CHAPTER - III

LARGE DEFLECTION OF A CIRCULAR PLATE ON ELASTIC FOUNDATION AND SUPPORTED AT SEVERAL POINTS ALONG THE BOUNDARY *

PAPER - I

Introduction :

Small deflections of thin plates placed on elastic foundations have been examined by S. Timoshenko and S. Woinowsky - Krieger (1959) and several other authors on the assumption that strain due to stretching of the middle surface of the plate is negligible. When the deflections are moderately large, that is, on the order of the thickness of the plate, then the forces in the middle surface of the plate must be taken into account. In the case of such large deflections of plates, three differential equations for displacements and deflections may be written, but it is usually difficult to obtain the solutions of these equations because of their nonlinear character.

On the other hand, various problems of large deflections of plates not resting on elastic foundations have been examined by S. Way (1934), S. Levy (1942) and many other authors. But the methods used by them require considerable computation. A simple and approximate, yet fairly accurate, method of analysing large deflections of plates was suggested by H. M. Berger (1955). The method uses the technique

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of neglecting the second strain invariant of the middle surface strains in the expression corresponding to the total potential energy of the system. Berger's method reduces computation and although no complete explanation of this method is offered in, the stresses and deflections obtained for both rectangular and circular plates are in good agreement with those found in practical analysis. This approximate method has been applied successfully by Nowinski (1958) to his plate problems. Nash and Modeer (1959) investigated the problems having no axial symmetry following this method.

The technique of neglecting the second strain invariant has been successfully applied by Sinha (1963) to determine large deflection of circular and rectangular plates placed on elastic foundations and under uniform lateral loads.

In this paper large deflection of a circular plate placed on elastic foundation and supported at several points along the boundary has been solved. The load is assumed to be uniformly distributed and the foundation is of the Winkler type. A complete analysis of a particular case, where the number of supports is two, is given.

Formulation of problem :

For moderately large deflections, the strain displacement relationships and the strain energy of the middle plane of the plate are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \ldots (3.1)$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \ldots (3.2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \qquad \ldots (3.3)$$

$$V_1 = \frac{D}{2} \iint \Big[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \Big\{ \frac{12}{h^2} e_2 + \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \Big(\frac{\partial^2 w}{\partial x \partial y} \Big)^2 \Big\} \Big] \, dx \, dy \qquad \ldots (3.4)$$

in which e_1 and e_2 are the first and second middle surface strain invariants, respectively. Neglecting e_2 and by adding the potential energy of the transverse load and of the foundation reaction, K the modified energy equation becomes [Sinha (1963)].

$$V = \frac{D}{2} \iint \Big[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu) \Big\{ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \Big(\frac{\partial^2 w}{\partial x \partial y} \Big)^2 \Big\} + \frac{K}{D} w^2 - \frac{2 q w}{D} \Big] \, dx \, dy \qquad \ldots (3.5)$$

Applying Euler's variational method to Eq.(3.5) the following differential equations in polar co-ordinates are obtained

$$\nabla^4 w - \alpha^2 \nabla^2 w + \frac{K}{D} w = \frac{q}{D} \qquad \ldots (3.6)$$

where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{2} \Big(\frac{\partial w}{\partial r} \Big)^2 + \frac{u}{r} + \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} + \frac{1}{2 r^2} \Big(\frac{\partial w}{\partial \theta} \Big)^2 \qquad \ldots (3.7)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Solution of problem :

Let the circular plate (Fig. 3.1) be of radius a, supported at several points along the boundary and placed on the elastic foundation. Let the centre of the plate be the origin and a diameter as the initial line, $\theta = 0$. Let the general solution of Eq.(3.6) be in the form

$$W = W_0 + W_1 \qquad \text{... (3.8)}$$

in which W_0 is the large deflection of a plate placed on elastic foundation and simply supported along the entire boundary and W_1 satisfies the equation

$$\nabla^4 W_1 - \alpha^2 \nabla^2 W_1 + \frac{K}{D} W_1 = 0 \qquad \text{... (3.9)}$$

Eq. (3.9) can be written in the form

$$(\nabla^2 - \rho_1^2)(\nabla^2 - \rho_2^2) W_1 = 0 \qquad \text{... (3.10)}$$

where

$$\rho_1^2 + \rho_2^2 = \alpha^2 \qquad \text{... (3.11)}$$

$$\rho_1^2 \rho_2^2 = \frac{K}{D} \qquad \text{... (3.12)}$$

Considering the number of points of support is i, and denoting the concentrated reactions at these points N_1, N_2, ... N_i, the expression for each reaction N_i is

$$\frac{N_i}{\pi a}\left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta_i\right] \qquad \text{... (3.13)}$$

where $\theta_i = \theta - \psi_i$, ψ_i is the angle defining the position of the support i.

The intensily of the reactive forces at any point of the boundary is then given by the expression

$$\sum_{i=1}^{i} \frac{N_i}{\pi a}\left[\frac{1}{2}+\sum_{m=1}^{\infty} \cos m\theta_i\right] \qquad \ldots (3.13a)$$

in which the summation is extended over all the concentrated reactions. Assuming that the plate is solid and considering that deflections and moments at the centre must be finite, the appropriate solution of Eq.(3.9) can be taken in the form

$$\begin{aligned} W_1 = {} & A_0 I_0(P_1 r) + B_0 I_0(P_2 r) + \\ & + \sum_{m=1}^{\infty}\left[A_m I_m(P_1 r) + B_m I_m(P_2 r)\right]\cos m\theta + \\ & + \sum_{m=1}^{\infty}\left[A'_m I_m(P_1 r) + B'_m I_m(P_2 r)\right]\sin m\theta \end{aligned} \qquad \ldots (3.14)$$

in which I_0 is the modified Bessel function of the first kind and zero order, and I_m is of the first kind and mth order.

For determining the constants we have the following conditions at the boundary.

$$(W)_{\substack{r=a \\ \theta=0,\pi}} = 0 \qquad \ldots (3.15)$$

$$\left[\frac{\partial^2 w_1}{\partial r^2} + \frac{\nu}{r}\frac{\partial w_1}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right]_{r=a} = 0 \qquad \ldots (3.16)$$

$$\left[Q_r - \frac{1}{r}\cdot\frac{\partial}{\partial\theta} M_{r\theta}\right]_{r=a} = -\sum_{i=1}^{i}\frac{N_i}{\pi a}\left[\frac{1}{2}+\sum_{m=1}^{\infty}\cos m\theta_i\right] \qquad \ldots (3.17)$$

where

$$Q_r = -D\frac{\partial}{\partial r}\left[(\nabla^2-\alpha^2)w_1\right] \qquad \ldots (3.17a)$$

$$M_{r\theta} = (1-\nu)D\left[\frac{1}{r}\cdot\frac{\partial^2 w_1}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w_1}{\partial\theta}\right] \qquad \ldots (3.17b)$$

consider a particular case when the plate is supported at two points which are the two end points of the diameter taken as the initial line from which θ is measured. Then

$$\psi_1 = 0\,, \quad \psi_2 = \pi$$

Considering the above boundary conditions and remembering that deflection is zero at these two points of support one gets after solving for the constants

$$A_0 = \frac{P}{\pi D a}\beta\psi_0(a) \qquad \ldots (3.18)$$

$$B_0 = -\frac{P}{\pi D a}\beta\phi_0(a) \qquad \ldots (3.19)$$

$$A_m = -\frac{P}{\pi D a}\cdot\frac{\mu_m(a)}{\beta_m(a)\mu_m(a)-\lambda_m(a)\eta_m(a)} \qquad \ldots (3.20)$$

$$B_m = \frac{P}{\pi D a}\cdot\frac{\lambda_m(a)}{\beta_m(a)\mu_m(a)-\lambda_m(a)\eta_m(a)} \qquad \ldots (3.21)$$

$$A'_m = 0 = B'_m \qquad \ldots (3.22)$$

where $P = \pi a^2 q =$ total load on the plate

$$\beta = \frac{\left[\lambda_m(a) I_m(P_2 a) - \mu_m(a) I_m(P_1 a)\right]}{\left[\beta_m(a)\mu_m(a) - \lambda_m(a)\eta_m(a)\right] \times \left[I_0(P_2 a)\Phi_0(a) - I_0(P_1 a)\Psi_0(a)\right]} \qquad \ldots (3.23)$$

$$\Psi_0(a) = P_2^2 I_0''(P_2 a) + \frac{\nu}{a} P_2 I_1(P_2 a) \qquad \ldots (3.24)$$

$$\Phi_0(a) = P_1^2 I_0''(P_1 a) + \frac{\nu}{a} P_1 I_1(P_1 a) \qquad \ldots (3.25)$$

$$\mu_m(a) = P_2^2 I_m''(P_2 a) + \frac{\nu}{a} P_2 I_m'(P_2 a) - \frac{\nu m^2}{a^2} I_m(P_2 a) \qquad \ldots (3.26)$$

$$\beta_m(a) = P_2^2 P_1 I_m'(P_1 a) - (1-\nu)\left\{\frac{m^2}{a^3} I_m(P_1 a) - \frac{P_1 m^2}{a^2} I_m'(P_1 a)\right\} \qquad \ldots (3.27)$$

$$\lambda_m(a) = P_1^2 I_m''(P_1 a) + \frac{\nu}{a} P_1 I_m'(P_1 a) - \frac{\nu m^2}{a^2} I_m(P_1 a) \qquad \ldots (3.28)$$

$$\eta_m(a) = P_1^2 P_2 I_m'(P_2 a) - (1-\nu)\left\{\frac{m^2}{a^3} I_m(P_2 a) - \frac{P_2 m^2}{a^2} I_m'(P_2 a)\right\} \qquad \ldots (3.29)$$

Thus the complete solution of Eq.(3.6) is obtained in the following form

$$W = W_0 + A_0 I_0(p_1 r) + B_0 I_0(p_2 r) + \sum_{m=2,4,6,\dots}^{\infty} \left[A_m I_m(p_1 r) + B_m I_m(p_2 r)\right] \cos m\theta \qquad \dots (3.30)$$

where

$$W_0 = \frac{q}{K} + A_0' I_0(p_1 r) + B_0' I_0(p_2 r) \qquad \dots (3.31)$$

$$A_0' = -\frac{q}{K}\left[\frac{p_2^2 I_0''(p_2 a) + p_2 \frac{\nu}{a} I_1(p_2 a)}{\Phi(pa)}\right] \qquad \dots (3.31a)$$

$$B_0' = \frac{q}{K}\left[\frac{p_1^2 I_0''(p_1 a) + p_1 \frac{\nu}{a} I_1(p_1 a)}{\Phi(pa)}\right] \qquad \dots (3.31b)$$

$$\Phi(pa) = \left\{I_0(p_1 a)\, p_2^2 I_0''(p_2 a) - I_0(p_2 a)\, p_1^2 I_0''(p_1 a)\right\} + \frac{\nu}{a}\left\{p_2 I_1(p_2 a) I_0(p_1 a) - p_1 I_1(p_1 a) I_0(p_2 a)\right\} \qquad \dots (3.31c)$$

Substituting the values of the constants A_0' , B_0' , A_0 , B_0 , A_m , and B_m into Eq.(3.30) one gets taking $K_F = \frac{Ka^4}{D}$

$$\frac{w}{h} = \left(\frac{qa^4}{Dh}\right)\left[\frac{1}{K_F}\left\{1 + \frac{\left[P_1^2 I_0''(P_1 a) + P_1 \frac{\nu}{a} I_1(P_1 a)\right] I_0(P_2 r)}{\phi(Pa)} - \right.\right.$$

$$\left. - \frac{\left[P_2^2 I_0''(P_2 a) + P_2 \frac{\nu}{a} I_1(P_2 a)\right] I_0(P_1 r)}{\phi(Pa)}\right\} +$$

$$+ \frac{1}{a^3}\left\{\beta \psi_0(a) I_0(P_1 r) - \beta \phi_0(a) I_0(P_2 r) - \right.$$

$$\left.\left. - \sum_{m=2,4,6,\ldots}^{\infty}\left[\frac{\mu_m(a) I_m(P_1 r) - \lambda_m(a) I_m(P_2 r)}{\beta_m(a)\mu_m(a) - \lambda_m(a)\eta_m(a)}\right]\cos m\theta\right\}\right]$$

... (3.32)

As $P_1 \to 0$ and $P_2 \to 0$, Eq.(3.32) reduces to

$$W = W_0 + \frac{Pa^2}{2\pi D}\frac{1}{(3+\nu)}\left\{2\log 2 - 1 + \frac{1+\nu}{1-\nu}\left(2\log 2 - \frac{\pi^2}{12}\right) - \right.$$

$$\left. - \sum_{m=2,4,6,\ldots}^{\infty}\left[\frac{1}{m(m-1)} + \frac{2(1+\nu)}{m^2(m-1)(1-\nu)} - \frac{\left(\frac{r}{a}\right)^2}{m(m+1)}\right]\left(\frac{r}{a}\right)^m \cos m\theta\right\}$$

... (3.33)

as obtained by Timoshenko (1959) in the corresponding small deflection problem for a plate without foundation and supported at two points on the boundary.

The normalised constant α can be determined from Eqs.(3.7) and (3.3
Since we are interested only in the lateral displacement w, the radia
and cross radial displacements u and v have been eliminated by choosin
suitable expressions for u and v, compatible with their boundary conditions and integrating over the whole area of the plate. The radia and cross radial displacements have been assumed in the forms

$$u = \sum U(r) \cos m\theta \qquad \ldots (3.34)$$

$$v = \sum V(r) \sin m\theta \qquad \ldots (3.35)$$

subject to the boundary conditions $U(a) = V(a) = 0$

Multiplying both sides of the Eq.(3.7) by $r\,dr\,d\theta$ and integrating between the limits 0 to a and 0 to 2π, one gets

$$\int_0^a\int_0^{2\pi} r\,U'(r)\cos m\theta\,dr\,d\theta + \int_0^a\int_0^{2\pi} U(r)\cos m\theta\,dr\,d\theta$$

$$+ \int_0^a\int_0^{2\pi} m\,V(r)\cos m\theta\,dr\,d\theta + \frac{1}{2}\int_0^a\int_0^{2\pi} r\left(\frac{\partial w}{\partial r}\right)^2 dr\,d\theta$$

$$+ \frac{1}{2}\int_0^a\int_0^{2\pi} \frac{1}{r}\left(\frac{\partial w}{\partial \theta}\right)^2 dr\,d\theta = \int_0^a\int_0^{2\pi} \frac{\alpha^2 h^2}{12}\, r\,dr\,d\theta$$

After evaluating the integrals the following equation leading to α is obtained.

$$-\frac{1}{2}A_0'^2 p_1^2 a^2 \left\{\frac{1}{4}\left[I_0(p_1 a)+I_2(p_1 a)\right]^2-\left[1+\frac{1}{p_1^2 a^2}\right]I_1^2(p_1 a)\right\}-$$

$$-\frac{1}{2}B_0'^2 p_2^2 a^2 \left\{\frac{1}{4}\left[I_0(p_2 a)+I_2(p_2 a)\right]^2-\left[1+\frac{1}{p_2^2 a^2}\right]I_1^2(p_2 a)\right\}+$$

$$+2A_0' B_0' p_1 p_2 \frac{a}{p_2^2-p_1^2}\left[-\frac{1}{2}p_1 I_1(p_2 a)\left\{I_0(p_1 a)+I_2(p_1 a)\right\}+\right.$$

$$\left.+\frac{1}{2}p_2 I_1(p_1 a)\left\{I_0(p_2 a)+I_2(p_2 a)\right\}\right]+$$

$$+\sum_{m=2,4,6,\ldots}^{\infty}\left[A_m^2 p_1^2\left\{-\frac{1}{8}a^2\left[\frac{1}{4}\left\{I_{m-2}(p_1 a)+I_m(p_1 a)\right\}^2-\right.\right.\right.$$

$$\left.-\left\{1+\frac{(m-1)^2}{p_1^2 a^2}\right\}I_m^2(p_1 a)\right]-$$

$$-\frac{1}{8}a^2\left[\frac{1}{4}\left\{I_m(p_1 a)+I_{m+2}(p_1 a)\right\}^2-\left\{1+\frac{(m+1)^2}{p_1^2 a^2}\right\}I_{m+1}^2(p_1 a)\right]+$$

$$+\frac{1}{2}\sum_{\substack{n=0\\ n\neq t}}^{\infty}\sum_{t=0}^{\infty}\left(\frac{p_1}{2}\right)^{2m+2n+2t}\cdot\Phi \quad \Big\}\frac{1}{2}+$$

$$+ B_m^2 P_2^2 \Big\{ -\frac{1}{8} a^2 \Big[\frac{1}{4} \big\{ I_{m-2}(P_2 a) + I_m(P_2 a) \big\}^2 + \Big\{ 1 + \frac{(m-1)^2}{P_2^2 a^2} \Big\} I_{m-1}^2 (P_2 a) \Big] -$$

$$- \frac{1}{8} a^2 \Big[\frac{1}{4} \big\{ I_m(P_2 a) + I_{m+2}(P_2 a) \big\}^2 + \Big\{ 1 + \frac{(m+1)^2}{P_2^2 a^2} \Big\} I_{m+1}^2 (P_2 a) \Big] +$$

$$+ \frac{1}{2} \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_2}{2} \right)^{2m+2n+2t} \cdot \phi \Big\} \frac{1}{2} +$$

$$+ \frac{1}{2} A_m B_m P_1 P_2 \Big\{ \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \Big[\frac{\left(\frac{P_1}{2}\right)^{m+2n-1} \left(\frac{P_2}{2}\right)^{m+2t-1} a^{2m+2n+2t}}{(2m+2n+2t) \lfloor n \lfloor t \, \Gamma(m+n) \Gamma(m+t)} +$$

$$+ \left(\frac{P_1}{2} \right)^{m+2n-1} \left(\frac{P_2}{2} \right)^{m+2t+1} \cdot \phi + \frac{\left(\frac{P_1}{2}\right)^{m+2n+1} \left(\frac{P_2}{2}\right)^{m+2t-1} a^{2m+2n+2t+2}}{(2m+2n+2t+2) \lfloor n \lfloor t \, \Gamma(m+t) \Gamma(m+n+2)} +$$

$$+ \frac{\left(\frac{P_1}{2}\right)^{m+2n+1} \left(\frac{P_2}{2}\right)^{m+2t+1} a^{2m+2n+2t+4}}{(2m+2n+2t+4) \lfloor n \lfloor t \, \Gamma(m+n+2) \Gamma(m+t+2)} \Big] \Big\} \frac{1}{2} +$$

$$+ A_m^2 \Big\{ \sum_{n=0}^{\infty} \left(\frac{P_1}{2} \right)^{2m+4n} \cdot \varphi_1 + \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_1}{2} \right)^{2m+2n+2t} \cdot \psi \Big\} \frac{m^2}{2} +$$

$$+ B_m^2 \Big\{ \sum_{n=0}^{\infty} \left(\frac{P_2}{2} \right)^{2m+4n} \cdot \varphi_1 + \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_2}{2} \right)^{2m+2n+2t} \cdot \psi \Big\} \frac{m^2}{2} +$$

$$+ 2 A_m B_m \Big\{ \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \left(\frac{P_1}{2} \right)^{m+2n} \left(\frac{P_2}{2} \right)^{m+2t} \cdot \psi \Big\} \frac{m^2}{2} \Big] = \frac{\alpha^2 h^2 a^2}{12}$$

... (3.36)

where

$$\Phi = \frac{a^{2m+2n+2t+2}}{(2m+2n+2t+2)\lfloor n \lfloor t \, \Gamma(m+n)\Gamma(m+t+2)}$$

$$\Phi_1 = \frac{a^{2m+4n}}{(2m+4n)\left\{\lfloor n \, \Gamma(m+n+1)\right\}^2}$$

$$\Psi = \frac{a^{2m+2n+2t}}{(2m+2n+2t)\lfloor n \lfloor t \, \Gamma(m+n+1)\Gamma(m+t+1)}$$

Thus the deflection, w is completely determined. The expressions for the bending and twisting moments are

$$M_r = -D\Big[p_1^2(A_0'+A_0)I_0''(p_1 r) + p_2^2(B_0'+B_0)I_0''(p_2 r)$$

$$+ \sum_{m=2,4,6}^{\infty}\left[p_1^2 A_m I_m''(p_1 r) + p_2^2 B_m I_m''(p_2 r)\right]\cos m\theta$$

$$+ \nu\Big\{\frac{p_1}{r}(A_0'+A_0)I_1'(p_1 r) + \frac{p_2}{r}(B_0'+B_0)I_1'(p_2 r)$$

$$+ \frac{1}{r}\sum_{m=2,4,6}^{\infty}\left[p_1 A_m I_m'(p_1 r) + p_2 B_m I_m'(p_2 r)\right]\cos m\theta$$

$$- \frac{1}{r^2}\sum_{m=2,4,6}^{\infty} m^2\left[A_m I_m(p_1 r) + B_m I_m(p_2 r)\right]\cos m\theta\Big\}\Big]$$

... (3.37)

$$M_\theta = -D\Big[\frac{P_1}{r}(A_0'+A_0)I_1'(P_1 r) + \frac{P_2}{r}(B_0'+B_0)I_1'(P_2 r)$$

$$+ \frac{1}{r}\sum_{m=2,4,6}^{\infty}\left[P_1 A_m I_m'(P_1 r) + P_2 B_m I_m'(P_2 r)\right]\cos m\theta$$

$$- \frac{1}{r^2}\sum_{m=2,4,6}^{\infty} m^2\left[A_m I_m(P_1 r) + B_m I_m(P_2 r)\right]\cos m\theta$$

$$+ \nu\Big\{P_1^2(A_0'+A_0)I_0''(P_1 r) + P_2^2(B_0'+B_0)I_0''(P_2 r)$$

$$+ \sum_{m=2,4,6}^{\infty}\left[P_1^2 A_m I_m''(P_1 r) + P_2^2 B_m I_m''(P_2 r)\right]\cos m\theta\Big\}\Big]$$

... (3.38)

$$M_{r\theta} = (1-\nu)D\Big[-\frac{1}{r}\sum_{m=2,4,6}^{\infty} m\left[P_1 A_m I_m'(P_1 r) + P_2 B_m I_m'(P_2 r)\right]\sin m\theta$$

$$+ \frac{1}{r^2}\sum_{m=2,4,6}^{\infty} m\left[A_m I_m(P_1 r) + B_m I_m(P_2 r)\right]\sin m\theta\Big]$$

... (3.39)

The stresses due to M_r, M_θ and $M_{r\theta}$ can be calculated from the expressions

$$(\sigma_r) = \frac{6M_r}{h^2}, \quad (\sigma_\theta) = \frac{6M_\theta}{h^2}, \quad \tau_{r\theta} = \frac{6M_{r\theta}}{h^2} \qquad \text{... (3.40)}$$

RESULTS

To obtain deflection for a given value of plate radius 'a' and foundation modulus 'K_F' one has to start from the Eq.(3.36) with an assumed value of ' α ' in order to obtain the corresponding value of the load function $\frac{qa^4}{Dh}$. Once this relationship is obtained the corresponding deflection $\frac{w}{h}$ can be calculated from Eq.(3.32). For $a = 50$ mm, $h = 0.75$ mm, $\nu = 0.3$, and $K_F = 80$ deflections have been presented in Fig. 3.2.

