Chapter 1

INTRODUCTION

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The theory of the distribution of values of entire and meromorphic functions was first developed by *R.* Nevanlinna in the year 1926. It was one of the most outstanding achievements in function theory in this century. Nevanlinna's fundamental theory, the famous First Fundamental theorem, the Second Fundamental theorem, as well as the related formulae of deficiencies are the foundation of the whole thesis. Before starting the discussion on Nevanlinna theory we state the following definitions.

Definition 1.1 A single valued function of one complex variable which has no singularities other than poles in the open complex plane (i.e., excluding the point at infinity) is called a meromorphic function.

Definition 1.2 A meromorphic function which has an essential singularity at the point at infinity is known as a transcendental meromorphic function.

Definition 1.3 A single valued function which is analytic in the open complex plane is defined as an entire function.

Definition 1.4 An entire function which has an essential singularity at the point at infinity is known as a transcendental entire function.

Let f be a meromorphic function in the finite complex plane $\mathbb{C}$. Also let $n(r, a\ ; f) \equiv n(r, a)$ which is a non-negative integer for each r, denote the number of a-points of f in $|z| \leq r$, counted with proper multiplicities, for a complex number a, finite or infinite. Obviously $n(r, \infty) \equiv n(r, f)$ represents the number of poles of f in $|z| \leq r$ counted with proper multiplicities. The function $N(r, a\ ; f) \equiv N(r, a)$ is defined as follows:

$$N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} dt + n(0,a)\log r \text{ and } N(r, \infty\ ; f) \equiv N(r, f).$$

Next let us define

$$\begin{aligned} \log^+ x &= \log x \text{ if } x \geq 1 \\ &= 0 \text{ if } 0 \leq x < 1. \end{aligned}$$

The following properties are then obvious

$$\begin{aligned} &(i) \log^+ x \geq 0 \text{ if } x \geq 0, \\ &(ii) \log^+ x \geq \log x \text{ if } x > 0, \\ &(iii) \log^+ x \geq \log^+ y \text{ if } x > y, \\ &(iv) \log x = \log^+ x - \log^+ \frac{1}{x} \text{ if } x > 0. \end{aligned}$$

The quantity $m(r, f)$ is defined as follows:

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta.$$

The term $m(r, f)$ is called the proximity function of f and is a sort of averaged magnitude of $\log |f(z)|$ on the arcs of $|z| = r$ where $|f(z)|$ is large. Now let us write

$$T(r, f) = m(r, f) + N(r, f).$$

The function $T(r, f)$ is called the Nevanlinna's characteristic function of f {p.4, [34]}. It plays an important role in the theory of meromorphic functions. Since for any positive integer p and complex number a_ν,

$$\log^+ |\Pi_{\nu=1}^p a_\nu| \leq \sum_{\nu=1}^p \log^+ |a_\nu|$$

$$\text{and } \log^+ \left| \sum_{\nu=1}^p a_\nu \right| \leq \log^+ (p \max_{\nu=1,2,\ldots p} |a_\nu|) \leq \sum_{\nu=1}^p \log^+ |a_\nu| + \log p,$$

it is easy to show that {p.5, [34] } for p meromorphic functions $f_1, f_2, \ldots f_p$

$$m(r, \Pi_{\nu=1}^p f_\nu) \leq \sum_{\nu=1}^p m(r, f_\nu)$$

$$\text{and } m(r, \sum_{\nu=1}^p f_\nu) \leq \sum_{\nu=1}^p m(r, f_\nu) + \log p.$$

Also one can easily verify that

$$N(r, \Pi_{\nu=1}^{p} f_\nu) \leq \sum_{\nu=1}^{p} N(r, f_\nu)$$

$$\text{and } N(r, \sum_{\nu=1}^{p} f_\nu) \leq \sum_{\nu=1}^{p} N(r, f_\nu).$$

$$\text{So } T(r, \sum_{\nu=1}^{p} f_\nu) \leq \sum_{\nu=1}^{p} (T(r, f_\nu) + \log p$$

$$\text{and } T(r, \Pi_{\nu=1}^{p} f_\nu) \leq \sum_{\nu=1}^{p} T(r, f_\nu).$$

Now we state the Poisson-Jensen formula {p.1, [34] } in the form of the following theorem:

