ON SOME PROBLEMS OF THERMO-ELASTICITY

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CONTENTS

CHAPTER - III

GENERATION OF THERMOELASTIC WAVES

CHAPTER - IV

STRESSES DUE TO HEAT EXPOSURE ON THE BOUNDING SURFACE OF ELASTIC SEMI-SPACE

INTRODUCTION

The theory of thermo-elasticity is concerned with the influence of the thermal state upon the distribution of stress and strain and with the inverse effect ,that of deformation upon the thermal state of an elastic medium. Duhamel [46] in 1838, initiated the subject deriving equation for the distribution of strain in an elastic medium containing temperature gradients. Subsequently, these results were rediscovered by several authors but Neumann gave the present form known as Duhamel-Neumann relations. The basic theory was applied by Duhamel to a number of problems and later used by Neumann and other authors as the basis of the detailed study. These investigations were instrumental in developing techniques for the solution of thermo-elastic problems but not until the present century did the subject received the practical stimulus. There have been a rapid development of thermo-elasticity stimulated by various engineering sciences in the post war years. A considerable progress in the field of air-craft and machine structures,mainly with gas and steam turbines,highway engineering especially in the preparation of air base,and the emergence of new topics in chemical and nuclear engineering have given rise to numerous problems in which thermal stress play an important role and frequently even a primary role.

For most practical problems,the effect of the stresses and deformations upon the temperature distribution is quite small and can be neglected. The procedure allows the determination of the temperature distribution in the solid resulting from prescribed thermal condition to become first,an independent step of a thermal stress analysis;the second step is then the determination of the stresses and deformation of the body due to this temperature distribution.Before proceeding further,it will be worthwhile mentioning briefly equation of heat conduction and steady state, dynamic state of thermo-elasticity.

EQUATION OF HEAT CONDUCTION

Let in the space (X_r) a solid body B be bounded by the surface S and $T(X_r, t)$ denote the temperature at the point (X_r) and at the time t. Then temperature differences between the points of the region B results in a flow of heat. Across a surface element $d\sigma$ at the point(X_r) the quantity of heat flowing in the time interval Δt is

$$\nabla Q = -\lambda T,_n d\sigma \, \Delta t$$

where λ is the coefficient of internal heat conduction, $T,_n = \frac{\partial T}{\partial n}$ is the normal derivative of the temperature at the point $(X_r)$of the surface element , in the direction of heat flow.

Now we investigate the equilibrium due to heat in a region B_1

bounded by S_1, constituting a part of B. The quantity of heat flowing into the region B_1 across the boundary S_1 in the time Δt is given by

$$\Delta Q' = \lambda \int_{S_1} T,_n d\sigma \; \Delta t$$

If W denotes the quantity of heat generated in unit volume in unit time, then the quantity of heat generated inside the region under consideration is

$$\Delta Q'' = \int_{B_1} W \, dv \, \Delta t.$$

On the other hand , $\Delta Q = \Delta Q' + \Delta Q''$ can be determined from

$$\Delta Q = \int_{B_1} c\rho \dot{T} \, dv \, \Delta t$$

where ρ is the density and c is the specific heat of the body. The condition $\Delta Q = \Delta Q' + \Delta Q''$ implies the equation

$$\int_{B_1} (c\rho\dot{T} - W) dv - \lambda \int_{S_1} T,_n \, d\sigma = 0$$

which by divergence theorem becomes

$$\int_{B_1} (c\rho\dot{T} - W - \lambda T,_{kk})\, dv = 0$$

Since this is true for all arbitrary region B_1, hence

$$T,_{kk} - \dot{T}/s = - Q/s \qquad (1)$$

where $s = \lambda/\rho c$, $W = Qc\rho$.

We have used tensor notation , i.e

$$T,_i = \frac{\partial T}{\partial X_i},\ T,_{kk} = \nabla^2 T$$

in a cartesian coordinate system. Dots represent derivatives with respect to time.

Solution of equations (1) determine temperature as a function of position and time. If the temperature is independent of time and if there are no heat sources inside the region B, then (1) can be by Laplace equation

$$T,_{kk} = 0 \qquad (2)$$

and hence in this case, temperature function is a potential function.

EQUATIONS OF THERMO-ELASTICITY.

Generation of stress and strain in a body takes place due to non-uniform distribution of temperature. The temperature T represents the increment of the temperature from the initial stress less state. We assume that the change in temperature is small and therefore it has no influence on the mechanical and thermal properties of the body.

We shall confine ourselves to an isotropic homogeneous body with respect to both its mechanical and thermal properties. Let u_i (i=1,2,3) be the components of displacement vector $\vec{u}$, e_{ij}(i,j=1,2,3) be the components of displacement of strain tensor and σ_{ij}(i,j=1,2,3), the components of stress tensor.

In the linear theory of elasticity, the strain tensor e_{ij} is considered with the displacement vector by the relation

$$e_{ij} = (u_{i,j} + u_{j,i})/2, \qquad i,j = 1,2,3 \tag{3}$$

The strain tensor is symmetric, i.e, $e_{ij} = e_{ji}$, The components of strain tensor can not be arbitrary, since they should have the following six relations— the so called comparability conditions:

$$e_{ij,kl} + e_{kl,ij} - e_{jl,ik} - e_{ik,jl} = 0, \qquad i,j,k,l = 1,2,3 \tag{4}$$

which are satisfied identically if e_{ij} is expressed by u_i in accordance with (3) when the displacement field is continuous.

In thermo-elasticity strain tensors are made up of two parts. The first part e^o_{ij} is a uniform expansion proportional to the temperature rise T. Since this expansion is the same in all directions for an isotropic body, only normal strains and no shearing strains arise in this manner. If α_t is the coefficient of linear expansion and δ_{ij} is the Kronecker's symbol, then

$$e^o_{ij} = \alpha_t T \delta_{ij}, \quad i,j=1,2,3 \tag{5}$$

The second part e'_{ij} comprises the strains required to maintain the continuity of the body as well as those arising because of external loads. These strains are related to the stresses by means of the Hooke's law of linear isothermal elasticity. Hence

$$e'_{ij} = \left[\sigma_{ij} - \frac{\nu}{1+\nu}\theta\delta_{ij}\right]/2\mu_1, \quad i,j=1,2,3 \tag{6}$$

where μ_1 is the shear modulus, ν is the Poission's ratio and $\theta = \sigma_{kk}$ is the sum of the normal stresses. Hence finally we have

$$e_{ij} = e^o_{ij} + e'_{ij} = \alpha_t T \delta_{ij} + \left[\sigma_{ij} - \frac{\nu}{1+\nu}\theta\delta_{ij}\right]/2\mu_1 \tag{7}$$

the so called Duhamel-Neumann relation.

Denoting $\theta = e_{kk}$, we have from (7)

$$\theta - 3\alpha_t T = \frac{1-2\nu}{E}\,\theta, \quad E = 2\mu_1(1+\nu) \tag{8}$$

where E is the Young's modulus

Solving (7) for stresses, we have

$$\sigma_{ij} = 2\mu_1 e_{ij} + (\lambda\theta - \gamma T)\delta_{ij}, \qquad i,j=1,2,3 \tag{9}$$

where λ, γ are Lame's elastic constants given by the relations

$$\nu = \frac{\lambda}{2(\lambda+\mu_1)}, \quad \gamma = (3\lambda+2\mu_1)\alpha_t$$

Now, in order to find the equations of elastic equilibrium, let us consider a body B with boundary S loaded in an arbitrary way and placed in a stationary temperature field. Let us consider the equilibrium of a sub-domain B_1 with boundary S_1. If F_i denotes the components of the body force per unit volume and p_i the components of surface tractions acting on the surface S_1, then from the condition of equilibrium we obtain the following three equations for the region B_1:

$$\int_{B_1} F_i \, dv + \int_{S_1} p_i \, d\sigma = 0, \qquad i = 1,2,3.$$

Taking into account that $p_i = \sigma_{ij} n_j$, where n_j denotes the components of unit normal vector of surface S_1, we get, on making use of divergence theorem

$$\int_{B_1} (F_i + \sigma_{ij,j}) \, dv = 0$$

Since this is true for an arbitrary region B_1 , the equilibrium equations take the form

$$\sigma_{ij,j} + F_i = 0 \quad , \qquad i = 1,2,3 \tag{10}$$

If in these equilibrium equations, we express stresses by strains and then by displacements, we obtain a system of three equations in which the unknown functions are the components of displacement vector:

$$\mu_1 u_{i,kk} + (\lambda + \mu) u_{k,ki} + F_i - \gamma T_{,i} = 0 \tag{11}$$

$$i,k = 1,2,3.$$

In cylindrical coordinates (r,θ,z), let u,v,w represent the components of displacement vector $\vec{u}$, σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} represent normal stresses and $\tau_{rz}, \tau_{\theta z}, \tau_{r\theta}$ represent shear stresses. In the case of axial symmetry about the z-axis, equations (11) reduce to two equations.

$$\nabla^2 u - r^{-2} u + \frac{1}{(1-2\nu)} \Theta_{,r} - \frac{2(\nu+1)}{(1-2\nu)} \alpha_t T_{,r} = 0$$

$$\nabla^2 w + \frac{1}{1-2\nu} \Theta_{,z} - \frac{2(\nu+1)}{(1-2\nu)} \alpha_t T_{,z} = 0 \tag{12a,b}$$

where

$$\theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}, \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

To solve the equations (11) in the absence of body forces i.e. $F_i=0$, Goodier [35] introduced a thermoelastic potential ϕ in terms of which the displacement vector is defined by the relation

$$u_i = \partial\phi/\partial x_i \qquad (13)$$

and ϕ is a particular solution of the Poission's equation

$$\nabla^2\phi = T(x_r) \qquad (14)$$

A well known particular integral of (14) is

$$\phi(x_r) = -\frac{1}{4\pi}\int_V \frac{T(\xi_r)\, dV(\xi_r)}{R(x_r,\xi_r)} \qquad (15)$$

where $R(x_r,\xi_r)$ is the distance between the points (x_r) and (ξ_r).

Integrals of the type (15) were employed by Borchardt [78] in a general discussion of the theory of thermo elasticity and also to solve certain special problems involving asymmetric distribution of temperature in solids with spherical on circular boundaries. Problems concerning spheres and cylinders are dealt in [15, pp.

362-67]. the problems of thin elastic plates, under fairly general distributions of temperatures have been considered by Galarkin, Nadai, Marguerre [55], Sokolnikoff [73] and Pell [74]. Several approximate solutions of the engineering problems concerned with thermal stresses in plates and rods are discussed in chapter 14 of Timoshenko and Goodier's " Theory of Elasticity" [17].

The calculation of the steady-state thermal stresses in an isotropic elastic half-space or slab with traction free faces has been the subject of several investigations. The distribution of thermal stress due to special temperature distribution in infinite and semi-infinite solids have been discussed by a variety of authors, i.e. Mindlin and Cheng [48], Myklestad [52] , Sternberg and Mc'Dowell [70], using an extension of Boussinesq-Papkovich method of isothermal elasticity solved the problem of half-space, The basis of the method is that the solution of the equation of equilibrium (11) may be expressed in terms of the four Boussinesq-Papkovich functions, one of which is the solution of Poisson's equation and remaining three are of Laplace equation. These equations have been studied extensively, particularly in potential theory, and general procedures of their solutions are known. Sneddon and Locket [72] approached this class of problems by direct solution of the equations of thermo-elasticity using a double Fourier integral transform method, the results being transformed to Hankel type integral in the case of axial symmetry. A further approach due to

Nowinski [75] exploits the fact that in steady-state thermo-elasticity each component of the displacement vector is a bi-harmonic function which can be expressed as a combination of harmonics. Possibly the most economical method of solutions of the type of problems is that of Williams [77] who expressed the displacement vector in terms of two scalar potential functions, one of which is directly related to the temperature field. Further, Muki [20] has introduced the displacement and stress components in the form of Hankel transform for the particular solution of the thermo -elastic equations.

It is to note that Nowacki [13] has made thorough survey of the problems of both elasto-static and elasto-dynamic in presence of the temperature excellently.

ABOUT THE THESIS

In the present age of science and technology it is inevitable to have a study on the problems of thermo-elasticity because of the increasing rarge of applications of the theory and analysis of the thermal stresses in industry, and especially in advanced technologies such as Aerospace Engineering, Laser Engineering, Design of Turbines, Micro electronics Industry. The subject has tremendous importance in compliance with its application in

Geophysical and Seismological problems. The interest in this field of science has been increasing among mechanical engineers, Semi-conductor engineers and chemical engineers.

The work of this thesis is occupied with some important and interesting problems of thermo-elasticity. In this elastic problems displacements and stresses have been derived in the presence of temperature with endeavor to obtain results which shall be practically important in applications to applied mathematics, engineering and technology in which the material of construction is solid. Here, both statical and dynamical types of problems of thermo-elasticity have been dealt with. The complete work is divided into four chapters and the problems in each chapter are relevant to each other.

In the process chapter I contains two highly interesting inclusion problems of thermo elasticity each of which is treated satisfying composite boundary conditions. The first one is a plane problem and the second one is considered in the case of solid body.

A general series form of stress function in bi-polar coordinate system was given by G.B.Geffrey [38]. It has been applied to the problems of a semi-infinite plate with a concentrated force at any point [51], a semi-infinite region with a circular hole under tension parallel to the straight edge or plane boundary [38] and under its own weight, and to the infinite plate with two holes [60], or a hole formed by two intersecting circles [45]. Solutions have been given for the circular disk subject to concentrated forces at any point to its own weight when suspended at a point [49], or in rotation about an eccentric axis [50], with and without [62] the use of bipolar coordinates, and for the effect of a circular hole in a semi-infinite plate with a concentrated force on the straight edge [79]. The equilibrium problem of thermo-elasticity for region bounded by two non-concentric circular inserts does not appear to have received any previous attention. In this particular instance, the region is infinite and contains two circular inserts of the same radius. The nucleus of heat is placed in the middle of the line joining two centers of the inserts, moreover both are assumed to be symmetric with respect to the common axis of symmetry. Bipolar coordinates [14] are used to obtain thermal stresses in the form of series following method of G.B.Geffrey [38] and then the approach of Das [30]. The series solution is based upon the boundary stress function approach. Numerical evaluations of the distribution of the plane thermal stresses have been performed by the method due to L.N.G.Fillon [84]. To begin with the second problem it is found that in a paper, E.Sternber and M.A.Sadowsky [67] determine a solution in series for the stress distribution in an infinite elastic medium which possess two spherical cavities of the same size and both are assumed to be symmetric with respect to the common axis of symmetry of the cavities and with respect to the plane of geometric symmetry perpendicular to this axis, the solution is based upon the boundary stress function approach and apparently constitutes the first application of spherical dipolar coordinates in the theory of elasticity. The thermal stress problem in an infinite elastic solid at zero temperature except for a heated region has been solved by Goodier [35]. In a paper, B.D.Sharma [69] has derived stresses and displacements on the surface of a spherical cavity when the heated element is at some finite distance from it and a solid sphere at zero temperature having a heated nucleus inside it in an infinite solid. Chatterjee and Dutta [37] have determined stresses due to a nucleus in the form of a center of dilatation in an infinite elastic solid with rigid infinite inclusion. An axially-symmetric thermo-elastic problem of the infinite cylinder has been solved due to nuclei of thermo elastic strain of unit intensity. Regarding the above solutions as the Green function, a number of particular cases of discontinuous temperature fields were investigated in detail in the paper of M. Sokolowski [63]. E.Sternberg, R.A.Eubanks, M.A.Sadowsky [64] have considered a problem where they have used spherical harmonics corresponding to either to the exterior or to the interior problem for the sphere. W.Piechocki, J.Ignaczak [66] have derived thermal stresses due to a thermal inclusion in a circular ring and a spherical shell using Heviside function for the temperature distribution. The problem of action of a nucleus of thermoelastic strain in a solid circular disk was solved in another way by B.Sen [80]. J.Ignaczak [41] has solved a problem with a hemispherical pit at the free surface in an elastic half-space in the form of series of spherical functions and analogous subsequences of solution for the problem of a hemispherical pit was given by R.A.Eubanks [42]. In a paper, stress have been determined due to a nucleus of thermoelastic strain in an infinite elastic solid with a spherical elastic inclusion of a different material by S.C.Bose [25] using spherical harmonics. The present problem is not only interesting but also important from physical point of view. Here an infinite elastic solid is considered to have two spherical inclusions of different materials while the nucleus of heat is placed in the axis of symmetry. This rotationally symmetric torsion free and mixed boundary value problem is solved by the application of spherical dipolar harmonics [64] employing Boussinesq [76] approach. Numerical calculations are made to show the distribution of displacements and stresses of this problem.

Chapter II contains two very useful problems of thermoelasticity (i) a double layered problem (ii) a three layered problem.

The design of highways and airport runways as well as the foundation problems in soil mechanics, especially when the earth mass supporting a heavy structure has different soil strata over it, it is highly needful to look to the endurance of the solid or land on which there is generated a thermal stress either due to impulse shock or owing to some local heating nucleus. Investigation of the stress distribution in a layered system was made by Burnister [27] in a series of papers. In a later paper, Acum and Fox [22] attacked a problem in a three layered system only by the method of Burnister. R.D.Mindlin and D.H.Cheng [48] who employed Galerkin vector for the center of dilatation to obtain stresses for semi-space. Paria [59] in his paper, determined elastic stress distribution in a three layered system due to a concentrated force. But when a plane bombs from above on the surface of the two layered system or a three layered system such as highway or airport land, an emmense heat is generated on the surface and as the heat is assumed to be distributed through layers, therefore the stresses are generated and in each layer and the underlying mass also. The upper layer of the double layered system, as in high ways is considered to be concrete pavement and the underlying mass is natural soil and in case of three layered system, the upper layer is of concrete pavement, the middle layer is of gravel base course and the lower mass is the natural soil as in the case of airport runways. The method of solution consists in taking the Hankel transform [18] of the stress function instead of stress function itself. Stresses in each layer due to the distribution of temperature and ultimately total stress in the underlying mass have been determined for both the cases. The type of heat flux function, if possible from physical point of view, is taken to be linear and graphical representation of the stress -distributions in the underlying mass are shown in the figures. Experimental results of the elastic constants are taken from international critical tables [8].

Chapter III is concerned with two dynamical problems of thermo-elasticity. In the first problem, components of displacement and stress are determined due to disturbance produced by a periodic heat nucleus and the second problem is on the generation of waves produced by an impulsive heat nucleus.

The problem of calculating the components of stress at a point in an elastic solid when it is deformed by the application of surface tractions which vary with time is of considerable interest in soil mechanics, in the theory of foundations and in the branch of applied mathematics. There has been extensive discussion of the corresponding statical problems. But Sneddon [71] in his palerno lecture discusses dynamical problems of this type in a systematic way. Special problems have been solved by Lambs [42] .

In a problem, Eason, Fulton and Sneddon [33] have dealt with the determination of distribution of stresses in an infinite elastic solid when the time dependent body force act upon certain region of the solid. Assuming strains to be small, the general solution of the equation of motion for any distribution of body forces is derived by the four-dimensional Fourier transforms [14] and from that general solution is derived for the isotropic solid. The solution of the equation of motion in the case in which the distribution of the body force is symmetrical about an axis is also derived. the solutions of some typical two dimensional and three dimensional problems are considered and exact analytical expressions are found for the components of displacement and stress. In the present discussion, at first detailed solution of the three dimensional thermo-elastic problem is obtained and the distribution of displacement and stress have been derived when the time dependent body force and temperature act on certain region of the solid. Then the problem consists of deducing the displacements and stresses due to the disturbance produced by the insertion of a periodic heat nucleus in the solid. Weber's Bessels function [19] of the second kind is ultimately realized for those cases. Capitatising the above procedure another interesting three dimensional problem is taken into account to determine components of displacement and stress when an impulsive heat nucleus act in the solid. Dirac Delta functions [4] are utilized in the time dependent body force and temperature acting at the origin in order to obtain the solution of the equation of motion. In each case, strains are assumed to be infinitesimal so that the equations of the classical theory of elasticity [7] are applicable. Fourier transform technique [14] is applied in both the cases separately.

In the last chapter, thermal stresses have been derived for the problems in the elastic semi-space due to heat exposure on the bounding plane of isotropic media and assume that there are no heat sources inside the semi-space. If $T(x_1, x_2, 0) = f(x_1, x_2)$ is prescribed then the determination of the state of stress due to a heating of the plane $x_3 = 0$ has been the subject of many investigations. E.Melan and H.Parkus [81] investigated the action of a concentrated heat source situated in the plane $x_3 = 0$ of a thermally insulated semi-space and proved that in this case a plane state of stress exists. The same conclusion was obtained by A.I.Luyre [82] who applied a different method, with respect to both the semi-space and a layer. E.Sternberg and E.L.MacDowell [68] in their investigation presented a solution to the problem by means of a method which was an extension of the Boussinseq-Papkovioh method [76] to thermal problems. A different procedure employing the Fourier integral transform was chosen by I.N.Sneddon and F.J.Lockett [72]. The solution can also be derived by introducing the thermoelastic displacement potential and satisfying the boundary condition by means of the Love or Galarkin function [54].

In the first problem, thermal stresses in a semi-infinite solid have been obtained for an interesting problem in which there is a constant supply of heat over an elliptic area on the bounding plane surface, the rest being kept at a constant temperature. In a paper, Nowacki [54] has solved the problem of thermal stresses in an ealstic half-space, the bounding plane surface of which is kept at a constant temperature $T = T_0$ inside a circle of radius 'a', the exterior of the circle being thermally insulated. B.R.Das [32] found thermal stresses in a semi infinite elastic solid with a constant heat flow over a circular area on the plane boundary. The analysis involves dual integral equation [3] and Bessel function [11] for its solution. Goodier [35] has given a complete solution of the thermo-elastic problem of an infinite solid at temperature zero except for a heated (or cooled) region. The semi infinite solid is considered by R.D.Mindlin and D.H.Cheng [48]. The cases of such a region in the form of an ellipsoid of revolution and semi-infinite circular cylinder, uniformly hot, have been worked out by N.O.Myklested [52]. In this problem, elliptic coordinates are used and the bounding surface of the semi-infinite elastic solid is given by $z=0$, the axis of z being drawn into the body. Temperature and the potential of the thermo elastic displacement ψ are obtained in terms of Matheieu function [10]. Love function is considered for the solution of biharmonic equation [9]. Components of displacement and stresses are obtained in curvilinear coordinates as given by C.B.Ling [44]. Numerical evaluations have been made for a suitable case collecting experimental results from Bickley and Mclachlan [2].

The second problem is a thermoelastic boundary value problem of three dimensions when these thermal stresses are produced in a body by unequal distribution of temperature which may be regarded as a specified function of coordinates and time. In this paper, stresses due to periodic supply of heat produced by the blow of a jet flame on the straight edge of a semi-infinite isotropic elastic thick plate distributed over a finite portion of it, have been obtained.

The third problem deals with the determination of thermal stresses due to prescribed flux of heat on the surface of a thick plate. The thermal stress problem of a thick circular plate at zero temperature except for a heated regions on the plane faces was considered by Nowacki [53]. The solution obtained by him satisfied the boundary conditions concerned with the stress on the edge surface in an approximate manner only. The object of this paper is to find the exact solution of the thermo elastic problem of a thick plate of infinite radius of an isotropic material, with stress free edges subjected to two different temperature distributions. In the first case, we assume a constant flux of heat within a circular region of exposure, the exterior of the circular region being free of any flux of heat. Secondly we assume a paraboloidal distribution of temperature within the circular region, the exterior being insulated. Numerical calculations for the variation of $(rr + \theta\theta)$ on the free surface have also been obtained in the second case.

The last problem is concerned with the distribution of stresses and displacements in a semi-infinite isotropic elastic solid when a prescribed flux of heat is applied on a circular region of the upper surface. The problem of determining the steady-state thermal stresses and displacements in a semi-infinite elastic medium was treated by Sternberg and McDowell [70] by the use of Green's functions. They proved that the stress field induced by an arbitrary distribution of surface temperature is plane and parallel to the boundary and obtained the solutions in closed forms for a circular region of exposure with uniform or hemispherical distribution of temperature. This problem was discussed by B.D.Sharma [83] by using integral transform methods. He discussed the same problem in case of isotropic material. Nowacki [54] solved the problem of thermal stresses in an elastic half-space , the bounding plane surface of which is kept at a constant temperature inside a circle of radius 'a', the exterior of the circle being thermally insulated. Sneddon and Locket [72] discussed the same problem by using double Fourier transform methods and arrived at the same result. In this problem two types of flux function, one being constant and other parabolic have been prescribed in a circular region of the bounding plane, the rest of the surface being kept free of flux of heat to determine thermal stresses and numerical results have been obtained.

NOMENCLATURE

x, y, z	Rectangular coordinates
r, θ, ϕ	Polar coordinates
r, θ, z	Cylindrical coordinates
α, β, γ	Orthogonal curvilinear coordinates
ξ, η	Elliptical coordinates
X, Y, Z	Components of a body force per unit volume
$\bar{X}, \bar{Y}, \bar{Z}$	Components of a distributed surface force per unit area.
$(\sigma_x)_T, (\sigma_y)_T, (\sigma_z)_T$	Normal components of stress parallel to x-, y- and z- axes in the presence of temperature.
$(\widehat{xx})_T, (\widehat{yy})_T, (\widehat{zz})_T$	Normal components of stress parallel to x-, y- and z- axes in the presence of temperature.

$(\widehat{rr})_T, (\widehat{\theta\theta})_T, (\widehat{\phi\phi})_T$	Normal components of stress in polar coordinates in the presence of temperature.
$(\widehat{rr})_T, (\widehat{\theta\theta})_T, (\widehat{zz})_T$	Normal components of stress in cylindrical coordinates in the presence of temperature.
$(\widehat{\alpha\alpha})_T, (\widehat{\beta\beta})_T$	Normal components of stress in curvilinear coordinates in the presence of temperature.
$(\widehat{\xi\xi})_T, (\widehat{\eta\eta})_T$	Normal components of stress in elliptical coordinates in the presence of temperature.
$(\sigma_x)_C, (\sigma_y)_C, (\sigma_z)_C$	Normal stress components in cartesian coordinates in the absence of temperature.
$(\widehat{xx})_C, (\widehat{yy})_C, (\widehat{zz})_C$	Normal stress components in cartesian coordinates in the absence of temperature.
$(\widehat{rr})_C, (\widehat{\theta\theta})_C, (\widehat{\phi\phi})_C$	Normal components of stress in polar coordinates in the absence of temperature.

$(\widehat{rr})_C, (\widehat{\theta\theta})_C, (\widehat{zz})_C$	Normal components of stress in cylindrical coordinates in the absence of temperature.
$(\widehat{\alpha\alpha})_C, (\widehat{\beta\beta})_C$	Normal components of stress in curvilinear coordinates in the absence of temperature.
$(\widehat{\xi\xi})_C, (\widehat{\eta\eta})_C$	Normal components of stress in elliptical coordinates in the absence of temperature.
$(\sigma_{xy})_T, (\sigma_{xz})_T, (\sigma_{zy})_T$	Shearing stress components in rectangular coordinates in the presence of temperature
$(\widehat{xy})_T, (\widehat{yz})_T, (\widehat{zx})_T$	Shearing stress components in rectangular coordinates in the presence of temperature
$(\widehat{r\theta})_T, (\widehat{\theta\phi})_T, (\widehat{r\phi})_T$	Shearing stress components in polar coordinates in the presence of temperature
$(\widehat{r\theta})_T, (\widehat{\theta z})_T, (\widehat{zr})_T$	Shearing stress components in cylindrical coordinates in the presence of temperature.

$(\widehat{\alpha\beta})_T, (\widehat{\beta\gamma})_T, (\widehat{\gamma\alpha})_T$	Shearing stress components in curvilinear coordinates in the presence of temperature.
$(\widehat{\xi\eta})_T, (\widehat{\eta z})_T, (\widehat{\xi z})_T$	Shearing stress components in elliptical coordinates in the presence of temperature.
$(\sigma_{xy})_C, (\sigma_{xz})_C, (\sigma_{zy})_C$	Shearing stress components in cartesian coordinates in the absence of temperature.
$(\widehat{r\theta})_C, (\widehat{\theta\phi})_C, (\widehat{r\phi})_C$	Shearing components of stress in polar coordinates in the absence of temperature.
$(\widehat{r\theta})_C, (\widehat{\theta z})_C, (\widehat{rz})_C$	Shearing components of stress in cylindrical coordinates in the absence of temperature.
$(\widehat{\alpha\beta})_C, (\widehat{\beta\gamma})_C, (\widehat{\gamma\alpha})_C$	Shearing components of stress in curvilinear coordinates in the absence of temperature.
$(\widehat{\xi\eta})_C$	Shearing components of stress in elliptical coordinates in the absence of temperature.
u_T, v_T, w_T	Components of displacements in the presence

of temperature.

$u_\alpha, u_\beta, u_\gamma$ — Components of displacements in curvilinear coordinates.

u_r, u_θ, u_ϕ — Components of displacements in polar coordinates.

u_r, u_θ, u_z — Components of displacements in cylindrical coordinates.

u_c, v_c, w_c — Components of displacements in the absence of temperature.

$(\varepsilon_x)_T, (\varepsilon_y)_T, (\varepsilon_z)_T$ — Components of strain in the presence of temperature in the x,y z directions respectively.

$(\varepsilon_x)_c, (\varepsilon_y)_c, (\varepsilon_z)_c$ — Components of strain in the absence of temperature in the x,y z directions respectively.

$\varepsilon_r, \varepsilon_\theta, \varepsilon_\phi$	Components of strain in polar coordinates.
$\varepsilon_r, \varepsilon_\theta, \varepsilon_z$	Components of strain in cylindrical coordinates.
E	Young's modulus.
G	Modulus of elasticity
σ, γ	Poission's ratio.
$\mu = G, \ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$	Lame's constants.
ϕ	Stress function
t	time
T_o	Absolute temperature
T	Temperature
α	Coefficients of linear thermal expansion.
E_1, E_2	Young's moduli in the x and y directions respectively.

ν_1 Ratio of the contraction parallel to x axis to the extension parallel to y axis.

ν_2 Ratio of the contraction parallel to y axis to the extension parallel to x axis.

α_1 Coefficient of linear thermal expansion in the x axis.

α_2 Coefficient of linear thermal expansion in the y axis.

α_3 Coefficient of linear thermal expansion in the z axis.

CHAPTER - I

STRESSES DUE TO NUCLEUS OF THERMOELASTIC STRAIN IN PRESENCE OF ELASTIC NUCLEUS

Paper 1 : Thermoelastic stress concentration due to nucleus of thermoelastic strain in an infinite elastic solid with two rigid circular inserts.

Paper 2 : Thermoelastic stress concentration due to nucleus of thermoelastic strain in an infinite solid with two spherical elastic inclusions

THERMOELASTIC STRESS CONCENTRATION DUE TO NUCLEUS OF THERMOELASTIC STRAIN IN AN INFINITE ELASTIC SOLID WITH TWO RIGID CIRCULAR INSERTS.

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.1Solution

Let us consider an infinite elastic solid at zero temperature. If there be situated a nucleus of thermoelastic strain at (0,0) the corresponding displacements can be derived as the gradient of a potential function [7] ψ where

$$\psi = Q_o r^2 \log r \tag{1}$$

in which

$$Q_o = \frac{1}{2\pi}\,\frac{1+\nu}{1-\nu}\,\alpha_o T_o\, ds \tag{2}$$

To obtain the particular solution we introduce bipolar co-ordinates by means of the transformation [14,38]

$$x = \frac{a \sinh\alpha}{\cosh\alpha + \cos\beta}, \qquad y = \frac{a \sin\beta}{\cosh\alpha + \cos\beta} \tag{3}$$

$$r^2 = a^2\,\frac{\cosh\alpha - \cos\beta}{\cosh\alpha + \cos\beta} \tag{4}$$

in which the ranges of the curvilinear co-ordinates α and β are $-\infty<\alpha<\infty$, $-\pi \le \beta \le \pi$.

As it is convenient to deal with $h\psi$ instead of ψ itself in these particular co-ordinates, we calculate displacements and stresses for the function [35]

$$h\psi = Q_o a\left[\sum_{n=0}^{\infty} n^{-2}\cos n\beta + \frac{1}{2}\log a\sum_{n=2}^{\infty} n^{-1}e^{n\alpha}\right] \quad (5)$$

where

$$h^{-2} = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2$$

Now, the displacements and stresses are

$$(u_\alpha)_T = (\cosh\alpha + \cos\beta)\,\frac{Q_o}{2}\,\log a\sum_{n=2}^{\infty} e^{n\alpha}$$

$$(u_\beta)_T = -(\cosh\alpha + \cos\beta)Q_o\sum_{n=0}^{\infty}\sin(n\beta)/n \quad (6)$$

$$(a\widehat{\alpha\alpha})_T = -Q_o a\Big[(\cosh\alpha + \cos\beta)\sum\cos(n\beta) + \frac{1}{2}\log a\,\sinh\alpha\sum e^{n\alpha} -$$

$$\sin\beta\sum\sin(n\beta)/n - \cos\alpha\left\{\sum n^{-2}\cos n\beta + \frac{1}{2}\log a\sum n^{-1}e^{n\alpha}\right\}\Big]$$

$$(a\widehat{\beta\beta})_T = Q_o a\Big[(\cosh\alpha + \cos\beta)\frac{1}{2}\log a\sum ne^{n\alpha} + \frac{1}{2}\log a\,\sinh\alpha\sum e^{n\alpha}$$

$$- \sin\beta \sum \sin(n\beta)/n + \cos\beta\left\{\sum n^{-2}\cos n\beta + \frac{1}{2}\log a \sum n^{-1} e^{n\alpha}\right\}\Big]$$

$$(\widehat{a\alpha\beta})_T = 0 \tag{7}$$

The stresses obtained in (7) are obviously produced by the thermal expansion. This expansion gives rise to certain stresses on the surfaces of the two circular inserts $\alpha = \pm\alpha_1$. It is to be noted that each of the circular inserts lies entirely on either side of the line α=0. We shall, therefore. make the boundaries free from stresses by the addition of the extra terms obtained on the hypothesis that there is no temperature distribution.

