Chapter-III

DISPERSION OF SOLUTE

PART ONE

DISPERSION OF SOLUTE IN OSCILLATING HYDROMAGNETIC COUETTE FLOW*

3.1 INTRODUCTION

Taylor [1] showed that if a solute is injected into a solvent flowing steadily in a tube, the combined action of the lateral molecular diffusion and the variation of velocity over the cross-section would cause the solute ultimately to spread diffusively with the effective molecular diffusivity given by Deff.= $a^2\omega_m{}^2/48D_m$, where D_m is the molecular diffusivity, ω_m is the mean velocity and a is the radius of the tube. Some restrictions in this model of Taylor [1] was overcome partially by Aris [2] using a statistical approach. Gill and Sankarasubramanian [3] constructed a dispersion model for the steady flow of fluid in a tube which is valid for all time after the injection of solute by allowing the diffusion coefficients to vary with time. Although authors [4] extended the scope of their model to study dispersion of solute in a time-dependent laminar flow which in principle, valid for all values of time, they confined their analysis only to the case of dispersion in a fully developed steady flow. Recently Hazra *et al.* [5] studied dispersion of solute in pulsatile flow of viscous fluid in a parallel plate channel.

The aim of the present paper is to study the effects of transversely applied uniform magnetic field on the dispersion of solute in oscillatory Couette flow. The analysis is based on the generalized dispersion model introduced by Gill and Sankarasubramanian. The effect of magnetic field and oscillation of plate on diffusion coefficients are investigated. The interesting part of the analysis is that $K_2(\tau)$, second dispersion coefficient, consists of a steady part S and a fluctuating part $D_2(\tau)$ due to the oscillation of plate. Dispersion of solute in oscillatory hydro-magnetic Couette flow is of interest in the field of chemical engineering, biomedical engineering and environmental sciences and in such areas as chemical

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Published in "Far East J. Appl. Math." Vol. 21, Issue 1, (2005), 43-63 reaction design and studies on flow transients using probes based on diffusion-controlled electrode reaction.

3.2 MATHEMATICAL ANALYSIS

The dispersion of passive solute in the form of a slug of finite extent (in the region $-x_s/2 \leq x \leq x_s/2$) in a fully developed oscillatory flow between two parallel flat plates $z=\pm\delta$ is considered. The lower plate is stationary and the upper one is oscillating in its own plane with a velocity $U(t)$ about a non zero constant mean velocity U_0. The z-axis is taken perpendicular to the plates and a transverse magnetic field of uniform strength B_0 is applied along z-axis. Since the plates are infinite in length, all physical quantities except pressure depend on z and t only.

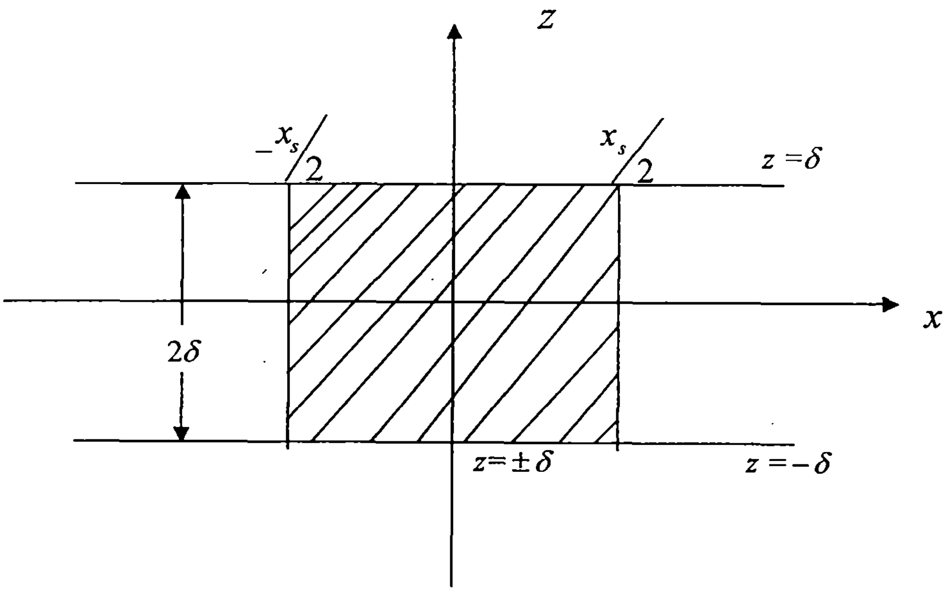

***Fig.* 3.1. Physical sketch of the problem**

The resulting velocity field $u(z,t)$, which is along x -axis, satisfies the Navier–Stokes' equation:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} - \frac{\sigma B_0^2}{\rho}u, \qquad \text{..... (3.1)}$$

with boundary conditions

$$\left.\begin{array}{lll} u=0 & \text{at} & z=-\delta \\ u=U(t)=U_0(1+\varepsilon\cos\beta t) & \text{at} & z=\delta \end{array}\right\}, \qquad \text{..... (3.2)}$$

where β is the frequency of oscillation.

The solution of (3.1) satisfying (3.2) is given as

$$u(z,\mathrm{t})=U_0\left[\ u_0(z)+\frac{\varepsilon}{2}\left\{u_1(z)e^{i\beta t}+u_2(z)e^{-i\beta t}\right\}\right], \qquad \text{..... (3.3)}$$

where,

$$u_0(z)=1-\frac{\sinh\left(\delta-z\right)M}{\sinh 2M\delta}, \qquad \text{..... (3.4)}$$

$$u_1(z)=1-\frac{\sinh(\delta-z)L}{\sinh 2L\delta}, \qquad \text{..... (3.5)}$$

$$u_2(z)=1-\frac{\sinh(\delta-z)L'}{\sinh 2L'\delta} \qquad \text{..... (3.6)}$$

and $L=\sqrt{M^2+i\omega}$, $L'=\sqrt{M^2-i\omega}$, ... (3.7)

where $M=\dfrac{\sigma B_0{}^2}{\rho}$, $\qquad \omega=\dfrac{\beta\delta^2}{\nu}$. ..... (2.8)

If a solute diffuses in the above fully developed flow, then the concentration $\mathrm{C}(\mathrm{t},x,z)$ of solute satisfies

$$\frac{\partial c}{\partial t}+u(z,\mathrm{t})\frac{\partial c}{\partial x}=\mathrm{D}\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial z^2}\right), \qquad \text{..... (3.9)}$$

where D is the molecular diffusivity.

The initial and boundary conditions are

$$\left.\begin{array}{ll} c(0,x,z)=c_0 & \text{for } |x| \le \dfrac{x_s}{2} \\ \quad =0 & \text{for } |x| > \dfrac{x_s}{2} \end{array}\right\}, \qquad \text{.... (3.10a)}$$

$$\frac{\partial c}{\partial z}=0 \qquad \text{at } z=\pm\delta, \qquad \text{.... (3.10b)}$$

$$c(\mathrm{t},\infty,z)=0, \qquad \text{..... (3.10c)}$$

where (3.10c) expresses the condition of zero mass flux at the plate walls.

We introduce the dimensionless quantities as

$$\theta=\frac{c}{c_0},U(\eta,\tau)=\frac{u(z,\mathrm{t})}{\bar{u}},X=\frac{x_s}{\delta^2\bar{u}},\tau=\frac{\mathrm{D}t}{\delta^2},Pe=\frac{\bar{u}\delta}{\mathrm{D}},\eta=\frac{z}{\delta}, \qquad \text{.... (3.11)}$$

where $\bar{u}$ is the time-averaged axial velocity on the central line $z=0$ given by

$$\bar{u}=\frac{2\pi}{\beta}\int_0^{\beta/2\pi} u(t,0)\,dt \qquad \text{.... (3.12)}$$

using (3.11) in (3.9) and (3.10) we get

$$\frac{\partial\theta}{\partial\tau}+U(\eta,\tau)\frac{\partial\theta}{\partial X}=\frac{1}{Pe^2}\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial\eta^2} \qquad \text{.... (3.13)}$$

with the initial and boundary conditions are as follows

$$\left.\begin{aligned}\theta(0,X,\eta)&=1 \qquad \text{for } |X|\le\frac{1}{2}X_s\\ &=0 \qquad \text{for } |X|\rangle\ \frac{1}{2}X_s\end{aligned}\right\}, \qquad \text{.... (3.14a)}$$

$$\theta(\tau,\infty,\eta)=0, \qquad \text{.... (3.14b)}$$

$$\frac{\partial\theta}{\partial\eta}(\tau,X,\pm1)=0. \qquad \text{.... (3.14c)}$$

Following Gill and Sankarasubramanian [3], the solution of (3.13) subject to (3.14) is formulated as

$$\theta(\tau,X,\eta)=\sum_{\mathrm{k}=0}^{\infty} f_{\mathrm{k}}(\tau,\eta)\frac{\partial^k\theta_m}{\partial X^k}, \qquad \text{.... (3.15)}$$

where

$$\theta_m = \frac{1}{2}\int_{-1}^{1} \theta d\eta. \qquad \text{.... (3.16)}$$

Substituting (3.15) in (3.13) we get

$$\frac{\partial\theta_m}{\partial\tau}+U(\eta,\tau)\frac{\partial\theta_m}{\partial X}-\frac{1}{Pe^2}\frac{\partial^2\theta_m}{\partial X^2}+\sum_{k=0}^{\infty}[(\frac{\partial f_k}{\partial\tau}-\frac{\partial^2 f_k}{\partial\eta^2})\frac{\partial^k\theta_m}{\partial X^k}$$

$$+U(\tau,\eta)f_k\frac{\partial^{k+1}\theta_m}{\partial X^{k+1}}-\frac{1}{Pe^2}f_k\frac{\partial^{k+2}\theta_m}{\partial X^{k+2}}+f_k\frac{\partial^{k+1}\theta_m}{\partial\tau\partial X^k}]=0. \qquad \text{.... (3.17)}$$

Integration of (3.13) gives upon using (3.16),

$$\frac{\partial\theta_m}{\partial\tau}=\frac{1}{Pe^2}\frac{\partial^2\theta_m}{\partial X^2}-\frac{1}{2}\int_{-1}^{1}U(\tau,\eta)\frac{\partial\theta}{\partial X}d\eta. \qquad \text{.... (3.18)}$$

We assume that the process of distribution of θ_m is diffusive in nature right from time zero, then following Gill and Sankarasubramanian's approach. They generalized dispersion model with time dependent dispersion coefficients can be written as

$$\frac{\partial\theta_m}{\partial\tau}=\sum_{i=1}^{\infty}K_i(\tau)\frac{\partial^i\theta_m}{\partial X^i}, \qquad \text{.... (3.19)}$$

where

$$K_1(\tau)=-\frac{1}{2}\int_{-1}^{1}U(\tau,\eta)f_0(\tau,\eta)d\eta, \qquad \text{.... (3.20a)}$$

$$K_2(\tau)=\frac{1}{Pe^2}-\frac{1}{2}\int_{-1}^{1}U(\tau,\eta)f_1(\tau,\eta)d\eta, \qquad \text{.... (3.20b)}$$

$$\vdots$$

$$K_{i+2}(\tau)=-\frac{1}{2}\int_{-1}^{1}U(\tau,\eta)f_{i+1}d\eta \qquad (i=1,2,3,............). \qquad \text{.... (3.20c)}$$

Substituting (3.19) in (3.17) and equating the coefficients $\partial^k\theta_m/\partial X^k$, we obtain the following equations for $f_k(\tau, \eta)$:

$$\frac{\partial f_0}{\partial \tau} = \frac{\partial^2 f_0}{\partial \eta^2}, \qquad \text{.... (3.21a)}$$

$$\frac{\partial f_1}{\partial \tau} = \frac{\partial^2 f_1}{\partial \eta^2} - \left[U(\tau,\eta) + K_1(\tau)\right] f_0. \qquad \text{.... (3.21b)}$$

$$\frac{\partial f_2}{\partial \tau} = \frac{\partial^2 f_2}{\partial \eta^2} + \left[\frac{1}{Pe^2} - K_2(\tau)\right] f_0 - \left[U(\tau,\eta) + K_1(\tau)\right] f_1. \qquad \text{.... (3.21c)}$$

$$\vdots$$

$$\frac{\partial f_k}{\partial \tau} = \frac{\partial^2 f_k}{\partial \eta^2} - \left[U(\tau,\eta) + K_1(\tau)\right] f_{k-1}$$

$$+ \left[\frac{1}{Pe^2} - K_2(\tau)\right] f_{k-2} - \sum_{i=3}^{k} K_i f_{k-i} \qquad (k=3,4,5,\ldots\ldots\ldots). \qquad \text{.... (3.21d)}$$

Equations (3.14a) and (3.16) give

$$\left.\begin{array}{lll} \theta_m(0,X) = 1 & \text{for} & |X| \le \dfrac{X_s}{2} \\ \theta_m(0,X) = 0 & \text{for} & |X| > \dfrac{X_s}{2} \end{array}\right\}, \qquad \text{.... (3.22a)}$$

$$\theta_m(\tau,\infty) = 0. \qquad \text{.... (3.22b)}$$

Now from (3.15), the initial conditions for f_k can be taken as

$$f_0(0,\eta) = 1, \quad f_k(0,\eta) = 0 \quad \text{for } k=1,2,3,\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \qquad \text{.... (3.23)}$$

Similarly the boundary conditions for f_k are derived from (3.14) and (3.15) as

$$\frac{\partial f_k}{\partial \eta} = 0 \quad \text{at } \eta = \pm 1 \qquad (k=0,1,2,\ldots\ldots\ldots\ldots\ldots\ldots). \qquad \text{.... (3.24)}$$

Further Equations (3.15) and (3.16) are consistent if

$$\int_{-1}^{1} f_0 d\eta = 2 \text{ and } \int_{-1}^{1} f_k d\eta = 0 \qquad \text{for } k=1,2,3,\ldots\ldots\ldots\ldots\ldots. \qquad \ldots. (3.25)$$

With the initial and boundary conditions given by (3.23)–(3.25), we determine the functions $f_k(\tau, \eta)$ by solving the system of equations (3.21a)–(3.21d) and then obtain the diffusion coefficients $K_i(\tau)$ from (3.20a)–(3.20c).Using (3.11) and (3.12), the dimensionless velocity distribution is given as

$$U(\tau,\eta)=\frac{1}{P}\left[1-\frac{\sinh(1-\eta)\delta M}{\sinh 2\delta M}+\varepsilon\cos 2\zeta\tau\right.$$

$$\left.-\frac{\varepsilon}{2}\left(\frac{\sinh(1-\eta)L\delta}{\sinh 2\delta L}e^{i\zeta\tau}+\frac{\sinh(1-\eta)L'\delta}{\sinh 2L'\delta}e^{-i\zeta\tau}\right)\right], \qquad \ldots. (3.26)$$

where

$$\zeta=\frac{\beta\delta^2}{\mathrm{D}}, \qquad \mathrm{Sc}=\frac{\nu}{\mathrm{D}}, \quad \omega=\frac{\zeta}{\mathrm{Sc}}. \qquad \ldots. (3.27)$$

