 3I^{+}_{1}(\tau)\mu\, P_1(2\mu - 1) + 5I^{+}_{2}(\tau)\mu P_2(2\mu - 1)]\ 0 \le \mu \ge 1 \tag{4.1a}$$

$$I^{-}(\tau,\mu) = I(0,0)\,[\,\phi(\tau) + \psi(\mu) + I^{-}_{0}(\tau)\mu + 3I^{-}_{1}(\tau)\mu\, P_1(2\mu + 1) + 5I^{-}_{2}(\tau)\mu P_2(2\mu+1)]\ -0 \le \mu \ge 0 \tag{4.1b}$$

Therefore when l=0 we have from (2.11)

l=0 :

$$\left(\frac{4}{3}I^{+\prime}_0 + 2I^{\prime}_1 + \frac{2}{3}I^{+\prime}_2\right) - [I^{+}_0 + I^{-}_0 + I^{+}_1 - I^{-}_1] = 2 - 2\phi'(\mu) \tag{4.2}$$

and from (2.12)

1=0 :

$$\left(\frac{4}{3}I^{-\prime}_0 - 2I^{-\prime}_1 + \frac{2}{3}I^{-\prime}_2\right) + [I^{+}_0 + I^{-}_0 + I^{+}_1 - I^{-}_1] = -2 - 2\phi'(\tau) \tag{4.3}$$

For l=1 Equation (2.11) gives

l=1 :

$$(2I^{+\prime}_0 + \frac{24}{5}I^{\prime}_1 + 4I^{+\prime}_2) - 2[I^{+}_0 + 3I^{+}_1 + 2I^{+}_2] = -2\phi'(\tau) \tag{4.4}$$

and from (2.14)

l=1 :

$$(-2I^{-\prime}_0 + \frac{24}{5}I^{-\prime}_1 + 4I^{-\prime}_2) - 2[I^{-}_0 + 3I^{-}_1 + 2I^{-}_2] = -2\phi'(\tau) \tag{4.5}$$

Finally for 1=2 from (2.11) and (2.12)

l = 2 :

$$\left(\frac{1}{3}I^{+\prime}_0 + 2I^{\prime}_1 + \frac{80}{21}I^{-\prime}_2\right) - [2I^{+\prime}_1 + 5I^{+}_2] = -5\phi'(\tau) \tag{4.6}$$

and from (2.12)

l = 2 :

$$\left(\frac{1}{3}I^{-\prime}_0 - 2I^{-\prime}_1 + \frac{80}{21}I^{-\prime}_2\right) - [2I^{-\prime}_1 - 5I^{-}_2 = -5\phi'(\tau) \tag{4.7}$$

We will take a trial solution of the form

$$I^{+}_1(\tau) = g_{1,\alpha}e^{-k\tau} + g_{1,\beta}e^{-\tau} + g_{1\gamma}e^{\tau} \tag{4.8a}$$

$$I^{-}_1(\tau) = h_{1,\alpha}e^{-k\tau} + h_{1,\beta}e^{-\tau} + h_{1\gamma}e^{\tau} \tag{4.8b}$$

If we insert (4.8a) and (4.8b) in (4.2) to (4.7) and then compare the coefficiens of $e^{-k\tau}$ on each side we get.

$$\frac{4K}{3}g_{0,\alpha} + (2K+1)g_{1,\alpha} + \frac{2K}{3}g_{2,\alpha} + h_{0,\alpha} - h_{1,\alpha} = 0 \tag{4.9a}$$

$$(2+2K)g_{0,\alpha} + 6\left(1+\frac{4K}{5}\right)g_{1,\alpha} + (4+4K)g_{2,\alpha} = 0 \tag{4.9b}$$

$$\frac{K}{3}g_{0,\alpha} + (2K+2)g_{1,\alpha} + 5\left(1+\frac{16K}{21}\right)g_{2,\alpha} = 0 \tag{4.9c}$$

$$g_{0,\alpha} + g_{0,\alpha} + \left(1-\frac{4K}{3}\right)h_{0,\alpha} + (2K-1)h_{1,\alpha} - \frac{2K}{3}h_{2,\alpha} = 0 \tag{4.9d}$$

$$(2K-2)h_{0,\alpha} + 6\left(1-\frac{4K}{5}\right)h_{1,\alpha} + (4K-4)h_{2,\alpha} = 0 \tag{4.9e}$$

$$\frac{K}{3}h_{0,\alpha} + (2K-2)h_{1,\alpha} + 5\left(1-\frac{16K}{5}\right)g_{2,\alpha} = 0 \tag{4.9f}$$