Let us consider a stress function which gives no stress at infinity and no stress over α=0 and such that on the surface $\alpha = \pm\alpha_1$

$$(\widehat{a\alpha\alpha})_C = -(\widehat{a\alpha\alpha})_T$$

$$(\widehat{a\alpha\beta})_C = -(\widehat{a\alpha\beta})_T$$

$$(u_\alpha)_C = -(u_\alpha)_T$$

$$(u_\beta)_C = -(u_\beta)_T \tag{8}$$

Now, considering stress function in the form

$$h\chi = \sum_{n=2}^{\infty} \left\{\phi_n(\alpha)\cos(n\beta) + \psi_n(\alpha)\sin(n\beta)\right\} \tag{9}$$

which is a solution of the biharmonic equation

$$\left[\frac{\partial^4}{\partial\alpha^4} + 2\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - 2\frac{\partial^4}{\partial\alpha^2} + 2\frac{\partial^4}{\partial\beta^2} + 1\right](h\chi) = 0 \tag{10}$$

with

$$\phi_n(\alpha) = A_n\cosh(n+1)\alpha + B_n\cosh(n-1)\alpha + C_n\sinh(n+1)\alpha + D_n\sinh(n-1)\alpha \tag{11}$$

and

$$\psi_n(\alpha) = A'_n\cosh(n+1)\alpha + B'_n\cosh(n-1)\alpha + C'_n\sinh(n+1)\alpha + D'_n\sinh(n-1)\alpha \tag{12}$$

It may be readily seen that the conditions $(\widehat{a\alpha\alpha})_c$, $(\widehat{a\alpha\beta})_c$ shall vanish over $\alpha=0$ is satisfied, if $\phi_n(0)=0$, $\phi'_n(0)=0$ and $\psi_n(0)=0$, $\psi'_n(0)=0$ for $n \geq 2$, and hence from (11) and (12),

$$A_n + B_n = 0$$

$$(n+1)C_n + (n-1)D_n = 0$$

$$A'_n + B'_n = 0$$

$$(n+1)C'_n + (n-1)D'_n = 0$$

So,

$$h\chi = \sum_{n=2}^{\infty}\left\{A_n\left[\cosh(n+1)\alpha - \cosh(n-1)\alpha\right] + E_n\left[(n-1)\sinh(n+1)\alpha - (n+1)\sinh(n-1)\alpha\right]\right.$$

$$\Big\}\cos(n\beta)+\sum_{n=2}^{\infty}\Big\{A'_n\big[\cosh(n+1)\alpha-\cosh(n-1)\alpha\big]+E'_n\big[(n-1)\sinh(n+1)\alpha-(n+1)\times$$

$$\sinh(n-1)\alpha\big]\Big\}\sin(n\beta) \qquad (13)$$

where

$$E_n=C_n/(n-1) \text{ and } E'_n=C'_n/(n-1)$$

So, the complementary displacements and stresses are given by

$$(u_\alpha)_c = a^{-1}(\cosh\alpha+\cos\beta)\sum_{n=2}^{\infty}\Big\{[A'_n\sin n\beta+A_n\cos n\beta]\big[(n+1)\sinh(n+1)\alpha-$$

$$(n-1)\sinh(n-1)\alpha\big]+(n^2-1)[E'_n\sin n\beta+E_n\cos n\beta]\big[\cosh(n+1)\alpha-\cosh(n-1)\alpha\big]\Big\}$$

$$(u_\beta)_c = a^{-1}(\cosh\alpha+\cos\beta)\sum_{n=2}^{\infty}n\Big\{[A'_n\cos n\beta-A_n\sin n\beta]\big[\cosh(n+1)\alpha-$$

$$\cosh(n-1)\alpha\big]+[E'_n\cos n\beta-E_n\sin n\beta]\big[(n-1)\sinh(n+1)\alpha-(n+1)\sinh(n-1)\alpha\big]\Big\}$$

(14)

$$(a\widehat{\alpha\alpha})_c =$$

$$-(\cosh\alpha+\cos\beta)\sum_{n=2}^{\infty}n^2\Big\{[A_n'\sin n\beta+A_n\cos n\beta]\big[\cosh(n+1)\alpha-\cosh(n-1)\alpha\big]+$$

$$[E_n'\sin n\beta+E_n\cos n\beta]\big[(n-1)\sinh(n+1)\alpha-(n+1)\sinh(n-1)\alpha\big]\Big\}+\sin\beta\times$$

$$\sum_{n=2}^{\infty}n\Big\{[A_n\sin n\beta-A_n'\cos n\beta]\big[\cosh(n+1)\alpha-\cosh(n-1)\alpha\big]+(E_n\sin n\beta-E_n'\cos n\beta]$$

$$\big[(n-1)\sinh(n+1)\alpha-(n+1)\sinh(n-1)\alpha\big]\Big\}-\sinh\alpha\sum_{n=2}^{\infty}\Big\{[A_n'\sin n\beta+A_n\cos n\beta]$$

$$\big[(n+1)\sinh(n+1)\alpha-(n-1)\sinh(n-1)\alpha\big]+(n^2-1)[E_n'\sin n\beta+E_n\cos n\beta]$$

$$\big[\cosh(n+1)\alpha-\cosh(n-1)\alpha\big]\Big\}+\cosh\alpha\sum_{n=2}^{\infty}\Big\{[A_n'\sin n\beta+A_n\cos n\beta]\big[\cosh(n+1)\alpha-$$

$$\cosh(n-1)\alpha\big]++[E_n\cos n\beta+E_n'\sin n\beta]\big[(n-1)\sinh(n+1)\alpha-(n+1)\sinh(n-1)\alpha\big]\Big\}$$

$$(a\widehat{\beta\beta})_c = (\cosh\alpha + \cos\beta)\sum_{n=2}^{\infty}\Big\{[A'_n \sin n\beta + A_n \cos n\beta]\Big[(n+1)^2 \cosh(n+1)\alpha -$$

$$(n-1)^2 \cosh(n-1)\alpha\Big] + (n^2-1)[E'_n \sin n\beta + E_n \cos n\beta]\Big[(n+1)\sinh(n+1)\alpha -$$

$$(n-1)\sinh(n-1)\alpha\Big]\Big\} - \sin\beta\sum_{n=2}^{\infty} n\Big\{[A_n \sin n\beta - A'_n \cos n\beta]\Big[\cosh(n+1)\alpha -$$

$$\cosh(n-1)\alpha\Big] + [E_n \sin n\beta - E'_n \cos n\beta]\Big[(n-1)\sinh(n+1)\alpha - (n+1)\sinh(n-1)\alpha\Big]\Big\}$$

$$-\sinh\alpha\sum_{n=2}^{\infty}\Big\{[A'_n \sin n\beta + A_n \cos n\beta]\Big[(n+1)\sinh(n+1)\alpha - (n-1)\sinh(n-1)\alpha\Big]$$

$$+ (n^2-1)[E'_n \sin n\beta + E_n \cos n\beta]\Big[\cosh(n+1)\alpha - \cosh(n-1)\alpha\Big]\Big\} + \cos\beta \times$$

$$\sum_{n=2}^{\infty}\Big\{[A'_n \sin n\beta + A_n \cos n\beta]\Big[\cosh(n+1)\alpha - \cosh(n-1)\alpha\Big] + [E_n \cos n\beta + E'_n$$

$$\times \sin n\beta \,)\left[(n-1)\sinh(n+1)\alpha-(n+1)\sinh(n-1)\alpha\right]\Big\}$$

$$(a\widehat{\alpha\beta})_c = (\cosh\alpha + \cos\beta)\sum_{n=2}^{\infty} n\Big\{[A_n \sin n\beta - A'_n \cos n\beta]\left[(n+1)\sinh(n+1)\alpha -\right.$$

$$\left.(n-1)\sinh(n-1)\alpha\right] + (n^2-1)[E_n \sin n\beta - E'_n \cos n\beta]\left[\cosh(n+1)\alpha - \cosh(n-1)\alpha\right]\Big\}$$

(15)

Using (8) on the surface $\alpha = \pm\alpha_1$, we have

$$E_n = -aQ_o \log a \frac{\cosh(n\alpha_1)\cos(n\beta)}{2(n^2-1)\left[\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1\right]}$$

$$E'_n = -aQ_o \log a \frac{\cosh(n\alpha_1)\sin(n\beta)}{2(n^2-1)\left[\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1\right]}$$

$$A_n = aQ_o\Big\{2\left[(1-n^{-2})\cosh\alpha_1 + \cos\beta\right]\cos^2 n\beta + \log a\Big(\sinh\alpha_1 \sinh n\alpha_1 - \frac{1}{n}\times$$

$$\cosh\alpha_1 \cosh n\alpha_1\Big)\cos n\beta\Big\}\Big\{\left[\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1\right]\left[2\cosh\alpha_1 - 2n^2\times\right.$$

$$(\cosh \alpha_1 + \cos \beta)\Big] - 2\sinh \alpha_1 \Big[(n+1) \sinh(n+1)\alpha_1 - (n-1) \sinh(n-1)\alpha_1\Big]\Big\}^{-1}$$

$$- aQ_o \frac{\cos^2 n\beta}{n^2 \left[\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1\right]}$$

$$A'_n = aQ_o \Big\{\Big[(1-n^{-2})\cosh \alpha_1 + \cos \beta\Big] \sin 2n\beta + \log a \Big(\sinh \alpha_1 \sinh n\alpha_1 - n^{-1}$$

$$\cosh \alpha_1 \cosh n\alpha_1\Big) \sin n\beta\Big\} \Big\{\Big[\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1\Big]\Big[2\cosh \alpha_1 - 2n^2 \times$$

$$(\cosh \alpha_1 + \cos \beta)\Big] - 2\sinh \alpha_1 \Big[(n+1) \sinh(n+1)\alpha_1 - (n-1) \sinh(n-1)\alpha_1\Big]\Big\}^{-1}$$

$$- aQ_o \frac{\cos(n\beta)\sin(n\beta)}{n^2 \left[\cosh(n+1)\alpha_1 - \cosh(n-1)\alpha_1\right]} \qquad (16)$$

With these values of the constants complementary stresses are known,. Thus the components of resultant displacement and stress on the surface $\alpha = \pm\alpha_1$ are given by

$$(a\widehat{\alpha\alpha}) = (a\widehat{\alpha\alpha})_C + (a\widehat{\alpha\alpha})_T$$

$$(a\widehat{\beta\beta}) = (a\widehat{\beta\beta})_C + (a\widehat{\beta\beta})_T$$

$$(a\widehat{\alpha\beta}) = (a\widehat{\alpha\beta})_C + (a\widehat{\alpha\beta})_T \qquad (17)$$

$$u_\alpha = (u_\alpha)_T + (u_\alpha)_C$$

$$u_\beta = (u_\beta)_T + (u_\beta)_C \qquad (18)$$

From (17) we have

$$a(\widehat{\alpha\alpha}+\widehat{\beta\beta}) = Q_o a(\cosh\alpha_1+\cos\beta)\left(n\log a \cosh n\alpha_1 - 2\cos n\beta\right) + Q_o a(\cosh\alpha_1 +$$

$$\cos\beta)\left\{\left[(1-n^{-2})\cosh\alpha_1+\cos\beta\right]\cos n\beta + \log a\left[\sinh\alpha_1 \sinh n\alpha_1 - n^{-1}\cosh\alpha_1\right.\right.$$

$$\left.\left.\cosh n\alpha_1\right]\right\} \left\{\left[(2n+2)\cosh(n+1)\alpha_1-(2n-2)\cosh(n-1)\alpha_1\right]\right\} \left\{\left[\cosh(n+1)\alpha_1 -\right.\right.$$

$$\cosh(n-1)\alpha_1\left.\right]\left[\cosh\alpha_1 - n^2(\cosh\alpha_1+\cos\beta)\right] - \sinh\alpha_1\left[(n+1)\sinh(n+1)\alpha_1 -\right.$$

$$(n-1)\sinh(n-1)\alpha_1]\Big\}^{-1} \qquad (19)$$

2.NUMERICAL RESULTS

The numerical results of equation (19) when $\alpha_1=1, a=1, n=2$ for different values of β between 0° and 180° are tabulated below:

β	0°	30°	60°
$\frac{(\widehat{\alpha\alpha}+\widehat{\beta\beta})}{Q_o}$	-0.5508	-0.1234	-0.4558

β	90°	120°	150°	180°
$\frac{(\widehat{\alpha\alpha}+\widehat{\beta\beta})}{Q_o}$	-2.905	-2.565	3.418	-2.580

Thermoelastic Stress Concentration Due To Nucleus Of Thermoelastic Strain In An Infinite Elastic Solid With Two Spherical Elastic Inclusions

Communicated for publications.

1.SOLUTION:

Let an infinite elastic solid at zero temperature contain an element of volume $d\tau$ at (a,0,0) which is heated to a temperature T. In this case, the displacement is given by the gradient of the scalar function χ [17] where

$$\chi = \frac{A}{2G} \frac{1}{\sqrt{r^2 - 2ra\cos\theta + a^2}} \tag{1}$$

in which

$$A = -\frac{2G}{4\pi} \frac{\alpha_1(1+\nu)}{1-\nu} T \, d\tau$$

α_1, G, ν being respectively coefficient of linear thermal expansion, coefficient of elasticity in shear and Poisson's ratio.

The components of displacement and stress in space polar coordinates due to χ are given by [86].

$$2Gu_r = -\frac{A(r - a\cos\theta)}{(r^2 - 2ar\cos\theta + a^2)^{3/2}}$$

$$2Gu_\theta = -\frac{Aa\sin\theta}{(r^2 - 2ar\cos\theta + a^2)^{3/2}}$$

$$u_{\phi} = 0 \tag{2}$$

$$\widehat{rr} = A\,\frac{2r^2+3a^2\cos\theta-4ra\cos\theta-a^2}{(r^2-\ 2ar\cos\theta\ +\ a^2)^{5/2}}$$

$$\widehat{\theta\theta} = A\left[\frac{3a^2\sin^2\theta}{(r^2-\ 2ar\cos\theta\ +\ a^2)^{5/2}} - \frac{1}{(r^2-\ 2ar\cos\theta\ +\ a^2)^{3/2}}\right]$$

$$\widehat{\phi\phi} = -\ \frac{A}{(r^2-\ 2ar\cos\theta\ +\ a^2)^{3/2}}$$

$$\widehat{r\theta} = A\ \frac{3a\sin\theta(r-a\cos\theta)}{(r^2-\ 2ar\cos\theta\ +\ a^2)^{5/2}},$$

$$\widehat{r\phi} = \widehat{\theta\phi} = 0. \tag{3}$$

We denote this displacement and stress fields derived from χ by $\overline{\chi}$.

In case of torsion free, rotational symmetry and in the absence of body forces, the general solution of the displacement equation of equilibrium following Boussinseq, is represented as the sum of the two displacement fields [64]

$$2G[u,v,w] = \text{grad}\ \phi \tag{4}$$

$$2G[u,v,w] = \text{grad}\ (z,\psi) - [0,0,4(1-\nu)\psi] \tag{5}$$

provided $\phi(r,z)$ and $\psi(r,z)$ with $r = \sqrt{x^2+y^2+z^2}$, are arbitrary harmonic functions [64], i.e.

$$\nabla^2\phi = 0, \qquad \nabla^2\psi = 0.$$

Now, introducing Spherical Dipolar Coordinates by means of the transformations [67]

$$x = \frac{\bar{p}}{q-p}\cos\gamma$$

$$y = \frac{\bar{p}}{q-p}\sin\gamma$$

$$z = -\frac{\bar{q}}{q-p}$$

where

$$q = \cosh\alpha, \qquad \bar{q} = \sinh\alpha$$

$$p = \cos\beta, \qquad \bar{p} = \sin\beta \tag{7}$$

The ranges of the curvilinear coordinates (α,β,γ) are $-\infty<\alpha<\infty$, $0 \le \beta \le \pi$, $0 \le \gamma \le 2\pi$ so that

$$1 \le q < \infty, \qquad -\infty < \bar{q} < \infty$$

$$-1 \le p \le 1, \qquad 0 \le \bar{p} \le 1.$$

To obtain the particular solution, we introduce the exterior and interior spherical dipolar harmonics of integral order as generating stress functions.

We consider the dipolar harmonic stress functions [64]

$$\left.\begin{aligned} \phi_n(\alpha,\beta) &= \mu C_n(\alpha) P_n(p) \\ \psi_n(\alpha,\beta) &= \mu S_n(\alpha) P_n(p), \quad n=0,1,2,\ldots\ldots \end{aligned}\right\} \text{(exterior)} \tag{7}$$

and

$$\left.\begin{aligned} \phi'_n(\alpha,\beta) &= \mu C_n(\alpha) P_n(p) \\ \psi'_n(\alpha,\beta) &= \mu S_n(\alpha) P_n(p), \quad n=0,1,2,\ldots\ldots \end{aligned}\right\} \text{(interior)} \tag{8}$$

where

$$\mu = \sqrt{q-p}, \quad C_n(\alpha) = \cosh(n+1/2)\alpha, \quad S_n(\alpha) = \sinh(n+1/2)\alpha$$

and $P_n(p)$ is the Legendre polynomial of degree n.

The particular solution generated by ϕ_n, ψ_n and ϕ'_n, ψ'_n will be distinguished by $\bar{\phi}_n, \bar{\psi}_n$ and $\bar{\phi}'_n, \bar{\psi}'_n$. Solutions for $\bar{\phi}_n$ are

$$u_\alpha = \frac{\mu}{2G}\left[\,\bar{q}C_n(\alpha)P_n(p)+k\mu^2 S_n(\alpha)P_n(p)\right],$$

$$u_\beta = \frac{\bar{p}\mu}{4G}\left[\,C_n(\alpha)P_n(p)-2\mu^2 C_n(\alpha)P'_n(p)\right],$$

$$u_\gamma = 0. \tag{9}$$

$$\sigma_\alpha = \mu\left\{\left[\frac{k^2+1}{4}\,\mu^4+\frac{\bar{q}^2}{2}-\frac{\bar{p}^2}{4}\right]C_n(\alpha)P_n(p)+\bar{p}^2\mu^2 C_n(\alpha)P'_n(p)+k\bar{q}\mu^2 S_n(\alpha)P_n(p)\right\},$$

$$\sigma_\beta = \mu\left\{\left[-\frac{k^2}{4}\,\mu^4-\frac{\bar{q}^2}{4}+\frac{\bar{p}^2}{2}\right]C_n(\alpha)P_n(p) + (p\mu^2-2\bar{p}^2)\mu^2 C_n(\alpha)P'_n(p) - \frac{1}{2}\,k\bar{q}\mu^2 S_n(\alpha)P_n(p)\right\},$$

$$\sigma_\gamma = -\mu^3\left\{\frac{q}{2}C_n(\alpha)P_n(p)+(pq-1)C_n(\alpha)P'_n(p)+\frac{1}{2}\,k\bar{q}S_n(\alpha)P_n(p)\right\},$$

$$\tau_{\alpha\beta} = \bar{p}\mu\Big[\frac{3}{4}\bar{q}C_n(\alpha)P_n(p) - \frac{3}{2}\bar{q}\mu^2 C_n(\alpha)P'_n(p) + \frac{3}{4}k\mu^2 S_n(\alpha)P_n(p) -$$

$$\frac{1}{2}k\mu^4 S_n(\alpha)P'_n(p)\Big] \qquad (10)$$

where $k = 2n+1$, $n=0,1,2,\ldots\ldots$ and

$\tau_{\beta\gamma} = \tau_{\gamma\alpha} = 0.$

Solutions for $\bar{\psi}_n$ are

$$u_\alpha = -\frac{1}{4G\mu}\Big[\Big\{(7-8\nu)(qp-1)+q\mu^2\Big\}S_n(\alpha)P_n(p)+k\bar{q}\mu^2 C_n(\alpha)P_n(p)\Big],$$

$$u_\beta = -\frac{\bar{p}\ \bar{q}}{4G\mu}\Big[(7-8\nu)S_n(\alpha)P_n(p)-2\mu^2 S_n(\alpha)P'_n(p)\Big],$$

$$u_\gamma = 0. \qquad (11)$$

$$\sigma_\alpha = -\mu^{-1}\Big\{\Big[\frac{k^2-4+8\nu}{4}q\bar{q}\ \mu^2 + \frac{k^2+2-4\nu}{4}\bar{q}p\ \mu^2 + \frac{7-8\nu}{4}\bar{q}^3\Big]S_n(\alpha)P_n(p)$$

$$+ (1-2\nu)\bar{q}\bar{p}^{2}\mu^{2} S_n(\alpha)P'_n(p) + k\left[\bar{q}^{2}+(1-\nu)(qp-1)\right]\mu^{2} C_n(\alpha)P_n(p)\Bigg\}$$

$$\sigma_\beta = \mu^{-1} \left\{\left[\frac{k^2-6+8\nu}{4} q\bar{q}\, \mu^2 - \frac{k^2+6-4\nu}{4} \bar{q}p\, \mu^2 + \frac{7-8\nu}{4} \bar{q}^{3}\right] S_n(\alpha)P_n(p)\right.$$

$$\left. + \left[(3-2\nu)\bar{p}^{2}-qp+1\right]\bar{q}\mu^2 S_n(\alpha)P'_n(p) + \frac{1}{2}k\left[\bar{q}^{2}-2\nu(qp-1)\right]\mu^2 C_n(\alpha)P_n(p)\right\}$$

$$\sigma_\gamma = \mu \left\{ (q-2\nu p)\frac{\bar{q}}{2} S_n(\alpha)P_n(p) + (pq-1+2\nu\bar{p}^{2})\bar{q}S_n(\alpha)P'_n(p) + \right.$$

$$\left. \frac{1}{2}k \left[\bar{q}^{2}-2\nu(qp-1)\right]C_n(\alpha)P_n(p)\right\}$$

$$\tau_{\alpha\beta} = \bar{p}\mu^{-1}\left\{\left[\frac{1-2\nu}{2} q\mu^2 - \frac{7-8\nu}{4} \bar{q}^{2}\right]S_n(\alpha)P_n(p) + \frac{1}{2}\left[3\bar{q}^{2}+(2-4\nu)(qp-1)\right]\right.$$

$$\left.\mu^2 S_n(\alpha)P'_n(p) - \frac{k(5-4\nu)}{4} \bar{q}\mu^2 C_n(\alpha)P_n(p) + \frac{1}{2} k\bar{q}\mu^2 C_n(\alpha)P'_n(p)\right\} \qquad (12)$$

where $k = 2n+1$, $n=0,1,2,\ldots\ldots.$, and

$\tau_{\beta\gamma} = \tau_{\gamma\alpha} = 0.$

Similar solutions will be found for $\bar{\phi}'_n, \bar{\psi}'_n$.

2·Solution Of The Problem

Infinite elastic solid with two axisymmetric spherical inclusions, the heated element being situated outside the spherical surface is considered in this section.

We consider two spherical inclusions of the same radius and of different materials whose surfaces are taken as

$$\alpha = \alpha_o, \quad \alpha = -\alpha_o, \quad \alpha_o > 0.$$

The heated element is at (0,0,0).

In order to fit the conditions on the spherical surfaces, we must have on $\alpha = \pm\alpha_o$

$$(\sigma_\alpha)_i = (\sigma_\alpha)_e,$$

$$(\tau_{\alpha\beta})_i = (\tau_{\alpha\beta})_e ,$$

$$(u_\alpha)_i = (u_\alpha)_e ,$$

$$(u_\beta)_i = (u_\beta)_e.$$

$\alpha = \pm\alpha_o$, $0 \le \beta \le \pi$, $0 \le \gamma \le 2\pi$, i and e stand for interior and exterior respectively (13)

Also, all the components of displacement and stress must be finite when $\alpha \le \pm\alpha_o$ and zero at infinity. In the region $\alpha \ge \pm\alpha_o$ the stress systems $\bar{\phi}_n$ and $\bar{\psi}_n$ are uniform and hence we can superpose them on the solutions S and S′ to the problem subject to the boundary conditions (13) in the forms

$$\bar{S}' = \Sigma\left[a'_n\bar{\phi}_n' + b'_n\bar{\psi}_n' \right] \qquad \text{(inside)} \qquad (14)$$

$$\bar{S} = \bar{\chi} + \Sigma\left[a_n\bar{\phi}_n + b_n\bar{\psi}_n \right] \qquad \text{(outside)} \qquad (15)$$

Now, for the solution $\bar{\chi}$ on $\alpha = \pm\alpha_o$

$$u_\alpha = \frac{A\ \mu_o}{2\sqrt{2}G^2}\ \Sigma\left[-\bar{q}_o S_n(\alpha_o)P_n(p) - \mu_o^2 k C_n(\alpha_o)P_n(p)\right]\cos(n\pi),$$

$$u_\beta = \frac{A\ \mu_o}{2\sqrt{2}G^2}\ \bar{P}\ \Sigma\left[C_n(\alpha_o)P_n(p) - 2\ \mu_o^2 C_n(\alpha_o)P'_n(p)\right]\cos(n\pi),$$

$$\sigma_\alpha = \frac{\sqrt{2}A\ \mu_o}{G} \Sigma\Bigg\{\Big[\ \frac{k^2+1}{4}\ \mu_o^4 + \frac{\bar{q}_o^2}{2} - \frac{\bar{p}^2}{4}\Big] C_n(\alpha_o)P_n(p) + \ \bar{p}^2\ \mu_o C_n(\alpha_o)P'_n(p) +$$

$$k\bar{q}_o\ \mu_o^2 S_n(\alpha_o)P_n(p)\Bigg\}\cos(n\pi)$$

$$\tau_{\alpha\beta} = \frac{\sqrt{2}A\ \mu_o}{G}\ \bar{p}\ \Sigma\Big[-\ \frac{3}{4}\ \bar{q}_o S_n(\alpha_o)P_n(p)\ +\ \frac{3}{2}\ \bar{q}_o\ \mu_o^2 S_n(\alpha_o)P'_n(p)$$

$$-\ \frac{3}{4}\ k\ \mu_o^2 C_n(\alpha_o)P_n(p) + \ \frac{1}{2}\ k\ \mu_o^4 C_n(\alpha_o)P'_n(p)\Big]\cos(n\pi) \tag{16}$$

So, if we satisfy the four conditions of (13) the values of the four constants a_n, a'_n, b_n, b'_n of (14) and (15) are determined. Thus

$$a'_n\mu\Bigg\{\Big[\frac{k^2+1}{4}\ \mu^4 + \ \frac{\bar{q}}{2}^{\ 2} - \ \frac{\bar{p}}{4}^{\ 2}\Big] C_n(\alpha)P_n(p) + \ \bar{p}^2\mu^2 C_n(\alpha)P'_n(p) + k\bar{q}\ \mu^2 S_n(\alpha)P_n(p)\Bigg\}\ -$$

$$-\ b'_n\mu^{-1}\ \Bigg\{\Big[\ \frac{k^2-4+8\nu'}{4}\, q\bar{q}\ \mu^2 + \ \frac{k^2+2-4\nu'}{4}\,\bar{q}p\ \mu^2 + \ \frac{7-8\nu'}{4}\,\bar{q}^3\Big] S_n(\alpha)P_n(p)\ +$$

$$+ (1-2\nu')\bar{q}\ \bar{p}^2\mu^4 S_n(\alpha)P'_n(p) + k\left[\bar{q}^2+(1-\nu')(qp-1)\right]\mu^2 C_n(\alpha)P_n(p)\Big\}$$

$$= \frac{\sqrt{2}A\ \mu_o}{G}\left\{\left[\frac{k^2+1}{4}\mu_o^4 + \frac{\bar{q}_o^2}{2} - \frac{\bar{p}^2}{4}\right]C_n(\alpha_o)P_n(p) + \bar{p}^2\mu_o C_n(\alpha_o)P'_n(p) +\right.$$

$$\left. + k\bar{q}_o\mu_o^2 S_n(\alpha_o)P_n(p)\right\}\cos(n\pi) + a_n\mu\left\{\left[\frac{k^2+1}{4}\mu^4 + \frac{\bar{q}^2}{2} - \frac{\bar{p}^2}{4}\right]C_n(\alpha)P_n(p) +\right.$$

$$\left. + \bar{p}^2\mu^2 C_n(\alpha)P'_n(p) + k\bar{q}\mu^2 S_n(\alpha)P_n(p)\right\} - b_n\mu^{-1}\left\{\left[\frac{k^2-4+8\nu}{4}q\bar{q}\ \mu^2 + \frac{k^2+2-4\nu}{4}\right.\right.$$

$$\left.\bar{q}p\ \mu^2 + \frac{7-8\nu}{4}\bar{q}^3\right]S_n(\alpha)P_n(p) + (1-2\nu)\bar{q}\ \bar{p}^2\mu^2 S_n(\alpha)P'_n(p) + k\left[\bar{q}^2+(1-\nu)(qp-1)\right]$$

$$\mu^2 C_n(\alpha)P_n(p)\Big\} \qquad (17)$$

$$a'_n\bar{p}\mu\left[\frac{3}{4}\bar{q}C_n(\alpha)P_n(p) - \frac{3}{2}\bar{q}\mu^2 C_n(\alpha)P'_n(p) + \frac{3}{4}k\mu^2 S_n(\alpha)P_n(p) - \frac{1}{2}k\ \mu^4 S_n(\alpha)P'_n(p)\right]$$

$$+ b'_n \bar{p}\mu^{-1}\left\{\left[\frac{1-2\nu'}{2} q\mu^2 - \frac{7-8\nu'}{4}\bar{q}^{-2}\right]S_n(\alpha)P_n(p) + \frac{1}{2}\left[3\bar{q}^{2} + (2-4\nu')(qp-1)\right]\mu^2 S_n(\alpha)\right.$$

$$\left. P'_n(p) - \frac{k(5-4\nu')}{4}\bar{q}\mu^2 C_n(\alpha)P_n(p) + \frac{1}{2}k\bar{q}\mu^4 C_n(\alpha)P'_n(p)\right\}$$

$$= \frac{\sqrt{2}A\,\mu_o}{G}\,\bar{p}\left[-\frac{3}{4}\bar{q}_o S_n(\alpha_o)P_n(p) + \frac{3}{2}\bar{q}_o\,\mu_o^2 S_n(\alpha_o)P'_n(p) - \frac{3}{4}k\,\mu_o^2 C_n(\alpha_o)P_n(p) + \right.$$

$$\left.\frac{1}{2}k\,\mu_o^4 C_n(\alpha)P'_n(p)\right]\cos(n\pi) + a_n\bar{p}\mu\left[\frac{3}{4}\bar{q}C_n(\alpha)P_n(p) - \frac{3}{2}\bar{q}\,\mu^2 C_n(\alpha)P'_n(p) + \right.$$

$$\left.\frac{3}{4}k\,\mu^2 S_n(\alpha)P_n(p) - \frac{1}{2}k\,\mu^4 S_n(\alpha)P'_n(p)\right] + b_n\bar{p}\mu^{-1}\left\{\left[\frac{1-2\nu}{2}q\,\mu^2 - \frac{7-8\nu}{4}\bar{q}^{-2}\right]S_n(\alpha)\right.$$

$$P_n(p) + \frac{1}{2}\left[3\bar{q}^{-2} + (2-4\nu)(qp-1)\right]\mu^2 S_n(\alpha)P'_n(p) - \frac{k(5-4\nu)}{4}\bar{q}\,\mu^2 C_n(\alpha)P_n(p) +$$

$$\left.\frac{1}{2}k\bar{q}\,\mu^4 C_n(\alpha)P'_n(p)\right\} \qquad (18)$$

$$a'_n \frac{\mu}{2G_1}\left[\bar{q}C_n(\alpha)P_n(p)+k\,\mu^2 S_n(\alpha)P_n(p)\right]-\frac{b'_n}{4G_1\mu}\left[\left\{(7-8\nu')(qp-1)+q\mu^2\right\}S_n(\alpha)\right.$$

$$\left.P_n(p)+k\bar{q}\,\mu^2 C_n(\alpha)P_n(p)\right]=\frac{A\,\mu_o}{2\sqrt{2}G^2}\left[\bar{q}C_n(\alpha_o)P_n(p)-k\,\mu_o^2 C_n(\alpha_o)P_n(p)\right]$$

$$\cos(n\pi)+a_n\frac{\mu}{2G}\left[\bar{q}C_n(\alpha)P_n(p)+k\,\mu^2 S_n(\alpha)P_n(p)\right]-\frac{b_n}{4G\mu}\left[\left\{(7-8\nu)(qp-1)+q\mu^2\right\}\right.$$

$$\left.S_n(\alpha)P_n(p)+k\bar{q}\,\mu^2 C_n(\alpha)P_n(p)\right] \qquad (19)$$

and

$$a'_n \frac{\bar{p}\mu}{4G_1}\left[C_n(\alpha)P_n(p)-2\mu^2 C_n(\alpha)P'_n(p)\right]-b'_n\frac{\bar{p}\,\bar{q}}{4G_1\mu}\left[(7-8\nu')S_n(\alpha)P_n(p)-\right.$$

$$\left.2\mu^2 S_n(\alpha)P'_n(p)\right]=\frac{A\,\mu_o}{2\sqrt{2}G^2}\,\bar{p}\left[C_n(\alpha_o)P_n(p)-2\,\mu_o^2 C_n(\alpha_o)P'_n(p)\right]\cos(n\pi)+$$

$$\frac{\bar{p}\mu}{4G}\,a_n\left[C_n(\alpha)P_n(p)-2\mu^2 C_n(\alpha)P'_n(p)\right]-\frac{\bar{p}\bar{q}}{4G\mu}\,b_n\left[(7-8\nu)S_n(\alpha)P_n(p)-2\mu^2\right.$$

$$S_n(\alpha)P'_n(p)\Big] \tag{20}$$

where

ν' = Poission's ratio for the material of the inclusion,

G_1 = Coefficients of elasticity in shear for the material of inclusion.