We next proceed to determine $K_1(\tau)$. It can be readily shown that the solution of (3.21a) subject to (3.23)–(3.25) is given by

$$f_0(\tau,\eta)=1. \qquad \ldots. (3.28)$$

Substitution from (3.26) and (3.28) in (3.20a) then gives

$$K_1(\tau)=-\frac{1}{P}\left[2+2\varepsilon\cos\zeta\tau\right.$$

$$\left.+\frac{1-\cosh 2\delta M}{\delta M\sinh 2\delta M}+\frac{\varepsilon}{\delta}\left(A\cos\zeta\tau-B\sin\zeta\tau\right)\right], \qquad \ldots. (3.29)$$

where

$$P=\frac{\sinh 2\delta M-\sinh\delta M}{\sinh 2\delta M};$$

$$A=\frac{A'\frac{\alpha}{\sqrt{2}}+B'\frac{\omega}{\sqrt{2}\alpha}}{\frac{\alpha^2}{2}+\frac{\omega^2}{2\alpha^2}};\qquad B=\frac{B'\frac{\alpha}{\sqrt{2}}-A'\frac{\omega}{\sqrt{2}\alpha}}{\frac{\alpha^2}{2}+\frac{\omega^2}{2\alpha^2}};$$

$$A'=\frac{\sinh\sqrt{2}\alpha\delta\cos\frac{\sqrt{2}\omega\delta}{\alpha}-\sinh\sqrt{2}\alpha\delta\cosh\sqrt{2}\alpha\delta}{\sinh^2\sqrt{2}\alpha\delta\cos^2\frac{\sqrt{2}\omega\delta}{\alpha}+\cosh^2\sqrt{2}\alpha\delta\sin^2\frac{\sqrt{2}\omega\delta}{\alpha}}; \qquad \text{.... (3.30)}$$

$$B'=\frac{\sin\frac{\sqrt{2}\omega\delta}{\alpha}\cos\frac{\sqrt{2}\omega\delta}{\alpha}-\cosh\sqrt{2}\alpha\delta\sin\frac{\sqrt{2}\omega\delta}{\alpha}}{\sinh^2\sqrt{2}\alpha\delta\cos^2\frac{\sqrt{2}\omega\delta}{\alpha}+\cosh^2\sqrt{2}\alpha\delta\sin^2\frac{\sqrt{2}\omega\delta}{\alpha}};$$

$$\alpha=\left\{M^2+\sqrt{M^4+\omega^2}\right\}^{\frac{1}{2}}.$$

To determine $f_1(\tau, \eta)$, one has to solve (3.21b) subject to the initial and boundary conditions (3.23) – (3.25) with $f_0=1$. From Duhamel's theorem, it follows that

$$f_1(\tau,\eta)=\frac{\partial}{\partial\tau}\int_0^{\tau}F(\tau-\xi,\eta,\xi)d\xi, \qquad \text{.... (3.31)}$$

where $F(\tau, \eta, \xi)$ satisfies

$$\frac{\partial F}{\partial\tau}=\frac{\partial^2 F}{\partial\eta^2}-\left[U(\xi,\eta)+K_1(\xi)\right]. \qquad \text{.... (3.32)}$$

Subject to

$$\left.\begin{array}{l} F(0,\eta,\tau)=0 \\ \left(\dfrac{\partial F}{\partial\eta}\right)_{\eta=\pm1}=0\,, \qquad \int_{-1}^{1}F\,d\eta=0 \end{array}\right\}. \qquad \text{.... (3.33)}$$

Note that ξ behaves like a parameter while solving (3.32). Substituting for U and K_I from (3.26) and (3.29) in (3.32) and solving the resulting equation subject to (3.33), we get

$$F(\tau,\eta,\xi)=\frac{1}{\delta PM\sinh 2\delta M}\left[\frac{\sinh(1-\eta)\delta M}{-\delta M}-\frac{(1-\cosh 2\delta M)}{4}\eta^2\right]$$

$$+\frac{\varepsilon}{2P\delta}\left[\frac{\sinh(1-\eta)\delta L}{\delta L^2\sinh 2\delta L}e^{i\zeta\xi}+\frac{\sinh(1-\eta)\delta L'}{\delta L'^2\sinh 2\delta L'}e^{-i\zeta\xi}-(A\cos\zeta\xi-B\sinh\zeta\xi)\frac{\eta^2}{2}\right]$$

$$-\left[\frac{(1+\cosh 2\delta M)}{2\delta PM\sinh 2\delta M}+\frac{\varepsilon}{2P\delta}\left\{(2C+A)\cos\zeta\xi+(2\bar{D}-B)\sin\zeta\xi\right\}\right]\eta$$

$$+\frac{1}{2\delta PM\sinh 2\delta M}\left[\frac{(\cosh 2\delta M-1)}{\delta^2M^2}+\frac{1}{6}(1-\cosh 2\delta M)\right]+$$

$$\frac{\varepsilon}{2P\delta}\left[\frac{(\cosh 2\delta L-1)}{\delta^2L^3\sinh 2\delta L}e^{i\zeta\xi}+\frac{(\cosh 2\delta L'-1)}{\delta^2L'^3\sinh 2\delta L'}e^{-i\zeta\xi}+\frac{(A\cos\zeta\xi-B\sin\zeta\xi)}{3}\right]$$

$$+\sum_{n=1}^{\infty} d_n e^{-\lambda_n^2\tau}\cos\lambda_n\eta, \qquad \text{.....}\ (3.34)$$

where

$$\lambda_n=n\pi;\qquad C=\frac{\frac{\alpha}{\sqrt{2}}C'-\frac{\omega}{\sqrt{2}\alpha}D'}{\frac{\alpha^2}{2}+\frac{\omega^2}{2\alpha^2}};\qquad \bar{D}=\frac{\frac{\omega}{\sqrt{2}\alpha}C'+\frac{\alpha}{\sqrt{2}}D'}{\frac{\alpha^2}{2}+\frac{\omega^2}{2\alpha^2}};$$

$$C'=\frac{\sinh\sqrt{2}\delta\alpha\cosh\sqrt{2}\delta\alpha}{\sinh^2\sqrt{2}\delta\alpha\cos^2\frac{\sqrt{2}\omega\delta}{\alpha}+\cosh^2\sqrt{2}\delta\alpha\sin^2\frac{\sqrt{2}\omega\delta}{\alpha}};$$

$$D' = \frac{\sin\frac{\sqrt{2}\omega\delta}{\alpha}\cos\frac{\sqrt{2}\omega\delta}{\alpha}}{\sinh^2\sqrt{2}\delta\alpha\cos^2\frac{\sqrt{2}\omega\delta}{\alpha} + \cosh^2\sqrt{2}\delta\alpha\sin^2\frac{\sqrt{2}\omega\delta}{\alpha}};$$

$$d_n = \frac{\cos\lambda_n(1-\cosh 2\delta M)}{\delta PM\sinh 2\delta M}\left[\frac{\delta M}{\lambda_n{}^2+\delta^2 M^2}+\frac{1}{\lambda_n{}^2}\right] - \frac{\varepsilon}{2}\frac{\cos\lambda_n}{P}$$

$$\times\left[\frac{(\cosh 2\delta L-1)}{L^2(\lambda_n{}^2+\delta^2L^2)\sinh 2\delta L}e^{i\zeta\xi} + \frac{(\cosh 2\delta L'-1)}{L'^2(\lambda_n{}^2+\delta^2L'^2)\sinh 2\delta L'}e^{-i\zeta\xi}\right.$$

$$\left.-2\frac{(A\cos\zeta\xi - B\sin\zeta\xi)}{\lambda_n{}^2}\right]. \qquad (n = 1,2,3,\ldots\ldots\ldots\ldots). \qquad \ldots.\ (3.35)$$

Equations (3.31) and (3.34) now give

$$f_1(\tau,\eta) = \frac{1}{\delta PM\sinh 2\delta M}\left[\frac{\sinh(1-\eta)\delta M}{-\delta M} - \frac{(1-\cosh 2\delta M)}{4}\eta^2\right]$$

$$+\frac{\varepsilon}{2P\delta}\left[\frac{\sinh(1-\eta)\delta L}{\delta L^2\sinh 2\delta L}e^{i\zeta\tau} + \frac{\sinh(1-\eta)\delta L'}{\delta L'^2\sinh 2\delta L'}e^{-i\zeta\tau} - (A\cos\zeta\tau - B\sin\zeta\tau)\frac{\eta^2}{2}\right]$$

$$-\frac{\varepsilon}{2P\delta}\left\{(2C+A)\cos\zeta\tau + (2\bar{D}-B)\sin\zeta\tau\right\}\eta +$$

$$\frac{1}{2\delta PM\sinh 2\delta M}\left[\frac{(\cosh 2\delta M-1)}{\delta^2M^2} + \frac{1}{6}(1-\cosh 2\delta M)\right] - \frac{(1+\cosh 2\delta M)}{2\delta PM\sinh 2\delta M}\eta$$

$$+\frac{\varepsilon}{2P\delta}\left[\frac{(\cosh 2\delta L-1)}{\delta^3L^3\sinh 2\delta L}e^{i\zeta\tau} + \frac{(\cosh 2\delta L'-1)}{\delta^2L'^3\sinh 2\delta L'}e^{-i\zeta\tau} + \frac{1}{3}(A\cos\zeta\tau - B\sin\zeta\tau)\right]$$

$$+\sum_{n=1}^{\infty}\left[\frac{(1-\cosh 2\delta M)\cos\lambda_n}{\delta PM\sinh 2\delta M}\left(\frac{\delta M}{\lambda_n{}^2+\delta^2M^2}+\frac{1}{\lambda_n{}^2}\right)e^{-\lambda_n{}^2\tau} - \frac{\varepsilon}{2}\frac{\cos\lambda_n}{P\delta}\times\right.$$

$$\left\{\frac{(\cosh 2\delta L-1)}{L^2\sinh 2\delta L}\frac{(i\zeta e^{i\zeta\tau}+\lambda_n{}^2 e^{-\lambda_n{}^2\tau})}{(\lambda_n{}^2+\delta^2L^2)(\lambda_n{}^2+i\zeta)}+\frac{(\cosh 2\delta L'-1)}{L'^2\sinh 2\delta L'}\right.$$

$$\times\frac{(\lambda_n{}^2 e^{-\lambda_n{}^2\tau}-i\zeta e^{-i\zeta\tau})}{(\lambda_n{}^2+\delta^2L'^2)(\lambda_n{}^2-i\zeta)}-\frac{2}{\lambda_n{}^2}\left(\frac{A(\zeta^2\cos\zeta\tau-\lambda_n{}^2\zeta\sin\zeta\tau+\lambda_n{}^4e^{-\lambda_n{}^2\tau})}{\lambda_n{}^4+\zeta^2}-\right.$$

$$\left.\left.\frac{B(\lambda_n{}^2\zeta\cos\zeta\tau+\zeta^2\sin\zeta\tau-\lambda_n{}^2\zeta e^{-\lambda_n{}^2\tau})}{\lambda_n{}^4+\zeta^2}\right)\right\}\Bigg]. \qquad \ldots.(3.36)$$

Substituting (3.26) and (3.36) in (3.20b), we obtain the diffusion coefficient $K_2(\tau)$ as follows:

$$K_2(\tau)=\frac{1}{Pe^2}+\mathrm{S}+D_2(\tau), \qquad \ldots.(3.37)$$

where S and $D_2(\tau)$ represent the steady and time dependent part of the diffusion coefficient, respectively.

The expressions for S and $D_2(\tau)$ are given as

$$\mathrm{S}=-\frac{1}{P\delta\left\{\sinh 2\delta M-\sinh\delta M\right\}}\left[-\frac{2\sinh^2\delta M}{\delta^2M^3}-\frac{(1-\cosh 2\delta M)}{2M}\left\{\frac{1}{3}-\right.\right.$$

$$\left.\frac{1}{2\sinh 2\delta M}\left(\frac{2\sinh^2\delta M}{\delta M}+\frac{4\sinh^2\delta M}{M^3\delta^3}-\frac{4\cosh\delta M\sinh\delta M}{M^2\delta^2}\right)\right\}-$$

$$\frac{\varepsilon^2\sinh 2\delta M}{8\delta^2\sinh 2\delta L\sinh 2\delta L'}\left(\frac{\sinh 2(L+L')\delta}{L+L'}-\frac{\sinh 2(L-L')\delta}{L-L'}\right)\left(\frac{1}{L^2}+\frac{1}{L'^2}\right)+$$

$$\frac{(1+\cosh 2\delta M)\cosh\delta M}{\delta M^2\sinh 2\delta M}\left(\frac{\sinh\delta M}{M\delta}-\cosh\delta M\right)+\frac{(\cosh 2\delta M-1)}{M}$$

$$\times\left(\frac{1}{\delta^2 M^2}-\frac{1}{6}\right)\left(1-\frac{\sinh^2\delta M}{\delta M\sinh 2\delta M}\right)-\frac{\varepsilon^2\sinh 2\delta M}{2\delta^3\sinh 2\delta L\sinh 2\delta L'}$$

$$\times\left\{\frac{(\cosh 2\delta L-1)\sinh^2\delta L'}{L'L^3}+\frac{(\cosh 2\delta L'-1)\sinh^2\delta L}{LL'^3}\right\}\Bigg]\Bigg].$$

$$D_2(\tau)=-\frac{1}{\delta P^2 M\sinh 2\delta M}\Bigg[\frac{\varepsilon e^{i\zeta\tau}}{2\delta M\sinh\delta L}\left(\frac{\sinh 2(L+M)\delta}{2(L+M)\delta}-\frac{\sinh 2(L-M)\delta}{2(L-M)\delta}\right)$$

$$+\frac{\varepsilon e^{-i\zeta\tau}}{2\delta M\sinh\delta L'}\left(\frac{\sinh 2(M+L')\delta}{2(M+L')\delta}-\frac{\sinh 2(M-L')\delta}{2(M-L')\delta}\right)+\varepsilon\cos\zeta\tau$$

$$\times\left\{\frac{(1-\cosh 2\delta M)}{6}-\frac{2\sinh^2\delta M}{\delta^2 M^2}\right\}+\frac{\varepsilon e^{i\zeta\tau}(1-\cosh 2\delta M)}{8\sinh\delta L}\left\{\frac{\sinh^2\delta L}{\delta L}\right.$$

$$\left.+\frac{4\sinh^2\delta L}{\delta^3 L^3}-\frac{2\sinh 2\delta L}{L^2\delta^2}\right\}+\frac{\varepsilon e^{-i\zeta\tau}(1-\cosh 2\delta M)}{8\sinh\delta L'}\left\{\frac{4\sinh^2\delta L'}{\delta^3 L'^3}\right.$$

$$\left.-\frac{2\sinh 2\delta L'}{\delta^2 L'^2}+\frac{\sinh^2\delta L'}{\delta L'}\right\}\Bigg]-\frac{\varepsilon}{2\delta P^2}\Bigg[\frac{2e^{i\zeta\tau}\sinh^2\delta L}{\delta^2 L^3\sinh 2\delta L}+\frac{2e^{-i\zeta\tau}\sinh^2\delta L'}{\delta^2 L'^3\sinh 2\delta L'}$$

$$-\frac{e^{i\zeta\tau}}{\delta L^2\sinh 2\delta M\sinh 2\delta L}\left\{\frac{\sinh 2(L+M)\delta}{2(L+M)\delta}-\frac{\sinh 2(L-M)\delta}{2(L-M)\delta}\right\}$$

$$-\frac{e^{-i\zeta\tau}}{\delta L'^2\sinh 2\delta M\sinh 2\delta L'}\left\{\frac{\sinh 2(L'+M)\delta}{2(L'+M)\delta}-\frac{\sinh 2(M-L')\delta}{2(M-L')\delta}\right\}$$

