Chapter 8

SCALE FREE ORDINARY DIFFERENTIAL EQUATION: NOVEL SOLUTIONS

8.1 Introduction

Here, we argue that the finitely differentiable scale free solutions to the simplest scale free initial value problem (IVP) [30, 31, 32, 33]

$$t\frac{\mathrm{d}\tau}{\mathrm{d}t} = \tau,\ \tau(1) = 1 \tag{8.1}$$

should be able to offer an ideal framework for many complex phenomena. We present a novel dynamical treatment of linear ODEs when the time (i.e. the independent real) variable t is assumed to have a random element [31]. We show how a judicious use of the golden mean partition of unity, $\nu^2 + \nu = 1$, $\nu = (\sqrt{5} - 1)/2$, not only allows time to undergo random changes (flips) by inversions, $t_- \to t_-^{-1} = t_+$, $t_\pm = 1 \pm \eta$, in the vicinity of an instant $t = 1$ (say), but also unveils the possible existence of a class of random, second derivative discontinuous, scale free solutions to equation (8.1). One of the major aim is to extend a framework of calculus accommodating inversions as a valid mode of changes (increments) besides ordinary translations. The freedom of random inversions provides a dynamic, evolutionary character to the second derivative discontinuous solutions, with a privileged sense of time's arrow. This solution, though approximate ($\sim O(\eta^2) = 0$), in the ordinary real number system R, is, however, exact in an nonstandard real number set **R**. Finally, we show that the 'approximate' solution is in fact generic, in the sense that the more accurate, in fact the exact solution, derived by generating successive self- similar corrections to an initially approximate solution, fails to yield the exact solution even in the limit of infinite number of iterations.

The new solution breaks the reflection symmetry ($t \to -t$) of the ODE. We also show here that besides these finitely differentiable (C^{2^n-1}) time asymmetric solutions as well as the infinitely differentiable, time (reflection) symmetric standard solution of equation (8.1), possesses another new class of fluctuating solutions which are both infinitely differentiable and time symmetric. Because of these nontrivial classes of finitely and infinitely differentiable fluctuating solutions, a real variable t can undergo changes not only by linear translations, but by inversions ($t \to 1/t$), in the neighbourhood of each real t. We next discuss how this could define a nonstandard extension of the real number system. This also clarifies the origin of an intrinsic randomness at as fundamental a level as the real number system. Consequently, every real number is identified with an equivalence class of a continuum of new, infinitesimally separated elements, which are in a state of "random fluctuations" (c.f., Sec.7.2).

8.2 Mathematical results

Because of the novelty of the result, it is instructive to give a fairly complete derivation of such solutions [30]. To this end, let us first construct the solution in the neighbourhood of $t = 1$. We need to introduce following notations.

Let $t_{n\pm} = 1 \pm \eta_n$, $t_0 \equiv t$, $0 < \eta_n << 1$, $\alpha_n = 1 + \epsilon_n$, $n = 0, 1, 2, \ldots,$

and $\epsilon_0 = 0$, $0 < \epsilon_n < 1 (n \neq 0)$, such that $\epsilon_n \to 0$, as $n \to \infty$ (we retain the symbol α_0 for the sake of symmetry). Next, we write $t'_{n\pm} = 1 \pm \alpha_n \eta'_n$, so that $\alpha_n t_{n-} = t'_{n-}$. Consequently, $\eta'_n = \eta_n - \frac{\epsilon_n}{\alpha_n}$. Here, α_n (and ϵ_n) are scaling parameters. A useful choice, however, is $\epsilon_n = \epsilon^{2^n}$, $\epsilon = \epsilon_1$ (the reason will become clear later). As will become evident, $\eta_{n+1} = \alpha_n^2 \eta_n'^2$.

To construct a nontrivial solution (with the initial condition $\tau(1) = 1$), we begin with an initial approximate solution in the small scale variable η_0. To this end, let

$$\tau(t) = \begin{cases} \tau_- \; if \; t \lessapprox 1 \\ \tau_+ \; if \; t \gtrapprox 1 \end{cases}, \quad \tau_-(t_-) = (1/t'_+) f_{1-}(\eta_0), \; \tau_+(t_+) = t_+ \tag{8.2}$$

be an exact solution of equation (8.1). This is obviously true for the right hand component τ_+. To verify the same for the nontrivial component τ_-, we differentiate it with respect to t_-, and use the scale invariance of equation (8.1). Utilizing $\alpha_0 t_- = t'_-$, one obtains

$$t'_- \frac{d\tau_-}{dt'_-} = \tau_-\left(\frac{t'_-}{t'_+} - t'_- \frac{f'_{1-}}{f_{1-}}\right) \tag{8.3}$$

where $f'_{1-} = \frac{df_{1-}}{d\bar{\eta}_0}$, $\bar{\eta}_0 = \alpha_0 \eta'_0$. Consequently, equation (8.3) would be an exact solution if and only if f_{1-} solves exactly the self-similar equation

