

*PREFACE*

## PREFACE

Structural members such as plates and shells are frequently used in various machine parts and, as such, the scope of study of bending properties of such members is sufficiently broad. Within elastic limit, various plate and shell problems have been dealt with by numerous eminent scientists and research workers. It is apparent from these studies that all these problems may be classified as

- \* Static Problems,
- \* Dynamic Problems,
- and
- \* Thermal Problems.

Any elastic behaviour (whether 'Static' or 'Dynamic' or 'Thermal') of plates and shells is influenced by the following factors :-

- (i) **Material properties** defined by Young's Modulus 'E' and Poisson's Ratio 'v', both of which may be constants or variables.
- (ii) **Geometry of the Plates and Shells** : Geometry of the plates may be simple such as Triangular, Rectangular, Circular, Elliptic etc., or may be complex like Parabolic, Polygonal or Skewed. Shells, on the other hand, may be Cylindrical, Spherical, Conical etc.
- (iii) **Thickness (h)** of the plate or shell may be either constant or variable.
- (iv) **Types of loading** may be Transverse (or Lateral) Loading, In-plane Loading, Concentrated loading, Edge Loading, Combined Loading etc.
- (v) **Nature of support** : Edge conditions of the plates and shells may be 'Simply-Supported' or 'Clamped' having 'Movable' or 'Immovable' states.

## THEORY OF PLATES

Whenever the normal deflection 'w' of a thin plate is small in comparison with its thickness 'h', a very approximate but satisfactory theory of bending of the plate by lateral load can be developed by making the following assumptions :

- \* There is no deformation in the middle plane of the plate and hence this plane remains '**Neutral**' during bending.
- \* Points of the plate lying initially on a normal to the middle plane of the plate remain on the normal to the middle surface of the plate after bending.
- \* The stresses normal to the middle plane of the plate, arising from the applied loading, can be disregarded.

These assumptions constitute the simplest and widely used "**Classical Small Deflection Theory**" or "**Linear Theory**", developed by Joseph Louis Lagrange in 1811. According to this theory, all the stress components can be expressed by the normal deflection 'w' of the plate, which is a function of the two co-ordinates in the plane of the plate. This function has to satisfy a linear partial differential equation which together with the boundary conditions, completely defines 'w'. Thus, the solution of this equation gives all the necessary information for calculating stresses at any point of the plate. Following the above mentioned linear theory, numerous works on the bending of thin plates have been carried out during the first half of the last century by many research workers and the Bibliography of these works have been nicely incorporated in the book, "Theory of Plates and Shells" by Timoshenko and Woinowski – Krieger [1959].

In most of these cases, bending of a plate is accompanied by strain in the middle plane, but, careful calculations show that the corresponding stresses in the middle plane are negligible, only if the deflections of the plate are small in comparison with its thickness. If the deflections are not small in comparison with the plate thickness, these supplementary stresses in the middle plane of the plate must be taken into account in

deriving the governing differential equations of the plate. The differential equations so obtained become non-linear in character and their solutions are much more complicated.

With the advent of modern technology and systems exposed to oppressive operation conditions, the linear hypothesis could no longer be 'retained'. Whenever Forces, Deformations, Velocities, Temperatures and other factors become excessive, non-linear effects come into play and their influence can no longer be ignored. This situation occurs also in the particular field of Applied Mechanics involving plates and shallow shells. These elements when used in modern structures, such as 'High-speed Aeroplanes', "Missiles" and "Space-Vehicles", are often subject to large transverse deflections and reveal clearly non-linear response.

## **METHODS OF TACKLING FOR THE ANALYSIS OF THE NON-LINEAR PROBLEMS OF THIN PLATES :**

So far there has been a wide application of differential equations for the non-linear analysis of thin plates, thick plates and sandwich plates. They are

- \* von-Kármán's Equations,

- \* Berger's Equations,

and

- \* Banerjee's Equations.

