CHAPTER II

SOME LINEAR AND NON-LINEAR PROBLEMS OF POLYGONAL PLATES

CHAPTER II

SOME LINEAR AND NON-LINEAR PROBLEMS OF POLYGONAL PLATES

ABSTRACT

This chapter, containing seven problems, is devoted to study the static, dynamic and thermal behaviours of polygonal plates using conformal mapping technique. These investigations are based on the linear and non-linear theory.

DIFFERENTIAL EQUATIONS

Following Timoshenko and Woinowsky-Krieger (Theory of plates and shells page 372) the differential equation for the deflection of an orthotropic plate except at the load point in rectangular co-ordinate system is

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = 0 \qquad \text{(II.1)}$$

The differential equation of the vibrations of orthotropic plates takes the following form

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} + \rho h \frac{\partial^2 W}{\partial t^2} = 0 \qquad (II.2)$$

For isotropic plate material the equation for vibrations of such plate takes the following form

$$D \nabla^4 W + \rho h \frac{\partial^2 W}{\partial t^2} = L \qquad (II.3)$$

where L is the load function applied to the plate.

Following the model offered by Biot the differential equation for viscoelastic plates can be put as

$$B_1 \nabla^4 W + \rho h \frac{\partial^2 W}{\partial t^2} = f \qquad (II.4)$$

where $B_1 = \dfrac{h^3 Q(3kP + Q)}{3P(3kP + 4Q)}$

and f is the load applied to the plate.

P and Q are linear operators as given by [Maso, Transient response of Linear Viscoelastic Plates, ASME, Dec.1960, P.588]

$$P = \sum_n a_n \frac{\partial^n}{\partial t^n} \quad \text{and} \quad Q = \sum_n b_n \frac{\partial^n}{\partial t^n}$$

For Kelvin type of material $P = 1$; $Q = 2G + 2\eta\rho$.

A. DEFLECTIONS OF ORTHOTROPIC POLYGONAL PLATES UNDER CONCENTRATED LOAD AT THE CENTRE*

Let us consider a clamped edged orthotropic plate under the action of concentrated load 'P' at the centre.

For an orthotropic material such as reinforced concrete

$$H = (D_x D_y)^{\frac{1}{2}}$$

Thus equation (II.1) reduces to

$$\frac{\partial^4 W}{\partial z_1^2 \partial \bar{z}_1^2} = 0 \qquad \text{(II.5)}$$

with the substitution

$$z_1 = x + \varrho_1 y.$$

where ϱ_1 is the root of the eqn. $D_y \varrho_1^4 + 2H \varrho_1^2 + D_x = 0.$

Clearly $\varrho_1 = i\omega$ where $\omega^2 = (D_x / D_y)^{\frac{1}{2}}$.

The solution of Eq. (II.5) will have two parts

$$W = W_0 + W_1 \qquad \text{(II.6)}$$

where W_0 is the particular solution of Eq. (II.5) at the load point and W_1 is the general solution of Eq. (II.5).

---

* Accepted for publication in the "Indian Journal of Technology", CSIR, (India).

In case of single load P at the centre, the solution W_o will be of the form

$$W_o = \frac{P}{16 \pi D_x} Z_1 \bar{Z}_1 \log \frac{Z_1 \bar{Z}_1}{L^2} \qquad (II.7)$$

where L is a dimension of the plate.

The general solution of Eq. (II.5) will be of the form

$$W_1 = Z_1 \phi(Z_1) + Z_1 \phi(\bar{Z}_1) + \chi (Z_1) + \chi (\bar{Z}_1) \qquad (II.8)$$

Eq. (II.7) and (II.8) together give the complete solution of Eq. (II.5). Now the expression for Z_1 in terms of $Z = x + iy$ is

$$Z_1 = \lambda_1 Z + \lambda_2 \bar{Z} \qquad (II.9)$$

where $\lambda_1 = \frac{1 + \omega}{2}$ and $\lambda_2 = \frac{1 - \omega}{2}$

Let $Z = L (\xi + \delta \xi^5)$ be the mapping function which maps the domain under consideration on to a unit circle where $\xi = e^{i\theta}$.

Let us assume

$$\phi(Z_1) = \sum_{n=1}^{\infty} a_n \xi^n, \quad \phi(\bar{Z}_1) = \sum_{n=1}^{\infty} \bar{a}_n \xi^n,$$

$$\mathcal{X}(Z_1) = \sum_{n=0}^{\infty} b_n \xi^n, \quad \mathcal{X}(\bar{Z}_1) = \sum_{n=0}^{\infty} \bar{b}_n \xi^n$$

and

$$\mathcal{X}'(\bar{Z}_1) = \sum_{n=0}^{\infty} \bar{E}_n \xi^n$$

For the clamped edge we have on the boundary

$$W = 0 \tag{II.10}$$

and

$$\frac{\partial W}{\partial \bar{Z}_1} = 0 \tag{II.11}$$

Now putting the values of $\phi(Z_1)$, $\phi'(\bar{Z}_1) = \frac{d}{d\bar{Z}_1}\left\{ \phi(\bar{Z}) \right\}$,

$\mathcal{X}(Z_1)$, & Z_1 we have from Eqs. (II.6), (II.7), (II.8) and (II.11)

$$\sum_{n=1}^{\infty} a_n \xi^n + \frac{(\lambda_1 L\xi + \lambda_1 L\delta\xi^5 + \lambda_2 L\bar{\xi} + \lambda_2 L\delta\bar{\xi}^5)}{L(1 + 5\delta\xi^4)\lambda_1} \sum_{n=1}^{\infty} n\,\bar{a}_n \xi^{n-1}$$

$$+ \sum_{n=0}^{\infty} \bar{E}_n \xi^n$$

$$= -\frac{PL}{16\pi D_x}\Big[(\lambda_1\xi+\lambda_1\delta\xi^5+\lambda_2\bar{\xi}+\lambda_2\delta\bar{\xi}^5)\,\mathrm{Log}\Big\{(\lambda_1^2+\lambda_2^2).$$

$$.(1+\delta\xi^4+\delta\bar{\xi}^4+\delta^2)+\lambda_1\lambda_2(\xi^8+\bar{\xi}^2+2\delta\xi^8$$

$$+2\delta\bar{\xi}^6+\delta^2\xi^{10}+\delta^2\bar{\xi}^{10})\Big\}$$

$$+\lambda_1\xi+\lambda_1\delta\xi^5+\lambda_2\bar{\xi}+\lambda_2\delta\bar{\xi}^5\Big] \qquad \text{(II.12)}$$

Equating the co-efficients of different powers of ξ on both sides and since the co-efficients of different powers of ξ on right hand side are purely real, we have

$$a_1 = \bar{a}_1 = -\frac{PL}{32\pi D_x}\Big[\lambda_1\Big\{1+A_1+\log(\lambda_1^2+\lambda_2^2)\Big\}$$

$$-\lambda_1\delta\frac{4Q_1}{1-5\delta^2}+\lambda_2\frac{B_1-5\delta F_1}{1-5\delta^2}$$

$$+\lambda_2\delta\frac{F_1-5\delta K_1}{1-5\delta^2}\Big] \qquad \text{(II.13)}$$

$$a_3 = \bar{a}_3 = -\frac{PL}{16\pi D_x}\Big[\lambda_1\frac{B_1(1+\delta)}{1+3\delta}+\lambda_2\frac{Q_1+\delta I_1}{1+3\delta}\Big] \qquad \text{(II.14)}$$

$$a_5 = \bar{a}_5 = -\frac{PL}{16\pi D_x}\Big[\lambda_1\delta\left\{1+A_1+\log(\lambda_1^2+\lambda_2^2)\right\} + \lambda_1 Q_1$$

$$+\lambda_2(F_1+\delta K_1)\Big] - a_1\delta \qquad (II.15)$$

$a_2 = \bar{a}_2 = a_4 = \bar{a}_4 = 0$ and so on.

Where

$$A_1 = -R^2\left(1 + 4\delta^2 + \delta^4 + \frac{\delta^4}{2R^2}\right)$$

$$B_1 = R(1-\delta-3\delta^2)$$

$$Q_1 = \left(\delta - \delta^3 - \frac{R^2}{2} - 2R\delta^2 - 2R^2\delta^3\right)$$

$$F_1 = R(\delta - 3\delta^3) \qquad (II.16)$$

$$I_1 = -\left(\frac{\delta^2}{2} + 2R^2\delta + R^2\delta^2\right)$$

$$K_1 = -R(\delta^2 + \delta^4)$$

and $$R = \frac{\lambda_1\lambda_2}{\lambda_1^2+\lambda_2^2}$$

Thus $\phi(Z_1)$ and $\phi(\bar{Z}_1)$ are completely known.

Now from Eqs. (II.6), (II.7), (II.8) and (II.10), we have

$$(\lambda_1\bar{Z}+\lambda_2 Z)\sum_{n=1}^{\infty} a_n \xi^n+(\lambda_1 Z+\lambda_2\bar{Z})\sum_{n=1}^{\infty}\bar{a}_n\xi^n+\sum_{n=0}^{\infty} b_n\xi^n+\sum_{n=0}^{\infty}\bar{b}_n\xi^n$$

$$= -\frac{P}{16\pi D_x}(\lambda_1 Z+\lambda_2\bar{Z})(\lambda_1\bar{Z}+\lambda_2 Z)\log\frac{(\lambda_1 Z+\lambda_2\bar{Z})(\lambda_1\bar{Z}+\lambda_2 Z)}{L^2}$$

... (II. 17)

From Eq. (II. 17) [after putting the values of Z and $\bar{Z}$ and then equating the co-efficients of different powers of ξ on both sides] we have

$$b_0+\bar{b}_0 = -\frac{PL^2}{16\pi D_x}\Big[(1+\delta^2)(\lambda_1^2+\lambda_2^2)\left\{A_1+\log(\lambda_1^2+\lambda_2^2)\right\}$$

$$+2\lambda_1\lambda_2 B_1+2\delta(\lambda_1^2+\lambda_2^2)Q_1+2\delta^2\lambda_1\lambda_2 K_1$$

$$+4\delta\lambda_1\lambda_2 F_1\Big]-2\lambda_1 L(a_1+\delta a_5) \qquad \text{(II. 18)}$$

$$b_2 = - \frac{PL^2}{16\pi D_x} \Big[(1+\delta)^2 (\lambda_1^2+\lambda_2^2) B_1 + \lambda_1\lambda_2 \Big\{ A_1 + \log(\lambda_1^2+\lambda_2^2)$$

$$+ (1+2\delta) Q_1 + 2\delta I_1 \Big\} \Big] - \lambda_1 L \Big\{ (1+\delta) a_3 + \delta a_7 \Big\} \qquad (II.19)$$

$b_1 = b_3 = 0$ and so on.

Thus $\chi(Z_1)$ and $\chi(\bar{Z}_1)$ are known where A_1, B_1, C_1 ... etc. are given by relations (II.16).

Hence the complete solution of Eq. (II.5) is known. The maximum deflection at the centre is given by

$$(W)_{max.} = b_0 + \bar{b}_0 \qquad (II.20)$$

From Eq. (II.20) and (II.18) the maximum deflection for isotropic plate can be deduced putting $D_x = D$, $\lambda_1 = 1$ and $\lambda_2 = 0$. Thus

$$(W)_{max.;iso.} = \frac{PL^2}{16\pi D} \Big[(1+\delta^2) + 2\delta Q_1' - \frac{4 Q_1' \delta}{1 - 5\delta^2} (1-\delta^2)$$

$$- 2\delta Q_1' \Big] \qquad (II.21)$$

where $Q_1' = \delta(1-\delta^2)$.

NUMERICAL RESULTS

The maximum deflections of different polygonal plates both for weaker and stronger orthotropy have been calculated and given in Table II.2. The maximum deflections for isotropic square and circular plates have also been calculated and given in Table II.3. For weaker orthotropy

$$\lambda_1 = 1.096, \quad \lambda_2 = -0.096 \;\left(\frac{E_y'}{E_x'} = \frac{1}{2} \right).$$

For stronger orthotropy

$$\lambda_1 = 1.557 \;\text{ and }\; \lambda_2 = -0.557 \;\left(\frac{E_y'}{E_x'} = \frac{1}{20} \right).$$

TABLE II.1

Mapping function $Z \simeq L\,(\xi + \delta\xi^5)$

Polygons of side 'a'	Mapping function co-efficients	
	L	δ
Square	0.54 a	- 0.102
Pentagon	0.5265 a	-
Hexagon	0.519 a	-
Heptagon	0.514 a	-
Octagon	0.5109 a	-
Circle of radius 'a'	a	0

TABLE II.2

Maximum deflection of orthotropic plates

Polygons of side 'a'	$\left(\frac{W}{h}\right)_{max.;orth.} = \alpha \frac{Pa^2}{D_x h}$	
	Weaker orthotropy α	Stronger orthotropy α
Square	0.0069	0.0118
Pentagon	0.00656	0.01128
Hexagon	0.006377	0.01096
Heptagon	0.006255	0.01075
Octagon	0.006179	0.0106
Circle of radius 'a'	0.0237	0.041

TABLE II.3

Maximum deflection of isotropic plates

Polygons of side 'a'	$\left(\frac{W}{h}\right)_{max.;iso.} = \alpha \frac{Pa^2}{Dh}$	
	Present study α	Known results from Timoshenko & Woinowsky-Krieger α
Square	0.005615	0.0056
Pentagon	0.00552	-
Hexagon	0.005359	-
Heptagon	0.005256	-
Octagon	0.005193	-
Circle of radius 'a'	0.02	0.02

CONCLUSION

In case of calculation of maximum deflection of orthotropic plates of more than 4 sides, only one term of the mapping function has been considered.

In case of square and circular isotropic plates the results of this study are in excellent agreement with the known results.

The deflections of orthotropic plates are greater than those of isotropic plates.

In case of orthotropic plates the deflections decrease as the number of sides of the polygon increases. The deflection is maximum for a circular plate.

B. VIBRATIONS OF ORTHOTROPIC POLYGONAL PLATES*

Let us consider a clamped orthotropic plate of thickness 'h'. The deflection of such plate is given by

$$W(x,y,t) = e^{i\omega t} W(x,y) \tag{II.22}$$

with this deflection function W (x,y,t) equation (II.2) reduces to

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} - \rho h \omega^2 W = 0 \tag{II.23}$$

Taking $H^2 = D_x D_y$ for orthotropic material such as reinforced concrete equation (II.23) is

$$16 \frac{\partial^4 W}{\partial z_1^2 \partial \bar{z}_1^2} - \frac{\rho h \omega^2}{D_x} W = 0 \tag{II.24}$$

with the substitution

$$z_1 = x + \rho_1 y \tag{II.25}$$

---

* Accepted for publication in the Journal of the "Indian Institute of Science" (India), Vol. 65, No. 2, 1984.

where ϱ_1 is the root of the equation

$D_y \varrho_1^4 + 2H \varrho_1^2 + D_x = 0$. Clearly $\varrho_1 = i \beta$ where

$$\beta^2 = \sqrt{\frac{D_x}{D_y}} \qquad \text{(II.26)}$$

From equation (II.24) we have

$$\left(\frac{\partial^2}{\partial Z_1 \partial \bar{Z}_1} - \frac{K^2}{4} \right) \left(\frac{\partial^2}{\partial Z_1 \partial \bar{Z}_1} + \frac{K^2}{4} \right) W = 0 \qquad \text{(II.27)}$$

where $K^4 = \dfrac{\varrho h \omega^2}{D_x}$.