$$t_{1-} \frac{df_{1-}}{dt_{1-}} = f_{1-} \tag{8.4}$$

in the smaller logarithmic variable $\ln t_{1-}^{-1}$, where $t_{1-} = 1 - \alpha_0^2 \eta_0'^2 \equiv 1 - \eta_1$. The self-similar replica equation (8.4) follows from the equality

$$\frac{t'_-}{t'_+} - t'_- \frac{f'_{1-}}{f_{1-}} = 1 \tag{8.5}$$

so that τ_- is an exact solution of equation (8.1). The exact (nontrivial part of the) solution could thus be written recursively as

$$\tau_- = C \frac{1}{t_+} \frac{1}{t'_{1+}} \cdots \frac{1}{t'_{(n-1)+}} f_{n-}(\eta'_n) \tag{8.6}$$

where f_n satisfies the nth generation self-similar equation

$$t_{n-} \frac{\mathrm{d} f_{n-}}{\mathrm{d} t_{n-}} = f_{n-} \tag{8.7}$$

and $t_{n-} = 1 - \alpha^2_{n-1} \eta'^2_{n-1} \equiv 1 - \eta_n$. We also note that $t'_+ = t_+$, since $\alpha_0 = 1$.

Plugging in the initial condition $\tau_\pm = 1$ at $t_\pm = 1$ (viz., $\eta_0 = 0$), one obtains finally the desired solution as

$$\tau_- = C \frac{1}{t_+} \frac{1}{t'_{1+}} \frac{1}{t'_{2+}} \cdots, \quad \tau_+ = t_+ \tag{8.8}$$

where $C = t'_{1+}(0) t'_{2+}(0) \ldots$. Notice that $C \neq 1$, since $\eta'_1 = -\epsilon_1/\alpha_1$, $\eta'_2 = \epsilon_1^2 - \epsilon_2/\alpha_2$, etc, when $\eta_0 = 0$.

A remark is in order here [27, 30].

The solution (8.8) follows from equation (8.6) only if the sequence $\{f_{n-}\}$ is convergent. In fact, we prove that $f_\infty = \lim_{n\to\infty} f_{n-} = 1$. Let $\tau_n = \frac{1}{t_+}\frac{1}{t'_{1+}} \cdots \frac{1}{t'_{n+}}$. Then for η_0 sufficiently small and $\epsilon_n \to 0$ for $n \to \infty$, the sequence $\{\tau_n\}$ is convergent (to a nonzero value), since $t'_{n+} \to 1$ as $n \to \infty$. Accordingly, for $\epsilon > 0$, $\exists N_1$ such that $|\tau_m - \tau_n| < \epsilon$ for $m, n > N_1\,(m > n)$. As a result, $0 < k_1 < \tau_n < k_2$, $k_1, k_2 \sim O(1)$, for

$n > N_2$ for a sufficiently large N_2. Again, f_{n-}, being defined by equation (8.7), is uniformly bounded in a neighbourhood of $t = 1$, so that $|f_{n-}| < k$ for $n > N_2$. The desired convergence now follows from the Cauchy convergence criterion, since $|f_{n-} - f_{m-}| = |\tau_n^{-1}||f_{m-}||\tau_m - \tau_n| < k_1^{-1}k\epsilon$ $\forall m, n > N$, $N = \max(N_1, N_2)$. Finally, equation (8.7), in the asymptotic limit $n \to \infty$, yields $f_\infty = \frac{\mathrm{d}f_{n-}}{\mathrm{d}t_{n-}}|_\infty = \frac{\mathrm{d}\tau}{\mathrm{d}t}|_{t=1} = \tau(1) = 1$.

Now to test the continuity of the derivatives of the solution (8.8) at $t_\pm = 1$, i.e., at $\eta_0 = 0$, we note that η'_n is a polynomial in η_0, of degree 2^n, being defined recursively by $\eta'_n = \eta_n - \frac{\epsilon_n}{\alpha_n}$, $\eta_n = \alpha_{n-1}^2\eta'^2_{n-1}$. As a result $\frac{\mathrm{d}\eta'_n}{\mathrm{d}\eta_0} = 0$, but $\frac{\mathrm{d}^2\eta'_n}{\mathrm{d}\eta_0^2} \neq 0$, at $\eta_0 = 0$. One thus obtains

$$\frac{\mathrm{d}\tau_-}{\mathrm{d}\eta_0} = -\tau_-\{\frac{1}{1+\eta_0} + (\frac{\alpha_1}{1+\alpha_1\eta'_1})\frac{\mathrm{d}\eta'_1}{\mathrm{d}\eta_0} + (\frac{\alpha_2}{1+\alpha_2\eta'_2})\frac{\mathrm{d}\eta'_2}{\mathrm{d}\eta_0} + \ldots\} \tag{8.9}$$