An examination of the Eq.3.32 will reveal that the deflection $\frac{w}{h}$ depends on K_F, the plate radius 'a', and on the value of the angle, θ. For a given value of the load function Eq.(3.32) can be written as

$$\left(\frac{w}{h}\right)_{\substack{r=0\\ \theta=0}} = K_1\left(\frac{qa^4}{Dh}\right), \quad \left(\frac{w}{h}\right)_{\substack{r=a\\ \theta=\pi/2}} = K_2\left(\frac{qa^4}{Dh}\right) \qquad \ldots (3.41)$$

where K_1 and K_2 are two numerical constants, K_2 being greater than K_1. Because of the reactive forces at the two points of support, deflection on the diameter at $\theta = 0$ will be less than those on the diameter at $\theta = \pi/2$. Maximum deflection will occur at the boundary at $\theta = \pm\pi/2$. Deflections according to the linear theory have also been plotted in Fig.3.2 and it is clear that the errors of the linear theory increases as the load increases. In order to study the variation of moments, Eqs.(3.37),(3.38) and (3.39) are plotted in Fig.3.3 for various values of (r/a) and for the angles at which they become maximum. It is observed that the maximum bending moments, their magnitudes being unequal, are developed at $r = 3a/4$, $\theta = \pm \pi/2$ and the twisting moment is maximum at $r = a$, $\theta = \pm \pi/4$, $\pm 3\pi/4$.

As the plate must be in equilibrium on the supports, the foregoing analysis for two simple supports represents the worst condition when the deflections and stresses are maximum for a given load function. With the increase in the number of supports, w_1 in Eq.3.8 decreases. For an infinitely large number of supports, w in Eq.3.8 will approach to w_o in the limit and the point of maximum bending moments will shift to the centre of the plate, $(M_r)_{max.}$ being equal to $(M_\theta)_{max.}$ in that case.

The present study can be extended to any number of supports, provided the supports are so chosen as not to disturb the equilibrium of the plate. For example, if three equidistant supports are chosen $\psi_1 = 0$, $\psi_2 = 120°$, $\psi_3 = 240°$, the differential equations together with the boundary conditions remaining unchanged. If the plate is clamped on the supports, the boundary conditions and the concentrated reactions at the supports will change totally demanding a separate investigation.

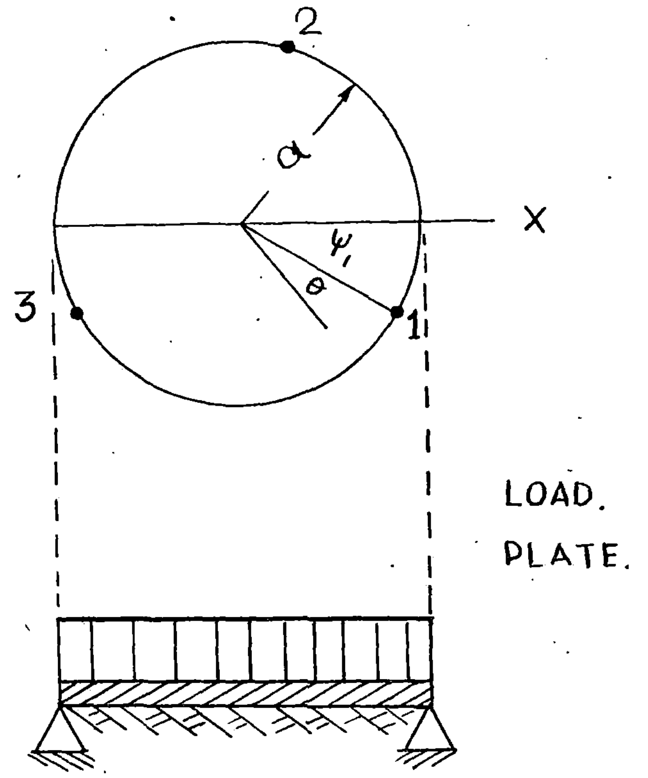

FIG.3.1 CIRCULAR PLATE ON FOUNDATION.

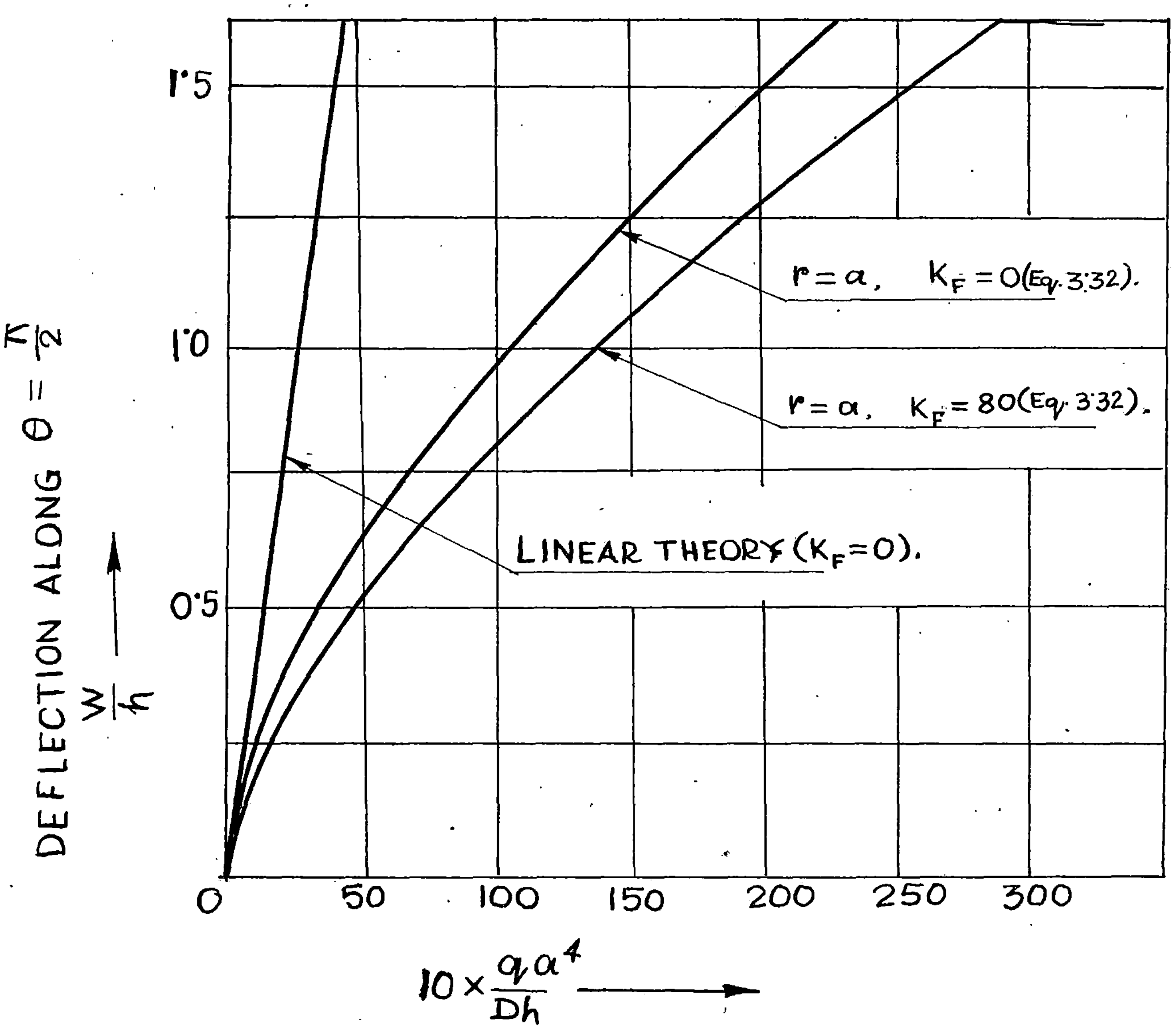

FIG. 3·2 LOAD DEFLECTION CURVE.

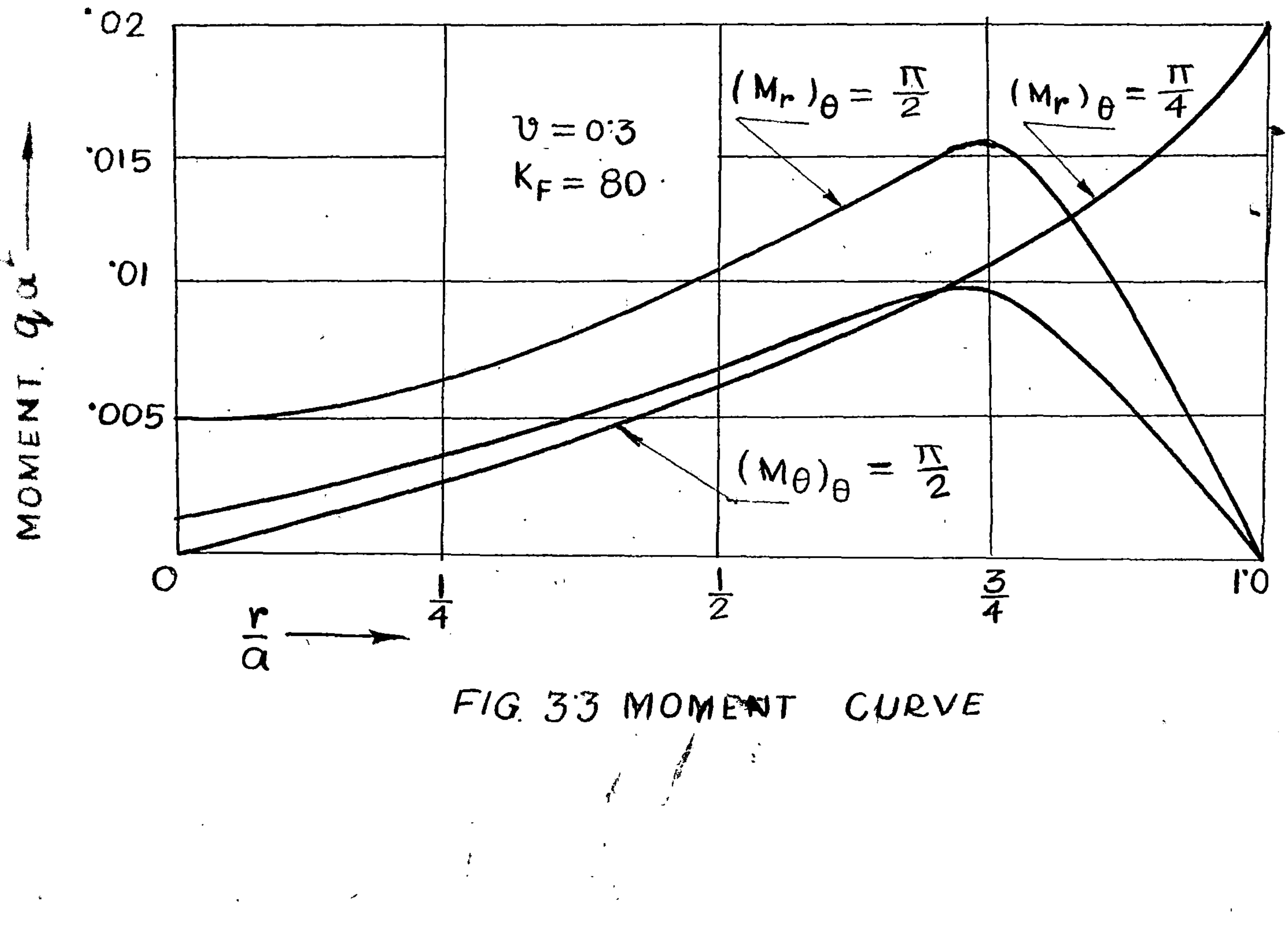

FIG. 33 MOMENT CURVE

LARGE DEFLECTION OF A CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER A CONCENTRATED LOAD AT THE CENTRE*

PAPER - II

INTRODUCTION :

In this paper the large deflection of a clamped circular plate on elastic foundation of Winkler type under a concentrated load at the centre of the plate has been investigated following Berger's approximate method. The deflections are obtained involving Bessel's functions and the theoretical results have been verified experimentally. Graphs are plotted both for theoretical and experimental values for a given value of foundation modulus. The theoretical results have been compared with other known results.

FORMULATION OF PROBLEM

For moderately large deflections of plates under a concentrated load Eq.(3.4) is modified to

$$V = \frac{D}{2}\iint \left[(\nabla^2 w)^2 + \frac{12}{h^2} e_1^2 - 2(1-\nu)\left\{ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} + \frac{K}{D} w^2 \right] dxdy \qquad \ldots (3.42)$$

*Published in the Journal of Applied Mechanics, Trans. A.S.M.E., Vol. 42, No. 2, 1975.

Applying Euler's variational method to Eq.(3.42) and using polar co-ordinates the following differential equations have been obtained

$$\nabla^2 (\nabla^2 - \alpha^2) w + \frac{K}{D} w = 0 \quad \text{except at the load point.} \qquad \ldots (3.43)$$

$$\frac{u}{r} + \frac{du}{dr} + \tfrac{1}{2} \left(\frac{dw}{dr}\right)^2 = \frac{\alpha^2 h^2}{12} \qquad \ldots (3.44)$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \cdot \frac{d}{dr} \quad \text{and}$$

α = a constant given by Eq.(3.44)

Considering the radial stress and shearing stress on a concentric circular area of radius r, the concentrated load P at the centre, and since u and $\frac{dw}{dr}$ are both zero at the centre, one gets

$$\lim_{r \to 0} \; Dr \frac{d}{dr} \left[(\nabla^2 - \alpha^2) \right] = \frac{P}{2\pi} \qquad \ldots (3.45)$$

SOLUTION OF PROBLEM

To solve the problem for a circular plate of radius 'a' and thickness 'h' solution of Eq.(3.43) can be taken in the following convenient form

$$W = A\, I_0 (P_1 r) + B\, I_0 (P_2 r) + C \left[K_0(P_1 r) - K_0(P_2 r) \right] \qquad \ldots (3.46)$$

where $P_1^2 + P_2^2 = \alpha^2$ ... (3.47)

$$P_1^2 \; P_2^2 = \frac{K}{D} \qquad \ldots (3.48)$$

A, B, and C are constants to be determined from the boundary conditions. Boundary conditions for clamped edge are

$$(W)_{r=a} = 0 = \left(\frac{dw}{dr} \right)_{r=a} \qquad \ldots (3.49)$$

Imposing Eq.(3.45) on Eq.(3.43) one gets

$$C = \frac{P}{2 \pi D (P_2^2 - P_1^2)} \qquad \ldots (3.50)$$

Considering Eq.(3.46) and Eq.(3.49) one gets

$$A = C \left[\frac{-\frac{1}{a} - K_1(P_1a)P_1I_0(P_2a) - K_0(P_1a)P_2I_1(P_2a)}{P_2I_1(P_2a)I_0(P_1a) - P_1I_1(P_1a)I_0(P_2a)} \right] \qquad \ldots (3.51)$$

$$B = C \left[\frac{\frac{1}{a} - K_0(P_2a)P_1I_1(P_1a) - K_1(P_2a)P_2I_0(P_1a)}{P_2I_1(P_2a)I_0(P_1a) - P_1I_1(P_1a)I_0(P_2a)} \right] \qquad \ldots (3.52)$$

Thus

$$W = C \left[\left\{ \frac{\frac{1}{a} - K_1(P_1a)P_1I_0(P_2a) - K_0(P_1a)P_2I_1(P_2a)}{P_2I_1(P_2a)I_0(P_1a) - P_1I_1(P_1a)I_0(P_2a)} \right\} I_0(P_1r) \right.$$

$$+ \left\{ \frac{\frac{1}{a} - K_0(P_2a)P_1I_1(P_1a) - K_1(P_2a)P_2a)P_2I_0(P_1a)}{P_2I_1(P_2a)I_0(P_1a) - P_1I_1(P_1a)I_0(P_2a)} \right\} I_0(P_2r)$$

$$\left. + \left\{ K_0(P_1r) - K_0(P_2r) \right\} \right] \qquad \ldots (3.53)$$

is determined completely.

Setting $r \rightarrow 0$ in Eq.(3.46) one gets the maximum deflection at the centre of the plate.

Thus $W_{max.} = A + B + C \log_e \left[\frac{P_2}{P_1} \right]$ ... (3.54)

If $P_1 \rightarrow 0$, $P_2 \rightarrow \infty$ or $P_1 \rightarrow \infty$, $P_2 \rightarrow 0$, one gets the corresponding large deflection for isotropic circular plate not resting on elastic foundation as obtained by Basuli (1961) in the form

$$W = - \frac{P}{2\pi D \alpha^3 \, a I_1(\alpha a)} \Big[I_0(\alpha a) - 1 - I_0(\alpha r) + I_0(\alpha r) \alpha a K_1(\alpha a) + \alpha a I_1(\alpha a) \log_e\left(\frac{r}{a}\right) + K_0(\alpha r) \times \alpha a I_1(\alpha a) \Big] \quad \text{... (3.55)}$$

If $r \rightarrow 0$, Eq.(3.55) leads to

$$W_{max.} = - \frac{P}{2\pi D \alpha^3 a \, I_1(\alpha a)} \Big[I_0(\alpha a) - 2 + \alpha a K_1(\alpha a) - \alpha a I_1(\alpha a) \log_e\left(\frac{\alpha a}{2}\right) \Big] \quad \text{... (3.56)}$$

To determine the displacement u, one gets from Eq.(3.44)

$$ur = \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{2} \int \left(\frac{dw}{dr} \right)^2 r dr + K^1 \quad \text{... (3.57)}$$

where K^1 is the constant of integration.

After evaluating the integrals and using the boundary condition $u \to 0$ as $r \to a$, the following equation determining u is obtained.

$$
\begin{aligned}
ur = {} & \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{2}\Big[A^2 P_1^2 \left\{ \frac{r\, I_o(P_1 r) I_1(P_1 r)}{P_1} + \frac{1}{2} I_1^2(P_1 r) r^2 - \frac{1}{2} I_o^2(P_1 r) r^2 \right\} + \\
& + B^2 P_2^2 \left\{ \frac{r I_o(P_2 r) I_1(P_2 r)}{P_2} + \frac{1}{2} I_1^2(P_2 r) r^2 - \frac{1}{2} I_o^2(P_2 r) r^2 \right\} + \\
& + C^2 P_1^2 \left\{ \frac{1}{2} r^2 K_1^2(P_1 r) - \frac{1}{2} K_o^2(P_1 r) r^2 - \frac{r}{P_1} K_o(P_1 r) K_1(P_1 r) \right\} + \\
& + C^2 P_2^2 \left\{ \frac{1}{2} r^2 K_1^2(P_2 r) - \frac{1}{2} K_o^2(P_2 r) r^2 - \frac{r}{P_2} K_o(P_2 r) K_1(P_2 r) \right\} + \\
& + \frac{2ABP_1 P_2\, r}{P_2^2 - P_1^2} \left\{ P_2 I_1(P_1 r) I_o(P_2 r) - P_1 I_1(P_2 r) I_o(P_1 r) \right\} - \\
& - \frac{2C^2 P_1 P_2\, r}{P_2^2 - P_1^2} \left\{ P_1 K_1(P_2 r) K_o(P_1 r) - P_2 K_1(P_1 r) K_o(P_2 r) \right\} - \\
& - \frac{2ACP_1 P_2\, r}{P_2^2 - P_1^2} \left\{ P_2 I_1(P_1 r) K_o(P_2 r) + P_1 K_1(P_2 r) I_o(P_1 r) \right\} - \\
& - \frac{2BC\, P_1 P_2 r}{P_2^2 - P_1^2} \left\{ P_1 I_1(P_2 r) K_o(P_1 r) + P_2 K_1(P_1 r) I_o(P_2 r) \right\} + \\
& + BC \Big\{ P_2^2 r^2 \big[I_1(P_2 r) K_1(P_2 r) + I_o(P_2 r) K_o(P_2 r) \big] -
\end{aligned}
$$

$$- P_2r \left[I_1(P_2r) K_0(P_2r) - I_0(P_2r) K_1(P_2r) \right] \}$$

$$- AC \left\{ P_1^2 r^2 \left[I_1(P_1r) K_1(P_1r) + I_0(P_1r) K_0(P_1r) \right] - \right.$$

$$\left. - P_1r \left[I_1(P_1r) K_0(P_1r) - I_0(P_1r) K_1(P_1r) \right] \right\}] +$$

$$+ A^2 P_1^2 \left[\frac{aI_0(P_1a) I_1(P_1a)}{P_1} + \tfrac{1}{2} I_1^2(P_1a) a^2 - \tfrac{1}{2} I_0^2(P_1a) a^2 \right] +$$

$$+ B^2 P_2^2 \left[\frac{aI_0(P_2a) I_1(P_2a)}{P_2} + \tfrac{1}{2} I_1^2(P_2a) a^2 - \tfrac{1}{2} I_0^2(P_2a) a^2 \right] +$$

$$+ C^2 P_1^2 \left[\tfrac{1}{2} a^2 K_1^2(P_1a) - \tfrac{1}{2} K_0^2(P_1a) a^2 - \frac{a}{P_1} K_0(P_1a) K_1(P_1a) \right] +$$

$$+ C^2 P_2^2 \left[\tfrac{1}{2} a^2 K_1^2(P_2a) - \tfrac{1}{2} K_0^2(P_2a) a^2 - \frac{a}{P_2} K_0(P_2a) K_1(P_2a) \right] +$$

$$+ \frac{2ABP_1P_2a}{P_2^2 - P_1^2} \left[P_2 I_1(P_1a) I_0(P_2a) - P_1 I_1(P_2a) I_0(P_1a) \right] -$$

$$- \frac{2C^2 P_1 P_2 a}{P_2^2 - P_1^2} \left[P_1 K_1(P_2a) K_0(P_1a) - P_2 K_1(P_1a) K_0(P_2a) \right] -$$

$$- \frac{2AC\, P_1P_2a}{P_2^2 - P_1^2} \left[P_2 I_1(P_1a) K_0(P_2a) + P_1 K_1(P_2a) I_0(P_1a) \right] -$$

$$- \frac{2BC\, P_1P_2a}{P_2^2 - P_1^2} \left[P_1 I_1(P_2a) K_0(P_1a) + P_2 K_1(P_1a) I_0(P_2a) \right] +$$

$$+ \; BC \left[P_2^2 a^2 \left\{ I_1(P_2a)K_1(P_2a) + I_o(P_2a)\, K_o(P_2a) \right\} - \right.$$

$$- \; P_2a \left\{ I_1(P_2a)\, K_o\, (P_2a) - I_o(P_2a)\, K_1(P_2a) \right\} \Big] -$$

$$- \; AC \left[P_1^2 a^2 \left\{ I_1\, (P_1a)K_1(P_1a) + I_o(P_1a)K_o(P_1a) \right\} - \right.$$

$$- \; P_1a \left\{ I_1(P_1a)K_o\, (P_1a) - I_o(P_1a)\, K_1\, (P_1a) \right\} \Big] - \frac{\alpha^2 h^2 a^2}{12} \qquad \ldots (3.58)$$