The solution of equation (II.27) will be of the form

$$W = AI_o \left(K \sqrt{Z_1 \bar{Z}_1} \right) + BJ_o \left(K \sqrt{Z_1 \bar{Z}_1} \right) \qquad \text{(II.28)}$$

where I_o and J_o are Bessel functions of zeroth order. A and B are constants to be evaluated from the boundary conditions of clamped edged plates namely

$$\text{and} \quad \left. \begin{aligned} W &= 0 \\ \frac{\partial W}{\partial Z} &= 0 \end{aligned} \right\} \text{ on the boundary} \qquad \text{(II.29)}$$

Let

$$Z = f(\xi) \simeq L\xi \qquad \text{(II.30)}$$

be the mapping function which maps the domain under consideration on to a unit circle where $\xi = e^{i\theta}$.
Now with the boundary condition for clamped edged plate as given by (II.29). One gets, from equation (II.28) and (II.30) (equating the coefficients of the term independent of ξ on both sides), the required frequency equation

$$\begin{vmatrix} I_0(K') & J_0(K') \\ I_1(K') & -J_1(K') \end{vmatrix} = 0 \qquad \text{(II.31)}$$

where

$$K'^2 = \frac{1}{2}\,\omega\,\sqrt{\frac{\rho h}{D_x}}\;L^2(1+\beta^2) \qquad \text{(II.32)}$$

The solution of equation (II.33) is $K' = 3.2$. Hence the frequency of a polygonal orthotropic clamped plate is

$$\omega = 2\left(\frac{3.2}{L}\right)^2 \cdot \frac{1}{1+\beta^2}\sqrt{\frac{D_x}{\rho h}} \qquad \text{(II.33)}$$

NUMERICAL CALCULATIONS AND DISCUSSIONS

The frequencies for free vibrations of orthotropic clamped plates of different shapes have been calculated and are given in Table II.6. The frequency for clamped orthotropic square plate has been compared with the known result.
The corresponding frequencies of clamped isotropic polygonal plates have been deduced from the present study and have been compared with known results (Table II.5).

TABLE II.4

Mapping function coefficients

$Z = f(\xi) \simeq L\xi.$

Polygons of side '2a'	Coefficient, L
Equilateral triangle	1.1353 a
Square	1.08 a
Pentagon	1.0586 a
Hexagon	1.0376 a
Heptagon	1.0279 a
Octagon	1.0219 a
Circle of radius 'a'	a

TABLE II.5

Fundamental frequency coefficients for isotropic clamped polygonal plates
($\beta^2 = 1$)

Polygon of side '2a'	$\omega \sqrt{\frac{\rho h}{D}} \cdot a^2$	
	Present Study	Known Results
Equilateral triangle	7.95	8.01 (Banerjee & Dutta, 1979)
Square	8.78	8.75 (" " ")
Pentagon	9.24	9.32 (" " ")
Hexagon	9.51	9.59 (" " ")
Heptagon	9.7	9.77 (Shahady : Pasarelli & Laura, 1967)
Octagon	9.81	9.85 (" " ")
Circle of radius 'a'	10.24	10.22 (Leissa, 1962)

TABLE II.6

Fundamental frequency coefficients for orthotropic clamped polygonal plates
($\beta^2 = 1.326$)

Polygon of side '2a'	$\omega \sqrt{\frac{\rho h}{D}} \; a^2$	
	Present Study	Known Results
Equilateral triangle	5.62	-
Square	6.22	6.64 (Banerjee, 1982)
Pentagon	6.54	-
Hexagon	6.73	-
Heptagon	6.86	-
Octagon	6.94	-
Circle of radius 'a'	7.25	-

From Table II.5 and II.6 it is observed that the results obtained by the present study are in good agreement with known results from the engineering point of view.

Thus a single equation (II.33) can be used with a good accuracy for predicting vibrations of plates of any shape with less computational labour.

C. VIBRATIONS OF POLYGONAL PLATES DUE TO SONIC BOOM *

Let us consider an elastic plate of uniform thickness 'h'. Considering the cartesian co-ordinate system, the Z-axis is taken along the thickness of the plate and perpendicular to both X and Y axes.

In case of sonic boom loading, it is well-known that under idealized, uniform atmospheric conditions, the sonic boom, for high altitude supersonic aircraft, can be described as N - shaped impulse (Figure II.1). Mathematically, the N - wave may be written as

$$L(x,y,t) = P_o \left\{ \left(1 - \frac{2t}{T}\right) + \left(\frac{2t}{T} - 1\right) H(t-T) \right\}$$

$$= P_o\, q(t) \qquad \text{(II.34)}$$

Here P_o and T are the maximum boom pressure and the boom period respectively, $H(t-T)$ is the Heaviside unit step function with values given by

$$H(t-T) = 0 \quad \text{for} \quad t \leqslant T$$

and (II.35)

$$H(t-T) = 1 \quad \text{for} \quad t > T$$

From equations (II.3) and (II.34) one gets,

$$\nabla^4 W + \frac{\rho h}{D} \frac{\partial^2 W}{\partial t^2} = \frac{P_o}{D} q(t) \qquad \text{(II.36)}$$

---

* Accepted for publication in the Defence Science Journal, Govt. of India.

Now changing equation (II.36) into complex co-ordinates by the transformation $Z = x + iy$ and $\bar{Z} = x - iy$

we have,

$$16 \frac{\partial^4 W}{\partial Z^2 \partial \bar{Z}^2} + \frac{\rho h}{D} \ddot{W} = \frac{P_o}{D} q(t) \qquad (II.37)$$

Let

$$Z = f(\xi) \simeq (L\xi + \partial \xi^5) \qquad (II.38)$$

be the mapping function which maps the domain under consideration on to a unit circle in the complex plane, where $\xi = re^{i\theta}$.

With this transformation one gets from equation (II.37)

$$16 \left[\frac{\partial^4 W}{\partial \xi^2 \partial \bar{\xi}^2} \frac{dZ}{d\xi} \frac{d\bar{Z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \bar{\xi}^2 \partial \xi} \frac{d^2 Z}{d\xi^2} \frac{d\bar{Z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \xi^2 \partial \bar{\xi}} \frac{d^2 \bar{Z}}{d\bar{\xi}^2} \frac{dZ}{d\xi} \right.$$

$$\left. + \frac{\partial^2 W}{\partial \xi \partial \bar{\xi}} \frac{d^2 Z}{d\xi^2} \frac{d^2 \bar{Z}}{d\bar{\xi}^2} \right] + \frac{\rho h}{D} \ddot{W} \left(\frac{dZ}{d\xi}\right)^3 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^3$$

$$= \frac{P_o}{D} q(t) \left(\frac{dZ}{d\xi}\right)^3 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^3 \qquad (II.39)$$

Let

$$W = W_o(t) \left[1 - 2P\xi\bar{\xi} + Q\xi^2\bar{\xi}^2 \right] \qquad (II.40)$$

be the solution of W which clearly satisfies the simply supported edge conditions when

$$P = \frac{2}{3}$$

and

$$Q = \frac{1}{3}$$

Again equation (II.40) satisfies the clamped edge conditions when

$$P = Q = 1$$

Substituting (II.40) and (II.38) in equation (II.39) one gets the error function $\epsilon(r, \theta)$: Galerkin's procedure requires that

$$\int_{c} \epsilon(r, \theta)\, W r \, dr \, d\theta = 0 \qquad \text{(II.41)}$$

After evaluating the integrals in (II.41) one gets the required differential equation determining $W_0(t)$.

The equation on simplyfication takes the following form,

$$W_0(t) + v^2 W_0(t) = C^2 q(t) \qquad \text{(II.42)}$$

where

$$w^2 = \frac{16\, A_{11}}{\Psi_4} \; \frac{D}{\rho h}$$

$$c^2 = \frac{P_o}{\rho h} \; \frac{\Psi_5}{\Psi_4}$$

$$A_{11} = \left[4\, Q \Psi_1 - 800\, Q \delta^2 \Psi_2 + 400\, \delta^2 \Psi_3 \right]$$

$$\Psi_1 = \left(.5 - .5P + \frac{Q}{6} \right) L^2 + \left(2.5 - \frac{25}{6} P + \frac{25}{14} Q \right) \delta^2$$

$$\Psi_2 = \left(.1 - \frac{P}{6} + \frac{Q}{14} \right)$$

$$\Psi_3 = \left(.25\, P + .4\, P^2 - \frac{5}{6} PQ + .4Q + \frac{2}{7} Q^2 \right)$$

$$\Psi_4 = \left[\left(.5 - P + \frac{Q}{3} - .5\, PQ + \frac{2}{3} P^2 + .1Q^2 \right) L^6 \right.$$

$$+ \left(\frac{1}{26} - \frac{P}{7} + \frac{Q}{15} - \frac{PQ}{3} + \frac{2}{15} P^2 + \frac{1}{34} Q^2 \right) \times 15625\, \delta^6$$

$$+ \left(.1 - \frac{P}{3} + \frac{Q}{7} - .25\, PQ + \frac{2}{7} P^2 + \frac{1}{18} Q^2 \right) \times 75\, L^4 \delta^2$$

$$+ \left(\frac{1}{18} - .2P + \frac{1}{11} Q - \frac{1}{6} PQ + \frac{2}{11} P^2 + \frac{1}{26} Q^2 \right) \times 1875\, L^2 \delta^4$$

And

$$_5 = (L^6 + 75\,\delta^2 L^4 + 1875\,\delta^4 L^2 + 15625\,\delta^6) \, (.5 - .5P + \frac{Q}{6})$$

Taking Laplace transformation of equation (II.42) and remembering $W_o(o) = W_o(o) = 0$ we have,

$$s^2 \, \overline{W}_o(t) + w^2 \, \overline{W}_o(t) = C^2 \, \overline{q}\,(t)$$

Thus

$$W_o(t) = \frac{C^2}{s^2 + w^2} \, \overline{q}\,(t) \qquad \text{(II.43)}$$

Performing inverse transformation of equation (II.43) one gets

$$W_o(t) = \frac{C^2}{w} \int_o^t q\,(t - t')\ \text{Sin}\ wt'\ dt'$$

$$= \frac{C^2}{w^2} \Bigg\{ (1 - \text{Cos}\ wt) - \frac{2\,(wt - \text{Sin}\ wt)}{wT}$$

$$+ \frac{2 \left[w\,(t - T) - \text{Sin}\ w\,(t - T) \right]}{wT} \, H\,(t - T)$$

$$+ \left[1 - \text{Cos}\ w\,(t - T) \right] H\,(t - T) \Bigg\} \qquad \text{(II.44)}$$

Thus W is completely determined.

NUMERICAL RESULTS AND CONCLUSION

Graphs have been plotted for both clamped (Fig.II.2) and simply supported (Fig. II.3) plates of different shape showing variation of $\frac{W_{max}}{W'}$ Vs. period ratio.

$W_{max.}$ is the maximum deflection of the plates in the present study, W' is the corresponding deflection of the plates in the static case and the period ratio is defined as T/T_f where T is the sonic boom period and T_f is the fundamental period of the plate.

The expressions for W' in the clamped edge and in simply supported edge conditions are given in Timoshenko and Woinowsky-Krieger, Theory of Plates and Shells, P. 180.

It is evedient from Fig. II.2 and Fig. II.3 that the deflection ratio (W_{max}/W') attains a maximum value when $T = T_f$ i.e. when the period ratio is unity. It is also observed from the figures that the deflection ratio attains local maximum when the sonic boom period is an integral multiple of the fundamental period of the plate. The oscillation of the deflection ratio is found to damp out for large period ratio.

The most interesting feature of both Fig. II.2 and Fig. II.3 is that the graphs are identical for any shape of the plate and the maximum value of the deflection ratio in both the cases are in excellent agreement with those of Masumdar J. and Coleby J., Sound Vib. (1976), 48 (4), P. 503 - 512.

Further it is observed that the deflection ratio will increase with the boom pressure. Thus we conclude that the boom pressure and boom period have a significant effect on the vibrations of the elastic plate.

The results in the present study are simple and have been obtained with ease and much less computational labour.