so that $\frac{\mathrm{d}\tau_-}{\mathrm{d}t_-} = 1 = \frac{\mathrm{d}\tau_+}{\mathrm{d}t_+}$ at $\eta_0 = 0$ which means that the first derivative of the solution is indeed continuous for all η_0. However, as is verified easily from equation (8.9), the second derivative of τ_- at $\eta_0 = 0$ is not zero, as one expects on the basis of the standard solution $\tau_s = t$. Indeed, one can verify that $\frac{\mathrm{d}^2\tau_-}{\mathrm{d}\eta_0^2} = 2(1 - \frac{1+\epsilon_1}{1-\epsilon_1} - \ldots) \neq 0$ at $\eta_0 = 0$, unless $\epsilon_n = 0$, for all n. In this special case, i.e., when $\epsilon_n = 0$, $\forall n$, our solution (8.8) reduces to the standard solution, since $\tau_- = \frac{1}{1+\eta_0}\frac{1}{1+\eta_0^2}\frac{1}{1+\eta_0^4}\cdots = 1 - \eta_0 = t_-$.

It thus follows that the solution (8.8), with nonzero scaling parameters, is indeed nontrivial, because of this second derivative discontinuity at $\eta_0 = 0$, that is at $t = 1$. In fact, the scaling invariance of equation (8.1) tells also that, $t = 1$ could be realized as $t \to t/t_0 = 1$, so that the nontrivial solution (8.8) actually holds in the neighbourhood of every real number t_0, the 2nd derivative being discontinuous at $t = t_0$. Let us note here that $\tau_- = \tau_{s-}(1 + \mathrm{O}(\eta_0^2))$, besides the arbitrariness of the scaling parameters ϵ_n. Combining the standard and the new solutions together, one can finally write down a more general one parameter class of solutions

$$\tau_g(t) = t(1 + \phi(t)),\ \phi(t) = \epsilon t^{-1}\tau(t). \tag{8.10}$$

Note that

$$t\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0 \tag{8.11}$$

because τ is an exact solution of equation (8.1). The 2nd derivative discontinuity of τ, however, tells that ϕ can not be considered simply as an ordinary constant.

8.3 Salient features

The salient features of this solution are the following.

1. The solution has discontinuous second derivative at $t = 1$. The said discontinuity is an effect of an infinity of nonzero rescaling parameters ϵ_n. For a finite set of ϵ_n (or in the special case when $\epsilon_n = 0$, $\forall n$), one gets back the standard solution. Moreover, the scale invariance is realized only in a one sided manner. The scaling $\alpha_n t_- = t'_-$ does not mean $\alpha_n t_+ = t'_+$.

2. It also follows that the solution (8.8) is indeed an exact solution of equation (8.1) when the ordinary real variable $t = 1 - \eta_0$ (near $t = 1$) is replaced by the fat real variable $\mathbf{t}_-^{-1} = \Pi_0^\infty t'_{n+}$. The fat variable $\mathbf{t}$ leaves in $\mathbf{R}$, a nonstandard extension of the ordinary real number set R, inhabiting

infinitesimal scales (variables) $\ln t'_n \approx \alpha_n \eta'_n$. All these variables can be treated as independent because of the arbitrary scaling parameters ϵ_n.

3. Finally, it is easy to verify that equation (8.1) possesses yet another (nontrivial) class of infinitely differentiable solutions of the form

$$\tau'(t) = \begin{cases} \tau'_- \ if \ t \lessapprox 1 \\ \tau'_+ \ if \ t \gtrapprox 1 \ , \end{cases} \tag{8.12}$$

$\tau'_-(t_-) = (1/t_+)f(\eta_0)$, $\tau_+(t_+)' = (1/t_-)(f(\eta_0), f(\eta_0) = \frac{1}{t'_{1+}}\frac{1}{t'_{2+}}\cdots$

which is, however, distinct from the standard solution. Note that the infinite differentiability is restored because of identical self similar corrections in $\tau'_\pm$. However, as it should be evident from the above derivations, the iteration schemes for both τ'_- and τ'_+ could be run independently with different sets of scaling factors ϵ_n and ϵ'_n respectively, leading again to second derivative discontinuity. Besides these second derivative discontinuous solutions, equation (8.1), also accommodates a larger class of C^{2^n-1} solutions. Consequently, the simple ODE (8.1) accommodates indeed an astonishingly rich set of solutions belonging to different differentiability classes.

We have presented new families of higher derivative discontinuous solutions of the ODE (8.1), which apparently do not respect the Picard's theorem. The origin of this violation could be traced to the fact that a variable in **R** may undergo changes (increments) via the extended SL(2, R)-like group actions. These solutions break explicitly the parity symmetry of the underlying ODE (for details, see Sec.9.3).