Also as $r \to 0$, $u \to 0$ from symmetry. Thus the equation for α is given by

$$\frac{\alpha^2 h^2 a^2}{12} = A^2 P_1^2 \left[\frac{aI_o\,(P_1a)I_1(P_1a)}{P_1} + \tfrac{1}{2}\, I_1^2(P_1a)a^2 - \tfrac{1}{2}\, I_o^2(P_1a)\, a^2 \right] +$$

$$+ \; B^2 P_2^2 \left[\frac{aI_o(P_2a)\; I_1(P_2a)}{P_2} + \tfrac{1}{2}\, I_1^2(P_2a)a^2 - \tfrac{1}{2}\, I_o^2(P_2a)\, a^2 \right] +$$

$$+ \; C^2 P_1^2 \left[\tfrac{1}{2}\, a^2 K_1^2\, (P_1a) - \tfrac{1}{2}\, K_o^2(P_1a)a^2 - \frac{a}{P_1}\, K_o(P_1a)K_1(P_1a) \right] +$$

$$+ \; C^2 P_2^2 \left[\tfrac{1}{2}\, a^2 K_1^2(P_2a) - \tfrac{1}{2}\, K_o^2(P_2a)a^2 - \frac{a}{P_2}\, K_o(P_2a)K_1\, (P_2a) \right] +$$

$$+ \; \frac{2AB\; P_1P_2\; a}{P_2^2 - P_1^2} \left[P_2I_1(P_1a)I_o(P_2a) - P_1I_1(P_2a)\; I_o(P_1a) \right] -$$

$$- \; \frac{2C^2 P_1 P_2\; a}{P_2^2 - P_1^2} \left[P_1K_1(P_2\, a)K_o(P_1a) - P_2K_1(P_1a)\; K_o(P_2a) \right] -$$

$$- \frac{2AC\, P_1 P_2\, a}{P_2^2 - P_1^2} \left[P_2 I_1(P_1 a) K_o(P_2 a) + P_1 K_1(P_2 a)\, I_o(P_1 a) \right] -$$

$$- \frac{2BC\, P_1 P_2\, a}{P_2^2 - P_1^2} \left[P_1 I_1(P_2 a) K_o(P_1 a) + P_2 K_1(P_1 a)\, I_o(P_2 a) \right] +$$

$$+ BC \left[P_2^2\, a^2 \left\{ I_1(P_2 a)\, K_1(P_2 a) + I_o(P_2 a)\, K_o(P_2 a) \right\} - \right.$$

$$\left. - P_2 a \left\{ I_1(P_2 a)\, K_o(P_2 a) - I_o(P_2 a)\, K_1\, (P_2 a) \right\} \right] -$$

$$- AC \left[P_1^2\, a^2 \left\{ I_1(P_1 a)\, K_1(P_1 a) + I_o(P_1 a)\, K_o(P_1 a) \right\} - \right.$$

$$\left. - P_1\, a \left\{ I_1(P_1 a)\, K_o(P_1 a) - I_o\, (P_1 a)\, K_1\, (P_1 a) \right\} \right] +$$

$$+ \frac{2ACP_1^2}{P_2^2 - P_1^2} + \frac{2BC\, P_2^2}{P_2^2 - P_1^2} + AC$$

$$- BC - C^2 \left[1 + \frac{P_2^2 + P_1^2}{P_2^2 - P_1^2}\ \log_e \frac{P_1}{P_2} \right] \qquad \ldots (3.59)$$

If $P_1 \to 0,\ P_2 \to \alpha$ or $P_1 \to \alpha,\ P_2 \to 0$, Eq.(3.59) leads to

$$\left[\frac{P\, a^2}{\lambda\, Dh} \right]^2 = \frac{\frac{1}{3}(\alpha\, a)^6}{\left[\gamma + \log_e \frac{\alpha a}{2} - \frac{I_o(\alpha\, a) + \alpha a\, K_1(\alpha\, a) - 2}{\alpha a\quad I_1(\alpha\, a)} \right.}$$

$$\left. - \frac{1}{4} \left\{ \frac{I_o(\alpha\, a) - 1}{I_1(\alpha\, a)} \right\}^2 \right] \qquad \ldots (3.60)$$

The above result was obtained by Basuli (1961) in determining the large deflection of a clamped circular plate under a concentrated load at the centre without any elastic foundation.

EXPERIMENTAL VERIFICATION

Deflection was measured experimentally by an apparatus shown in Fig.3.4. The thin circular mild steel plate (A) of 0.75 mm thickness was placed between two washers and rigidly clamped over the base (B) by means of eight bolts and a thick mild steel ring (M) of 6.0 mm thickness, the diameter of the cavity (H) being 160 mm. Load was applied at the centre of the plate through the load spindle (C). The load spindle was accurately finished and made to move vertically through the bush (D) which was closely fitted at the centre of the frame (E). The lower face of the load spindle collar (F) was accurately ground and polished. Loads were applied over the load pan (G) and the corresponding deflection was measured by means of the dial indicator (J) placed against the lower face of the collar. The dial indicator used can read upto 12 mm with an accuracy of 0.0125 mm. The readings were taken first with the cavity empty and these readings correspond to $K_F = 0$. The cavity was then completely filled in with sand and the experiment was repeated. The value of the nondimensional modulus for sand used was determined experimentally to be $K_F = 430$.

RESULTS

To calculate deflections for plates resting on elastic foundation one has to start from Eq.(3.59) with assumed values of $\frac{K}{D}$ and α leading to the corresponding values of the load function.

Once the load function is obtained, the deflection is determined from Eq.(3.54). The theoretical results for plates with elastic foundation and without elastic foundation have been verified by the experimental results. The corresponding graphs are shown in the Fig. 3.5. Results according to Timoshenko and Kireger (1959, P.415) and Schmidt (1968) corresponding to $K_F=0$ have also been presented for comparison in the same graph.

It is observed from Fig. 3.5 that the deflections calculated both for plates with and without any elastic foundation are in good agreement with the values obtained experimentally. This justifies the assumptions Berger made that the second strain invariant of the middle plane can be neglected for practical purpose. It is also observed that the deviation of the experimental curve from the theoretical one is mo with higher values of the load function. This is due to the fact that the assumption of K to be proportional to the deflection is not strictly correct as K varies nonlinearly with the deflection at the higher values of loads.

The equation for bending moment is given by

$$M_r = -D\left[A P_1^2 I_0(P_1 r) + BP_2^2 I_0(P_2 r) - (1-\nu)\frac{1}{r} AP_1 I_1(P_1 r) - (1-\nu)\frac{1}{r} BP_2 I_1(P_2 r) + C\left\{P_1^2 K_0(P_1 r) - P_2^2 K_0(P_2 r) + (1-\nu)\frac{1}{r} P_1 K_1(P_1 r) - (1-\nu)\frac{1}{r} P_2 K_1(P_2 r)\right\}\right] \quad \ldots (3.61)$$

Bending moment at a point not very close to the centre of the plate may be calculated with the help of Eq.(3.61) and the corresponding bendin stress is given by

$$\sigma_n = -\frac{6}{h^2}(M_r) \quad \ldots (3.62)$$

On examination of the Eq.(3.61) it is observed that the bending moment becomes infinite at the centre of the plate. Hence Eq.(3.61) cannot be used to determine the maximum bending stress which occurs at the centre of the plate. The concentrated load P at the centre of the plate may be assumed to be uniformly distributed over a concentric circular area of a very small radius, c. The shear stress is given by

$$(\tau)_{rz} = \frac{6\ Q}{h^3}\left(\frac{h^2}{4} - z^2\right) \qquad \ldots (3.63)$$

$$Q \;=\; \text{Shear force} \;=\; - D \frac{d}{dr} \left[(\nabla^2 - \alpha^2) \right] w.$$

From the theory of bending of rectangular beams $(\tau)_{rz}$ is zero on the outer surfaces of the plate. The maximum compressive stress at the centre of the upper face of the plate is then given by

$$\sigma = -\sigma_1 - \frac{P}{\pi c^2} \left[\frac{1+2\nu}{2} - (1+\nu)\beta \right] \qquad \ldots (3.64)$$

where σ_1 is the bending stress calculated by Eq.(3.62) for points very close to the load point and β is a numerical factor depending on $\frac{2c}{h}$, the ratio of the diameter of the loaded area to the thickness of the plate [Timoshenko (1959), P.70]. The maximum tensile stress will occur at the centre of the inner surface of the plate and its value will be less than the compressive stress at the corresponding point on the upper surface. There will be a high concentration of stress surrounding the load point.

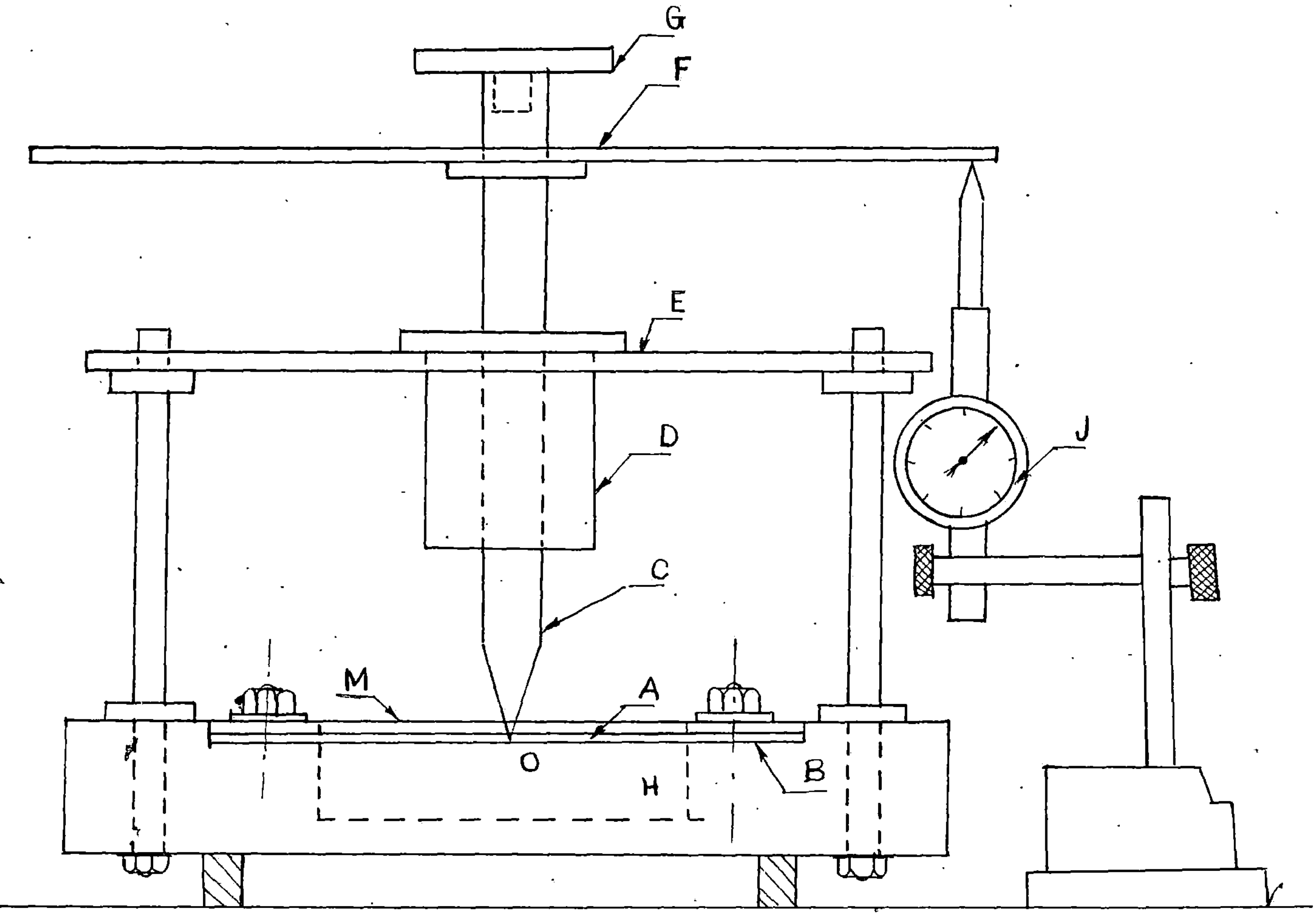

FIG. 3·4

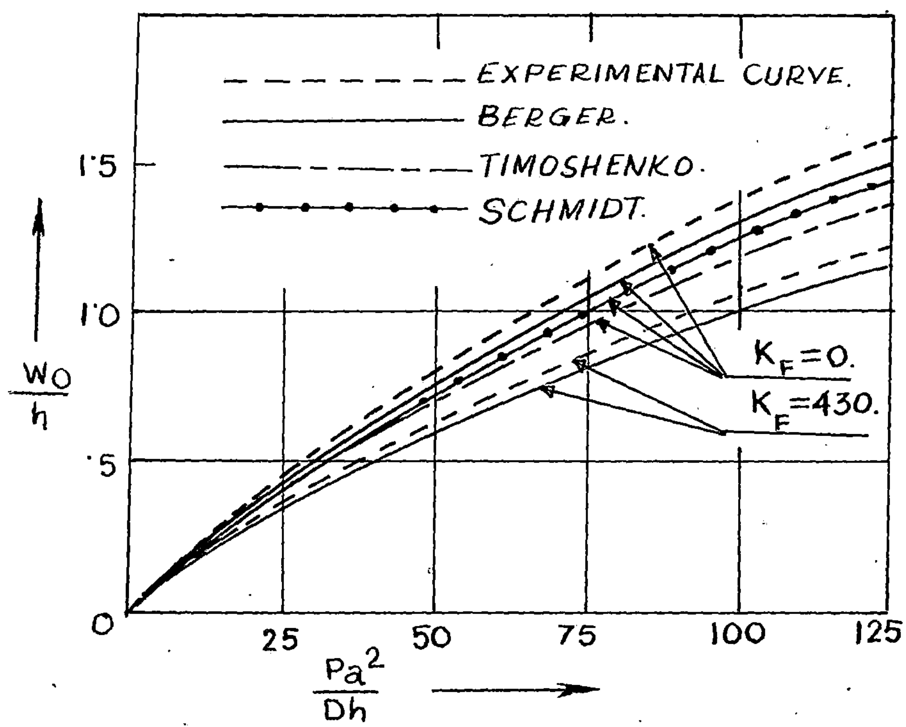

FIG.3·5 DEFLECTION CURVE.

LARGE DEFLECTION OF A CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER SYMMETRICAL LOADS *

PAPER - III

INTRODUCTION

Following Berger's approximate method, large deflections of clamped circular plates on elastic foundation and subjected to some special types of symmetrical transverse loads, which are functions of the distance from the centre of the plates, distributed over a concentric circular portion of the plates have been investigated in this paper. Deflections, bending moments and bending stresses are calculated for different values of foundation modulus and these are presented in the form of graphs. The results have been compared with other known results.

SOLUTION OF PROBLEM

Let us consider a clamped circular plate of radius 'a'. The centre of the plate is taken as the origin. For moderately large deflection of plates the governing differential equations in polar co-ordinates are

$$\nabla^4 W - \alpha^2 \nabla^2 W + \frac{K}{D} W = \frac{q}{D} \qquad \ldots (3.65)$$

$$\frac{u}{r} + \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = \frac{\alpha^2 h^2}{12} \qquad \ldots (3.66)$$

*Published in Journal of Structural Mechanics, University of Illinois, U.S.A., Vol.3(4) No. 000-000, 1974-75.

where

$$\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

If there be a symmetrical distribution of transverse load varying as $(b^2-r^2)^\lambda$, $(\lambda > -1)$ over a concentric circular area of radius $b < a$,

$$\frac{q}{D} = \frac{\Phi(r)}{D} = f(r) = c(b^2-r^2)^\lambda \text{ when } r < b < a \qquad \ldots (3.67)$$

$$= 0 \qquad \text{when } b < r < a$$

Eq.(3.65) now is written as

$$(\nabla^2 - \alpha^2)\ \nabla^2 W + \frac{K}{D} W = f\ (r) \qquad \ldots (3.68)$$

The boundary conditions for clamped edges are

$$(W)_{r=a} = 0 = \left(\frac{dW}{dr}\right)_{r=a} \qquad \ldots (3.69)$$

Let us assume the deflection W in the following form

$$W = \sum_{S=1}^{\infty} A_S \left[J_0\ (P_S r) - J_0\ (P_S a) \right] \qquad \ldots (3.70)$$

where J_0

is the Bessel function of the first kind and zero order and P_S is the S-th root of J_1 (Pa) = 0, J_1 being the Bessel function of the first kind and first order.

It is evident that the boundary conditions for clamped edges are satisfied by the above configurations of W.

Since

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] J_0(P_s r) = - P_s^2 J_0(P_s r),$$

putting Eq.(3.70) into Eq.(3.68) one gets,

$$\sum_{S=1}^{\infty} A_s \left[P_s^4 J_0(P_s r) + \alpha^2 P_s^2 J_0(P_s r) + \frac{K}{D} \left\{ J_0(P_s r) - J_0(P_s a) \right\} \right] = f(r) \qquad \ldots (3.71)$$

If it is possible to expand f(r) in a series of Bessel function, one gets,

$$A_s \int_0^a \left[P_s^2 (P_s^2 + \alpha^2) J_0^2 (P_s r) + A_s \frac{K}{D} \left\{ J_0^2 (P_s r) - J_0 (P_s a) J_0(P_s r) \right\} \right] r dr = \int_0^a f(r) J_0(P_s r) \, r dr$$

Or

$$A_s \frac{a^2}{2} \left[P_s^2 (P_s^2 + \alpha^2) + \frac{K}{D} \right] J_0^2 (P_s a) = \int_0^a f(r) \times J_0 (P_s r) r dr \qquad \ldots (3.72)$$

Putting $r = b \sin\theta$ and $f(r) = C (b^2 - r^2)^{\lambda}$ in the integral of Eq.(3.72), one gets

$$\int_0^a f(r) J_0(P_s r) r dr = \int_0^b Cr (b^2 - r^2)^{\lambda} J_0(P_s r) dr$$

$$= Cb^{2(\lambda + 1)} \int_0^{\frac{\pi}{2}} \sin\theta \cos^{2\lambda + 1}\theta \times J_0 (P_s b \sin\theta) d\theta$$

$$= \frac{Cb^{2(\lambda+1)} J_{\lambda+1}(P_s b)\, 2^{\lambda}\, \Gamma(\lambda+1)}{(P_s\, b)^{\lambda+1}} \qquad \ldots (3.73)$$

This is a special form of Sonaine's first definite integral containing Bessel function [Watson (1952)], where $\lambda > -1$.

Substituting the value obtained from Eq.(3.73) in Eq.(3.72) one gets after simplification

$$A_s = \frac{C(2b)^{\lambda+1} J_{\lambda+1}(P_s b)\,\Gamma(\lambda+1)}{a^2 \left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_o^2(P_s a)} \qquad \ldots (3.74)$$

Thus

$$W = \frac{C(2b)^{\lambda+1}\Gamma(\lambda+1)}{a^2} \sum_{S=1}^{\infty} \frac{J_{\lambda+1}(P_s b) \left[J_o(P_s r) - J_o(P_s a) \right]}{\left[P_s^{\lambda+3} (P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_o^2(P_s a)}$$

is completely determined. $\qquad \ldots (3.75)$

The deflection will be maximum at the centre. From Eq.(3.75) maximum deflection is obtained putting $r = 0$.

Thus

$$W_{max.} = \frac{C(2b)^{\lambda+1}\Gamma(\lambda+1)}{a^2} \sum_{S=1}^{\infty} \frac{J_{\lambda+1}(P_s b) \left[1 - J_o(P_s a) \right]}{\left[P_s^{\lambda+3}(P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_o^2(P_s a)}$$

$\ldots (3.76)$

To determine the displacement u, one gets from Eq.(3.66) and Eq.(3.70)

$$\frac{du}{dr} + \frac{u}{r} = \frac{\alpha^2 h^2}{12} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2$$

$$= \frac{\alpha^2 h^2}{12} - \frac{1}{2}\sum_{S=1}^{\infty} A_s^2 P_s^2 J_1^2 (P_s r) - \frac{1}{2}\sum_{S=1}^{\infty}\sum_{\substack{m=1,\\ S\neq m}}^{\infty} A_s A_m P_s P_m \times$$

$$\times J_1 (P_s r) J_1(P_m r) \qquad \ldots (3.77)$$

Multiplying Eq.(3.77) by $r\,dr$ and integrating with respect to r, one gets,

$$u r = \frac{\alpha^2 h^2 r^2}{24} - \frac{1}{2} \sum_{S=1}^{\infty} A_s^2 P_s^2 \left[\frac{r^2}{2} \left\{ \left(1 - \frac{1}{P_s^2 r^2}\right) \times \right.\right.$$

$$\left.\left. \times J_1'^2(P_s r) + J_1'^2 (P_s r)\right\} \right] - \frac{1}{2} \sum_{S=1}^{\infty} \sum_{\substack{m=1\\ S\neq m}}^{\infty} A_s A_m P_s P_m \times$$

$$\times \left[r \left\{ \frac{P_s J_2(P_s r) J_1(P_m r) - P_m J_1(P_s r) J_2(P_m r)}{P_s^2 - P_m^2} \right\} \right] +$$

$+ K$............. ....Eq.(3.78) where K is the constant of integration. Boundary condition imposed on u is

$$(u)_{r = a} = 0$$

Thus

$$K = \frac{1}{4} \sum_{S=1}^{\infty} A_s^2 P_s^2 a^2 J_o^2 (P_s a) - \frac{\alpha^2 h^2 a^2}{24} \qquad \ldots (3.79)$$

Also as $r \to 0$, $u \to 0$, from symmetry.