Theorem 1.1 Suppose that f is meromorphic in $|z| \leq R (0 < R < \infty)$. Also let $a_\mu (\mu = 1, 2, ...M)$ and $b_\nu (\nu = 1, 2, ...N)$ denote the zeros and poles of f respectively in $|z| < R$. Then if $z = re^{i\theta} (0 < r < R)$ and if $f(re^{i\theta}) \neq 0, \infty$ we have

$$\log |f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(\text{R}e^{i\phi}) \right| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$

$$+ \sum_{\mu=1}^{M} \log \left| \frac{R(z - a_\mu)}{R^2 - \bar{a}_\mu z} \right| - \sum_{\nu=1}^{N} \log \left| \frac{R(z - b_\nu)}{R^2 \bar{b}_\nu z} \right|.$$

The theorem holds good also when f has zeros and poles on $|z| = R$. When $z = 0$, we obtain Jensen's formula

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(\text{R}e^{i\phi}) \right| d\phi + \sum_{\mu=1}^{M} \log \frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log \frac{|b_\nu|}{R},$$

provided that $f(0) \neq 0, \infty$.

If f has a zero of order λ or a pole of order $-\lambda$ at $z = 0$ such that $f = C_\lambda Z^\lambda + ...$then Jensen's formula takes the form

$$\log |C_\lambda| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(\text{R}e^{i\phi}) \right| d\phi + \sum_{\mu=1}^{M} \log \frac{|a_\mu|}{R} - \sum_{\nu=1}^{N} \log \frac{|b_\nu|}{R} - \lambda \log R.$$

The complicated modification is one of the minor irritations of the theory. Generally we shall assume that our function behave in such a way that the

terms in the Jensen's formula do not become infinite in our use of that formula knowing that exceptional cases can be treated.
When f has no a-points (i.e., the roots of the equation $f = a$) at $z = 0$, then it follows from Riemann-Stieltjes integral that

$$\sum_{0<|a_\nu|<r} \log \frac{r}{|a_\nu|} = \int_0^r \frac{n(t,a)}{t} dt, \text{ where } a_\nu\text{'s are the } a\text{-points of } f \text{ in } |z| \leq r.$$

Again since $N(r, 0\ ; f) = N(r, \frac{1}{f})$, from Jensen's formula we get that

$$\log|f(0)| = m(R, f) - m(R, \frac{1}{f}) + N(R, f) - N(R, \frac{1}{f})$$

$$\text{i.e., } T(R, f) = T(R, \frac{1}{f}) + \log|f(0)|.$$

For any finite complex number a let us denote by $m(r, a)$ the function $m(r, \frac{1}{f-a})$ and $m(r, \infty) = m(r, f)$.
Now we express Nevanlinna's First Fundamental theorem in the following form:
Theorem 1.2 {p.6, [34] }. Let f be a meromorphic function in $|z| < \infty$ and a be any complex number, finite or infinite, then

$$m(r, a) + N(r, a) = T(r, f) + O(1).$$

This result shows the remarkable symmetry exhibited by a meromorphic function in its behavior relative to different complex number a, finite or infinite. The sum $m(r, a) + N(r, a)$ for different values of a maintains a total, given by the quantity $T(r, f)$ which is invariant upto a bounded additive term involving r.
One part of this invariant sum, the quantity $N(r, a\ ; f)$ hints how densely the roots of the equation $f = a$ are distributed in the average in the disc $|z| < r$. The large the number of a-points the faster this counting function for a-points grows with r.
The first term $m(r, a)$ which is defined to be the mean value of

$$\log^+|1/f - a| \ \ (\text{or } \log^+|f| \text{ if } a = \infty)$$

on the circle $|z| = r$, receives a remarkable contribution only from those arcs on the circle where the functional values differ very little from the given value a. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle $|z| = r$ of the functional value f from the value a.