$$-\frac{\varepsilon e^{2i\zeta\tau}}{2\delta L^2\sinh^2 2\delta L}\left\{\frac{\sinh 4\delta L}{4\delta L}-1\right\}-\frac{\varepsilon e^{-2i\zeta\tau}}{2\delta L'^2\sinh^2 2\delta L'}\left\{\frac{\sinh 4\delta L'}{4\delta L'}-1\right\}$$

$$+\frac{2\varepsilon\cos\zeta\tau\sinh^2\delta L}{\delta^2L^3\sinh2\delta L}e^{i\zeta\tau}+\frac{2\varepsilon\cos\zeta\tau\sinh^2\delta L'}{\delta^2L'^3\sinh2\delta L'}e^{-i\zeta\tau}-\left(A\cos\zeta\tau-B\sin\zeta\tau\right)$$

$$\times\left\{\frac{2}{3}-\frac{1}{\sinh2\delta M}\left(\frac{\sinh^2\delta M}{\delta M}+\frac{2\sinh^2\delta M}{M^3\delta^3}-\frac{\sinh2\delta M}{M^2\delta^2}\right)\right\}-\frac{\varepsilon e^{i\zeta\tau}}{2\sinh2\delta L}$$

$$\times\left\{\frac{2\sinh^2\delta L}{\delta L}+\frac{4\sinh^2\delta L}{L^3\delta^3}-\frac{2\sinh2\delta L}{L^2\delta^2}\right\}-\frac{\varepsilon e^{-i\zeta\tau}}{2\sinh2\delta L'}\left\{\frac{2\sinh^2\delta L'}{\delta L'}\right.$$

$$\left.+\frac{4\sinh^2\delta L'}{L'^3\delta^3}-\frac{2\sinh2\delta L'}{L'^2\delta^2}\right\}+\frac{2}{3}\varepsilon\cos\zeta\tau\Bigg]-\frac{1}{P}\left[\frac{(1+\cosh2\delta M)}{2P\delta M\sinh2\delta M}\right.$$

$$\times\left\{\frac{\varepsilon e^{i\zeta\tau}}{2\sinh2\delta L}\left(\frac{\sinh2\delta L}{L^2\delta^2}-\frac{2\cosh^2\delta L}{\delta L}\right)+\frac{\varepsilon e^{-i\zeta\tau}}{2\sinh2\delta L'}\left(\frac{\sinh2\delta L'}{L'^2\delta^2}\right.\right.$$

$$\left.\left.-\frac{2\cosh^2\delta L'}{\delta L'}\right)\right\}+\frac{\varepsilon}{2P\delta\sinh2\delta M}\left(\frac{\sinh2\delta M}{M^2\delta^2}-\frac{2\cosh^2\delta M}{M\delta}\right)$$

$$\times\left\{(2C+A)\cos\zeta\tau+(2\bar{D}-B)\sin\zeta\tau\right\}+\frac{\varepsilon^2}{2P\delta}\{(2C+A)\cos\zeta\tau$$

$$+\left(2\bar{D}-B\right)\sin\zeta\tau\}\left\{\frac{e^{i\zeta\tau}}{2\sinh2\delta L}\left(\frac{\sinh2\delta L}{L^2\delta^2}-\frac{2\cosh^2\delta L}{\delta L}\right)+\frac{e^{-i\zeta\tau}}{2\sinh2\delta L'}\right.$$

$$\times\left.\left(\frac{\sinh2\delta L'}{L'^2\delta^2}-\frac{2\cosh^2\delta L'}{\delta L'}\right)\right\}\Bigg]+\frac{1}{P}\left[\frac{\varepsilon(\cosh2\delta M-1)}{2\delta PM\sinh2\delta M}\left(\frac{1}{\delta^2M^2}-\frac{1}{6}\right)\right.$$

$$\times\left\{\frac{e^{i\zeta\tau}\sinh^2\delta L}{\delta L\sinh2\delta L}+\frac{e^{-i\zeta\tau}\sinh^2\delta L'}{\delta L'\sinh2\delta L'}-2\cos\zeta\tau\right\}-\frac{\varepsilon}{P\delta}\left(1-\frac{\sinh^2\delta M}{\delta M\sinh2\delta M}\right)$$

$$\times\left\{\frac{(\cosh 2\delta L-1)}{\delta^2 L^3 \sinh 2\delta L}e^{i\zeta\tau}+\frac{(\cosh 2\delta L'-1)}{\delta^2 L'^3 \sinh 2\delta L'}e^{-i\zeta\tau}+\frac{1}{3}\left(A\cos\zeta\tau-B\sin\zeta\tau\right)\right\}$$

$$+\frac{\varepsilon^2}{2P\delta}\left\{\frac{(\cosh 2\delta L-1)}{\delta^2 L^3 \sinh 2\delta L}e^{i\zeta\tau}+\frac{(\cosh 2\delta L'-1)}{\delta^2 L'^3 \sinh 2\delta L'}e^{-i\zeta\tau}+\frac{1}{3}\left(A\cos\zeta\tau-B\sin\zeta\tau\right)\right\}$$

$$\times\left\{\frac{\sinh^2\delta L}{\delta L\sinh 2\delta L}e^{i\zeta\tau}+\frac{\sinh^2\delta L'}{\delta L'\sinh 2\delta L'}e^{-i\zeta\tau}-2\cos\zeta\tau\right\}\Bigg]\Bigg]+\sum_{n=1}^{\infty}\frac{1}{P}\Bigg[\frac{\cos\lambda_n}{\delta PM\sinh 2\delta M}$$

$$-\frac{\cos\lambda_n\cosh 2\delta M}{\delta PM\sinh 2\delta M}\left(\frac{\delta M}{\lambda_n^2+\delta^2 M^2}+\frac{1}{\lambda_n 2}\right)e^{-\lambda_n^2\tau}-\frac{\varepsilon\cos\lambda_n}{2P\delta}$$

$$\times\Bigg\{\frac{(\cosh 2\delta L-1)(i\zeta e^{i\zeta\tau}+\lambda_n^2 e^{-\lambda_n^2\tau})}{L^2(\lambda_n^2+\delta^2 L^2)(\lambda_n^2+i\zeta)\sinh 2\delta L}+\frac{(\cosh 2\delta L'-1)(\lambda_n^2 e^{-\lambda_n^2\tau}-i\zeta e^{-i\zeta\tau})}{L'^2(\lambda_n^2+\delta^2 L'^2)(\lambda_n^2-i\zeta)\sinh 2\delta L'}$$

$$-\frac{2}{\lambda_n^2}\Bigg(A\frac{(\zeta^2\cos\zeta\tau-\lambda_n^2\zeta\sin\zeta\tau+\lambda_n^4 e^{-\lambda_n^2\tau})}{\lambda_n^4+\zeta^2}-$$

$$B\frac{(\lambda_n^2\zeta\cos\zeta\tau+\zeta^2\sin\zeta\tau-\lambda_n^2\zeta e^{-\lambda_n^2\tau})}{\lambda_n^4+\zeta^2}\Bigg)\Bigg\}\Bigg]\times\Bigg[\frac{M\delta\cos\lambda_n(\cosh 2\delta M-1)}{(\lambda_n^2+M^2\delta^2)\sinh 2\delta M}$$

$$+\frac{\varepsilon\delta L\cos\lambda_n(\cosh 2\delta L-1)}{2(\lambda_n^2+L^2\delta^2)\sinh 2\delta L}e^{i\zeta\tau}+\frac{\varepsilon\delta L'\cos\lambda_n(\cosh 2\delta L'-1)}{2(\lambda_n^2+L'^2\delta^2)\sinh 2\delta L'}e^{-i\zeta\tau}\Bigg].$$

The foregoing procedure may be repeated and higher order dispersion coefficients K_3, K_4,.............. , may be determined with the help of (3.20c). It is also noted from previous work [3, 4] that higher order dispersion coefficients decrease rapidly in magnitude. Neglecting K_3 and higher order dispersion coefficients equation (3.19) can be written as

$$\frac{\partial\theta_m}{\partial\tau} = K_1(\tau)\frac{\partial\theta_m}{\partial X} + K_2(\tau)\frac{\partial^2\theta_m}{\partial X^2}. \qquad \text{.... (3.38)}$$

The solution of equation (3.38) subject to (3.22a) and (3.22b) is obtained as

$$\theta_m = \frac{1}{2}\left[erf\frac{\frac{1}{2}X_s + X_1}{2T_0^{\frac{1}{2}}} + erf\frac{\frac{1}{2}X_s - X_1}{2T_0^{\frac{1}{2}}} \right],$$

where

$$X_1 = X + \int_0^\tau K_1(\eta)d\eta \;; \qquad T_0 = \int_0^\tau K_2(\eta)d\eta.$$

3.3 RESULTS AND DISCUSSION

The most striking result of this analysis is that the dispersion of solute in the unsteady flow arising out of oscillation of upper plate gives rise to dispersion coefficient which consists of both steady and unsteady part. We have computed S, the steady part of dispersion coefficient, for various values of ω (oscillation parameter) and M (magnetic parameter) with Sc=1000 (for liquid Sc is very large) and plotted in Fig.–3.2.and Fig.-3.3. It can be seen that from Fig.–3.2 that for fixed M, S decreases with increases in ω and becomes more or less constant for $\omega \geq 2$.It may be remarked that Fig.–3.2 shows no variation for higher values of oscillation parameter, which is quite natural. In Fig.–3.3 it is noted that S decreases slowly with increase in M for fixed ω but decreases rapidly for $M \geq 0.5$ and this steepness is prominent for $\omega=1$.The time-dependent part $D_2(\tau)$ of dispersion coefficient $K_2(\tau)$ is evaluated for different values of τ, ω and M for Sc=1000. Fig.–3.4 shows the plot of $D_2(\tau)$ with τ for various values of ω with fixed M. Fig.–3.4 shows irregular oscillation for higher values of ω. Similarly Fig.–3.5 depicts the graph of $D_2(\tau)$ with τ for different values of M with fixed ω. It is interesting to note that no. of oscillation is more for large ω but amplitude of oscillation decreases as M increases. Fig.–3.6 shows the variation of $D_2(\tau)$ with ω for different values of M.

The amplitude of oscillation is less for higher values of ω and the nature of oscillation becomes regular as ω increases. Thus we can conclude that frequency of oscillation has great influence on steady and unsteady part of dispersion coefficient.