Therefore the equation to determine α leads to

$$\frac{\alpha^2 h^2 a^2}{6} = \sum_{S=1}^{\infty} A_s^2 P_s^2 a^2 J_o^2 (P_s a) \qquad \ldots (3.80)$$

Putting in Eq.(3.75)

$$\lambda = \tfrac{1}{2}\ , \quad K = 0$$

the deflection W is given by,

$$W = \sum_{S=1}^{\infty} A_S \left[J_o(P_s r) - J_o(P_s a) \right]$$

$$= \frac{2b^3 c}{a^2} \sum_{S=1}^{\infty} \frac{P(P_s b)}{P_s^2 (P_s^2 + \alpha^2) J_o^2(P_s a)} \left[J_o(P_s r) - J_o(P_s a) \right] \qquad \ldots (3.81)$$

where $P(P_s b) = \frac{1}{2} \left[1 - \frac{P_s^2 b^2}{2.5} + \frac{P_s^4 b^4}{2.4.5.7} - \ldots \right]$

As α tends to zero Eq.(3.81) leads to

$$W = \frac{2b^3 c}{a^2} \sum_{S=1}^{\infty} \frac{P(P_s b) \left[J_o(P_s r) - J_o(P_s a) \right]}{P_s^4 J_o^2(P_s a)} \qquad \ldots (3.82)$$

Eq.(3.82) is the result obtained by Sen (1935) in his corresponding small deflection problem.

Let us examine another type of transverse load function varying as $(r^4 - b^4)$ over a Concentric Circular area of $b < a$.

In this case $f(r) = \begin{cases} C(r^4 - b^4) & (0 \leq r \leq b < a) \\ 0 & (b \leq r \leq a) \end{cases}$ $\qquad \ldots (3.83)$

Expanding $f(r)$ in the series of Bessel function and proceeding in the same manner one gets,

$$A_s = \frac{32bc\,(4 - P_s^2 b^2)\, J_1(P_s b)}{a^2 \left[P_s^7 (P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_o^2(P_s a)} \qquad \ldots (3.84)$$

Thus

$$W = \frac{32bc}{a^2} \sum_{S=1}^{\infty} \frac{(4-P_s^2 b^2) J_1(P_s b) \left[J_o(P_s r) - J_o(P_s a) \right]}{\left[P_s^7 (P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_o^2 (P_s a)} \quad \ldots (3.85)$$

is completely determined. Deflection is maximum at the centre. The central deflection is obtained putting $r = 0$.

Thus

$$W_{max.} = \frac{32bc}{a^2} \sum_{S=1}^{\infty} \frac{(4-P_s^2 b^2) J_1(P_s b) \left[1-J_o(P_s a) \right]}{\left[P_s^7 (P_s^2 + \alpha^2) + P_s^5 \frac{K}{D} \right] J_o^2 (P_s a)} \quad \ldots (3.86)$$

By substituting the Eq.(3.84) in Eq.(3.78) and Eq.(3.80) one gets the equation for u and α respectively for the type of load function in Eq.(3.83).

The equation for bending moment is given by

$$M_r = -D \left[\frac{\partial^2 W}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 W}{\partial \theta^2} \right) \right] \quad \ldots (3.87)$$

Since W is a function of r only, the equation for bending moment becomes

$$M_r = -D \left[\frac{d^2 W}{d r^2} + \nu \left(\frac{1}{r} \frac{dw}{dr} \right) \right] \quad \ldots (3.88)$$

Considering Eq.(3.75) and Eq.(3.88) the value for bending moment for the type of loading in Eq.(3.67) is obtained as

$$M_r = \frac{DC(2b)^{\lambda+1} \Gamma(\lambda+1)}{a^2} \sum_{S=1}^{\infty} \frac{P_s J_{\lambda+1} (P_s b)}{\left[P_s^{\lambda+3} (P_s^2 + \alpha^2) + P_s^{\lambda+1} \frac{K}{D} \right] J_o^2 (P_s a)}$$

$$X \left[P_s J_o (P_s r) + (\nu - 1) \frac{1}{r} J_1 (P_s r) \right] \quad \ldots (3.89)$$

For clamped edge the bending moment will be maximum at the centre. Thus

$$(M_r)_{max.} = \frac{DC(2b)^{\lambda+1}\,\Gamma(\lambda+1)}{a^2}$$

$$\times \sum_{S=1}^{\infty} \frac{P_s^2\, J_{\lambda+1}(P_s b)}{\left[P_s^{\lambda+3}(P_s^2+\alpha^2) + P_s^{\lambda+1}\frac{K}{D}\right] J_o^2(P_s a)} \times \left[1 + \frac{\nu - 1}{2}\right] \qquad \ldots (3.90)$$

The maximum bending stress is given by

$$(\sigma r)_{max.} = -\frac{6}{h^2}\,(M_r)_{max.} \qquad \ldots (3.91)$$

For the load function given by Eq.(3.67), the maximum deflections, maximum bending moments and maximum bending stress for $\lambda = 1$ are given as follows :

$$W_{max.} = \frac{4b^2c}{a^2}\sum_{S=1}^{\infty} \frac{J_2(P_s b)\left[1 - J_o(P_s a)\right]}{\left[P_s^4(P_s^2+\alpha^2) + P_s^2\frac{K}{D}\right] J_o^2(P_s a)} \qquad \ldots (3.92)$$

$$(M_r)_{max.} = \frac{4DCb^2}{a^2}\sum_{S=1}^{\infty} \frac{J_2(P_s b)\left[1 + \frac{\nu-1}{2}\right]}{\left[P_s^2(P_s^2+\alpha^2) + \frac{K}{D}\right] J_o^2(P_s a)} \qquad \ldots (3.93)$$

$$(\sigma r)_{max.} = -\frac{24DCb^2}{a^2 h^2}\sum_{S=1}^{\infty} \frac{J_2(P_s b)\left[1 + \frac{\nu-1}{2}\right]}{\left[P_s^2(P_s^2+\alpha^2) + \frac{K}{D}\right] J_o^2(P_s a)} \qquad \ldots (3.94)$$

If $\alpha \to 0$ and $K = 0$, the corresponding results for small deflections problems are obtained.

Thus for small deflections

$$W_{max.} = \frac{4b^2c}{a^2} \sum_{S=1}^{\infty} \frac{J_2(P_s b)\left[1 - J_o(P_s a)\right]}{P_s^6 \; J_o^2 \; (P_s a)} \qquad \ldots (3.95)$$

$$(M_r)_{max.} = \frac{4DC\, b^2}{a^2} \sum_{S=1}^{\infty} \frac{J_2\,(P_s\, b)}{P_s^4\, J_o^2\, (P_s\, a)} \left[1 + \frac{\nu - 1}{2}\right] \qquad \ldots (3.96)$$

$$(\sigma r)_{max.} = -\frac{24DCb^2}{a^2h^2} \sum_{S=1}^{\infty} \frac{J_2(\,P_s\, b\,)}{P_s^4\, J_o^2(P_s a)} \times \left[1 + \frac{\nu - 1}{2}\right] \qquad \ldots (3.97)$$

The above results will be utilised for numerical computations and for side by side comparison.

R E S U L T S

Numerical results are presented here for the case of the circular plates with clamped edge. The type of load function considered is as in Eq.(3.67) and the value of λ is assumed to be 1, radius of the plate being $a = 2b$. The maximum deflections, maximum bending stresses are calculated for various values of the load functions and for various values of the foundation modulus. These are presented in the form of graphs. Central deflection and maximum bending stresses are also calculated for small deflections and these are also presented in the form of graphs for comparison. Variation of the bending moment along the radius is also calculated both for small deflection and large deflection.

In calculating the central deflection, one has to start from Eq.(3.80) with an assumed value of (α a) leading to a particular value of the load function. Once this relationship is obtained the maximum value of the deflection can be obtained from Eq.(3.92) for various values of the foundation modulus. These results are presented in Fig.3.6 to 3.9. On examination of the Eq.(3.92), it is revealed that as the radius of the plate increases, the central deflection also increases for a given value of the load function. For small deflection Eq.(3.95) is to be used for calculation of the central deflection.

In calculating the bending moment for various values of $(\frac{r}{a})$, Eq.(3.89) is to be used putting the value of $\lambda = 1$. The variation of the bending moment along the radius of the plate is presented in Fig. 3.10. Variation of the bending moment along the radius according to the linear theory can be calculated with the help of the Eq.(3.89) putting the value of $\alpha = 0$ and $\lambda = 1$. The maximum bending stresses both for large and small deflection and for various values of foundation modulus are presented in the Fig. 3.11.

For the type of loading in Eq.(3.83) the central deflection for various values of load function and foundation modulus can be calculated with the help of Eq.(3.86) in conjunction with the corresponding equation for α . Values of the bending moment and bending stresses can also be easily calculated.

From Eq.(3.92) it is observed that the central deflection of the circular plate depends on the radius 'a' of the plate and on the value of the foundation modulus. But for a given value of the foundation modulus, deflection increases mainly due to increase of radius, because the effect of the value of $P_S^2 \cdot \frac{K}{D}$ is little in comparison with other terms when the plate radius is increased. For a given value of the plate radius, as the foundation modulus increases the Eq.(3.65) behaves as the linear equation of the type

$$\nabla^4 W + \frac{K}{D} W = q/D \qquad \ldots (3.98)$$

This is also seen from the Fig.3.6 through 3.9 where the deflection curves for higher values of the foundation modulus K_F tend towards linearity. The nature of the curves of the Fig.3.6 through 3.9 for $K_F = 0$ are in good agreement with those as found by other authors. For value of $\lambda = \frac{1}{2}$ in Eq.(3.67) and for $K_F = 0$, the deflection curve is presented in Fig. 3.12. This is in good agreement with the result obtained by Banerjee B [(1967), 5].

From the Eq. (3.92) and also from the Fig.3.6 to 3.9 it is seen that with the increase of foundation modulus, the deflections of the plates decrease. This is expected. Because better foundations will give larger upward pressures to reduce the effect of applied loads.

For the type of loading in Eq.(3.67), the load on the plate at $r = \pm b$ is zero and the deflected shape of the plate is shown in the Fig.3.13. From the Fig.3.10 it is seen that the bending moment varies from + ive to - ive value along the radius. The deflection of the plate along the radius will be as shown in Fig.3.13. This shows that the direct stresses in the upper and lower fibres of the plates will change from tensile to compressive with the value of zero stress at the point of no deflection. Maximum bending stress will occur at the centre of the plate. With the increase of load maximum bending stresses will vary linearly with the load and the stresses will vary nonlinearly with the deflection. The effect of the foundation modulus is to reduce the bending stresses. Errors in the results of the deflection obtained from the two approximate equations (3.65) and (3.66) will be less for the values of K_F other than zero. But the results for the bending moments and bending stresses will not be as accurate as the deflection because their values depend on the derivative of the displacement and deflection.

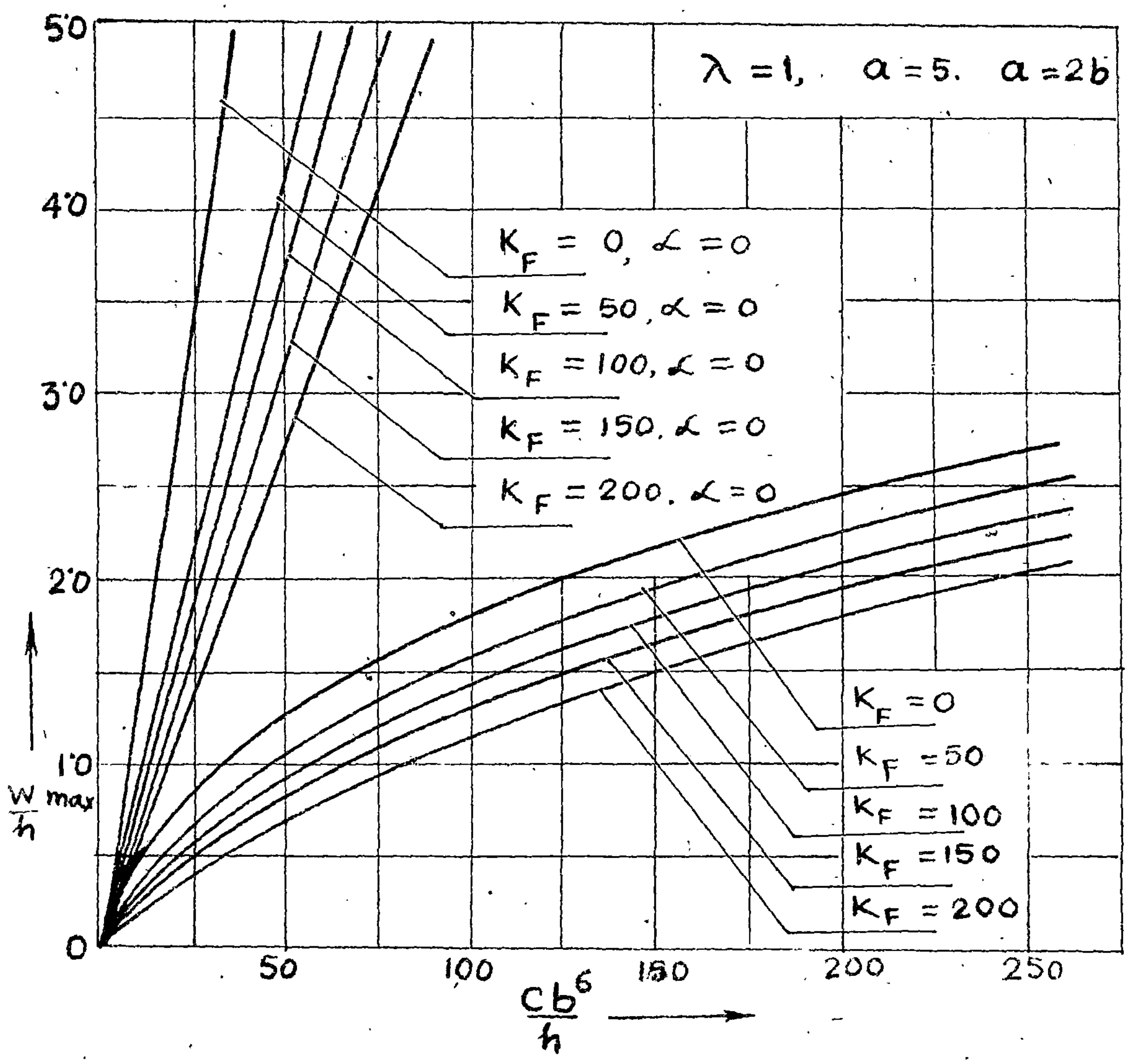

FIG. 3.6 CURVES FOR DEFLECTION.

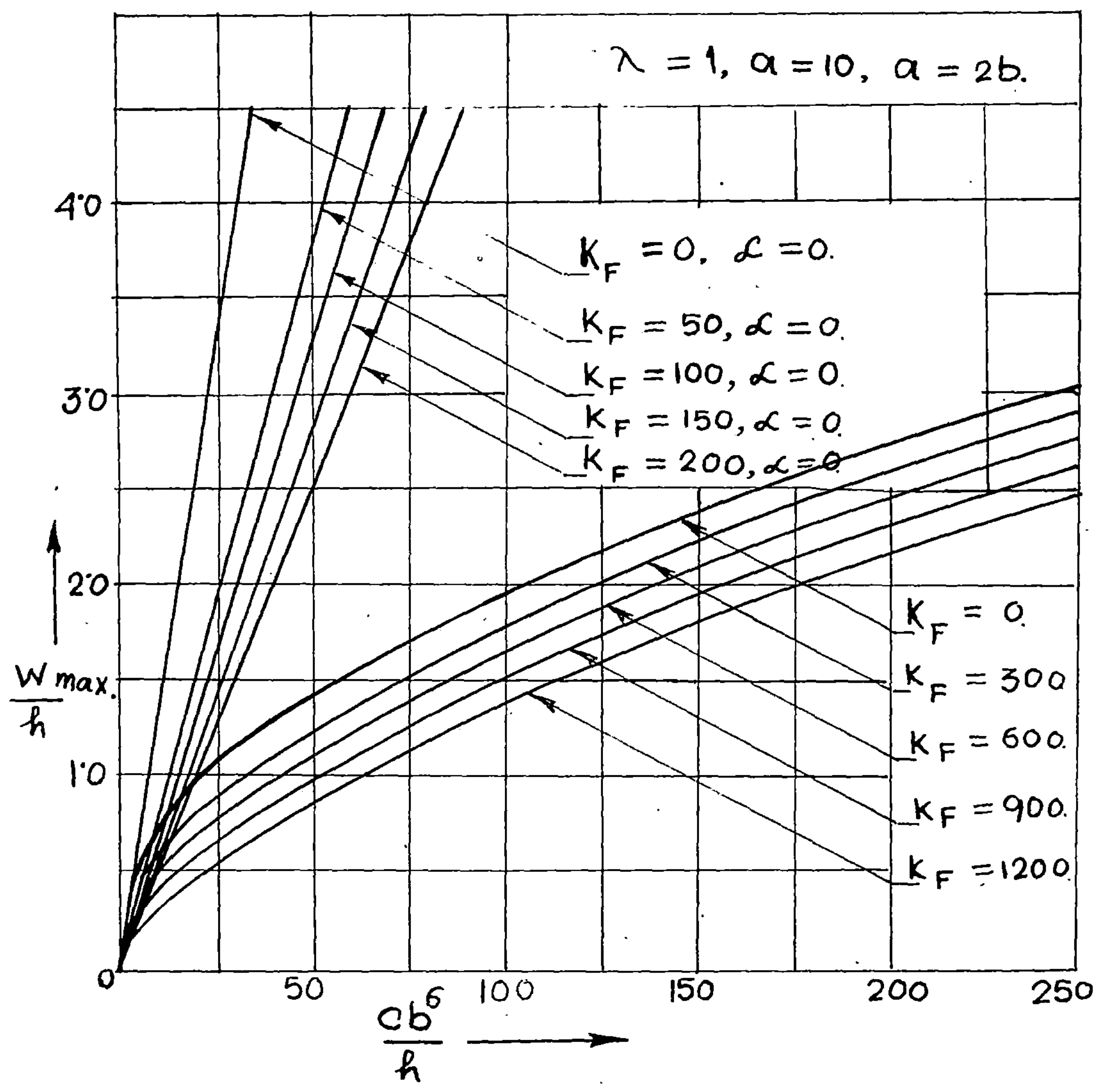

FIG.3·7 CURVES FOR DEFLECTION.

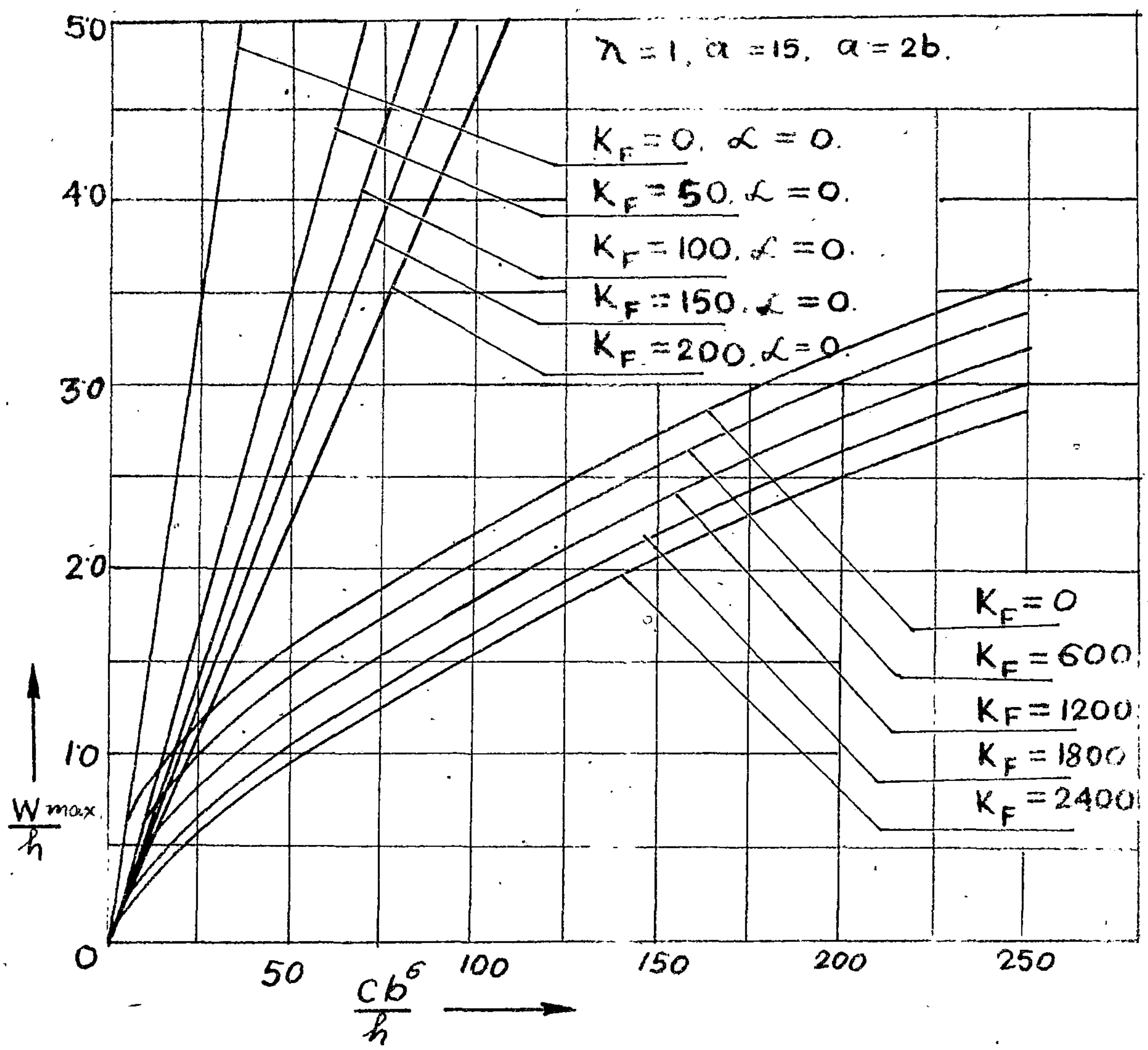

FIG. 3.8 CURVES FOR DEFLECTION

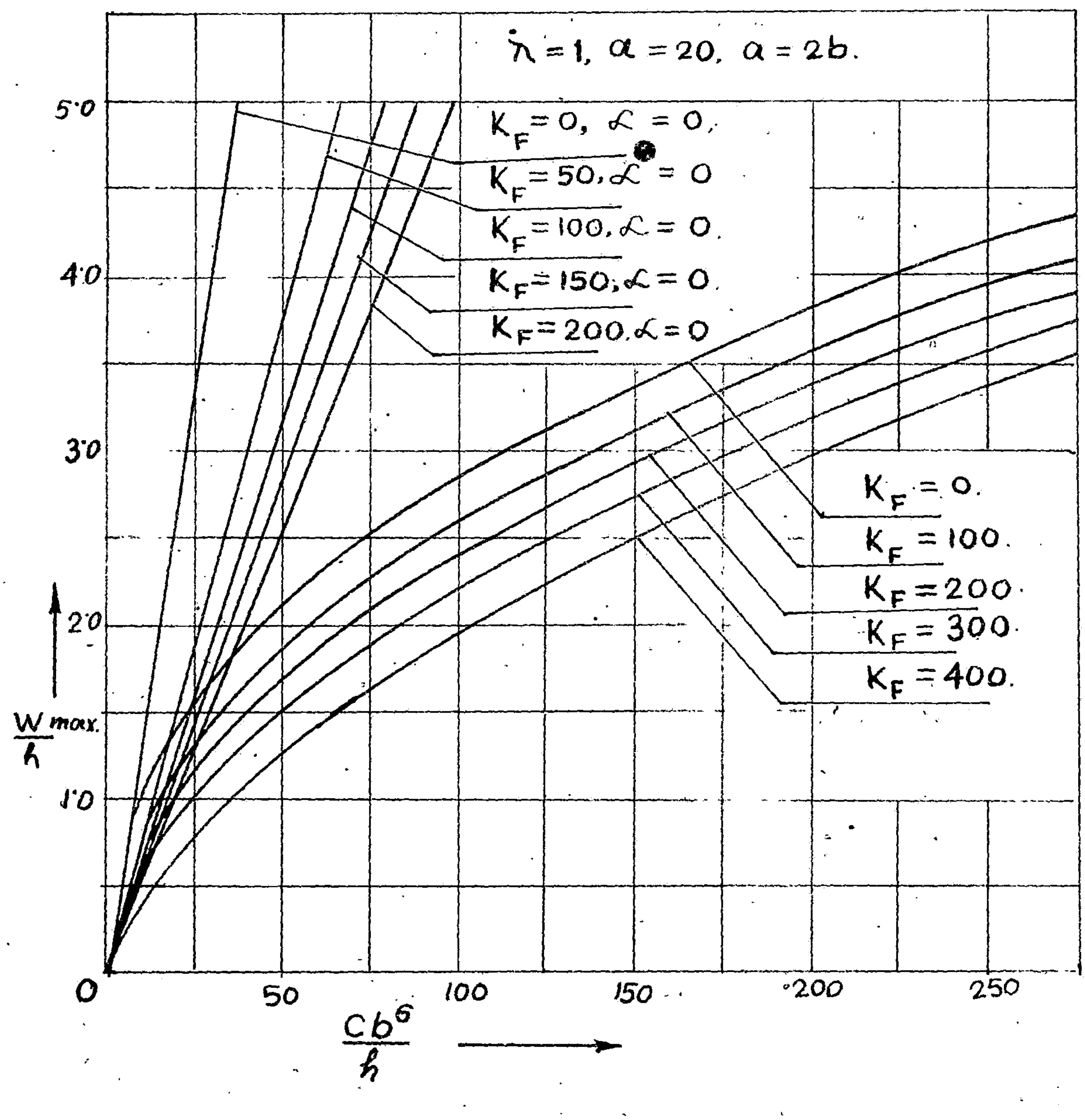

FIG. 3·9 CURVES FOR DEFLECTION.