Similarly inserting (4.8a) and (4.8b) in (4.2) to (4.7) and then equating the coefficients of $e^{-\varsigma}$ and e^{ς} on each side we get respectively,

$$\frac{7}{3}g_{0.\beta}+3g_{1.\beta}+\frac{2}{3}g_{2.\beta}+h_{0.\beta}-h_{1.\beta} = -2A \qquad (4.10a)$$

$$4g_{0.\beta}+\frac{54}{5}g_{1.\beta}+8g_{2.\beta} = -2A \qquad (4.10b)$$

$$\frac{1}{3}g_{0.\beta}+4g_{1.\beta}+\frac{185}{21}g_{2.\beta} = -5A \qquad (4.10c)$$

$$g_{0.\beta}+g_{1.\beta}-\frac{1}{3}h_{0.\beta}+h_{1.\beta}-\frac{2}{3}h_{2.\beta} = 2A \qquad (4.10d)$$

$$\frac{6}{5}h_{1.\beta} = 2A \qquad (4.10e)$$

$$\frac{1}{3}h_{0.\beta}+\frac{25}{21}h_{2.\beta} = -5A \qquad (4.10f)$$

and

$$\frac{1}{3}g_{0.\gamma}+g_{1.\gamma}+\frac{2}{3}g_{2.\gamma}+h_{0.\gamma}-h_{1.\gamma} = -2B \qquad (4.11a)$$

$$\frac{6}{5}g_{1.\gamma} = -2B \qquad (4.10b)$$

$$\frac{1}{3}g_{0\gamma}-\frac{25}{21}g_{2.\gamma} = -5B \qquad (4.11c)$$

$$g_{0.\gamma}+g_{1.\gamma}+\frac{7}{3}h_{0.\upsilon}-3h_{1.\gamma}+\frac{2}{3}h_{2.\gamma} = -2B \qquad (4.11d)$$

$$-4h_{0.\gamma}-\frac{54}{45}h_{2.\gamma}-8h_{2.\gamma} = -2B \qquad (4.11e)$$

$$\frac{1}{3}h_{0.\gamma}-4h_{1.\gamma}+\frac{185}{21}h_{2\gamma} = 5B \qquad (4.11f)$$

The determinant corresponding to (4.9) is

$$D_2(K) = \begin{vmatrix} \frac{4k}{3} & 2K+1 & \frac{2k}{3} & 1 & -1 & 0 \\ 2+2K & 6(1+\frac{4k}{5}) & 4+4K & 0 & 0 & 0 \\ \frac{k}{3} & 2+2K & 5(1+\frac{16k}{3}) & 0 & 0 & 0 \\ 1 & 1 & 0 & 1-\frac{4k}{3} & 2K-1 & \frac{2k}{3} \\ 0 & 0 & 0 & 2K-2 & 6(1-\frac{4k}{5}) & 4K-4 \\ 0 & 0 & 0 & -\frac{k}{3} & 2K-2 & 5(1-\frac{16k}{5}) \end{vmatrix}$$

when $D_2(K) = 0$, we have

$$8K^2\,[.9404K^4 - 14.1061K^2 + 24] = 0 \text{ so that}$$

$$K = 0,\ 0,\ \pm 3.6116,\ \pm 1.3988$$

Let us take $K_1 = 3.6116$, $K_2 = 1.3988$ (4.12)

The boundary conditions may be restated as

$$I^-_1(\tau) \equiv 0 \tag{4.13a}$$

$$I^+_1(\tau)\, e^{-\tau} \to 0 \text{ as } \tau \to \alpha \text{ and} I^-_1(\tau) e^{-\tau} \to 0 \text{ as } \tau \to \alpha \tag{4.13b}$$

Using the (4.19a) and (4.19b) we can write (See Wilson and Sen 1963)

$$\sum_{l=1}^{n} h^{(r)}_{l,\alpha} + h_{l\beta} = 0 \text{ (for all l)} \tag{4.14}$$

Solving (4.9), (4.10), (4.11), (4.12) and using (4.14) we obtain

$g^{(1)}_{0,\alpha} = -180.8898$

$g^{(1)}_{1,\alpha} = 101.9557$

$g^{(1)}_{2,\alpha} = 38.5290$

$g^{(2)}_{0,\alpha} = -11.5438$

$g^{(2)}_{1,\alpha} = -6.1007$

$g^{(2)}_{2,\alpha} = 13.8553$

$g_{0,\beta} = -18.944322$

$g_{1,\beta} = 11.727565$

$g_{2,\beta} = -7.7392102$

$g_{0,\gamma} = -36.22498$

$g_{1,\gamma} = -7.5277$

$g_{2,\gamma} = -29.112963$

$h^{(1)}_{0,\alpha} = -9.3888$

$h^{(1)}_{1,\alpha} = -12.2123$

$h^{(1)}_{2,\alpha} = 13.6550$

$h^{(2)}_{0,\alpha} = -2.9256$

$h^{(2)}_{1,\alpha} = 3.0180$

$h^{(2)}_{2,\alpha} = 6.0737$

$h_{1,\beta} = -12.290109$

$h_{1,\beta} = 9.1943$

$h_{2,\beta} = -19.728719$

$h_{0,\gamma} = -155.71057$

$h_{1,\gamma} = -8.869875$

$h_{2,\gamma} = -33.825808$

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