The first formulation of the theory of plates with a stretching of the middle plane and moderately large deflections was developed by Gehring [1860] and was improved upon by Kirchhoff [1883]. The potential energy per unit area of the plate was written as a sum of a quadratic function of the quantities defining the **Extension** of the middle plane and a quadratic function of the quantities defining **Bending**. The equations of motion with finite displacements were deduced by the Principle of Virtual Work. The stress function satisfying the in-plane force equilibrium equations was introduced by *Föppl* [1907]. The currently popular form of the two governing equations, in the rectangular Cartesian co-ordinates, was given by von-Kármán's [1910]. The equations

of von-Kármán's are in the coupled form and involve Transverse Deflection and the Membrane Stress Function as unknown functions. These are difficult to solve. Several numerical methods have been employed by different investigator for solutions of von-Kármán's equations.

### INTERESTING WORKS ON VON-KÁRMÁN'S EQUATIONS :

Among the authors who initially treated the non-linear analysis of plates, the works of Timoshenko [1937] and Way [1934, 1938] need special mention. Since then many authors investigated the different plate problems using von-Kármán's Equations. Useful works in this field are due to Levy [1942], Levy and Greenman [1942], Chien [1947], Chi-Teh Wang [1948], Chu and Herrmann [1955, 1956], Weil and Newmark [1956], Medwadowski [1958], Nash and Modeer[1959], Nash and Cooley [1959], Nowinski [1962, 1963, 1964], Yamaki [1964], Alwan [1964], Nowinski and. Ismail [1965], Harris and Mansfield [1967], Kennedy and Simon Ng. [1967], Schmidt [1968], Bauer [1968], Whitney and Leissa [1969], Sathyamoorthy and Pandalai [1972], Bolton [1972], Nowinski and Obnabe [1973], Kanakeraju and Rao [1976], Dutta [1976, 1978], Sayed and Schmidt [1977], Karmakar [1978, 1979], Satyamoorthy [1978, 1979], Sathyamoorthy and Chia [1979, 1980], Banerjee and Dutta [1980], Reddy and Chao [1981], Reddy and Huang [1981], Banerjee [1982, 1983, 1984], Chaudhuri [1982, 1984], Das [1984], Kanakarju and Rao [1986], Dumir [1988], Ostiguy and Nguyeu [1988], Yeh Kai-Yuan, Zheng Xiao-Jing and Zhou You-he [1989], Gorji [1989], Kobayashi and Sonoda [1989], Liew and Lam [1990], Zhang [1991], Singh and Chakravorty [1991], Kobayashi and Sonoda [1991], Gupta, Lal and Jain [1991], Ganapathi, Varadan and Sarma [1991], Chai [1991] and Gorman[1991].

All these works solved the coupled form of von-Kármán's equations by different numerical methods which are elegant but laborious.

Berger [1955] offered an approximate method for non-linear analysis of thin elastic plates by neglecting the so-called second strain invariant in the expression for the total potential energy of the system. He investigated the large deflections of circular and

rectangular plates both for clamped **immovable** and simply supported **immovable** edges. Although no physical justification of his assumption was given in the paper, the results obtained in both the cases are in very good agreement with other known results. The speciality of Berger's equations is that the equations have been uncoupled so that the solutions of the differential equations have been simplified and obtained in the closed form. Actually Berger's equations are quasi-linear in character. The essential non-linearity depends on the coupling parameter ' $\hat{\alpha}$ '. These specialities made Berger's equations very popular and many useful works on plates of immovable edges have been done following Berger's equations.

### USEFUL WORKS ON BERGER'S EQUATIONS :-

The comprehensive works on Berger's Equations which need special mention are due to Iwinski and Nowinski [1957], Williams [1958], Nash and Modeer [1959, 1960], Basuli [1961], Hassert and Nowinski [1962], Wah [1963], Sinhã [1963], Bera [1964], Nowinski and Ismail [1965], Pal [1967], Banerjee [1967, 1968, 1969], Cheng-Ih-Wu and Vinson [1968, 1969], Pal [1969, 1970, 1973], Sathyamoorthy and Pandalai [1970, 1973, 1974], Mazumdar and Jones [1974], Biswas [1975], Dutta [1975, 1976], Banerjee [1976], Kamiya [1976], Karmakar [1977], Mazumdar and Jones [1977], Dutta [1976, 1977], Sathyamoorthy [1977, 1978], Kamiya [1978], Banerjee and Dutta [1979], Banerjee [198], Das [1984], Das [1986], Bera and Sinharay [1992], and Bera and Mukhopadhyay [1994].