If the a-points of a meromorphic function are relatively scarce for a certain 'a', this fact finds expression analytically in the relatively slow growth of the function $N(r,a)$ as $r \to \infty$; in the extreme case where a is a Picard's exceptional value of the function (so that $f \neq a$ in $|z| < \infty$), $N(r,a)$ is identically zero. But this fact on a-points finds a compensation.

The function deviates in the mean slightly from the value a in question, the corresponding proximity function $m(r,a)$ will be relatively large, so that the sum $m(r,a) + N(r,a)$ reaches the magnitude $T(r,f)$, characteristic function of f.

For an entire function f, $N(r,f) = 0$ and so $T(r,f) = m(r,f)$, i.e., in the case of an entire function, the Nevanlinna's characteristic function and the proximity function are same.

Let us consider that f be an entire function, i.e., a function of a complex variable regular in the whole finite complex plane $\mathbb{C}$. By Taylor's theorem such a function has an everywhere convergent power series expansion as

$$f = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n + \ldots \tag{1.1}$$

which forms a natural generalization of the polynomials.

The degree of a polynomial which is equal to its number of zeros estimates the rate of growth of the polynomial as the independent variable moves without bound. So the more zeros, the greater is the growth.

An analogous property that relate the set of zeros and the growth of a function can be developed for arbitrary entire functions.

Establishing relations between the distribution of the zeros of an entire function and its asymptotic behavior as z tending to infinity enriched most of the classical results of the theory of entire functions. The classical investigations of Borel, Hadamard and Lindelof are of this kind.

To characterise the growth of an entire function and the distribution of its zeros a special growth scale called maximum modulus function of f on $|z| = r$ is introduced as $M(r) = M(r,f) = \max_{|z|=r} |f(z)|$.

It plays an important role in the theory of entire functions. Since by Li-

ouville's theorem a bounded entire function is constant, it follows that for non-constant f the maximum modulus function $M(r)$ is unbounded.
The following theorem is due to Cauchy.
Theorem 1.3 {Theorem 1, p. 5, [66] }. The maximum of the modulus of a function f, which is regular in a closed connected region D, bounded by one or more curves C, is attained on the boundary.
This theorem implies that when f is an entire function, $M(r)$ is a non-decreasing function of r for all values of r.Using the uniform continuity of f in any closed region and the above theorem, i.e., the value $M(r)$ is attained by f on $|z| = r$, it follows that $M(r)$ is a continuous function of r. Also $M(r)$ is differentiable in adjacent intervals {Theorem 10, p. 27, [66] }. In view of Hadamard's theorem {Theorem 9, p. 20, [66]} we know that $\log M(r)$ is a continuous, convex and ultimately increasing function of $\log r$.
For an entire function f the study of the comparative growth properties of $T(r, f)$ and $\log M(r, f)$ is a popular problem among the researchers. Now we express a fundamental inequality relating $T(r, f)$ and $\log M(r, f)$.
Theorem 1.4 {p. 18, [34] }. If f is regular for $|z| \leq R$ then

$$T(r, f) \leq \log^{+} M(r, f) \leq \frac{R+r}{R-r} T(R, f), 0 \leq r < R.$$

In case of a transcendental entire function f, $M(r)$ grows faster than any positive power of r.Thus in order to estimate the growth of transcendental entire functions we choose a comparison function $e^{r^k}, k > 0$ that grows more rapidly than any positive power of r.
More precisely f is said to be a function of finite order if there exists a positive constant k such that $\log M(r) < r^k$ for all sufficiently large values of $r(r > r_0(k)$; say).The infimum of such k's is called the order of f. If no such k exists, f is said to be of infinite order.
For example the order of the function e^z is 1 i.e., finite but that of e^{e^z} is infinite.
Let ρ be the order of f. It can be easily shown that the order ρ of f has the following alternative definition

