Chapter 1

Introduction

The study of fractals and other complex systems is of considerable interest in contemporary time [1]. The relevance and usefulness of fractal geometric concepts and constructions in modeling various natural structures and processes are now well established. Fractals are generally considered to be sets or geometric figures that are highly irregular in nature, so that the theory of Classical Calculus and Analysis fail to yield any satisfactory result. Historically, such non-smooth sets and figures were studied in past, for instance, Cantor sets, Von Koch continuous but no-where differentiable curves, Sierpinski triangle or gasket etc. However, these objects were considered mostly as general mathematical curiosities and counter-examples only, having almost no relevance in general mathematics and its real world applications and were even regarded as *pathological and/ or mathematical monsters*. The scenario, however, is changed dramatically with the ground breaking work of Benoit Mandelbrot [2] who systematically initiated and established the theory of fractal geometry as a new field of mathematical and scientific research.

The salient features of fractals and other nonlinear complex systems involve mainly of *fine structures*, that is, details on arbitrarily small scales, *self-similarity, scale invariance* and so on. Self-similarity generally means "a smaller part (subset) resembling exactly (or approximately) similar to the whole (set)". More analytical definition is given in Chapter 2. Scale invariance, on the other hand, means that the relevant physical quantity has properties that remain unaltered under scale changes. Consequently, the functional form of a scale invariant quantity has generally a power law form: $f(t) \sim t^a$, so as to respect an equation of the form $f(\lambda t) = \lambda^a t, \ \lambda > 0, \ a > 0$. It turns out that scale invariance and self similarity are closely related concepts and enjoyed by a host of fractal like objects.

The theory of fractals is a very active area of contemporary research in the field of nonlinear sciences, and may be said to represent an example of a class of complex systems. Although there is still no generally acceptable definition, fractals are generally considered to be those subsets of the Euclidean space $\mathbb{R}^n$ which are not only irregular (rough) and nonsmooth but also enjoy some sort of self-similarity and scale invariance. Further, generation of such fractal subsets of $\mathbb{R}^n$ involve some non-linear iteration or recursion processes. Various natural objects and processes are known to reflect fractal like self-similarity and scale invariance. For example, large scale galaxy distribution, cloud boundaries, topographical surfaces of the planet earth, coastlines, turbulence in fluid and plasma states, stock market fluctuations, structures of mammalian hearts and lungs and so on [3]. The interest in the study of self-similarity and scale invariance of the global and local structures of the nature - ranging from the macroscopic cosmological scales down to the finer microscopic scales - is gaining momentum over the last few decades from extensive works of several mathematicians and physicists throughout the world [3, 4].

A Cantor set is a totally disconnected, compact and perfect subset of the real line. Cantor set is an example of a self-similar fractal set that arises in various fields of applications. The chaotic attractors of a number of one dimensional maps; such as the logistic maps, tent map, turn out to be topologically equivalent to Cantor sets [5]. Cantor set also arises in electrical communications [6], in biological systems [4], and diffusion processes [7, 8]. Recently there have been a lot of interest in developing a framework of analysis on a Cantor like fractal sets [9–12]. Because of the disconnected nature, methods of ordinary real analysis break down on a Cantor set. Various approaches based on the fractional derivatives [13–15] and the measure theoretic harmonic analysis [16], functional analysis, probability theory [17] have already been considered at length in the literature [18]. However, a simpler intuitively appealing approach is still considered to be welcome. The present thesis is a part of an ongoing project that aims at developing a general framework of scale invariant analytical framework that would be suitable to construct not only a rigorous analysis on fractal subsets of $\mathbb{R}^n$, but would also be suitable for having new insights into the dynamics of general nonlinear dynamical systems [19–21].

In [22], Parvate and Gangal formulated an elegant framework of fractal calculus on a Cantor subset C of real line involving integral mass functions and the associated integral staircase functions of the form $S_C^\alpha(x)$ that is related to the singular Lebesgue-Cantor staircase function and also proportional to the corresponding Hausdorff measure on the compact set C, α being the Hausdorff dimension of C. Moreover, the construction of an integral mass function was argued to be algorithmically more advantageous compared to the Hausdorff measure, so as to be more appropriate for physical applications.

On the hand, in [23, 24], Datta and co-workers studied an alternative formulation of a scale invariant analysis on Cantor like fractal subsets of the form $C \subset \mathbb{R}$. Instead of going directly for defining a measure theoretic approach as in [18], the approach is based on a larger conceptual framework involving relative infinitesimals and associated non-archimedean ultrametric valuations (norms) respecting some scaling and inversion properties [24] . The relative infinitesimals are argued to reside non-trivially in the *gaps* of an arbitrarily small neighbourhood of a point, for instance 0, in the said Cantor set C. More interestingly, the associated nontrivial norm $v_C(x)$ for a relative infinitesimal x is shown to have the properties of the corresponding Lebesgue-Cantor staircase function, and hence proportional to the integral staircase function $S_C^{\alpha}(x)$, that is subsequently utilized systematically to define limit, continuity and differentiability of a function that is defined on a subset of $\mathbb{R}$, but having nontrivial support (that is, experiencing nontrivial changes only) on the Cantor set C. Consequently, two independent approaches, namely [25] and [26] respectively, though offer almost identical final outcomes towards having a consistent calculus on a Cantor like fractal set, the later approach appear to have greater potentiality in having more general extensions and applications, not only in fractal sets but also in other nonlinear complex systems.