It is to be noted that Berger's equations are meaningful for immovable edge conditions only. It leads to meaningless results for **movable edge** conditions. This has been pointed out by Nowinski and Obnabe [1972] in an excellent research work.

In 1981 Banerjee and Dutta offered a modified strain energy expression to investigate non-linear problems of thin elastic plates. A set of uncoupled differential equations is obtained under this modified strain energy expression. The authors have tested the accuracy of their equations by solving different non-linear plate problems. They have obtained sufficiently accurate results, both for **movable** as well as for **immovable** edge conditions. The equivalent hypothesis of Banerjee's equations is that

the radial stretching of the plate is proportional to  $\left(\frac{dw}{dr}\right)^2$ . This is reasonable because under any type of loading and under any boundary condition the extra strain imposed by bending is represented by the term  $\left(\frac{dw}{dr}\right)^2$ . It is believed that Banerjee's equations are more welcome from the practical point of view because unlike von-Kármán's equation's they are uncoupled and unlike Berger's equations they give reasonable results both for **movable** as well as for immovable edge conditions.

### **IMPORTANT WORKS ON BANERJEE'S EQUATIONS :-**

So far many useful works have been carried out on Banerjee's equations. Important works are due to Banerjee and Dutta [1981], Banerjee [1984], Banerjee and Sinharay [1985, 1986] and Ghosh [1991] who have utilized Banerjee's equations in different non-linear thin plate problems under different types of loading and achieved satisfactory results.

Employing Banerjee's hypothesis of thick plates, Bhattacharjee and Banerjee [1988] have achieved satisfactory results towards non-linear behaviour of thick plates under different types of loading.

Utilizing Banerjee's equations on skewed plates, excellent results were also obtained by Roy, Banerjee and Bhattacharjee [1991, 1992, 1993, 1994] for rhombic plates under different types of loading. Numerical results obtained in the above cases are interesting and useful from the practical point of view.

### **SANDWICH PLATES**

Investigations on finite deformations of sandwich plates are gaining importance day by day due to its wide applications in modern designs. Outstanding research workers who carried out interesting investigations in this field are Reissner [1948], Alwan [1964] and Nowinski and Ohnabe [1973].

Reissner proposed the basic differential equations for finite transverse deflections of sandwich plates under the assumptions that the plate consists of a core layer and two face layers of such construction that the face parallel stresses in the core and the variations of the face stresses over the thickness of the face layers are negligible.

The resultant equations permit the analysis of the effect of transverse shear stress deformation and transverse normal stress deformation in the core on the overall behaviour of the plate. It is shown that, in general, the effect of the transverse normal stresses in the core is negligibly small as compared to the effect of the transverse shear stresses. The equations, when simplified by the omission of the transverse normal stress terms are brought into a form suitable for the solution of rectangular plate problems. It was further shown that the range of deflections for which the linear theory is adequate decreases in accordance with a simple explicit formula as the core is made softer relative to the faces. He used the finite deflection equations to obtain plate-buckling equations that include the effect of transverse shear stress deformation on buckling loads. This is an excellent work.

Alwan proposed differential equations for large deflection of sandwich plates with orthotropic core in terms of the Airy stress function. This work is very interesting and useful for modern design.

Nowinski and Ohnabe presented a generalization of Reissner equations for the isotropic plates to the case when the material of both the core and the faces is isotropic and having different elastic characteristics. They derived a system of four governing equations involving the deflection  $w$ , stress function  $\phi$ , and two functions,  $\alpha$  and  $\beta$ , characterizing the shearing of the core. A transformation enables one to reduce the system to two equations involving two functions,  $w$  and  $\phi$  only. For the isotropic faces the system reduces to two equations obtained by Alwan for large deflections of plates and to the equations of Cheng [1968] for small deflections. The integral analysis based on the application of Reissner-Hellinger variational principles is simpler than the five equations differential approach of Wempner and Bayler [1965]. The authors obtained an explicit

solution for a simple supported rectangular plate prevented from in-plane motions on the contour, by means of Galerkin's procedure. Numerical calculations using UNIVAC 1108 computer were performed for various aspect ratios and the thickness of the plate as well as for several contrasting sets of elastic constants of orthotropic and isotropic materials. This paper is an attractive work on orthotropic sandwich plates. All these investigations involve non-linear partial differential equations in the coupled form. These equations are difficult to solve. Different numerical techniques have been employed to get desired solutions.