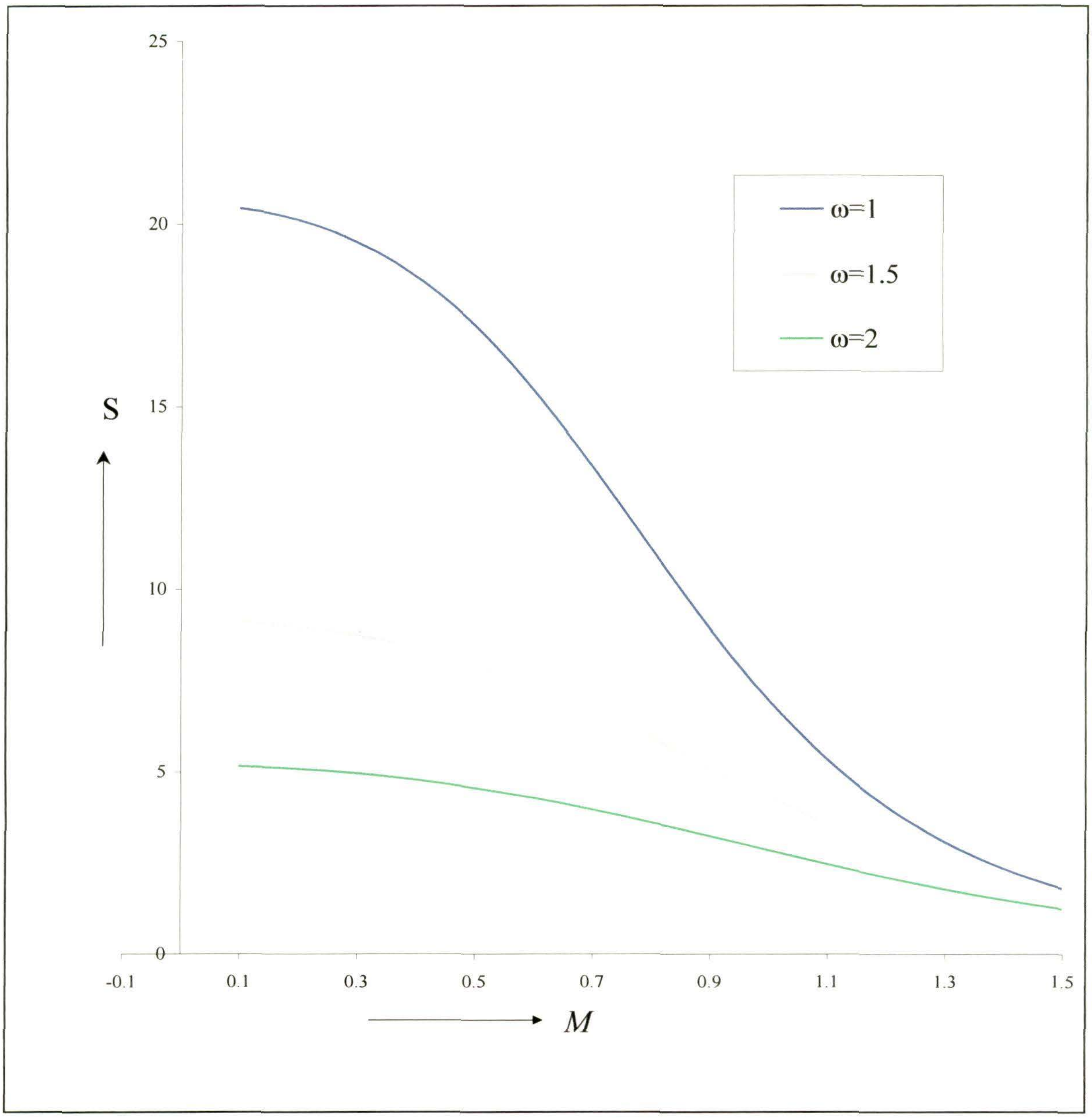

Fig. **3.2 S decreases with increase in ω for fixed M.**

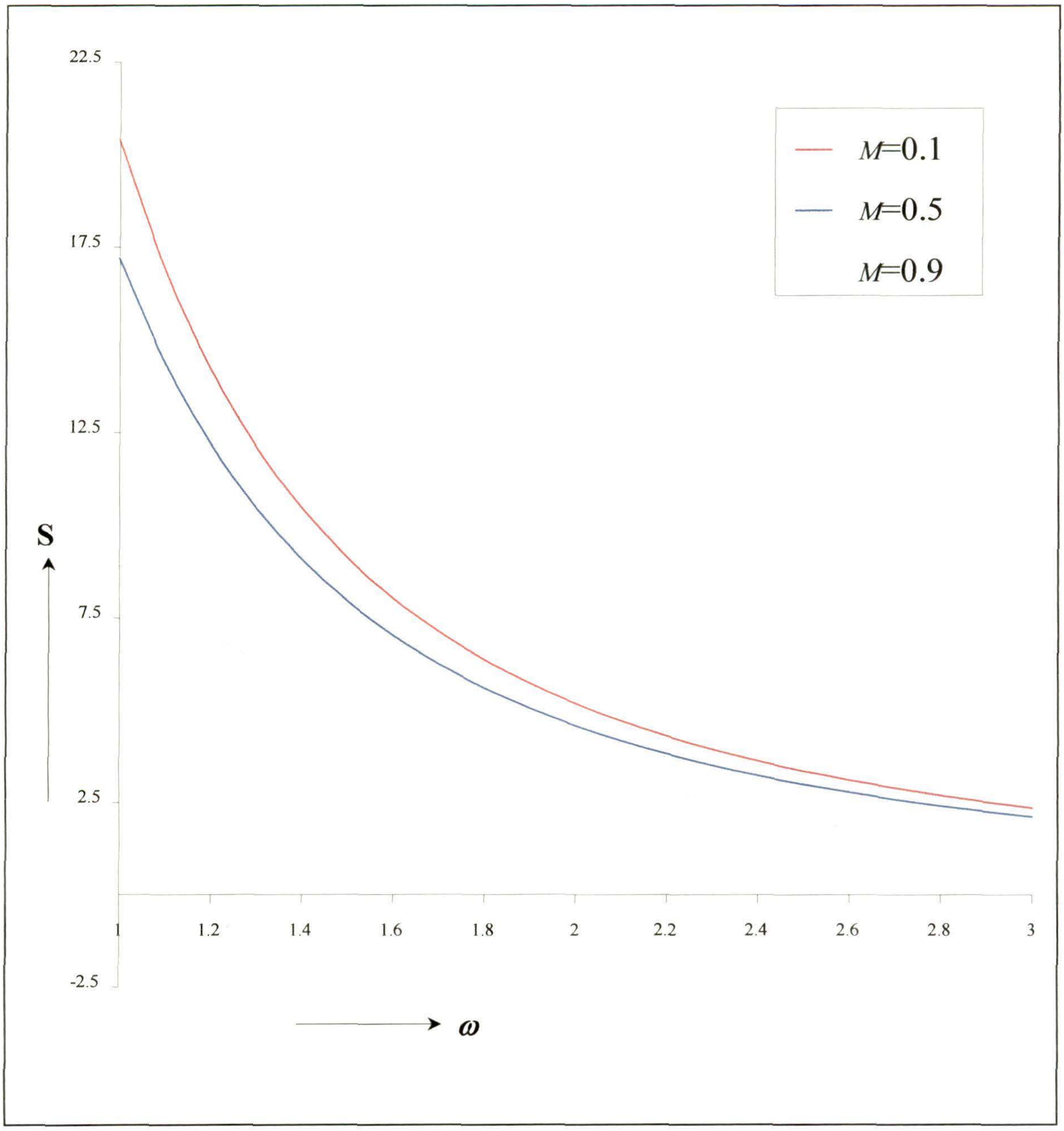

***Fig*. 3.3 S decreases slowly with increase in M forω, but decreases rapidly for M> 0.5.**

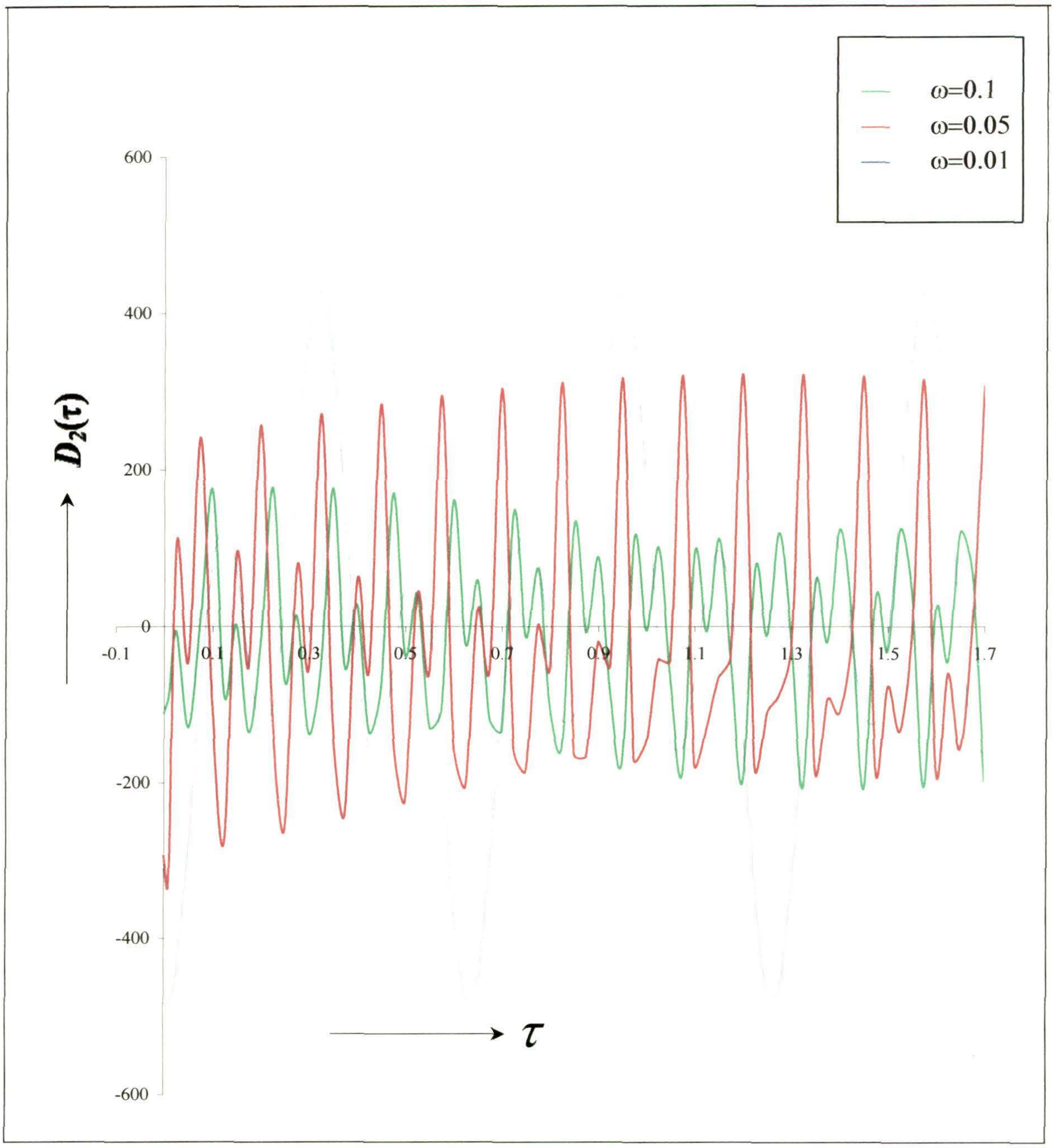

***Fig.* 3.4 Variation of $D_2(\tau)$ vs. τ for different values of ω with M =0.3, Sc=1000.**

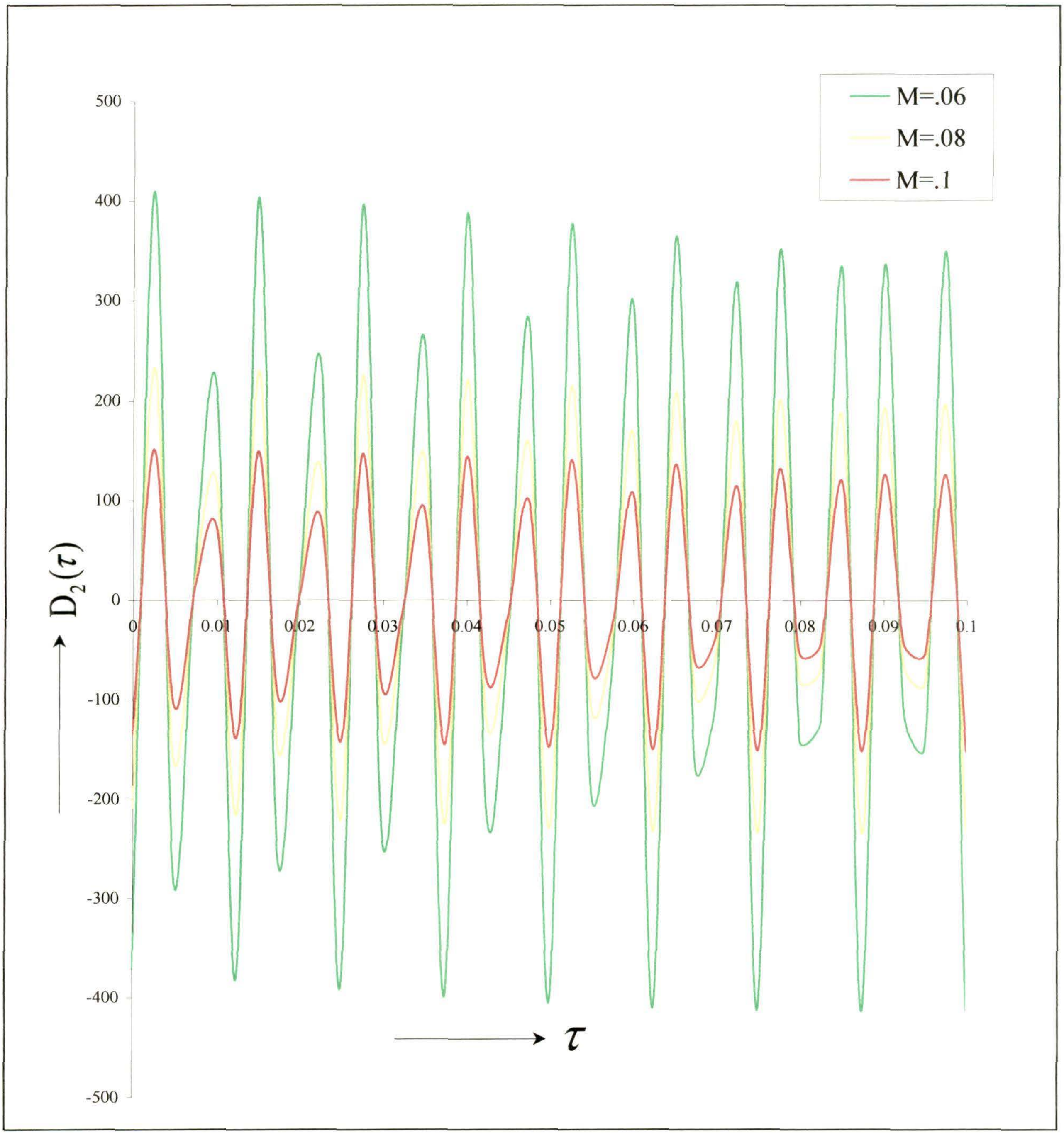

***Fig.* 3.5 Variation of D_2 (τ) vs. τ for different values of M with ω=1, Sc=1000**

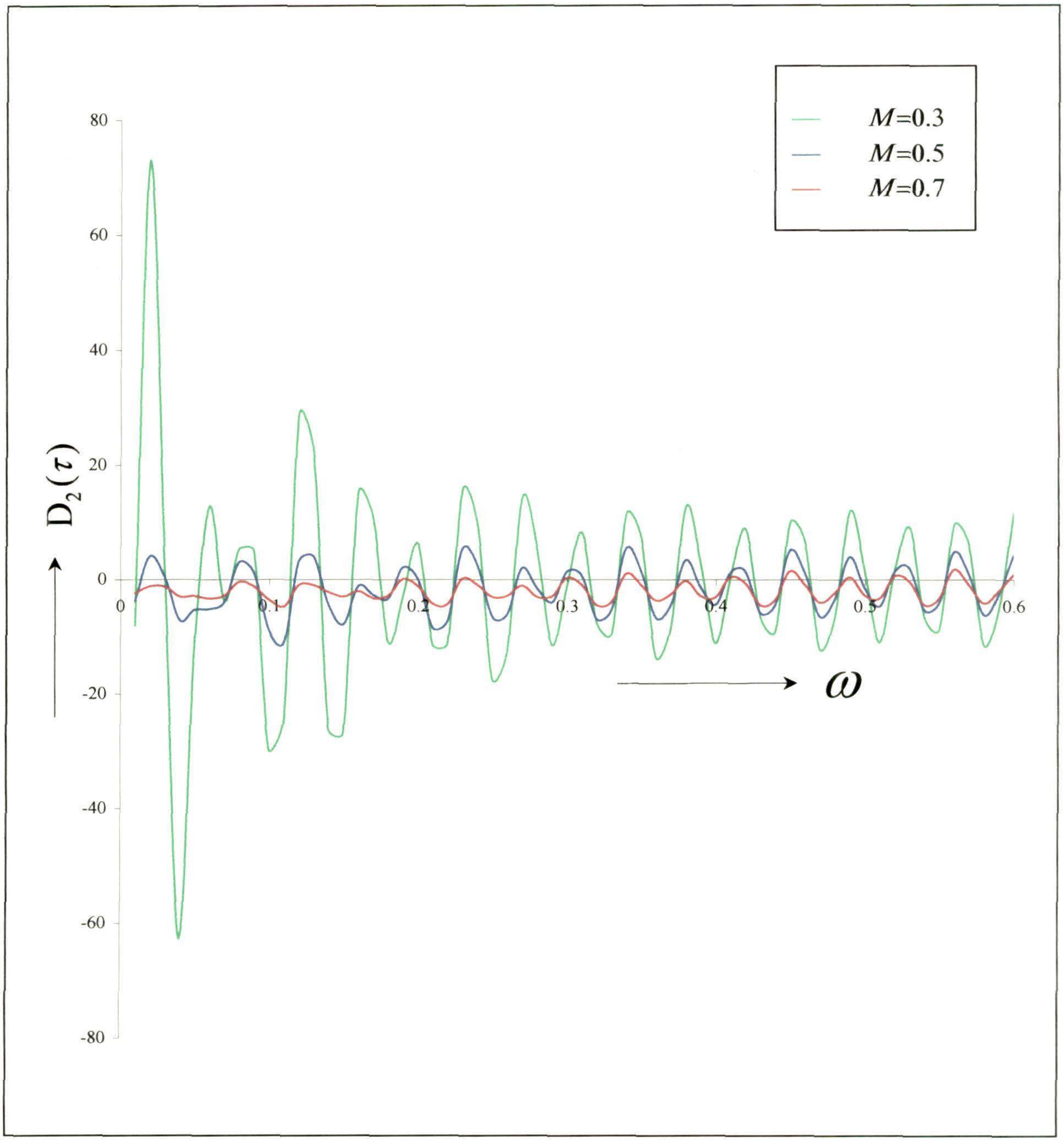

***Fig*. 3.6 Variation of D_2 (τ) vs. ω for different values of M with τ=0.8, Sc=1000**

PART TWO

EXACT ANALYSIS OF UNSTEADY CONVECTIVE DIFFUSION OF SOLUTE IN TWO-LAYERED MHD FLOW THROUGH PARALLEL PLATES*

3.4 INTRODUCTION

The longitudinal dispersion of solute in a solvent flowing in a conduit is a phenomenon of wide application in chemical engineering, biomedical engineering, physiological fluid dynamics and environmental science. The basic principle underlying the dispersion theory is the spreading of a passive species in a flowing fluid due to combined effect of molecular diffusion and non- uniform velocity distribution.