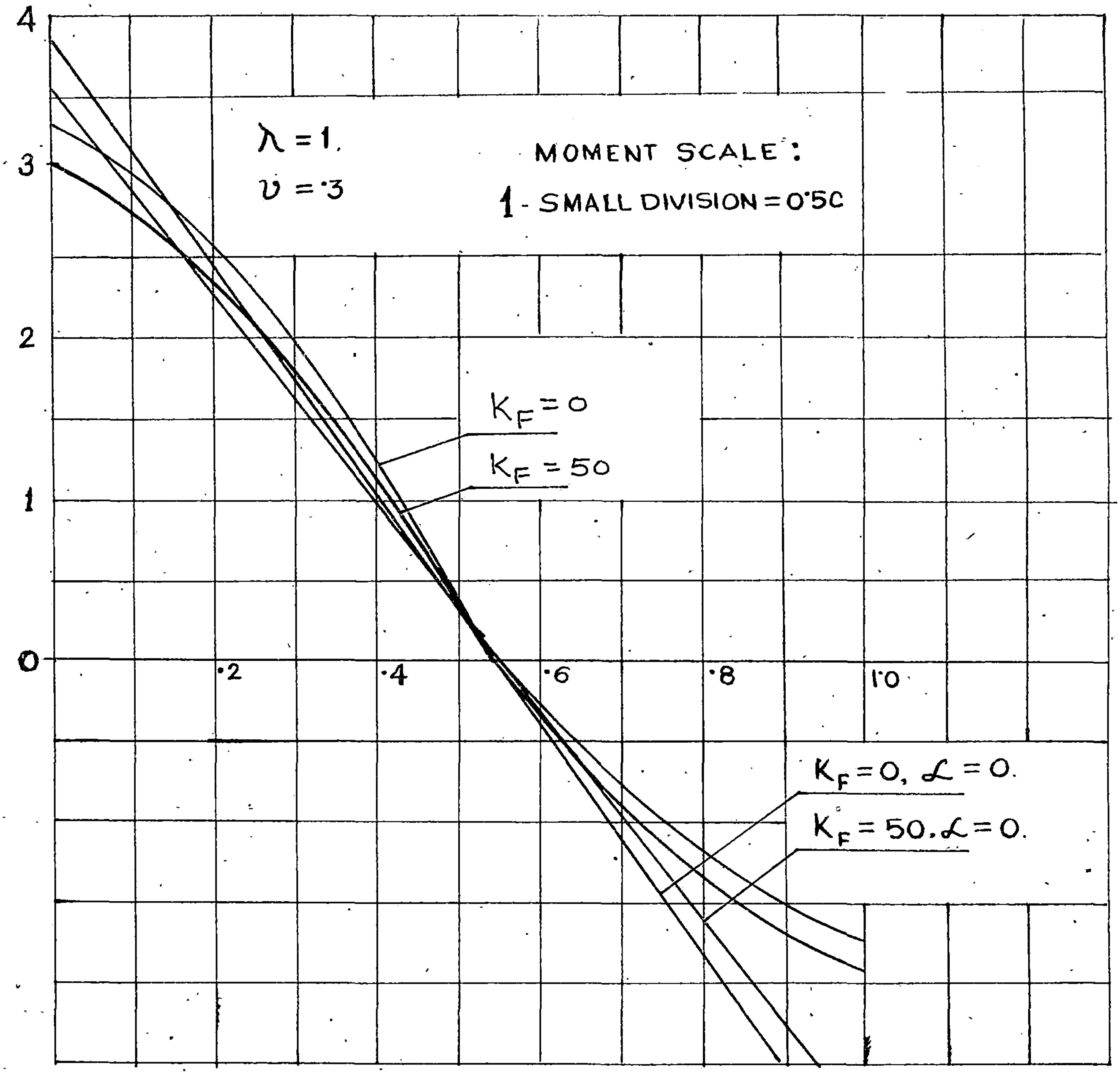

FIG. 3·10 CURVES FOR BENDING MOMENT.

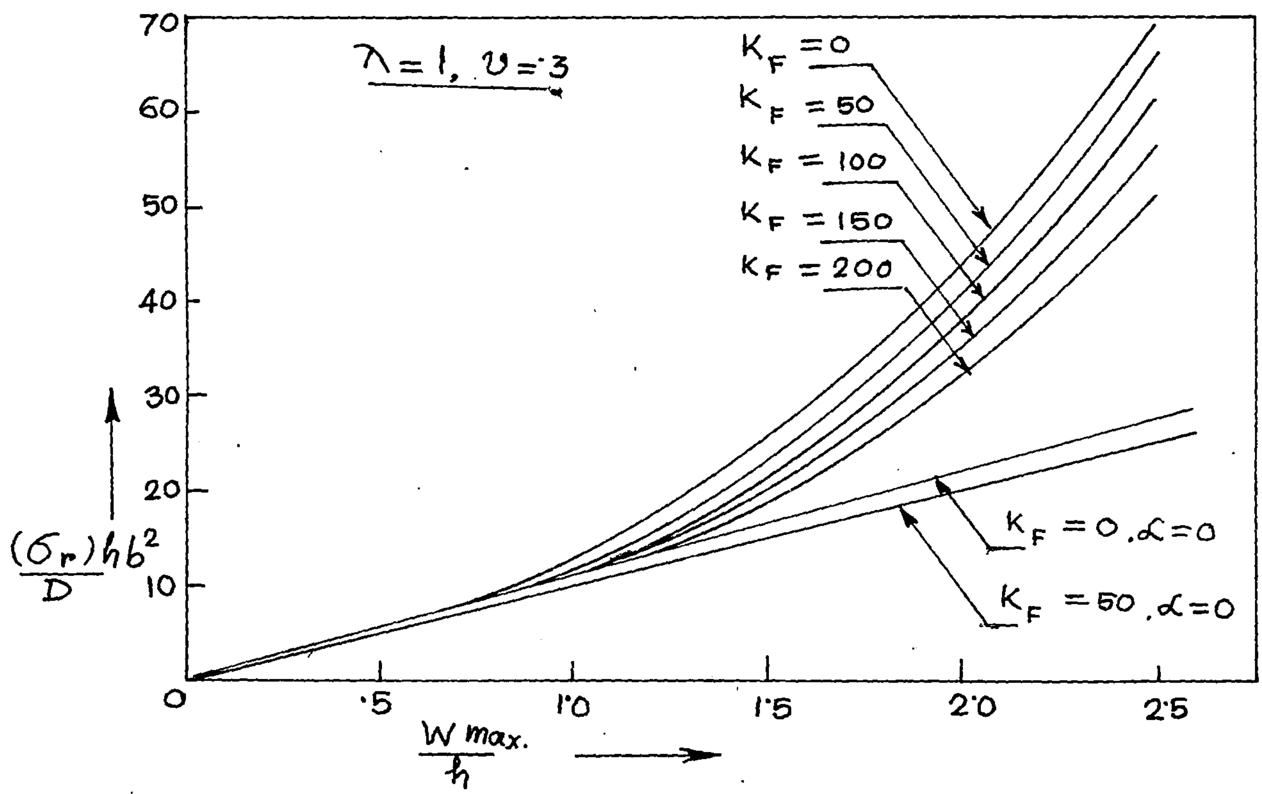

FIG. 3·11 CURVES FOR STRESSES.

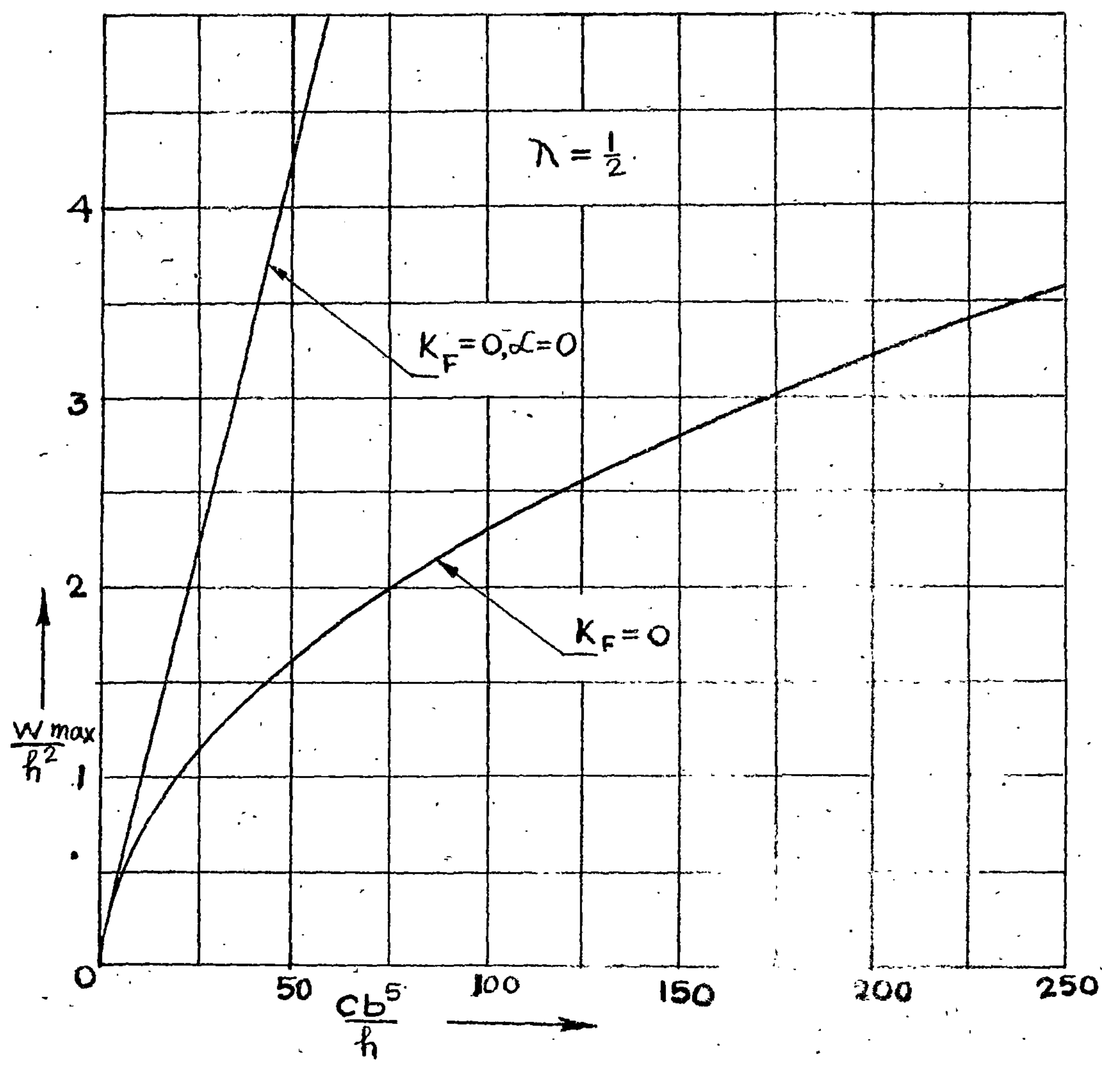

FIG. 3·12 CURVE FOR DEFLECTION.

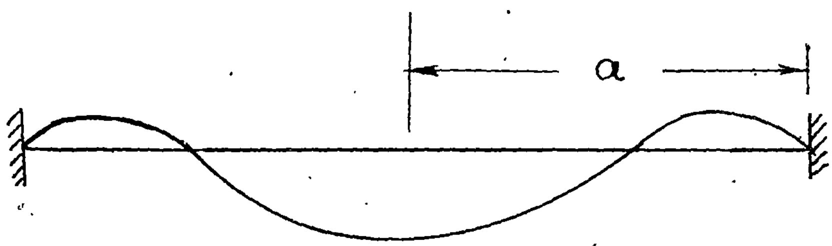

FIG. 3.13 DEFLECTED PLATE SHAPE.

LARGE DEFLECTION OF A SEMI-CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER A UNIFORM LOAD*

PAPER - IV

INTRODUCTION

Large deflection of a simply supported semicircular plate placed on elastic foundation of Winkler type and subjected to a uniform load has been investigated following Berger's approximate method. Expressions for the deflections and bending moments are obtained and the theoretical results have been presented in the form of graphs.

SOLUTION OF PROBLEM

Let us take a plate in the form of a semicircle, Fig.3.14 and let it be simply supported and placed on an elastic foundation having the reaction K per unit area per unit deflection. Let the centre be the origin, the bounding diameter be the initial line and the plate be uniformly loaded. The governing differential equations in polar coordinates are

$$\nabla^4 w - \alpha^2 \nabla^2 w + \frac{K}{D} W = \frac{q}{D} \qquad \ldots (3.99)$$

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where α is a constant given by

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad \ldots (3.100)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

To solve Eq.(3.99) let us put it in the following form

$$(\nabla^2 - p_1^2)(\nabla^2 - p_2^2)\,w = \frac{q}{D} \qquad \ldots (3.101)$$

$$p_1^2 + p_2^2 = \alpha^2 \qquad \ldots (3.102)$$

$$p_1^2\, p_2^2 = \frac{K}{D} \qquad \ldots (3.103)$$

Expanding the load into the appropriate Fourier series,

$$q = \frac{4q}{\pi} \sum_{m=1,3,5.}^{\infty} \frac{\sin m\theta}{m} \qquad \ldots (3.104)$$

and assuming

$$W = \sum_{m=1,3,5.}^{\infty} R_m \sin m\theta \qquad \ldots (3.105)$$

where R_m is a function of r only, and substituting Eq.(3.104) and (3.105) in (3.101) one gets

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - P_1^2 \right) \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - P_2^2 \right) R_m$$

$$= \frac{4q}{\pi mD} \qquad \ldots (3.106)$$

The appropriate solution of Eq.(3.101) is given by

$$W = \sum_{m=1,3,5.}^{\infty} \left[A_m I_m (P_1 r) + B_m I_m (P_2 r) + \frac{4q}{\pi Dm} \times \right.$$

$$\left. \times \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s,m}(i P_2 r) \right] \sin m\theta \qquad \ldots (3.107)$$

where $\lambda_s = \dfrac{(-1)^s (i P_1)^{2s}}{(2^2 - m^2)(4^2 - m^2) \cdots \{(2+2s)^2 - m^2\}}$

and $S_{3+2s,m}(i P_2 r) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (i P_2 r)^{4+2s+2n}}{\{(4+2s)^2 - m^2\} \cdots \{(4+2s+2n)^2 - m^2\}}$

is the Lommel's function which is uniformly convergent.

The required boundary conditions are

$$(W)_{r=a} = 0 \qquad \ldots (3.108)$$

$$(u)_{r=a} = 0 \qquad \ldots (3.108a)$$

$$\left[\frac{\partial^2 w}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right]_{r=a} = 0 \qquad \ldots (3.109)$$

Considering eqs.(3.107),(3.108) and (3.109) and solving for the constants Am and Bm one gets

$$A_m = \frac{4q}{\pi D m} \cdot \frac{\begin{aligned} & \left[\left\{ a P_2^2 I_m''(P_2 a) + \nu P_2 I_m'(P_2 a) \right\} \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S_{3+2s,m}(i P_2 a) - \right. \\ & - I_m(P_2 a) \left\{ a \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S''_{3+2s,m}(i P_2 a) + \right. \\ & \left. \left. + \nu \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S'_{3+2s,m}(i P_2 a) \right\} \right] \end{aligned}}{\begin{aligned} & \left[I_m(P_2 a) \left\{ a P_1^2 I_m''(P_1 a) + \nu P_1 I_m'(P_1 a) \right\} - \right. \\ & \left. - I_m(P_1 a) \left\{ a P_2^2 I_m''(P_2 a) + \nu P_2 I_m'(P_2 a) \right\} \right] \end{aligned}} \quad \ldots (3.110)$$

$$B_m = -\frac{4q}{\pi D m} \cdot \frac{\begin{aligned} & \left[\left\{ a P_1^2 I_m''(P_1 a) + P_1 I_m'(P_1 a) \right\} \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} \cdot S_{3+2s,m}(i P_2 a) - \right. \\ & - I_m(P_1 a) \left\{ a \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S''_{3+2s,m}(i P_2 a) + \right. \\ & \left. \left. + \nu \sum_{s=0}^{\infty} \frac{\lambda_s}{(i P_2)^{4+2s}} S'_{3+2s,m}(i P_2 a) \right\} \right] \end{aligned}}{\begin{aligned} & \left[I_m(P_2 a) \left\{ a P_1^2 I_m''(P_1 a) + \nu P_1 I_m'(P_1 a) \right\} - \right. \\ & \left. - I_m(P_1 a) \left\{ a P_2^2 I_m''(P_2 a) + \nu P_2 I_m'(P_2 a) \right\} \right] \end{aligned}} \quad \ldots (3.111)$$

where dashes represent differentiations with respect to r.