To get over the difficulties in the solutions of coupled differential equations for sandwich plates, Kamiya [1976] has employed Berger's well known technique to solve non-linear problems of sandwich plates. The author has offered a new set of governing equations by using Berger's approximation to study the non-linear static behaviour of sandwich plates. The accuracy of his method depends on a correction factor  $F\left(\frac{b}{a}\right)$ . Thus although Kamiya has been able to simplify the coupled differential equations of sandwich plates proposed by earlier authors, his attempt has been restricted to a particular plate geometry due to introduction of this correction factor. Moreover, Berger's method fails completely for movable edge conditions.

Later Karmakar [1978] also employed Berger's line of thought to investigate free vibrations of rectangular sandwich plates. The numerical results obtained in this study are interesting.

Large deflection analysis of heated sandwich plates is very important in modern design. Unfortunately work in this area has not received much attention so far. An interesting work in this field is due to Kamiya [1978] who has quite elegantly analyzed the large deflections of rectangular sandwich plates under thermal loading by using Berger-type equations. The author's analysis is useful. He has compared the results with those obtained from conventional governing equations solved by the Ritz-Galerkin method. These results are approximate due to Berger's approximation which is fairly

accurate for immovable simply supported edges and fails for movable edges. Ritz-Galerkin's method also involves approximations.

Banerjee and Dutta [1991] proposed a new set of differential equations to study the non-linear static and dynamic behaviours of sandwich plates following Banerjee's hypothesis [1981]. A new set of uncoupled differential equations have been formed in rectangular Cartesian co-ordinates. Accuracy of the results of different sandwich plates both for movable as well as immovable edge conditions has been tested under different types of loading and compared with the other known results.

Later applying the hypothesis of the above study, Roy, Dutta and Banerjee [1993] have offered a new set of uncoupled differential equations to solve the non-linear thermal behaviours of sandwich plates. The case of a rectangular plate has been carried out in detail. Numerical results of a simply-supported rectangular plate with movable as well as immovable edges have been given in graphs. The results obtained by the authors are interesting and useful for modern design.

The above work seems to be more advantageous because of the following considerations:

- i) Results of the immovable as well as movable edge conditions can be obtained from the same cubic equations with ease and accuracy.
- ii) The proposed differential equations are in uncoupled form and hence results can be obtained employing minimum computational labour.
- iii) The uncoupled differential equations yield sufficiently accurate and useful results acceptable for the practical purpose, and
- iv) Previous investigations by Kamiya [1976] show that accuracy of the results depend on a correction factor, which is a function of the ratio of the sides of plate geometry. But the above study does not depend on any correction factor. This is an additional advantage of the above work.

Bhattacharjee and Banerjee [1988,1993] and Chakrabarty and Bera (2002) carried out some interesting investigation on sandwich plates and shallow shells of irregular shapes following Banerjee's hypothesis along with constant deflection contour line method opined by Mazumdar [ 1970 ]. The authors investigated the buckling of elastic sandwich plates and shells and the non-linear behaviours of elastic plates and shells under mechanical, dynamic and thermal loading. The advantage of constant deflection contour line method is that problem of any geometrical shape can be analyzed by changing the equations of level curves  $u(x, y) = \text{constant}$  for different contours.

## THEORY OF SHELLS

The main suppositions of the theory of thin plates also form the basis for the usual theory of the thin shells. There exists, however, a substantial difference in the behaviour of plates and shells under the action of external loading. The static equilibrium of a plate element under a lateral load is only possible by action of bending and twisting moments, usually accompanied by shearing forces, while a shell, in general is able to transmit the surface load by "membrane" stresses which act parallel to the tangential plane at a given point of the middle surface and are distributed uniformly over the thickness of the shell. This property of shells makes them, as a rule, a much more rigid and a more economical structure than a plate would be under the same conditions.