Solving (17)-(20), we get

$$a'_n = \frac{A}{G}\frac{\Delta_1}{\Delta} \; ; \qquad b'_n = \frac{A}{G}\frac{\Delta_2}{\Delta}$$

$$a_n = \frac{A}{G}\frac{\Delta_3}{\Delta} \; ; \qquad b_n = \frac{A}{G}\frac{\Delta_4}{\Delta}$$

where $\Delta, \Delta_1, \Delta_2, \Delta_3$ and Δ_4 denote the determinants, the elements of which are the coefficients of a'_n, b'_n, a_n, b_n and the constant term of the equations (17)-(20). The values of these coefficients together with the equations (14),(15),(2),(3) and (9)–(12) constitute the complete solution of the problem.

The components of displacement and stress on the surface $\alpha = \pm\alpha_o$ are

$$\frac{[u_\alpha]_{\alpha}}{A/G} = \pm\alpha_o = \Sigma \frac{a'_n \; \mu_o}{A/G \;\; 4G_1}\left[\bar{q}_o C_n(\alpha_o) + k\mu_o^2 S_n(\alpha_o)\right]P_n(p) - \Sigma \frac{1}{A/G}\frac{b'_n}{4G_1\mu_o}$$

$$\left[\left\{(7-8\nu')(q_o p-1)+ q_o\mu_o^2\right\} S_n(\alpha_o)P_n(p)+ k\bar{q}_o\mu_o^2 C_n(\alpha_o)P_n(p)\right],$$

$$\frac{[u_\beta]_\alpha}{A/G} = \pm\alpha_o = \Sigma\, \frac{a'_n}{A/G}\, \mu_o\bar{p}\left[C_n(\alpha_o)P_n(p) - 2\mu_o^2 C_n(\alpha_o)P'_n(p)\right] - \Sigma\, \frac{b'_n}{A/G}\, \frac{\bar{q}_o\bar{p}}{\mu_o}$$

$$\left[(7-8\nu')S_n(\alpha_o)P_n(p)-2\, \mu_o^2 S_n(\alpha_o)P'_n(p)\right],$$

$$\frac{[\sigma_\alpha]_\alpha}{A/G} = \pm\alpha_o = \Sigma\, \frac{a'_n}{A/G}\, \mu_o\left\{\left[\frac{k^2+1}{4}\, \mu_o^4 + \frac{\bar{q}_o^2}{2} - \frac{\bar{p}^2}{4}\right] C_n(\alpha_o)P_n(p)+\bar{p}^2\mu_o^2 C_n(\alpha)P'_n(p)\right.$$

$$\left. k\bar{q}_o\mu_o^2 S_n(\alpha_o)P_n(p)\right\} - \Sigma\, \frac{b'_n}{A/G}\, \mu_o^{-1}\left\{\left[\frac{k^2-4+8\nu'}{4}\, q_o\bar{q}_o\mu_o^2 + \frac{k^2+2-4\nu'}{4}\, \bar{q}_o p\mu_o^2 + \frac{7-8\nu'}{4} \times\right.\right.$$

$$\left. q_o^3\right] S_n(\alpha_o)P_n(p) + (1-2\nu')\bar{q}_o p^2\mu_o^2 S_n(\alpha_o)P'_n(p) + k\left[\bar{q}_o^2 + (1-\nu')(q_o p-1)\right]\times$$

$$\left. \mu_o^2 C_n(\alpha_o)P_n(p)\right\}$$

$$\frac{[\tau_{\alpha\beta}]_{\alpha}}{A\ /G} = \pm\alpha_o = \Sigma \frac{a'_n}{A/G} \mu_o \bar{p}\Big[\frac{3}{4}\,\bar{q}_o C_n(\alpha_o)P_n(p) - \frac{3}{2}\,\bar{q}_o\mu_o^2 C_n(\alpha_o)P'_n(p) + \frac{3}{4}k\,\mu_o^2$$

$$S_n(\alpha_o)P_n(p) - \frac{1}{2}k\,\mu_o^4 S_n(\alpha_o)P'_n(p)\Big] + \Sigma \frac{b'_n}{A/G} \mu_o^{-1}\bar{p}\,\Big\{\Big[\,\frac{1-2\nu}{2}\,q_o\mu_o^2 - \frac{7-8\nu'}{4}\,\bar{q}_o^{-2}\Big]$$

$$S_n(\alpha_o)P_n(p) + \frac{1}{2}\Big[\,3\bar{q}_o^{-2} + (2-4\nu')(q_o p-1)\Big]\mu_o^2 S_n(\alpha_o)P'_n(p) - \frac{k(5-4\nu')}{4}\,\bar{q}_o\mu_o^2$$

$$C_n(\alpha_o)P_n(p) + \frac{1}{2}\,k\bar{q}_o\mu_o^4 C_n(\alpha_o)P'_n(p)\Big\}$$

where ν' and G_1 are Poission's ratio and coefficients of elasticity in shear respectively for the material of the inclusion.

3. Numerical Results

Here the numerical calculations are made for the case n=0, on the surface $\alpha = \pm\alpha_o = 1$ of the spherical inclusions taking numerical values [6] $\nu' = 1/3$, $\nu = 1/4$ and $G/G_1 = 2$.

Numerical values of u_α, u_β and σ_α for different values of β between 0° and 180° are given in the table below :

β	0°	30°	60°	120°	150°	180°
$\dfrac{u_\alpha}{A/8G_1^2}$	-10.453	-1.037	3.6904	-8.429	-17.371	-9.117
$\dfrac{u_\beta}{A/8G_1^2}$	-9.967	-4.241	1.381	3.629	-3.709	9.252
$\dfrac{\sigma_\alpha}{A/2G_1}$	-8.241	-5.975	3.333	-12.146	-35.153	13.541

CHAPTER - II

THERMAL STRESSES IN LAYERED MEDIA

Paper 1 : Stress concentration in an elastic layered media (Two layered case)

Paper 2 : Stress concentration in an elastic layered media (Three layered case)

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STRESS CONCENTRATION IN ELASTIC LAYERED MEDIA DUE TO THERMAL EFFECT (TWO LAYERED CASE)

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1.Statement Of The Problem

Let the material under consideration occupy the lower half of the plane z=0 and the axis of z being taken positive when drawn into the material. Physical quantities involved here are all symmetrical about z-axis. Lower boundary of the upper layer is given by the plane $z=h_1$ and the underlying mass extended to infinity. interfaces are supposed to be perfectly rough so that stresses and displacements are continuous across it.

2.Method Of Solution

Since the physical quantities involved in the problem are all symmetrical about z-axis, so four non-zero stress components $\sigma_r, \sigma_\theta, \sigma_z, \tau_{zr}$ and two non-zero displacements u_r, u_z are retained in terms of the stress function ϕ satisfying the differential equation [9]

$$\nabla^4\phi = \left\{ \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right\}\phi = 0 \tag{1}$$

The stress and displacement components are given by [17]

$$\sigma_r = \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right] \tag{2}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - r^{-1}\frac{\partial\phi}{\partial r}\right] \tag{3}$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{4}$$

$$\tau_{zr} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{5}$$

$$u_r = -\frac{1+\nu}{E}\frac{\partial^2\phi}{\partial r\partial z} \tag{6}$$

$$u_z = \frac{1+\nu}{E}\left[(1-2\nu)\nabla^2\phi + \frac{\partial^2\phi}{\partial r^2} + r^{-1}\frac{\partial\phi}{\partial r}\right] \tag{7}$$

Here the stress function ϕ due to Hankel transform [14] will take the form G, defined as

$$G(\xi, z) = \int_0^\infty r\phi(r,z)\, J_o(\xi r)\, dr, \quad (\xi>0, z>0) \tag{8}$$

where $J_o(\xi r)$ denotes the Bessel function [19] of order zero, whose inverse is given by

$$\phi(r,z) = \int_0^\infty \xi G(\xi, z)\, J_o(\xi r)\, d\xi \tag{9}$$

Applying Hankel transform on (1) and (4)-(7) we get

$$\left(\frac{d^2}{dz^2} - \xi^2 \right)^2 G(\xi, z) = 0 \tag{10}$$

$$\int_0^\infty r\sigma_z \, J_o(\xi r) \, dr = (1-\nu) \frac{d^3 G}{dz^3} - (2-\nu)\xi^2 \frac{dG}{dz} \tag{11}$$

$$\int_0^\infty r\tau_{zr} \, J_1(\xi r) \, dr = \left[\nu \frac{d^2 G}{dz^2} + (1-\nu)\xi^2 \, G \right] \tag{12}$$

$$\int_0^\infty ru_r \, J_1(\xi r) \, dr = \frac{1+\nu}{E} \, \xi \, \frac{dG}{dz} \tag{13}$$

$$\int_0^\infty ru_z \, J_o(\xi r) \, dr = \frac{1+\nu}{E} \left[(1-2\nu)\frac{d^2 G}{dz^2} - 2(1-\nu)\xi^2 \, G \right] \tag{14}$$

Computing inverses of (11)-(14) we get

$$\sigma_z = \int_0^\infty \xi \left[(1-\nu) \frac{d^3 G}{dz^3} - (2-\nu)\xi^2 \frac{dG}{dz} \right] J_o(\xi r) \, d\xi \tag{15}$$

$$\tau_{zr} = \int_0^\infty \xi^2 \left[\nu \frac{d^2 G}{dz^2} + (1-\nu)\xi^2 \, G \right] J_1(\xi r) \, d\xi \tag{16}$$

$$u_r = \frac{1+\nu}{E}\int_0^{\infty} \xi^2 \frac{dG}{dz} J_1(\xi r)\, d\xi \tag{17}$$

$$u_z = \frac{1+\nu}{E}\int_0^{\infty} \left[\xi(1-2\nu)\frac{d^2G}{dz^2} - 2(1-\nu)\xi^2 G\right] J_0(\xi r)\, d\xi \tag{18}$$

Let the solution of (10) be given by

$$G(\xi, z) = (A+Bz)(2\sinh\xi z + e^{-\xi z}) + (C+Dz)(2\cosh\xi z - e^{\xi z})$$

where A, B, C and D are functions of ξ to be determined from the suitable boundary conditions.

3. Solution Of The Problem

To determine the stresses, the potential of thermoelastic displacement ψ, related by the equations

$$\frac{\partial\psi}{\partial r} = (u_r)_T$$

$$\frac{\partial\psi}{\partial z} = (u_z)_T \tag{19}$$

is considered in the steady state of the temperature field given by

[3]

$$\nabla^2 T = 0 \tag{20}$$

Since upper layer and the underlying mass have different thermal properties, different potential displacement functions are chosen. From the stress-strain relations of thermo-elasticity and the equation of equilibrium [17], we have

$$\nabla^2 \psi_i = \beta_i T \tag{21}$$

where

$$\beta_i = \frac{1+\nu_i}{1-\nu_i}\,\alpha_i$$

Applying Hankel transform on (20) and (21) we obtain

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) M(\xi, z) = 0 \tag{22}$$

and

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) L_i(\xi, z) = \beta_i M(\xi, z), \quad i=1,2 \tag{23}$$

where $$M(\xi, z) = \int_0^\infty r T J_o(\xi r)\, dr \tag{24}$$

$$L_i(\xi, z) = \int_0^\infty r \psi_i J_o(\xi r)\, dr \tag{25}$$

The corresponding thermal stress components $(\sigma_z)_T, (\tau_{zr})_T$ and displacement components $(u_r)_T, (u_z)_T$ are given by

$$(\sigma_z)_{T_i} = 2\mu_i \left\{ \frac{\partial^2 \psi_i}{\partial z^2} - \nabla^2 \phi_i \right\} \tag{26}$$

$$(\tau_{zr})_{T_i} = 2\mu_i \frac{\partial^2 \psi_i}{\partial r \partial z} \tag{27}$$

$$(u_r)_{T_i} = \frac{\partial \psi_i}{\partial r} \tag{28}$$

$$(u_z)_{T_i} = \frac{\partial \psi_i}{\partial z} \tag{29}$$

Now we consider the transformations of (26)-(29) into relation involving $L_i(\xi, z)$ and $M(\xi, z)$. The results of computations are

$$\int_0^\infty r(\sigma_z)_{T_i} J_o(\xi r)\, dr = 2\mu_i \xi^2 L_i(\xi, z) \tag{30}$$

$$\int_0^\infty r(\tau_{zr})_{T_i} J_1(\xi r)\, dr = -2\mu_i \xi \frac{d}{dz} L_i(\xi, z) \tag{31}$$

$$\int_0^\infty r(u_r)_{T_i} J_1(\xi r)\ dr = -\xi\ L_i(\xi, z) \tag{32}$$

$$\int_0^\infty r(u_z)_{T_i} J_o(\xi r)\ dr = \frac{d}{dz}\ L_i(\xi, z) \tag{33}$$

Applying Hankel inverse transform, we get

$$(\sigma_z)_{T_i} = 2\mu_i \int_0^\infty \xi^3 L_i(\xi, z) J_o(\xi r)\ d\xi \tag{34}$$

$$(\tau_{zr})_{T_i} = -2\mu_i \int_0^\infty \xi^2\ \frac{d}{dz}\ L_i(\xi, z) J_1(\xi r)\ d\xi \tag{35}$$

$$(u_r)_{T_i} = -\int_0^\infty \xi^2 L_i(\xi, z) J_1(\xi r)\ d\xi \tag{36}$$

$$(u_z)_{T_i} = \int_0^\infty \xi\ \frac{d}{dz}\ L_i(\xi, z) J_o(\xi r)\ d\xi \tag{37}$$

Let the solution of (23) be

$$L_i(\xi, z) = \frac{\beta_i}{2\xi} A_o(1-z)e^{-\xi z} \tag{38}$$

then the solution of (22) is

$$M(\xi,z) = A_o e^{-\xi z} \tag{39}$$

where A_o is a function of ξ.

Applying (38) we have from (30)-(33)

$$\int_0^\infty r(\sigma_z)_{T_i} J_o(\xi r)\ dr = \mu_i \beta_i \xi A_o (1-z) e^{-\xi z} \tag{40}$$

$$\int_0^\infty r(\tau_{zr})_{T_i} J_1(\xi r)\ dr = -\mu_i \beta_i A_o (z\xi - \xi - 1) e^{-\xi z} \tag{41}$$

$$\int_0^\infty r(u_r)_{T_i} J_1(\xi r)\ dr = -\frac{1}{2} \beta_i A_o (1-z) e^{-\xi z} \tag{42}$$

$$\int_0^\infty r(u_z)_{T_i} J_o(\xi r)\ dr = \frac{1}{2\xi} \beta_i (z\xi - \xi - 1) A_o e^{-\xi z} \tag{43}$$

Taking solution of (10) for different layers [59] as:

For the upper layer

$$G_1(\xi,z) = (A_1 + B_1 z)(2\sinh \xi z + e^{-\xi z}) + (C_1 + D_1 z)(2\cosh \xi z - e^{\xi z}) \tag{44}$$

For the underlying mass

$$G_2(\xi,z) = (A_2 + B_2 z) e^{-\xi z} + (C_2 + D_2 z) e^{\xi z}, \quad (\xi > 0, z > 0) \tag{45}$$

It is to be noted that stress and displacements in the underlying mass vanish as z tends to infinity. So, the components of stress and displacement for the upper layer obtained from (11)-(14) as

$$\int_0^\infty r(\sigma_z)_1 \, J_o(\xi r) \, dr =$$

$$= (1-2\nu_1)\xi^2 B_1(2\sinh \xi z + e^{-\xi z}) + (1-2\nu_1)\xi^2 D_1(2\cosh \xi z - e^{\xi z})$$

$$+\xi^3(A_1+B_1 z)(e^{-\xi z} - 2\cosh \xi z) + \xi^3(C_1+D_1 z)(e^{\xi z} - 2\sinh \xi z) \qquad (46)$$

$$\int_0^\infty r(\tau_{zr})_1 J_1(\xi r)dr = 2\nu_1\xi^2 B_1(2\cosh \xi z - e^{-\xi z}) + 2\nu_1\xi^2 D_1(2\sinh \xi z - e^{\xi z})$$

$$+\xi^3(A_1+B_1 z)(e^{-\xi z} + 2\sinh \xi z) + \xi^3(C_1+D_1 z)(2\cosh \xi z - e^{\xi z}) \qquad (47)$$

$$\int_0^\infty r(u_r)_1 \, J_1(\xi r)dr = \frac{1+\nu_1}{E_1} \xi \Big[B_1(2\sinh \xi z + e^{-\xi z}) + D_1(2\cosh \xi z - e^{\xi z})$$

$$+\xi(A_1+B_1 z)(2\cosh \xi z - e^{\xi z}) + \xi(C_1+D_1 z)(2\sinh \xi z - e^{-\xi z}) \qquad (48)$$

$$\int_0^\infty r(u_z)_1\, J_o(\xi r)dr = \frac{1+\nu_1}{E_1}\,\xi\Big[2(1-2\nu_1)B_1(2\cosh\xi z - e^{-\xi z}) + 2(1-2\nu_1)D_1$$

$$(2\sinh\xi z - e^{\xi z}) - \xi(A_1+B_1 z)(2\sinh\xi z + e^{-\xi z}) - \xi(C_1+D_1 z)(2\cosh\xi z - e^{\xi z})$$

(49)

Stresses and displacements for the underlying mass

$$\int_0^\infty r(\sigma_z)_2\, J_o(\xi r)\, dr = \Big[(A_2+B_2 z)e^{-\xi z} - (C_2+D_2 z)e^{\xi z}\Big]\xi^3 +$$

$$+(1-2\nu_2)\xi^2\Big[B_2 e^{-\xi z} + D_2 e^{\xi z}\Big] \qquad (50)$$

$$\int_0^\infty r(\tau_{zr})_2\, J_1(\xi r)\, dr = \Big[(A_2+B_2 z)e^{-\xi z} + (C_2+D_2 z)e^{\xi z}\Big]\xi^3$$

$$-2\nu_2\xi^2\Big[B_2 e^{-\xi z} - D_2 e^{\xi z}\Big] \qquad (51)$$

$$\int_0^\infty r(u_r)_2\, J_1(\xi r)\, dr = -\frac{1+\nu_2}{E_2}\Big\{\Big[\xi(A_2+B_2 z)+B_2\Big]e^{-\xi z} - \Big[\xi(C_2+D_2 z)+D_2\Big]e^{\xi z}\Big\}$$

(52)

$$\int_0^{\infty} r(u_z)_3\ J_o(\xi r)\ dr = -\frac{1+\nu_2}{E_2}\left\{\left[(A_2+B_2 z)e^{-\xi z}+(C_2+D_2 z)e^{\xi z}\right]\xi^2+\right.$$

$$\left.+2(1-2\nu_2)\xi\left[B_2 e^{-\xi z}-D_2 e^{\xi z}\right]\right\} \qquad (53)$$

4.Boundary Condition

In order to nullify the stresses on the boundaries the following conditions are to be satisfied:

At $z=0$,

$$-(u_r)_{T_1} = (u_r)_1$$

$$-(u_z)_{T_1} = (u_z)_1$$

$$-(\sigma_z)_{T_1} = (\sigma_z)_1$$

$$-(\tau_{zr})_{T_1} = (\tau_{zr})_1 \qquad (54)$$

At $z=h_1$,

$$-(u_r)_{T_2} = (u_r)_2$$

$$-(u_z)_{T_2} = (u_z)_2$$

$$-(\sigma_z)_{T_2} = (\sigma_z)_2$$

$$-(\tau_{zr})_{T_2} = (\tau_{zr})_2 \qquad (55)$$

Conditions (54),(55) with the help of (40)-(43) and (46)-(49) give the values of constants as

$$A_1 = A'_1 A_o \; , \; A'_1 = S_1 - S_2 + S_3$$

$$B_1 = B'_1 A_o , \; B'_1 = S_2 - S_3$$

$$C_1 = C'_1 A_o \; , \; C'_1 = S_1 - S_2 - S_3$$

$$D_1 = D'_1 A_o \; , \; D'_1 = - S_3$$

where

$$S_1 = \frac{2\xi + 1}{\xi} \beta_1 \mu_1$$

$$S_2 = \frac{\alpha_1}{(1-\nu_1)^2 \xi^2} \left[\mu_1 (1+\nu_1) - \xi E_1 \right]$$

$$S_3 = \frac{\alpha_1}{(1-\nu_1) \xi^3} \left[\mu_1 (1+\nu_1) + (2\xi + 1) E_1 \right]$$

$$D_2 = \frac{\beta_2 r_1}{4(1-\nu_2^2)\xi^2 s_1} \left[\mu_2 (1+\nu_2)(\xi h_1 - h_1 - \xi) + (1 + 2\xi - 2\xi h_1) E_2 \right] A_o = D'_2 A_o$$

$$B_2 = \frac{\beta_2}{4(1-\nu_2^2)\xi^2} \left[\mu_2 (1+\nu_2)(2+\xi - \xi h_1 - h_1) + E_2 \right] A_o = B'_2 A_o$$

$$A_2 = \frac{\beta_2 E_2}{4(1+\nu_2) \; \xi^3} \Big\{ \xi^3 (\xi + 1)(h_1 - 1) - \xi^2 \mu_2 (1+\nu_2)$$

$$+ \frac{1}{2\xi r_1}\left[(2h_1\xi - 4\nu_2 + 1) r_1 B_2' + (1-2\nu_2) s_1 D_2'\right]\Bigg\} A_o$$

$$C_2 = \left\{ \frac{\beta_2 E_2}{(1-\nu_2) s_1} (1-h_1) r_1 + \frac{r_1}{s_1}\left[\xi A_2' + (\xi h_1 - 1) B_2'\right] - (\xi h_1 + 1) D_2' \right\} A_o$$

Thus the constants A_i', B_i', C_i', D_i', $i=1,2$ are independent of A_o.
So, on the surface $z=0$, the resultant displacement and stress components are

$$\int_0^\infty r(u_r)_{R_1} J_1(\xi r)\, dr = \int_0^\infty r[(u_r)_1 + (u_r)_{T_1}] J_1(\xi r)\, dr =$$

$$= \left[\frac{1+\nu_1}{E_1} (\xi A_1' + B_1' - \xi C_1' + D_1') - \beta_1 \right] A_o \tag{56}$$

$$\int_0^\infty r(u_z)_{R_1} J_o(\xi r)\, dr = \int_0^\infty r[(u_z)_1 + (u_z)_{T_1}] J_o(\xi r)\, dr =$$

$$= - \left[\frac{1+\nu_1}{E_1} \xi \left\{ \xi (A_1' + C_1') - 2(1-2\nu_1)(B_1' + D_1') \right\} + \frac{\xi + 1}{\xi} \beta_1 \right] A_o \tag{57}$$

$$\int_0^\infty r(\sigma_z)_{R_1} J_o(\xi r)\, dr = \int_0^\infty r[(\sigma_z)_1+(\sigma_z)_{T_1}] J_o(\xi r)\, dr =$$

$$= \xi\left[\mu_1\beta_1-\xi^2\left\{(A'_1-C'_1)+(1-2\nu_1)(B'_1+D'_1)\right\}\right]A_o \qquad (58)$$

$$\int_0^\infty r(\tau_{zr})_{R_1} J_1(\xi r)\, dr = \int_0^\infty r[(\tau_{zr})_1+(\tau_{zr})_{T_1}] J_1(\xi r)\, dr =$$

$$= \left[(1+\xi)\mu_1\beta_1+\xi^2\left\{(A'_1+C'_1)+2\nu_1(B'_1-D'_1)\right\}\right]A_o \qquad (59)$$

At $z=h_1$,

$$\int_0^\infty r(u_r)_{R_2} J_1(\xi r)dr = \int_0^\infty r[(u_r)_2+(u_r)_{T_2}] J_1(\xi r)dr =$$

$$=\left[\beta_2(h_1-1)\xi r_1+\frac{1+\nu_2}{E_2}\left\{\left[-\xi A'_2+(1-h_1)B'_2\right]r_1+\left[\xi C'_2+(1+h_1)D'_2\right]s_1\right\}\right]A_o \qquad (60)$$

$$\int_0^\infty r(u_z)_{R_2} J_o(\xi r)dr = \int_0^\infty r[(u_z)_2+(u_z)_{T_2}] J_o(\xi r)dr =$$

$$= -\left[\frac{\xi h_1 - \xi - 1}{\xi}\beta_2 r_1 + \frac{1+\nu_2}{E_2}\left\{(4\nu_2 - \xi h_1 - 2)(r_1 B'_2 + s_1 D'_2) - (A'_2 - C'_2)\right\}\right] A_o \quad (61)$$

$$\int_0^\infty r(\sigma_z)_{R_2} J_o(\xi r)\, dr = \int_0^\infty r[(\sigma_z)_2 + (\sigma_z)_{T_2}] J_o(\xi r)\, dr = \Big[\mu_2\beta_2(1-h_1)\xi r_1 +$$

$$\xi^2(A'_2 r_1 - C'_2 s_1) + r_1\xi(\xi h_1 - 2\nu_2 + 1)B'_2 - s_1\xi(\xi h_1 + 2\nu_2 - 1)D'_2\Big] A_o \quad (62)$$

$$\int_0^\infty r(\tau_{zr})_{R_2} J_1(\xi r)\, dr = \int_0^\infty r[(\tau_{zr})_2 + (\tau_{zr})_{T_2}] J_1(\xi r)\, dr = \Big[\mu_2\beta_2(\xi h_1 - \xi - 1) r_1 +$$

$$\xi^3(A'_2 r_1 + C'_2 s_1) + r_1\xi^2(\xi h_1 - 2\nu_2)B'_2 + s_1\xi(\xi h_1 + 2\nu_2)D'_2\Big] A_o \quad (63)$$

5. Flux Of Heat On The Boundaries

Let the flux of heat in a region of the surface $z=0$, distributed through layers in the underlying mass be

$$\frac{\partial T}{\partial z} = f(r/a), \quad 0<r<a \quad (64)$$

$$= 0, \quad r>a$$

using dimensionless variables

$$\xi A_o(\xi) = aX(\xi a), \quad \eta=\xi a, \quad \rho=r/a, \quad \zeta=z/a$$

where a is some length and η, a new variable of integration, we get from (24) and (39), on z=0,

$$\frac{\partial T}{\partial z} = -a^{-1} \int_0^\infty \eta X(\eta) J_o(\rho\eta) \, d\eta \tag{65}$$

By Hankel inversion theorem

$$X(\eta) = -a^{-1} \int_0^1 \rho f(\rho) J_o(\rho\eta) \, d\rho \tag{66}$$

For a simple physical situation we consider a linear temperature distribution $f(\rho) = K\rho$, K = constant, then

$$X(\eta) = - \frac{K}{a\eta^2} J_1(\eta) \tag{67}$$

So, the unknown $A_o(\xi)$ being known the problem is completely solved.

6·Numerical Results

If the upper layer be concrete pavement, and the underlying mass be natural soil, then elastic constants for those materials are [8]

$E_1 = 2.18\times10^{8}$ gms/cm^2 , $\alpha_1 = -5\times10^{-6}/0^{\circ}C$

$E_2 = 1.1\times10^{8}$ gms/cm^2 , $\alpha_2 = 7.5\times10^{-6}/0^{\circ}C$

$\nu_1 = 0.15$, $\mu_1 = 0.94\times10^{8}$ gms/cm^2

$\nu_2 = 0.25$, $\mu_2 = 0.43\times10^{8}$ gms/cm^2

$K_1 = 6.4\times10^{-3}$, $K_2 = 6.7\times10^{-3}$

So, evaluating constants for a given value of η when a=1 and h_1=2, we have

$$S_1 = 10.8\times10^{2}; \quad S_2 = 7.605\times10^{2}; \quad S_9 = -50.6\times10^{2}$$

Thus

$$A'_1 = -32.1845\times10^{2}, B'_1 = -42.9945\times10^{2}, C'_1 = 53.8045\times10^{2}, D'_1 = 50.6\times10^{2},$$

$$A'_2 = 0.9794\times10^{2}, B'_2 = 1.8723\times10^{2}, C'_2 = 1.2163\times10^{2}, D'_2 = -7.559\times10^{2},$$

So, applying dimensionless variables on (56)-(63) substituting the value of $X(\eta)$ from (67) and the values of constants , we get

$$\int_0^{\infty} r(u_r)_R \, J_1(\eta r)dr = \int_0^{\infty} r[(u_r)_{R_1} + (u_r)_{R_2}] \, J_1(\eta r)dr$$

$$= -0.19593\times10^{-3}K\eta^{-2}J_1(\eta) \tag{68}$$

$$\int_0^\infty r(u_z)_R\ J_o(\eta r)dr = 0.00324\times10^{-3}\ K\eta^{-2}J_1(\eta) \tag{69}$$

$$\int_0^\infty r(\sigma_z)_R\ J_o(\eta r)dr = -99.2487\times10^{2}\ K\eta^{-2}J_1(\eta) \tag{70}$$

Inversions gives

$$|(u_r)_R| = 0.19593\times10^{-3}\ K\int_0^\infty \frac{J_1(\eta)J_1(\rho\eta)}{\eta}\,d\eta \tag{71}$$

$$|(u_z)_R| = 0.00324\times10^{-3}\ K\int_0^\infty \frac{J_1(\eta)J_o(\rho\eta)}{\eta}\,d\eta \tag{72}$$

$$|(\sigma_z)_R| = 99.2483\times10^{2}\ K\int_0^\infty \frac{J_1(\eta)J_o(\rho\eta)}{\eta}\,d\eta \tag{73}$$

Thus

$$|(u_r)_R| = 0.19593\times10^{-3}K \begin{Bmatrix} (2\rho)^{-1}F(1,0;2;\rho^{-2}), & \rho>1 \\ \frac{1}{2}, & \rho=1 \\ .5\rho\ F(1,0;2;\rho^{2}), & \rho<1 \end{Bmatrix} \tag{74}$$

$$|(u_z)_R| = 0.00324\times10^{-3}K \begin{Bmatrix} F(.5,.5;1;\rho^2) & , \rho<1 \\ \frac{2}{\pi}\ , & , \rho=1 \\ .5F(.5,.5;2;\rho^{-2}), & \rho>1 \end{Bmatrix} \tag{75}$$

$$|(\sigma_z)_R| = 99.2483\times10^{2}\ K \begin{Bmatrix} F(.5,.5;1;\rho^2) & , \rho<1 \\ \frac{2}{\pi}\ , & , \rho=1 \\ .5F(.5,.5;1;\rho^{-2}), & \rho>1 \end{Bmatrix} \tag{76}$$

F denotes hypergeometric function.

DISCUSSIONS

Solution of the system of equations (40)-(43) and (46)-(49) for the unknowns A_i, B_i, i=1,2,3,4, depend on A_o, where A_o is a function of ξ. So existence of uniqueness of the solutions depends on the uniqueness of A_o. A_o is unique for a definite type of distribution of heat flux on the boundary. So, the solutions A_i, B_i are unique subject to the condition that A_o is unique.

Here the distribution of heat flux is taken to be linear but it would have been physically more interesting and suitable if it is considered other that linear.

Figures (1) and (2) show how the radial component of displacement and component of stress in the z- direction vary with ρ.