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}.$$

The lower order λ of f is defined as follows

$$\lambda = \liminf_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

Clearly $\lambda \leq \rho$.
If in particular for a function f, $\lambda = \rho$, then f is said to be of regular growth. For example a polynomial or the functions $e^z, \cos z$ etc. are of regular growth. With known order $\rho (0 < \rho < \infty)$ the growth of an entire function can be characterised more precisely by the type of the function. The number τ given by

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho}, 0 < \rho < \infty$$

is called the type of f.
Between two functions of same order one can be characterised to be of greater growth if its type is greater. The quantities ρ, λ and τ are extensively used to the study of growth properties of f. At this stage we note the following definition.
Definition 1.5 {p.16, [34]}. Let S be a real and non-negative function increasing for $0 < r_0 \leq r < \infty$. The order k and the lower order λ of the function $S(r)$ are defined as

$$k = \limsup_{r\to\infty} \frac{\log S(r)}{\log r}$$
$$\text{and } \lambda = \liminf_{r\to\infty} \frac{\log S(r)}{\log r}.$$

Moreover if

$$0 < k < \infty, \text{ we set } C = \limsup_{r\to\infty} \frac{S(r)}{r^k}$$

and distinguish the following possibilities:

(a) $S(r)$ has maximal type if $C = +\infty$;

(b) $S(r)$ has mean type if $0 < C < +\infty$;

(c) $S(r)$ has minimal type if $C = 0$;

(d) $S(r)$ has convergence class if $\int_{r_0}^{\infty} \frac{S(t)}{t^{k+1}} dt$ converges

Now we state the following theorem.

Theorem 1.5 {p.18, [34]}. If f is an entire function then the order k of the function $S_1(r) = \log^+ M(r, f)$ and $S_2(r) = T(r, f)$ is the same. Further if $0 < k < \infty$, $S_1(r)$ and $S_2(r)$ belong to the same classes (a), (b), (c) and (d). Also we note that $S_1(r)$ and $S_2(r)$ have the same lower order.

A function f meromorphic in the plane is said to have order ρ, lower order λ and maximal, minimal, mean type or convergence class if the function $T(r, f)$ has this property. For entire functions these coincide by the above theorem with the corresponding definition in terms of $M(r, f)$ which is classical. The type of a meromorphic function f is defined by

$$\tau = \limsup_{r \to \infty} \frac{T(r, f)}{r^\rho}, 0 < \rho < \infty.$$

We know that the order and the lower order of an entire function f and its derivative are equal. The same result holds for a meromorphic function also. After revealing the important symmetry property of a meromorphic function f, which is expressed in the first fundamental theorem through the invariance of the sum $m(r, a) + N(r, a)$, it is natural to attempt for a more careful investigation of the relative strength of two terms in the sum, of the proximity component $m(r, a)$ and of the counting component $N(r, a)$. Individual results have been obtained in this direction {p.234, [34]}.

1, Picard's theorem shows that the counting function for a non-constant meromorphic function in the finite complex plane vanish for almost two values of a.

2. For a meromorphic function of finite non-integral order there is almost one Picard's exceptional value.

3. That the counting function $N(r, a)$ is in general i.e., for the great majority of the values of a, large in comparison with the proximity function.

We now state Nevanlinna's Second Fundamental theorem.

Theorem 1.6 {p.31, [34]}. Suppose that f is a non-constant meromorphic function in $|z| \leq r$. Let $a_1, a_2, ...a_q (q \geq 2)$ be distinct finite complex numbers, $\delta > 0$ and suppose that $|a_\mu - a_\gamma| \geq \delta$ for $1 \leq \mu < \gamma \leq q$. Then

$$m(r, \infty) + \sum_{\gamma=1}^{q} m(r, a_\gamma) \leq 2T(r, f) - N_1(r) + S(r),$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N(r, \frac{1}{f'}) + 2N(r, f) - N(r, f')$$