In principle, the membrane forces are independent of bending and are wholly defined by the conditions of static equilibrium. The method of determination of these forces represent the so-called "membrane theory of shells". However, the reactive forces and deformation obtained by the use of the membrane theory at the shell's boundary usually become incompatible with the actual boundary conditions. To remove this discrepancy, the bending of the shell in the edge zone has to be considered, which may affect slightly the magnitude of initially calculated membrane forces. This bending, however, usually has a very localized character and may be calculated on the basis of the same assumptions which were used in the case of small deflection of thin plates. But there are problems, especially those concerning the elastic stability of shells, in which the

assumption of small deflections should be discontinued and the “Large Deflection Theory” should be used.

Non-linear analyses of shells are receiving considerable attention from design engineers for their wide application in modern engineering design. Interesting works in this field are due to Nash and Modeer [1959], Nawacki [1962], Nowinski and Ismail [1964], Basuli [1964], Bhattacharya [1976], Biswas [1978, 1980], Ramchandran [1976], Sinharay and Banerjee [1985], Bera [1993] and Chakrabarty and Bera (2002).

Nash and Modeer studied the non-linear behaviour of shallow spherical shells by using Berger’s method. Nowacki offered some interesting works on thermal deflections and stresses of cylindrical and spherical shells based on the linear as well as non-linear theory in thermoelasticity. Nowinski and Ismail used Berger’s line of thought and investigated large deflections in cylindrical shell panels. The authors’ analyses also are of interest from a practical point of view. An interesting paper on large deflection of shell panel by Basuli needs special mention where the author studied elegantly cylindrical shell panel following Berger approximation. Bhattacharya’s analysis is also based on Berger’s method. He studied non-linear dynamic behaviours of cylindrical shells. Large deflection analysis of a heated cylindrical shell has been carried out by Biswas where he applied Berger’s approximation. In another paper the author studied the large deflection of heated orthotropic cylindrical shallow shell. Ramchandran, on the other hand, used Marguerre’s shallow shell equations to analyze successfully the large amplitude vibrations of shallow spherical shells. Sinharay and Banerjee have offered a modified strain energy expression to study the non-linear behaviour of spherical and cylindrical shells. The authors have obtained significant results both for movable as well as immovable edge conditions.

Non-linear analysis of sandwich shell is attractive to design engineers for its wide application in space industry. Useful works in this field are due to Bera [1996], Bhattacharya and Banerjee [1996] and Chakrabarty and Bera (2002). Bera studied the non-linear behaviour of a shallow unsymmetrical sandwich shell of double curvature following a new approach. Chakrabarty and Bera studied the nonlinear vibration and stability of a shallow unsymmetrical orthotropic sandwich shell while Bhattacharya and

Banerjee studied large deflection of sandwich shells following displacement formulations.

A suitable analysis of large deflection of heated spherical shell has been given by Sircar [1978] where he used well known Berger's method. Behaviours of cylindrical shells with circular holes subject to uniform impulsive pressure have been studied by Andrev, Antsiferov, Maslov and Pavlenko [1989]. The authors' analysis is noteworthy. Suleimanova and Panova [1990] offered a suitable analysis on stress and strain of non-linear notched shell. This paper is also interesting. Non-linear deformation of cylindrical shell with local heat shock has been carried out by Obodan and Makarenko [1990], Kumar, Olson and Anderson [1991] studied the large deflection elastic-plastic analysis of cylindrical shells by using the finite strip method. All these papers are useful in designs.

Another interesting paper on vibrations of shallow shells could be located where Bera [1993] has successfully investigated non-linear vibrations of shallow shells of arbitrary shape using constant deflection contour lines theory. A useful paper on snapping of a thin spherical cap by Pal and Bera [1993] can be located where non-linear analysis of spherical cap has been studied following a new approach.