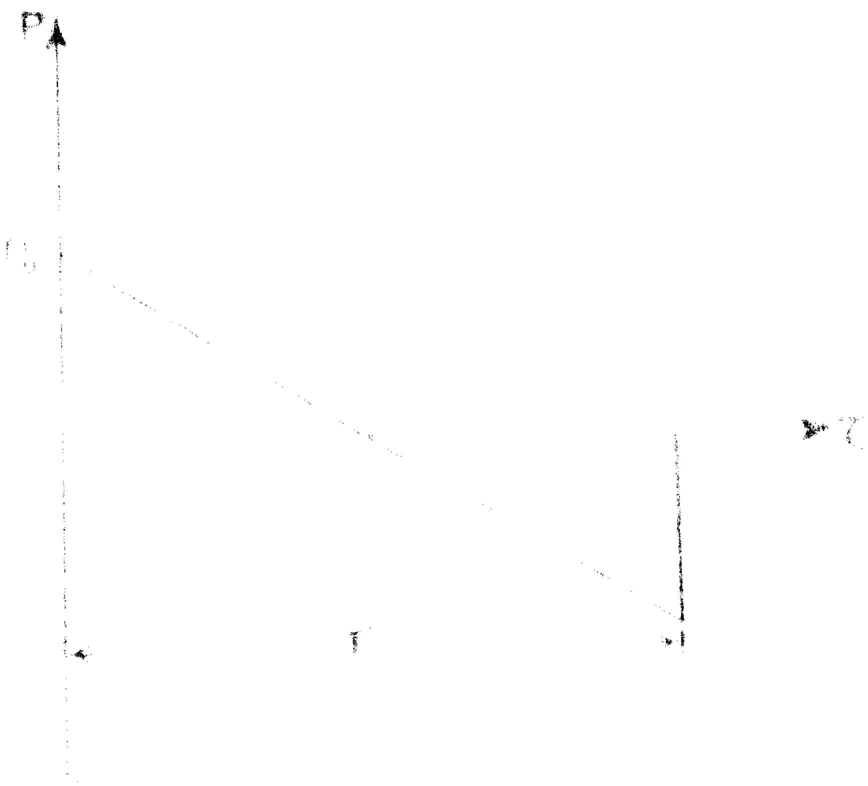


Fig. II 1

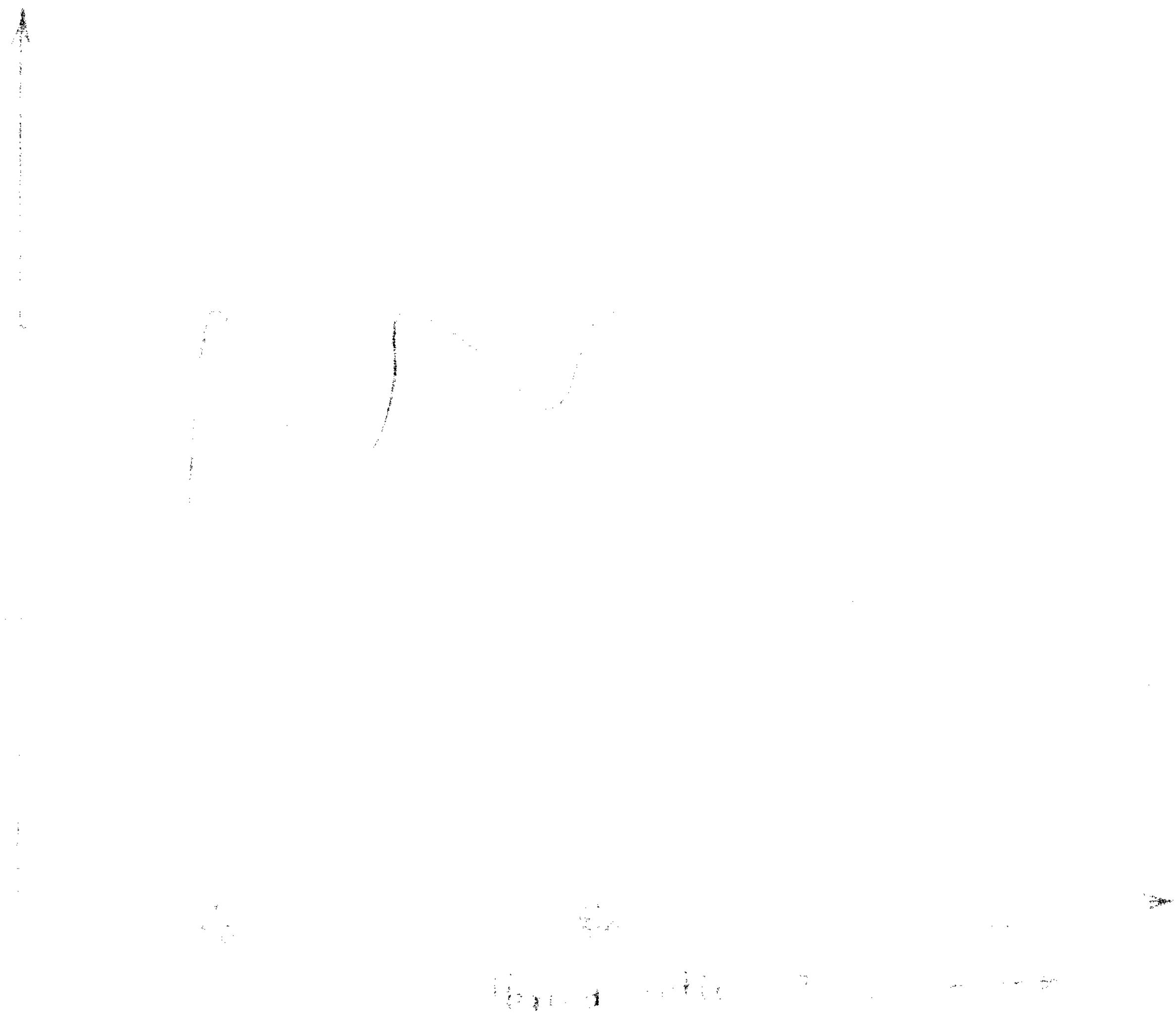


FIG. II.2

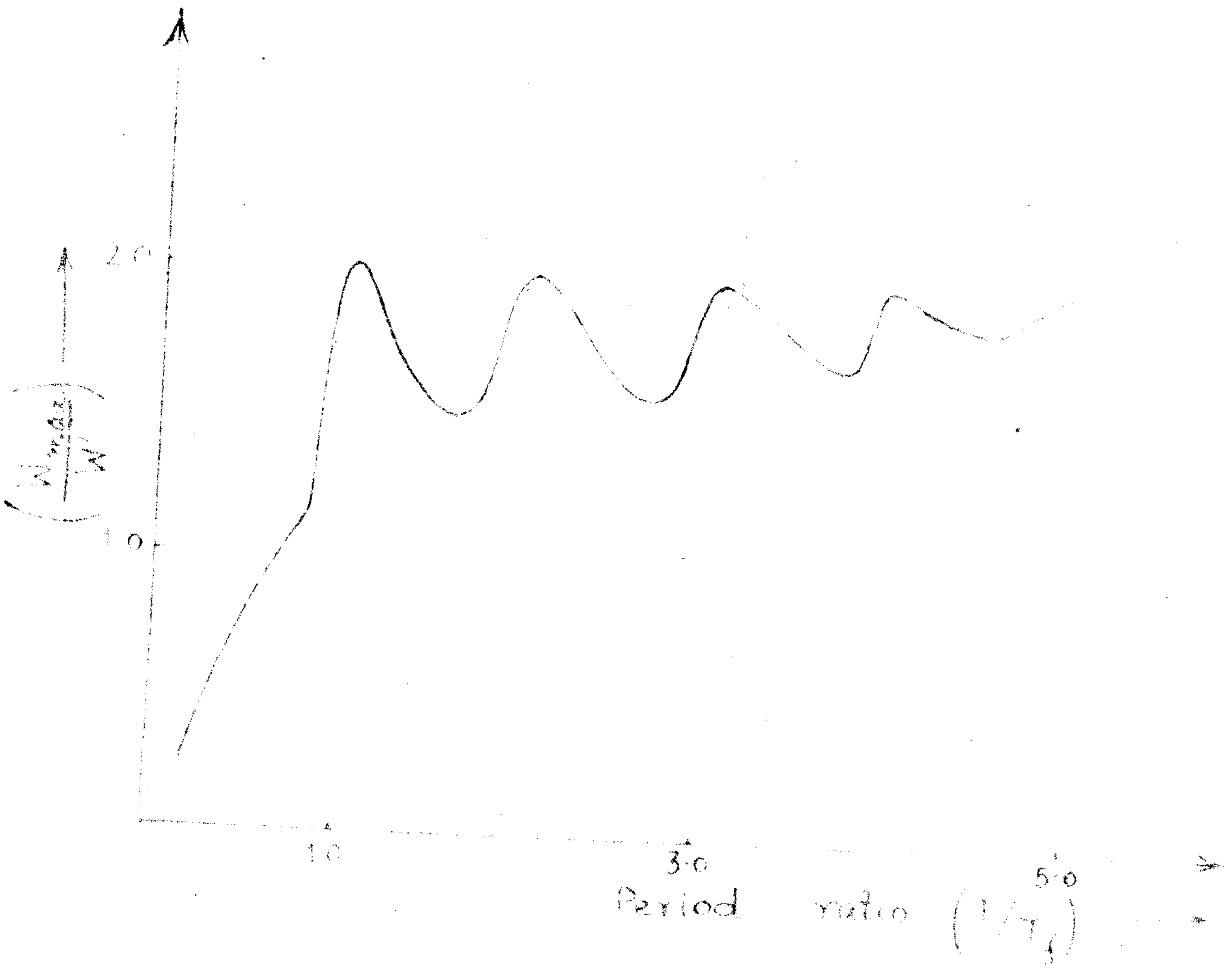

2.0
1.0
$\left(\frac{W_{max}}{W}\right)$
1.0
3.0
5.0
Period ratio $\left(T/T_f\right)$

FIG II . 3

D. DEFLECTIONS OF REGULAR POLYGONAL VISCOELASTIC PLATES OF SIMPLY SUPPORTED EDGES

Let us consider an elastic plate of uniform thickness 'h'. In a cartesian co-ordinate system, the Z- axis is taken along the thickness of the plate and perpendicular to both x and y axes.

Following the model offered by Biot the differential equations for viscoelastic plates are given by eqn. (II.4).

Let $f(x,y,t) = f_0\, u(t)$ be the load applied to the plate, where $u(t)$ is the unit step function.

For Kelvin type of material equation (II.4) reduces to after taking Laplace transformation

$$\nabla^4 \bar{W} = \frac{f_0}{\frac{h^3}{3}(G+\eta s)\, s} - \frac{\rho h s^3 \bar{W}}{\frac{h^3}{3}(G+\eta s)\, s} \qquad (II.45)$$

where

$$\bar{W} = \int_0^\infty e^{-st}\, W(x,y,t)\, dt$$

In complex co-ordinate system $Z = x + iy$ equation (II.45) reduces to

$$16 \frac{\partial^4 \bar{W}}{\partial z^2 \, \partial \bar{z}^2} + K_1 \bar{W} = K_2 f_o \qquad \text{(II.46)}$$

where $K_1 = \dfrac{3 \rho s^2}{h^2 (G + \eta s)}$ and $K_2 = \dfrac{3}{h^3 s (G + \eta s)}$

Let $Z = f(\xi)$ be the mapping function which maps the domain under consideration on to a unit circle where $\xi = re^{i\theta}$. Thus in $(\xi, \bar{\xi})$ co-ordinates equation (II.46) reduces to

$$16 \left[\frac{\partial^4 \bar{W}}{\partial \xi^2 \partial \bar{\xi}^2} \frac{dZ}{d\xi} \frac{d\bar{Z}}{d\bar{\xi}} - \frac{\partial^3 \bar{W}}{\partial \xi^2 \partial \bar{\xi}} \frac{d^2 Z}{d\xi^2} \frac{d\bar{Z}}{d\bar{\xi}} - \frac{\partial^3 \bar{W}}{\partial \xi^2 \partial \bar{\xi}} \frac{d^2 \bar{Z}}{d\bar{\xi}^2} \frac{dZ}{d\xi} \right.$$

$$\left. + \frac{\partial^2 \bar{W}}{\partial \xi \partial \bar{\xi}} \frac{d^2 Z}{d\xi^2} \frac{d^2 \bar{Z}}{d\bar{\xi}^2} \right] + K_1 \bar{W} \left(\frac{dZ}{d\xi} \right)^3 \left(\frac{d\bar{Z}}{d\bar{\xi}} \right)^3$$

$$= K_2 f_o \left(\frac{dZ}{d\xi} \right)^3 \left(\frac{d\bar{Z}}{d\bar{\xi}} \right)^3 \qquad \text{(II.47)}$$

Since the solution of (II.47) is at best difficult we seek approximate method to solve it. Let us assume

$$\overline{W}(\xi,\bar{\xi},t)=\overline{W}_0(t)(1-\xi\bar{\xi})\left[1-\frac{1}{3}\xi\bar{\xi}+\frac{1}{2}(\xi^2+\bar{\xi}^2)(1-\xi\bar{\xi})^2\right]$$

... (II.48)

Clearly $\overline{W}(\xi, \bar{\xi}, t)$ is θ dependent and satisfies the simply supported edge conditions $\overline{W} = 0$ and $\frac{\partial^2 \overline{W}}{\partial\xi\partial\bar{\xi}} = 0$ at $r = 1$.

Let $Z \simeq L\xi + L_1\xi^5$ (II.49)

be the mapping function where L and L_1 are mapping function co-efficients and given in Table I for different plates.

Substituting (II.48) and (II.49) in (II.47) one gets the error function $\epsilon(r, \theta)$.

For minimising the error, Galerkin's procedure requires that

$$\int_{r=0}^{1}\int_{\theta=0}^{2\pi} \epsilon(r,\theta).\ \overline{W}.r.dr.d\theta = 0 \qquad (II.50)$$

After evaluating the integrals in (II.50), $\overline{W}_0(t)$ is obtained as follows

$$\overline{W}_0(t) = \frac{f_0 \dfrac{\Psi_5}{\Psi_4}}{h^2GAS + h^3\eta AS^2 + \rho hS^3} \qquad (II.51)$$

Taking inverse transformation of (II.51) the central deflection is obtained as

$$W_o(t) = \frac{\Psi_5}{\Psi_4} \cdot \frac{f_o}{Gh^3A} \left[1 + \frac{1}{(1 - \omega_o)^{1/2}} e^{-Bt} \sin(\omega t + \phi) \right] \qquad (II.52)$$

where

$$\Psi_1 = 0.2963\, L^2 + 0.4233\, L_1^2 ;$$

$$\Psi_2 = 3.39\, L^2 ; \quad \Psi_3 = -\ 3.597\, L_1^2$$

$$\Psi_4 = 0.141\, L^6 + 0.225\, L^4 L_1^2 + 1.0634\, L^2 L_1^4 + 244.175\, L_1^6$$

$$\Psi_5 = 0.222\, L^6 + 0.9624\, L^4 L_1^2 + 7.576\, L^2 L_1^4 + 30.525\, L_1^6$$

$$A = \frac{16 \left[\Psi_1 - \Psi_2 + \Psi_3 \right]}{3\Psi_4}$$

$$\omega_o = \frac{\eta^2 A h^2}{4 \rho G} \qquad B = \frac{\eta A h^2}{2\rho}$$

$$\omega = \left(\frac{GAh^2}{\rho} - \frac{2 A^2 h^4}{4 \rho^2} \right)^{\frac{1}{2}}$$

and $\phi = \tan^{-1} \left(\frac{4 \rho G}{A \eta^2 h^2} - 1 \right)^{\frac{1}{2}}$

As $t \to \infty$, $$W_o(t) = \frac{f_o \Psi_5}{Gh^3 A \Psi_4} \qquad (II.53)$$

thus a steady state is attended.

Let us now check the accuracy of relation (II.52) for central deflection of a square simply supported viscoelastic plate of side 'a'. Only two terms of the mapping function have been considered.

For square plate of side 'a'

$L = 0.54\ a$ and $L_1 = -\ 0.055\ a$

In that case equation (II.52) reduces to

$$W_o(t) = 0.01627\ \frac{f_o\ a^4}{Gh^3} \left[1 + \frac{1}{(1-w_o)^{\frac{1}{2}}}\ e^{-Bt} \sin(wt + \phi) \right] \qquad (II.54)$$

Result (II.54) is in excellent agreement with that given by Mase (1960).

NUMERICAL CALCULATIONS

The central deflection $\frac{W_o(t)}{h}$ of different polygons have been calculated and are given in Table II.8 against different values of the time parameter 'Bt'. Following datas have been assumed for numerical calculations

$$\frac{f_o a^4}{Gh^4} = 10\ ; \quad w_o = \frac{1}{4} \quad \text{and} \quad (\ wt + \phi\ ) = \pi/2$$

TABLE II.7

Mapping function coefficients

$$Z \simeq L\xi + L_1\xi^5$$

Polygon of side 'a'	L	L_1
Square	0.54 a	-0.055 a
Pentagon	0.5263 a	-0.035 a
Hexagon	0.5188 a	-0.025 a

TABLE II.8

Central deflections of different plates

Bt	Square	Pentagon	Hexagon
0	0.35062	0.2627	0.2336
2	0.18813	0.14393	0.12534
4	0.16614	0.14479	0.11069
8	0.16276	0.12195	0.10844
10	0.16271	0.12191	0.10841
15	0.1627	0.1219	0.1084
20	0.1627	0.1219	0.1084

CONCLUSION

If the mapping function is known the deflection function can be easily calculated for plates of any shape. Only two terms of the mapping function yield sufficiently accurate results.

From the Table II.8 it is clear that the deflection function becomes stable when 'Bt' reaches the value 10. For values of 'Bt' less than 10 the deflection function falls sharply and the viscoelastic property of the plates is prominent and thus this region is of very much importance to design engineers.

E. VIBRATIONS OF POLYGONAL PLATES DUE TO THERMAL SHOCK*

Let us consider an elastic plate of uniform thickness 'h'. In a cartesian co-ordinate system, the Z - axis is taken along the thickness of the plate and perpendicular to both x and y axes. The face $Z = + h/2$ is subjected to a sudden heating while the other face $Z = - h/2$ together with all edges is insulated.

The equation for vibrations of such a plate is given by Nowacki, Thermoelasticity page 499

$$\nabla^4 W(x,y,t) + (1+\nu)\alpha_t \nabla^2 \tau(x,y,t) + \frac{\rho h}{D} \ddot{W}(x,y,t) = 0 \qquad \text{(II.55)}$$

where $W(x,y,t)$ is the displacement of the plate and $\tau(x,y,t)$ is the temperature field given by

$$\tau(x,y,t) = \frac{12}{h^2} \int_{-h/2}^{+h/2} z\, T(z,t)\, dz$$

$$= \frac{q_1}{2\lambda}\left(1 - \frac{96}{\pi^4} \sum_{j=1,3..} \frac{1}{j^4} e^{-j^2 \cdot \beta t}\right) \qquad \text{(II.56)}$$

where $\lambda \dfrac{\partial T}{\partial Z} = q_1$ at $Z = \dfrac{h}{2}$

---

* Published in the "Journal of Sound and Vibration" (England), Vol. 89(4), 1983.

and $\tau(x,y,t) = \frac{q_1}{2\lambda} K(t)$ (say)

Now changing equation (II.55) into complex co-ordinates by the transformation $z = x + iy$ and $\bar{Z} = x - iy$, one has,

$$16 \frac{\partial^4 W}{\partial z^2 \partial \bar{Z}^2} + 4(1+\nu)\alpha_t \frac{\partial^2 \tau}{\partial z \partial \bar{Z}} + \frac{\rho h}{D} \ddot{W} = 0 \qquad \text{(II.57)}$$

Let

$$Z = f(\xi) \simeq L\xi + L_1 \xi^5 \qquad \text{(II.58)}$$

be the mapping function which maps the domain under consideration on to a unit circle in the complex plane, where $\xi = re^{i\theta}$.