$$\text{and } S(r) = m(r, \frac{f'}{f}) + m\{r, \sum_{\gamma=1}^{q} \frac{f'}{(f - a_\gamma)}\} + q \log^+ \frac{3q}{\delta} + \log 2 + \log \frac{1}{|f'(0)|},$$

with modifications if $f(0) = 0$ or ∞ and $f'(0) = 0$.
The quantity $S(r)$ will in general play the role of an unimportant error term. The combination of this fact with the above theorem yields the second fundamental theorem.
The following theorem gives an estimation of $S(r)$.
Theorem 1.7{p.34, [34]}. Let f be a meromorphic function and not constant in $|z| < R_0 \leq \infty$ and that $S(r) \equiv S(r, f)$ is defined as in the above theorem. Then we have
(i) If $R_0 = +\infty$, $S(r, f) = O\{\log T(r, f)\} + O(\log r)$ as $r \to \infty$ through all values if f has finite order and as $r \to \infty$ outside a set E of finite linear measure otherwise
(ii) If $0 < R_0 < \infty, S(r, f) = O\{\log^+ T(r, f) + \log 1/(R_0 - r)\}$ as $r \to R_0$ outside a set E such that $\int_E \frac{dr}{R_0 - r} < +\infty$.
Further there is a point r outside E for which $\rho < r < \rho'$ provided that $0 < R - \rho' < e^{-2}(R - \rho)$.
Consequently we get the following theorem.
Theorem 1.8{p.41, [34]}. Let f be meromorphic and non-constant in $|z| \leq R_0$. Then

$$\frac{S(r, f)}{T(r, f)} \to 0 \text{ as } r \to R_0 \text{ with the following provisions:} \qquad (*)$$

$\frac{S(r,f)}{T(r,f)} \to 0$ as $r \to R_0$ with the following provisions:
(a)(*) holds without restrictions if $R_0 = +\infty$ and f is of finite order in the plane.
(b)(*) If f has infinite order in the plane, (*) still holds as $r \to \infty$ outside a certain exceptional set E_0 of finite length.
Here E_0 depends only on f.
(c) If $R_0 < +\infty$ and $\limsup_{r \to \infty} \frac{T(r,f)}{\log\{\frac{1}{(R_0 - r)}\}} = +\infty$,
then (*) holds as $r \to R_0$ through a suitable sequence r_n, which depends on f only. This theorem points out why $S(r)$ plays the role of an unimportant

error term.
Let f be meromorphic and not constant in the plane. We shall denote by $S(r,f)$ any quantity $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set r of finite linear measure. Also we shall denote by a, a_0, a_1 etc. functions meromorphic in the plane and satisfying $T(r,a)=S(r,f)$ as $r\to\infty$. Now we introduce Milloux's theorem which is important in studying the properties of the derivatives of meromorphic functions.

Theorem 1.9{p.55, [34]}. Let l be a positive integer and $\psi=\sum\limits_{\gamma=0}^{l}a_\gamma f^{(\gamma)}$. Then $m(r,\frac{\psi}{f})=S(r,f)$ and $T(r,\psi)\le(l+1)T(r,f)+S(r,f)$.
Milloux showed that in the second fundamental theorem we can replace the counting functions for certain roots of $f=a$ by roots of the equation $\psi=b$, where ψ is given as in the above theorem. In this connection we state the following theorem.
Theorem 1.10{p.57, [34]}. Let f be meromorphic and non-constant in the plane and $\psi=\sum\limits_{\gamma=0}^{l}a_\gamma f^{(\gamma)}$, where l is a positive integer. If ψ is non-constant then

$$T(r,f)<\overline{N}(r,f)+N(r,\frac{1}{f})+\overline{N}(r,\frac{1}{\psi-1})-N_0(r,\frac{1}{\psi'})+S(r,f)$$

where in $N_0(r,\frac{1}{\psi'})$ only zeros of ψ' not corresponding to the repeated roots of $\psi=1$ are to be considered.
Now we set

$$\delta(a)=\delta(a;f)=\liminf_{r\to\infty}\frac{m(r,a)}{T(r,f)}=1-\limsup_{r\to\infty}\frac{N(r,a)}{T(r,f)},$$
$$\Theta(a)=\Theta(a;f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,a)}{T(r,f)},$$

where $\overline{N}(r,a\,;f)\equiv\overline{N}(r,a)$ is the counting function for distinct a-points,

$$\theta(a)=\theta(a;f)=\liminf_{r\to\infty}\frac{N(r,a)-\overline{N}(r,a)}{T(r,f)}.$$