STRESS CONCENTRATION IN ELASTIC LAYERED MEDIA DUE TO THERMAL EFFECT (THREE LAYERED CASE)

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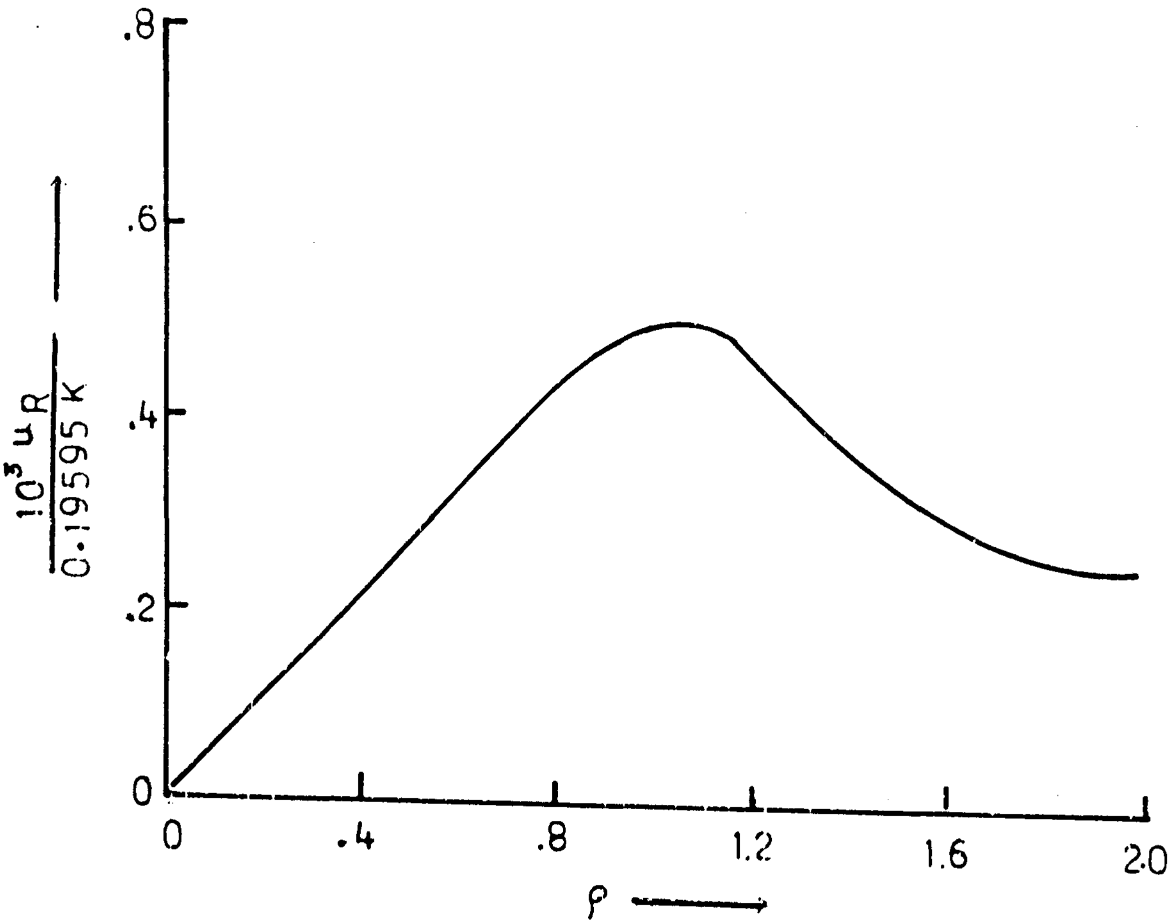

Fig.1

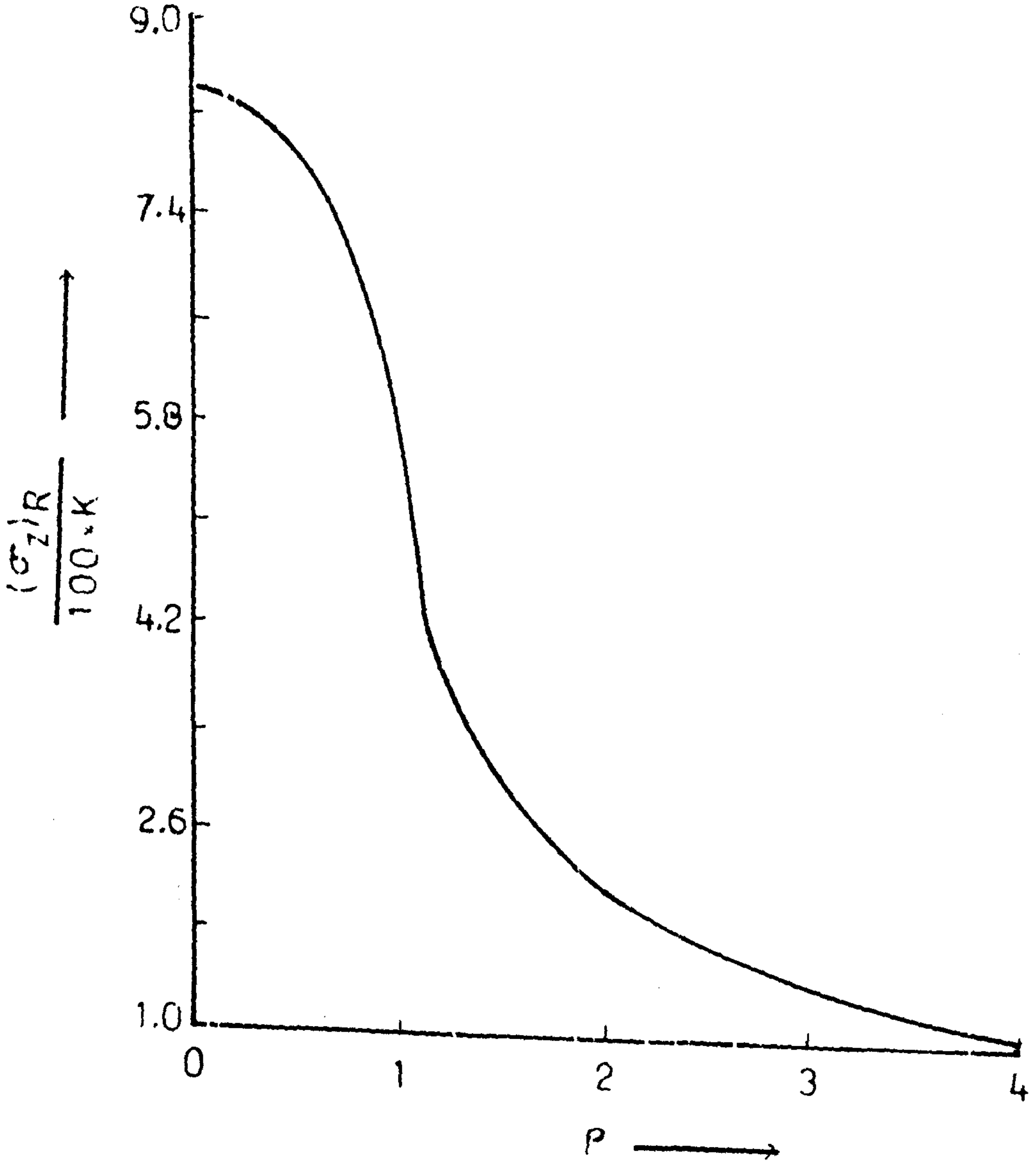

FIG.2 : DISTRIBUTION OF THERMOELASTIC STRESS $(\sigma_z)_R$ IN THE UNDERLYING MASS FOR A TWO LAYERED SYSTEM.

1.Statement Of The Problem

Material under consideration occupy the lower half of the plane $z=0$ and the axis of z being taken positive when drawn into the material. Physical quantities involved here are all symmetrical about z-axis. Lower boundaries of the first and second layers are given by the planes $z=h_1$ and $z=h_2$ and the underlying mass extended to infinity. Interfaces are supposed to be perfectly rough so that stresses and displacements are continuous across it.

2.Method Of Solution

Physical quantities involved in the problem are all symmetrical about z-axis, so four non-zero stress components $\sigma_r, \sigma_\theta, \sigma_z, \tau_{zr}$ and two non-zero displacements u_r, u_z are retained in terms of the stress function ϕ satisfying the differential equation [9]

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)\phi = 0 \tag{1}$$

The stress and displacement components are given by [17]

$$\sigma_r = \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right] \tag{2}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - r^{-1}\frac{\partial\phi}{\partial r}\right] \tag{3}$$

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{4}$$

$$\tau_{zr} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \tag{5}$$

$$u_r = -\frac{1+\nu\partial^2\phi}{E\quad\partial r\partial z} \tag{6}$$

$$u_z = \frac{1+\nu}{E}\left[(1-2\nu)\nabla^2\phi + \frac{\partial^2\phi}{\partial r^2} + r^{-1}\frac{\partial\phi}{\partial r}\right] \tag{7}$$

Here the stress function ϕ due to Hankel transform [14] will take form G, defined as

$$G(\xi, z) = \int_0^\infty r\phi(r,z)\, J_o(\xi r)\, dr, \quad (\xi > 0, z > 0) \tag{8}$$

where $J_o(\xi r)$ denotes the Bessel function [19] of order zero, whose inverse is given by

$$\phi(r,z) = \int_0^\infty \xi G(\xi, z)\, J_o(\xi r)\, d\xi \tag{9}$$

Applying Hankel transform [14] on (1) and (4)-(7) we get

$$\left(\frac{d^2}{dz^2} - \xi^2\right)^2 G(\xi, z) = 0 \tag{10}$$

$$\int_0^\infty r\sigma_z J_o(\xi r)\, dr = (1-\nu)\frac{d^3G}{dz^3} - (2-\nu)\xi^2 \frac{dG}{dz} \tag{11}$$

$$\int_0^\infty r\tau_{zr} J_1(\xi r)\, dr = \left[\nu \frac{d^2G}{dz^2} + (1-\nu)\xi^2\, G\right] \tag{12}$$

$$\int_0^\infty r u_r J_1(\xi r)\, dr = \frac{1+\nu}{E}\,\xi\,\frac{dG}{dz} \tag{13}$$

$$\int_0^\infty r u_z J_o(\xi r)\, dr = \frac{1+\nu}{E}\left[(1-2\nu)\frac{d^2G}{dz^2} - 2(1-\nu)\xi^2\, G\right] \tag{14}$$

Computing inverses of (11)-(14) we get

$$\sigma_z = \int_0^\infty \xi\left[(1-\nu)\frac{d^3G}{dz^3} - (2-\nu)\xi^2\frac{dG}{dz}\right] J_o(\xi r)\, d\xi \tag{15}$$

$$\tau_{zr} = \int_0^\infty \xi^2\left[\nu\frac{d^2G}{dz^2} + (1-\nu)\xi^2\, G\right] J_1(\xi r)\, d\xi \tag{16}$$

$$u_r = \frac{1+\nu}{E}\int_0^\infty \xi^2 \frac{dG}{dz} J_1(\xi r)\, d\xi \tag{17}$$

$$u_r = \frac{1+\nu}{E}\int_0^\infty \left[\xi(1-2\nu)\frac{d^2G}{dz^2} - 2(1-\nu)\xi^2 G\right] J_o(\xi r)\, d\xi \tag{18}$$

Let the solution of (10) be given by

$$G(\xi, z) = (A+Bz)\cosh \xi z + (C+Dz)\sinh \xi z$$

where A,B,C and D are functions of ξ to be determined from the suitable boundary conditions.

3.Solution Of The Problem

To determine the stresses, the potential of thermoelastic displacement ψ, related by the equations

$$\frac{\partial\psi}{\partial r} = (u_r)_T$$

$$\frac{\partial\psi}{\partial z} = (u_z)_T \tag{19}$$

is considered in the steady state of the temperature field given by [3]

$$\nabla^2 T = 0 \tag{20}$$

Since different layers have different thermal properties, different potential displacement functions are chosen. From the stress-strain relations of thermo-elasticity and the equation of equilibrium [17], we have

$$\nabla^2 \psi_i = \beta_i T \ , \quad i=1,2,3 \tag{21}$$

where

$$\beta_i = \frac{1+\nu_i}{1-\nu_i} \alpha_i$$

Applying Hankel transform on (20) and (21) we obtain

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) M(\xi, z) = 0 \tag{22}$$

and

$$\left(\frac{d^2}{dz^2} - \xi^2 \right) L_i(\xi, z) = \beta_i M(\xi, z) \tag{23}$$

where
$$M(\xi, z) = \int_0^\infty r T J_o(\xi r) \, dr \tag{24}$$

$$L_i(\xi, z) = \int_0^\infty r \psi_i J_o(\xi r) \, dr \tag{25}$$

The corresponding thermal stress components $(\sigma_z)_T, (\tau_{zr})_T$ and displacement components $(u_r)_T, (u_z)_T$ are given by

$$(\sigma_z)_{T_i} = 2\mu_i \left\{ \frac{\partial^2 \psi_i}{\partial z^2} - \nabla^2 \phi_i \right\} \tag{26}$$

$$(\tau_{zr})_{T_i} = 2\mu_i \frac{\partial^2 \psi_i}{\partial r \partial z} \tag{27}$$

$$(u_r)_{T_i} = \frac{\partial \psi_i}{\partial r} \tag{28}$$

$$(u_z)_{T_i} = \frac{\partial \psi_i}{\partial z} \tag{29}$$

Now we consider the transformations of (26)-(29) into relation involving $L_i(\xi,z)$ and $M(\xi,z)$. The results of computations are

$$\int_0^\infty r(\sigma_z)_{T_i} J_o(\xi r)\, dr = 2\mu_i \xi^2 L_i(\xi,z) \tag{30}$$

$$\int_0^\infty r(\tau_{zr})_{T_i} J_1(\xi r)\, dr = -2\xi\mu_i \frac{d}{dz} L_i(\xi,z) \tag{31}$$

$$\int_0^\infty r(u_r)_{T_i} J_1(\xi r)\, dr = -\xi\ L_i(\xi,z) \tag{32}$$

$$\int_0^\infty r(u_z)_{T_i} J_o(\xi r)\, dr = \frac{d}{dz} L_i(\xi,z) \tag{33}$$

Applying Hankel inverse transform, we get

$$(\sigma_z)_{T_i} = 2\mu_i \int_0^\infty \xi^3 L_i(\xi, z) J_o(\xi r)\, d\xi \tag{34}$$

$$(\tau_{zr})_{T_i} = -2\mu_i \int_0^\infty \xi^2 \frac{d}{dz} L_i(\xi, z) J_1(\xi r)\, d\xi \tag{35}$$

$$(u_r)_{T_i} = -\int_0^\infty \xi^2 L_i(\xi, z) J_1(\xi r)\, d\xi \tag{36}$$

$$(u_z)_{T_i} = \int_0^\infty \xi \frac{d}{dz} L_i(\xi, z) J_o(\xi r)\, d\xi \tag{37}$$

Let the solution of (23) be

$$L_i(\xi, z) = -\frac{\beta_i}{2\xi^2} A_o (1+\xi z) e^{-\xi z} \tag{38}$$

then the solution of (22) is

$$M(\xi, z) = A_o e^{-\xi z} \tag{39}$$

where A_o is a function of ξ.

Applying (38) we have from (30)-(33)

$$\int_0^\infty r(\sigma_z)_{T_i} J_o(\xi r)\, dr = \mu_i \beta_i A_o (1+\xi z) e^{-\xi z} \tag{40}$$

$$\int_0^{\infty} r(\tau_{zr})_{T_i} J_1(\xi r)\; dr = -\mu_i \beta_i A_o \xi z e^{-\xi z} \tag{41}$$

$$\int_0^{\infty} r(u_r)_{T_i} J_1(\xi r)\; dr = \frac{1}{2\xi} \beta_i A_o (1+\xi z) e^{-\xi z} \tag{42}$$

$$\int_0^{\infty} r(u_z)_{T_i} J_o(\xi r)\; dr = \frac{1}{2} \beta_i z A_o e^{-\xi z} \tag{43}$$

Taking solution of (10) for different layers [59] as:

For the upper layer

$$G_1(\xi, z) = (A_1+B_1 z)\cosh \xi z + (C_1+D_1 z)\sinh \xi z \tag{44}$$

For the middle layer

$$G_2(\xi, z) = (A_2+B_2 z)\cosh \xi z + (C_2+D_2 z)\sinh \xi z \tag{45}$$

For the underlying layer

$$G_3(\xi, z) = (A_3+B_3 z)e^{-\xi z}, \quad \xi>0, \quad z>h_2 \tag{46}$$

It is to be noted that stress and displacements in the underlying mass vanish as z tends to infinity. So, the components of stress and displacement for the upper and middle layers obtained from (11)-(14) when $j=1,2$ are

$$\int_0^\infty r(\sigma_z)_j \, J_o(\xi r) \, dr = (1-2\nu_j)\xi^2 B_j \cosh \xi z + (1-2\nu_j)\xi^2 D_j \sinh \xi z$$

$$- (A_j + B_j z)\xi^2 \sinh \xi z - (C_j + D_j z)\xi^2 \cosh \xi z \qquad (47)$$

$$\int_0^\infty r(\tau_{zr})_j \, J_1(\xi r) \, dr = \xi^2 \left[2\nu_j B_j \sinh \xi z + 2\nu_j D_j \cosh \xi z \right]$$

$$+ \xi^3 \left[(A_j + B_j z) \cosh \xi z + (C_j + D_j z) \sinh \xi z \right] \qquad (48)$$

$$\int_0^\infty r(u_r)_j \, J_1(\xi r) \, dr = \frac{1+\nu_j}{E_j} \xi^2 \left[(A_j + B_j z + \frac{D_j}{\xi}) \sinh \xi z \right.$$

$$\left. + (C_j + D_j z + \frac{B_j}{\xi}) \cosh \xi z \right] \qquad (49)$$

$$\int_0^\infty r(u_z)_j \, J_o(\xi r) \, dr = \frac{1+\nu_j}{E_j} \xi^2 \left[\left[\frac{2D_j}{\xi}(1-2\nu_j) - (A_j + B_j z) \right] \cosh \xi z \right.$$

$$\left. + \left[\frac{2B_j}{\xi} (1-2\nu_j) - (C_j + D_j z) \right] \sinh \xi z \right] \qquad (50)$$

For the underlying mass

$$\int_0^\infty r(\sigma_z)_3 J_o(\xi r)\ dr = \left[A_3 + B_3 z + \frac{B_3}{\xi}(1-2\nu_3)\right]\xi^3\ e^{-\xi z} \tag{51}$$

$$\int_0^\infty r(\tau_{zr})_3 J_1(\xi r)\ dr = \left[A_3 + B_3 z + \frac{B_3}{\xi}\ 2\nu_3\right]\xi^3\ e^{-\xi z} \tag{52}$$

$$\int_0^\infty r(u_r)_3 J_1(\xi r)\ dr = \frac{1+\nu_3}{E_3}\ \xi^2\ \left[\ \frac{B_3}{\xi}\ -A_3 - B_3 z\ \right]\ e^{-\xi z} \tag{53}$$

$$\int_0^\infty r(u_z)_3 J_o(\xi r)\ dr = \frac{1+\nu_3}{E_3}\ \xi^3\left[A_3 + B_3 z + \frac{2B_3}{\xi}\ (1-2\nu_3)\right]e^{-\xi z} \tag{54}$$

4. Boundary Conditions

In order to nullify the stresses on the boundaries the following conditions are to be satisfied:

At z=0, $\quad -(\sigma_z)_{T_i} = (\sigma_z)_1$

$$-(\tau_{zr})_{T_i} = (\tau_{zr})_1 \tag{55}$$

At the interfaces,

$z=h_1$, $|(\sigma_z)_2| = |(\sigma_z)_1|$, $|(\tau_{zr})_2| = |(\tau_{zr})_1|$.

So at $z=h_1$,

$$-(\sigma_z)_{T_2} = (\sigma_z)_1$$

$$-(\tau_{zr})_{T_2} = (\tau_{zr})_1$$

and

$$-(\sigma_z)_{T_2} = (\sigma_z)_2$$

$$-(\tau_{zr})_{T_2} = (\tau_{zr})_2 \qquad (56)$$

Since at the interfaces

$z=h_2$, $|(\sigma_z)_2| = |(\sigma_z)_3|$ and $|(\tau_{zr})_2| = |(\tau_{zr})_3|$.

So at $z=h_2$,

$$-(\sigma_z)_{T_3} = (\sigma_z)_3$$

$$-(\tau_{zr})_{T_3} = (\tau_{zr})_3$$

and

$$-(\sigma_z)_{T_3} = (\sigma_z)_2$$

$$-(\tau_{zr})_{T_3} = (\tau_{zr})_2 \qquad (57)$$

Boundary conditions relating to the continuities of displacement are assumed to be identically satisfied at the interfaces.

Conditions (55) with the help of (40),(41) and(47),(48) we have

$$A_1 = -\frac{2\nu_1 D_1}{\xi} \qquad (58)$$

$$B_1 = \frac{\beta_1 \mu_1 A_o + \xi^2 C_1}{(1-2\nu_1)\xi^2} \tag{59}$$

Using (40),(41),(51) and (52) on the first two conditions of (57) we get

$$A_3 = \frac{2\nu_3 A_o}{\xi^3} \beta_3 \mu_3 \tag{60}$$

$$B_3 = A_o \beta_3 \mu_3 \xi^{-2} \tag{61}$$

From the boundary conditions (56) and last two of (57)

$$(q_1 - \xi h_1 p_1) D_1 - \frac{h_1 q_1 \xi^2}{1-2\nu_1} C_1 = A_o \xi^{-2} \left[\beta_2 \mu_2 (1+\xi h_1) e^{-\xi h_1} - \mu_1 \beta_1 \left\{p_1 - \frac{h_1 q_1 \xi}{1-2\nu_1}\right\}\right] = S_5 \tag{62}$$

$$\xi q_1 h_1 D_1 + \frac{(q_1 + \xi h_1 p_1)\xi}{1-2\nu_1} C_1 = -A_o \xi^{-2} \beta_1 \mu_1 \frac{2\mu_1 q_1 + \xi h_1 p_1}{1-2\nu_1} + S_5 = S_6 \tag{63}$$

$$\left[(1-2\nu_2) p_1 - \xi h_1 q_1\right] B_2 + \left[(1-2\nu_2) q_1 - \xi h_1 p_1\right] D_2 - \xi q_1 A_2 - \xi p_1 C_2 - S_1 = 0 \tag{64}$$

$$\left[2\nu_2 q_1 + \xi h_1 p_1\right] B_2 + \left[2\nu_2 p_1 + \xi h_1 q_1\right] D_2 + \xi p_1 A_2 + \xi q_1 C_2 - S_3 = 0 \tag{65}$$

$$\left[(1-2\nu_2)p_2-\xi h_2 q_2\right]B_2+\left[(1-2\nu_2)q_2-\xi h_2 p_2\right]D_2-\xi q_2 A_2-\xi p_2 C_2-S_2=0 \qquad (66)$$

$$\left[2\nu_2 q_2+\xi h_2 p_2\right]B_2+\left[2\nu_2 p_2+\xi h_2 q_2\right]D_2+\xi p_2 A_2+\xi q_2 C_2-S_4=0 \qquad (67)$$

where

$$S_1 = S_1' A_o ; \quad S_1' = \frac{\mu_2\beta_2}{\xi^2}(1+\xi h_1)e^{-\xi h_1}$$

$$S_2 = S_2' A_o ; \quad S_2' = \frac{\mu_3\beta_3}{\xi^2}(1+\xi h_2)e^{-\xi h_2}$$

$$S_3 = S_3' A_o ; \quad S_3' = \frac{\mu_2\beta_2}{\xi^2} h_1 e^{-\xi h_1}$$

$$S_4 = S_4' A_o ; \quad S_4' = \frac{\mu_3\beta_3}{\xi^2} h_2 e^{-\xi h_2}$$

$$S_5 = S_5' A_o ; \quad S_5' = S_1' - \frac{\mu_1\beta_1}{\xi^2}\left\{p_1 - \frac{h_1 q_1 \xi}{1-2\nu_1}\right\}$$

$$S_6 = S_6' A_o ; \quad S_6' = S_3' - \xi^{-2}\beta_1\mu_1\frac{2\nu_1 q_1+\xi h_1 p_1}{1-2\nu_1}$$

Solving (62) and (63)

$$C_1 = C_1' A_o \quad \text{and} \quad D_1 = D_1' A_o$$

where

$$C_1' = \frac{1-2\nu_1}{\xi} \; \frac{\xi h_1 q_1 S_5' - (q_1 - \xi h_1 p_1) S_6'}{q_1^2 - \xi^2 h_1^2 p_1^2 - \xi^2 h_1^2 q_1^2}$$

and

$$D_1' = \frac{\xi h_1 q_1 S_6' + (q_1 + \xi h_1 p_1) S_5'}{q_1^2 - \xi^2 h_1^2 p_1^2 - \xi^2 h_1^2 q_1^2}$$

Putting the values of C_1 and D_1 in (58) and (59),

$$A_1 = A_1' A_o \quad \text{and} \quad B_1 = B_1' A_o$$

where

$$A_1' = -2\nu_1 \xi^{-1} \; \frac{\xi h_1 q_1 S_6' + (q_1 + \xi h_1 p_1) S_5'}{q_1^2 - \xi^2 h_1^2 p_1^2 - \xi^2 h_1^2 q_1^2}$$

$$B_1' = \frac{1}{(1-2\nu_1)\xi^2} \; (\beta_1 \mu_1 + \xi^3 C_1')$$

Equations (64)-(67) are then solved to find A_2, B_2, C_2 and D_2 and are written in the form

$$A_2 = A_2' A_o \;, \; B_2 = B_2' A_o, \; C_2 = C_2' A_o \;, \; D_2 = D_2' A_o$$

Subsequently numerical values of A_i', B_i', $i=1,2,3$ and C_i', D_i', $i=1,2$ are obtained. Here $A_3 = A'_3, B_3 = B'_3$. So the relations (58)-(67) give 10 constants $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3$ and B_3 in terms of A_o. So the formal solution of the problem is complete.

The resultant stresses in the direction of z-axis are

On z=0,

$$\int_0^\infty r(\sigma_z)_{R_1} J_o(\xi r)dr = \int_0^\infty r[(\sigma_z)_1 + (\sigma_z)_{T_1}] \, J_o(\xi r)dr =$$

$$\left[(1-2\nu_1)\xi^2 B'_1 - C'_1\xi^3 - \beta_1\mu_1\right] A_o \tag{68}$$

At $z=h_1$,

$$\int_0^\infty r(\sigma_z)_{R_2} J_o(\xi r)dr = \int_0^\infty r[(\sigma_z)_2 + (\sigma_z)_{T_2}] \, J_o(\xi r)dr =$$

$$= \Big[(1-2\nu_1)\xi^2 B'_2 \cosh \xi h_1 + (1-2\nu_1)\xi^2 D'_2 \sinh \xi h_1 - (A'_2 + B'_2 h_1)\xi^3 \sinh \xi h_1$$

$$- (C'_2 + D'_2 h_1)\xi^3 \cosh \xi h_1 - \beta_2\mu_2 (1+\xi h_1) e^{-\xi h_1}\Big] A_o \tag{69}$$

At $z=h_2$

$$\int_0^\infty r(\sigma_z)_{R_3} J_o(\xi r)dr = \int_0^\infty r[(\sigma_z)_3 + (\sigma_z)_{T_3}] \, J_o(\xi r)dr =$$

$$= \left[(1-2\nu_3)\xi^2 B'_3 \; e^{-\xi h_2} + (A'_3 + B'_3 h_2)\xi^3 \; e^{-\xi h_2} - \beta_3 \mu_3 (1+\xi h_2) e^{-\xi h_2} \right] A_o \qquad (70)$$

Constants $A'_1, B'_1, C'_1, D'_1, A'_2, B'_2, C'_2, D'_2, A'_3$ and B'_3 are independent of A_o. Thus the total thermoelastic stress $(\sigma_z)_R$ in the underlying mass is

$$\int_0^\infty r(\sigma_z)_R \; J_o(\xi r)dr = \int_0^\infty r[(\sigma_z)_{R_1} + (\sigma_z)_{R_2} + (\sigma_z)_{R_3}] \; J_o(\xi r)dr \qquad (71)$$

5. Flux Of Heat On The Boundaries

Let the flux of heat in a region of the surface $z=0$, distributed through layers in the underlying mass be

$$\frac{\partial T}{\partial z} = f(r/a), \quad 0<r<a \qquad (72)$$

$$= 0, \quad r>a$$

using dimensionless variables

$$\xi A_o(\xi) = aX(\xi a), \; \eta=\xi a, \; \rho=r/a, \; \zeta=z/a$$

where a is some length and η, a new variable of integration, we get from (24) and (39), on $z=0$,

$$\frac{\partial T}{\partial z} = -a^{-1} \int_0^\infty \eta X(\eta) J_o(\rho\eta) \, d\eta \tag{73}$$

By Hankel inversion theorem

$$X(\eta) = -a^{-1} \int_0^1 \rho f(\rho) J_o(\rho\eta) \, d\rho \tag{74}$$

For a simple physical situation we consider a linear temperature distribution $f(\rho) = K\rho$, K = constant, then from (74)

$$X(\eta) = -\frac{K}{a\eta} J_1(\eta) \tag{75}$$

With this value of $X(\eta)$, the problem is completely solved since only unknown $A_o(\xi)$ is now known.

6. NUMERICAL RESULTS

If the upper layer be concrete pavement, the middle layer be gravel base and the underlying mass be natural soil, then elastic constants for those materials are [8]

$E_1 = 2.18\times10^{8}$ gms/cm^2 , $\alpha_1 = -5\times10^{-6}/0^\circ C$

$E_2 = 1.1\times10^{8}$ gms/cm^2 , $\alpha_2 = 7.5\times10^{-6}/0^\circ C$

$E_3 = 0.4\times10^{8}$ gms/cm^2 , $\alpha_3 = 2.3\times10^{-6}/0^{\circ}C$

$\nu_1 = 0.15$, $\mu_1 = 0.94\times10^{8}$ gms/cm^2

$\nu_2 = 0.25$, $\mu_2 = 0.43\times10^{8}$ gms/cm^2

$\nu_3 = 0.50$, $\mu_3 = 0.14\times10^{3}$ gms/cm^2

$K_1 = 6.4\times10^{-8}$, $K_2 = 6.7\times10^{-3}$

$K_3 = 2.9\times10^{-3}$

So, evaluating constants for a given value of η when a=1 and h_1=2, h_2=4 we have

$$A'_1 = 3.504\times10^{2}\ ,\ B'_1 = 77.2671\times10^{2},\ C'_1 = 47.73\times10^{2}\ ,D'_1 = -166.8\times10^{2},$$

$$A'_2 = 0.005829\times10^{2},\ B'_2 = 0.01207\times10^{-2},\ C'_2 = 0.007071\times10^{2},$$

$$D'_2 = -0.02578\times10^{2},\ A'_3 = -14.49\times10^{2},\ B'_3 = 0.966\times10^{2}$$

So, applying dimensionless variables on (71) substituting the value of $X(\eta)$ from (75) and the values of constants in (71), we get

$$\int_0^{\infty} r(\sigma_z)_R\, J_o(\rho\eta)\ d\rho = 4.94423\times10^{2}\ \frac{K}{\eta}\ J_1(\eta)$$

whose Hankel transform is

$$(\sigma_z)_R = 4.94423\times10^2 \; K\int_0^\infty \frac{J_1(\eta)J_o(\rho\eta)}{\eta}\,d\eta$$

$$= 4.94423\times10^2 \; K\; F\left[1/2,\; 1/2;1;\rho^2\right] \qquad ,\; \rho<1$$

$$= 4.94423\times10^2 \; 2K/\pi, \qquad \rho=1$$

$$= 4.94423\times10^2 \; \frac{1}{2}\; K\; F\left[1/2,\; 1/2;1;\rho^{-2}\right], \qquad \rho>1$$

F denotes hypergeometric function.

7. DISCUSSION

It is important to study the existence of uniqueness of the solutions of the system of equations (62)-(67) for the unknowns A_i', B_i', $i=1,2,3$ and C_i', D_i', $i=1,2$. Here, it is found that all the above solutions depend on the quantities A_o. But A_o is unique for a particular type of distribution of heat flux on the boundary. So the solutions A_i', B_i', $i=1,2,3$ and C_i', D_i', $i=1,2$ are unique subject to the condition that A_o is unique for a definite kind of heat flux used in this problem.

For simplicity, linear heat flux has been applied on the boundary z=0 in this problem. It will be more interesting and physically suitable if the heat flux is considered in the form other than linear one.

The numerical results of this paper have been compared with the works of Paria [59] who solved a problem of this nature in absence of temperature and the results tally completely with the results of Paria.

Fig.3 displays the nature of distribution of thermoelastic stress $(\sigma_z)_a$ against ρ.