Since we are interested only in the lateral displacement w, let us determine α by eliminating u and v from Eq.(3.100)

Let

$$u = \sum U(r) \cos m\theta \qquad \ldots (3.112)$$

$$v = \sum V(r) \sin m\theta \qquad \ldots (3.113)$$

Multiplying Eq.(3.100) by $r d\theta dr$ and integrating within the limits 0 to a and 0 to π one gets

$$\int_0^a \int_0^{\pi} r \sum U'(r) \cos m\theta \, d\theta \, dr + \frac{1}{2} \int_0^a \int_0^{\pi} r \left(\frac{\partial w}{\partial r}\right)^2 d\theta \, dr +$$

$$+ \int_0^a \int_0^{\pi} \sum U(r) \cos m\theta \, d\theta \, dr + \int_0^a \int_0^{\pi} \sum m \, V(r) \cos m\theta \, d\theta \, dr +$$

$$+ \frac{1}{2} \int_0^a \int_0^{\pi} \frac{1}{r} \left(\frac{\partial w}{\partial \theta}\right)^2 d\theta \, dr = \frac{\alpha^2 h^2}{12} \int_0^a \int_0^{\pi} r \, d\theta \, dr$$

After evaluating the integrals the following equation leading to α is obtained.

$$\sum_{m=1,3,5.}^{\infty} \Big[A_m^2 P_1^2 \Big(-\frac{a^2}{8} \Big\{ \frac{1}{2} \big[I_{m-2}(P_1a) + I_m(P_1a) \big]^2 - \Big[1 + \frac{(m-1)^2}{P_1^2 a^2} \Big] \times$$

$$\times I_{m-2}^2(P_1a) \Big\} - \frac{a^2}{8} \Big\{ \frac{1}{2} \big[I_m(P_1a) + I_{m+2}(P_1a) \big]^2 - \Big[1 + \frac{(m+1)^2}{P_1^2 a^2} \Big] \times$$

$$\times I_{m+1}^2(P_1a) \Big\} + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \phi \Big(\frac{P_1}{2} \Big)^{2m+2n+2t} \Big) + B_m^2 P_2^2 \times$$

$$\times \Big(\frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty} \phi \Big(\frac{P_2}{2} \Big)^{2m+2n+2t} - \frac{a^2}{8} \Big\{ \frac{1}{2} \big[I_{m-2}(P_2a) + I_m(P_2a) \big]^2 +$$

$$+ \Big[1 + \frac{(m-1)^2}{P_2^2 a^2} \Big] I_{m-1}^2(P_2a) \Big\} - \frac{a^2}{8} \Big\{ \frac{1}{2} \big[I_m(P_2a) + I_{m+2}(P_2a) \big]^2 +$$

$$+ \Big[1 + \frac{(m+1)^2}{P_2^2 a^2} \Big] I_{m+1}^2(P_2a) \Big\} \Big) + \frac{16 \; q^2}{\pi^2 D^2 m^2} \Big\{ \sum_{s=0}^{\infty} \frac{\lambda^2{}_s}{(i \, P_2)^{8+4s}} \times$$

$$\times \Big[\sum_{n=0}^{\infty} \frac{\mu_n^2 (4+2s+2n)^2 \, a^{8+4s+4n}}{8+4s+4n} + \sum_{n=0}^{\infty} \sum_{\substack{t=0 \\ n \neq t}}^{\infty}$$

$$\frac{\mu_n \mu_t (4+2s+2n)(4+2s+2t) a^{8+4s+2n+2t}}{8+4s+2n+2t} \Big] \Big\} + \frac{1}{2} A_m B_m P_1 P_2 \times$$

$$\times \left\{ \sum_{\substack{n=0\\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \left[\frac{\left(\frac{P_1}{2}\right)^{m-1+2n} \left(\frac{P_2}{2}\right)^{m-1+2t} a^{2m+2n+2t}}{(2m+2n+2t)\,\lfloor n\,\lfloor t\,\Gamma(m+n)\,\Gamma(m+t)} + \right.\right.$$

$$\frac{\left(\frac{P_1}{2}\right)^{m+1+2n} \left(\frac{P_2}{2}\right)^{m+1+2t} a^{2m+2n+2t+4}}{(2m+2n+2t+4)\,\lfloor n\,\lfloor t\,\Gamma(m+n+2)\,\Gamma(m+t+2)} +$$

$$+ \phi \cdot \left(\frac{P_1}{2}\right)^{m-1+2n} \left(\frac{P_2}{2}\right)^{m+1+2t} +$$

$$\left.\left. + \phi \cdot \left(\frac{P_1}{2}\right)^{m+1+2n} \left(\frac{P_2}{2}\right)^{m-1+2t} \right] \right\} +$$

$$+ 4\,A_m\,P_1 \frac{q}{\pi\,Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \left\{ \sum_{\substack{n=0\\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \left[\mu_n \left(\frac{P_1}{2}\right)^{m-1+2t} \psi \right.\right. +$$

$$\left.\left. + \mu_n\,\psi_1 \left(\frac{P_1}{2}\right)^{m+1+2t} \right] \right\} +$$

$$+ 4\,B_m P_2 \frac{q}{\pi\,Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \left\{ \sum_{\substack{n=0\\ n\neq t}}^{\infty} \sum_{t=0}^{\infty} \mu_n\,\psi \left(\frac{P_2}{2}\right)^{m-1+2t} \right. +$$

$$\left. + \mu_n\,\psi_1 \left(\frac{P_2}{2}\right)^{m+1+2t} \right] \Big\} + m^2 \Big(A_m^2 \sum_{n=0}^{\infty} \psi_2 \left(\frac{P_1}{2}\right)^{2m+4n} +$$

$$+ A_m^2 \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \psi_3 \left(\frac{P_1}{2}\right)^{2m+2n+2t} + B_m^2 \sum_{n=0}^{\infty} \psi_2 \left(\frac{P_2}{2}\right)^{2m+4n} +$$

$$+ B_m^2 \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \psi_3 \left(\frac{P_2}{2}\right)^{2m+2n+2t} + \frac{16\, q^2}{\pi^2 D^2 m^2} \sum_{s=0}^{\infty} \frac{\lambda_s^2}{(iP_2)^{8+4s}} \times$$

$$\times \left\{ \sum_{n=0}^{\infty} \frac{\mu_n^2\, a^{8+4s+4n}}{8+4s+4n} + \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \frac{\mu_n\, \mu_t\, a^{8+4s+2n+2t}}{8+4s+2n+2t} \right\} +$$

$$+ 2A_mB_m \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \frac{\left(\frac{P_1}{2}\right)^{m+2n} \left(\frac{P_2}{2}\right)^{m+2t} a^{2m+2n+2t}}{(2m+2n+2t)\, \underline{/n}\, \underline{/t}\; \Gamma(m+n+1)\, \Gamma(m+t+1)} +$$

$$+ 8\, A_m \frac{q}{\pi Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \mu_n\, \phi_1 \left(\frac{P_1}{2}\right)^{m+2t} +$$

$$+ 8\, B_m \frac{q}{\pi Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \sum_{\substack{n=0 \\ n \neq t}}^{\infty} \sum_{t=0}^{\infty} \mu_n \cdot \phi_1 \left(\frac{P_2}{2}\right)^{m+2t}$$

$$= \frac{\alpha^2 h^2 a^2}{6} \qquad \ldots (3.114)$$

where

$$\mu_n = \frac{(-1)^n \, (iP_2)^{4+2s+2n}}{\left\{(4+2s)^2 - m^2\right\} \ldots \left\{(4+2s+2n)^2 - m^2\right\}}$$

$$\mu_t = \frac{(-1)^n \, (iP_2)^{4+2s+2t}}{\left\{(4+2s)^2 - m^2\right\} \ldots \left\{(4+2s+2t)^2 - m^2\right\}}$$

$$\Phi = \frac{a^{2m+2n+2t+2}}{(2m+2n+2t+2) \angle n \, \angle t \, \Gamma(m+n) \, \Gamma(m+t+2)}$$

$$\Phi_1 = \frac{a^{4+2s+2n+2t+m}}{(4+2s+2n+2t+m) \angle t \, \Gamma(m+t+1)}$$

$$\Psi = \frac{(4+2s+2n) \, a^{4+2s+2n+2t+m}}{(4+2s+2n+2t+m) \angle t \, \Gamma(m+t)}$$

$$\Psi_1 = \frac{(4+2s+2n) \, a^{6+2s+2n+2t+m}}{(6+2s+2n+2t+m) \angle t \, \Gamma(m+t+2)}$$

$$\Psi_2 = \frac{a^{2m+4n}}{(2m+4n) \left\{\angle n \, \Gamma(m+n+1)\right\}^2}$$

$$\psi_3 = \frac{a^{2m+2n+2t}}{(2m+2n+2t)\,\underline{|n}\;\underline{|t}\;\Gamma(m+n+1)\,\Gamma(m+t+1)}$$

As $P_1 \rightarrow 0$, $P_2 \rightarrow 0$, Eq.(3.107) reduces to

$$W = \frac{qa^4}{D} \sum_{m=1,3,5}^{\infty} \left\{ \frac{4r^4}{a^4} \; \frac{1}{m\pi\,(16-m^2)\,(4-m^2)} \right.$$

$$+ \frac{r^m}{a^m} \; \frac{m+5+\nu}{m\pi(16-m^2)\,(2+m)\left[m+\frac{1}{2}(1+\nu)\right]}$$

$$\left. - \frac{r^{m+2}}{a^{m+2}} \; \frac{m+3+\nu}{m\pi(4+m)(4-m^2)\left[m+\frac{1}{2}(1+\nu)\right]} \right\} \sin m\theta$$

as obtained by Timoshenko (1959) for the corresponding problem of small deflection without any elastic foundation.

R E S U L T S

To obtain deflection for a given value of plate radius 'a' and foundation modulus 'K_F' one has to start from Eq.(3.114) with an assumed value of ' α ' in order to obtain the corresponding value of the load function ($\frac{qa^4}{Dh}$). Once this relationship is obtained the corresponding deflection $(\frac{W}{h})$ can be calculated from Eq.(3.107) with the help of Eq.(3.110) and (3.111). For $a=30mm$, $\nu = .3$, and $K_p = 350$ deflections have been plotted in Fig.3.15 for various values of load function $(\frac{qa^4}{Dh})$.

On examination of the Eq.(3.107), it is clear that the radius of symmetry of the plate undergoes the maximum deflection with respect to other radii. The expression for the deflection at a given point on the radius of symmetry can be expressed in the form $W = \beta \frac{qa^4}{D}$, where β is a numerical factor. Deflections at various points on the radius of symmetry are plotted in Fig.3.16 for a given value of load function. From Fig.3.16 it is observed that maximum deflection occurs at the centre of gravity of the plate.

The plate is subjected to bending moments in radial and tangential directions, as well as to a twisting moment. The moments can be easily computed, because the deflection w is known. The expressions for bending and twisting moments are

$$M_r = \sum_{m=1,3,5}^{\infty} - D \left[\frac{A_m P_1^2}{4} \left\{ I_{m+2}(P_1 r) + 2 I_m(P_1 r) + I_{m-2}(P_1 r) \right\} + \right.$$

$$+ \frac{B_m P_2^2}{4} \left\{ I_{m+2}(P_2 r) + 2 I_m(P_2 r) + I_{m-2}(P_2 r) \right\} +$$

$$+ \nu \left\{ \frac{A_m P_1}{2r} \left[I_{m-1}(P_1 r) + I_{m+1}(P_1 r) \right] + \frac{B_m P_2}{2r} \right. \times$$

$$\times \left[I_{m-1}(P_2 r) + I_{m+1}(P_2 r) \right] - \frac{m^2}{r^2} \left[A_m I_m(P_1 r) \right. +$$

$$+ \; C_m \, I_m(P_2 r) \,\rfloor \Big\} + \frac{4q}{\pi Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \; x$$

$$x \left\{ S''_{3+2s,m}(iP_2 r) + \frac{\nu}{r} S'_{3+2s,m}(iP_2 r) - \frac{m^2 \nu}{r^2} S_{3+2s,m}(iP_2 r) \right\} \rfloor \; \text{Sin } m\theta$$

$$\ldots (3.115)$$

$$M_\theta = \sum_{m=1,3,5}^{\infty} - D \,\llcorner\, \frac{A_m P_1^2 \nu}{4} \Big\{ I_{m+2}(P_1 r) + 2\, I_m(P_1 r) + I_{m-2}(P_1 r) \Big\} +$$

$$+ \frac{B_m P_2^2 \nu}{4} \Big\{ I_{m+2}(P_2 r) + 2\, I_m(P_2 r) + I_{m-2}(P_2 r) \Big\} +$$

$$+ \frac{A_m P_1}{2r} \Big\{ I_{m-1}(P_1 r) + I_{m+1}(P_1 r) \Big\} + \frac{B_m P_2}{2r} \Big\{ I_{m-1}(P_2 r) +$$

$$+ I_{m+1}(P_2 r) \Big\} - \frac{m^2}{r^2} \Big\{ A_m \, I_m(P_1 r) + B_m \, I_m(P_2 r) \Big\} +$$

$$+ \frac{4q}{\pi Dm} \cdot \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \cdot \Big\{ \nu \, S''_{3+2s,m}(iP_2 r) +$$

$$+ \frac{1}{r} S'_{3+2s,m}(iP_2r) - \frac{m^2}{r^2} S_{3+2s,m}(iP_2r)\Big\} \Big] \sin m\theta \qquad \ldots (3.116)$$

$$M_{r\theta} = (1-\nu) D \sum_{m=1,3,5}^{\infty} m \Big[\frac{A_m P_1}{2r} \Big\{ I_{m-1}(P_1r) + I_{m+1}(P_1r) \Big\} +$$

$$+ \frac{B_m P_2}{2r} \Big\{ I_{m-1}(P_2r) + I_{m+1}(P_2r) \Big\} - \frac{1}{r^2} \Big\{ A_m I_m(P_1r) + B_m I_m(P_2r) \Big\} +$$

$$+ \frac{4q}{\pi Dm} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \cdot \Big\{ \frac{1}{r} S'_{3+2s,m}(iP_2r) -$$

$$- \frac{1}{r^2} S_{3+2s,m}(iP_2r) \Big\} \Big] \cos m\theta \qquad \ldots (3.117)$$

Eqs.(3.115), (3.116) and (3.117) show that the bending moments M_r and M_θ are both maximum on the radius of symmetry, the twisting moment $M_{r\theta}$ is maximum on the bounding diameter. The bending moments can be expressed in the form

$$M_r = \beta_1 qa^2 \qquad M_\theta = \beta_2 qa^2 \qquad \ldots (3.118)$$

The stresses can be calculated from the expressions

$$(\sigma_r)_{max.} = \frac{6\,(M_r)_{max.}}{h^2}, \quad (\sigma_\theta)_{max.} = \frac{6\,(M_\theta)_{max.}}{h^2} \qquad \ldots (3.119)$$

Expressions for shearing forces can be obtained with the help of Eq.(3.107) from the expressions

$$Q_r = -D\frac{\partial}{\partial r}\left(\nabla^2 w - \alpha^2 w\right), \quad Q_\theta = -\frac{D}{r}\frac{\partial}{\partial\theta}\left(\nabla^2 w - \alpha^2 w\right) \qquad \ldots (3.120)$$

For the simicircular plate Eq.(3.120) can be expressed in the form

$$Q_r = -D\sum_{m=1,3,5}^{\infty}\Big\{\Big[A_m P_1^3 I_m'''(P_1 r) + B_m P_2^3 I_m'''(P_2 r) + \frac{4q}{\pi Dm} \times$$

$$\times \sum_{s=0}^{\infty}\frac{\lambda_s}{(iP_2)^{4+2s}}\cdot S_{3+2s,m}'''(i P_2 r)\Big] + \frac{1}{r}\Big[A_m P_1^2 I_m''(P_1 r) + B_m P_2^2 I_m''(P_2 r) +$$

$$+ \frac{4q}{\pi Dm}\sum_{s=0}^{\infty}\frac{\lambda_s}{(iP_2)^{4+2s}}\cdot S_{3+2s,m}''(i P_2 r)\Big] - \left(\alpha^2 + \frac{m^2}{r^2}\right)\Big[A_m P_1 I_m'$$

$$(P_1 r) + B_m P_2 I_m'(P_2 r) + \frac{4q}{\pi Dm}\sum_{s=0}^{\infty}\frac{\lambda_s}{(iP_2)^{4+2s}}\cdot S_{3+2s,m}'(i P_2 r)\Big]\Big\} \times$$

$$\times \sin m\theta \qquad \ldots (3.121)$$

$$Q_\theta = - \frac{D}{r} \sum_{m=1,3,5}^{\infty} m \Big[A_m P_1^2 I_m''(P_1 r) + B_m P_2^2 I_m''(P_2 r) + \frac{4q}{\pi D m} \times$$

$$\times \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \cdot S''_{3+2s,m}(i P_2 r) + \frac{1}{r} \Big\{ A_m P_1 I_m'(P_1 r) + B_m P_2 I_m'(P_2 r) +$$

$$+ \frac{4q}{\pi D m} \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \cdot S'_{3+2s,m}(i P_2 r) \Big\} - \left(\alpha^2 + \frac{m^2}{r^2} \right) \times$$

$$\times \Big\{ A_m I_m(P_1 r) + B_m I_m(P_2 r) + \frac{4q}{\pi D m} \cdot \sum_{s=0}^{\infty} \frac{\lambda_s}{(iP_2)^{4+2s}} \cdot S_{3+2s,m}(iP_2 r) \Big\} \Big] \times$$

$$\times \cos m\theta \qquad \ldots (3.122)$$

The shearing stresses can be calculated from the expressions

$$\tau_{r\theta} = \frac{6 M_{r\theta}}{h^2}, \quad \tau_{rz} = \frac{3}{2} \frac{Q_r}{h}, \quad \tau_{\theta z} = \frac{3}{2} \frac{Q_\theta}{h}$$

$$\ldots (3.123)$$

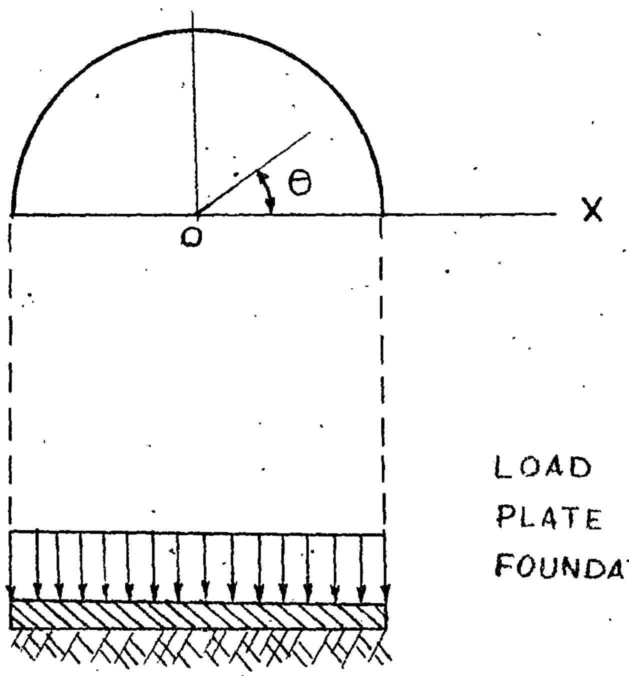

FIG. 3.14 SEMICIRCULAR PLATE ON FOUNDATION.

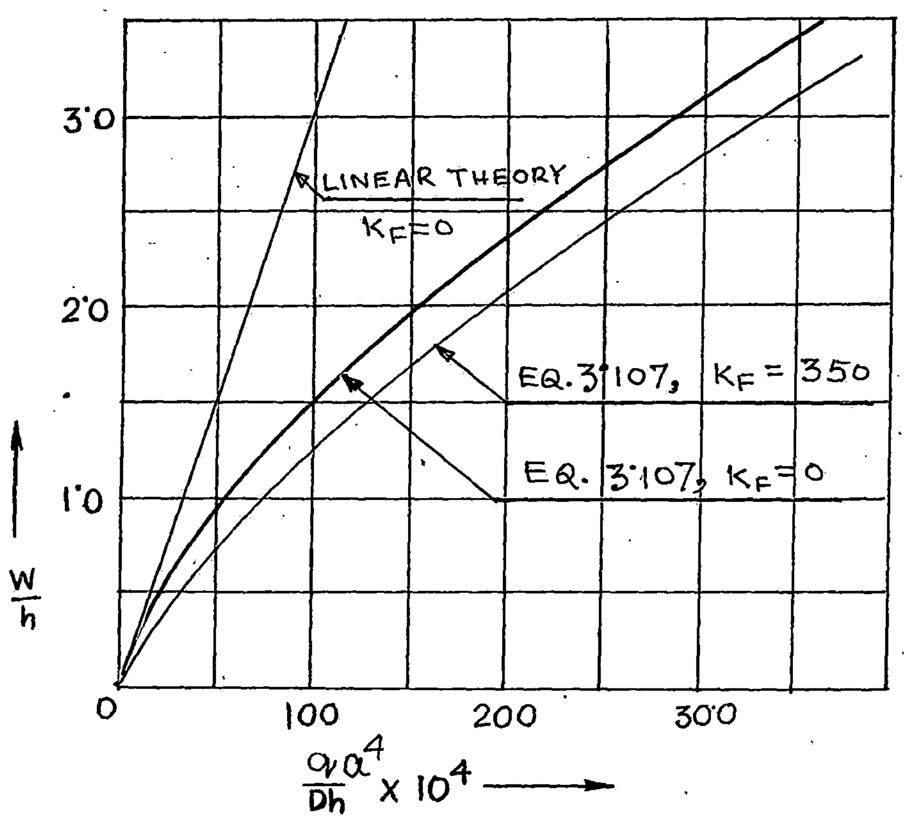

FIG. 3·15 DEFLECTION CURVE.

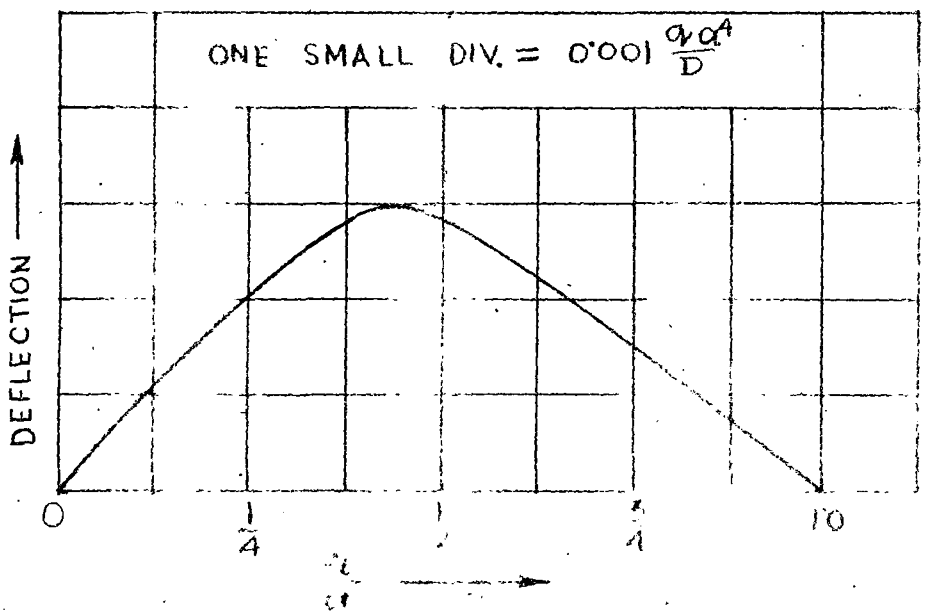

FIG. 3·16 DEFLECTION

LARGE DEFLECTION OF A TRIANGULAR ORTHOTROPIC PLATE ON ELASTIC FOUNDATION*

PAPER - V

INTRODUCTION

Triangular reinforced concrete slabs are sometimes used as bottom slabs of bunkers. Thus the design of this type of structure is of practical interest for Defence. These slabs may rest freely on soil or sand and generally are subjected to a uniform load. If the thickness of the slab is small compared to the other dimensions, then it may be regarded as a thin orthotropic plate resting on elastic foundation and subjected to a uniform load. Following Berger's method numerous isotropic plate problems have been solved with ease and accuracy. Iwinski and Nowinski (1957) generalised the procedure of Berger to Orthotropic plate and found the deflections of circular and rectangular plates under uniform load and under various boundary conditions.

In this Paper large deflection of an equilateral triangular orthotropic plate, such as reinforced concrete, resting on elastic foundation has been solved following Berger's method. The plate is under uniform load and the foundation is assumed to be such that its reaction is proportional to the deflection of the plate.