Evidently, given $\varepsilon(>0)$, we have for sufficiently large values of r,

$$N(r,a)-\overline{N}(r,a)>\{\theta(a)-\varepsilon\}T(r,f),$$
$$N(r,a)<\{1-\delta(a)+\varepsilon\}T(r,f)$$

and hence

$$\overline{N}(r,a) < \{1 - \delta(a) - \theta(a) + 2\varepsilon\}T(r,f)$$

so that

$$\Theta(a) \geq \delta(a) + \theta(a).$$

The quantity $\delta(a)$ is known as the deficiency of the value 'a' and $\theta(a)$ is called the index of multiplicity. Evidently $\delta(a)$ is positive only if there are relatively few roots of the equation $f = a$, while $\theta(a)$ is positive if there are relatively many multiple roots.

Let us now state a fundamental theorem called Nevanlinna's theorem on deficient values.

Theorem 1.11 {p.43, [34]}. Let f be a non-constant meromorphic function defined on the plane. Then the set of values a for which $\Theta(a) > 0$ is countable and we have, on summing over all such values a

$$\sum_a \{\delta(a) + \theta(a)\} \leq \sum_a \Theta(a) \leq 2.$$

The magnitude of the deficiency $\delta(a)$ lies in the closed unit interval $[0,1]$ and it gives us a very accurate measure for the relative density of the points where the function f assumes the value 'a' in question. The larger the deficiency is, the more rare are latter points. The deficiency reaches its maximum value 1 when the latter have been very sparsely distributed, as for example, in the extreme case where the value 'a' is a Picard exceptional value i.e., a complex number which is not assumed by the function. We shall call every value of vanishing deficiency $\delta(a)$, a normal value in contrast to the deficient values for which $\delta(a)$ is positive.

We know from Picard's theorem that a meromophic function can have at most two Picard exceptional values. This theorem follows easily from Nevanlinna's theorem on deficient value because as we have stated before that for a Picard exceptional value a, $\delta(a) = 1$.

The quantity

$$\Delta(a\,;f) = 1 - \liminf_{r \to \infty} \frac{N(r,a)}{T(r,f)} = \limsup_{r \to \infty} \frac{m(r,a)}{T(r,f)}$$

gives another measure of deficiency and is called the Valiron deficiency. Clearly

$$0 \leq \delta(a\,;f) \leq \Delta(a\,;f) \leq 1.$$

An entire function $f(z)$ has a Taylor's expansion about any point a in the complex plane of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Since this series is absolutely convergent everywhere in the plane, the terms $|a_n|$ must approach 0. Consequently, there exists for each a an index $n_0 = n(a)$ for which $|a_n|$ is a maximal coefficient. B. Lepson [41]raised the problem of characterising entire functions for which $n(a)$ is bounded. The latter are called functions of bounded index.

Definition 1.6. An entire function f is said to be of bounded index if and only if there exists an integer N, such that for all z

$$\max\left(|f|,\ \left|f^{(1)}\right|),\ \frac{\left|f^{(2)}\right|}{2!}, \ldots, \frac{\left|f^{(N)}\right|}{N!}\right) \geq \frac{\left|f^{(j)}\right|}{j!},$$

$$\text{where } j = 0,1,2,3 \ldots \text{and } f^{(0)} \text{denotes } f. \tag{1.2}$$

We shall say that f is of index N, if N is the smallest integer for which (1.2) holds.

An entire function which is not of bounded index is said to be of unbounded index.