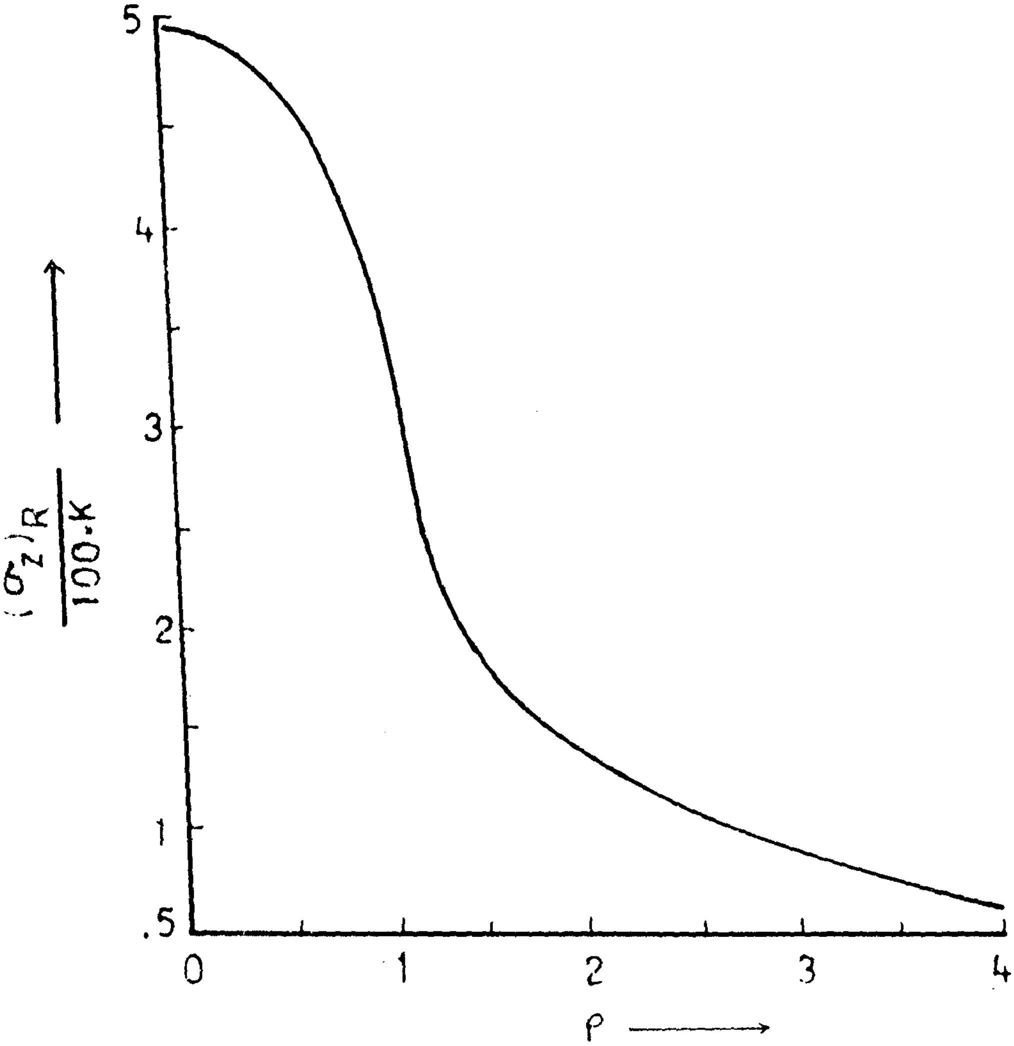

FIG.3 : DISTRIBUTION OF THERMOELASTIC STRESS $(\sigma_z)_R$ IN THE UNDERLYING MASS FOR A THREE LAYERED SYSTEM.

CHAPTER - III

GENERATION OF THERMOELASTIC WAVES

Paper 1 : On the generation of thermoelastic waves due to the distribution of stress produced by a periodic heat nucleus

(ACCEPTED FOR PUBLICATION IN INT. J. MATH. AND MATHL. SC.,U.S.A.)

Paper 2 : On the generation of thermoelastic waves due to the distribution of stress produced by an impulsive heat nucleus

On The Generation Of Thermoelastic Waves Due To The Distribution Of Stresses Produced By A Periodic Heat Nucleus

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1.Basic Formulation

If τ_{mn} be the stress tensor, F_m, components of the body force and f_m, the components of acceleration of the infinitesimal element centered at (x_1, x_2, x_3) then from

$$f_m = \frac{\partial^2 u_m}{\partial t^2} = c^2 \frac{\partial^2 u_m}{\partial \tau^2} \qquad (A)$$

where u_m, denotes components of displacement, t, time c, some characteristic velocity and $\tau = ct$ is a space-time coordinate determined by time. The tensor form of well known equations of motion in three dimensions with density ρ of the medium in vector form [85] gives

$$\tau_{mn,n} + \rho F_m = \rho f_m, \qquad m, n = 1, 2, 3$$

which by (A) becomes

$$\tau_{mn,n} + \rho F_m = \rho c^2 \frac{\partial^2 u_m}{\partial \tau^2} \qquad (1)$$

To solve equations (1) by Fourier transform technique, let us denote the Fourier transform [14] of a function ϕ by $\overline{\phi}$, in other words

$$(2\pi)^{-2} \int_{S_4} \phi(x, x, x, \tau)\, e^{i(x_l \xi_l + \omega\tau)}\, dS_4 = \overline{\phi}(\xi_1, \xi_2, \xi_3, \omega)$$

$$(2\pi)^{-2}\int_{S_4}\left[\frac{\partial\phi}{\partial x_l},\ \frac{\partial^2\phi}{\partial\tau^2}\right]e^{i(x_l\xi_l+\omega\tau)}\,dS_4 = -\,(i\xi_l,\omega^2)\bar{\phi} \qquad (2)$$

where $l=1,2,3$; $dS_4 = dx_1dx_2dx_3d\tau$ and S_4 denotes entire $x_1x_2x_3\tau$ space.

Applying (2) in (1), we get

$$i\xi_m\bar{\tau}_{mn} - \rho\bar{F}_m = \rho c^2\omega^2\,\bar{u}_m \qquad (3)$$

Reducing strains in terms of displacement components u_m the stress-strain relations in three dimensional thermoelastic problem for a temperature distribution T [13,17], we have

$$\tau_{11} = E'(u_{1,1}+\nu' u_{2,2}+\nu' u_{3,3} - \alpha' T)$$

$$\tau_{12} = \mu(u_{1,2}+u_{2,1}) \qquad (4)$$

where

$$E' = \frac{E(1-\nu)}{(1+2\nu)(1+\nu)},\quad \nu' = \frac{\nu}{1-\nu},\quad \alpha' = \frac{1-\nu}{1+\nu},$$

$$\mu = \frac{E}{2(1+\nu)},\quad u_{m,n} = \frac{\partial u_m}{\partial x_n}$$

$\tau_{22}, \tau_{33}, \tau_{23}$ and τ_{31} are derived by cyclic permutation of 1,2,3.

Using (2) over (4),

$$\bar{\tau}_{11} = -iE'(\xi_1\bar{u}_1+\nu'\xi_2\bar{u}_2+\nu'\xi_3\bar{u}_3 - i\alpha'\bar{T})$$

$$\bar{\tau}_{12} = -i\mu(\xi_2\bar{u}_1+\xi_1\bar{u}_2) \tag{5}$$

$\bar{\tau}_{22}, \bar{\tau}_{33}, \bar{\tau}_{23}$ and $\bar{\tau}_{31}$ are derived by cyclic permutation of 1,2,3. Substituting (5) in (3) and solving for u_m,

$$\bar{u}_1 = \left[\beta^2 R_o \bar{F}_1 - \xi_1(BL + QR_o\bar{T})\right]\left[C_1^2\ R\right]^{-1} \tag{6}$$

$$\bar{u}_2 = \left[\beta^2 R_o \bar{F}_2 - \xi_2(BL + QR_o\bar{T})\right]\left[C_1^2\ R\right]^{-1} \tag{7}$$

$$\bar{u}_3 = \left[\beta^2 R_o \bar{F}_3 - \xi_3(BL + QR_o\bar{T})\right]\left[C_1^2\ R\right]^{-1} \tag{8}$$

where $R = R_o R'$, $R_o = \xi_1^2+\xi_2^2+\xi_3^2-\beta^2$, $R' = \xi_1^2+\xi_2^2+\xi_3^2-\beta^2\omega^2$, $\beta^2 = 4(\lambda+\mu)/(\lambda+2\mu)$, $B = \beta^2-1$, $C_1^2 = \mu\beta^2/\rho$ is chosen for the characteristic velocity C, $Q = 2iC_1^2 B\alpha/\beta^2$ and $L = \xi_1\bar{F}_1+\xi_2\bar{F}_2+\xi_3\bar{F}_3$.

So, (5) takes the form

$$\bar{\tau}_{11} = -i\rho\left[QR'\bar{T}\left\{(\nu'-1)(\xi_2^2+\xi_3^2)+\omega^2\right\}+\beta^2 R_o M_1 - BN_1 L\right]R^{-1} \tag{9}$$

$$\bar{\tau}_{12} = -i\rho\beta^{-2}\left[QR'\bar{T}(\xi_2^2+\xi_1^2)+\beta^2 R_o(\xi_2\bar{F}_1+\xi_1\bar{F}_2)-2B\,\xi_1\xi_2(\xi_1\bar{F}_1+\xi_2\bar{F}_2)\right]R^{-1} \quad (10)$$

where $M_1 = \xi_1\bar{F}_1 + \nu'(\xi_2\bar{F}_2+\xi_3\bar{F}_3)$, $N_1 = \xi_1^2 + \nu'(\xi_2^2+\xi_3^2)$.

$\bar{\tau}_{22}, \bar{\tau}_{33}, \bar{\tau}_{23}$, $\bar{\tau}_{31}$ and M_2, N_2, M_3, N_3 are obtained by cyclic permutation of 1,2,3.

Applying Fourier inverse transform [14] over (6) - (10),

$$u_1 = (2\pi C_1)^{-2}\int_{W_4}\left[\beta^2 R_o\bar{F}_1-\xi_1(BL+QR_o\bar{T})\right]R^{-1}e^{-i(x_l\xi_l+\omega\tau)}\,dW_4 \quad (11)$$

$$\tau_{11} = \frac{i\rho}{2\pi^2}\int_{W_4}\left[QR'\bar{T}\left\{(\nu'-1)(\xi_2^2+\xi_3^2)+\omega^2\right\}+\beta^2 R_o M_1 - BN_1 L\right]R^{-1}e^{-i(x_l\xi_l+\omega\tau)}\,dW_4 \quad (12)$$

$$\tau_{12} = i\rho(2\pi\beta)^{-2}\int_{W_4}\left[QR'\bar{T}(\xi_2^2+\xi_1^2)+\beta^2 R_o(\xi_2\bar{F}_1+\xi_1\bar{F}_2)-2B\,\xi_1\xi_2(\xi_1\bar{F}_1+\xi_2\bar{F}_2)\right]R^{-1}\times$$

$$\times e^{-i(x_l\xi_l+\omega\tau)}\,dW_4 \quad (13)$$

where W_4 is the $\xi_1\xi_2\xi_3\omega$ space and $dW_4 = d\xi_1 d\xi_2 d\xi_3 d\omega$.

$u_2, u_3, \tau_{22}, \tau_{33}, \tau_{23}$ and τ_{31} are obtained by cyclic permutation of 1,2,3. keeping C_1 fixed.

2.SOLUTION

For the solution of the equation of motion when the body force X and temperature T, acting at the origin in the direction of x_1 increasing, which vary harmonically with time period $2\pi/p$ may be written as [43]

$$X = F\rho^{-1}\delta(x_1)\delta(x_2)\delta(x_3)e^{i\lambda\tau} \tag{14}$$

and $$T = T_o(2\mu)^{-1}\delta(x_1)\delta(x_2)\delta(x_3)e^{i\lambda\tau} \tag{15}$$

where $\lambda = p/C_1$.

The result of transformation [77] gives

$$\bar{X} = (F/\rho)\ (2\pi)^{-1/2}\ \delta(\omega+\lambda) \tag{16}$$

and $$\bar{T} = (T_o/2\mu)\ (2\pi)^{-1/2}\ \delta(\omega+\lambda) \tag{17}$$

To obtain the components of displacement and stress produced by the insertion of the periodic heat nucleus by means of the formula deduced in the last section, we shall adopt usual notation [13] viz.

$$u_1 = u_T,\ u_2 = v_T,\ u_3 = w_T,\ \tau_{11} = (\sigma_1)_T,\ \tau_{22} = (\sigma_2)_T,\ \tau_{33} = (\sigma_3)_T,$$

$$\tau_{12} = (\tau_{12})_T,\ \tau_{23} = (\tau_{23})_T \text{ and } \tau_{31} = (\tau_{31})_T.$$

3.Method Of Solution

For the sake of simplicity, we assume F_1 only x_1- component of the body force, non-zero [33] i.e. F_1=X, F_2=F_2=0. So, equations (11), u_1, u_2, u_3 take the form

$$u_T = (2\pi C_1)^{-2}\int_{W_4} \left[(\beta^2 R_o - B\,\xi_1^2) R^{-1}\overline{X} + \xi_1 Q R'^{\,-1}\overline{T}\right] e^{-i(x_l \xi_l + \omega\tau)} dW_4$$

$$v_T = (2\pi C_1)^{-2}\int_{W_4} \left[\xi_2 Q R'^{\,-1}\overline{T} - B\,\xi_1 \xi_2 R^{-1}\overline{X}\right] e^{-i(x_l \xi_l + \omega\tau)} dW_4$$

$$w_T = (2\pi C_1)^{-2}\int_{W_4} \left[\xi_3 Q R'^{\,-1}\overline{T} - B\,\xi_1 \xi_3 R^{-1}\overline{X}\right] e^{-i(x_l \xi_l + \omega\tau)} dW_4$$

Rewriting u_T, v_T, w_T,

$$u_T = (2\pi C_1)^{-2}\int_{W_4} \Big[(\xi_1^2+\xi_2^2+\xi_3^2)^{-1}\ \overline{X}\left\{\xi_1^2 R_o^{-1} + \beta^2(\xi_2^2+\xi_3^2) R'^{\,-1}\right\} +$$

$$+\ \xi_1 Q R'^{\,-1}\overline{T}\Big] e^{-i(x_l \xi_l + \omega\tau)} dW_4 \qquad (18)$$

$$v_T = (2\pi C_1)^{-2}\int_{W_4} \left[\xi_2 Q R'^{\,-1}\overline{T} + \xi_1 \xi_2 (\xi_1^2+\xi_2^2+\xi_3^2)^{-1}\ \overline{X}(R_o^{-1} - \beta^2 R'^{\,-1})\right]\times$$

$$\times e^{-i(x_l \xi_l + \omega\tau)} dW_4 \qquad (19)$$

$$w_T = (2\pi C_1)^{-2} \int_{W_4} \left[\xi_3 QR'^{-1} \bar{T} + \xi_1 \xi_3 (\xi_1^2 + \xi_2^2 + \xi_3^2)^{-1} \bar{X}(R_o^{-1} - \beta^2 R'^{-1}) \right] \times$$

$$\times e^{-i(x_l \xi_l + \omega\tau)} dW_4 \qquad (20)$$

Substituting the expressions (16) and (17) for $\bar{X}$ and $\bar{T}$ in (18)-(20) and performing W- integration [77],

$$u_T = -Q_1 e^{i\lambda\tau} F\left[\frac{\partial^2 I}{\partial x_1^2} + \beta^2 \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) I_1 \right] + BQ_1 \frac{T_o}{2} e^{i\lambda\tau} \alpha \frac{\partial}{\partial x_1}\left[\nabla^2 I_1\right] \qquad (21)$$

$$v_T = -Q_1 e^{i\lambda\tau} F\left[\frac{\partial^2 I}{\partial x_1^2 \partial x_2^2} - \beta^2 I_1 \right] + BQ_1 \frac{T_o}{2} e^{i\lambda\tau} \alpha \frac{\partial}{\partial x_2}\left[\nabla^2 I_1\right] \qquad (22)$$

$$w_T = -Q_1 e^{i\lambda\tau} F\left[\frac{\partial^2 I}{\partial x_1^2 \partial x_3^2} - \beta^2 I_1 \right] + BQ_1 \frac{T_o}{2} e^{i\lambda\tau} \alpha \frac{\partial}{\partial x_3}\left[\nabla^2 I_1\right] \qquad (23)$$

where

$$I_1 = I_1(x_1, x_2, x_3, \beta\lambda) = \iiint_{-\infty}^{\infty} \left[(\xi_1^2+\xi_2^2+\xi_3^2)(\xi_1^2+\xi_2^2+\xi_3^2 - \beta^2\lambda^2) \right]^{-1} \times$$

$$\times e^{-i(x_1\xi_1+x_2\xi_2+x_3\xi_3)}\, d\xi_1 d\xi_2 d\xi_3$$

$$I = I(x_1, x_2, x_3, \lambda) = \iiint_{-\infty}^{\infty} \left[(\xi_1^2+\xi_2^2+\xi_3^2)(\xi_1^2+\xi_2^2+\xi_3^2 - \lambda^2) \right]^{-1} \times$$

$$\times e^{-i(x_1\xi_1+x_2\xi_2+x_3\xi_3)}\, d\xi_1 d\xi_2 d\xi_3$$

$Q_1 = \left[(2\pi)^{5/2}\, \mu\beta^2\right]^{-1}$ and ∇^2 is the Laplacian operator.

Now changing the variables by $\xi_1 = \cos\phi \sin\theta$, $\xi_2 = \sin\phi \sin\theta$, $\xi_3 = \cos\theta$, $\chi_1 = r\cos\theta \sin\phi$, $\chi_2 = r\sin\theta \sin\phi$, $\chi_3 = r\cos\phi$, and performing the integration with respect to θ and ϕ, we find that $I(x_1, x_2, x_3, \lambda)$ is a function of r and λ alone and that [87]

$$\frac{\partial I}{\partial r} = 2\pi\lambda^{-2} \int_0^{\infty} \left\{ 1 - \frac{2}{\rho^2 - \lambda^2} \right\} J_1(\rho r)\, d\rho$$

Making use of the well known result in the theory of Bessel function [19], we find that

$$\frac{\partial I}{\partial r} = 2\pi\lambda^{-2}\left[r^{-1} - \frac{i\pi\lambda}{2} H^1_\nu(\lambda r)\right]$$

where $H^1_\nu(\lambda r) = J_\nu(\lambda r) + iY_\nu(\lambda r)$, $Y_\nu(\lambda r)$ denotes Weber Bessel function of the second kind.

Applying (24) in(21),(22) and (23) and writing the result in terms of t, the time

$$u_T = (Q_2/r)\left\{B\alpha x_1 T_o\left[\frac{2}{r} H^1_2(s_2) - \frac{p}{c_2} H^1_3(s_2)\right] + \frac{F}{p}\left[c_1 H^1_1(s_1) + 2\beta^2 c_2 H^1_1(s_2)\right] - \right.$$

$$\left. - \frac{F}{r}\left[x_1^2 H^1_2(s_1) + \beta^2 (x_2^2 + x_3^2) H^1_3(s_2)\right]\right\}$$

$$v_T = (Q_2/r)\left\{B\alpha x_2 T_o\left[\frac{2}{r} H^1_2(s_2) - \frac{p}{c_2} H^1_2(s_2)\right] - \frac{x_1 x_2}{r} F\left[H^1_2(s_1) - \beta^2 H^1_2(s_2)\right]\right\}$$

$$w_T = (Q_2/r)\left\{B\alpha x_3 T_o\left[\frac{2}{r} H^1_2(s_2) - \frac{p}{c_2} H^1_2(s_2)\right] - \frac{x_1 x_3}{r} F\left[H^1_2(s_1) - \beta^2 H^1_2(s_2)\right]\right\}$$

where $c_2 = c_1/\beta$ is the second elastic velocity, $Q_2 = Q_1 ie^{ipt}$, $s_1 = pr/c_1$ and $s_2 = pr/c_2$.

So thermal stress components can be easily obtained form (4) using derivatives of u_T, v_T, w_T just derived.

So, algebraic sum of thermoelastic normal stresses as calculated thus:

$$(\sigma_1)_T+(\sigma_2)_T+(\sigma_3)_T = \frac{E}{1-2\nu} Q_2\left\{ \frac{Fx_1}{r}\left[\frac{p\beta^2}{c_2 r^2} (x_1^2+x_2^2)[H_1^1(s_2)+ 4s_2^{-1}H_2^1(s_2)+\right.\right.$$

$$\left. H_3^1(s_1)]\right]-\left[\frac{p}{c_1} H_1^1(s_1) + s_1 H_2^1(s_1)\right]+ B\alpha T_o\left[\frac{2p}{rc_2} [H_1^1(s_2) + H_2^1(s_2)]- \frac{p^2}{c_2 r^2} \times\right.$$

$$(x_3^2+x_2^2)[H_1^1(s_2) + 3s_2^{-1}H_2^1(s_2)] - \frac{2}{r^2} [H_2^1(s_2)+ .5s_2 H_3^1(s_2)] - \frac{p^2 x_1^2}{c_2 r^2} \times$$

$$\left.\left.[H_2^1(s_2)- 4s_2^{-1} H_3^1(s_2)]\right]\right\}$$

$$(\tau_{12})_T = Q_2 B\alpha T_o p x_1 x_2 (c_2 r^2)^{-1}\left\{\frac{4}{r}\left[H_1^1(s_1)- 4s_2^{-1}H_2^1(s_2)+ H_3^1(s_1)\right]-\right.$$

$$\left. - \frac{p}{c_2} \left[H_1^1(s_2)+H_2^1(s_2)\right] +\frac{3}{r} H_2^1(s_2)\right\} +Q_2 F x_2 r^{-2}\left\{\frac{-2px_1}{c_1 r} H_1^1(s_1)-(8x_1^2 r^{-2}-2)H_2^1(s_1)\right.$$

$$-p\beta^2 s_2 H_1^1(s_2) + r^{-2}\left[2(x_2^2-x_1^2-x_3^2) - \beta^2(5x_1^2+x_2^2-3x_3^2)\right]H_2^1(s_2)\Big\}$$

$$(\tau_{23})_T = Q_2 B\alpha T_o x_3 x_2 (2r^2)^{-1}\left\{ \frac{p}{c_2}\left(\frac{2}{r} - p\right) H_1^1(s_2) - r^{-1}\left(\frac{3p}{c_2} - \frac{2}{r}\right) H_2^1(s_2)\right\}$$

$$+ 2Q_2 F x_1 x_2 x_3 r^{-3}\left\{ \frac{4}{r}\left[H_2^1(s_1) - \beta^2 H_2^1(s_2)\right] + \frac{p}{r}\left[\beta^2 c_2^{-1} H_1^1(s_2) - c_1^{-1} H_1^1(s_1)\right]\right\}$$

$$(\tau_{13})_T = Q_2 B\alpha T_o x_1 x_3 p(c_2 r^2)^{-1}\Big\{ \frac{4}{r}\left[H_1^1(s_2) - 4s_2^{-1}H_2^1(s_2) + H_3^1(s_2)\right] - \frac{p}{c_2}$$

$$\left[H_1^1(s_2) + H_2^1(s_2)\right] + \frac{3}{r}H_2^1(s_2)\Big\} + Q_2 F x_3 r^{-2}\Big\{\frac{-2px_1^2}{c_1 r} H_1^1(s_1) - (8x_1^2 r^{-2} - 2)H_2^1(s_1)$$

$$-(p\beta^2/c_2 r)(x_2^2 + x_3^2 - x_1^2)H_1^1(s_2) + \left[r^{-2}(\beta^2 x_2^2 - \beta^2 x_1^2 + x_3^2) - \beta^2 - 2\right]H_2^1(s_2)\Big\}$$

Thus the algebraic sum of normal thermal stresses and components of shear stress are determined. Each of these expressions contain characteristic velocities c_1 and c_2. It is clear from the analysis of the above expressions that the disturbance is propagated outwards from the centre with velocities c_1 and $c_2 = c_1/\beta$.

On The Generation Of Thermoelastic Waves Due To The Distribution Of Stresses Produced By An Impulsive Heat Nucleus

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Communicated for publications.

1. Abstract

This paper is concerned with the determination of three dimensional thermo-elastic waves in a solid disturbed by the distribution of stresses when the time dependent body force and an impulsive heat nucleus act upon certain region of the solid.

2 .Basic Formulation

Equation of motion in three dimensions with density ρ of the medium are

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho F^{x} = \rho f^{x}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho F^{y} = \rho f^{y}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho F^{z} = \rho f^{z}$$

where F^{x}, F^{y}, F^{z} denotes components of body force at (x,y,z). The acceleration of the infinitesimal element centered at this point is denoted by f^{x}, f^{y}, f^{z}. Introducing displacement vector components v^{x}, v^{y}, v^{z} at such a point, we have

$$f^x = \frac{\partial^2 v^x}{\partial t^2} = C^2 \frac{\partial^2 v^x}{\partial \tau^2}$$

$$f^y = \frac{\partial^2 v^y}{\partial t^2} = C^2 \frac{\partial^2 v^y}{\partial \tau^2}$$

$$f^z = \frac{\partial^2 v^z}{\partial t^2} = C^2 \frac{\partial^2 v^z}{\partial \tau^2}$$

where t denotes time, C is some characteristic velocity and τ = ct is a space-time coordinate determined by time. The equations of motion may therefore be written in the form

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho F^x = \rho C^2 \frac{\partial^2 v^x}{\partial \tau^2}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho F^y = \rho C^2 \frac{\partial^2 v^y}{\partial \tau^2}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho F^z = \rho C^2 \frac{\partial^2 v^z}{\partial \tau^2} \tag{1}$$

To solve equations (1) by Fourier transform technique [14] , let us denote the Fourier transform of a function ϕ by $\bar{\phi}$, in other words

$$(2\pi)^{-2}\int_{S_4} \phi(x_1, x_2, x_3, \tau)\ e^{i(x_l\xi_l+\omega\tau)}\ dS_4 = \overline{\phi}(\xi_1, \xi_2, \xi_3, \omega)$$

$$(2\pi)^{-2}\int_{S_4} \left[\frac{\partial\phi}{\partial x_l}\ ,\ \frac{\partial^2\phi}{\partial\tau^2} \right] e^{i(x_l\xi_l+\omega\tau)}\ dS_4 = -\ (i\xi_l, \omega^2)\overline{\phi} \qquad (2)$$

where $l=1,2,3$; $dS_4 = dx_1dx_2dx_3d\tau$ and S_4 denotes entire $x_1x_2x_3\tau$ space.

Applying (2) in (1), we get

$$i\xi_x\overline{\tau}_{xx} + i\xi_y\overline{\tau}_{xy} + i\xi_z\overline{\tau}_{xz} - \rho\overline{F}^x = \rho C^2\omega^2\ \overline{v}^x$$

$$i\xi_x\overline{\tau}_{xy} + i\xi_y\overline{\tau}_{yy} + i\xi_z\overline{\tau}_{yz} - \rho\overline{F}^y = \rho C^2\omega^2\ \overline{v}^y$$

$$i\xi_x\overline{\tau}_{zx} + i\xi_y\overline{\tau}_{zy} + i\xi_z\overline{\tau}_{zz} - \rho\overline{F}^z = \rho C^2\omega^2\ \overline{v}^z \qquad (3)$$

From stress-strain relation in three dimensional thermo-elastic problem for a temperature distribution T , we have

$$\tau_{xx} = E_1\left(\frac{\partial v^x}{\partial x} + \nu_1 \frac{\partial v^y}{\partial y} + \nu_1 \frac{\partial v^z}{\partial z} - \alpha_1 T\right)$$

$$\tau_{yy} = E_1\left(\nu_1 \frac{\partial v^x}{\partial x} + \frac{\partial v^y}{\partial y} + \nu_1 \frac{\partial v^z}{\partial z} - \alpha_1 T\right)$$

$$\tau_{zz} = E_1\left(\nu_1 \frac{\partial v^x}{\partial x} + \nu_1 \frac{\partial v^y}{\partial y} + \frac{\partial v^z}{\partial z} - \alpha_1 T\right)$$

$$\tau_{xy} = \mu\left(\frac{\partial v^x}{\partial y} + \frac{\partial v^y}{\partial x} \right)$$

$$\tau_{zy} = \mu\left(\frac{\partial v^z}{\partial y} + \frac{\partial v^y}{\partial z} \right)$$

$$\tau_{xz} = \mu\left(\frac{\partial v^x}{\partial z} + \frac{\partial v^z}{\partial x} \right) \qquad (4)$$

where

$$E_1 = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)}, \quad \nu_1 = \frac{\nu}{1-\nu}, \quad \alpha_1 = \frac{1+\nu}{1-\nu}, \quad \mu = \frac{E}{2(1+\nu)}$$

Using (2) over (4),

$$\bar{\tau}_{xx} = -iE_1(\xi_x\bar{v}^x + \nu_1\xi_y\bar{v}^y + \nu_1\xi_z\bar{v}^z - i\alpha_1\bar{T})$$

$$\bar{\tau}_{yy} = -iE_1(\nu_1\xi_x\bar{v}^x + \xi_y\bar{v}^y + \nu_1\xi_z\bar{v}^z - i\alpha_1\bar{T})$$

$$\bar{\tau}_{zz} = -iE_1(\nu_1\xi_x\bar{v}^x + \nu_1\xi_y\bar{v}^y + \xi_z\bar{v}^z - i\alpha_1\bar{T})$$

$$\bar{\tau}_{xy} = -i\mu(\xi_y \bar{v}^x + \xi_x \bar{v}^y)$$

$$\bar{\tau}_{zy} = -i\mu(\xi_y \bar{v}^z + \xi_z \bar{v}^y)$$

$$\bar{\tau}_{xz} = -i\mu(\xi_z \bar{v}^x + \xi_x \bar{v}^z) \qquad (5)$$

Substituting (5) in (3) and solving for $\bar{v}^x, \bar{v}^y, \bar{v}^z$ we get

$$\bar{v}^x = \Big[\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\bar{F}^x - (\beta^2-1)(\xi_x^2\,\bar{F}^x+\xi_y\xi_x\bar{F}^y+\xi_z\xi_x\bar{F}^z) + \frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\xi_x$$

$$\times(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\Big]\Big[C_1^2(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\Big]^{-1} \qquad (6)$$

$$\bar{v}^y = \Big[\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\bar{F}^y - (\beta^2-1)(\xi_x\xi_y\bar{F}^x+\xi_y^2\bar{F}^y+\xi_z\xi_y\bar{F}^z) + \frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\xi_y$$

$$\times(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\Big]\Big[C_1^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\Big]^{-1} \qquad (7)$$

$$\bar{v}^z = \Big[\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\bar{F}^z - (\beta^2-1)(\xi_x\xi_z\bar{F}^x+\xi_y\xi_z\bar{F}^y+\xi_z^2\bar{F}^z) + \frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\xi_z$$

$$\times(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\Big]\Big[C_1^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\Big]^{-1} \qquad (8)$$

where $\beta^2=4(\lambda+\mu)/(\lambda+2\mu)$ and $C_1^2=\mu\beta^2/\rho$ is chosen for our characteristic velocity C.