With this transformation one gets

$$16 \left[\frac{\partial^4 W}{\partial \xi^2 \partial \bar{\xi}^2} \frac{dZ}{d\xi} \frac{d\bar{Z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \bar{\xi}^2 \partial \xi} \frac{d^2 Z}{d\xi^2} \frac{d\bar{Z}}{d\bar{\xi}} - \frac{\partial^3 W}{\partial \xi^2 \partial \bar{\xi}} \frac{d^2 \bar{Z}}{d\bar{\xi}^2} \frac{dZ}{d\xi} \right.$$

$$\left. + \frac{\partial^2 W}{\partial \xi \partial \bar{\xi}} \frac{d^2 \bar{Z}}{d\bar{\xi}^2} \frac{d^2 Z}{d\xi^2} \right] + \frac{\rho h}{D} \left(\frac{dZ}{d\xi}\right)^3 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^3 \ddot{W}$$

$$= -4(1+\nu)\alpha_t \frac{\partial^2 \tau}{\partial \xi \partial \bar{\xi}} \left(\frac{dZ}{d\xi}\right)^2 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2 \qquad \text{(II.59)}$$

Let

$$W = B_0(t)\left[1 - 2P\xi\bar{\xi} + Q\xi^2\bar{\xi}^2\right] \tag{II.60}$$

which clearly satisfies the simply supported edge condition when

$$P = \frac{2}{3}$$

and

$$Q = \frac{1}{3}$$

Also let

$$\tau = \frac{q_1}{2\lambda} K(t)\left[1 - \xi\bar{\xi}\right] \tag{II.61}$$

This states that τ vanishes on the boundary $r = 1$.
Substituting equations (II.60), (II.61) and (II.58) in equation (II.50) one gets the error function $\epsilon(r, \theta)$; Galerkin's procedure requires that

$$\int_c \epsilon(r, \theta)\, W r\, dr d\theta = 0 \tag{II.62}$$

After evaluating the integrals in expression (II.62) one gets the required differential equation determing $B_0(t)$.

This equation, on simplification, takes the form

$$\ddot{B}_0(t) + w^2 B_0(t) = c^2 K(t) \tag{II.63}$$

where

$$v^2 = \frac{16\, A_{11}}{\Psi_4} \; \frac{D}{\rho h} \, , \quad C^2 = \frac{4(1+\nu)\alpha_t \Psi_5}{\Psi_4} \; \frac{D}{\rho h} \; \frac{q_1}{2\lambda} \, ,$$

$$A_{11} = (4\, Q\, \Psi_1 - 800\, Q L_1^2\, \Psi_2 + 400\, L_1^2\, \Psi_3),$$

$$\Psi_1 = (0.5 - 0.5P + Q/6)\, L^2 + (2.5 - \frac{25}{6} P + \frac{25}{14} Q)\, L_1^2 ,$$

$$\Psi_2 = 0.1 - \frac{1}{6} P + \frac{1}{14} Q ,$$

$$\Psi_3 = -\,0.25P + 0.4P^2 - \frac{5}{5} PQ + 0.4Q + \frac{2}{7} Q^2 ,$$

$$\Psi_4 = (0.5 - P + \frac{1}{3} Q - 0.5\, PQ + \frac{2}{3} P^2 + 0.1\, Q^2)\, L^6$$

$$+ (\frac{1}{26} - \frac{1}{7} P + \frac{1}{15} Q - \frac{1}{8} PQ + \frac{2}{15} P^2 + \frac{1}{34} Q^2)\, 15625\, L_1^6$$

$$+ (0.1 - \frac{1}{3} P + \frac{1}{7} Q - 0.25\, PQ + \frac{2}{7} P^2 + \frac{1}{18} Q^2)\, 75\, L^4 L_1^2$$

$$+ (\frac{1}{18} - 0.2P + \frac{1}{11} Q - \frac{1}{6} PQ + \frac{2}{11} P^2 + \frac{1}{26} Q^2)\, 1875\, L^2 L_1^4 ,$$

$$\Psi_5 = (0.5 - 0.5P + \frac{1}{6} Q)\, L^4 + 50\, (0.1 - \frac{1}{6} P + \frac{1}{14} Q)\, L^2 L_1^2$$

$$+ 625\, (\frac{1}{18} - 0.1\, P + \frac{1}{22} Q)\, L_1^4$$

Taking the Laplace transformation of equation (II.63) one has

$$s^2 \bar{B}_o(4) - s B_o(0) - \dot{B}_o(0) + w^2 \bar{B}_o(t) = C^2 \bar{K}(t) \qquad (II.64)$$

In the case of a sudden heating of the plate $W(x,y,o) = 0$, $\dot{W}(x,y,o) = 0$ and hence $B_o(0) = \dot{B}_o(0) = 0$. Thus, from equation (II.64), one gets

$$\bar{B}_o(t) = C^2 \bar{K}(t) / (s^2 + w^2) \qquad (II.65)$$

Performing the inverse transformation one gets

$$B_o(t) = \frac{C^2}{w^2} \int_o^t K(t - t') \sin wt' \, dt'$$

$$= \frac{C^2}{w^2} \Big[(1 - \cos wt) - \frac{96 \beta^2}{\pi^4} \sum_{j=1,3\ldots}^{\infty} \frac{1}{j^4 \beta^2 + w^2}$$

$$\left\{ \frac{w}{j^2 \beta} \sin wt - \frac{w^2}{j^4 \beta^2} \cos wt + \frac{w^2}{j^4 \beta^2} e^{-j^2 \beta t} \right\} \Big] \qquad (II.66)$$

Thus W is completely determined.

Now, from equation (II.66), one has the maximum displacement as

$$(W)_{max.} = \Big| B_o(t) \Big|_{max} = \frac{2C^2}{w^2} \qquad (II.67)$$

Neglecting the inertia term in equation (II.63) one gets the maximum displacement for the quasi-static part as

$$(W_s)_{max} = \frac{c^2}{v^2} \qquad \text{(II.68)}$$

This value of $(W_s)_{max}$ is in good agreement with that given in Nowacki.

Unfortunately there appears to be a printing mistake in Nowacki [Thermoelasticity page 503] in the solution for the dynamic displacement. That solution should read correctly as

$$B_{mn}(t) = \frac{16\, q_1 (1+\nu) \alpha_t}{2\lambda \alpha_n \beta_n (\alpha_n^2 + \beta_n^2)\, ab} \left\{ \frac{12\, \beta}{\pi^2 v} \sin vt - \cos vt \right.$$

$$- \frac{96\, \beta^2}{\pi^4} \sum_{j=1,3\ldots}^{\infty} \frac{1}{j^4 \beta^2 + v^2} \left[\frac{v}{j^2 \beta} \sin vt - \frac{v^2}{j^4 \beta^2} \cos vt \right.$$

$$\left.\left. - e^{j^2 \beta t} \right\} \right] \qquad \text{(II.69)}$$

Thus the maximum value of $B_{mn}(t)$, which is also the maximum dynamic displacement, is given by

$$(W_d)_{max} = \left| B_{mn}(t) \right|_{max}$$

$$= \frac{16\, q_1 (1+\nu) \alpha_t}{2\lambda \alpha_n \beta_n (\alpha_n^2 + \beta_n^2)\, ab} \simeq \frac{c^2}{v^2}$$

(in present notation).

NUMERICAL RESULTS

Table II.9 contains the first term coefficient of the mapping function for elastic plates of different shapes.

TABLE II.9

Mapping function $Z = f(\xi) \simeq L\xi$

Polygon of side 2a	L
Equilateral triangle	1.1352 a
Square	1.08 a
Pentagon	1.0526 a
Hexagon	1.0376 a
Heptagon	1.0279 a
Octagon	1.0219 a
Circle of radius a	a

In the present study W_{max} is the total displacement and in Table II.10, it is compared with the quasi-steady displacement as given by Nowacki in his book Thermoelasticity (Page 499) in the form W_{max} (quasi-steady) = $W_{d\ max}$ (quasi-steady) + $W_{s\ max}$ (quasi-steady). For B = 0 the two separate solutions yield the same result for any shape of the plate.

The numerical results for different plates have been plotted to show the variation of $\left| \frac{W_{max}}{K'a^2} \times 10^2 \right|$ Vs. B in Figure II.4. A graph showing variation of $\left| \frac{W_{max}}{K'a^2} \times 10^2 \right|$ Vs the number of sides of the regular polygons for B = 0 is shown in Figure II.5.

TABLE II.10

Comparison of results for total displacement (all results in units of $K'a^2$)

B	Equilateral triangular plate		Square plate		Circular plate	
	W_{max} (Present study)	W_{max} (Quasi-steady solution)	W_{max} (Present study)	W_{max} (Quasi-steady solution)	W_{max} (Present study)	W_{max} (Quasi-steady solution)
0	0.3282	0.3282	0.2927	0.2927	0.2548	0.2548
0.5	0.2376	0.2134	0.2150	0.1932	0.1843	0.1656
1.0	0.1846	0.1706	0.1671	0.1545	0.1433	0.1325
1.5	0.1735	0.1668	0.1570	0.1510	0.1318	0.1296
2.0	0.1697	0.1658	0.1536	0.1501	0.1289	0.1287
2.5	0.1680	0.1655	0.1521	0.1499	0.1276	0.1285
3.0	0.1672	0.1653	0.1514	0.1497	0.1270	0.1283
3.5	0.1668	0.1652	0.1510	0.1496	0.1267	0.1282
4.0	0.1666	0.1651	0.1508	0.1495	0.1265	0.1281

For all the calculations $K' = q_1 (1+\nu)\alpha_t$.
Only one term of the mapping function has been taken, for simplicity of calculation, and that has yielded sufficiently accurate results.

DISCUSSION

From table it is evedient that the role of inertia is much greater than that given in the book Thermoelasticity by Nowacki, and this is quite expected when thermal shock loading is applied to any elastic plate. It is also interesting to note from Table II.9 that the maximum displacement obtained from the quasi-steady solution rapidly falls, whereas, the maximum displacement as obtained here only steadily decreases. This is due to the fact that in the quasi-steady solution the inertia term takes a very significant role.

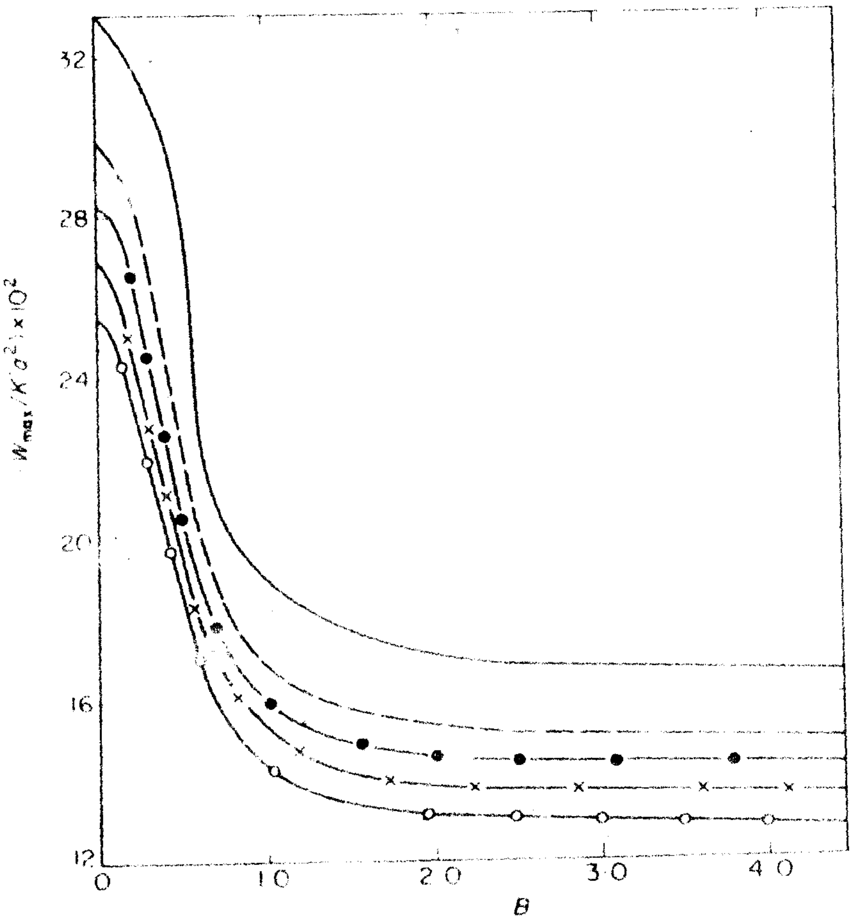


Figure II.4 $(W_{max}/K'a^2)$ *vs.* B. ——, Equilateral triangle; – – –, square; –●–, pentagon; –×–, heptagon; –○–, circle.

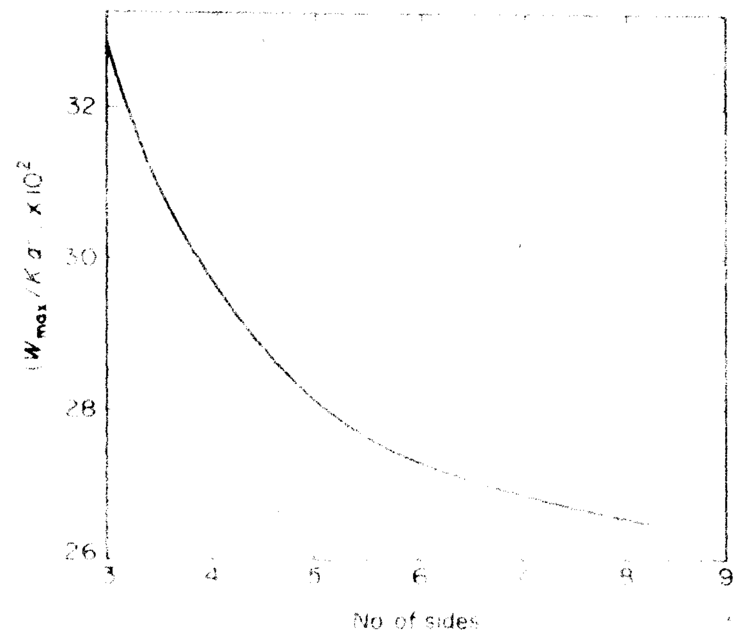


Figure II.5 $(W_{max}/K a^2)$ vs. number of sides of regular polygons, for $B = 0$.

F. THERMAL DEFLECTION OF REGULAR POLYGONAL VISCOELASTIC PLATES *

Let us consider a plate of medium thickness in a stationary temperature field $T(x_1, x_2, x_3)$; the origin of the co-ordinate system is situated in the middle plane of the plate which also contains the axes x_1 and x_2, the x_3 being normal to the middle plane (Figure II.6).

In the plane state of stress occuring in a plate of medium thickness we have the following relations between the components of the stress and strain tensors [Nowacki, Thermoelasticity, page 439] :

$$\sigma_{ij} = \frac{E}{1-\nu^2}\left\{(1-\nu)\epsilon_{ij} + \left[\nu\epsilon_{kk} - (1+\nu)\alpha_t x_3\right]\delta_{ij}\right\} \qquad \text{(II.70)}$$

The deformations are related to the deflection of the plate as follows :

$$\epsilon_{ij} = -x_3 W_{,ij} \quad , \quad i, j = 1, 2 \qquad \text{(II.71)}$$

where

$$W_{,ij} = \frac{\partial^2 W}{\partial x_i \partial x_j}$$

---

* Published in the "International Journal of Mechanical Sciences" (England), Vol. 23, No. 6, 1981.