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FORMULATION OF PROBLEM

For moderately large deflections, the strain displacement relationships are

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}$$

Neglecting the second middle surface strain invariant, the strain energy due to bending and stretching of the middle surface of the plate of thickness, h, can be written as

$$V_1 = \frac{1}{2}\iint \left[D_x \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1 \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4 D_{xy} \left(\frac{\partial^2 w}{\partial x . \partial y}\right)^2 \right.$$

$$\left. + D_x \frac{12}{h^2} e_1^2 \right] dxdy \qquad \ldots (3.124)$$

in which

$$D_x = \frac{E'_x h^3}{12}, \quad D_y = \frac{E'_y h^3}{12}, \quad D_1 = \frac{E'' h^3}{12}, \quad D_{xy} = \frac{G h^3}{12} \qquad \ldots (3.124a)$$

$$e_1 = \frac{\partial u}{\partial x} + K_1 \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{K_1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \ldots (3.124b)$$

$$K_1^2 = \frac{D_y}{D_x} \qquad \ldots (3.124c)$$

and E_x', E_y', E'', and G are constants to characterise the elastic properties of the material. By adding the potential energy of the uniform normal load, q, and of the foundation reaction, K, to the energy expression of Eq.(3.124) the modified energy expression is obtained as follows :

$$V = \frac{1}{2}\iint \Big[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{xy}\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 +$$

$$+ D_x\frac{12}{h^2}e_1^2\Big]dxdy - \iint qw\,dxdy + \frac{1}{2}\iint K w^2\,dxdy \qquad \ldots (3.125)$$

According to the principle of minimum potential energy, the displacements satisfying the equilibrium conditions make the potential energy, V, minimum. In order for the integral of Eq.(3.125) to be an extremum, its integrand, F, must satisfy the following Euler's variational principle :

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{yy}}\right)$$

$$+ \frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial F}{\partial w_{xy}}\right) = 0 \qquad \ldots (3.126)$$

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) = 0 \qquad \ldots (3.127)$$

and

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0 \qquad \ldots (3.128)$$

Application of Eqs.(3.127) and (3.128) to Eq.(3.125) yields

$$\frac{\partial}{\partial x}(e_1) = 0 \qquad \ldots (3.129)$$

$$\frac{\partial}{\partial y}(e_1) = 0 \qquad \ldots (3.130)$$

Thus

$$e_1 = C \qquad \ldots (3.131)$$

a normalised constant of integration to be determined. Applying Eq.(3.126) to Eq.(3.125) and considering Eq.(3.131) one gets

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + \frac{2(D_1 + 2 D_{xy})}{D_x} \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12C}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2}\right) +$$

$$+ \frac{K}{D_x} = \frac{q}{D_x} \qquad \ldots (3.132)$$

Introducing the notation

$$H = D_1 + 2 D_{xy}$$

Eq.(3.132) can be written as

$$\frac{\partial^4 w}{\partial x^4} + K_1^2 \frac{\partial^4 w}{\partial y^4} + 2\frac{H}{D_x} \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{12C}{h^2}\left(\frac{\partial^2 w}{\partial x^2} + K_1 \frac{\partial^2 w}{\partial y^2}\right) +$$

$$+ \frac{K}{D_x} = \frac{q}{D_x} \qquad \ldots (3.133)$$

For a slab with two way reinforcement in the directions x and y, H can be taken as [Timoshenko (1959), P.366]

$$H = \sqrt{D_x \ D_y}$$

Introducing now

$$x_1 = x$$

$$y_1 = y\left(\frac{D_x}{D_y}\right)^{\frac{1}{4}} \qquad \ldots (3.134)$$

Eq.(3.133) is reduced to the form

$$(\nabla^2 - \alpha^2)\ \nabla^2 w + \frac{K}{D_x} w = \frac{q}{D_x} \qquad \ldots (3.135)$$

in which

$$\alpha^2 = \frac{12C}{h^2}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$$

SOLUTION OF PROBLEM

To solve equation (3.135) tri-linear co-ordinates as shown by Sen, B. (1968) has been used. Let the plate be in the form of an equilateral triangle, ABC (Fig.3.17) having each side of length, 2a. Let the centroid, O, be the origin, X - axis and Y - axis perpendicular and parallel to the base BC, respectively. If x_1, y_1 be the cartesian co-ordinates of any point, P, within the triangle, P_1, P_2, P_3 be the three perpendiculars from P on CA, AB, and BC respectively, and r, the radius of the inscribed circle, then

$$P_1 = r + \frac{x_1}{2} - \frac{y_1 \sqrt{3}}{2} ,$$

$$P_2 = r + \frac{x_1}{2} + \frac{y_1 \sqrt{3}}{2} ,$$

$$P_3 = r - x_1 ,$$

$$P_1 + P_2 + P_3 = \sqrt{3}a = K_2 = \text{constant},$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}$$

$$= \frac{\partial^2}{\partial P_1^2} + \frac{\partial^2}{\partial P_2^2} + \frac{\partial^2}{\partial P_3^2} - \frac{\partial^2}{\partial P_1 \partial P_2} - \frac{\partial^2}{\partial P_2 \partial P_3} - \frac{\partial^2}{\partial P_3 \partial P_1}$$

Using the trilinear co-ordinates (P_1, P_2, P_3) the deflection, w can be taken in the form

$$W = \sum_{n=1}^{\infty} A_n \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \qquad \ldots (3.136)$$

where

A_n = a constant.

The above form of W satisfies the following boundary conditions of simply supported edges :

$$\left.\begin{matrix} W = 0 \\ \nabla^2 W = 0 \end{matrix}\right\} \text{at } P_1 = 0, \quad P_2 = 0, \quad P_3 = 0$$

Expanding the transverse uniform load, q, into Fourier sine series

$$q = \sum_{n=1}^{\infty} \frac{2q}{n\pi} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n\pi P_3}{K_2} \right] \qquad \ldots (3.137)$$

and substituting Eqs.(3.136) and (3.137) into Eq.(3.135) one gets

$$A_n = \sum_{n=1}^{\infty} \frac{2q}{n\pi D_x} \cdot \frac{1}{\left[\left(\frac{2n\pi}{K_2}\right)^4 + \alpha^2 \left(\frac{2n\pi}{K_2}\right)^2 + \frac{K}{D_x} \right]} \qquad \ldots (3.138)$$

Thus w is determined.

To determine α , Eq.(3.124b) is transformed into x_1 , y_1 co-ordinates in the following form

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial x_1} + \sqrt{K_1}\,\frac{\partial v}{\partial y_1} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2 + \tfrac{1}{2}\left(\frac{\partial w}{\partial y_1}\right)^2 \qquad \ldots (3.139)$$

The boundary conditions on u and v are

$$u = 0 \qquad \text{at} \quad P_3 = 0 \qquad \ldots (3.140)$$

$$\sqrt{3}\,v + u = 0 \text{ at } \quad P_2 = 0 \qquad \ldots (3.141)$$

$$\sqrt{3}\,v - u = 0 \text{ at } \quad P_1 = 0 \qquad \ldots (3.142)$$

The following forms of u and v satisfy the above boundary conditions.

$$u = \sum_{m=1}^{\infty} \sqrt{3}\; B_m \left[\sin \frac{2m\pi\,(P_2 + P_3)}{K_2} + \sin \frac{2m\pi\,(P_1 + P_3)}{K_2} \right] \qquad \ldots (3.143)$$

$$v = \sum_{m=1}^{\infty} \frac{1}{\sqrt{K_1}} B_m \left[\sin \frac{2m\pi\,(P_1 + P_3)}{K_2} - \sin \frac{2m\pi\,(P_2 + P_3)}{K_2} \right] \qquad \ldots (3.144)$$

in which B_m is a constant.

Substituting the expressions for u, v and w into Eq.(3.139) and integrating over the whole area of the plate, the following equation determining α is obtained.

$$\frac{\alpha^2 h^2}{12} = \sum_{n=1}^{\infty} \frac{3 A_n^2 n^2 \pi^2}{K_2^2} \qquad \ldots (3.145)$$

Thus W is completely determined in the following form in x, y co-ordinates

$$W = A_n \left[2 \sin 2n\pi \left(\tfrac{1}{3} + \frac{x}{2\sqrt{3}a}\right) \cos \frac{2n\pi y}{2\sqrt{K_1}\, a} + \sin 2n\pi \left(\tfrac{1}{3} - \frac{x}{\sqrt{3}a}\right) \right]$$

$$\ldots (3.146)$$

If $D_x = D_y = D, \alpha \longrightarrow 0$, and $K = 0$, Eqs.(3.136) and (3.138) give the small deflection result for an isotropic plate not resting on the elastic foundation in the following form :

$$W = \sum_{n=1}^{\infty} \frac{q K_2^4}{8 n^5 \pi^5 D} \left[\sin \frac{2n\pi P_1}{K_2} + \sin \frac{2n\pi P_2}{K_2} + \sin \frac{2n \quad P_3}{K_2} \right]$$

$$\ldots (3.147)$$

The corresponding equation as obtained by S. Woinowsky - Krieger [1959, P.313] for a plate having each side of length $\frac{2a}{\sqrt{3}}$ is

$$W = \frac{q}{64aD} \left[x^3 - 3y^2x - a(x^2 + y^2) + \frac{4}{27} a^3 \right] \left(\frac{4}{9} a^2 - x^2 - y^2 \right)$$

$$\ldots (3.147a)$$

At the origin ($P_1 = P_2 = P_3$), w is given by Eq.(3.147)

$$W = \frac{27\ q\ a^4}{8\ \pi^5 D} \sum_{n=1}^{\infty} \frac{1}{n^5} \sin\frac{2n\pi}{3} = \cdot 009\ \frac{qa^4}{D} \qquad \ldots (3.147b)$$

which is numerically equal to that obtained from Eq.(3.147a) for the plate having each side of length 2a as

$$(W)_{x=y=0} = \frac{qa^4}{108D} = \cdot 009\ \frac{qa^4}{D}$$

RESULTS

To calculate deflection at any point within the plate, one has to start from Eq.(3.145) with an assumed value of (α a) leading to the corresponding value of the load function $\frac{qa^4}{D_x h}$. Once this relationship is obtained, the corresponding deflection can be obtained from the Eq.(3.136) and with the help of Eq.(3.138).

At the origin maximum deflection is obtained and is given by

$$\frac{W_{max.}}{h} = \frac{6}{\pi} \left(\frac{qa^4}{D_x\ h}\right) \sum_{n=1}^{\infty} \frac{\sin\frac{2n\pi}{3}}{n\left[\frac{16\pi^4 n^4}{9} + \frac{4\ \pi^2 n^2\ \alpha^2\ a^2}{3} + K_F\right]}$$

$$\ldots (3.148)$$

in which the nondimensional foundation modulus

$$K_F = \frac{Ka^4}{D_x}$$

For $K_F = 0$ and $K_F = 100$ graphs are plotted in Fig.3.18 showing the deflection $\frac{W}{h}$ at the centroid of the plate against the load. Fig.3.18 also contains a graph plotted according to the linear theory.
From Fig.3.18 it is clear that design calculations should be made according to the nonlinear theory, because deflections calculated according to small deflection theory will be far from the actual values for higher values of load function. The effect of the foundation is to reduce the deflection for a given value of load function.

Because the deflection, W, has been determined, bending moments and stresses can be computed easily. The bending moments M_x, and M_y at the centroid of the plate are obtained as

$$M_x = 4 (1+ \nu_c)qa^2 \sum_{n=1}^{\infty} \frac{n \, \mathrm{Sin} \, \frac{2n\pi}{3}}{\left[\frac{16 \, \pi^4 n^4}{9} + \frac{4 \, \pi^2 n^2 \alpha^2 a^2}{3} + K_F \right]}$$

... (3.149)

$$M_y = K_1 M_x$$ ... (3.150)

in which ν_c is the Poisson's ratio for concrete.
For isotropic plate without elastic foundation and undergoing small deflection $\nu_c = \nu$, $K_1 = 1$, $K_F = 0$, $\alpha \to 0$ and for a plate having each side of length $\frac{2a}{\sqrt{3}}$, Eq.(3.149) and (3.150) lead to

$$M_x = M_y = (1 + \nu) \frac{qa^2}{54}$$ ... (3.151)

which is the same result obtained by Timoshenko [1959, P. 314].

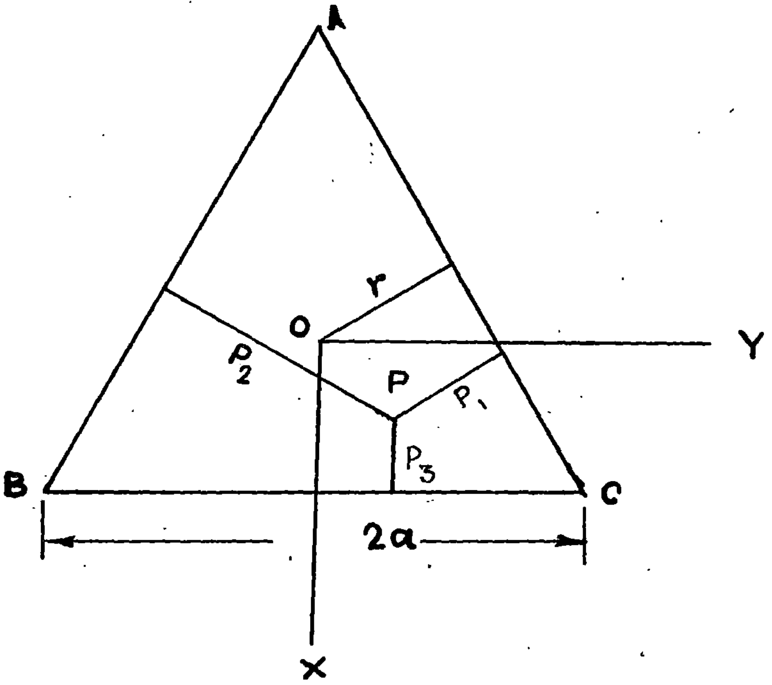

FIG. 3.17

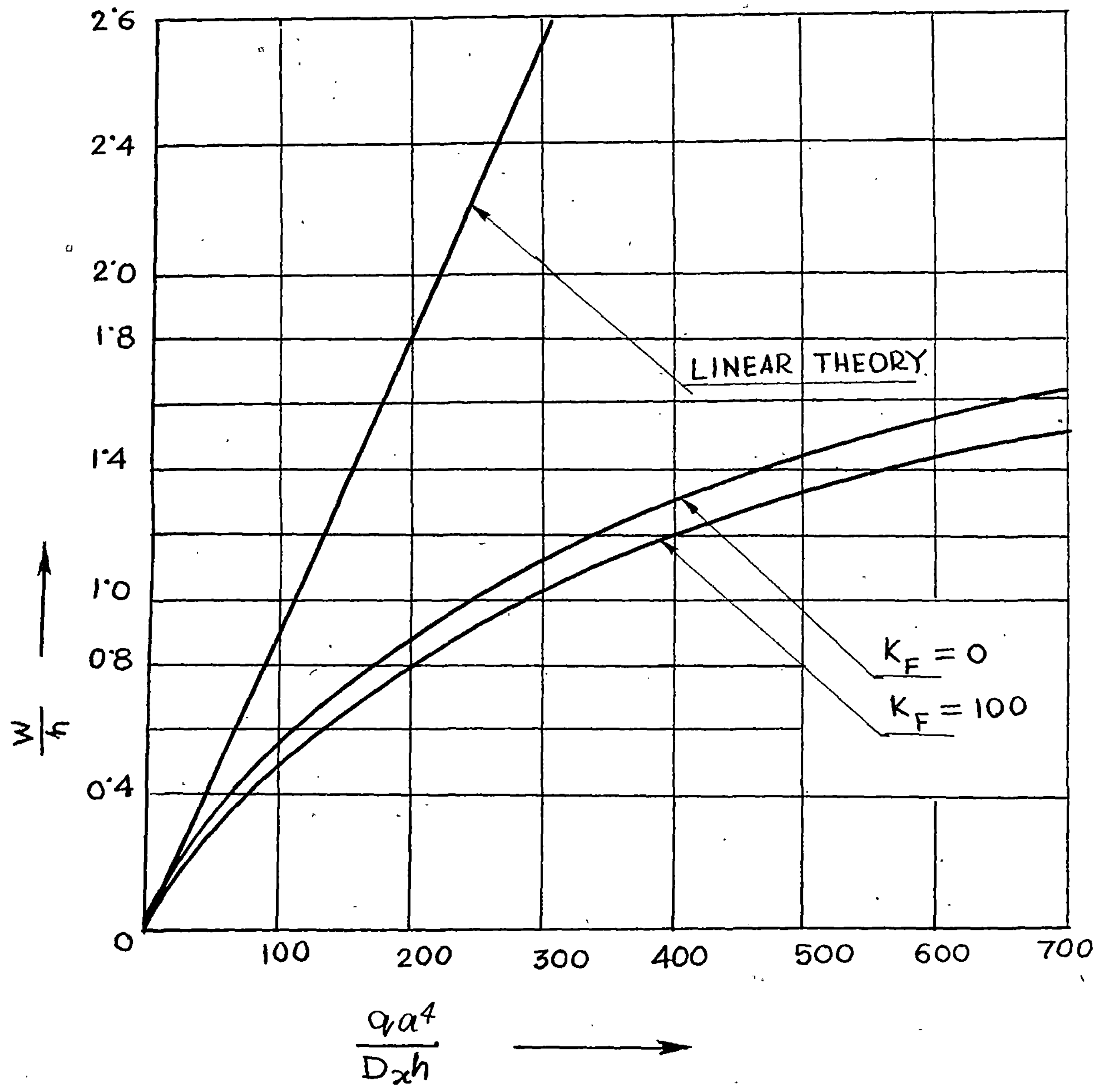

FIG. 3·18 DEFLECTION CURVE.

LARGE DEFLECTION OF A HEATED ELLIPTIC PLATE ON ELASTIC FOUNDATION *

PAPER- VI

INTRODUCTION

In recent years there has been a rapid development of thermoelasticity stimulated by various engineering sciences. In the field of machine structures, mainly with air-craft, steam and gasturbines and in chemical and nuclear engineering, thermal stresses play an important and frequently even a primary role. Determination of thermal deflections of plates, espacially of thin plates, is of vital importance in the design of machine structures, because excessive deflections may cause heavy undesirable thermal stresses.

Following Berger many non-linear plate problems have been solved under various edge conditions and different types of loads. Berger's technique of neglecting the second invariant of the middle surface strains has been extended by Basuli (1968) to the large deflection problems of heated plates to obtain the large deflections of heated rectangular, circular and right-angled triangular plates without any elastic foundation and under uniform load and stationary temperature

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distribution. Sinha [1963] has extended this method of Berger to investigate the large deflections of circular and rectangular plates on elastic foundation of Winkler type.

In this paper the author has applied the method of Basuli and Sinha to investigate the large deflection of an elliptic plate placed on elastic foundation and heated under stationary temperature distribution. The deflection is obtained in terms of Mathieu function of the first kind and of zero order. Numerical results have been presented in the form of graphs.

FORMULATION OF PROBLEM

The strain energy due to bending and stretching of the middle surface of the plate is given by

$$V_1 = \frac{D}{2}\iint\Big[(\nabla^2 w)^2 + \frac{12}{h^2}e_1^2 - 2(1-\nu)\Big\{\frac{12}{h^2}e_2 + \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\Big\}\Big]\,dx\,dy \qquad \ldots (3.152)$$

Combining the potential energy of the foundation reaction and also the potential energy due to heating with Eq.(3.152) and neglecting e_2, the modified energy expression for the total energy becomes

$$V = \frac{D}{2}\iint\Big[(\nabla^2 w)^2 + \frac{12}{h^2}e_1^2 - 2(1-\nu)\Big\{\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \Big(\frac{\partial^2 w}{\partial x \partial y}\Big)^2\Big\} +$$

$$+ \frac{K}{D}w^2\Big]dx\,dy - \iiint_{-h/2}^{h/2}\frac{E\alpha T'}{1-\nu}(e_1 - z\nabla^2 w)\,dx\,dy\,dz \qquad \ldots (3.153)$$

in which T' is the temperature distribution at any point given by [Basuli, (1968)]

$$T'(x,y,z) = T_0(x,y) + g(z)\,T(x,y) \qquad \ldots (3.154)$$

and

$$\int_{-h/2}^{h/2} z\,g(z)\,dz = f(h) \quad ; \quad \int_{-h/2}^{h/2} g(z)\,dz = 0 \qquad \ldots (3.155)$$

Combining Eqs. (3.153), (3.154) and (3.155) one gets

$$V = \frac{D}{2}\iint\Big[(\nabla^2 w)^2 + \frac{12}{h^2}e_1^2 - 2(1-\nu)\Big\{\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} - \Big(\frac{\partial^2 w}{\partial x \partial y}\Big)^2\Big\} +$$

$$+ \frac{K}{D}w^2\Big]dx\,dy - \iint\frac{E\alpha}{1-\nu}\Big\{T_0 e_1 h - f(h)\,T\,\nabla^2 w\Big\}dx\,dy \qquad \ldots (3.156)$$

According to the principle of minimum potential energy, the displacements that satisfy the equilibrium conditions make the potential energy, V, minimum. In order for the integral of Eq.(3.156) to be an extremum, the integrand, F, must satisfy the following Euler's equations of the calculus of variation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \quad \text{... (3.157a)}$$

$$\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0 \quad \text{... (3.157b)}$$

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xx}}\right) +$$

$$+ \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{yy}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial w_{xy}}\right) = 0 \quad \text{... (3.157c)}$$

Application of the Eqs. (3.157a), (3.157b) and (3.157c) to Eq.(3.156) yields

$$\frac{\partial}{\partial x}\left\{e_1 - (1+\nu)\alpha T_0\right\} = 0 \quad \text{... (3.158a)}$$

$$\frac{\partial}{\partial y}\left\{e_1 - (1+\nu)\alpha T_0\right\} = 0 \quad \text{... (3.158b)}$$

$$\nabla^4 w - \frac{12}{h^2}\left\{e_1 - (1+\nu)\alpha T_0\right\}\nabla^2 w + \frac{K}{D} w + \frac{E\alpha f(h)}{D(1-\nu)}\nabla^2 T = 0 \quad \text{... (3.158c)}$$

Eqs.(3.158a) and (3.158b) prove that

$\{e_1-(1+\nu)\alpha T_0\}$ is independent of x and y and therefore

$$e_1-(1+\nu)\alpha T_0 = \text{constant} = \frac{\beta^2 h^2}{12} \quad \text{... (3.159a)}$$

in which β is a normalised constant of integration, and

$$e_1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \quad \text{... (3.159b)}$$

Considering Eq.(3.159a), Eq.(3.158c) reduces to

$$\nabla^2(\nabla^2-\beta^2)w + \frac{K}{D}w = -\frac{E\alpha f(h)}{D(1-\nu)}\nabla^2 T \quad \text{... (3.160)}$$

SOLUTION OF PROBLEM

Let us take an elliptic plate of thickness, h. The centre of the plate in the middle surface is taken as the origin and the Z - axis downwards.

If there is no source of heat inside the plate the following differential equations must be satisfied for stationary temperature distribution [Nowacki (1962)]

$$\nabla^2 T_0 - \epsilon T_0 = -\frac{\epsilon_0}{2}(\theta_1+\theta_2) \quad \text{... (3.161)}$$

$$\nabla^2 T - \frac{12}{h^2}(1+\epsilon)T = -\frac{12\epsilon}{h^3}(\theta_1-\theta_2) \quad \text{... (3.162)}$$

in which Θ_1 and Θ_2 denote temperatures at the upper and lower media of the plate respectively.