A function of bounded index satisfies

$$\sum_{i=0}^{N} \frac{\left|f^{(i)}\right|}{i!} \geq \frac{\left|f^{(j)}\right|}{j!}, j = 0,1,2,3, \ldots\ldots$$

Furthermore, if (1.3) holds then

$$\max\left(|f|,\ \left|f^{(1)}\right|,\ \frac{\left|f^{(2)}\right|}{2!}, \ldots\ldots, \frac{f^{(N)}}{N!}\right) \geq \frac{1}{(N+1)} \cdot \frac{\left|f^{(j)}\right|}{j!}; j = 0,1,2,3, \ldots\ldots$$

These facts suggests

Definition 1.7. An entire function $f(z)$ is said to be of non-uniform bounded index if and only if there exist integers N_j, such that

$$\sum_{i=0}^{N} \frac{\left|f^{(i)}(z)\right|}{i!} \geq c. \frac{\left|f^{(j)}(z)\right|}{j!} \text{ for } |z| > N_j,$$

where $j = 0,1,2,3, \ldots\ldots$ and c is any fixed constant.

For bounded regions Lepson [41]proved that

Theorem 1.12. If $f(z) \neq 0$ is an entire function and R is any bounded set, then there is an integer N such that for any z in R and any non-negative integer n

$$\frac{\left|f^{(n)}(z)\right|}{n!} \leq \max\left(\left|f(z)\right|, \left|f^{(1)}(z)\right|, \frac{\left|f^{(2)}(z)\right|}{2!}, \ldots\ldots, \frac{\left|f^{(n)}(z)\right|}{N!}\right).$$

Example 1.1. The function $\exp z$ is obviously of bounded index.

Example 1.2. Let k be a positive integer. The entire function $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(kn)!}$ is of non-uniform bounded index which can be proved by the following theorem of F. Gross [31].

Theorem 1.13. Let t be a positive integer. If $g(z) = f(z^{\frac{1}{t}})$ is entire and $f(z)$ is of bounded index (say of index k) then $g(z)$ is of non-uniform bounded index. We now discuss two special type of orders of entire functions.

During the past decades, several authors {cf. [54], [57], [58] and [61]} made close investigations on the properties of entire Dirichlet series related to Ritt order. Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined by everywhere absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1.4}$$

where $0 < \lambda_n < \lambda_{n+1} (n \geq 1)$, $\lambda_n \to \infty$ as $n \to \infty$ and a_n's are complex constants.

If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (1.4) then in this clearly $\sigma_c = \sigma_a = \infty$.

$$\text{Let } F(\sigma) = \underset{-\infty<t<\infty}{lub} \left|f(\sigma + it)\right|. \tag{1.5}$$

Then the Ritt order [59] of $f(s)$ denoted by $\rho(f)$ is given by

$$\rho(f) = \limsup_{\sigma\to\infty} \frac{\log\log F(\sigma)}{\sigma} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} F(\sigma)}{\sigma}.$$

In other words

$$\rho(f) = \inf\{\mu > 0 : \log F(\sigma) < \exp(\sigma\mu) \text{ for all } \sigma > R(\mu)\}.$$

Similarly the lower Ritt order of $f(s)$ denoted by $\lambda(f)$ may be defined.

Let us denote by C^n and R^n the complex and real $n-$space respectively. We

indicate the point $(z_1, \dots z_n), (m_1, \dots m_n)$ of C^n or I^n by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. The modulus of z, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + \dots + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then z^m will denote $z_1^{m_1} \dots z_n^{m_n}$ and $||m|| = m_1 + \dots + m_n$.
Let $D \subset C^n$ be an arbitrary bounded complete n-circular domain with center at the origin of coordinates. Then for the analytic function f and $R > 0$, $M_{f,D}(R) = \sup_{z \epsilon D_R} |f(z)|$, where a point $z \epsilon D_R$ if and only if $\frac{z}{R} \epsilon D$.
Definition 1.8 {[30], p. 339} The Gol'dberg order (briefly G-order) ρ_D^f of f with respect to the domain D is defined as

$$\rho_D(f) \equiv \rho_D^f = \limsup_{R \to \infty} \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \dots$and $\log^{[0]} x = x$.