Inserting equation (II.71) into equation (II.70), we have

$$\sigma_{ij} = - \frac{E x_3}{1-\nu^2} \left\{ (1-\nu) W,_{ij} + \left[\nu W,_{kk} + (1+\nu) \alpha_t \tau \right] \delta_{ij} \right\} \qquad \text{(II.72)}$$

Now introducing into the analysis the resultants of the state of stress — bending moments and torsional moments which are defined by the integrals

$$M_{ij} = \int_{-h/2}^{+h/2} x_3 \, \sigma_{ij} \, d x_3$$

We have from (II.72) after carrying out the above integration

$$M_{ij} = -D \left\{ (1-\nu) W,_{ij} + \left[\nu W,_{kk} + (1+\nu) \alpha_t \tau \right] \delta_{ij} \right\} \qquad \text{(II.73)}$$

where

$$D = \frac{E h^3}{12(1-\nu^2)}$$

Making use of the equilibrium condition which states that the sum of all forces acting on an element of the plate $h \, dx_1 \, dx_2$ should vanish, and using the equation

$$M_{ij}\, ,_{ij} = 0$$

i.e., $$\frac{\partial^2 M_{11}}{\partial x_1^2} - 2\frac{\partial^2 M_{21}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial x_2^2} = 0$$

We arrive at the differential equation to the deflection of thermally loaded viscoelastic plates as

$$\nabla^4 W + (1+\nu)\alpha_t \nabla^2 \tau = 0$$

or, $$\nabla^4 W + m(t)\nabla^2 \tau = 0 \qquad \text{(II.74)}$$

where W denotes deflection and ν Poisson's ratio which is a function of time. α_t is the temperature coefficient and

$$m(t) = (1+\nu)\alpha_t$$

Now considering the equation of heat conduction [vide Nowacki] we have

$$(\partial_1^2 + \partial_2^2 + \partial_3^2) T(x_1, x_2, x_3) = -\frac{1}{\lambda} W'(x_1, x_2, x_3) \qquad \text{(II.75)}$$

where, $$\partial_i^2 = \frac{\partial^2}{\partial x_i^2}, \quad i = 1,2,3.$$

Multiplying (II.75) by x_3 and integrating from $-\frac{h}{2}$ to $\frac{h}{2}$ with respect to x_3, we obtain

$$(\partial_1^2 + \partial_2^2)\int_{-h/2}^{h/2} T x_3 dx_3 + \left[x_3 \frac{\partial T}{\partial x_3} - T \right]_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{1}{\lambda}\int_{-h/2}^{h/2} W' x_3 dx_3$$

... (II.76)

Now introducing Newton's boundary conditions

$$\lambda \left[\frac{\partial T}{\partial x_3} \right]_{x_3 = \frac{h}{2}} = \lambda_1 (\theta_1 - T_1),$$

$$\lambda \left[\frac{\partial T}{\partial x_3} \right]_{x_3 = -\frac{h}{2}} = - \lambda_1 (\theta_2 - T_2), \tag{II.77}$$

where θ_1, θ_2 denote the temperatures of the lower and upper media respectively (Fig. II.6) and

$$T_1(x_1,x_2) = T(x_1,x_2,\frac{h}{2}), \quad T_2(x_1,x_2) = T(x_1,x_2,-\frac{h}{2})$$

we arrive at the relation

$$(\partial_1^2 + \partial_2^2)\tau + \frac{12}{h^3} \epsilon \frac{\theta_1 - \theta_2}{h} - \frac{12}{h^2}(1+\epsilon) \frac{T_1 - T_2}{h} = - \frac{q}{\lambda} \tag{II.78}$$

where,

$$\tau = \frac{12}{h^3} \int_{-h/2}^{h/2} T x_3 \, dx_3 ; \quad q = \frac{12}{h^3} \int_{-h/2}^{h/2} W' x_3 \, dx_3$$

and

$$\epsilon = \frac{h \lambda_1}{2 \lambda}$$

The relation (II.78) is independent of the variable x_3. If we assume that the temperature varies linearly with respect to x_3, we have

$$T(x_1,x_2,x_3) = \frac{T_1+T_2}{2} + \frac{T_1-T_2}{h} x_3 \qquad (II.79)$$

This assumption is closer to the rigorous solution of equation (II.75) when the thickness of the plate is smaller in comparison with its other dimensions.

Let,

$$\tau_o = \frac{T_1+T_2}{2} \quad \text{and} \quad \tau = \frac{T_1-T_2}{h} \qquad (II.80)$$

Then we have from relation (II.79)

$$T = \tau_o + x_3 \tau$$

Now in a plate which can freely expand in its plane, the temperature distribution τ_o results in no stresses. Under the above assumptions the stresses in the plate are due only to the stationary temperature field $x_3 \tau(x_1,x_2)$. So we have from relation (II.78) making use of (II.80)

$$\nabla^2 \tau - \frac{12}{h^2}(1+\epsilon)\tau = -\frac{12\epsilon}{h^3}(\theta_1 - \theta_2) - \frac{q}{\lambda} \qquad (II.81)$$

The equation (II.81) is considerably simplified if there is no source of heat i.e. $q = 0$, in that case equation (II.81) is reduced to

$$\nabla^2 \tau - \frac{12}{h^2}(1+\epsilon)\tau = -\frac{12}{h^3}(\theta_1 - \theta_2) \qquad \text{(II.82)}$$

Thus τ in equation (II.74) is given by temperature equation (II.82) i.e.

$$\nabla^2 \tau - k^2 \tau = -\beta \qquad \text{(II.83)}$$

where

$$k^2 = \frac{12}{h^2}(1+\epsilon)$$

and

$$\beta = \frac{12}{h^3}(\theta_1 - \theta_2) = \text{constant.}$$

Now it is advantageous to replace the differential equation (II.74) of the fourth order developed for a plate by two equations of the second order.

The expressions for bending moments along x_1 and x_2 axes aro given by

$$M_{11} = -D\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\right)$$

and

$$M_{22} = -D\left(\frac{\partial^2 w}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_1^2}\right)$$

So

$$M_{11} + M_{22} = - D(1+\nu)\left(\frac{\partial^2 W}{\partial x_1^2} + \frac{\partial^2 W}{\partial x_2^2}\right)$$

$$= - D(1+\nu)\nabla^2 W.$$

Or, $$\nabla^2 W = - \frac{M}{D} \qquad \text{(II.84)}$$

where

$$M = \frac{M_{11} + M_{22}}{1+\nu}$$

Thus we have form equation (II.74)

$$\nabla^2 M = m(t)\, D \nabla^2 \tau \qquad \text{(II.85)}$$

The solution of the equations (II.84) and (II.85) is very much simplified in case of a simply supported plate of polygonal shape.

The solution of equation (II.85) is as follows

$$M = m(t)\, D\tau \qquad \text{(II.86)}$$

since M and τ both are zero at the boundary of a simply supported plate.

So we have from equation (II.84) and (II.86),

$$\nabla^2 W = - m(t)\,\tau$$

Thus for a simply supported plate it is sufficient to solve

$$\nabla^2 W + m(t)\tau = 0 \tag{II.87}$$

subject to the condition $W = 0$ at the boundary.

Equation (II.87) after making a Laplace transformation reduces to,

$$\nabla^2 \bar{W} + \frac{\bar{m}(s)}{s}\tau = 0 \tag{II.88}$$

where

$$\bar{W} = \int_0^\infty W e^{-st} dt,$$

$$\bar{m}(s) = (1 + \bar{\nu})\alpha_t$$

and s is a transformation parameter.

From equation (II.79) we have,

$$\nabla^2 \bar{W} + \left\{\frac{\bar{m}(s)}{m_o s}\right\} m_o \tau = 0 \tag{II.89}$$

In a complex co-ordinate system $x_3 = x_1 + ix_2$ $\bar{x}_3 = x_1 - ix_2$ and equations (II.83) and (II.89) reduce to,

$$4\frac{\partial^2 \tau}{\partial x_3 \partial \bar{x}_3} - k^2\tau = -\beta \tag{II.90}$$

and

$$4 \frac{\partial^2 \bar{W}}{\partial x_3 \partial \bar{x}_3} + \left\{ \frac{\bar{m}(s)}{m_o s} \right\} m_o \tau = 0 \qquad (II.91)$$

Let $x_3 = f(\xi)$ be the function which maps the domain under consideration on to a unit circle. Thus in $(\xi, \bar{\xi})$ co-ordinates equations (II.90) and (II.91) reduce respectively to

$$4 \frac{\partial^2 \tau}{\partial \xi \partial \bar{\xi}} - K^2 \tau \frac{dx_3}{d\xi} \frac{d\bar{x}_3}{d\bar{\xi}} = - \beta \frac{dx_3}{d\xi} \frac{d\bar{x}_3}{d\bar{\xi}} \qquad (II.92)$$

and

$$4 \frac{\partial^2 \bar{W}}{\partial \xi \partial \bar{\xi}} + \left\{ \frac{\bar{m}(s)}{m_o s} \right\} m_o \tau \frac{dx_3}{d\xi} \frac{d\bar{x}_3}{d\bar{\xi}} = 0 \qquad (II.93)$$

where

$$\xi = re^{i\theta} \quad \text{and} \quad \bar{\xi} = re^{-i\theta}$$

To solve equation (II.92) let us take the first term of the mapping function which yields,

$$\frac{dx_3}{d\xi} = \frac{d\bar{x}_3}{d\xi} = a_1 = \text{constant.}$$

Thus we have from equation (II.92),

$$\frac{\partial^2 \tau}{\partial \xi \partial \bar{\xi}} - \alpha^2 \tau = - \frac{\beta a_1^2}{4} \qquad (II.94)$$

where

$$\alpha^2 = \frac{k^2 a_1^2}{4}$$

The closed form solution of the equation (II.94) is given by,

$$\tau = A\, I_0 \left(2\alpha \sqrt{\xi \bar{\xi}} \right) + \frac{\beta a_1^2}{4\alpha^2}$$

$$= A\, I_0 (2\alpha r) + \frac{\beta a_1^2}{4\alpha^2} \qquad \text{(II.95)}$$

$I_0 (2\alpha r)$ is the modified Bessel function of zeroth order and A is a constant to be evaluated for the prescribed boundary condition

$$\tau = 0 \quad \text{at} \quad r = 1.$$

Thus,

$$A = - \frac{\beta a_1^2}{4\alpha^2} \cdot \frac{1}{I_0 (2\alpha)}$$

Hence,

$$\tau = \frac{\beta a_1^2}{4\alpha^2} \left[1 - \frac{I_0 (2\alpha r)}{I_0 (2\alpha)} \right], \qquad \text{(II.96)}$$

is determined completely.

To solve the final equation (II.93), let

$$\frac{dx_3}{d\xi} = \frac{d\bar{x}_3}{d\bar{\xi}} = a_1 = \text{constant}.$$

As before with this approximate equation (II.93) reduces to

$$\frac{\partial^2 \bar{W}}{\partial \xi \partial \bar{\xi}} + \lambda'^2 \tau = 0, \qquad \text{(II.97)}$$

where

$$\lambda'^2 = \left\{ \frac{\bar{m}\,(s)}{4\, m_o\, s} \right\} m_o\, a_1^2$$

From equation (II.97) after substitution of the values for τ as given in equation (II.96), we have

$$\frac{\partial^2 \bar{W}}{\partial \xi \partial \bar{\xi}} = - \lambda'^2 \frac{\beta a_1^2}{4 \alpha^2} \left[1 - \frac{I_o(2\alpha r)}{I_o(2\alpha)} \right]$$

$$= P + Q\, I_o\,(2\alpha r), \qquad \text{(II.98)}$$

where,

$$P = - \frac{\lambda'^2 \beta a_1^2}{4 \alpha^2} \quad \text{and} \quad Q = - \frac{P}{I_o(2\alpha)}$$

Equation (II.98) will have a solation of the form

$$\bar{W} = B + Pr^2 + \frac{Q}{\alpha^2} I_o(2\alpha r)$$

where B is a constant to be evaluated under the boundary condition $\bar{W} = 0$ at $r = 1$.

Thus,

$$B = -P - \frac{Q}{\alpha^2} I_0(2\alpha) \quad \text{and}$$

$$\bar{W} = -P - \frac{Q}{\alpha^2} I_0(2\alpha) + Pr^2 + \frac{Q}{\alpha^2} I_0(2\alpha r),$$

is determined.

It is known that $\bar{W}$ is maximum at $r = o$ and thus

$$(\bar{W})_{max} = -P - \frac{Q}{\alpha^2} I_0(2\alpha) + \frac{Q}{\alpha^2}$$

and putting in the values of P and Q,

$$(\bar{W})_{max} = \frac{\lambda^2 \beta a_1^2}{4\alpha^2} \left[1 - \frac{1}{\alpha^2} + \frac{1}{\alpha^2 I_0(2\alpha)} \right]$$

or $$(\bar{W})_{max} = \left\{ \frac{\bar{m}(s)}{m_0 s} \right\} m_0 \frac{\beta a_1^4}{16\alpha^2} \left[1 - \frac{1}{\alpha^2} + \frac{1}{\alpha^2 I_0(2\alpha)} \right]$$

... (II.99)

Taking the inverse transformation of equation (II.99), the deflection is obtained as,

$$(W)_{max} = \mathcal{L}^{-}\left\{ \frac{\bar{m}(s)}{m_0 s} \right\} m_0 \frac{\beta a_1^4}{16\alpha^2} \left[1 - \frac{1}{\alpha^2} + \frac{1}{\alpha^2 I_0(2\alpha)} \right]$$

$$= h(t) \frac{m_0 \beta a_1^4}{16\alpha^2} \left[1 - \frac{1}{\alpha^2} + \frac{1}{\alpha^2 I_0(2\alpha)} \right]$$

$$= \mu \beta m_0 a^4 h(t), \qquad \text{(II.100)}$$

where,

$$h(t) = \mathcal{L}^{-1}\left\{\frac{\bar{m}(s)}{m_o\, s}\right\}$$

NUMERICAL CALCULATIONS

The central deflection coefficient (μ) for different polygons has been calculated in Table II.11 from equation (II.100) taking different values of a_1, assigned for different plates. Ka has been assumed to be unity in the present study.

TABLE II.11

Mapping function $x_3 = f(\xi) \simeq a_1 \xi$

Polygons of side 'a'	a_1	μ	μ [Nowacki, Thermo-elasticity]
Square	.54 a	.00381	.00389
Pentagon	.5265 a	.0034	-
Hexagon	.519 a	.0028	-
Circle of radius a	a	.0396	-

CONCLUSION

If the mapping function is known the deflection function for thermally loaded viscoelastic plates of any shape can be easily calculated using the present study. It is interesting to note that only one term of the mapping function yields sufficiently accurate results.

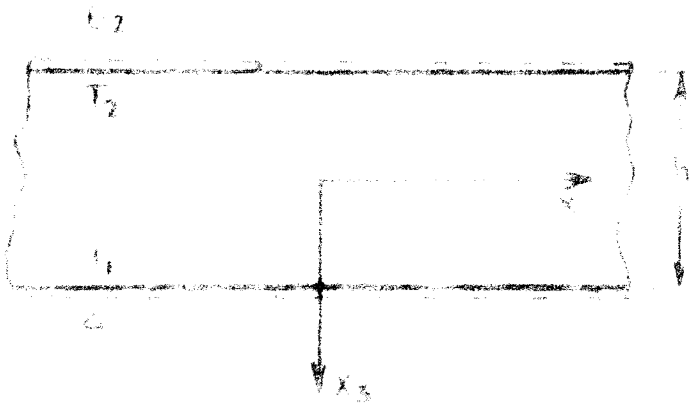


Fig. II 6

Special notations used in this problem

σ_{ij} = Component of the stress tensor

ϵ_{ij}, ϵ_{kk} = Components of the strain tensor

M_{ij} = Component of the bending moment

$m(t) = (1 + \nu)\alpha_t$

λ = Co-efficient of internal heat conduction

W' = Quantity of heat generated in unit volume and unit time

λ_1 = Thermal constant

θ_1, θ_2 = Temparatures of the upper and lower media respectively

$$\epsilon = \frac{h\lambda_1}{2\lambda}, \qquad k = \frac{12}{h^2}(1 + \epsilon)$$

$$\beta = \frac{12}{h^3}(\theta_1 - \theta_2) = \text{constant}$$

s = Laplace transformation parameter

$$\alpha^2 = \frac{K^2 a_1^2}{4} ; \qquad \lambda'^2 = \frac{\bar{m}(s)}{4 m_o s} \quad m_o a_1^2$$

a_1 = Mapping function co-efficient.