The first fundamental study on dispersion was that of Taylor [1] who showed that if a solute is injected into a solvent flowing steadily in a straight tube, the combined effect of lateral molecular diffusion and variation of velocity over the cross-section would cause the solute ultimately to spread diffusively with the effective molecular diffusivity Deff. [Deff. = $a^2\omega_m^2/48D_m$], where D_m is molecular diffusivity, ω_m is the mean axial velocity and a is the radius of the tube. Aris [2] using the method of moments, showed that the effective molecular diffusivity would be Deff. $=D_m + a^2\omega_m^2/48D_m$ when the contribution of axial molecular diffusion is also taken into account.Gill [3] generalized Taylor-Aris's work by proposing a series expansion about the mean concentration to describe the local concentration distribution. In a subsequent analysis, Gill and Sankarasubramanian showed that the method of series solution mentioned earlier provides an exact solution for the unsteady convective diffusion problem for laminar flow in a circular tube if the coefficients in the dispersion model are obtained as suitable function of time t. This model is widely referred to as the generalized dispersion

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Published in the journal "Acta Ciencia Indica", Vol. XXXII M, No.2, 659 (2006).

model. Lightfoot [6] represented the flow of blood as suspension of cells in plasma and the flow of blood tends to be two-layered. Moreover, it is shown by Woodcock [7] that red cells have negative electric charge and blood can be considered as an electrically conducting fluid. Also from MHD studies, it is well known that magnetic field could be used to control the movement of charged particles. Hence, it is of interest to study dispersion of solute in two-layered model of the flow of blood between two parallel plates under the influence of magnetic field using generalized dispersion model proposed by Gill and Sankarasubramanian [3].

3.5 MATHEMATICAL FORMULATION

Consider the steady flow of blood between two horizontal parallel plates separated by a distance 2h apart, taking x-axis along the plates and y-axis perpendicular to it and a constant uniform magnetic field is applied along y-axis. At a large distance from the entry section, the flow will be fully developed and in

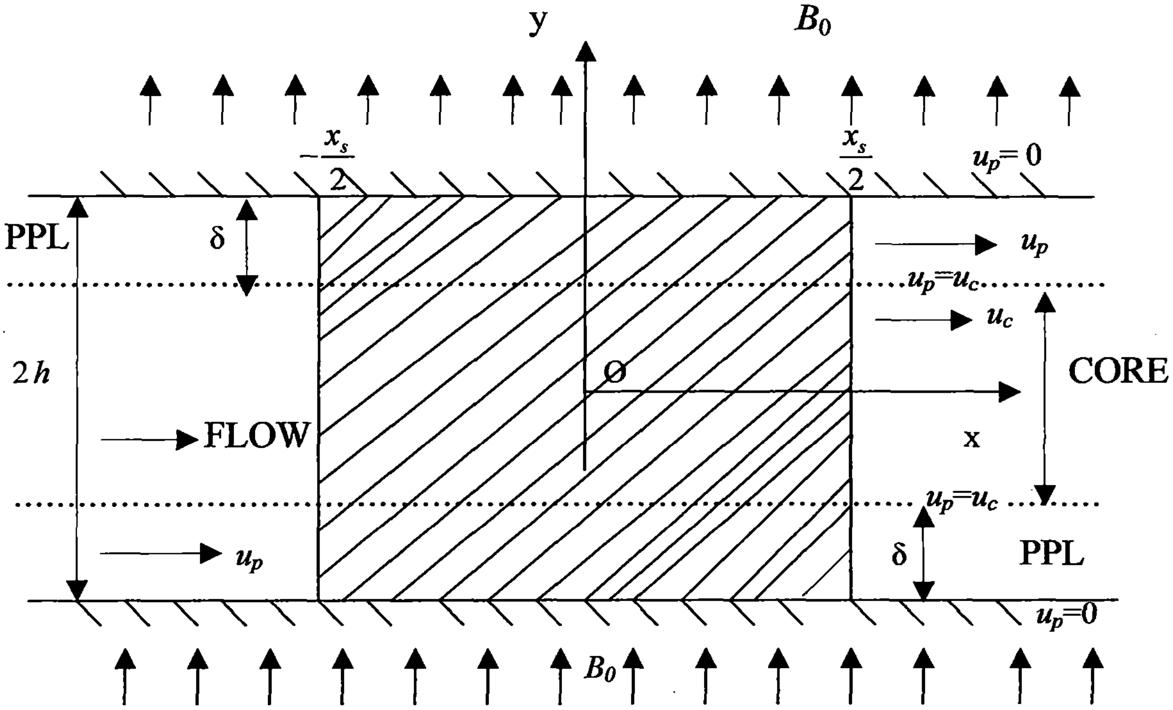

***Fig. 3.7.* Physical sketch of the problem**

The steady state all the physical variables except pressure will be function of y only. Further, blood is assumed to be Newtonian fluid. Since blood is a the suspension of red cells in plasma, the cells have a tendency to move away from the walls and this forms a peripheral plasma layer (PPL) near the wall and a core-region consisting of cells and plasma. Since the plasma is an electrically non-conducting fluid, the flow in PPL is not affected by the magnetic field; however, due to electric charges on red cells, the flow in the core region is influenced by the magnetic field.

The velocity field as obtained by Chaturani and Saxena [8] is given as

$$\begin{aligned} u_x &= u_c \qquad \text{when } -(h-\delta) \le y \le (h-\delta), \\ &= u_p \qquad \text{"} \quad -h \le y \le -(h-\delta),\ (h-\delta) \le y \le h, \end{aligned}$$

where $\eta = \dfrac{y}{h}$, $M = B_0 h\sqrt{\dfrac{\sigma}{\eta_c}}$ (M is a non-dimensional number, called Hartmann number);

$$u_c = \frac{P_0}{\sigma B_0^2} - \left\{ \frac{P_0}{\sigma B_0^2} + \frac{P_0 h^2}{2\eta_p}\left(\frac{\delta}{h}\right)\left(\frac{\delta}{h} - 2\right) \right\} \frac{\cosh M\eta}{\cosh M\left(1 - \dfrac{\delta}{h}\right)},$$

$$u_p = \frac{P_0 h^2}{2\eta_p}\left(\eta^2 - 1\right),$$

where u_c and u_p are velocities of fluid in the core-region and PPL respectively, σ is electrical conductivity of blood, B_0 is applied constant uniform magnetic field(Fig. 3. 7), η_c and η_p are viscosities of the fluid in the core-region and PPL respectively, $P_0(= -\partial p/\partial x)$ is pressure gradient in the x-direction and δ is the thickness of PPL.

If a solute diffuses in the above fully developed flow, the concentration $c(t,x,y)$ of the solute satisfies

$$\frac{\partial c}{\partial t} = D\nabla^2 c - u_x \frac{\partial c}{\partial x}, \qquad \text{.... (3.39)}$$

where D is the molecular diffusivity.

We introduce the dimensionless variables as

$$\left.\begin{array}{lll} \theta = \dfrac{c}{c_0}, & X = \dfrac{Dx}{h^2 \bar{u}}, & \tau = \dfrac{Dt}{h^2}, \\ \eta = \dfrac{y}{h}, & X_s = \dfrac{Dx_s}{h^2 \bar{u}}, & Pe = \dfrac{\bar{u}h}{D} \end{array}\right\}, \qquad \text{.... (3.40)}$$

where c_0 is the concentration of initial slag input and $\bar{u} = \frac{1}{2}\int_{-1}^{1} u_x d\eta$ is the average velocity.

We introduce a new axial co-ordinate moving with the average velocity $\bar{u}$ as $x_1 = x - \bar{u}t$ which in dimensionless form is given as

$$\xi = X - \tau, \quad \text{where } \xi = \frac{Dx_1}{h^2 \bar{u}}. \qquad \text{.... (3.41)}$$

Using (3.40) and (3.41) in (3.39), we get

$$\frac{\partial \theta}{\partial \tau} + \Psi(\eta)\frac{\partial \theta}{\partial \xi} = \frac{1}{Pe^2}\frac{\partial^2 \theta}{\partial \xi^2} + \frac{\partial^2 \theta}{\partial \eta^2}, \qquad \text{.... (3.42)}$$

where $\Psi(\eta) = \frac{u_x}{\bar{u}} - 1$.

$$\Psi(\eta) = \frac{LL_1 - L^2(1 - \frac{L}{3}) + \frac{L_2}{M}\tanh M(1-L) - L_2 \dfrac{\cosh M\eta}{\cosh M(1-L)}}{Q},$$

$$\text{when} \quad -(1-L) \le \eta \le (1-L). \qquad \text{.... (3.43a)}$$

$$= \frac{(1-\eta^2) - L^2(1-\frac{L}{3}) - L_1(1-L) + \frac{L_2}{M}\tanh M(1-L)}{Q},$$

when $-1 \le \eta \le -(1-L),\ (1-L) \le \eta \le 1.$ .... (3.43b)

Velocity field in non-dimensional form is

$$\left.\begin{aligned} U(\eta) &= L_1 - L_2\frac{\cosh M\eta}{\cosh M(1-L)} && \text{when } -(1-L) \le \eta \le (1-L) \\ &= 1-\eta^2 && \text{when } -1 \le \eta \le -(1-L), (1-L) \le \eta \le 1 \end{aligned}\right\}, \quad .... (3.44)$$

where $L = \frac{\delta}{h}$; $L_1 = \frac{2\eta_p}{\sigma B_0{}^2 h^2}$; $L_2 = L_1 + L(L-2)$;

$$Q = L^2\left(1-\frac{L}{3}\right) + L_1(1-L) - \frac{L_2}{M}\tanh M(1-L).$$

The initial and boundary conditions for (3.42) are

$$\left.\begin{aligned} &\theta(0,\xi,\eta) = 1 \text{ for } |\xi| \le \frac{1}{2}\xi_s, \\ &\theta(0,\xi,\eta) = 0 \ \text{ " } |\xi| > \frac{1}{2}\xi_s \\ &\theta(0,\infty,\eta) = 0 \\ &\text{and } \frac{\partial\theta}{\partial\eta}(\tau,\xi,-1) = \frac{\partial\theta}{\partial\eta}(\tau,\xi,1) = 0 \end{aligned}\right\}. \quad .... (3.45)$$

where the last two conditions are consistent with the no mass at the channel walls.

3.6 SOLUTION

We now assume that the solution of equation (3.42) is formulated as a series expansion in $\frac{\partial^k \theta_m}{\partial \xi^k}$

such that

$$\theta = \theta_m(\tau,\xi) + \sum_{k=1}^{\infty} f_k(\tau,\eta)\frac{\partial^k \theta_m}{\partial \xi^k}, \qquad \ldots.\ (3.46)$$

where $\theta_m = \frac{1}{2}\int_{-1}^{1} \theta d\eta.$

Substitution of (3.46) in (3.42) gives

$$\frac{\partial \theta_m}{\partial \tau} + \Psi(\eta)\frac{\partial \theta_m}{\partial \xi} - \frac{1}{Pe^2}\frac{\partial^2 \theta_m}{\partial \xi^2} + \sum_{k=1}^{\infty}\Bigg[\left(\frac{\partial f_k}{\partial \tau} - \frac{\partial^2 f_k}{\partial \eta^2}\right)\frac{\partial^k \theta_m}{\partial \xi^k} + \psi(\eta)\frac{\partial^{k+1}\theta_m}{\partial \xi^{k+1}},$$

$$- \frac{1}{Pe^2} f_k \frac{\partial^{k+2}\theta_m}{\partial \xi^{k+2}} + f_k \frac{\partial^{k+1}\theta_m}{\partial \tau \partial \xi^k}\Bigg] = 0. \qquad \ldots.\ (3.47)$$

Following Gill and Sankarasubramanian, we can introduce the generalized dispersion model with time dependent dispersion coefficients as

$$\frac{\partial \theta_m}{\partial \tau} = \sum_{i=1}^{\infty} K_i(\tau)\frac{\partial^i \theta_m}{\partial \xi^i}. \qquad \ldots.\ (3.48)$$

This gives

$$\frac{\partial^{k+1}\theta_m}{\partial \tau \partial \xi^k} = \sum_{i=1}^{\infty} K_i(\tau)\frac{\partial^{i+k}\theta_m}{\partial \xi^{i+k}}. \qquad \ldots.\ (3.49)$$

Introduction of (3.48) and (3.49) into (3.47) and rearrangement of terms give

$$\left[\frac{\partial f_1}{\partial \tau} - \frac{\partial^2 f_1}{\partial \eta^2} + \Psi(\eta) + K_1(\tau)\right]\frac{\partial \theta_m}{\partial \xi} + \Bigg[\frac{\partial f_2}{\partial \tau} - \frac{\partial^2 f_2}{\partial \eta^2} + \psi(\eta) f_1 + f_1 K_1 + K_2$$

$$- \frac{1}{Pe^2}\Bigg]\frac{\partial^2 \theta_m}{\partial \xi^2} + \sum_{k=1}^{\infty}\Bigg[\frac{\partial f_{k+2}}{\partial \tau} - \frac{\partial^2 f_{k+2}}{\partial \eta^2} + \psi(\eta) f_{k+1} + f_{k+1}K_1 + (K_2 - \frac{1}{Pe^2}) f_k$$

$$-\sum_{i=3}^{k+2} K_i f_{k+2-i}\Bigg]\frac{\partial^{k+2}\theta_m}{\partial\xi^{k+2}}=0 \qquad \text{.... (3.50)}$$

with the understanding that $f_0 = 1$.

Equating the coefficients of $\partial^k\theta_m/\partial\xi^k$ (k=1,2,3...........) to zero, an infinite set of differential equations are obtained as

$$\frac{\partial f_1}{\partial\tau}=\frac{\partial^2 f_1}{\partial\eta^2}-\psi(\eta)-K_1(\tau), \qquad \text{.... (3.51)}$$

$$\frac{\partial f_2}{\partial\tau}=\frac{\partial^2 f_2}{\partial\eta^2}-\psi(\eta)f_1-f_1K_1(\tau)+K_2(\tau)-\frac{1}{Pe^2}, \qquad \text{.... (3.52)}$$

$$\vdots$$

and $$\frac{\partial f_{k+2}}{\partial\tau}=\frac{\partial^2 f_{k+2}}{\partial\eta^2}-\psi(\eta)f_{k+1}-K_1(\tau)-f_{k+1}$$

$$-\left\{K_2(\tau)-\frac{1}{Pe^2}\right\}f_k-\sum_{i=3}^{k+2}K_i(\tau)f_{k+2-i}, \quad \text{where } k=1,2,3,\ldots\ldots\ldots. \qquad \text{.... (3.53)}$$

Now θ_m will be chosen to satisfy the initial conditions on θ given by (3.45).