So, (5) takes the form

$$\bar{\tau}_{xx} = -i\rho\Big\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\ [(\nu_1-1)(\xi_y^2+\xi_z^2)+\ \omega^2](\xi_x^2+\xi_y^2+\xi_z^2-\ \beta^2\omega^2)+\beta^2\times$$

$$(\xi_x^2+\xi_y^2+\xi_z^2-\ \omega^2)(\xi_x\bar{F}^x+\nu_1\xi_y\bar{F}^y+\nu_1\xi_z\bar{F}^z)-(\beta^2-1)(\xi_x^2+\nu_1\xi_y^2+\nu_1\xi_z^2)\times$$

$$(\xi_x\bar{F}^x+\xi_y\bar{F}^y+\xi_z\bar{F}^z)\Big\}\Big[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\ \beta^2\omega^2)\Big]^{-1} \qquad (9)$$

$$\bar{\tau}_{yy}= -i\rho\Big\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\ [(\nu_1-1)(\xi_x^2+\xi_z^2)+\ \omega^2](\xi_x^2+\xi_y^2+\xi_z^2-\ \beta^2\omega^2)+\beta^2\times$$

$$(\xi_x^2+\xi_y^2+\xi_z^2-\quad \omega^2)(\nu_1\xi_x\bar{F}^x+\xi_y\bar{F}^y+\nu_1\xi_z\bar{F}^z)-(\beta^2-1)(\nu_1\xi_x^2+\xi_y^2+\nu_1\xi_z^2)\times$$

$$(\xi_x F^x+\xi_y F^y+\xi_z F^z)\Big\}\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1} \qquad (10)$$

$$\bar{\tau}_{zz}=-i\rho\Big\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\ [(\nu_1-1)(\xi_y^2+\xi_x^2)+\omega^2](\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)+\beta^2\times$$

$$(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\nu_1\xi_x F^x+\nu_1\xi_y F^y+\xi_z F^z)-(\beta^2-1)(\nu_1\xi_x^2+\nu_1\xi_y^2+\xi_z^2)\times$$

$$(\xi_x F^x+\xi_y F^y+\xi_z F^z)\Big\}\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1} \qquad (11)$$

$$\bar{\tau}_{yx}=-\frac{i\rho}{\beta^2}\Big\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\ (\xi_y^2+\xi_x^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)+\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\times$$

$$(\xi_x F^y+\xi_y F^x)-2(\beta^2-1)\xi_y\xi_x(\xi_y F^y+\xi_x F^x)\Big\}\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1}$$

$$(12)$$

$$\bar{\tau}_{yz}=-\frac{i\rho}{\beta^2}\Big\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\ (\xi_y^2+\xi_z^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)+\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\times$$

$$(\xi_z \bar{F}^y+\xi_y \bar{F}^z)-2(\beta^2-1)\xi_y\xi_z(\xi_y \bar{F}^y+\xi_z \bar{F}^z)\Big\}\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1} \tag{13}$$

$$\bar{\tau}_{zx} = -\frac{i\rho}{\beta^2}\Big\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\;(\xi_x^2+\xi_z^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)+\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\times$$

$$(\xi_z \bar{F}^x+\xi_x \bar{F}^z)-2(\beta^2-1)\xi_x\xi_z(\xi_x \bar{F}^x+\xi_z \bar{F}^z)\Big\}\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1} \tag{14}$$

Applying Fourier inverse transform over (6) - (14),

$$v^x = (2\pi C_1)^{-2}\int_{W_4}\Big[\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\bar{F}^x-(\beta^2-1)(\xi_x^2\bar{F}^x+\xi_y\xi_x\bar{F}^y+\xi_z\xi_x\bar{F}^z)+$$

$$\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\;\xi_x(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\Big]\left[C_1^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1}\times$$

$$e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}\,dw \tag{15}$$

$$v^y = (2\pi C_1)^{-2}\int_{W_4}\Big[\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\bar{F}^y-(\beta^2-1)(\xi_x\xi_y\bar{F}^x+\xi_y^2\bar{F}^y+\xi_z\xi_y\bar{F}^z)+$$

$$\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\ \xi_y(\xi_x^2+\xi_y^2+\xi_z^2-\ \omega^2)\Big]\Big[C_1^2(\xi_x^2+\xi_y^2+\xi_z^2-\ \omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\ \beta^2\omega^2)\Big]^{-1}\times$$

$$e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}\ dw \qquad (16)$$

$$v^z = (2\pi C_1)^{-2}\int_{W_4}\Big[\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\ \omega^2)\overline{F}^z-(\beta^2-1)(\xi_x\xi_z\overline{F}^x+\xi_y\xi_z\overline{F}^y+\xi_z^2\ \ \overline{F}^z)+$$

$$\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\ \xi_z(\xi_x^2+\xi_y^2+\xi_z^2-\ \omega^2)\Big]\Big[C_1^2(\xi_x^2+\xi_y^2+\xi_z^2-\ \omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\ \beta^2\omega^2)\Big]^{-1}\times$$

$$e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}\ dw \qquad (17)$$

$$\tau_{xx} = -i\rho(2\pi)^{-2}\int_{W_4}\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\ \ [(\nu_1-1)(\xi_y^2+\xi_z^2)+\ \omega^2](\xi_x^2+\xi_y^2+\xi_z^2-\ \beta^2\omega^2)+$$

$$\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\ \ \omega^2)(\xi_x\overline{F}^x+\nu_1\xi_y\overline{F}^y+\nu_1\xi_z\overline{F}^z)-(\beta^2-1)(\xi_x^2+\nu_1\xi_y^2+\nu_1\xi_z^2)(\xi_x\overline{F}^x+\xi_y\overline{F}^y+$$

$$\left.\xi_z F^z)\right\}\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1} e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}\,dw \quad (18)$$

$$\tau_{yy}= -i\rho(2\pi)^{-2}\int_{W_4}\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\ \ [(\nu_1-1)(\xi_x^2+\xi_z^2)+\omega^2](\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)+$$

$$\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\nu_1\xi_x F^x+\xi_y F^y+\nu_1\xi_z F^z)-(\beta^2-1)(\nu_1\xi_x^2+\xi_y^2+\nu_1\xi_z^2)(\xi_x F^x+\xi_y F^y$$

$$\left.+\xi_z F^z)\right\}\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1} e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}\,dw \quad (19)$$

$$\tau_{zz}= -i\rho(2\pi)^{-2}\int_{W_4}\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T}\ \ [(\nu_1-1)(\xi_y^2+\xi_x^2)+\omega^2](\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)+$$

$$\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\nu_1\xi_x F^x+\nu_1\xi_y F^y+\xi_z F^z)-(\beta^2-1)(\nu_1\xi_x^2+\nu_1\xi_y^2+\xi_z^2)(\xi_x F^x+\xi_y F^y+$$

$$\left.\xi_z F^z)\right\}\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1} e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}\,dw \quad (20)$$

$$\tau_{yx} = \frac{-i\rho}{2\pi\beta^2} \int_{W_4} \left\{ \frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T} \quad (\xi_y^2+\xi_x^2)(\xi_x^2+\xi_y^2+\xi_z^2- \beta^2\omega^2) + \beta^2(\xi_x^2+\xi_y^2+\xi_z^2- \omega^2) \right.$$

$$\left. (\xi_x F^y+\xi_y F^x) - 2(\beta^2-1)\xi_y\xi_x(\xi_y F^y+\xi_x F^x) \right\} \left[(\xi_x^2+\xi_y^2+\xi_z^2- \omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1}$$

$$e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}\, dw \qquad (21)$$

$$\tau_{yz} = \frac{-i\rho}{2\pi\beta^2} \int_{W_4} \left\{ \frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T} \quad (\xi_y^2+\xi_z^2)(\xi_x^2+\xi_y^2+\xi_z^2- \beta^2\omega^2) + \beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2) \right.$$

$$\left. (\xi_z\bar{F}^y+\xi_y\bar{F}^z) - 2(\beta^2-1)\xi_y\xi_z(\xi_y\bar{F}^y+\xi_z\bar{F}^z) \right\} \left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2- \beta^2\omega^2)\right]^{-1}$$

$$e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}\, dw \qquad (22)$$

$$\tau_{zx} = \frac{-i\rho}{2\pi\beta^2} \int_{W_4} \left\{ \frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\bar{T} \quad (\xi_x^2+\xi_z^2)(\xi_x^2+\xi_y^2+\xi_z^2- \beta^2\omega^2) + \beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2) \right.$$

$$(\xi_z F^x + \xi_x F^z) - 2(\beta^2-1)\xi_x \xi_z (\xi_x F^x + \xi_z F^z)\Big\} \left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1}$$

$$e^{-i(x\xi_x + y\xi_y + z\xi_z + \omega\tau)} \, dw \qquad (23)$$

where W_4 is the $\xi_x \xi_y \xi_z \omega$ space and $dw = d\xi_x d\xi_y d\xi_z d\omega$.

2.Solution

We consider the solution of the equation of motion when the time dependent body force X and temperature T, acting at the origin in the direction of x_1 increasing, which varies harmonically with time period $2\pi/p$. For such cases we may write

$$X = F\rho^{-1}\delta(x)\delta(y)\delta(z)\delta(t) \qquad (24)$$

and

$$T = T_o(2\mu)^{-1}\delta(x)\delta(y)\delta(z)\delta(t) \qquad (25)$$

which gives us for $\bar{X}$ and $\bar{T}$, the relations

$$\bar{X} = (FC_1/\rho)\ (2\pi)^{-2} \qquad (26)$$

and

$$\bar{T} = (T_o C_1/2\mu)\ (2\pi)^{-2} \qquad (27)$$

since $\delta(t) = C_1\delta(\tau)$

For the sake of simplicity, we assume $F^x = x$, $F^y = 0$, $F^z = 0$. We shall adopt the usual notations for this problem i.e. $v^x = u_T, v^y = v_T, v^z = w_T$

$\tau_{xx} = (\sigma_x)_T, \tau_{yy} = (\sigma_y)_T, \tau_{zz} = (\sigma_z)_T, \tau_{xy} = (\tau_{xy})_T, \tau_{yz} = (\tau_{yz})_T, \tau_{xz} = (\tau_{xz})_T.$

Putting $F^x = x$, $F^y = 0$, $F^z = 0$ and using the usual notations we get from (15)-(17),

$$u_T = (2\pi C_1)^{-2}\int_{W_4}\left\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\ \xi_x(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)^{-1}\right] + \left[\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)\right.$$

$$\left.-(\beta^2-1)\xi_x^2\right]\overline{X}\ \left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1}\Big\}e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}dw$$

$$v_T = (2\pi C_1)^{-2}\int_{W_4}\left\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\quad \xi_y(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)^{-1}\right]-(\beta^2-1)\xi_x\xi_y\overline{X}$$

$$\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1}\Big\}e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}dw$$

$$w_T=(2\pi C_1)^{-2}\int_{W_4}\left\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\ \xi_z(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)^{-1}\right]-(\beta^2-1)\xi_x\xi_z\overline{X}$$

$$\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)\right]^{-1}\Bigg\}e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}dw$$

Rewriting u_T, v_T, w_T. We get

$$u_T=(2\pi C_1)^{-2}\int_{W_4}\left\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\ \xi_x(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)^{-1}\right]+\left[\xi_x^2(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)^{-1}+\right.$$

$$\left.+\ \beta^2(\xi_y^2+\xi_z^2)(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)^{-1}\right]\overline{X}\ \left[\xi_x^2+\xi_y^2+\xi_z^2\right]^{-1}\Bigg\}\ e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}dw$$

(28)

$$v_T=(2\pi C_1)^{-2}\int_{W_4}\left\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\ \xi_y(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)^{-1}\right]+\left[(\xi_x^2+\xi_y^2+\xi_z^2-\omega^2)^{-1}-\right.$$

$$\left.\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)^{-1}\right]\overline{X}\xi_x\xi_y\left[\xi_x^2+\xi_y^2+\xi_z^2\right]^{-1}\Bigg\}e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}dw \qquad (29)$$

$$w_T=(2\pi C_1)^{-2}\int_{W_4}\left\{\frac{2iC_1^2}{\beta^2}(\beta^2-1)\alpha\overline{T}\ \xi_z(\xi_x^2+\xi_y^2+\xi_z^2-\ \beta^2\omega^2)^{-1}\right]+\Big[(\xi_x^2+\xi_y^2+\xi_z^2-\ \omega^2)^{-1}-$$

$$\beta^2(\xi_x^2+\xi_y^2+\xi_z^2-\beta^2\omega^2)^{-1}\Big]\overline{X}\xi_x\xi_z\left[\xi_x^2+\xi_y^2+\xi_z^2\right]^{-1}\Big\}e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)}dw \tag{30}$$

Substituting the value of $\overline{X}$ and $\overline{T}$ from (26) and (27) into (28),(29) and (30)

$$u_T=\frac{-FC_1}{16\pi^4\mu\beta^2}\left[\frac{\partial^2}{\partial x^2}I_1+\beta^2\left\{\frac{\partial^2}{\partial y^2}\ +\ \frac{\partial^2}{\partial z^2}\right\}I_2\right]-\ \frac{(\beta^2-1)}{16\pi^4\mu\beta^2}\ \alpha T_o C_1\frac{\partial}{\partial x}\left\{\frac{\partial^2}{\partial x^2}+\ \frac{\partial^2}{\partial y^2}+\ \frac{\partial^2}{\partial z^2}\right\}I_2 \tag{31}$$

$$v_T=\frac{-FC_1}{16\pi^4\mu\beta^2}\ \frac{\partial^2}{\partial y\partial x}(I_1-\beta^2 I_2)-\ \frac{(\beta^2-1)}{16\pi^4\mu\beta^2}\ \alpha T_o C_1\frac{\partial}{\partial y}\left\{\frac{\partial^2}{\partial x^2}+\ \frac{\partial^2}{\partial y^2}+\ \frac{\partial^2}{\partial z^2}\right\}I_2 \tag{32}$$

$$w_T=\frac{-FC_1}{16\pi^4\mu\beta^2}\ \frac{\partial^2}{\partial z\partial x}(I_1-\beta^2 I_2)-\ \frac{(\beta^2-1)}{16\pi^4\mu\beta^2}\ \alpha T_o C_1\frac{\partial}{\partial z}\left\{\frac{\partial^2}{\partial x^2}+\ \frac{\partial^2}{\partial y^2}+\ \frac{\partial^2}{\partial z^2}\right\}I_2 \tag{33}$$

where

$$I_1 = \int_{W_4} \left[(\xi_x^2+\xi_y^2+\xi_z^2)(\xi_x^2+\xi_y^2+\xi_z^2- \omega^2)\right]^{-1} e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)} dw \tag{34}$$

$$I_2 = \int_{W_4} \left[(\xi_x^2+\xi_y^2+\xi_z^2)(\xi_x^2+\xi_y^2+\xi_z^2- \beta^2\omega^2)\right]^{-1} e^{-i(x\xi_x+y\xi_y+z\xi_z+\omega\tau)} dw \tag{35}$$

After making necessary substitution and integration [19,33]we find that $\partial I_2/\partial r$ is a function of r only, and that

$$\partial I_2/\partial r = -\frac{4\pi^2}{\beta}\int_0^\infty \frac{\sin(\rho\tau\beta^{-1})}{\rho} J_1(\rho r)d\rho$$

$$= -\frac{4\pi^2\tau}{\beta^2 r}, \qquad (\tau\le\beta r)$$

$$= -\frac{4\pi^2}{\beta^2 r}\left\{(\tau-\sqrt{\tau^2-\beta^2 r^2}\right\}, \quad (\tau\ge\beta r) \tag{36}$$

A similar expression can be obtained for $\partial I_1/\partial r$ by putting β=1 in equation (36). Substituting these values into equations (28)-(30) we obtain the formula

$$u_T = \begin{cases} 0, & r > \tau \\ \dfrac{FC_1}{4\mu\pi^2\beta^2}\left[\dfrac{x^2}{r^2}(\tau^2-r^2)^{-1/2} + \dfrac{x^2-y^2-z^2}{r^6}(\tau^2-r^2)^{1/2}\right], & (\tau' < r < \tau) \\ \dfrac{\alpha C_1}{4\mu\pi^2\beta^3}(\beta^2-1)T_o(\tau'^2-r^2)^{-2} + & \\ \dfrac{FC_1}{4\mu\pi^2\beta^2}\left[\dfrac{x^2}{r^2}(\tau^2-r^2)^{-1/2} + \dfrac{\beta}{r^2}(y^2+z^2)(\tau'^2-r^2)^{1/2} + \dfrac{x^2-y^2-z^2}{r^6}\times\right. & \\ \left.\left\{(\tau^2-r^2)^{1/2} - 2\beta(\tau'^2-r^2)^{1/2}\right\}\right] & r < \tau' \end{cases}$$

$$v_T = \begin{cases} 0, & r > \tau \\ \dfrac{FC_1}{4\mu\pi^2\beta^2}\,\dfrac{xy}{r^2}\left[(\tau^2-r^2)^{-1/2} + \dfrac{2}{r^2}(\tau^2-r^2)^{1/2}\right], & (\tau' < r < \tau) \\ \dfrac{\alpha C_1}{4\mu\pi^2\beta^3}(\beta^2-1)T_o\,y(\tau'^2-r^2)^{-2} + & \\ + \dfrac{FC_1}{4\mu\pi^2\beta^2}\,\dfrac{xy}{r^2}\left[(\tau^2-r^2)^{-1/2} - \beta(\tau^2-r^2)^{1/2} + \dfrac{2}{r^2}\right. & \\ \left.\left\{(\tau^2-r^2)^{1/2} - \beta(\tau'^2-r^2)^{1/2}\right\}\right] & r < \tau' \end{cases}$$

$$w_T = \begin{cases} 0, & r>\tau \\ \dfrac{FC_1}{4\mu\pi^2\beta^2}\,\dfrac{xz}{r^2}\left[(\tau^2-r^2)^{-1/2} + \dfrac{2}{r^2}(\tau^2-r^2)^{1/2}\right], & (\tau'<r<\tau) \\ \dfrac{\alpha C_1}{4\mu\pi^2\beta^3}(\beta^2-1)T_o z(\tau'^2-r^2)^{-2} + & \\ + \dfrac{FC_1}{4\mu\pi^2\beta^2}\,\dfrac{xz}{r^2}\left[(\tau^2-r^2)^{-1/2} - \beta(\tau^2-r^2)^{1/2} + \dfrac{2}{r^2}\right. & \\ \left.\left\{(\tau^2-r^2)^{1/2} - \beta(\tau'^2-r^2)^{1/2}\right\}\right] & r<\tau' \end{cases}$$

In these formulae $\tau = C_1 t$, $\tau' = C_2 t = C_1 t/\beta$

So, the components of displacement vector are known completely.

Differentiating u_T, v_T and w_T and substituting these values in expanded and simplified form of (12) we obtain the resultant of principal stresses as

$$\frac{(4-\beta^2)\pi^2\beta^2}{(\beta^2-1)FC_1}\left[(\sigma_x)_T+(\sigma_y)_T+(\sigma_z)_T\right] = \left[x(\tau^2-r^2)^{-3/2} - \frac{x^2-y^2-z^2}{r^6}\,x\left\{(\tau^2-r^2)^{-1/2} + \right.\right.$$

$$\left.\frac{6}{r^2}(\tau^2-r^2)^{1/2}\right\} - \frac{4x}{r^4}\,(y^2+z^2)\left\{(\tau^2-r^2)^{-1/2} + \frac{2}{r^2}\,(\tau^2-r^2)^{1/2}\right\} + \frac{2x}{r^4}(2+r^{-2})\times$$

$$(\tau^2-r^2)^{1/2}+\frac{2x}{r^2}(2-r^{-2})(\tau^2-r^2)^{-1/2}\Big], \qquad \tau'<r<\tau$$

$$= 0, \qquad r>\tau$$

$$= \frac{T_o\alpha(\beta^2-1)}{F\beta}\left[3(\tau'^2-r^2)^{-2}+\frac{r^2}{4}(\tau'^2-r^2)^{-3}\right]+x(\tau^2-r^2)^{-3/2}+\frac{2x^3}{r^4}\Big\{(\tau^2-r^2)^{1/2}$$

$$-\beta(\tau'^2-r^2)^{-1/2}\Big\}+\frac{x^2-y^2-z^2}{r^6}x\Big\{2\beta(\tau'^2-r^2)^{-1/2}-(\tau^2-r^2)^{-1/2}\Big\}+\frac{x^2-y^2-z^2}{r^6}$$

$$4x\Big\{(\tau^2-r^2)^{1/2}-\beta(\tau'^2-r^2)^{1/2}\Big\}+\frac{2x}{r^6}\Big\{(\tau^2-r^2)^{1/2}-2\beta(\tau'^2-r^2)^{1/2}\Big\}$$

$$+\frac{x^2-y^2-z^2}{r^6}6x\Big\{2\beta(\tau'^2-r^2)^{1/2}-(\tau^2-r^2)^{1/2}\Big\}, \qquad r<\tau' \qquad (37)$$

$$\frac{4\pi^2\beta^2}{yFC_1}(\tau_{xy})_T=2yx^2r^{-2}(\tau^2-r^2)^{-3/2}-\frac{x^2-y^2-z^2}{r^6}y\Big\{(\tau^2-r^2)^{-1/2}+\frac{6}{r^2}(\tau^2-r^2)^{1/2}\Big\}$$

$$- \frac{2x^2y}{r^4}\left\{3(\tau^2-r^2)^{-1/2} + \frac{4}{r^2}(\tau^2-r^2)^{1/2}\right\} + \frac{y}{r^2}\left\{(\tau^2-r^2)^{1/2} + \frac{2}{r^2}(1-r^{-2})\right.$$

$$\left.(\tau^2-r^2)^{-1/2}\right\}, \qquad \tau'<r<\tau$$

$$= 0, \qquad r>\tau$$

$$= \frac{2T_o\alpha(\beta^2-1)}{F\beta} x(\tau'^2-r^2)^{-3} + \frac{y}{r^2}\left\{2x^2(\tau^2-r^2)^{-3/2} + \beta(y^2+z^2-x^2)(\tau'^2-r^2)^{-3/2}\right\}$$

$$- \frac{x^2-y^2-z^2}{r^8} 6y\left\{(\tau^2-r^2)^{1/2} - 2\beta(\tau'^2-r^2)^{1/2}\right\} + \frac{2y}{r^6}\left\{(\tau^2-r^2)^{1/2} - 2\beta(\tau'^2-r^2)^{1/2}\right\}$$

$$+ \frac{y^2+z^2-3x^2}{r^6} 2y\left\{(\tau^2-r^2)^{1/2} - \beta(\tau'^2-r^2)^{1/2}\right\} + \frac{y}{r^2}\left\{(\tau^2-r^2)^{-1/2} + \beta(\tau'^2-r^2)^{-1/2}\right\}$$

$$- \frac{x^2-y^2-z^2}{r^6} y\left\{(\tau^2-r^2)^{-1/2} - 2\beta(\tau'^2-r^2)^{-1/2}\right\} - \frac{2y}{r^4}\left\{3x^2(\tau^2-r^2)^{-1/2} +\right.$$

$$\left.\beta(y^2+z^2-2x^2)(\tau'^2-r^2)^{-1/2}\right\}, \qquad r<\tau' \qquad (38)$$

$$\frac{4\pi^2\beta^2}{yzFC_1}(\tau_{yz})_\tau = 2yxzr^{-2}\left\{(\tau^2-r^2)^{-3/2} - \frac{4}{r^2}(\tau^2-r^2)^{-1/2} - \frac{8}{r^4}(\tau^2-r^2)^{1/2}\right\} \qquad \tau'<r<\tau$$

$$= 0, \qquad r>\tau$$

$$= \frac{2T_o\alpha(\beta^2-1)}{F\beta}(\tau'^2-r^2)^{-3} + \frac{2xyz}{r^2}\left\{(\tau^2-r^2)^{-3/2} - \beta(\tau'^2-r^2)^{-3/2}\right\} - \frac{16xyz}{r^6}\times$$

$$\left\{(\tau^2-r^2)^{1/2} - \beta(\tau'^2-r^2)^{1/2}\right\} - \frac{8xyz}{r^4}\left\{(\tau^2-r^2)^{-1/2} - \beta(\tau'^2-r^2)^{-1/2}\right\} \qquad r<\tau'$$

(39)

$$\frac{4\pi^2\beta^2}{zFC_1}(\tau_{zx})_\tau = -\frac{x^2-y^2-z^2}{r^6}z\left\{(\tau^2-r^2)^{-1/2} + \frac{6}{r^2}(\tau^2-r^2)^{1/2}\right\} - \frac{2x^2z}{r^4}\times$$

$$\left\{3(\tau^2-r^2)^{-1/2} + \frac{4}{r^2}(\tau^2-r^2)^{1/2}\right\} + \frac{z}{r^2}\left\{(\tau^2-r^2)^{-1/2} + \frac{2}{r^2}(1-r^{-2})(\tau^2-r^2)^{1/2} + \right.$$

$$\left. 2x(\tau^2-r^2)^{-3/2}\right\} \qquad \tau'<r<\tau$$

$$= 0, \qquad r>\tau$$

$$= \frac{2T_o\alpha(\beta^2-1)}{F\beta}(\tau'^2-r^2)^{-3} + \frac{z}{r^2}\left\{2x^2(\tau^2-r^2)^{-3/2}+\beta(y^2+z^2-x^2)(\tau'^2-r^2)^{-3/2}\right\} -$$

$$\frac{x^2-y^2-z^2}{r^8}6z\left\{(\tau^2-r^2)^{1/2}-2\beta(\tau'^2-r^2)^{1/2}\right\} - \frac{2z}{r^6}\left\{(\tau^2-r^2)^{1/2}-2\beta(\tau'^2-r^2)^{1/2}\right\}$$

$$+2z\frac{y^2+z^2-3x^2}{r^6}\left\{(\tau^2-r^2)^{1/2}-\beta(\tau'^2-r^2)^{1/2}\right\}+ \frac{z}{r^2}\left\{(\tau^2-r^2)^{-1/2}+\beta(\tau'^2-r^2)^{-1/2}\right\}$$

$$- \frac{x^2-y^2-z^2}{r^6}z\left\{(\tau^2-r^2)^{-1/2}-2\beta(\tau'^2-r^2)^{-1/2}\right\} - \frac{2z}{r^4}\left\{3x^2(\tau^2-r^2)^{-1/2} +\right.$$

$$\left.\beta(y^2+z^2-2x^2)(\tau'^2-r^2)^{-1/2}\right\}, \qquad r<\tau' \qquad (40)$$

From the analysis of the above expressions it is clear that the disturbance is propagated outwards from the centre with velocities C_1 and $C_2 = C_1/\beta$. These waves are known in seismology as the P-wave and S-waves respectively. The wave fronts are circles, centre the origin and radii $\tau=C_1t$, $\tau'=C_2t$. At the wave front the components of stress and of displacement have finite discontinuities. This is of course, an impossible situation to arise in a perfectly elastic solid; it is due to the representation of the impulsive applied force by the idealized Dirac Delta function.

3. Numerical Results

The fig.4 shows the displacement q in the r- direction at the point with polar coordinates produced by an impulsive heat nucleus and body force applied at the origin at time t=0 in an infinite elastic solid of Poission's ratio .25 (i.e. $\lambda=\mu$). the solid curve corresponds to $\theta=0^{\circ}$, the dotted curve to $\theta=45^{\circ}$ and the broken curve to $\theta=90^{\circ}$. In the figure, variation of displacement components is shown. The interesting fact is that the direction in which the temperature distribution and body force are acting the wave front of P- wave is an infinite discontinuity, but not so the wave front of S-wave, which in a direction perpendicular to this, the wave front of S-wave is an infinite discontinuity, but not the wave front of P-wave. In direction in between to these both wave fronts are infinite discontinuities. This fact may explain some the discrepancies existing in the interpretation of geophysical observations, since in the first case, the arrival of S- wave and in the second that of the P- wave would not be apparent.

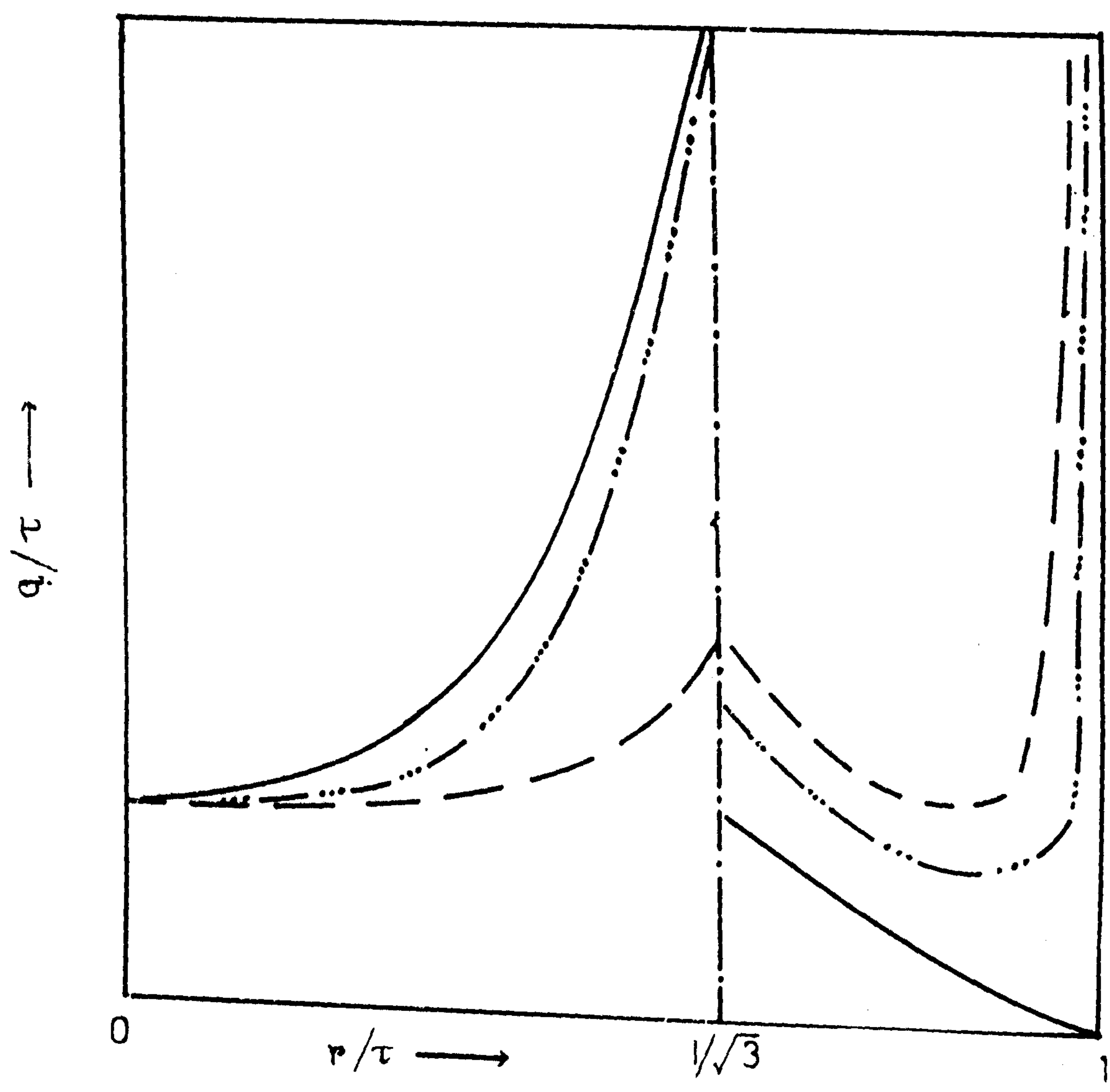

Fig.4

CHAPTER - IV

STRESSES DUE TO HEAT EXPOSURE ON THE BOUNDING SURFACE OF ELASTIC SEMI-SPACE

Paper 1 : Axisymmetric stress distribution in a semi-infinite elastic solid with constant heat flow over an elliptic area on the plane boundary.

(PUBLISHED IN IND J. THEO. PHY., 40(1), 1992)

Paper 2 : Three dimensional thermal stresses due to periodic supply of heat on the straight edges of a semi-infinite thick plate.

(PUBLISHED IN IND J. THEO. PHY., 41(4), 1993)

Paper 3 : Thermal stresses due to prescribed flux of heat on the surface of a thick plate.

Paper 4 : Thermal stresses due to prescribed flux of heat on the boundary of a semi-infinite elastic solid.

Axisymmetric Stress Distribution In A Semi-infinite Elastic Solid With Constant Heat-flow Over An Elliptic Area On The Plane Boundary

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1.INTRODUCTION

In this paper, axisymmetric thermal stresses in a semi-infinite elastic solid have been obtained when there is a constant supply of heat over an elliptic area on the bounding plane surface, the rest being kept at a constant temperature. Temperature and the potential of the thermo-elastic displacement are obtained in terms of Matheiu functions employing the curvilinear coordinates due to C.B.Ling [44].

2. METHOD OF SOLUTION

Let us introduce elliptical coordinates (ξ,η) connected with cartesian coordinates in the form

$$x+iy = h\cosh(\xi+i\eta) \tag{1}$$

where $0\le \eta \le 2\pi$, and 2h is the distance between the foci. Let the bounding surface of the semi-infinite elastic solid be given by $z=0$, the axis of z being drawn into the body. The temperature field in the steady state is given by the differential equation [3]

$$\nabla^2 T = 0 \tag{2}$$

and the boundary conditions are

$$-K \frac{\partial T}{\partial z} = Q, \quad \xi < \xi_o, \quad z=0$$

$$T = 0, \quad \xi > \xi_o, \quad z=0 \tag{3}$$

$$T = 0 \quad \text{at infinity} \tag{4}$$

where K = thermal conductivity

Q = rate of flow of heat per unit area.

Equations (2) and (4) are satisfied, if we take [10]

$$T = \sum_{n=0}^{\infty} c_{2n} c_{e_{2n}}(\xi, q)\, C_{e_{2n}}(\eta, q)\, e^{-nz} \tag{5}$$

where c_{2n} is a constant to be determined from the boundary conditions (3) and

$$c_{e_{2n}}(\xi, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh(2r\xi)$$

$$C_{e_{2n}}(\eta, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos(2r\eta)$$

are Mathieu functions of integral order [2], q being a constant obtainable from Mathieu's equation [10].

On the plane z=0, the following relations are to be satisfied

$$\sum_{n=0}^{\infty} n c_{2n} c_{e_{2n}}(\xi, q)\ C_{e_{2n}}(\eta, q) = Q/K\ ,\ \xi < \xi_o$$

$$\sum_{n=0}^{\infty} c_{2n} c_{e_{2n}}(\xi, q)\ C_{e_{2n}}(\eta, q) = 0\ ,\ \xi > \xi_o \qquad (6)$$

Multiplying both sides of(6) by $C_{e_{2n}}(\eta, q)$ and integrating w.r.t. η from 0 to 2π and w.r.t. ξ from 0 to ξ_o we obtain

$$\frac{2\pi Q}{K} A_o^{(2n)} = \pi n\ c_{2n} \int_0^{\xi_o} c_{e_{2n}}(\xi, q)\ d\xi$$

where in general [10]

$$A_{m+2r}^{(m)} \cong (-1)^r \frac{m!}{r!(m+r)!} t^r, \qquad r \geq 0,\ m > 0$$

and t is a function of q.