If $\Theta_1 = \Theta_2$, Eq.(3.162) becomes

$$\nabla^2 T - \beta_1^2 T = 0 \qquad \text{... (3.163)}$$

in which

$$\beta_1^2 = (1+\epsilon)\frac{12}{h^2} \qquad \text{... (3.164)}$$

Transferring to elliptic co-ordinates (ξ , η) defined by $x + iy = d\cosh(\xi + i\eta)$, where $2d$ is the interfocal distance of the ellipse, Eq.(3.163) reduces to

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} - \frac{\beta_1^2 d^2}{2}\left(\cosh 2\xi - \cos 2\eta\right) T = 0 \qquad \text{... (3.165)}$$

Solution of Eq.(3.165) can be taken in the following form

$$T = \sum_{m=0}^{\infty} C_{2m}\, Ce_{2m}(\xi, -q)\, ce_{2m}(\eta, -q) \qquad \text{... (3.166)}$$

in which $Ce_{2m}(\xi, -q)$ and $ce_{2m}(\eta, -q)$ are modified Mathieu function and ordinary Mathieu function of the first kind and of order $2m$ respectively, and

$$q = \frac{\beta_1^2 d^2}{4} \qquad \text{... (3.167)}$$

while solving a problem of bending of a plate with an elliptic hole, by taking a single Mathieu function of the second order instead of taking Mathieu functions of all orders, Naghdi (1955) has shown that the results are satisfactory for larger elliptic holes. In this paper also similar approximation is made by taking Mathieu function of zero order and on this assumption Eq.(3.166) reduces to

$$T = C_0 Ce_0(\xi,-q)\, ce_0(\eta,-q) \qquad \ldots (3.168)$$

The following boundary condition is imposed on T

T = Constant = K_1 on $\xi = \xi_0$

with the above boundary condition Eq.(3.168) yields

$$K_1 = C_0\, Ce_0(\xi_0,-q)\, ce_0(\eta,-q) \qquad \ldots (3.169)$$

Multiplying Eq.(3.169) by $ce_0(\eta,-q)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation [Mclachlan] one gets

$$C_0 = \frac{2A_0^{(0)} K_1}{Ce_0(\xi_0,-q)} \qquad \ldots (3.170)$$

in which $A_0^{(0)}$ is the first Fourier coefficient in the expansion of $ce_0(\eta,-q)$.

Therefore

$$T = \frac{2A_0^{(0)} K_1}{Ce_0(\xi_0,-q)} Ce_0(\xi,-q)\, ce_0(\eta,-q) \qquad \dots (3.171)$$

is determined.

Changing Eq.(3.160) to elliptic co-ordinates and substituting the expression for $\nabla^2 T$ one gets

$$(\nabla^2 - P_1^2)(\nabla^2 - P_2^2) W = \lambda\, Ce_0(\xi,-q)\, ce_0(\eta,-q) \qquad \dots (3.172)$$

in which

$$P_1^2 + P_2^2 = \beta^2 \qquad \dots (3.173)$$

$$P_1^2 P_2^2 = \frac{K}{D} \qquad \dots (3.174)$$

$$\lambda = -\frac{E\alpha f(h)}{D(1-\nu)} \cdot \frac{2\beta_1^2 A_0^{(0)} K_1}{Ce_0(\xi_0,-q)} \qquad \dots (3.175)$$

$$\nabla^2 = \frac{2}{d^2(\cosh 2\xi - \cos 2\eta)} \left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \qquad \dots (3.176)$$

Complimentary function of Eq.(3.172) is given by

$$W = B_0 Ce_0(\xi,-q_1)\, ce_0(\eta,-q_1) + D_0 Ce_0(\xi,-q_2)\, ce_0(\eta,-q_2)$$

$$\dots (3.177)$$

in which

$$q_1 = \frac{P_1^2 d^2}{4} \; ; \quad q_2 = \frac{P_2^2 d^2}{4} \qquad \ldots (3.178)$$

Clearly the particular integral of Eq.(3.172) is

$$\frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} \, Ce_0(\xi, -q) \, ce_0(\eta, -q) \qquad \ldots (3.179)$$

Thus the complete solution of Eq.(3.172) is

$$W = B_0 \, Ce_0(\xi, -q_1) \, ce_0(\eta, -q_1) + D_0 \, Ce_0(\xi, -q_2) \, ce_0(\eta, -q_2) +$$

$$+ \frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} \, Ce_0(\xi, -q) \, ce_0(\eta, -q) \qquad \ldots (3.180)$$

If the outer boundary of the plate $\xi = \xi_0$ be clamped, the boundary conditions are

$$(W)_{\xi=\xi_0} = \left(\frac{\partial w}{\partial \xi}\right)_{\xi=\xi_0} = 0 \qquad \ldots (3.181)$$

Using Eq.(3.181) in Eq.(3.180) one gets the following two conditional equations

$$B_0 \, Ce_0(\xi_0, -q_1) \, ce_0(\eta, -q_1) + D_0 \, Ce_0(\xi_0, -q_2) \, ce_0(\eta, -q_2) +$$

$$+ \frac{\lambda}{(\beta_1^2 - P_2^2)(\beta_1^2 - P_1^2)} \, Ce_0(\xi_0, -q) \, ce_0(\eta, -q) = 0 \qquad \ldots (3.182a)$$

$$B_0 Ce_0'(\xi_0,-q_1)\, ce_0(\eta,-q_1) + D_0 Ce_0'(\xi_0,-q_2)\, ce_0(\eta,-q_2) +$$

$$+ \frac{\lambda}{(\beta_1^2-p_2^2)(\beta_1^2-p_1^2)}\, Ce_0'(\xi_0,-q)\, ce_0(\eta,-q) = 0$$

... (3.182b)

Multiplying Eqs.(3.182a) and (3.182b) by $ce_0(\eta,-q_1)$ and integrating with respect to η from 0 to 2π and using the orthogonality relation and normalisation one gets

$$B_0 = -\frac{\psi\Phi_2}{\psi_1\pi}\left[Ce_0(\xi_0,-q_2)\, Ce_0'(\xi_0,-q) - Ce_0(\xi_0,-q)\, Ce_0'(\xi_0,-q_2)\right]$$

... (3.183)

$$D_0 = \frac{\psi\Phi_2}{\psi_1\Phi_1}\left[Ce_0(\xi_0,-q_1)\, Ce_0'(\xi_0,-q) - Ce_0(\xi_0,-q)\, Ce_0'(\xi_0,-q_1)\right]$$

... (3.184)

in which

$$\psi = \frac{\lambda}{(\beta_1^2-p_2^2)(\beta_1^2-p_1^2)}$$

$$\psi_1 = Ce_0(\xi_0,-q_2)\, Ce_0'(\xi_0,-q_1) - Ce_0(\xi_0,-q_1)\, Ce_0'(\xi_0,-q_2)$$

$$\Phi_1 = 2\bar{A}_0^{(0)}\bar{\bar{A}}_0^{(0)} + \sum_{n=1}^{\infty}\bar{A}_{2n}^{(0)}\bar{\bar{A}}_{2n}^{(0)}$$

$$\Phi_2 = 2A_0^{(0)}\bar{A}_0^{(0)} + \sum_{n=1}^{\infty}\bar{A}_{2n}^{(0)} A_{2n}^{(0)}$$

and $\bar{A}_{2n}^{(0)}$, $\bar{\bar{A}}_{2n}^{(0)}$ and $A_{2n}^{(0)}$ are the Fourier coefficients in the expression of $ce_0(\eta, -q_1)$, $ce_0(\eta, -q_2)$ and $ce_0(\eta, -q)$ respectively.

The constants B_0 and D_0 thus being determined, the deflection W is known.

Let T_0 = constant which is clearly solution of the differential equation (3.161).

To determine the constant β^2 , Eq.(3.159) is transformed into elliptic co-ordinates in the form

$$h_1 h_2 \left\{ \frac{\partial}{\partial \xi}\left(\frac{u_\xi}{h_2}\right) + \frac{\partial}{\partial \eta}\left(\frac{u_\eta}{h_1}\right) \right\} + \frac{1}{2} h_1 h_2 \left\{ \left(\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{\partial w}{\partial \eta}\right)^2 \right\}$$

$$= \frac{\beta^2 h^2}{12} + (1+\nu)\alpha T_0 \qquad \dots (3.185)$$

in which

$$h_1 = h_2 = \frac{1}{d\sqrt{\sinh^2\xi + \sin^2\eta}}$$

The boundary conditions for u_ξ and u_η are

$$u_\xi = 0 = u_\eta \quad at \quad \xi = \xi_0 \qquad \dots (3.186)$$

Let

$$u_\xi = \sum_{n=0}^{\infty} P(\xi) \cos 2n\eta \qquad \dots (3.187)$$

$$u_\eta = \sum_{n=1}^{\infty} Q(\xi) \sin 2n\eta \qquad \dots (3.188)$$

subject to the conditions

$$P(\xi_0) = g(\xi_0) = 0$$

Substituting Eqs.(3.180), (3.187) and (3.188) in Eq.(3.186) and integrating over the surface of the plate one gets

$$\int_0^{2\pi}\int_0^{\xi_0}\left\{\left(\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{\partial w}{\partial \eta}\right)^2\right\} d\xi\, d\eta$$

$$= d^2\left\{\frac{\beta^2 h^2}{6} + 2(1+\nu)\alpha T_0\right\}\int_0^{2\pi}\int_0^{\xi_0}\left(\sinh^2\xi + \sin^2\eta\right) d\xi\, d\eta$$

... (3.189)

After evaluating the integrals the following equation leading to β is obtained.

$$B_0^2\left[\left(2\left\{\bar{A}_0^{(0)}\right\}^2 + \sum_{n=1}^{\infty}\left\{\bar{A}_{2n}^{(0)}\right\}^2\right)\left\{\sum_{n=1}^{\infty} 4n^2\left\{A_{2n}'^{(0)}\right\}^2 \Psi_2 + \right.\right.$$

$$\left. + \sum_{n=1}^{\infty}\sum_{\substack{s=1\\ n\neq s}}^{\infty} 2ns(-1)^n(-1)^s A_{2n}'^{(0)} A_{2s}'^{(0)} \Psi_3\right\} +$$

$$+ \left(\sum_{n=1}^{\infty} 4n^2\left\{A_{2n}^{(0)}\right\}^2\right)\left\{\left(A_0'^{(0)}\right)^2 \xi_0 + A_0'^{(0)}\sum_{n=1}^{\infty} A_{2n}'^{(0)}(-1)^n \frac{\sinh 2n\xi_0}{n} + \right.$$

$$\left.\left. + \sum_{n=1}^{\infty} A_{2n}'^{(0)}\Psi_4 + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{\substack{s=1\\ n\neq s}}^{\infty}(-1)^n(-1)^s A_{2n}'^{(0)} A_{2s}'^{(0)}\Psi_5\right\}\right] +$$

$$+ D_0^2\Big[\Big(2\{\bar{\bar{A}}_0^{(0)}\}^2 + \sum_{n=1}^{\infty}\{\bar{\bar{A}}_{2n}^{(0)}\}^2\Big)\Big\{\sum_{n=1}^{\infty} 4n^2\{A_{2n}''^{(0)}\}^2\Psi_2 +$$

$$+ \sum_{\substack{n=1\\ n\neq s}}^{\infty}\sum_{s=1}^{\infty} 2ns(-1)^n(-1)^s A_{2n}''^{(0)} A_{2s}''^{(0)}\Psi_3\Big\} +$$

$$+ \Big(\sum_{n=1}^{\infty} 4n^2\{\bar{\bar{A}}_{2n}^{(0)}\}^2\Big)\Big\{\Big(A_0''^{(0)}\Big)^2\xi_0 + A_0''^{(0)}\sum_{n=1}^{\infty} A_{2n}''^{(0)}(-1)^n\frac{\sinh 2n\xi_0}{2n} +$$

$$+ \sum_{n=1}^{\infty} A_{2n}''^{(0)}\Psi_4 + \frac{1}{2}\sum_{\substack{n=1\\ n\neq s}}^{\infty}\sum_{s=1}^{\infty}(-1)^n(-1)^s A_{2n}''^{(0)} A_{2s}''^{(0)}\Psi_5\Big\}\Big] +$$

$$+ 2B_0D_0\Big[\Big(2\bar{A}_0^{(0)}\bar{\bar{A}}_0^{(0)} + \sum_{n=1}^{\infty}\bar{A}_{2n}^{(0)}\bar{\bar{A}}_{2n}^{(0)}\Big)\Big\{\sum_{n=1}^{\infty}4n^2 A_{2n}'^{(0)} A_{2n}''^{(0)}\Psi_2 +$$

$$+ \sum_{\substack{n=1\\ n\neq s}}^{\infty}\sum_{s=1}^{\infty}2ns(-1)^n(-1)^s A_{2n}''^{(0)} A_{2s}'^{(0)}\Psi_3\Big\} +$$

$$+ \Big(\sum_{n=1}^{\infty}4n^2\bar{A}_{2n}^{(0)}\bar{\bar{A}}_{2n}^{(0)}\Big)\Big\{A_0''^{(0)}A_0'^{(0)}\xi_0 + A_0''^{(0)}\sum_{n=1}^{\infty}(-1)^n A_{2n}''^{(0)}\frac{\sinh 2n\xi_0}{2n} +$$

$$+ A_0'^{(0)}\sum_{n=1}^{\infty}(-1)^n A_{2n}''^{(0)}\frac{\sinh 2n\xi_0}{2n} + \sum_{n=1}^{\infty}A_{2n}''^{(0)}A_{2n}'^{(0)}\Psi_4 +$$

$$+ \frac{1}{2}\sum_{\substack{n=1\\ n\neq s}}^{\infty}\sum_{s=1}^{\infty}(-1)^n(-1)^s A_{2n}''^{(0)}A_{2s}'^{(0)}\Psi_5\Big\}\Big] +$$

$$+ 2B_0\Psi\Big[\Big(2A_0^{(0)}\bar{A}_0^{(0)} + \sum_{n=1}^{\infty}A_{2n}^{(0)}\bar{A}_{2n}^{(0)}\Big)\Big\{\sum_{n=1}^{\infty}4n^2 A_{2n}'^{(0)}a_{2n}^{(0)}\Psi_2 +$$

$$+ \sum_{\substack{n=1 \\ n\neq s}}^{\infty} \sum_{s=1}^{\infty} 2ns(-1)^n(-1)^s \dot{A}_{2s}^{\prime(0)} a_{2n}^{(0)} \Psi_3 \Big\} +$$

$$+ \left(\sum_{n=1}^{\infty} 4n^2 A_{2n}^{(0)} \bar{A}_{2n}^{(0)} \right) \Big\{ a_0^{(0)} A_0^{\prime(0)} \xi_0 + a_0^{(0)} \sum_{n=1}^{\infty} (-1)^n A_{2n}^{\prime(0)} \frac{\sinh 2n\xi_0}{2n} +$$

$$+ A_0^{\prime(0)} \sum_{n=1}^{\infty} (-1)^n a_{2n}^{(0)} \frac{\sinh 2n\xi_0}{2n} + \sum_{n=1}^{\infty} a_{2n}^{(0)} A_{2n}^{\prime(0)} \Psi_4 +$$

$$+ \frac{1}{2} \sum_{\substack{n=1 \\ n\neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^n (-1)^s a_{2n}^{(0)} A_{2s}^{(0)} \Psi_5 \Big\} \Big] +$$

$$+ 2D_0\Psi \Big[\left(2A_0^{(0)} \bar{\bar{A}}_0^{(0)} + \sum_{n=1}^{\infty} A_{2n}^{(0)} \bar{\bar{A}}_{2n}^{(0)} \right) \Big\{ \sum_{n=1}^{\infty} 4n^2 A_{2n}^{\prime\prime(0)} a_{2n}^{(0)} \Psi_2 +$$

$$+ \sum_{\substack{n=1 \\ n\neq s}}^{\infty} \sum_{s=1}^{\infty} 2ns(-1)^n(-1)^s A_{2s}^{\prime\prime(0)} a_{2n}^{(0)} \Psi_3 \Big\} +$$

$$+ \left(\sum_{n=1}^{\infty} 4n^2 A_{2n}^{(0)} \bar{\bar{A}}_{2n}^{(0)} \right) \Big\{ a_0^{(0)} A_0^{\prime\prime(0)} \xi_0 + a_0^{(0)} \sum_{n=1}^{\infty} (-1)^n A_{2n}^{\prime\prime(0)} \frac{\sinh 2n\xi_0}{2n} +$$

$$+ A_0^{\prime\prime(0)} \sum_{n=1}^{\infty} (-1)^n a_{2n}^{(0)} \frac{\sinh 2n\xi_0}{2n} + \sum_{n=1}^{\infty} a_{2n}^{(0)} A_{2n}^{\prime\prime(0)} \Psi_4 +$$

$$+ \frac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s a_{2r}^{(0)} A_{2s}^{''(0)} \psi_5 \Big\} \Big] +$$

$$+ \psi^2 \Big[\Big(2 \{ A_0^{(0)} \}^2 + \sum_{r=1}^{\infty} \{ A_{2r}^{(0)} \}^2 \Big) \Big\{ \sum_{r=1}^{\infty} 4r^2 \{ a_{2r}^{(0)} \}^2 \psi_2 +$$

$$+ \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} 2rs (-1)^r (-1)^s a_{2r}^{(0)} a_{2s}^{(0)} \psi_3 +$$

$$+ \Big(\sum_{r=1}^{\infty} 4r^2 \{ A_{2r}^{(0)} \}^2 \Big) \Big\{ \big(a_0^{(0)} \big)^2 \xi_0 + a_0^{(0)} \sum_{r=1}^{\infty} a_{2r}^{(0)} (-1)^r \frac{\sinh 2r\xi_0}{r} +$$

$$+ \sum_{r=1}^{\infty} a_{2r}^{(0)} \psi_4 + \frac{1}{2} \sum_{\substack{r=1 \\ r \neq s}}^{\infty} \sum_{s=1}^{\infty} (-1)^r (-1)^s a_{2r}^{(0)} a_{2s}^{(0)} \psi_5 \Big\} \Big]$$

$$= \frac{d^2}{2} \Big\{ \frac{\beta^2 h^2}{6} + 2(1+\nu)\alpha T_0 \Big\} \sinh 2\xi_0$$

... (3.190)

in which

$a_{2r}^{(0)}$, $A_{2r}^{\prime(0)}$ and $A_{2r}^{\prime\prime(0)}$ are the Fourier coefficients in the expansions of $Ce_0(\xi,-q)$, $Ce_0(\xi,-q_1)$ and $Ce_0(\xi,-q_2)$ respectively,

and

$$\Psi_2 = \frac{\sinh 4r\xi_0}{8r} - \frac{\xi_0}{2} ,$$

$$\Psi_3 = \frac{\sinh \overline{2r+2s}\,\xi_0}{2r+2s} - \frac{\sinh \overline{2r-2s}\,\xi_0}{2r-2s} ,$$

$$\Psi_4 = \frac{\sinh 4r\xi_0}{8r} + \frac{\xi_0}{2} ,$$

$$\Psi_5 = \frac{\sinh \overline{2r+2s}\,\xi_0}{2r+2s} + \frac{\sinh \overline{2r-2s}\,\xi_0}{2r-2s}$$

Since β is determined, W is determined completely.

RESULTS

To find the deflection at a given point, one has to start from Eq.(3.190) with an assumed value of β leading to the correspondin value of $\frac{E\alpha f(h)K_1}{D(1-\nu)}$. With this value of $\frac{E\alpha f(h)K_1}{D(1-\nu)}$ and considering Eqs.(3.183) and (3.184) the deflection will be obtained from Eq.(3.180).

For numerical calculation the following values have been assumed.

$$\xi = 0, \quad \eta = \frac{\pi}{2}, \quad \xi_0 = 3, \quad d^2 = 2{\cdot}5, \quad h = 1, \quad f(h) = h,$$

$$K_F = \frac{K}{D}\,\xi_0^4 = 100, \quad \epsilon = 0{\cdot}03, \quad \nu = 0{\cdot}3, \quad \alpha T_0 = 2{\cdot}5 \times 10^{-3}$$

The interfocal distance being assumed and the values of β^2, P_1^2 and P_2^2 being known, the values of q, q_1 and q_2 are determined. q, q_1, and q_2 being known the corresponding values of the Fourier coefficients as well as those of Mathieu functions are determined. The maximum deflection W_0 is obtained at the centre of the plate. These deflections are graphically presented in Fig. 3.19 in which $\frac{W_0}{h}$ for $K_F = 0$ and $K_F = 100$ are plotted against the nondimensional load function λ. By setting $\beta \to 0$ the deflections according to the linear theory is obtained. For comparison Fig.3.19 also includes a straightline which represents small deflections for $K_F = 0$. The results obtained in this study could not be compared in absense of any known results.

From Fig. 3.19 it is observed that the error according to the linear theory increases progressively with the increase in load function. The solution proposed in this study is rapidly convergent and no computational difficulty other than computational effort is involved. The parameter q for the series $Ce(\xi, q)$ may be real or imaginary and the corresponding coefficients can be computed with accuracy. The numerical results presented in this study are obtained by taking the first two terms of the series and sufficient for practical purposes. Since the deflection at any point is known the corresponding stresses can now be easily estimated.

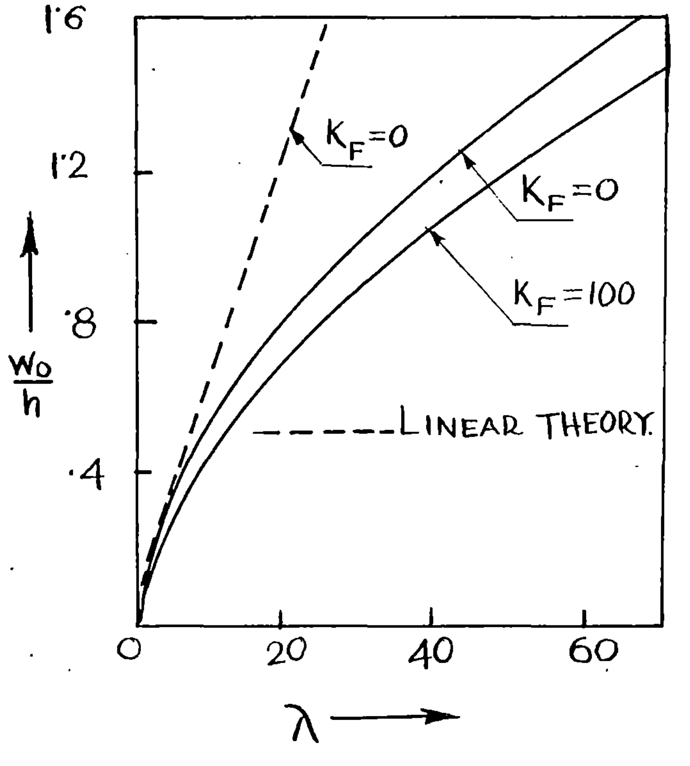

FIG. 3·19 LOAD - DEFLECTION CURVE.