G. NON-LINEAR VIBRATIONS OF ORTHOTROPIC POLYGONAL PLATES CARRYING CONCENTRATED MASS

Let us consider an elastic orthotropic plate of uniform thickness 'h'. Considering the cartesian co-ordinate system, the Z-axis is taken along the thickness of the plate and perpendicular to both X and Y axes.

Following Berger's approximation the equation for non-linear vibrations of such plates can be obtained by forming the Lagrangian function from the sum of the bending energies in conjunction with the expression for the Kinetic energy. Finally applying Hamilton's principle and Euler's variational equations the resulting equation (neglecting in-plane rotary inertia effects) takes the following form

$$D_x \frac{\partial^4 W}{\partial x^4} + 2H \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} - \alpha^2 \left(D_x \frac{\partial^2 W}{\partial x^2} + \sqrt{D_x D_y} \frac{\partial^2 W}{\partial y^2} \right)$$

$$+ \left[\rho h + M \delta(x). \delta(y) \right] \frac{\partial^2 W}{\partial t^2} = 0 \qquad (II.101)$$

where, W is the transverse deflection, M is the concentrated mass at the centre and $\delta(x)$, $\delta(y)$ are Dirac Delta functions

$$D_x = \frac{E'_x h^3}{12}, \quad D_y = \frac{E'_y h^3}{12}, \quad D_{xy} = \frac{Gh^3}{12}, \quad D_1 = \frac{E'' h^3}{12}$$

E_x', E_y', G and E'' being the constants characterising the elastic properties of the plate material. The normalised constant of integration α is given by [Berger's method]

$$\frac{\alpha^2 h^2}{12} = \frac{\partial u}{\partial x} + \sqrt{\frac{D_y}{D_x}}\,\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}\sqrt{\frac{D_y}{D_x}}\left(\frac{\partial W}{\partial y}\right)^2 \qquad \ldots \text{(II.102)}$$

where u and v are in-plane displacements along x and y-axes respectively.

Introducing the complex co-ordinates $Z = x + iy$, $\bar{Z} = x - iy$ and using the mapping function relation $Z = f(\xi)$ which maps the domain under consideration on to a unit circle, the equations (II.101) and (II.102) in (ξ, $\bar{\xi}$) co-ordinates reduce to

$$A\Big[\frac{\partial W}{\partial \xi}\left\{10\,\frac{d^3Z}{d\xi^3}\,\frac{d^2Z}{d\xi^2}\,\frac{dZ}{d\xi}\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7 - 15\left(\frac{d^2Z}{d\xi^2}\right)^3\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7 - \frac{d^4Z}{d\xi^4}\left(\frac{dZ}{d\xi}\right)^2\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7\right\}$$

$$+\frac{\partial W}{\partial \bar{\xi}}\left\{10\,\frac{d^3\bar{Z}}{d\bar{\xi}^3}\,\frac{d^2\bar{Z}}{d\bar{\xi}^2}\,\frac{d\bar{Z}}{d\bar{\xi}}\left(\frac{dZ}{d\xi}\right)^7 - 15\left(\frac{d^2\bar{Z}}{d\bar{\xi}^2}\right)^3\left(\frac{dZ}{d\xi}\right)^7 - \frac{d^4\bar{Z}}{d\bar{\xi}^4}\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)\left(\frac{dZ}{d\xi}\right)^7\right\}$$

$$+\frac{\partial^2 W}{\partial \xi^2}\left\{10\left(\frac{d^2Z}{d\xi^2}\right)^2\frac{dZ}{d\xi}\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7 - 3\,\frac{d^3Z}{d\xi^3}\left(\frac{dZ}{d\xi}\right)^2\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7 - 2\,\frac{d^2Z}{d\xi^2}\left(\frac{dZ}{d\xi}\right)^4\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7\right\}$$

$$+\frac{\partial^3 W}{\partial\bar{\xi}^3}\left\{10\left(\frac{d^2\bar{Z}}{d\bar{\xi}^2}\right)^2\frac{d\bar{Z}}{d\bar{\xi}}\left(\frac{dZ}{d\xi}\right)^7-3\,\frac{d^3\bar{Z}}{d\bar{\xi}^3}\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2\left(\frac{dZ}{d\xi}\right)^7-2\,\frac{d^2\bar{Z}}{d\bar{\xi}^2}\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^4\left(\frac{dZ}{d\xi}\right)^7\right\}$$

$$+\frac{\partial^3 W}{\partial\xi^3}\left\{3\left(\frac{d^2 Z}{d\xi^2}\right)^2\frac{dZ}{d\xi}\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7-6\,\frac{d^2 Z}{d\xi^2}\left(\frac{dZ}{d\xi}\right)^2\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7-\frac{d^3 Z}{d\xi^3}\left(\frac{dZ}{d\xi}\right)^2\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7\right\}$$

$$+\frac{\partial^3 W}{\partial\bar{\xi}^3}\left\{3\left(\frac{d^2\bar{Z}}{d\bar{\xi}^2}\right)^2\frac{d\bar{Z}}{d\bar{\xi}}\left(\frac{dZ}{d\xi}\right)^7-6\,\frac{d^2\bar{Z}}{d\bar{\xi}^2}\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2\left(\frac{dZ}{d\xi}\right)^7-\frac{d^3\bar{Z}}{d\bar{\xi}^3}\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2\left(\frac{dZ}{d\xi}\right)^7\right\}$$

$$+\frac{\partial^4 W}{\partial\xi^4}\left(\frac{dZ}{d\xi}\right)^3\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7+\frac{\partial^4 W}{\partial\bar{\xi}^4}\left(\frac{dZ}{d\xi}\right)^7\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^3\ \Big]$$

$$+B\Big[\frac{\partial^4 W}{\partial\xi^2\partial\bar{\xi}^2}\left(\frac{dZ}{d\xi}\right)^5\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^5-\frac{\partial^3 W}{\partial\xi^2\partial\bar{\xi}}\,\frac{d^2\bar{Z}}{d\bar{\xi}^2}\left(\frac{dZ}{d\xi}\right)^5\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^4$$

$$-\frac{\partial^3 W}{\partial\xi\partial\bar{\xi}^2}\,\frac{d^2 Z}{d\xi^2}\left(\frac{dZ}{d\xi}\right)^4\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^5+\frac{\partial^2 W}{\partial\xi\partial\bar{\xi}}\,\frac{d^2 Z}{d\xi^2}\,\frac{d^2\bar{Z}}{d\bar{\xi}^2}\left(\frac{dZ}{d\xi}\right)^4\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^4\Big]$$

$$+C\Big[\frac{\partial^2 W}{\partial\xi\partial\bar{\xi}}\left\{3\left(\frac{d^2 Z}{d\xi^2}\right)^2\left(\frac{dZ}{d\xi}\right)^2\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^6-\frac{d^3 Z}{d\xi^3}\left(\frac{dZ}{d\xi}\right)^3\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^6+3\left(\frac{d^2\bar{Z}}{d\bar{\xi}^2}\right)^2\left(\frac{dZ}{d\xi}\right)^6\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2\right.$$

$$\left.-\frac{d^3\bar{Z}}{d\bar{\xi}^3}\left(\frac{dZ}{d\xi}\right)^6\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^3\right\}-3\,\frac{\partial^3 W}{\partial\xi^2\partial\bar{\xi}}\,\frac{d^2 Z}{d\xi^2}\left(\frac{dZ}{d\xi}\right)^3\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^6-3\,\frac{\partial^3 W}{\partial\xi\partial\bar{\xi}^2}\,\frac{d^2\bar{Z}}{d\bar{\xi}^2}\left(\frac{dZ}{d\xi}\right)^6\left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^3$$

$$+ \frac{\partial^4 W}{\partial \xi^3 \partial \bar{\xi}} \left(\frac{dZ}{d\xi}\right)^4 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^6 + \frac{\partial^4 W}{\partial \xi \partial \bar{\xi}^3} \left(\frac{dZ}{d\xi}\right)^6 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^4 \Bigg]$$

$$+ D' \Bigg[\frac{\partial^2 W}{\partial \xi^2} \left(\frac{dZ}{d\xi}\right)^5 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7 + \frac{\partial^2 W}{\partial \bar{\xi}^2} \left(\frac{dZ}{d\xi}\right)^7 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^5 - \frac{\partial W}{\partial \xi} \frac{d^2 Z}{d\xi^2} \left(\frac{dZ}{d\xi}\right)^4 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7$$

$$- \frac{\partial W}{\partial \bar{\xi}} \frac{d^2 \bar{Z}}{d\bar{\xi}^2} \left(\frac{dZ}{d\xi}\right)^7 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^4 \Bigg] + 2E \frac{\partial^2 W}{\partial \xi \partial \bar{\xi}} \left(\frac{dZ}{d\xi}\right)^6 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^6$$

$$+ \Big[\rho h + M \delta \{ f_1(\xi) \} \, \delta \{ f_2(\xi) \} \Big] \left(\frac{dZ}{d\xi}\right)^7 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^7 = 0 \qquad \text{(II.103)}$$

and

$$\frac{\alpha^2 h^2}{12} \left(\frac{dZ}{d\xi}\right)^2 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2 = \frac{\partial u}{\partial \xi} \frac{dZ}{d\xi} \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2 + \frac{\partial u}{\partial \bar{\xi}} \left(\frac{d\bar{Z}}{d\bar{\xi}}\right) \left(\frac{dZ}{d\xi}\right)^2$$

$$+ \frac{\sqrt{D_y/D_x}}{i} \Bigg[\frac{\partial v}{\partial \xi} \frac{dZ}{d\xi} \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2 - \frac{\partial v}{\partial \bar{\xi}} \left(\frac{d\bar{Z}}{d\bar{\xi}}\right) \left(\frac{dZ}{d\xi}\right)^2 \Bigg]$$

$$+ \frac{1}{2} \left(1 - \sqrt{\frac{D_y}{D_x}}\right) \Bigg[\left(\frac{\partial W}{\partial \xi}\right)^2 \left(\frac{d\bar{Z}}{d\bar{\xi}}\right)^2 + \left(\frac{\partial W}{\partial \bar{\xi}}\right)^2 \left(\frac{dZ}{d\xi}\right)^2 \Bigg]$$

$$+ \left(1 + \sqrt{\frac{D_y}{D_x}}\right) \frac{\partial W}{\partial \xi} \frac{\partial W}{\partial \bar{\xi}} \frac{dZ}{d\xi} \frac{d\bar{Z}}{d\bar{\xi}} \qquad \text{(II.104)}$$

where

$A = D_x + D_y - 2H$, $B = 6D_x + 6D_y + 4H$, $C = 4\,(D_x - D_y)$

$2E = 2\alpha^2\,(D_x + \sqrt{D_x D_y})$, $D' = \alpha^2\,(D_x - \sqrt{D_x D_y})$, $H = D_1 + 2D_{xy}$

$$\xi = re^{i\theta},\ f_1(\xi) = \frac{f(\xi) + f(\bar{\xi})}{2},\ f_2(\xi) = \frac{f(\xi) - f(\bar{\xi})}{2i}$$

and $Z = f(\xi)$.

Let us take W (neglecting θ dependence) in the following form,

$$W = W_o(t)\left[1 - 2P\xi\bar{\xi} + Q\xi^2\bar{\xi}^2\right] \qquad \text{(II.105)}$$

as the solution of equation (II.103).

Now for clamped polygonal plates,

$$P = Q = 1 \qquad \text{(II.106)}$$

this clearly satisfies clamped edge conditions namely

$$W = 0 \ \text{ and } \ \frac{\partial W}{\partial \xi} = 0 \ \text{ at } \ r = 1 \qquad \text{(II.107)}$$

For simply supported polygonal plates

$$P = \frac{2}{3} \ \text{ and } \ Q = \frac{1}{3} \qquad \text{(II.108)}$$

and this clearly satisfies the simply supported edge conditions namely

$$W = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial \xi \, \partial \bar{\xi}} = 0 \quad \text{at} \quad r = 1 \qquad \text{(II.109)}$$

It is to be noted that in cartesian co-ordinates the simply supported edge conditions for polygonal plates are $W = 0$ and $\nabla^2 W = 0$ at the boundary.

Considering one term of the mapping the function, for simplicity we have

$$Z = f(\xi) \simeq L\xi \qquad \text{(II.110)}$$

Substituting (II.105) and (II.110) in equation (II.103) one gets the error function $\in (r,\theta)$; Galerkin's procedure requires that

$$\int_{c} \in (r, \theta) \, W r \; dr \, d\theta = 0 \qquad \text{(II.111)}$$

After evaluating the integrals in equation (II.111) we have,

$$4Q\,B\,L^{10}\,\Psi_1\,W_o(t) - 2EL^{12}\,\Psi_2\,W_o(t) + \left\{ \rho h \Psi_3 + M \frac{L^{12}}{2\pi} \right\} \ddot{W}_o(t) = 0 \qquad \text{(II.112)}$$

where

$$\Psi_1 = \left(.5 - .5P + \frac{Q}{6} \right) \qquad (II.113)$$

$$\Psi_2 = \left(.5\,Q^2 + P^2 - \frac{5}{3} PQ + Q - P \right) \qquad (II.114)$$

$$\Psi_3 = (.5 - P + .33\,Q - .5PQ + .66\,P^2 + .1\,Q^2)\,L^{14} \qquad (II.115)$$

Integrating equation (II.104) over the cycle 2π, the in-plane displacements are eliminated by considering suitable expressions compatible with the boundary conditions for such displacements and thus α is given by

$$\alpha^2 h^2 \int_{r=0}^{1} \int_{\theta=0}^{2} \frac{dZ}{d\xi} \frac{d\bar{Z}}{d\bar{\xi}}\, r\, dr\, d\theta = 12\left(1 + \sqrt{\frac{D_y}{D_x}}\right) \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \frac{\partial W}{\partial \xi} \frac{\partial W}{\partial \bar{\xi}}\, r\, dr\, d\theta$$

$$\alpha^2 = \frac{96\,(1 + \sqrt{D_y/D_x})}{L^2\, h^2}\, \Psi_4\, W_o^2(t) \qquad (II.116)$$

where $$\Psi_4 = (\,.25\,P^2 + .1\,Q^2 - .33\,PQ\,) \qquad (II.117)$$

From equation (II.112) we have putting the value of α^2 from equation (II.116)

$$\ddot{W}_o(t) + \gamma W_o(t) + \delta' W_o^3(t) = 0 \qquad (II.118)$$

$$\text{where} \quad \gamma = \frac{4\,QB\,\psi_1\,L^{10}}{\rho h \psi_3 + \dfrac{ML^{12}}{2\pi}} \qquad (II.119)$$

and

$$\delta' = \frac{-192\,(1+\sqrt{D_y/D_x}\;(D_x+\sqrt{D_y\,D_x}\;)\,\psi_2\,\psi_4\,L^{10}}{h^2\left\{\rho h \psi_3 + \dfrac{ML^{12}}{2\pi}\right\}} \qquad (II.120)$$

Now taking $\quad W_o(t) = A\,F(t) \qquad$ (II.121)

Equation (II.118) reduces to

$$\ddot{F} + \gamma F + \delta F^3 = 0 \qquad (II.122)$$

where

$$\delta = \delta' \beta^2 \quad \text{and} \quad \beta = \frac{A}{h}$$

$$\text{and} \quad v_1^{*2} = \gamma + \delta \qquad (II.123)$$

Insertion of the values of γ and δ from equations (II.119), (II.120) and (II.121) into the equation (II.123) gives the relation connecting the relative amplitude β with the frequency v_1^{*2}.