Therefore, $c_{2n} = 2QM_n/K$

where

$$M_n = \frac{A_o^{(2n)}}{n\int_0^{\xi_o} c_{e_{2n}}(\xi, q)\ d\xi} \qquad (7)$$

The temperature T is ,therefore, given by

$$T = \frac{2Q}{K} \sum_{n=0}^{\infty} M_n c_{e_{2n}}(\xi, q)\, C_{e_{2n}}(\eta, q)\, e^{-nz} \tag{8}$$

To determine the stresses, the potential of thermo-elastic displacement ψ will be used. This related to the displacement components u,v,w by the equations

$$\frac{\partial \psi}{\partial x} = u, \quad \frac{\partial \psi}{\partial y} = v, \quad \frac{\partial \psi}{\partial z} = w \tag{9}$$

From the stress-strain relation in problems of thermal stresses and the equation of equilibrium [17] we have

$$\nabla^2 \psi = \beta T$$

$$= \frac{2Q}{K} \beta \sum_{n=0}^{\infty} M_n c_{e_{2n}}(\xi, q)\, C_{e_{2n}}(\eta, q)\, e^{-nz} \tag{10}$$

where $\beta = \frac{1+\nu}{1-\nu}\alpha$, ν = Poission's ratio, α is the coefficient of linear thermal expansion.

A particular integral of equation (10) is given by

$$\psi = -\frac{Q\beta z}{K} \sum_{n=0}^{\infty} n^{-1} M_n c_{e_{2n}}(\xi, q)\, C_{e_{2n}}(\eta, q)\, e^{-nz} \tag{11}$$

Now the stress components $\widehat{(\xi\xi)}_T$, $\widehat{(\eta\eta)}_T$, $\widehat{(zz)}_T$, $\widehat{(\xi z)}_T$ are calculated as

$$\frac{\widehat{(\xi\xi)}_T}{2\mu} = 2[h(\cosh(2\xi)-\cos(2\eta))]^{-2}\frac{Q\beta z}{K}\sum_{n=0}^{\infty} n^{-1}M_n\Big[\sinh 2\xi\, c'_{e_{2n}}(\xi,q)C_{e_{2n}}(\eta,q)$$

$$-\sin(2\eta)c_{e_{2n}}(\xi,q)\; C'_{e_{2n}}(\eta,q)\Big]e^{-nz} - \frac{2Q\beta}{K}\sum_{n=0}^{\infty} M_n c_{e_{2n}}(\xi,q)\; C_{e_{2n}}(\eta,q)e^{-nz} +$$

$$+2[h^2(\cosh(2\xi)-\cos(2\eta))]^{-1}\frac{\partial^2\psi}{\partial\xi^2},$$

$$\frac{\widehat{(\eta\eta)}_T}{2\mu} = 2[h(\cosh(2\xi)-\cos(2\eta))]^{-2}\frac{Q\beta z}{K}\sum_{n=0}^{\infty} n^{-1}M_n\Big[\sin 2\eta\; c_{e_{2n}}(\xi,q)C'_{e_{2n}}(\eta,q)$$

$$-\sinh(2\xi)c'_{e_{2n}}(\xi,q)\; C_{e_{2n}}(\eta,q)\Big]e^{-nz} - \frac{2Q\beta}{K}\sum_{n=0}^{\infty} M_n c_{e_{2n}}(\xi,q)C_{e_{2n}}(\eta,q)e^{-nz} +$$

$$+2[h^2(\cosh(2\xi)-\cos(2\eta))]^{-1}\frac{\partial^2\psi}{\partial\eta^2}$$

(12)

$$\frac{\widehat{(zz)}_T}{2\mu} = -\frac{zQ\beta}{K}\sum_{n=0}^{\infty} nM_n c_{e_{2n}}(\xi,q)C_{e_{2n}}(\eta,q)e^{-nz} ,$$

$$\frac{\widehat{(\xi z)}_T}{2\mu} = \sqrt{\cosh^2 2\xi - \cos^2 2\eta}\ \frac{Q\beta}{K}\sum_{n=0}^{\infty} n^{-1}M_n(nz-1)\Big[\sinh(2\xi)c'_{e_{2n}}(\xi,q)C_{e_{2n}}(\eta,q)$$

$$- \sin(2\eta)c_{e_{2n}}(\xi,q)\ C'_{e_{2n}}(\eta,q)\Big]\ e^{-nz} \qquad (13)$$

$\widehat{(\xi\eta)}_T$ and $\widehat{(\eta z)}_T$ do not appear due to symmetry. Here prime stands for differentiation w.r.t. ξ and η.

It is observed that the normal stress $\widehat{(zz)}_T$ vanishes for z=0. The stress $\widehat{(\xi z)}_T$, however, does not vanish. In order to suppress it, the stress system $\widehat{(\xi\xi)}_C$, $\widehat{(\eta\eta)}_C$, $\widehat{(zz)}_C$, $\widehat{(\xi z)}_C$ obtained on the hypothesis that there is no temperature distribution is to be superposed.

3. Complimentary Stresses

In order to determine the complimentary stresses we use Love's function ϕ satisfying the biharmonic equation [9]

$$\nabla^4 \phi = 0 \qquad (14)$$

with the boundary conditions

$$\left|\widehat{(zz)}_c\right|_{z=0} = 0$$

$$\left|\widehat{(\xi z)}_T + \widehat{(\xi z)}_c\right|_{z=0} = 0$$

and $\phi = 0$ at infinity. (15)

Let us assume the function ϕ in the form

$$\phi = \sum_{n=0}^{\infty} (C+Dnz)\, ce_{2n}(\xi, q)\; Ce_{2n}(\eta, q)\; e^{-nz} \tag{16}$$

where C and D are functions of n.

The complimentary stresses are given by [16]

$$\widehat{(\xi\xi)}_c = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}\left[\nu\nabla^2\phi - h_1^{-1} e_{\xi\xi}\right]$$

$$\widehat{(\eta\eta)}_c = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}\left[\nu\nabla^2\phi - h_2^{-1} e_{\eta\eta}\right]$$

$$\widehat{(zz)}_c = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - e_{zz}\right]$$

$$\widehat{(\xi z)}_c = \frac{2\mu}{1-2\nu}\frac{\partial}{\partial z}\left[(1-\nu)\nabla^2\phi - e_{zz}\right] \tag{17}$$

where $h_1^2 = h_2^2 = 2[h^2(\cosh(2\xi)-\cos(2\eta))]^{-1}, \quad h_3 = 1.$

Using (16) in (17), we have

$$\frac{\widehat{(\xi\xi)}_c}{2\mu} = \frac{1}{1-2\nu}\left\{ \sum_{n=0}^{\infty} 2\nu n^3 D c_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) e^{-nz} - 2[h(\cosh(2\xi)-\cos(2\eta))]^{-2} \right.$$

$$\sum_{n=0}^{\infty} n(D-C-Dnz)\left[-\sinh 2\xi \; c'_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) + \sin 2\eta \; c_{e_{2n}}(\xi,q) C'_{e_{2n}}(\eta,q)\right]$$

$$\left. \times e^{-nz} - 2[h^2(\cosh(2\xi)-\cos(2\eta))]^{-1} \frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial\xi^2} \right\},$$

$$\frac{\widehat{(\eta\eta)}_c}{2\mu} = \frac{1}{1-2\nu}\left\{ \sum_{n=0}^{\infty} 2\nu n^3 D c_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) e^{-nz} - 2[h(\cosh(2\xi)-\cos(2\eta))]^{-2} \right.$$

$$\sum_{n=0}^{\infty} n(D-C-Dnz)\left[\sinh 2\xi \; c'_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) - \sin 2\eta \; c_{e_{2n}}(\xi,q) C'_{e_{2n}}(\eta,q)\right]$$

$$\left. \times e^{-nz} - 2[h^2(\cosh(2\xi)-\cos(2\eta))]^{-1} \frac{\partial}{\partial z}\frac{\partial^2\phi}{\partial\eta^2} \right\},$$

$$\frac{\widehat{(zz)}_c}{2\mu} = \frac{1}{1-2\nu} \sum_{n=0}^{\infty} n^3 [D-2\nu D+C+Dnz] c_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) e^{-nz}$$

$$\frac{\widehat{(\xi z)}_c}{2\mu} = [(1-2\nu)h]^{-1} \sqrt{2(\cosh^2 2\xi - \cos^2 2\eta)} \sum_{n=0}^{\infty} n^2 (2\nu D-C-Dnz)\Big[\sinh(2\xi) \times$$

$$c'_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) - \sin(2\eta) c_{e_{2n}}(\xi,q)\; C'_{e_{2n}}(\eta,q)\Big]\, e^{-nz} \qquad (18)$$

In view of the first boundary condition

$$C = -D(1-2\nu) \qquad (19)$$

The second boundary conditions gives

$$D = \frac{M_n \beta Q(1-2\nu)}{Kn^3} \qquad (20)$$

With these values of C and D, the components of complimentary stresses are known.

Therefore, the resultant principal stresses are given by

$$\widehat{(\xi\xi)} = \widehat{(\xi\xi)}_T + \widehat{(\xi\xi)}_C$$

$$\widehat{(\eta\eta)} = \widehat{(\eta\eta)}_T + \widehat{(\eta\eta)}_C \qquad (21)$$

$$\widehat{(\xi\xi)} = \frac{4\mu Q\beta(\nu-1)}{K} \sum_{n=0}^{\infty} M_n \Big\{ c_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) - 2[nh(\cosh(2\xi)-\cos(2\eta))]^{-2} \times$$

$$\Big[\sinh(2\xi) c'_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) - \sin(2\eta) c_{e_{2n}}(\xi,q) C'_{e_{2n}}(\eta,q) - (\cosh(2\xi)-$$

$$\cos(2\eta))\Big[1 - \frac{n^2h^2}{2}\cosh(2\xi)\Big] c_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q)\Big]\Big\} e^{-nz},$$

$$\widehat{(\eta\eta)} = \frac{4\mu Q\beta(\nu-1)}{K} \sum_{n=0}^{\infty} M_n \Big\{ c_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) + 2[nh(\cosh(2\xi)-\cos(2\eta))]^{-2} \times$$

$$\Big[\sinh(2\xi) c'_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q) - \sin(2\eta) c_{e_{2n}}(\xi,q) C'_{e_{2n}}(\eta,q) - (\cosh(2\xi)-$$

$$\cos(2\eta))\Big[1 - \frac{n^2h^2}{2}\cosh(2\eta)\Big] c_{e_{2n}}(\xi,q) C_{e_{2n}}(\eta,q)\Big]\Big\} e^{-nz}$$

(22)

On the plane $z = 0$, we have from (22)

$$[\widehat{(\xi\xi)} + \widehat{(\eta\eta)}]_{z=0} = \frac{4\mu Q\beta(\nu-1)}{K} \sum_{n=0}^{\infty} M_n\, c_{e_{2n}}(\xi,q)\, C_{e_{2n}}(\eta,q) \tag{23}$$

$$[\widehat{(\xi\xi)}-\widehat{(\eta\eta)}]_{z=0} =$$

$$= \frac{8\mu Q\beta(\nu-1)}{K}[\cosh(2\xi)-\cos(2\eta)]^{-1}\sum_{n=0}^{\infty} n^{-2}M_n\Bigg\{\Bigg[2-\frac{n^2h^2}{2}(\cosh(2\eta)+\cos(2\eta))\Bigg]\times$$

$$c_{e_{2n}}(\xi,q)C_{e_{2n}}(\eta,q)-(\cosh(2\xi)-\cos(2\eta))^{-1}\Bigg[\sinh(2\xi)c'_{e_{2n}}(\xi,q)C_{e_{2n}}(\eta,q)-$$

$$\sin(2\eta)c_{e_{2n}}(\xi,q)\ C'_{e_{2n}}(\eta,q)\Bigg]\Bigg\} \qquad (24)$$

So the stresses are determined and the problem is solved.

Three Dimensional Thermal Stresses Due To periodic Supply Of Heat On The Straight Edges Of A Semi Infinite Thick Plate

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1.Introduction

This is a thermoelastic boundary value problem of three dimensions when these thermal stresses are produced in a body by unequal distribution of temperature which may be regarded as a specified function of coordinates and time. In this paper stresses due to periodic supply of heat produced by the blow of a jet flame on the straight edges of a semi-infinite isotropic elastic thick plate distributed over a finite portion of it, have been considered.

2.Solution

1. If T denotes the temperature at the point (x,y,z) and α, the coefficient of linear thermal expansion, we have the three dimensional equation of heat conduction as [3]

$$\frac{\partial T}{\partial t} = k \nabla_1^2 T \tag{1.1}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We have the following stress-strain relation in three dimensional problems of thermal stresses [13]

$$\varepsilon_x - \alpha T = E^{-1}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y - \alpha T = E^{-1}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z - \alpha T = E^{-1}\left[\sigma_z - \nu(\sigma_y + \sigma_x)\right]$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}, \quad \gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz}, \quad \gamma_{xz} = \frac{2(1+\nu)}{E}\tau_{xz} \qquad (1.2)$$

Solving for the stresses we find

$$\sigma_x = \frac{E}{1+\nu}\varepsilon_x + \frac{E\nu}{(1-2\nu)(1+\nu)}e - \frac{E\alpha T}{1-2\nu}$$

$$\sigma_y = \frac{E}{1+\nu}\varepsilon_y + \frac{E\nu}{(1-2\nu)(1+\nu)}e - \frac{E\alpha T}{1-2\nu}$$

$$\sigma_z = \frac{E}{1+\nu}\varepsilon_z + \frac{E\nu}{(1-2\nu)(1+\nu)}e - \frac{E\alpha T}{1-2\nu}$$

$$\tau_{xy} = \frac{1}{2}\frac{E}{1+\nu}\gamma_{xy}, \quad \tau_{yy} = \frac{1}{2}\frac{E}{1+\nu}\gamma_{yz}, \quad \tau_{xz} = \frac{1}{2}\frac{E}{1+\nu}\gamma_{xz} \qquad (1.3)$$

where $e = \varepsilon_x + \varepsilon_y + \varepsilon_z$.

Hence from the equations of equilibrium

$$\frac{\partial}{\partial x}(\sigma_x)+\frac{\partial}{\partial y}(\tau_{xy})+\frac{\partial}{\partial z}(\tau_{xz}) = 0$$

$$\frac{\partial}{\partial x}(\tau_{xy})+\frac{\partial}{\partial y}(\sigma_y)+\frac{\partial}{\partial z}(\tau_{yz}) = 0$$

$$\frac{\partial}{\partial x}(\tau_{xz})+\frac{\partial}{\partial y}(\tau_{yz})+\frac{\partial}{\partial z}(\sigma_z) = 0 \qquad (1.4)$$

We get when expressed in terms of displacements

$$\frac{\partial e}{\partial x} +(1-2\nu)\nabla_1^2 u =2(1+\nu)\alpha \frac{\partial T}{\partial x}$$

$$\frac{\partial e}{\partial y} +(1-2\nu)\nabla_1^2 v =2(1+\nu)\alpha \frac{\partial T}{\partial y}$$

$$\frac{\partial e}{\partial z} +(1-2\nu)\nabla_1^2 w =2(1+\nu)\alpha \frac{\partial T}{\partial z} \qquad (1.5)$$

Assuming that [17]

$$u = \frac{\partial \psi}{\partial x} \quad , \quad v = \frac{\partial \psi}{\partial y} \quad , \quad w = \frac{\partial \psi}{\partial z} \qquad (1.6)$$

where ψ is a function of x,y,z and also of time t, the relation (1.5) reduces to

$$(1-\nu)\frac{\partial}{\partial x}\,\nabla_1^2\psi = (1+\nu)\alpha\,\frac{\partial T}{\partial x}$$

$$(1-\nu)\frac{\partial}{\partial y}\,\nabla_1^2\psi = (1+\nu)\alpha\,\frac{\partial T}{\partial y}$$

$$(1-\nu)\frac{\partial}{\partial z}\,\nabla_1^2\psi = (1+\nu)\alpha\,\frac{\partial T}{\partial z} \tag{1.7}$$

These three equations are evidently satisfied if we take the function ψ as a solution of the equations [17]

$$\nabla_1^2\psi = \frac{1+\nu}{1-\nu}\,\alpha T \tag{1.8}$$

Differentiating equations (1.8) with respect to t and substituting for $\partial T/\partial t$ from relation (1.1) we get

$$\nabla_1^2\,\frac{\partial\psi}{\partial t} = \frac{1+\nu}{1-\nu}\,\alpha K\,\nabla_1^2 T$$

We may therefore take

$$\frac{\partial\psi}{\partial t} = \frac{1+\nu}{1-\nu}\,\alpha\,KT \tag{1.9}$$

2. Considering a semi infinite thick plate bounded by the plane with edges $y=0, z=0$, the axes of y and z being into the plate we can write the solution of (1.1) as

$$T = \int_0^\infty A(m) e^{-\alpha_m (y+z)} \cos(pt - \beta_m y) \cos(pt - \beta_m z) \cos(mx) \, dm \tag{2.1}$$

where

$$\alpha_m = \left[\frac{1}{2}m^2 + \sqrt{\frac{m^4}{4} + \frac{p^2}{4k^2}}\right]^{1/2}$$

$$\beta_m = \left[\frac{1}{2}m^2 + \sqrt{\frac{m^4}{4} + \frac{p^2}{4k}}\right]^{1/2} \tag{2.2}$$

and A(m) being an arbitrary function of m. From equation (1.9) the function ψ corresponding to this temperature becomes

$$\psi = \frac{1+\nu}{1-\nu} \frac{\alpha K}{4p} \int_0^\infty A(m) \, e^{-\alpha_m (y+z)} \sin\left[2pt - \beta_m (y+z)\right] \cos(mx) \, dm \tag{2.3}$$

The relation (2.3) represents a particular solution of the general equation (1.5). The corresponding displacements and stresses can now be calculated from relations (1.3) and (1.6)

$$u = -\frac{1+\nu}{1-\nu} \frac{\alpha K}{4p} \int_0^\infty m A(m) e^{-\alpha_m (y+z)} \sin\left[2pt - \beta_m (y+z)\right] \sin(mx) \, dm$$

$$v = w = -\frac{1+\nu}{1-\nu}\frac{\alpha K}{4p}\int_0^\infty A(m)e^{-\alpha_m(y+z)}\left[\alpha_m\sin[2pt-\beta_m(y+z)]+\beta_m\cos[2pt-\beta_m(y+z)]\right]$$

$$\times\cos(mx)\,dm \qquad (2.4)$$

$$\sigma_x = \frac{-E}{1-2\nu}\frac{\alpha K}{4p}\int_0^\infty A(m)e^{-\alpha_m(y+z)}\left[m^2\sin[2pt-\beta_m(y+z)]+\frac{2\nu}{1-\nu}\left\{(\beta_m^2-\alpha_m^2)\sin[2pt-\beta_m\right.\right.$$

$$\left.\left.\times(y+z)]-2\alpha_m\beta_m\cos[2pt-\beta_m(y+z)]\right\}+4pK^{-1}\cos(pt-\beta_m y)\cos(pt-\beta_m z)\right]\cos(mx)dm$$

$$\sigma_y = \sigma_z = \frac{-E}{(1-2\nu)(1-\nu)}\frac{\alpha K}{4p}\int_0^\infty A(m)e^{-\alpha_m(y+z)}\left[\left\{(\beta_m^2-\alpha_m^2)\sin[2pt-\beta_m(y+z)]\right.\right.$$

$$\left.-2\alpha_m\beta_m\cos[2pt-\beta_m(y+z)]\right\}+\nu m^2\sin[2pt-\beta_m(y+z)]+$$

$$\left.+4p(1-\nu)K^{-1}\cos(pt-\beta_m y)\cos(pt-\beta_m z)\right]\cos(mx)\,dm$$

$$\tau_{xy} = \tau_{zx} = \frac{E}{1-\nu}\frac{\alpha K}{4p}\int_0^\infty mA(m)e^{-\alpha_m(y+z)}\left[\left\{\alpha_m\sin[2pt-\beta_m(y+z)]\right.\right.$$

$$+\beta_m \cos[2pt-\beta_m(y+z)]\Big\}\sin(mx)\ dm$$

$$\tau_{yz} = \frac{E}{1-\nu}\,\frac{\alpha K}{4p}\int_0^{\infty} A(m)e^{-\alpha_m(y+z)}\Big[\Big\{(\alpha_m^2-\beta_m^2)\sin[2pt-\beta_m(y+z)]-$$

$$+2\alpha_m\beta_m\cos[2pt-\beta_m(y+z)]\Big\}\cos(mx)dm \qquad (2.5)$$

The stresses obtained in relations (2.5) are produced by the thermal expansion. This expansion gives rise to certain stresses on the boundary of the plate. We shall therefore make the boundary free from stresses by the addition of the extra terms obtained on the hypothesis that there is no temperature distribution. In order to nullify the stresses on the boundary yz=0 we are to superimpose a complementary stress -system ($\sigma_{x_1},\sigma_{y_1},\sigma_{z_1},\tau_{xy_1},\tau_{yz_1},\tau_{zx_1}$) such that

$$(\sigma_{y_1})_0 = -(\sigma_y)_0,\quad (\sigma_{z_1})_0 = -(\sigma_z)_0,\quad (\tau_{xy_1})_0 = -(\tau_{xy})_0\ ,\quad (\tau_{yz_1})_0 = -(\tau_{xy})_0,$$

$$(\tau_{zx_1})_0 = (\tau_{zx})_0 \qquad (2.6)$$

Considering the stress function

$$\psi = \int_0^{\infty} \Big[C(m)+yD(m)+zE(m)\Big]e^{-m(y+z)}\cos(mx)\ dm \qquad (2.7)$$

which satisfies the biharmonic equation, C(m), D(m) and E(m) being arbitrary functions of m, we have stresses [9]

$$\sigma_{x_1} = \left\{ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right\}\psi , \quad \sigma_{y_1} = \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right\}\psi, \quad \sigma_{z_1} = \left\{ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right\}\psi$$

$$\tau_{xy_1} = -\frac{\partial^2}{\partial x \partial y}\psi , \quad \tau_{yz_1} = -\frac{\partial^2}{\partial y \partial z}\psi , \quad \tau_{zx_1} = -\frac{\partial^2}{\partial x \partial z}\psi$$

So we get the complementary stresses as

$$\sigma_{x_1} = \int_0^\infty \left\{ 2m^2 \left[C(m) + yD(m) + zE(m) \right] - 2m\left[D(m) + E(m) \right] \right\} e^{-m(y+z)} \cos(mx)\, dm$$

$$\sigma_{y_1} = -\int_0^\infty 2mE(m) e^{-m(y+z)} \cos(mx)\, dm$$

$$\sigma_{z_1} = -\int_0^\infty 2mD(m) e^{-m(y+z)} \cos(mx)\, dm$$

$$\tau_{xy_1} = -\int_0^\infty \left\{ m^2 \left[C(m) + yD(m) + zE(m) \right] - mD(m) \right\} e^{-m(y+z)} \sin(mx)\, dm$$

$$\tau_{yz_1} = -\int_0^\infty \left\{ m^2 \left[C(m) + yD(m) + zE(m) \right] - m\left[D(m) + E(m) \right] \right\} e^{-m(y+z)} \cos(mx)\, dm$$

$$\tau_{zx_1} = -\int_0^\infty \left\{ m^2\left[C(m)+yD(m)+zE(m)\right]-mE(m)\right\} e^{-m(y+z)} \sin(mx)\ dm$$

Using relations (2.5),(2.6) and (2.8) and solving we get

$$D(m) = E(m) = \frac{-E\alpha K}{(1-\nu)(1-2\nu)8pm} A(m)\left\{(\beta_m^2-\alpha_m^2+\nu m^2)\sin(2pt)-2\alpha_m\beta_m\cos(2pt)+\right.$$

$$\left. + \frac{1-\nu}{K} 4p \cos^2 pt \right\}$$

$$C(m)= \frac{E\alpha K}{(1-\nu)4pm^2}A(m)\left\{\left[m\alpha_m - \frac{(\beta_m^2-\alpha_m^2+\nu m^2)}{2(1-2\nu)}\right]\sin(2pt)+\left[m\beta_m + \frac{\alpha_m\beta_m}{(1-2\nu)}\right]\cos(2pt)\right.$$

$$\left. - \frac{2p(1-\nu)}{K(1-2\nu)} \cos^2 pt \right\} \qquad (2.9)$$

With the value of this constants substituted from relation (2.9)

$$\sigma_{x_1} = \int_0^\infty \left\{\frac{E\alpha Km}{(1-\nu)2p} A(m)\left[[\alpha_m\sin(2pt)+\beta_m\cos(2pt)] - \frac{(\beta_m^2-\alpha_m^2+\nu m^2)}{2(1-2\nu)m}(my+mz-1)\right.\right.$$

$$\left.\left.\sin(2pt) + \frac{\alpha_m\beta_m}{(1-2\nu)m}(my+mz-1)\cos(2pt) - \frac{4p(1-\nu)}{K(1-2\nu)}\cos^2 pt\right\} e^{-m(y+z)} \cos(mx)dm$$

$$\sigma_{y_1} = \sigma_{z_1} = \frac{E\alpha K}{(1-2\nu)(1-\nu)4p} \int_0^\infty A(m)\left\{(\beta_m^2 - \alpha_m^2 + \nu m^2)\sin(2pt) - 2\alpha_m\beta_m\cos(2pt) + 4p(1-\nu)K^{-1}\cos^2 pt\right\} e^{-m(y+z)} \cos(mx)\, dm$$

$$\tau_{xy_1} = \tau_{zx_1} = -\int_0^\infty \left\{ \frac{E\alpha K m}{(1-\nu)4p} A(m) \left[\alpha_m \sin(2pt) + \beta_m \cos(2pt) - \frac{(\beta_m^2 - \alpha_m^2 + \nu m^2)}{2(1-2\nu)m}(y+z)\sin(2pt) + \frac{\alpha_m\beta_m}{(1-2\nu)m}(y+z)\cos(2pt) - \frac{2p(1-\nu)}{K(1-2\nu)}(y+z)\cos^2 pt \right\} e^{-m(y+z)} \cos(mx)\, dm$$

$$\tau_{yz_1} = -\int_0^\infty \frac{E\alpha K A(m)}{(1-\nu)(1-2\nu)4p} \left[m(1-2\nu)[\alpha_m \sin(2pt) + \beta_m \cos(2pt)] - \frac{(\beta_m^2 - \alpha_m^2 + \nu m^2)}{2}(my+mz-1)\sin(2pt) + \alpha_m\beta_m(my+mz-1)\cos(2pt) - \frac{2}{K}\, p(1-\nu)(my+mz+2)\cos^2 pt \right] \times e^{-m(y+z)} \cos(mx)\, dm \qquad (2.10)$$

Thus the resultant stress components are given by

$$(\sigma_x)_R = \frac{E\alpha K}{(1-\nu)2p}\int_0^\infty\Big\{mA(m)\Big[[(\alpha_m\sin(2pt)+\beta_m\cos(2pt)] - \frac{\beta_m^2-\alpha_m^2+\nu m^2}{2(1-2\nu)m}(my+mz-1)$$

$$\sin(2pt) + \frac{\alpha_m\beta_m}{(1-2\nu)m}(my+mz-1)\cos(2pt) - \frac{4p(1-\nu)}{K(1-2\nu)}\cos^2 pt\Big\}e^{-m(y+z)}\cos(mx)dm$$

$$-\frac{E}{1-2\nu}\frac{\alpha K}{4p}\int_0^\infty A(m)\Big[m^2\sin[2pt-\beta_m(y+z)]\Big] + \frac{2\nu}{1-\nu}\Big\{(\beta_m^2-\alpha_m^2)\sin[2pt-\beta_m(y+z)]-$$

$$2\alpha_m\beta_m\cos[2pt-\beta_m(y+z)]\Big\} + \frac{4p}{K}\cos(pt-\beta_m y)\cos(pt-\beta_m z)\Big]e^{-\alpha_m(y+z)}\cos(mx)dm$$

$$(\sigma_y)_R = (\sigma_z)_R = \frac{E\alpha K}{(1-2\nu)(1-\nu)4p}\Big[\int_0^\infty A(m)\Big\{(\beta_m^2-\alpha_m^2+\nu m^2)\sin(2pt) - 2\alpha_m\beta_m\cos(2pt)$$

$$+4p(1-\nu)K^{-1}\cos^2 pt\Big\}e^{-m(y+z)}\cos(mx)dm - \int_0^\infty A(m)e^{-\alpha_m(y+z)}\Big[\Big\{(\beta_m^2-\alpha_m^2+\nu m^2)\times$$

$$\sin[2pt-\beta_m(y+z)] - 2\alpha_m\beta_m\cos[2pt-\beta_m(y+z)]\Big\} + 4p(1-\nu)K^{-1}\cos(pt-\beta_m y)$$

$$\cos(pt-\beta_m z)\Big]\cos(mx)\ dm\Bigg]$$

$$(\tau_{xy})_R=(\tau_{zx})_R=\frac{E}{1-\nu}\ \frac{\alpha K}{4p}\int_0^\infty mA(m)e^{-\alpha_m(y+z)}\Big[\Big\{\alpha_m\sin[2pt-\beta_m(y+z)]+\beta_m\cos[2pt$$

$$-\beta_m(y+z)]\Big\}\sin(mx)dm-\frac{E\alpha K}{(1-\nu)4p}\int_0^\infty\Big\{mA(m)\Big[\alpha_m\sin(2pt)+\beta_m\cos(2pt)-\sin(2pt)$$

$$\frac{(\beta_m^2-\alpha_m^2+\nu m^2)}{2(1-2\nu)}(y+z)+\frac{\alpha_m\beta_m}{(1-2\nu)}(y+z)\cos(2pt)-\frac{2p(1-\nu)}{K(1-2\nu)}(y+z)\cos^2 pt\Big\}e^{-m(y+z)}$$

$$\times\ \sin(mx)dm$$

$$(\tau_{yz})_R=-\frac{\alpha KE}{(1-\nu)(1-2\nu)4p}\int_0^\infty A(m)\Big[\ m(1-2\nu)[\alpha_m\sin(2pt)+\beta_m\cos(2pt)]-$$

$$\frac{\beta_m^2-\alpha_m^2+\nu m^2}{2}(my+mz-1)\sin(2pt)+\alpha_m\beta_m(my+mz-1)\cos(2pt)-\frac{2}{K}p(1-\nu)(my+mz+2)$$

$$\cos^2 pt\Big]e^{-m(y+z)}\cos(mx)dm+\frac{E}{1-\nu}\ \frac{\alpha K}{4p}\int_0^\infty A(m)e^{-\alpha_m(y+z)}\Big\{(\alpha_m^2-\beta_m^2)\sin[2pt-$$

$$\beta_m(y+z)] + 2\alpha_m\beta_m\cos[2pt-\beta_m(y+z)]\Big\}\cos(mx)dm \qquad (2.11)$$

3. Suppose on the plane surface yz=0 we have

$$T = P\cos^2 pt \qquad |x|<a$$

$$= 0, \qquad |x|>a$$

From the relation (2.1) we have on the edges y=0, z=0

$$T = \int_0^\infty A(m)\cos^2 pt\ \cos(mx)\ dm$$

Hence

$$P = \int_0^\infty A(m)\ \cos(mx)\ dm$$

Then by Fourier's cosine transform

$$A(m) = \frac{2P\sin(ma)}{\pi m}$$

with this value of a(m) the complete solution is given by the relation (2.11)

Thermal Stresses Due To Prescribed Flux Of Heat On The Surface Of A Thick Plate

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Communicated for publications.

1. Solution Of The Equations Of Thermoelasticity

We shall consider the temperature and displacement fields in an elastic plate of finite thickness but infinite radius which is conducting heat. It will be assumed that there is symmetry about the z- axis and any point of the solid may be expressed in terms of cylindrical coordinates (r, θ, z). For symmetrical deformation of the solid, the displacement vector will have components $(u, 0, w)$ and the only non-vanishing components of the stress tensor will be rr, $\theta\theta$, zz and rz.

The temperature field is given by Laplace's equation

$$\frac{\partial^2 T}{\partial r^2} + r^{-1}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad (1)$$

in the steady state and in absence of thermal sources.

Stress components are obtained by using the potential of thermo-elastic displacement ψ given by [17]

$$u_T = \frac{\partial \psi}{\partial r}, \quad w_T = \frac{\partial \psi}{\partial z} \qquad (2)$$

From the stress strain relations in problems of thermal stresses and the equations of equilibrium we have

$$\frac{\partial^2 \psi}{\partial r^2} + r^{-1}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = \beta T; \qquad \beta = \frac{1+\nu}{1-\nu}\alpha_1 \qquad (3)$$

Where T is the deviation of the absolute temperature from the temperature of the solid in a state of zero stress and strain, α_1 being the coefficient of linear thermal expansion of the solid and ν is the Poission's ratio.

A particular integral of the equation (3) is

$$\psi = \frac{\beta}{2}\int_0^\infty A \frac{J_o(\alpha r)}{\alpha}\left\{z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1} - d(1+e^{-2\alpha z})(1-e^{-2\alpha d})(1+e^{-2\alpha d})^{-2}\right\} \times e^{\alpha(z-d)}\, d\alpha \qquad (4)$$

where A is a function of α only and 2d is the thickness of the plate.

From the relations (3) and (4) we obtain,

$$T = \int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1} J_o(\alpha r) e^{\alpha(z-d)}\, d\alpha \qquad (5)$$

which satisfies equation (1).