The conditions on $f_k(\tau,\eta)$ are given as

$$f_k(0,\eta)=0, \quad \frac{\partial f_k}{\partial\eta}_{[\tau,\pm1]}=0, \quad \text{where } k=1,2,3,4,\ldots\ldots\ldots. \qquad \text{.... (3.54)}$$

Further equations (3.46) and (3.47) require that

$$\int_{-1}^{1} f_k\,d\eta=0, \qquad k=1,2,3,\ldots\ldots\ldots\ldots. \qquad \text{.... (3.55)}$$

Integrating (3.51) w. r. to η from -1 to 1 and using (3.54) and (3.55) we get

$$K_1(\tau) = 0. \qquad \text{.... (3.56)}$$

Following the same procedure, we find from (3.52), (3.54) and (3.55), the expression for $K_2(\tau)$ as

$$K_2(\tau) = \frac{1}{Pe^2} - \frac{1}{2}\int_{-1}^{1} \psi(\eta) f_1 d\eta. \qquad \text{.... (3.57)}$$

From (3.53), using $f_0 = 1$ we obtain in a similar manner

$$K_{k+2}(\tau) = -\frac{1}{2}\int \psi(\eta) f_{k+1}(\eta) d\eta\ , \qquad k = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots . \qquad \text{.... (3.58)}$$

with $K_1(\tau)=0$, we then solve (3.51) subject to (3.54) and (3.55) by using Duhamel's theorem and get the expression for f_1 as

$$\left.\begin{aligned} f_1(\tau,\eta) &= \frac{1}{Q}\left\{L_1\frac{\eta^2}{2} - \frac{L_2\cosh M\eta}{M^2\cosh M(1-L)}\right\} - \frac{\eta^2}{2} + C_1\eta \\ &+ C_2 + \sum_{n=1}^{\infty} d_n e^{-\lambda_n^2\tau}\cos\lambda_n\eta, \qquad \text{when } -(1-L) \le \eta \le (1-L). \\ &= \frac{1}{Q}\left\{\frac{\eta^2}{2} - \frac{\eta^4}{12}\right\} - \frac{\eta^2}{2} + C_3 + \sum_{n=1}^{\infty} d_n e^{-\lambda_n^2\tau}\cos\lambda_n\eta, \\ &\qquad \text{when } -1 \le \eta \le -(1-L), (1-L) \le \eta \le 1. \end{aligned}\right\} \qquad \text{.... (3.59)}$$

where

$$C_1 = \frac{1}{Q}\left[\frac{2}{3} - L^2 + \frac{L^3}{3} - L_1(1-L) + \frac{L_2}{M}\tanh M(1-L)\right],$$

$$\begin{aligned} C_2 = \frac{1}{Q}\Bigg[& L\left\{1 - \frac{(1-L)^2}{6} - L_1\right\}\frac{(1-L)^2}{2} + \frac{LL_2}{M^2} - \frac{3}{40} - \frac{L_1(1-L)^3}{12}\left\{\frac{(1-L)^2}{10} - 1\right\} \\ & - \frac{L_1(1-L)^3}{6} + \frac{L_2}{M^3}\tanh M(1-L)\Bigg] + \frac{1}{6} - C_1(1-L)L, \end{aligned}$$

$$C_3 = C_2 - \frac{1}{Q}\left[\left\{1-\frac{(1-L)^2}{6}-L_1\right\}\frac{(1-L)^2}{2}+\frac{L_2}{M^2}\right]+C_1(1-L),$$

$$d_n = \frac{1}{Q}\left[\left\{\frac{(1-L)^2}{\lambda_n}-\frac{2}{\lambda_n^3}-\frac{(1-L)^4}{6\lambda_n}+\frac{2(1-L)^2}{\lambda_n^3}+\frac{4}{\lambda_n^5}-\frac{L_1(1-L)^2}{\lambda_n}+\frac{2L_1}{\lambda_n^3}\right\}\right.$$

$$\times\sin\lambda_n(1-L)-\left\{\frac{2(1-L)^3}{3\lambda_n^2}-\frac{2(1-L)}{\lambda_n^2}-\frac{4(1-L)}{\lambda_n^4}+\frac{2L_1(1-L)}{\lambda_n^2}\right\}\cos\lambda_n(1-L)$$

$$+\frac{L_2\left\{\lambda_n\sin\lambda_n(1-L)\cosh M(1-L)+M\sinh M(1-L)\cos\lambda_n(1-L)\right\}}{M^2(\lambda_n^2+M^2)\cosh M(1-L)}$$

$$\left.-\left(\frac{4}{3\lambda_n^2}+\frac{4}{\lambda_n^4}\right)\cos\lambda_n\right]-\frac{2(C_2-C_3)}{\lambda_n}\sin\lambda_n(1-L).$$

Substituting (3.59) in (3.57), the expression for $K_2(\tau)$ is obtained as

$$K_2(\tau)-\frac{1}{Pe^2}=-\frac{1}{2Q}\left[\frac{2}{Q}\left\{\frac{13}{210}-\frac{(1-L)^3}{6}+\frac{7(1-L)^5}{60}-\frac{(1-L)^7}{84}\right\}+\frac{(1-L)^3}{3}\right.$$

$$-\frac{(1-L)^5}{5}+2C_3L+\frac{2C_3}{3}(1-L)^3-\frac{2C_3}{3}-\frac{2}{15}+\frac{2}{Q}\left\{\frac{L_1(1-L)^3}{6}\right.$$

$$\left.-\frac{L_1L_2}{M^3}\tanh M(1-L)\right\}-\frac{L_1(1-L)^3}{3}+2C_2L_1(1-L)-\frac{2L_2}{Q}$$

$$\times\left\{\frac{L_1(1-L)^2\tanh M(1-L)}{2M}-\frac{L_1(1-L)}{M^2}+\frac{L_1}{M^3\tanh M(1-L)}\right.$$

$$\left.\left.-\frac{L_2(1-L)\sec h^2M(1-L)}{2M^2}-\frac{L_2}{4M^3}\tanh M(1-L)\sec hM(1-L)\right\}\right]$$

$$-\frac{1}{Q}\sum_{n=1}^{\infty} d_n e^{-\lambda_n^2\tau}\left[\frac{2(1-L)}{\lambda_n^2}\cos\lambda_n(1-L)-\frac{2}{\lambda_n^2}\cos\lambda_n-\left\{\frac{1}{\lambda_n}-\frac{(1-L)^2}{\lambda_n}+\frac{2}{\lambda_n^3}\right\}\right.$$

$$\times\sin\lambda_n(1-L)+\sin\frac{\lambda_n(1-L)}{\lambda_n}-L_2\frac{M\lambda_n\cos\lambda_n(1-L)\tanh M(1-L)}{M^2+\lambda_n^2}$$

$$\left.+\frac{\lambda_n^2\sin\lambda_n(1-L)}{M^2+\lambda_n^2}\right]. \qquad ....\ (3.60)$$

For large τ, the equation (3.60) reduces to the form

$$K_2^*(\infty)=K_2(\infty)-\frac{1}{Pe^2}$$

$$=-\frac{1}{2Q}\left[\frac{2}{Q}\left\{\frac{13}{210}-\frac{(1-L)^3}{6}+\frac{7(1-L)^5}{60}-\frac{(1-L)^7}{84}\right\}+\frac{(1-L)^3}{3}-\frac{(1-L)^5}{5}\right.$$

$$+\frac{4C_3L}{3}+\frac{2C_3(1-L)^3}{3}-\frac{2}{15}+\frac{2}{Q}\left\{\frac{L_1(1-L)^3}{6}-\frac{L_1L_2}{M^3}\tanh M(1-L)\right\}$$

$$-\frac{L_1(1-L)^3}{3}+2C_2L_1(1-L)-\frac{2L_2}{Q}\left\{\frac{L_1(1-L)^2}{2M}\tanh M(1-L)-\frac{L_1(1-L)}{M^2}\right.$$

$$+\frac{L_1}{M^3}\tanh M(1-L)-\frac{L_2}{2M^2}(1-L)\sec h^2M(1-L)$$

$$\left.-\frac{L_2}{4M^3}\tanh M(1-L)\sec hM(1-L)\right\}+2L_2\left\{\frac{\tanh M(1-L)}{M^3}-\frac{(1-L)}{M^2}\right\}$$

$$\left.-\frac{2C_2L_2}{M}\tanh M(1-L)\right]. \qquad ....\ (3.61)$$

The asymptotic form for $f_2(\tau,\eta)$ as $\tau\to\infty$ is obtained from (3.52) as the solution of

$$\frac{d^2 f_2}{d\eta^2}-\psi(\eta)f_1-K_2(\tau)+\frac{1}{Pe^2}=0, \qquad \text{.... (3.62)}$$

where $K_1(\tau)=0$.

Substituting the limiting form of $f_1(\tau,\eta)$ and $K_2(\tau)$ as $\tau\to\infty$ form (3.59) and (3.60) respectively and using the conditions (3.54) and (3.55) for $k=2$,we get the solution of (3.62) as

$$f_2(\eta)=(\frac{1}{Q}-1)\left[\frac{1}{Q}\left\{\frac{\eta^4}{24}-\frac{\eta^6}{360}\right\}-\frac{\eta^4}{24}+C_3\frac{\eta^2}{2}\right]-\frac{1}{Q}\left[\frac{1}{Q}\left\{\frac{\eta^6}{60}-\frac{\eta^8}{672}\right\}-\frac{\eta^6}{60}\right.$$

$$\left.+\frac{C_3\eta^4}{12}\right]+K_2^*(\infty)+\alpha_1. \quad \text{when } -1\le\eta\le-(1-L),(1-L)\le\eta\le 1. \qquad \text{.... (3.63)}$$

$$=\frac{1}{Q}\left[\frac{L_1}{Q}\left\{\frac{L_1\eta^4}{24}-\frac{L_2\cosh M\eta}{M^4\cosh M(1-L)}\right\}-\frac{L_1}{24}\eta^4+\frac{L_1C_1}{6}\eta^3+\frac{L_1C_2}{2}\eta^2\right]$$

$$-\frac{L_2}{Q\cosh M(1-L)}\left[\frac{1}{Q}\left\{L_1\left(\frac{\eta^2}{2M^2}\cosh M\eta-\frac{2\eta}{M^3}\sinh M\eta+\frac{3\cosh M\eta}{M^4}\right)\right.\right.$$

$$\left.-\frac{L_2}{2M^2\cosh M(1-L)}\left(\frac{\eta^2}{2}+\frac{\cosh 2M\eta}{4M^2}\right)\right\}-\left(\frac{\eta^2}{2M^2}\cosh M\eta\right.$$

$$\left.\left.-\frac{2\eta}{M^3}\sinh M\eta+\frac{3\cosh M\eta}{M^4}\right)+C_1\left(\frac{\eta\cosh M\eta}{M^2}-\frac{2\sinh M\eta}{M^3}\right)+\frac{C_2}{M^2}\cosh M\eta\right]$$

$$-\left[\frac{1}{Q}\left\{\frac{L_1\eta^4}{24}-\frac{L_2\cosh M\eta}{M^4\cosh M(1-L)}\right\}-\frac{\eta^4}{24}+\frac{C_1}{6}\eta^3+\frac{C_2}{2}\eta^2\right]$$

$$+K_2^{*}(\infty)\frac{\eta^2}{2}+C''\eta+\alpha_2, \qquad \text{when } -(1-L)\le\eta\le(1-L). \qquad \ldots.\ (3.64)$$

where

$$\alpha_2=\frac{C_2(1-L)^3}{6}+\frac{1}{Q}\Bigg\{\frac{L_1(1-L)^5}{120}-\frac{L_2}{M^5}\tanh M(1-L)\Bigg\}+\frac{L_2}{Q}\Bigg[\bigg(\frac{L_1}{Q}-1\bigg)$$

$$\times\Bigg\{\frac{(1-L)^2}{2M^3}\tanh M(1-L)-\frac{3(1-L)}{M^4}+\frac{6}{M^5}\tanh M(1-L)\Bigg\}$$

$$-\frac{L_2}{2M^2Q\cosh^2 M(1-L)}\Bigg\{\frac{(1-L)^3}{6}+\frac{\sinh 2M(1-L)}{8M^3}\Bigg\}+\frac{C_2}{M^3}\tanh M(1-L)\Bigg]$$

$$+\frac{1}{Q}\Bigg[\frac{67}{30240Q}-\frac{1}{420}+\frac{C_3}{60}\Bigg]-\frac{(1-L)^5}{120}-\frac{1}{Q}\Bigg[\bigg(\frac{L_1}{Q}-1\bigg)\frac{L_1(1-L)^5}{120}$$

$$-\frac{L_1L_2}{M^5Q}\tanh M(1-L)+\frac{L_1C_2(1-L)^3}{3}\Bigg]-\frac{K_2^{*}(\infty)}{6}-\bigg(\frac{1}{Q}-1\bigg)$$

$$\times\Bigg[\frac{209}{2520Q}-\frac{1}{120}+\frac{C_3}{6}\Bigg]-\frac{1}{Q}\Bigg[\bigg(\frac{1}{Q}-1\bigg)\frac{(1-L)^7}{420}-\frac{(1-L)^9}{6048Q}+\frac{C_3(1-L)^5}{60}\Bigg]$$

$$+\bigg(\frac{1}{Q}-1\bigg)\Bigg[\bigg(\frac{1}{Q}-1\bigg)\frac{(1-L)^5}{120}-\frac{(1-L)^7}{2520Q}+\frac{C_3(1-L)^3}{6}\Bigg]-LA,$$

$$A=\frac{1}{Q}\Bigg[\bigg(\frac{L_1}{Q}-1\bigg)\frac{L_1}{24}(1-L)^4-\frac{L_1L_2}{M^4Q}+\frac{L_1C_1(1-L)^3}{6}+\frac{L_1C_2(1-L)^2}{2}\Bigg]$$

$$-\frac{L_2}{Q}\Bigg[\bigg(\frac{L_1}{Q}-1\bigg)\Bigg\{\frac{(1-L)^2}{2M^2}-\frac{2(1-L)}{M^3}\tanh M(1-L)+\frac{3}{M^4}\Bigg\}$$

$$-\frac{L_2}{2M^2Q\cosh^2 M(1-L)}\Bigg\{\frac{(1-L)^2}{2}+\frac{\cosh 2M(1-L)}{4M^2}\Bigg\}+C_1\Bigg\{\frac{(1-L)}{M^2}$$

$$-\frac{2}{M^3}\tanh M(1-L)\Bigg\}+\frac{C_2}{M^2}\Bigg]-\Bigg[\left(\frac{L_1}{Q}-1\right)\frac{(1-L)^4}{24}-\frac{L_2}{M^4Q}+C_1\frac{(1-L)^3}{6}$$

$$+\frac{C_2(1-L)^2}{2}\Bigg]+\frac{K_2^*(\infty)(1-L)^2}{2}+C''(1-L)-\left(\frac{1}{Q}-1\right)\Bigg[\left(\frac{1}{Q}-1\right)\frac{(1-L)^4}{24}$$

$$-\frac{(1-L)^6}{360Q}+\frac{C_3(1-L)^2}{2}\Bigg]+\frac{1}{Q}\Bigg[\left(\frac{1}{Q}-1\right)\frac{(1-L)^6}{60}-\frac{(1-L)^8}{672Q}+\frac{C_3(1-L)^4}{12}\Bigg]$$

$$-K_2^*(\infty)\frac{(1-L)^2}{2}$$

$$\alpha_1=\alpha_2+A,$$

$$C''=\left(\frac{1}{Q}-1\right)\Bigg[\frac{1}{Q}\left\{\frac{(1-L)^3}{6}-\frac{(1-L)^5}{60}\right\}-\frac{(1-L)^3}{6}+C_3(1-L)\Bigg]-\frac{1}{Q}\Bigg[\frac{1}{Q}\Bigg\{\frac{(1-L)^5}{10}$$

$$-\frac{(1-L)^7}{84}\Bigg\}-\frac{(1-L)^5}{10}+\frac{C_3(1-L)^3}{3}\Bigg]-\frac{1}{Q}\Bigg[\frac{L_1}{Q}\left\{\frac{L_1(1-L)^3}{6}-\frac{L_2}{M^3}\tanh M(1-L)\right\}$$

$$-\frac{L_1(1-L)^3}{6}+\frac{L_1C_1(1-L)^2}{2}+L_1C_2(1-L)\Bigg]+\frac{L_2}{Q}\Bigg[\frac{1}{Q}\Bigg\{\frac{L_1(1-L)^2}{2M}\tanh M(1-L)-\frac{L_1(1-L)}{M^2}$$

$$+\frac{L_1}{M^3}\tanh M(1-L)-\frac{L_2}{2M^2}(1-L)\sec hM(1-L)-\frac{L_2}{4M^3}\tanh M(1-L)\Bigg\}$$

$$-\frac{(1-L)^2}{2M}\sinh M(1-L)+\frac{(1-L)}{M^2}\cosh M(1-L)-\frac{\sinh M(1-L)}{M^3}$$

$$+C_1\left\{\frac{(1-L)}{M}\sinh M(1-L)-\frac{\cosh M(1-L)}{M^2}\right\}+\frac{C_2}{M}\sinh M(1-L)\Bigg]$$

$$+\left[\frac{1}{Q}\left\{\frac{L_1(1-L)^3}{6}-\frac{L_2}{M^3}\tanh M(1-L)\right\}-\frac{(1-L)^3}{6}+\frac{C_1(1-L)^2}{2}+C_2(1-L)\right].$$