NUMERICAL RESULTS

Frequencies of vibrations of different orthotropic and isotropic polygonal elastic plates have been calculated for both the edge conditions and are presented graphically showing variation of frequencies with amplitude (β). Figure II.7 shows the variation of frequencies (v_1^*) with the number of sides for given mass $4Ma^2/D_x = 1$ and $\beta = 0.4$. Figures II.8 to II.13 show the variations of frequencies with amplitude (β) for both simply-supported and clamped edge conditions for square (Figs. II.8 and II.9) pentagon (Figs. II.10 and II.11) and heptagon (Figs. II.12 and II.13).

In all calculations only one term of the mapping function has been considered.

TABLE II.12

$$Z = f(\xi) \simeq L\xi.$$

Shape of the plate of side 'a'	Mapping function co-efficient L
Equilateral triangle	0.5676 a
Square	0.54 a
Pentagon	0.5263 a
Hexagon	0.5188 a
Heptagon	0.5139 a
Octagon	0.51096a

Frequencies of vibrations of square orthotropic clamped plate of side 'a' for different values of the concentrated mass have been given in the following Table for comparison with known results.

TABLE II.13

v_1^* for square clamped plate of side 'a'

$H/D_x = 0.4$; $D_y/D_x = 0.3$; $ha^4/D_x = 1$; $\beta = 0.6$

$\epsilon = 4Ma^2/D_x$	v_1^* from Berger's Equation (Present study)	v_1^* from Von-Karman's Equation [B. Banerjee, 1982]
0	30.16	29.73
1	18.58	18.10
4	11.24	10.45
6	9.43	8.83

DISCUSSION

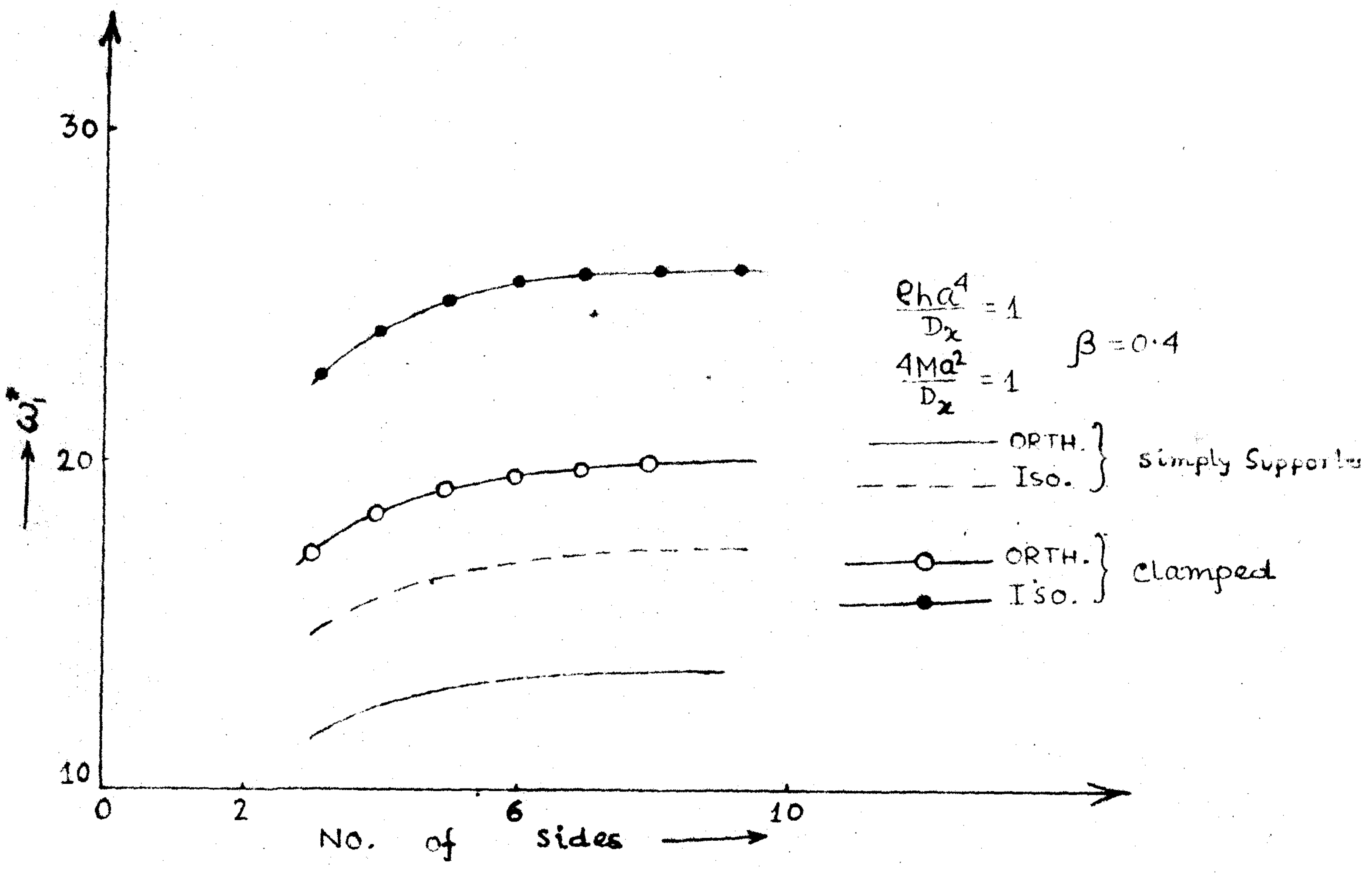
In the present study only one term of the mapping function has been considered and the solutions obtained are sufficiently accurate from the engineering point of view. From the present study following observations can be made :

I. For a given mass at the centre the frequencies of vibrations increase with the increase of number of sides of the polygon for both isotropic and orthotropic elastic plates. For a given polygon the frequencies of vibrations of simply supported plate are smaller than those of clamped plate.

II. It is also seen from the present study that the frequencies of vibrations of orthotropic and isotropic polygonal plates for both the edge conditions decrease with the increase of mass at the centre. The effect of the concentrated mass is to increase the potential energy of the system and thus the individual frequencies decrease. With larger values of $\in = 4Ma^2/D_x$ irregularity sets in the vibration because of the pronounced inertial effect due to mass [Lord Rayleigh, 1926].

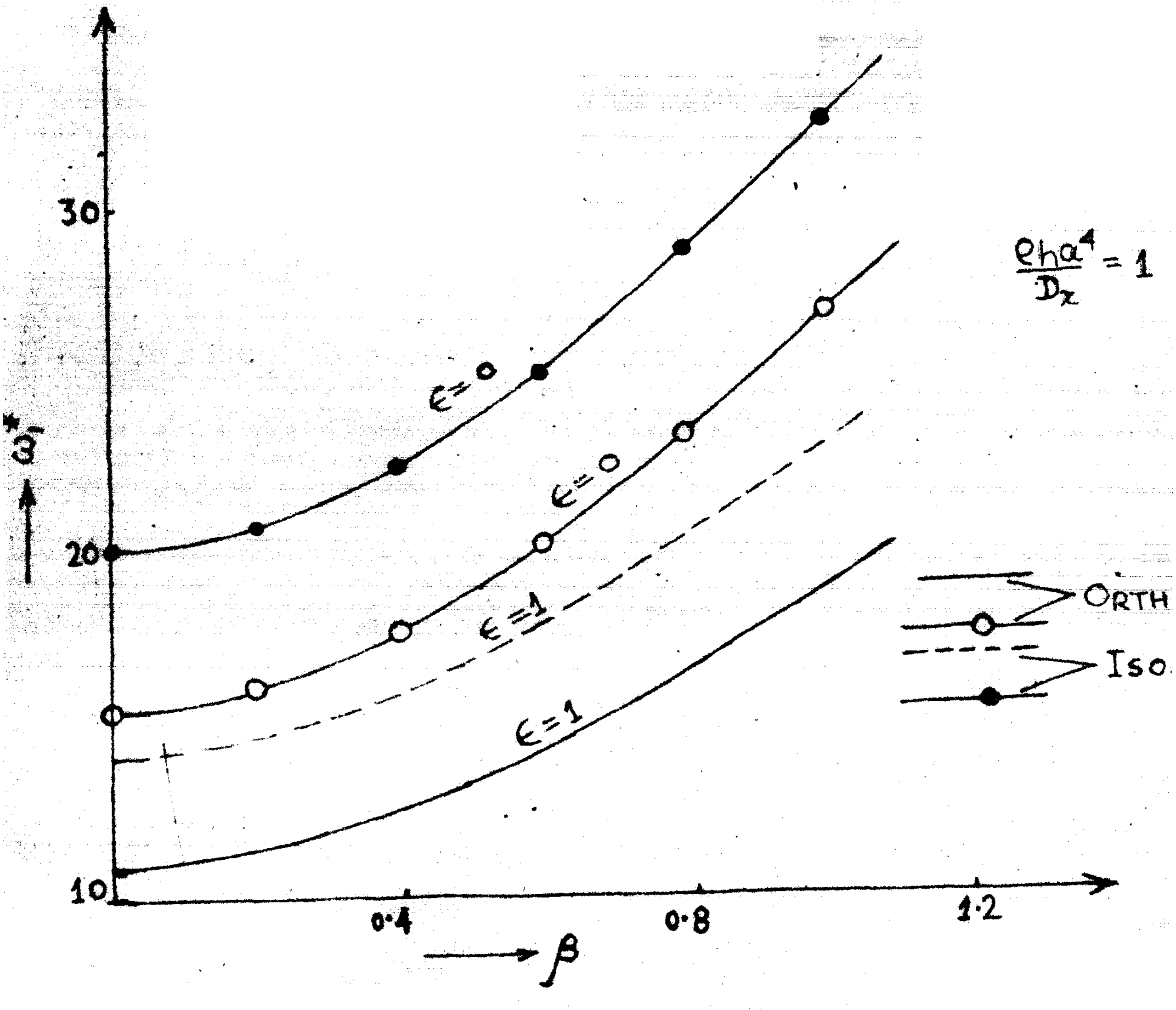
III. It is also observed from the Table II.13 that the results of the present study are in excellent agreement with those obtained by Banerjee B. [1982] who employed

Von-Karman's equations to obtain his solutions. Banerjee's method of solution is laborious and needs much computations where as the results of the present study have been obtained with ease and accuracy.



30
20
10
0
2
6
10
No. of sides
$\frac{\rho h a^4}{D_x} = 1$
$\frac{4Ma^2}{D_x} = 1$
$\beta = 0.4$
ORTH.
Iso.
simply Supported
ORTH.
Iso.
Clamped

FIG. II.7



30
20
10
$\bar{\omega}^*$
0·4
0·8
1·2
β
$\epsilon = 0$
$\epsilon = 0$
$\epsilon = 1$
$\epsilon = 1$
$\frac{\rho h a^4}{D_x} = 1$
ORTH
ISO

Fig. II. 8

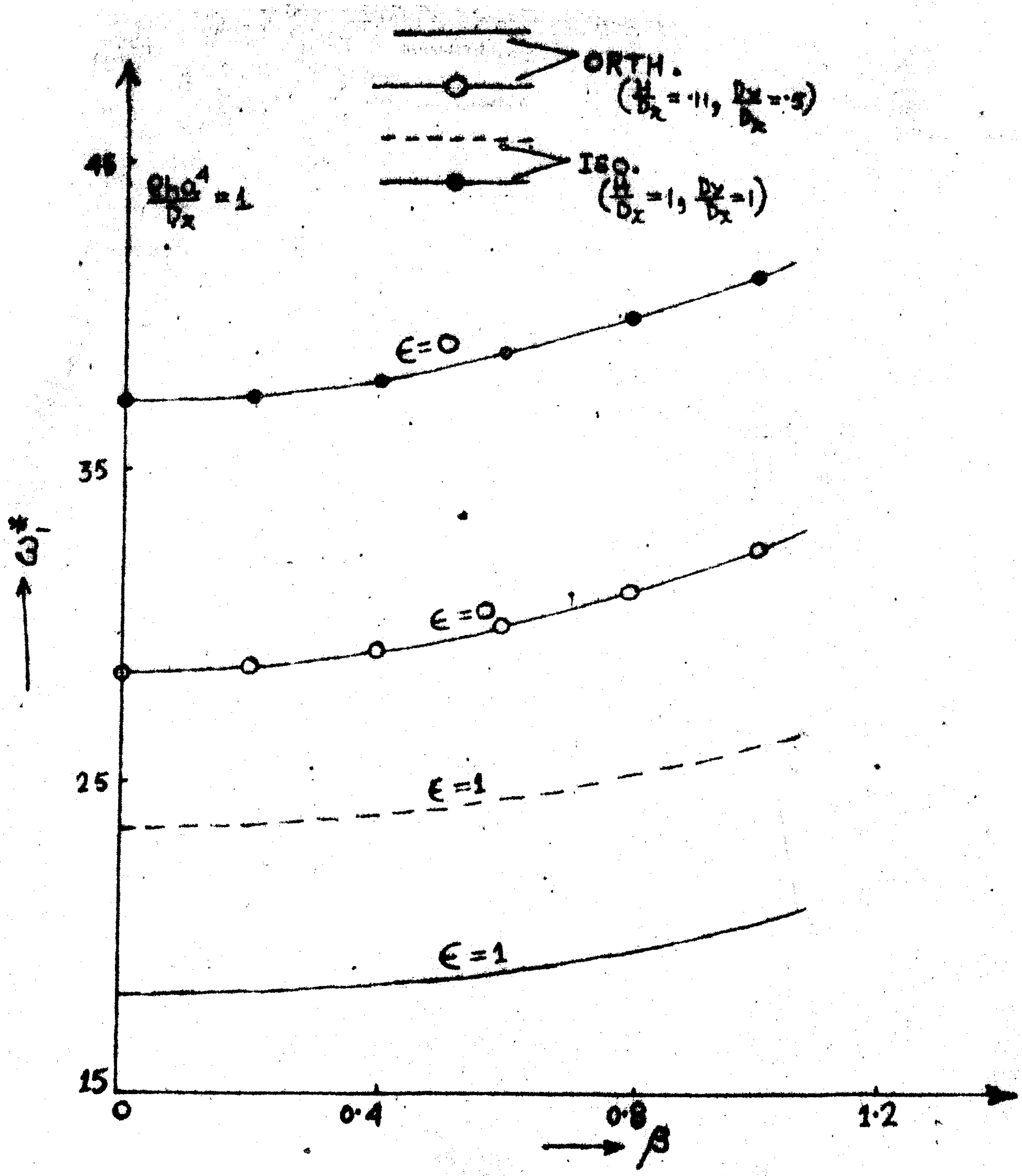
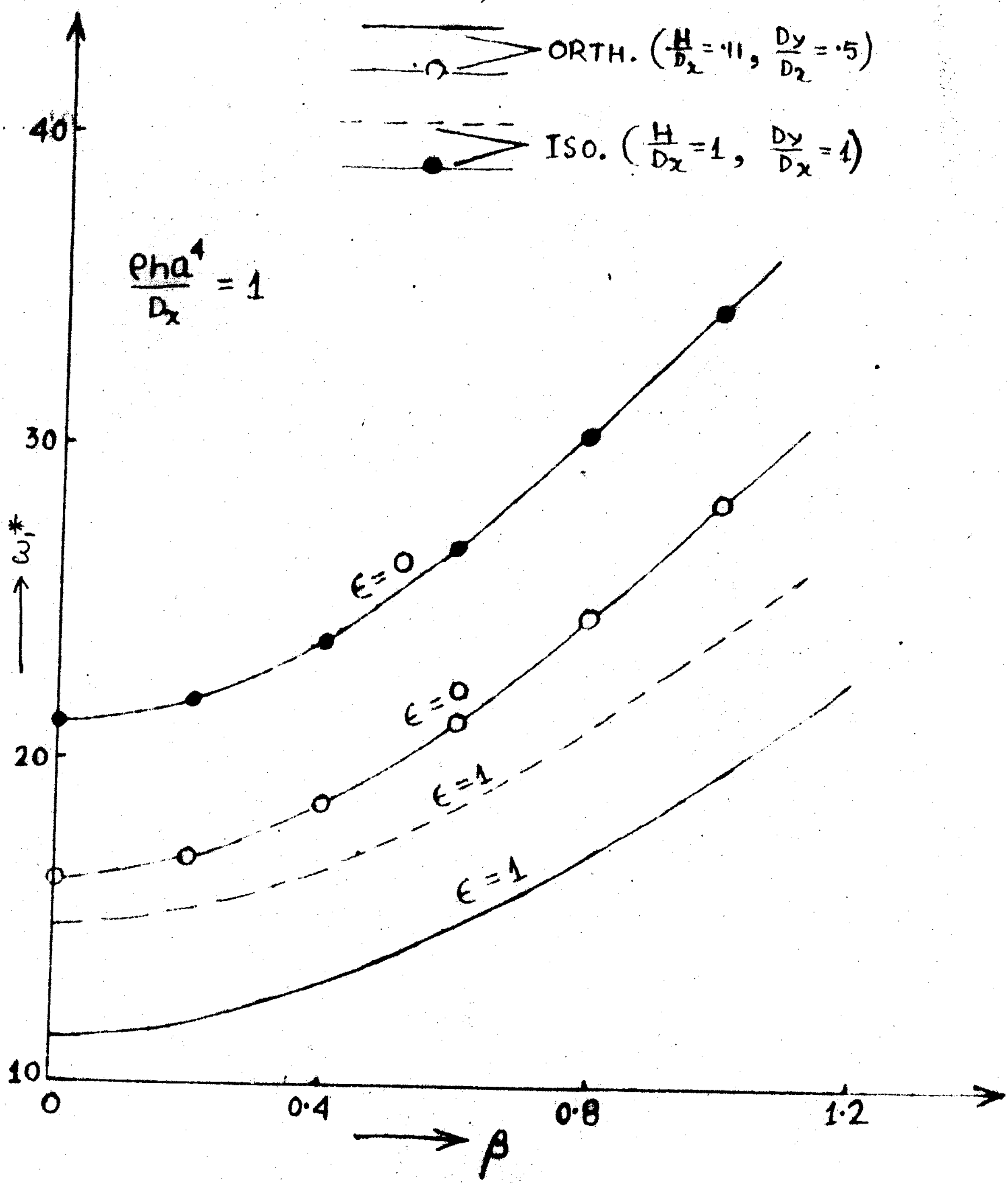