The stress components and the displacements can now be written as,

$$\widehat{rz}_T = 2G\frac{\partial^2\psi}{\partial r\partial z} = -G\beta\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\left\{1+\alpha z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1} - \alpha d(1-e^{-2\alpha d})(1+e^{-2\alpha d})^{-1}\right\} J_o(\alpha r) e^{\alpha(z-d)}\, d\alpha$$

$$\widehat{zz}_T = 2G\left(\frac{\partial^2\psi}{\partial z^2} - \nabla^2\psi\right) = G\beta\int_0^\infty A\alpha(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\Big\{z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1}$$

$$-d(1-e^{-2\alpha d})(1+e^{-2\alpha d})^{-1}\Big\}J_o(\alpha r)e^{\alpha(z-d)}\;d\alpha$$

$$\widehat{rr}_T = 2G\left(\frac{\partial^2\psi}{\partial r^2} - \nabla^2\psi\right) = G\beta\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\Big[\alpha\Big\{z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1}$$

$$-d(1-e^{-2\alpha d})(1+e^{-2\alpha d})^{-1}\Big\}\left\{\frac{J_1(\alpha r)}{\alpha r} - J_o(\alpha r)\right\} - 2J_o(\alpha r)\Big]e^{\alpha(z-d)}d\alpha$$

$$\widehat{\theta\theta}_T = 2G\left(\frac{1}{r}\frac{\partial\psi}{\partial r} - \nabla^2\psi\right) = -G\beta\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\Big[\alpha\Big\{z(1-e^{-2\alpha z})\times$$

$$(1+e^{-2\alpha z})^{-1} - d(1-e^{-2\alpha d})(1+e^{-2\alpha d})^{-1}\Big\}\frac{J_1(\alpha r)}{\alpha r} + 2J_o(\alpha r)\Big]e^{\alpha(z-d)}d\alpha \qquad (6)$$

$$u_T = \frac{\partial\psi}{\partial r} = -\frac{\beta}{2}\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\Big\{z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1} - d(1-e^{-2\alpha d})$$

$$\times(1+e^{-2\alpha d})^{-1}\Big\}J_1(\alpha r)e^{\alpha(z-d)}\quad d\alpha$$

$$w_T = \frac{\partial \psi}{\partial z} = \frac{\beta}{2}\int_0^{\infty} A\alpha^{-1}(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\Big\{1+\alpha z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1}-\alpha d \times$$

$$(1-e^{-2\alpha d})(1+e^{-2\alpha d})^{-1}\Big\}J_o(\alpha r)e^{\alpha(z-d)}\, d\alpha \qquad (7)$$

The subscript T denotes that the stresses and displacements are due to thermal expansions only, G being the modulus of elasticity in shear.

We observe that the normal stresses $\widehat{zz}_T$ vanishes at z=±d, the stress $\widehat{rz}_T$ however does not vanish. To satisfy the boundary conditions on the planes z=±d we superimpose a complimentary stress system. The components of stresses and displacements are expressed by means of Love's function ϕ by the relations [7]

$$\widehat{rr}_c = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right]$$

$$\widehat{\theta\theta}_c = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[\nu\nabla^2\phi - r^{-1}\frac{\partial\phi}{\partial r}\right]$$

$$\widehat{zz}_c = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]$$

$$\widehat{rz}_c = \frac{2G}{1-2\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \qquad (8)$$

$$u_c = \frac{1}{1-2\nu}\frac{\partial^2\phi}{\partial r\partial z}$$

$$w_c = \frac{1}{1-2\nu}\left[2(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \quad (9)$$

where ϕ satisfies the biharmonic equation [17]

$$\nabla^4\phi = 0 \quad (10)$$

A solution of equation can be assumed in the form

$$\phi = \frac{1}{2}\int_0^\infty \left\{B(1-e^{-2\alpha z})+C\alpha z(1+e^{-2\alpha z})\right\} J_o(\alpha r)e^{\alpha z}\, d\alpha \quad (11)$$

where B and C are functions of α to be determined.

The components of complementary stresses and displacements are given by

$$\widehat{zz}_c = \frac{G}{1-2\nu}\int_0^\infty \alpha^3\left[\left\{C(1-2\nu)-B\right\}(1+e^{-2\alpha z})-C\alpha z(1-e^{-2\alpha z})\right]J_o(\alpha r)e^{\alpha z}\, d\alpha$$

$$\widehat{rz}_c = \frac{G}{1-2\nu}\int_0^\infty \alpha^3\left[\left\{2C\nu+B\right\}(1-e^{-2\alpha z})-C\alpha z(1+e^{-2\alpha z})\right]J_1(\alpha r)e^{\alpha z}\, d\alpha \quad (12)$$

Now the boundary conditions to be satisfied are,

$$\left[\widehat{zz}_c\right]_{z=\pm d} = 0, \quad \left[\widehat{rz}_c + \widehat{rz}_T\right]_{z=\pm d} = 0 \tag{13}$$

Now the first relation of (13) will be satisfied, if

$$B = C\left\{1-2\nu-\alpha d(1-e^{-2\alpha d})(1+e^{-2\alpha d})^{-1}\right\} \tag{14}$$

Again, in view of the second relation of (13), we have from (6) and (12)

$$C = \frac{\beta(1-2\nu)}{\alpha^3(1+e^{-2\alpha d})} e^{-\alpha d} A \tag{15}$$

Consequently, the remaining stresses and displacements are given by

$$\widehat{rr}_c = G\beta\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\left[\left\{2+\alpha z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1}-\alpha d(1-e^{-2\alpha d})\right.\right.$$

$$\left.\left.\times(1+e^{-2\alpha d})^{-1}\right\}\left\{J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r}\right\} + 2\nu\frac{J_1(\alpha r)}{\alpha r}\right]e^{\alpha(z-d)}\, d\alpha$$

$$\widehat{\theta\theta}_c = G\beta\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\left[\left\{2+\alpha z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1}-\alpha d(1-e^{-2\alpha d})\right.\right.$$

$$\left.\left.\times(1+e^{-2\alpha d})^{-1}-2\nu\right\}\frac{J_1(\alpha r)}{\alpha r} + 2\nu J_0(\alpha r)\right]e^{\alpha(z-d)}\, d\alpha \tag{16}$$

$$u_c = \frac{\beta}{2}\int_0^\infty A\alpha^{-1}(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\left\{2+\alpha z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1}-\alpha d(1-e^{-2\alpha d})\right.$$

$$\left.\times(1+e^{-2\alpha d})^{-1}-2\nu\right\}J_1(\alpha r)e^{\alpha(z-d)}\,d\alpha$$

$$w_c = \frac{\beta}{2}\int_0^\infty A\alpha^{-1}(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\left\{1-\alpha z(1-e^{-2\alpha z})(1+e^{-2\alpha z})^{-1}+\alpha d(1-e^{-2\alpha d})\right.$$

$$\left.\times(1+e^{-2\alpha d})^{-1}-2\nu\right\}J_o(\alpha r)e^{\alpha(z-d)}\,d\alpha \qquad (17)$$

Applying (7) and (17) we have the final displacements given by,

$$u = \beta(1-\nu)\int_0^\infty A\alpha^{-1}(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}J_1(\alpha r)e^{\alpha(z-d)}\,d\alpha$$

$$w = \beta(1-\nu)\int_0^\infty A\alpha^{-1}(1-e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}J_o(\alpha r)e^{\alpha(z-d)}\,d\alpha \qquad (18a)$$

Also applying (6) and (16) we have finally,

$$\widehat{rr} = -2G\beta(1-\nu)\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\,\frac{J_1(\alpha r)}{\alpha r}\,e^{\alpha(z-d)}\,d\alpha$$

$$\widehat{\theta\theta} = 2G\beta(1-\nu)\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\left\{\frac{J_1(\alpha r)}{\alpha r} - J_o(\alpha r)\right\}e^{\alpha(z-d)}d\alpha \qquad (18b)$$

Hence we have,

$$\widehat{rr}+\widehat{\theta\theta} = -2G\beta(1-\nu)\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}J_o(\alpha r)e^{\alpha(z-d)}d\alpha$$

$$\widehat{rr}-\widehat{\theta\theta} = 2G\beta(1-\nu)\int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1}\left\{J_o(\alpha r) - 2\frac{J_1(\alpha r)}{\alpha r}\right\}e^{\alpha(z-d)}d\alpha$$

(19)

2. Temperature Distribution

We shall suppose that on the free surface z=d, there is a flux of heat within a circular region, the rest of the surface being free of any flux of heat. So the boundary conditions are, on the plane z=d,

$$\frac{\partial T}{\partial z} = f(r/a), \quad \text{for} \quad 0 \le r < a$$

$$= 0, \qquad \text{for} \qquad r > a \qquad (20)$$

Now from (5),

$$T = \int_0^\infty A(1+e^{-2\alpha z})(1+e^{-2\alpha d})^{-1} J_o(\alpha r) e^{\alpha(z-d)}\, d\alpha$$

Therefore

$$\frac{\partial T}{\partial z} = \int_0^\infty A\alpha(1-e^{-2\alpha z})(1+e^{-2\alpha d})^{-1} J_o(\alpha r) e^{\alpha(z-d)}\, d\alpha$$

Now we consider dimensionless coordinates ρ, η, ζ the new variable of integration being η, defined by the transformations

$$\alpha A(\alpha) = a\chi(\alpha a), \quad \eta=\alpha a, \quad \rho=r/a, \quad \zeta=z/a \tag{21}$$

Under these transformations , we have,

$$\frac{\partial T}{\partial z} = \int_0^\infty \chi(\eta)(1-e^{-2\eta\zeta})(1+e^{-2\eta d/a})^{-1} J_o(\rho\eta) e^{\eta(\zeta-d/a)}\, d\eta$$

So that on the surface z=d, we have

$$\int_0^\infty \chi(\eta)(1-e^{-2\eta d/a})(1+e^{-2\eta d/a})^{-1} J_o(\rho\eta)\, d\eta = f(\rho), \quad 0\le\rho<1$$
$$= 0\,, \qquad \rho>1$$

Hence by Hankel's inversion theorem [14]

$$\eta^{-1}\chi(\eta)(1-e^{-2\eta d/a})(1+e^{-2\eta d/a})^{-1} = \int_0^1 \rho f(\rho) J_o(\rho\eta)\, d\rho \tag{22}$$

Under the same set of transformations, we have on z=d,

$$u = \beta(1-\nu)a^2\int_0^\infty \frac{\chi(\eta)}{\eta^2}\ J_1(\rho\eta)\,d\eta$$

$$w = \beta(1-\nu)a^2\int_0^\infty \frac{\chi(\eta)}{\eta^2}\ (1-e^{-2\eta d/a})(1+e^{-2\eta d/a})^{-1}J_o(\rho\eta)\,d\eta$$

$$\widehat{rr} + \widehat{\theta\theta} = -2G\beta(1-\nu)a\int_0^\infty \frac{\chi(\eta)}{\eta}\ J_o(\rho\eta)\,d\eta$$

$$\widehat{rr} - \widehat{\theta\theta} = 2G\beta(1-\nu)a\int_0^\infty \frac{\chi(\eta)}{\eta}\left\{J_o(\rho\eta) - 2\frac{J_1(\rho\eta)}{\rho\eta}\right\}d\eta \tag{23}$$

Let us assume that $f(\rho)=k$. Then from (22), we have

$$\eta^{-1}\chi(\eta)(1-e^{-2\eta d/a})(1+e^{-2\eta d/a})^{-1} = k\int_0^1 \rho J_o(\rho\eta)\ d\rho = k\eta^{-1}J_1(\eta)$$

Hence

$$\chi(\eta) = kJ_1(\eta)(1-e^{-2\eta d/a})(1+e^{-2\eta d/a})^{-1} \tag{24}$$

This value of $\chi(\eta)$ substituted in the relations for stresses and displacements gives the complete solution.

We find the value of $[\widehat{rr} + \widehat{\theta\theta}]_{z=d}$ with this expression for $\chi(\eta)$ as

$$[\widehat{rr}+\widehat{\theta\theta}]_{z=d} = -2G\beta(1-\nu)ak\int_0^{\infty}\eta^{-1}(1-e^{-2\eta d/a})(1+e^{-2\eta d/a})^{-1}J_1(\eta)J_o(\rho\eta)d\eta$$

Writing,

$$(1-e^{-2\eta d/a})(1+e^{-2\eta d/a})^{-1} = 1+2\sum_{p=1}^{\infty}e^{-2p\eta d/a}$$, we obtain

$$[\widehat{rr}+\widehat{\theta\theta}]_{z=d} = \delta\int_0^{\infty}\eta^{-1}\left[1+2\sum_{p=1}^{\infty}e^{-2p\eta d/a}\right]J_1(\eta)J_o(\rho\eta)d\eta = \delta I_1+2\delta I_2 \tag{25}$$

where

$$I_1 = {}_2F_1\left(\frac{1}{2},-\frac{1}{2},1;\rho^2\right), \quad \text{for } \rho<1$$

$$= 2/\pi, \quad \text{for } \rho=1$$

$$= (2\rho)^{-1}{}_2F_1\left(\frac{1}{2},\ \frac{1}{2},2;\rho^{-2}\right), \quad \text{for } \rho>1 \tag{26a}$$

$$I_2 = \sum_{p=1}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(2s)!2^{-(2s+1)}}{s!(s+1)!(p^2+4p^2d^2/a^2)^{2s+1/2}}P_{2s}\left\{\frac{2pd/a}{(p^2+4p^2d^2/a^2)^{1/2}}\right\} \tag{26b}$$

and $\delta = -2G\beta(1-\nu)ak$ (26c)

3. Paraboloidal Distribution Of Temperature

We consider a paraboloidal distribution of temperature over a circular region of exposure $0 \le r \le a$ while the rest of the surface is kept at zero temperature. We have therefore on the surface $z=d$,

$$T = T_o(1-r^2/a^2), \qquad \text{for } 0 \le r \le a$$

$$= 0, \qquad \text{for } r>a \tag{27}$$

By Fourier-Bessel Representation on the surface $z=d$

$$T = \int_0^\infty \alpha J_o(\alpha r)\,d\alpha \int_0^\infty T_o u(1-u^2/a^2) J_o(\alpha u)\,du$$

$$= 2T_o \int_0^\infty \alpha^{-1} J_2(\alpha a) J_o(\alpha r)\,d\alpha \tag{28}$$

Again, from (5), we have on $z=d$,

$$T = \int_0^\infty A J_o(\alpha r)\,d\alpha \tag{29}$$

Comparing (28) with (29), we get,

$$A(\alpha) = 2T_o \alpha^{-1} J_2(\alpha a)$$

Writing in the dimensionless form, we get

$$\chi(\eta) = 2T_o a^{-1} J_2(\eta)$$

Substituting in (23), we get

$$\left[\ \widehat{rr} + \widehat{\theta\theta}\ \right]_{z=d} = -4G\beta(1-\nu)T_o\int_0^\infty \eta^{-1} J_2(\eta) J_o(\rho\eta)\,d\eta$$

and

$$\left[\ \widehat{rr} - \widehat{\theta\theta}\ \right]_{z=d} = 4G\beta(1-\nu)T_o\int_0^\infty \eta^{-1}\left\{J_2(\eta)J_o(\rho\eta) - 2(\eta\rho)^{-1}J_2(\eta)J_1(\rho\eta)\right\}d\eta$$

(31)

Thus we get finally,

$$\left[\ \widehat{rr} + \widehat{\theta\theta}\ \right]_{z=d} = -2G\beta(1-\nu)T_o \begin{cases} {}_2F_1(1,-1/2,1;\rho^2), & \rho<1 \\ 0\,, & \rho=1 \\ 0\,, & \rho>1 \end{cases} \qquad (32)$$

$$\left[\ \widehat{rr} - \widehat{\theta\theta}\ \right]_{z=d} = 2G\beta(1-\nu)T_o \begin{cases} {}_2F_1(1,-1/2,1;\rho^2) - {}_2F_1(1,-1,2;\rho^2), & \rho<1 \\ -1/2, & \rho=1 \\ -(2\rho^2)^{-1} & \rho>1 \end{cases}$$

(33)

4. Numerical Results

The variation of $-\left\{\left[\widehat{rr}+\widehat{\theta\theta}\right]_{z=d}\right\}/2G\beta(1-\nu)T_{o}$ for different values of ρ within the circle $\rho\le 1$ is given in the following table

ρ	$-\left\{\left[\widehat{rr}+\widehat{\theta\theta}\right]_{z=d}\right\}/2G\beta(1-\nu)T_{o}$
0.0	1.0000
0.2	0.9798
0.4	0.9185
0.6	0.8000
0.8	0.6002
1.0	0.0000

5. Conclusion

Thus, we note that the value of $-\left\{\left[\widehat{rr}+\widehat{\theta\theta}\right]_{z=d}\right\}/2G\beta(1-\nu)T_{o}$ is maximum at the origin, diminishes slowly at the initial stage, but rapidly near the edge of the circle of exposure and zero value at the edge of the exposure and outside it.

Thermal Stresses Due To Prescribed Flux Of Heat On The Boundary Of A Semi-Infinite Elastic Solid

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Communicated for publications.

1. Method Of Solution

Let the bounaary surface of the semi-infinite isotropic elastic solid be given by z=0, the axis of z being drawn into the body.The temperature field in the steady state is given by Laplace's equation

$$\nabla^2 T=0 \tag{1}$$

To determine the stresses , the potential of thermo-elastic displacement ψ will be used. This is related to the displacement components u,0,w by the equation, if axially symmetrical coordinates be assumed [17],

$$u_T = \frac{\partial\psi}{\partial r}, \quad w_T = \frac{\partial\psi}{\partial z} \tag{2}$$

In this problem of axially symmetrical temperature field, the nonvanising components of displacements are u and w and since we are considering axially symmetrical coordinates the general values of u,w are not taken.

From the stress strain relations in problems of thermal stresses and the equations of equilibrium we have [17]

$$\nabla^2\psi = \beta T; \tag{3}$$

where $$\beta = \frac{1+\nu}{1-\nu}\,\alpha_1 \tag{4}$$

α_1 being the coefficient of linear thermal expansion of the solid and ν is the Poission's ratio.

A particular integral of the equation (3) is

$$\psi = \frac{\beta}{2}\int_0^\infty A(\alpha)\frac{z}{\tanh\alpha z}\left\{\sinh\alpha z-\tanh^2\alpha z\ \cosh\alpha z\right\}J_o(\alpha r)d\alpha \tag{5}$$

Substituting in (3), we get,

$$T = \int_0^\infty \alpha A(\alpha)\frac{1}{\tanh\alpha z}\left\{\sinh\alpha z-\tanh^2\alpha z\ \cosh\alpha z\right\}J_o(\alpha r)d\alpha \tag{6}$$

where $A(\alpha)$ is a function of α only to be determined.

The stress components [17] can now be written as,

$$\widehat{rz}_T = 2G\frac{\partial^2\psi}{\partial r\partial z} = -G\beta\int_0^\infty \alpha A(\alpha)(1-\alpha z)\frac{1}{\tanh\alpha z}\left\{\sinh\alpha z-\tanh^2\alpha z\ \cosh\alpha z\right\}J_1(\alpha r)d\alpha$$

$$\widehat{zz}_T = 2G\left(\frac{\partial^2\psi}{\partial z^2}-\nabla^2\psi\right) = zG\beta\int_0^\infty \alpha^2 A(\alpha)\frac{1}{\tanh\alpha z}\left\{\sinh\alpha z-\tanh^2\alpha z\ \cosh\alpha z\right\}J_o(\alpha r)d\alpha$$

$$\widehat{rr}_T = 2G\left(\frac{\partial^2\psi}{\partial r^2}-\nabla^2\psi\right) = G\beta\int_0^\infty\left[\frac{z}{r}J_1(\alpha r)+(2-\alpha z)J_o(\alpha r)\right]\alpha A(\alpha)\frac{1}{\tanh\alpha z}\left\{\sinh\alpha z-\tanh^2\alpha z\ \cosh\alpha z\right\}d\alpha$$

$$\widehat{\theta\theta}_T = 2G\left(\frac{1}{r}\frac{\partial\psi}{\partial r} - \nabla^2\psi\right) = G\beta\int_0^\infty\left[2J_0(\alpha r) - \frac{z}{r}J_1(\alpha r)\right]\alpha A(\alpha)\frac{1}{\tanh\alpha z}\left\{\sinh\alpha z - \tanh^2\alpha z\cosh\alpha z\right\}d\alpha \qquad (7)$$

The subscript T denotes that the stresses are due to thermal expansion only, G being the modulus of elasticity in shear.

We observe that the normal stresses $\widehat{zz}_T$ vanishes at z=0, the stress $\widehat{rz}_T$ however does not vanish. In order to suppress it the stress system $(\widehat{rr}_c, \widehat{\theta\theta}_c, \widehat{zz}_c, \widehat{rz}_c)$ obtained on the hypothesis that there is no temperature distribution is to be superposed.

2. Complementary Stresses

In order to determine the complementary stresses we use Love's function satisfying biharmonic equation

$$\nabla^4\phi = 0 ,$$

with the boundary conditions

$$\left[\widehat{zz}_c\right]_{z=0} = 0, \quad \left[\widehat{rz}_c + \widehat{rz}_T\right]_{z=0} = 0 \qquad (8)$$

and $\phi=0$ at infinity.

Let us assume the function ϕ in the form,

$$\phi = \int_0^\infty [B+C\alpha z]\frac{1}{\tanh\alpha z}\left\{\sinh\alpha z-\tanh^2\alpha z\ \cosh\alpha z\right\}J_o(\alpha r)d\alpha \qquad (9)$$

where B and C are functions of α.

The complementary stresses are then given by [17]

$$\widehat{rr}_c = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right]$$

$$\widehat{\theta\theta}_c = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[\nu\nabla^2\phi - r^{-1}\frac{\partial\phi}{\partial r}\right]$$

$$\widehat{zz}_c = \frac{2G}{1-2\nu}\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]$$

$$\widehat{rz}_c = \frac{2G}{1-2\nu}\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \qquad (10)$$

In view of the first condition of (8), we have from (10)

$$B=-C(1-2\nu) \qquad (11)$$

Then from (9)-(11), we get

$$\widehat{zz}_c = \frac{2Gz}{1-2\nu}\int_0^\infty C\alpha^4 \frac{1}{\tanh\alpha z}\left\{\sinh\alpha z - \tanh^2\alpha z \cosh\alpha z\right\} J_o(\alpha r)d\alpha$$

$$\widehat{rz}_c = -\frac{2G}{1-2\nu}\int_0^\infty C\alpha^3 \frac{1}{\tanh\alpha z}\left\{\sinh\alpha z - \tanh^2\alpha z \cosh\alpha z\right\}(1-\alpha z) J_o(\alpha r)d\alpha$$

$$\widehat{\theta\theta}_c = \frac{2G}{1-2\nu}\int_0^\infty \left[2\nu J_o(\alpha r) - (2\nu-2+\alpha z)J_1(\alpha r)/\alpha r\right] C\alpha^3 \frac{1}{\tanh\alpha z}\left\{\sinh\alpha z - \tanh^2\alpha z \cosh\alpha z\right\}d\alpha$$

$$\widehat{rr}_c = \frac{2G}{1-2\nu}\int_0^\infty \left[(2-\alpha z)J_o(\alpha r) - (2\nu-2+\alpha z)J_1(\alpha r)/\alpha r\right] C\alpha^3 \frac{1}{\tanh\alpha z}\left\{\sinh\alpha z - \tanh^2\alpha z \cosh\alpha z\right\}d\alpha \qquad (12)$$

The second boundary condition (8) gives

$$C = -\frac{\beta(1-2\nu)}{2\alpha^2} A(\alpha) \qquad (13)$$

We have therefore the stresses $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{zz}$ etc. given by

$$\widehat{rr} = \widehat{rr}_C + \widehat{rr}_T = 2G\beta\int_0^\infty (1-\nu)\frac{\alpha A(\alpha)}{\tanh\alpha z}\left\{\sinh\alpha z - \tanh^2\alpha z\ \cosh\alpha z\right\}\frac{J_1(\alpha r)}{\alpha r}\,d\alpha$$

$$\widehat{\theta\theta} = \widehat{\theta\theta}_C + \widehat{\theta\theta}_T = 2G\beta\int_0^\infty (1-\nu)\frac{\alpha A(\alpha)}{\tanh\alpha z}\left\{\sinh\alpha z - \tanh^2\alpha z\ \cosh\alpha z\right\}\left[J_0(\alpha r) - \frac{J_1(\alpha r)}{\alpha r}\right]d\alpha \tag{14}$$

We write the solution in a dimensionless form on putting

$$\alpha A(\alpha) = a\chi(\alpha a), \quad \eta = \alpha a, \quad \rho = r/a, \quad \zeta = z/a \tag{15}$$

where a is some length and η a new variable of integration. We find that the solution may be written in the form

$$T = -\int_0^\infty \chi(\eta)\frac{1}{\tanh\eta\zeta}\left\{\sinh\eta\zeta - \tanh^2\eta\zeta\ \cosh\eta\zeta\right\}J_0(\rho\eta)\,d\eta \tag{16}$$

Correspondingly, we get,

$$\widehat{rr} = 2G\beta(1-\nu)\int_0^\infty \chi(\eta)\frac{1}{\tanh\eta\zeta}\left\{\sinh\eta\zeta - \tanh^2\eta\zeta\ \cosh\eta\zeta\right\}\frac{J_1(\rho\eta)}{\rho\eta}\,d\eta$$

$$\widehat{\theta\theta} = 2G\beta(1-\nu)\int_0^\infty \chi(\eta)\frac{1}{\tanh\eta\zeta}\left\{\sinh\eta\zeta - \tanh^2\eta\zeta\ \cosh\eta\zeta\right\}\left[J_0(\rho\eta) - \frac{J_1(\rho\eta)}{\rho\eta}\right]d\eta$$

3. Prescribed Flux Of Heat On The Boundary

We shall suppose that on the free surface z=0, there is a flux of heat within a circular region, the rest of the surface being free of any flux of heat. So the boundary conditions are, on the surface z=0,

$$\frac{\partial T}{\partial z} = f(r/a), \quad \text{for} \quad 0 \le r < a$$

$$= 0, \quad \text{for} \quad r > a \tag{17}$$

Now from (16),

$$\frac{\partial T}{\partial z} = \int_0^{\infty} a^{-1} \eta \chi(\eta) J_o(\rho\eta)\, d\eta \tag{18}$$

Hence by Hankel's inversion theorem [14]

$$\chi(\eta) = a^{-1} \int_0^1 \rho f(\rho) J_o(\rho\eta)\, d\rho \tag{19}$$

Then we have from the relation (14)

$$\widehat{rr}+\widehat{\theta\theta} = 2G\beta(1-\nu)\int_0^{\infty} \chi(\eta)\frac{1}{\tanh\eta\zeta}\left\{\sinh\eta\zeta - \tanh^2\eta\zeta \ \cosh\eta\zeta\right\} J_o(\rho\eta)\,d\eta$$

$$\widehat{rr}-\widehat{\theta\theta} = 2G\beta(1-\nu)\int_0^\infty \chi(\eta)\frac{1}{\tanh\eta\zeta}\left\{\sinh\eta\zeta - \tanh^2\eta\zeta\ \cosh\eta\zeta\right\}\left[2\frac{J_1(\rho\eta)}{\rho\eta} - J_0(\rho\eta)\right]d\eta \tag{20}$$

When $z=0$, take the forms

$$[\ \widehat{rr}+\widehat{\theta\theta}\ ]_{z=0} = 2G\beta(1-\nu)\int_0^\infty \chi(\eta)J_0(\rho\eta)\,d\eta$$

$$[\ \widehat{rr}-\widehat{\theta\theta}\ ]_{z=0} = 2G\beta(1-\nu)\int_0^\infty \chi(\eta)\left[2\frac{J_1(\rho\eta)}{\rho\eta} - J_0(\rho\eta)\right]d\eta \tag{21}$$

We shall obtain expressions for $[\ \widehat{rr}+\widehat{\theta\theta}\ ]_{z=0}$ and $[\ \widehat{rr}-\widehat{\theta\theta}\ ]_{z=0}$ for two particular functional value of $f(\rho)$.

Case 1.

Let us assume that $f(\rho)=k$. Then from (19), we have

$$\chi(\eta) = a^{-1}k\int_0^1 \rho J_0(\rho\eta)\ d\rho = k(a\eta)^{-1}J_1(\eta) \tag{22}$$

Substituting this value of $\chi(\eta)$ intothe relations (21), we obtain

$$[\widehat{rr}+\widehat{\theta\theta}]_{z=0} = \frac{\delta k}{a} F\left(\frac{1}{2},-\frac{1}{2},1;\rho^2\right), \qquad \text{for } \rho<1$$

$$= \frac{2\delta k}{a\pi}, \qquad \text{for } \rho=1$$

$$= \frac{\delta k}{2a\rho} F\left(\frac{1}{2},\frac{1}{2},2;\rho^{-2}\right), \qquad \text{for } \rho>1$$

$$[\widehat{rr}-\widehat{\theta\theta}]_{z=0} = \frac{\delta k}{a}\left[F\left(\frac{1}{2},-\frac{1}{2},2;\rho^2\right)-F\left(\frac{1}{2},-\frac{1}{2},1;\rho^2\right)\right] \qquad \text{for } \rho<1$$

$$= \frac{2\delta k}{3a\pi}, \qquad \text{for } \rho=1$$

$$= \frac{\delta k}{2a\rho}\left[2F\left(\frac{1}{2},-\frac{1}{2},2;\rho^{-2}\right)-F\left(\frac{1}{2},\frac{1}{2},2;\rho^{-2}\right)\right], \quad \text{for } \rho>1 \qquad (23)$$

where $\delta = 2G\beta(1-\nu)$

Case 2.

We take the flux function as a parabolic one, so that we take $f(\rho) = k(1-\rho^2)$. Then from (19)

$$\chi(\eta) = ka^{-1}\left[4\eta^{-3}J_1(\eta)-2\eta^{-2}J_o(\eta)\right] \qquad (24)$$

Substituting this value of $\chi(\eta)$ into the relations (21) and integrating we get [14]

$$[\widehat{rr}+\widehat{\theta\theta}]_{z=0} = \frac{\delta k}{3a}\left[6F\left(-\frac{1}{2},-\frac{1}{2},1;\rho^{2}\right)-4F\left(-\frac{1}{2},-\frac{3}{2},1;\rho^{2}\right)\right], \quad \text{for } \rho<1$$

$$= \frac{8\delta k}{9a\pi}, \quad \text{for } \rho=1$$

$$= \frac{2\delta k\rho}{a}\left[F\left(-\frac{1}{2},-\frac{1}{2},1;\rho^{-2}\right)-F\left(-\frac{1}{2},-\frac{1}{2},2;\rho^{-2}\right)\right], \quad \text{for } \rho>1$$

$$[\widehat{rr}-\widehat{\theta\theta}]_{z=0} = \frac{\delta k}{3a}\left[6F\left(-\frac{1}{2},-\frac{1}{2},2;\rho^{2}\right)+4F\left(-\frac{1}{2},\frac{3}{2},1;\rho^{2}\right)\right.$$

$$\left.-6F\left(-\frac{1}{2},-\frac{1}{2},1;\rho^{2}\right)-4F\left(-\frac{1}{2},\frac{3}{2},2;\rho^{2}\right)\right] \quad \text{for } \rho<1$$

$$= \frac{8\delta k}{15a\pi}, \quad \text{for } \rho=1$$

$$= \frac{\delta k\rho}{3a}\left[6F\left(-\frac{1}{2},-\frac{1}{2},-2;\rho^{-2}\right)+4F\left(-\frac{1}{2},-\frac{3}{2},1;\rho^{-2}\right)\right.$$

$$\left.-6F\left(-\frac{1}{2},-\frac{1}{2},1;\rho^{-2}\right)-4F\left(-\frac{1}{2},-\frac{3}{2},2;\rho^{-2}\right)\right] \quad \text{for } \rho>1$$

4. Numerical Results

Taking a=1, the variation of $\left\{[\widehat{rr}-\widehat{\theta\theta}]_{z=0}\right\}/2Gk(1+\nu)\alpha_1$ for different values of ρ , in the two cases is given in the following table

Table 1.

ρ	$\left\{ [\widehat{rr} - \widehat{\theta\theta}]_{z=0} \right\} / 2Gk(1+\nu)\alpha_1$
0.0	0.00000
0.2	0.00505
0.4	0.02087
0.6	0.04988
0.8	0.09868
1.0	0.21212
1.5	0.27138
2.0	0.22519
3.0	0.15955
4.0	0.12203

Table 2.

ρ	$\left\{ [\widehat{rr} - \widehat{\theta\theta}]_{z=0} \right\} / 2Gk(1+\nu)\alpha_1$
0.0	0.00000
0.2	0.00990
0.4	0.03836
0.6	0.08140
0.8	0.13125
1.0	0.16969
1.5	0.14538
2.0	0.11686
3.0	0.08098

5. CONCLUSION

In the first case of distribution of temperature we note that the value of $\left\{\left[\widehat{rr}-\widehat{\theta\theta}\right]_{z=0}\right\}/2Gk(1+\nu)\alpha_t$ is zero at the origin, reaches its maximum at a point shortly beyond the edge of the region of exposure and then diminishes continuously as ρ increases further.

In the second case of distribution of temperature we observe that the nature of $\left[\widehat{rr}-\widehat{\theta\theta}\right]_{z=0}$ remains unaltered but its value diminishes rapidly outside the region of exposure.

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