Shells of non-uniform thickness and non-homogeneous material are sometimes encountered in the design of machine parts and their stress analyses are imperative to design engineers. As far as it is known, works in this field are very rare. Only one paper could be located in this area, where Sinharay and Banerjee [1986] investigated large amplitude free vibrations of spherical and cylindrical shells of non-uniform thickness. They employed a modified strain energy expression in their investigation and although their results are interesting, some amount of approximation has entered because of the use of modified strain energy expression. Another interesting paper may also be located in this area where Mukhopadhyay and Bera [1995] studied quite elegantly non-linear thermal vibration of non-homogeneous elastic shell. A few more interesting papers on shells could be located where Ray and Banerjee analyzed quite elegantly non-linear analysis of a clamped cylindrical shell [1996], non-linear analyses of a heated cylindrical

and spherical shell [1996], large deflection of heated orthotropic thin cylindrical shell [1997] and large amplitude free vibration of spherical of variable thickness [1997]. The authors' observations are interesting for practical purpose.

A careful survey of above literature reveals that some works have been done on problems of sandwich plates and shells. But more attention is required for analyses of sandwich structure. Keeping in mind this void on sandwich literature, the present study aims at investigating some non-linear problems of thin elastic plates and shells. The major portion of the study is devoted to large deflection analyses of sandwich structures.

The work has been divided into four chapters. The first chapter deals with three interesting problems on large deflection of sandwich plates. The first paper of this chapter is devoted to a note on large deflection analysis of circular sandwich plate following a new approach. A new set of differential equations in polar co-ordinate system has been developed in uncoupled form which governs the behaviour of circular sandwich elastic plates with symmetrical faces. In deriving these equations, the idea of Dutta and Banerjee [1991] has been utilized. Numerical results for different types of loading have been computed and compared with other known results.

The second paper deals with bending of a sandwich circular plate under large deflection. Berger's technique has been utilized. Numerical results are computed and the corresponding graphs are drawn from the results thus obtained. The reduction of the present results to the classical results for the ordinary plate have been found to be in excellent agreement.

The third problem is concerned with a note on the large deflection of heated sandwich circular plate. A new set of uncoupled differential equations for the study of circular sandwich plates under thermal loading have been derived following the idea of Roy, Dutta and Banerjee [1993]. Numerical results obtained from the present study have been plotted graphically for some special values of the parameter involved. The results so obtained have been compared with those of open literature.

The second chapter deals with two problems on large amplitude free vibrations of parabolic plates. The first problem is concerned with a modified approach on the large amplitude free vibrations of parabolic plates. A simplified method is developed in two different approaches, one proposed by Mazumdar and other by Banerjee for the analysis of large amplitude transverse vibration of thin, isotropic, homogeneous elastic plates of arbitrary shapes. Numerical results are presented in tabular form and corresponding graphs are drawn and compared with available results.

The second paper deals with large amplitude free vibration of sandwich parabolic plates. The present method is a modification of Mazumdar and Banerjee's approaches for the analysis of large amplitude transverse vibration of thin plates. Numerical results are given in the form of graphs and compared with available results.

The third chapter deals with two problems on large deflection of sandwich spherical shells. The first problem is concerned with a new approach on large deflection analysis of spherical sandwich shell. In the present analysis the non-linear static and dynamic behaviour of spherical sandwich shell due to large deflection have been studied by a new set of uncoupled differential equations. Numerical results for different types of loading have been computed and compared with other known results.

The second problem is concerned with a note on large deflection analysis of heated sandwich spherical shell by a new approach. The aim of the present of analysis is to propose a new set of uncoupled differential equations for the study of spherical sandwich shells under thermal loading by using a new and simple approximation. Moreover, the failure of Berger's analysis have been overcome by the present method. The analysis of clamped and simply supported heated sandwich spherical shells with movable as well as immovable edges has been carried out in detail. Numerical results obtained from the present study have been plotted graphically for some special values of the parameters involved. The results have been compared with those available from the other methods.

The fourth and the last chapter deals with an interesting problem on non-linear analysis of a thin spherical cap. The aim of the present investigation is to show a simple and yet sufficiently accurate method for the non-linear analysis of snapping of a thin spherical cap. The method is based on the idea of Sinharay and Banerjee [1985]. The numerical results for the maximum value of the load and the minimum value of the load are obtained, plotted graphically and compared with available results.