The present thesis is a step forward towards one of such more possible general extensions of the scale invariant framework presented in [24]. In this thesis, we present some new results in this direction based on three papers of which two are published [25, 26] and one communicated paper [27].

1.1 Main Results of Thesis

The thesis is divided into two parts. In Part 1, the formalism of asymptotic duality is presented in details. In Part 2, we present two applications of the developed ideas (i) to formulate a self-consistent calculus on a Cantor like fractal set and (ii) to study a new type of deformation in the context of a nonlinear KdV type solitary wave equation.

In Chapter 3, the basic constructions towards realizing an extended scale invariant analytic framework are presented. Instead of working directly on a Cantor set like fractal set, we now intend to look at the ordinary real number system (continuum) $\mathbb{R}$ from an extended non-standard (and non-Archimedean) like extended continuum $\mathbb{R}^*$ accommodating relative infinitesimals and infinitely large new elements relative to a preassigned scale, that are supposed to be accessed, in actual applications, in an asymptotic limiting sense, involving a nontrivial, finitely valued, ultrametric norm, respecting some scaling and duality (inversion) transformations. As the ultrametric valuation turns out to be discretely valued, the ordinary linear neighbourhood of 0 in $\mathbb{R}$ gets extended into a Cantor set like structure in the extended set $\mathbb{R}^*$. Consequently, an arbitrarily small asymptotic real variable $x \to 0$ would get a relatively finite non-negative real value $x \mapsto v(x)$. Moreover, the nontrivial scale invariant function has the structure of a Lebesgue-Cantor staircase function that has nontrivial variations only on that preassigned Cantor set like extended neighbourhood. The entire constructions leading to the above scenario is called *asymptotic duality* transformations.

In Chapter 4, we present a measure theoretic realization of the asymptotic valuation $v(x)$ in an extended Cantor set like neighbourhood $\mathbf{O}^*$ of $0 \in \mathbb{R}$. It is shown that $v(x)$ can be interpreted as a regular measure that is absolutely continuous with respect to the associated Hausdorff measure in the sense of Radon-Nikodym theorem.

In Chapter 5, we continue our study of asymptotic duality and associated discretely valued valuation $v(x)$ in a deformed connected neighbourhood of the form $\mathcal{O} = v(\mathbf{O}^*) \subset \mathcal{R}$, deformed real continuum. The duality transformations and corresponding valuations are classified as self dual, weakly self dual and strictly dual asymptotics. Next, a geometric characterization of the deformed extended set $\mathcal{O}$ is carried out for self dual and strictly dual valuations. For self dual valuations, $\mathcal{O}$ can be realized either as a smooth or a piece-wise smooth broken curve (line), when a strictly dual valuation corresponds to a fractal like extension. Such distinct geometric features are shown to relate to cases when the original extended set $\mathbf{O}^*$ could be covered either by a finite or countable set of clopen balls, on each of which constant values of the valuation v are assigned, respecting continuity induced by the ultrametric norm.

In Chapter 6, we study an extension of asymptotic duality concepts in a function space. Introducing function space dependent valuation $v_F(f)$ for a given function $f(x),\ x \in \mathbb{R}$, we next study *asymptotic continuity* and *asymptotic differentiability* in the associated extended space. Some simple examples are worked out to show how classical discontinuity and nondifferentiability are realized as asymptotically continuous and differtentiable in a point-wise manner.

In Chapter 7, we make applications of above formulated ideas and results to the middle third Cantor set and develop a formal calculus on such sets.

Finally, in Chapter 8, an interesting application of this asymptotic duality formulations is studied in the context of the KdV type non-linear evolutionary equation [27]. It is shown explicitly how an intrinsically realized *seed deformation* in a neighbouhood of an initial solitary wave profile, aided by asymptotic duality principle, could subsequently induce a global deformation on the original solitary wave, so as to realize exotic wave forms such as rouge wave, breather wave, periodic singular wave and other more complex wave patterns [27], even in absence of of any external excitation. Consequently, the realizations of such exotic wave patterns can be interpreted as manifestations of another non-physical level of excitation that generally remain nascent at ordinary scales, but could become activated and realizable here as asymptotic duality principle.

To summarize, the present thesis represents a body of analytic results which are of interdisciplinary in nature involving various topics such as Cantor like fractal sets, non-standard analysis, non-Archimedean valuations, real analysis, measure theory and their interesting applications on a Cantor Set and a KdV like evolutionary systems.