Substituting the value of $f_2(\eta)$ in (3.58) with $k=1$, the asymptotic form of $K_3(\tau)$ as $\tau \to \infty$ is given by

$$K_3(\infty)=\left(\frac{1}{Q}-1\right)\Bigg[\left(\frac{1}{Q}-1\right)\Bigg\{\left(\frac{1}{Q}-1\right)\frac{(1-L)^5}{120}-\frac{(1-L)^7}{2520Q}+\frac{C_3(1-L)^3}{6}-\frac{1}{126Q}$$

$$+\frac{1}{120}-\frac{C_3}{6}\Bigg\}-\frac{1}{Q}\Bigg\{\left(\frac{1}{Q}-1\right)\frac{(1-L)^7}{420}-\frac{(1-L)^9}{6048Q}+\frac{C_3(1-L)^5}{60}+\frac{67}{30240Q}$$

$$-\frac{1}{420}+\frac{C_3}{60}\Bigg\}+K_2^*(\infty)\left\{\frac{(1-L)^3}{6}-\frac{1}{6}\right\}-\alpha_1 L\Bigg]-\left(\frac{L_1}{Q}-1\right)\Bigg[\frac{1}{Q}\Bigg\{(\frac{L_1}{Q}-1)$$

$$\times\frac{L_1(1-L)^5}{120}-\frac{L_1L_2}{M^5Q}\tanh M(1-L)+\frac{L_1C_2(1-L)^3}{6}\Bigg\}-\frac{L_2}{Q}\Bigg\{\left(\frac{1}{Q}-1\right)\Bigg(\frac{(1-L)^2}{2M^3}$$

$$\times\tanh M(1-L)-\frac{3(1-L)}{M^4}+\frac{6}{M^5}\tanh M(1-L)\Bigg)-\frac{L_2}{2M^2Q\cosh^2 M(1-L)}$$

$$\times\left(\frac{(1-L)^3}{6}+\frac{\sinh 2M(1-L)}{8M^3}\right)+\frac{C_2}{M^3}\tanh M(1-L)\Bigg\}-\Bigg(\left(\frac{1}{Q}-1\right)\frac{(1-L)^5}{120}$$

$$-\frac{L_2}{M^5Q}\tanh M(1-L)+\frac{C_2(1-L)^3}{6}\Bigg)+\frac{K_2^*(\infty)(1-L)^3}{6}+\alpha_2(1-L)\Bigg]$$

$$+\frac{L_2}{Q}\Bigg[\left(\frac{1}{Q}-1\right)\frac{L_1}{24Q}\Bigg\{\frac{(1-L)^4}{M}\tanh M(1-L)-\frac{4}{M^2}(1-L)^3+\frac{12(1-L)^2}{M^3}$$

$$\times\tanh M(1-L)-\frac{24}{M^4}(1-L)+\frac{24}{M^5}\tanh M(1-L)\Bigg\}-\frac{L_1L_2}{2M^4Q^2}$$

$$\times\left\{(1-L)\sec h^{2}M(1-L)+\frac{\sinh 2M(1-L)}{2M\cosh^{2}M(1-L)}\right\}+\frac{L_1C_2}{Q}\left\{\frac{(1-L)^{2}}{2M}\right.$$

$$\left.\times\tanh M(1-L)-\frac{(1-L)}{M^{2}}+\frac{\tanh M(1-L)}{M^{3}}\right\}-\frac{L_2}{Q\cosh^{2}M(1-L)}$$

$$\times\left(\frac{L_1}{Q}-1\right)\left\{\frac{(1-L)^{3}}{12M^{2}}+\frac{(1-L)^{2}\sinh 2M(1-L)}{8M^{3}}-\frac{5(1-L)}{8M^{4}}\cosh 2M(1-L)\right.$$

$$\left.+\frac{17}{16M^{5}}\sinh 2M(1-L)+\frac{3(1-L)}{2M^{4}}\right\}+\frac{L_2^{2}}{2M^{2}Q^{2}\cosh^{2}M(1-L)}$$

$$\times\left\{\frac{(1-L)^{2}}{2M}\tanh M(1-L)-\frac{(1-L)}{M^{2}}+\frac{9}{8M^{3}}\tanh M(1-L)\right.$$

$$\left.+\frac{\sinh 3M(1-L)}{24M^{3}\cosh M(1-L)}\right\}-\frac{L_2C_2}{Q\cosh^{2}M(1-L)}\left\{\frac{(1-L)}{2M^{2}}+\frac{\sinh 2M(1-L)}{4M^{3}}\right\}$$

$$-\frac{1}{24}\left(\frac{L_1}{Q}-1\right)\left\{\frac{(1-L)^{4}}{M}\tanh M(1-L)-\frac{4(1-L)^{3}}{M^{2}}+\frac{12(1-L)^{2}}{M^{3}}\tanh M(1-L)\right.$$

$$\left.-\frac{24}{M^{4}}(1-L)+\frac{24}{M^{5}}\tanh M(1-L)\right\}+\frac{L_2}{2M^{4}Q\cosh^{2}M(1-L)}\left\{1-L\right.$$

$$\left.+\frac{\sinh 2M(1-L)}{2M}\right\}-C_2\left\{\frac{(1-L)^{2}}{2M}\tanh M(1-L)-\frac{(1-L)}{M^{2}}+\frac{\tanh M(1-L)}{M^{3}}\right\}$$

$$+K_2^{*}(\infty)\left\{\frac{(1-L)^{2}}{2M}\tanh M(1-L)-\frac{(1-L)}{M^{2}}+\frac{\tanh M(1-L)}{M^{3}}\right\}$$

$$\left.+\frac{\alpha_2\tanh M(1-L)}{M}\right]+\frac{1}{Q}\left[\left(\frac{1}{Q}-1\right)\left\{\frac{16}{2385Q}-\frac{1}{168}+\frac{C_3}{10}\right\}-\frac{1}{Q}\right.$$

$$\times\left\{\frac{571}{332640Q}-\frac{1}{540}+\frac{C_3}{84}\right\}+\frac{K_2^*(\infty)}{10}+\frac{\alpha_1}{3}\Bigg]. \qquad \text{.... (3.65)}$$

The foregoing procedure may be repeated and higher order dispersion coefficients may be determined to any order although computations will be more and more difficult. It is also noted that higher order dispersion coefficients decrease rapidly in magnitude. Neglecting $K_3$and higher order dispersion coefficients, equation (3.48) reduces to

$$\frac{\partial\theta_m}{\partial\tau}=K_2(\tau)\frac{\partial^2\theta_m}{\partial\xi^2}. \qquad \text{.... (3.66)}$$

Since a slug is being considered, θ_m will have to satisfy

$$\left.\begin{aligned}\theta_m(0,\xi)&=1 \quad \text{when } |\xi|\le\frac{1}{2}\xi_s\\ &=0 \quad \text{"} \quad\ |\xi|>\frac{1}{2}\xi_s\end{aligned}\right\}, \qquad \text{.... (3.67)}$$

$$\theta_m(\tau,\eta)=0. \qquad \text{.... (3.68)}$$

The solution of equation (3.66) subject to (3.51), (3.52), (3.53) is obtained as

$$\theta_m=\frac{1}{2}\left[erf\,\frac{\frac{1}{2}X_s-\xi}{2T_0^{\frac{1}{2}}}+erf\,\frac{\frac{1}{2}X_s-\xi}{2T_0^{\frac{1}{2}}}\right],$$

where $T_0=\int_0^\tau K_2(\eta)d\eta.$

3.7 RESULTS AND DISCUSSIONS

The time dependent nature of the dispersion coefficient K_2 is illustrated in Fig.-3.8. and Fig.-3.9. The plot of [$K_2(\tau)-Pe^{-2}$] versus time τ for dispersion in a pipe with different values of L (plug flow parameter) is shown in Fig.-3.8. whereas the variation of [$K_2(\tau)-Pe^{-2}$] with τ for different values of magnetic parameter M is shown in Fig.-3.9. It is observed from Fig.-3.8. that initially (τ =0.1) the values of dispersion coefficient [$K_2(\tau)-Pe^{-2}$] increase considerably with the increasing values of L [for fixed value of M]. It is also observed that the amount of dispersion becomes slower with the increasing values of L for a certain period of time $\tau<0.2$ approx. and after that as the time increases ($\tau>$ 0.6 approx.), the nature of dispersion is nearly same for all values L and attains a constant value after $\tau\geq0.6$. The time τ required to reach the steady state is seen to depend on plug flow parameter L. It is observed from the Fig.-3.9 that initially the values of dispersion coefficient [$K_2(\tau)-Pe^{-2}$] decrease with the increasing values of M. It is further observed that the nature of the graph is nearly same for all values of M and for the increasing values of $\tau>$ 0.35 the rate of dispersion becomes steady. In Fig. - 3.10. It is observed that the dispersion coefficient $K_2^*(\infty)$ increases upto a certain value of $L<0.3$ and then for increasing values of L the dispersion coefficient decreases. It is further observed that there is no effect of magnetic parameter M on the dispersion coefficient for L=1. In Fig. - 3.11. It is observed that the dispersion coefficient increases upto a certain value of $L<0.48$ approx. and then for increasing values of L the dispersion coefficient $K_3(\infty)$ decreases. It is also observed that there is no effect of magnetic parameter M on the dispersion coefficient for L=1.

The objective of the present investigations is to study the unsteady dispersion of a solute in the two-layered MHD flow through parallel plates due to its wide practical application. This study may be relevant in understanding many physiological processes, which involve injecting a quantity of solute into the concentration at some downstream point. This analysis can also be applied to artificial blood handling devices such as blood oxy-generators.