The lower Gol'dberg order λ_D^f of f with respect to the domain D is defined as

$$\lambda_D(f) \equiv \lambda_D^f = \liminf_{R \to \infty} \frac{\log^{[2]} M_{f,D}(R)}{\log R}$$

We say that f is of regular growth if $\rho_D^f = \lambda_D^f$.
Further theories of entire and meromorphic functions can be found in [4] and [13] and those of analytic functions are available in [15] and [30]. Extensive works on growth properties of entire and meromorphic functions related to relative order, relative lower order, relative sharing of values, relative Ritt order, relative Gol'dberg order etc. have been done in [6], [7], [8], [9], [10], [11], [45], [49], [51], [52], [53], [54], [55], [56] and [60]. Also growth properties based on differential monomials, differential polynomials and wronskians, all generated by entire or meromorphic functions have been studied in[5], [47] and [50]. One can find the theories of Dirichlet series and their related problems in [33] and [61]. Further growth properties related to convex or entire functions as well as the potential theory in several complex variables can be found in [39] and [40]. For results related to deficiencies of meromorphic functions one can see [44] and [69].
Apart from **Chapter 1** the thesis consists of ten other chapters.

- **In Chapter 2** we study the growth properties of composite entire and meromorphic functions using L-order and L^*-order improving some earlier results where $L \equiv L(r)$ is a slowly changing function. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [19].

- **In Chapter 3** we consider the growth properties of composite entire and meromorphic functions using (p, q)th order which improve some earlier results, where p, q are positive integers and $p > q$. Some results in the form of remarks based upon minimum modulus of entire functions have also been stated in this chapter. The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences**, see [20].

- **In Chapter 4** we discuss the growth properties of entire functions on the basis of relative L-order where $L \equiv L(r)$ is a slowly changing function. Also some growth properties in terms of minimum modulus of entire functions have been discussed in this chapter. The results of this chapter have been published in **International Mathematical Forum**, see [21].

- **In Chapter 5** we study the comparative growth properties of entire functions on the basis of relative $L - (p, q)$th order where p, q are positive integers and $p > q$. A few results in the form of remarks on the basis of minimum modulus of entire functions have also been stated in this chapter. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [22].

- **In Chapter 6** we study the ${}_{(t)}L^* - (p, q)$th order of entire and meromorphic functions based on sharing of values of them. The results of this chapter have been published in **International Mathematical Forum**, see [23].

- **In Chapter 7** we generalise the results of Datta and Mondal [18]. The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences,** see [24].

- **In Chapter 8** we study regular L relative growth and regular L^*relative growth of an entire function with respect to another entire function and show that under certain conditions they are equal where $L \equiv L(r)$ is a slowly changing function. The results of this chapter have been published in **International Mathematical Forum**, see [25].

- **In Chapter 9** we establish some results on the comparative growth properties related to generalised L^*-Ritt order of entire Dirichlet series. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [26].

- **In Chapter 10** we study the comparative growth properties of composite entire functions of two complex variables. We also introduce the notion of zero order of entire functions of two complex variables and discuss some related results. The results of this chapter have been published in **International Journal of Contemporary Mathematical Sciences**, see [27].

- **In Chapter 11** we discuss about the comparative growth properties related to the Gol'dberg order of composition of two entire function of several complex variables. The results of this chapter have been published in **International Journal of Mathematical Analysis**, see [28].

From Chapter 2 onwards when we write Theorem a.b.c (Corollary a.b.c. etc.) where a,b and c are positive integers, we mean the c-th Theorem (c-th Corollary etc.) of the b-th Section in the a-th chapter. Also by equation number (a.b) we mean the b-th equation in the a-th chapter for positive integers a and b. The references to books and journals have been classified as bibliography and are given at the end of the thesis.

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