ORTH.
$\left(\frac{H}{D_x} = \cdot 11, \frac{D_y}{D_x} = \cdot 5\right)$
ISO.
$\left(\frac{H}{D_x} = 1, \frac{D_y}{D_x} = 1\right)$
$\frac{Q h a^4}{D_x} = 1$
45
35
25
15
$\epsilon = 0$
$\epsilon = 0$
$\epsilon = 1$
$\epsilon = 1$
σ^*
0
0·4
0·8
1·2
β

FIG. II.9



ORTH. $\left(\frac{H}{D_x}=\cdot 11,\ \frac{D_y}{D_x}=\cdot 5\right)$
ISO. $\left(\frac{H}{D_x}=1,\ \frac{D_y}{D_x}=1\right)$
$\frac{\rho h a^4}{D_x}=1$
40
30
20
10
ω_1^*
$\epsilon=0$
$\epsilon=0$
$\epsilon=1$
$\epsilon=1$
0
0·4
0·8
1·2
β

FIG. II · 10

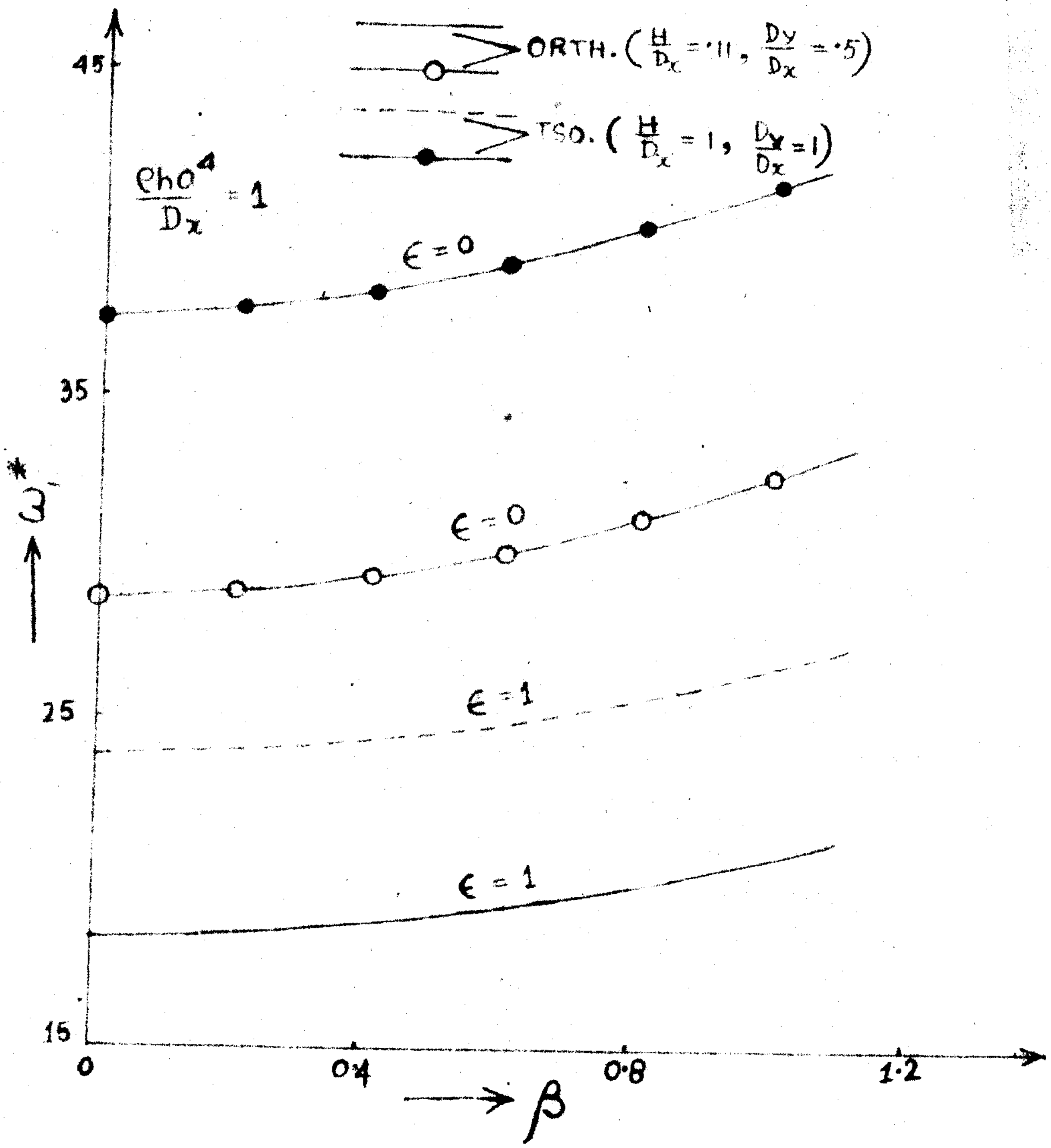

ORTH. (H/Dx = ·11, Dy/Dx = ·5)
ISO. (H/Dx = 1, Dy/Dx = 1)
ρho⁴/Dx = 1
ε = 0
ε = 0
ε = 1
ε = 1
45
35
25
15
ω*
0
0·4
0·8
1·2
β

FIG. II . 11

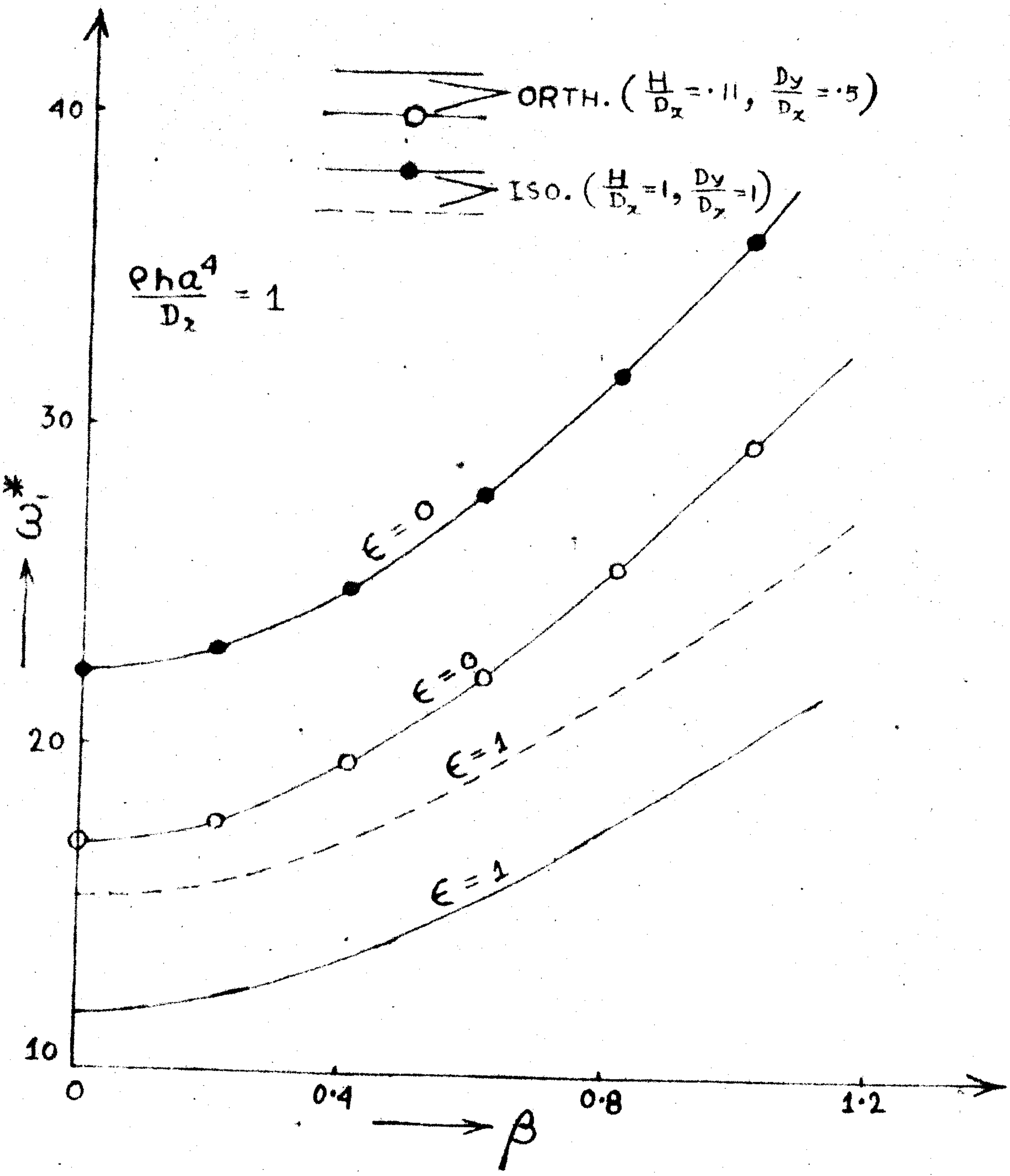

ORTH. ($\frac{H}{D_x}$ = ·11, $\frac{D_y}{D_x}$ = ·5)
ISO. ($\frac{H}{D_x}$ = 1, $\frac{D_y}{D_x}$ = 1)
$\frac{\rho h a^4}{D_x}$ = 1
40
30
20
10
ω^*
ε = 0
ε = 0
ε = 1
ε = 1
0
0·4
0·8
1·2
β

FIG. II . 12

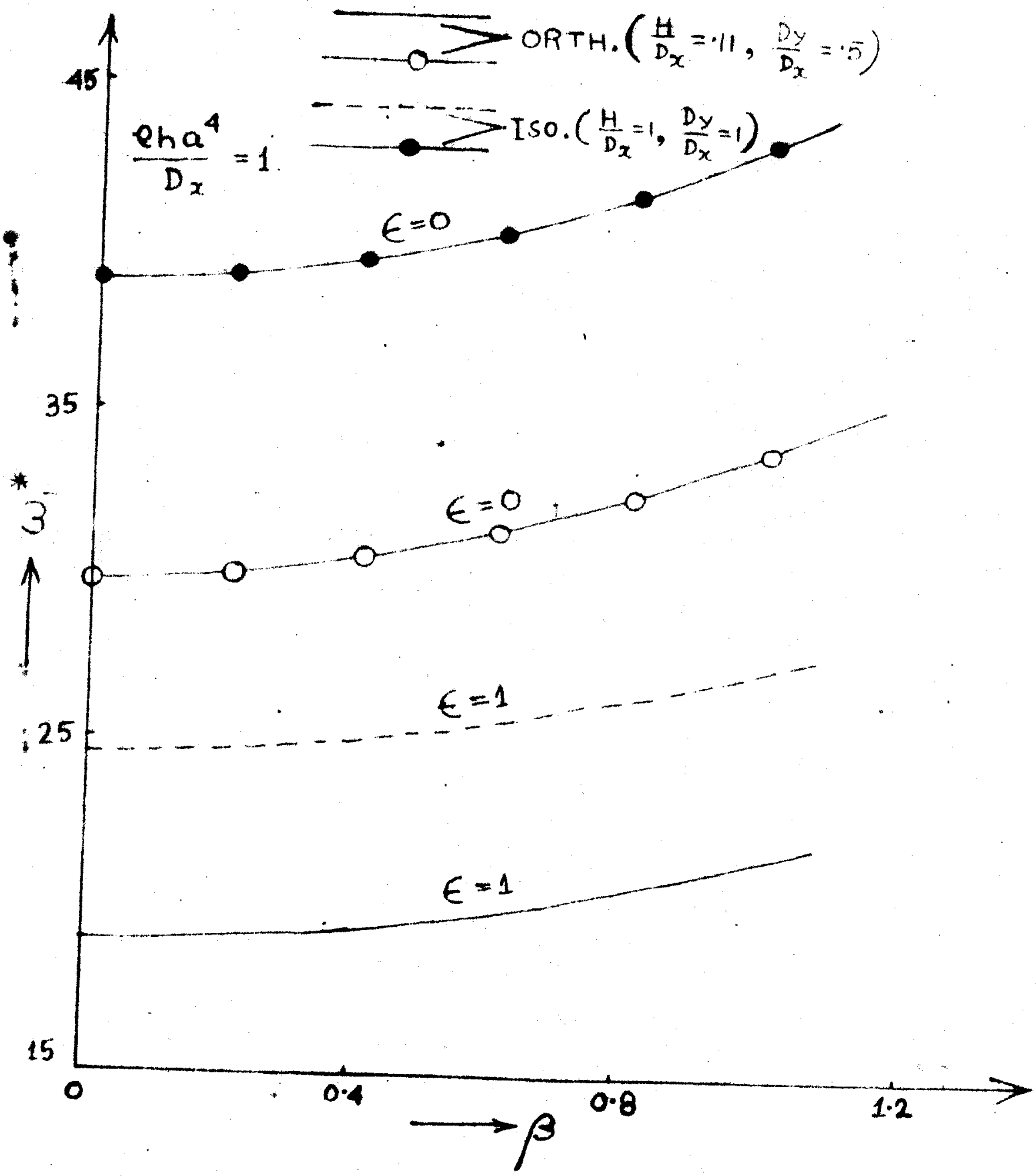

ORTH. (H/Dx = ·11, Dy/Dx = ·5)
Iso. (H/Dx = 1, Dy/Dx = 1)
ρha⁴/Dx = 1
∈=0
∈=0
∈=1
∈=1
45
35
25
15
0
0·4
0·8
1·2
β

FIG. II. 13