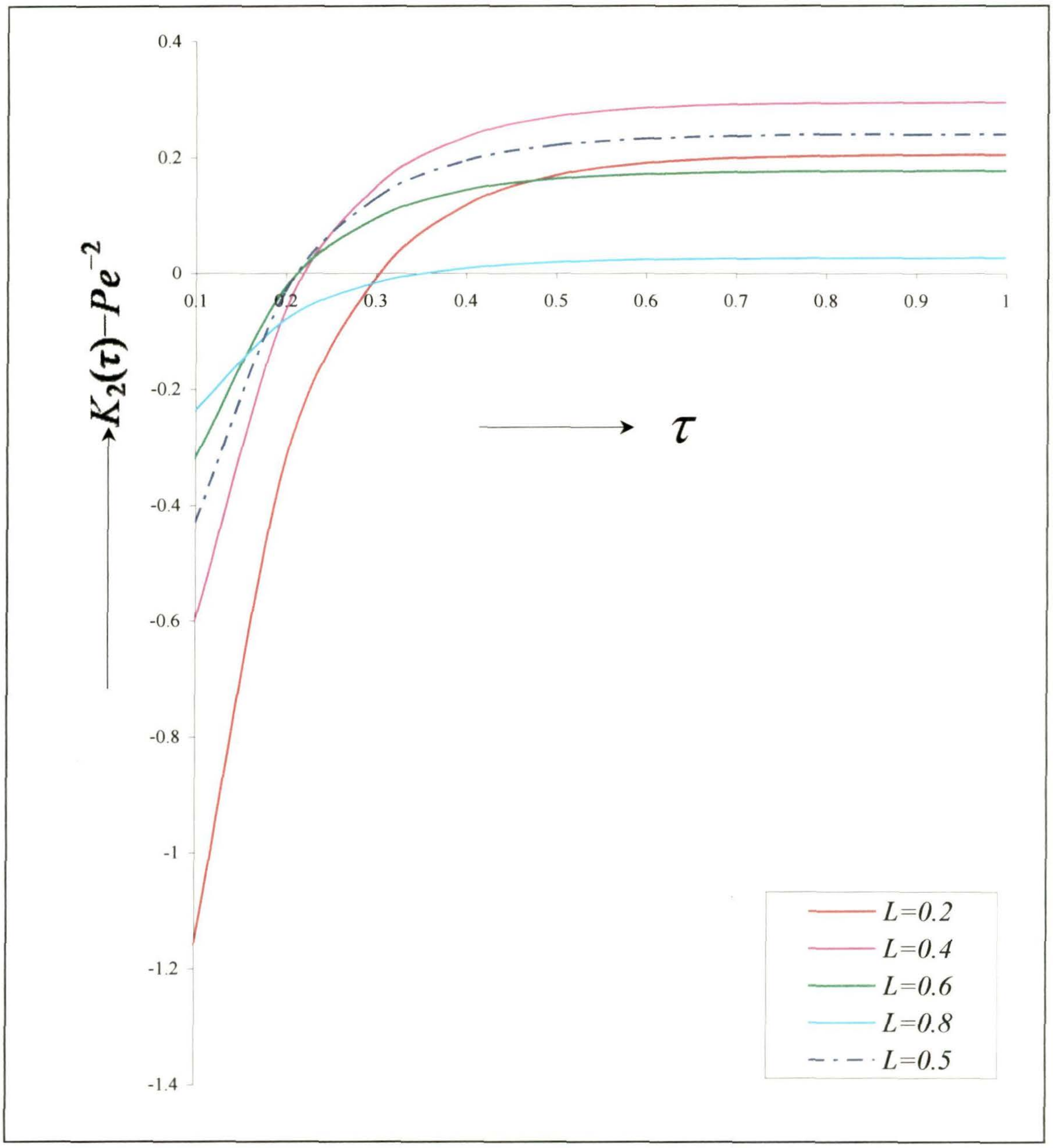

***Fig*. 3.8 Plot of $[K_2(\tau)-Pe^{-2}]$ against τ for different values of *L*.**

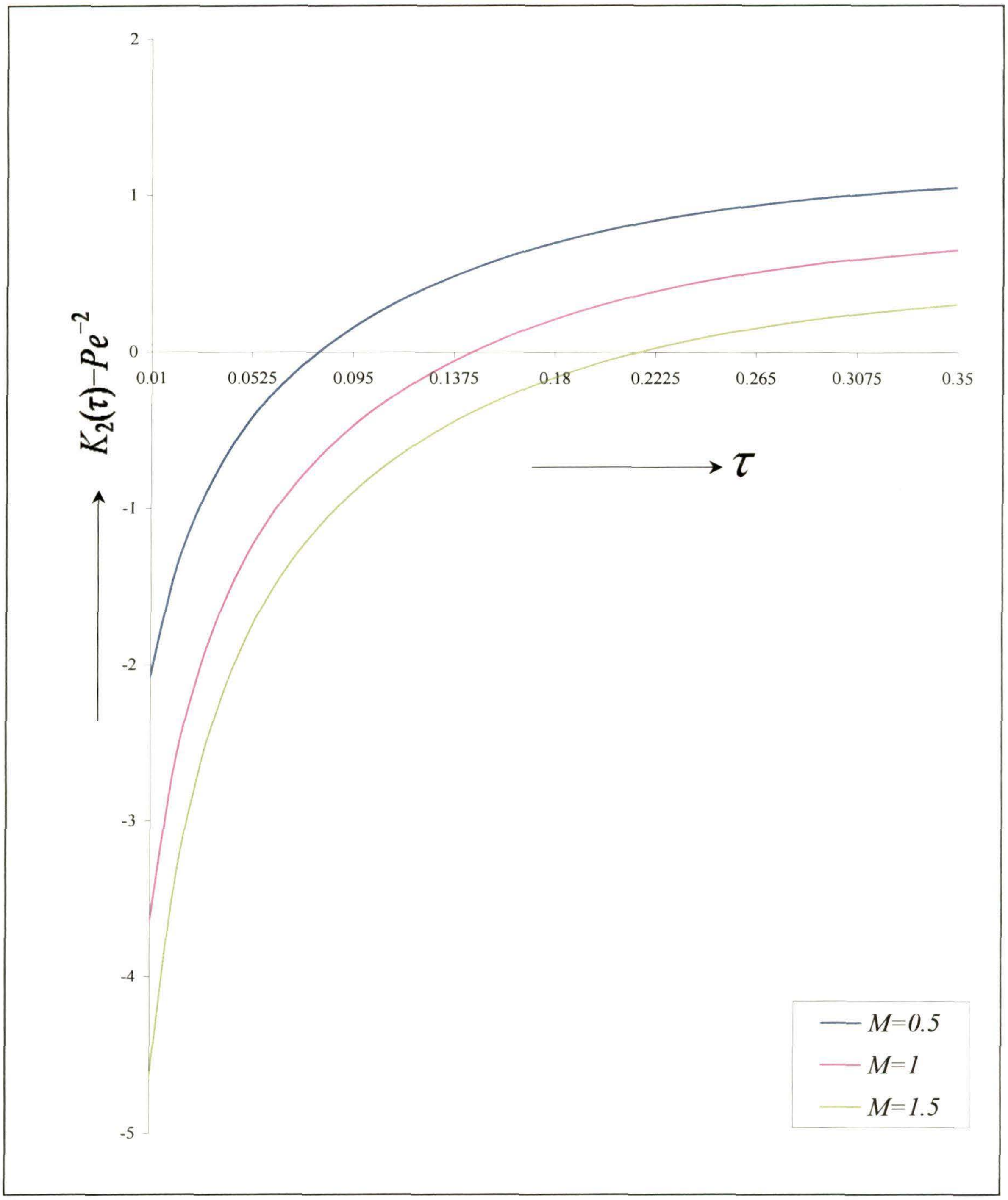

Fig. **3.9 Plot of $[K_2(\tau)-Pe^{-2}]$ against τ for different values of *M*.**

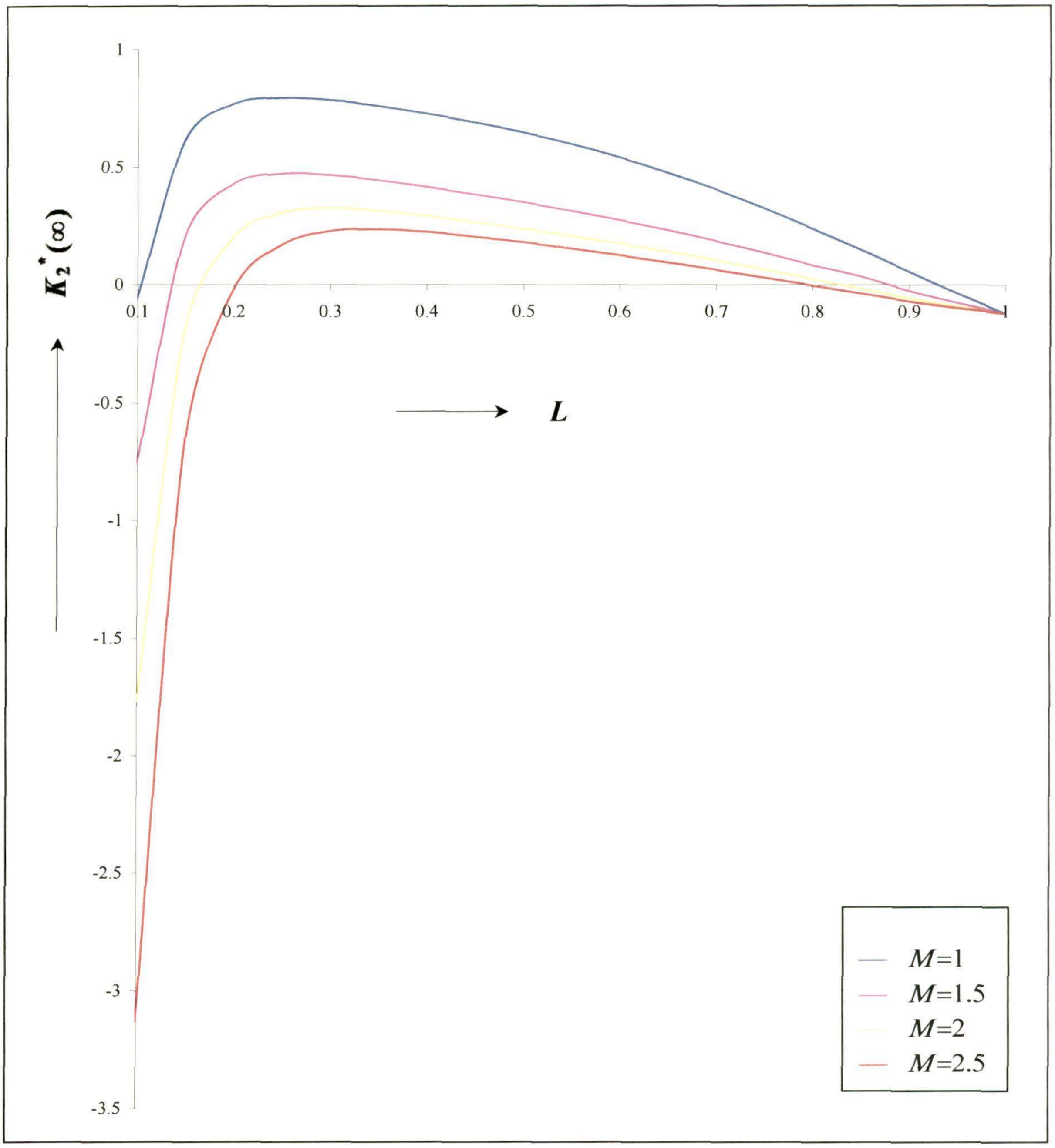

***Fig*. 3.10 Plot of $K_2^*(\infty)$ against *L* for different values of *M*.**

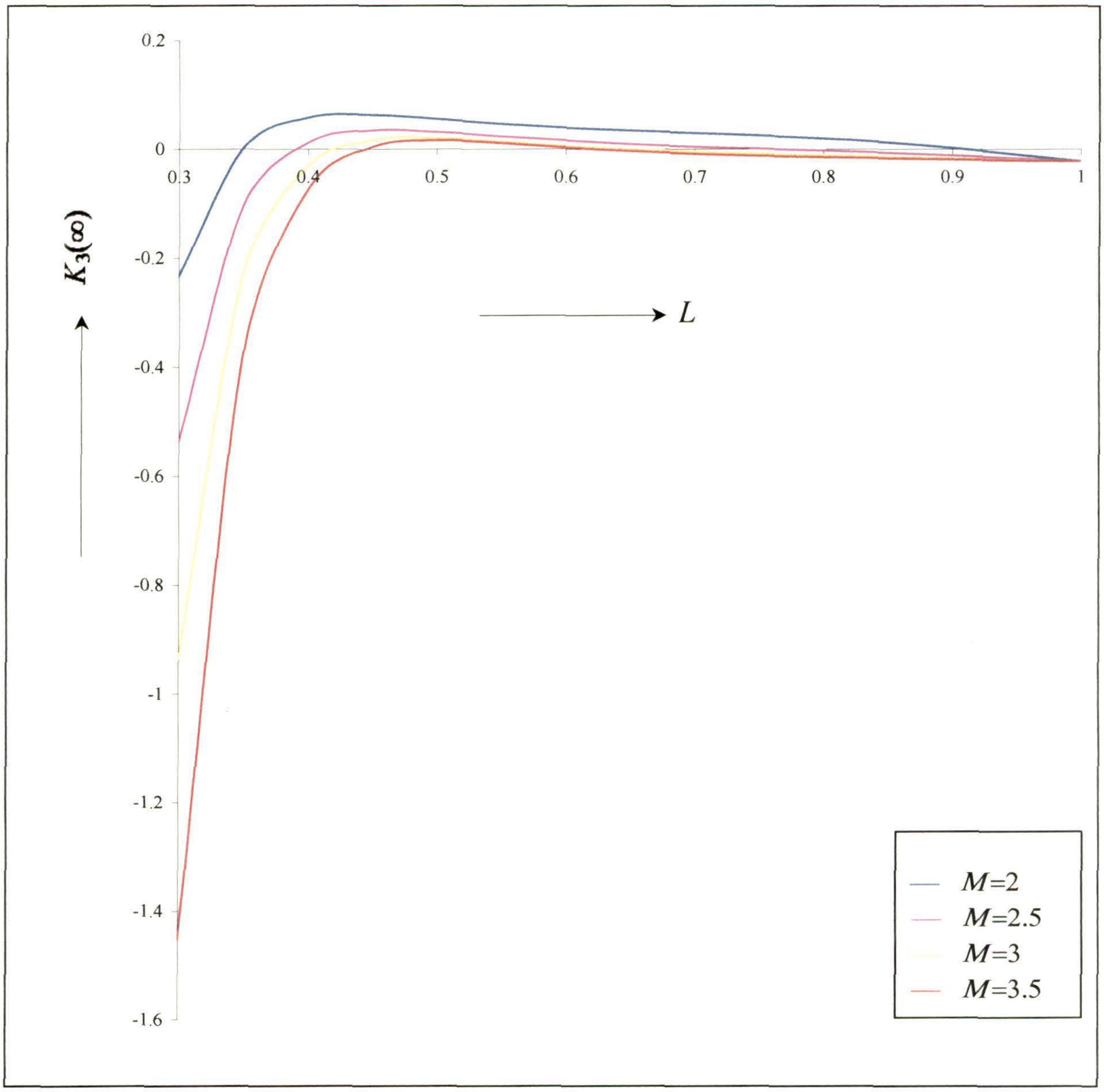

***Fig*. 3.11 Plot of $K_3(\infty)$ against L for different values of M.**

PART THREE

DISPERSION OF SOLUTE IN OSCILLATING HYDROMAGNETIC COUETTE FLOW IN A ROTATING SYSTEM*

3.8 INTRODUCTION

The dispersion of passive impurities or solute in an incompressible viscous fluid flowing in a circular pipe under laminar conditions was investigated by Taylor[1]. He showed that relative to a plane moving with the mean speed of the flow, the solute is dispersed with an effective longitudinal dispersion coefficient Deff. = $a^2\omega^2/48D_m$ Where D_m is the molecular diffusivity, ω_m is the mean velocity and a is the radius of the tube. However his conceptual model is asymptotically valid for large time. Gill and Sankarasubramanian [3]constructed a dispersion model for problem of convective diffusion, which is valid for all time, by allowing the dispersion coefficients to vary with time. This dependence on time might account for the considerable amount of scatter, which generally found in experimental data on dispersion. Following this generalized dispersion model, Gill and Sankarasubramanian [4], Annapurna and Gupta [9], Mukherjee *et al.* [10] and Layek *et al.* [11] studied the dispersion of solute in different geometrical conditions. Although the generalized dispersion of solute in time-dependent laminar flow which in principle valid for all value of time, they confined their analysis only to cases of dispersion in a fully developed steady flow. Moreover, to the best of our knowledge, the effect of rotation on the dispersion of solute in a time-dependent flowing fluid has not been studied in the literature. But it is well known that earth's rotation plays an important role in the dynamics of thin sheets of fluid. Any dispersion problem involving say, damping of waste materials in thin sheets of liquids (say river or lakes), one would expect that earth's rotation would have significant effect.

**Accepted for publication in "Acta Ciencia Indica"*

The above consideration provides the motivation for our present study. Here we have studied the effect of rotation and transversely applied magnetic field on the dispersion of solute in oscillating hydromagnetic Couette flow. The interesting part of the analysis is that $K_2(\tau)$, second dispersion coefficient consists of a steady part S and a fluctuating part $D_2(\tau)$ due to rotation and oscillation of plate.

3.9 MATHEMATICAL ANALYSIS

An unsteady Couette flow of an electrically conducting, viscous, incompressible fluid is considered between two parallel plates of distance 'δ' apart. The lower plate is stationary and the upper one is oscillating in its own plane with a velocity U(t) about a non-zero constant mean velocityU_0. Choose the origin on the lower plate and x-axis parallel to the direction of the upper plate .The z-axis taken perpendicular to the plate, which is the axis of rotation about which the entire system is rotating with constant angular velocity Ω. A transverse magnetic field of uniform strength B_0 is applied along the axis of rotation. Since the plates are infinite in extent, all physical quantities, except the pressure, depend on z and t only. If a solute diffuses in the above fully developed flow, then the concentration c(t,x,z) of solute satisfies

$$\frac{\partial c}{\partial t}+u(z,\mathrm{t})\frac{\partial c}{\partial x}=\mathrm{D}\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial z^2}\right), \qquad \text{.... (3.70)}$$

where D is the molecular diffusivity.

The initial and boundary conditions are

$$\left.\begin{aligned} c(0,x,z)&=c_0 && \text{for } |x| \le \frac{x_s}{2} \\ &=0 && \text{for } |x| > \frac{x_s}{2} \end{aligned}\right\}, \qquad \text{.... (3.70a)}$$

$$\frac{\partial c}{\partial z}=0 \qquad \text{at } z=\pm\delta, \qquad \text{.... (3.70b)}$$

$$c(\mathrm{t},\infty,z)=0, \qquad \text{.... (3.70c)}$$

where (3.70c) expresses the condition of zero mass flux at the plate walls.

We introduce the dimensionless quantities as

$$\theta=\frac{c}{c_0},\Psi(\eta,\tau)=\frac{u(z,\mathrm{t})}{\bar{u}},X=\frac{x_s}{\delta^2\bar{u}},\tau=\frac{\mathrm{D}t}{\delta^2},Pe=\frac{\bar{u}\delta}{\mathrm{D}},\eta=\frac{z}{\delta}, \qquad \text{.... (3.71)}$$

where $\bar{u}$ is the time-averaged axial velocity on the central line z =0 given by

$$\bar{u}=\frac{2\pi}{\omega}\int_0^{\omega/2\pi}u(t,0)dt. \qquad \text{.... (3.72)}$$

Using (3.71) in (3.69) and (3.70) we get

$$\frac{\partial\theta}{\partial\tau}+\Psi(\eta,\tau)\frac{\partial\theta}{\partial X}=\frac{1}{Pe^2}\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial\eta^2}. \qquad \text{.... (3.73)}$$

The complete velocity field in this flow was determined by Singh [12] which is given as follows

$$\Psi(\eta,\tau)=\frac{U_0}{\bar{u}}\left[q_0(\eta)+\frac{\varepsilon}{2}\left\{q_1(\eta)e^{i\lambda\tau}+q_2(\eta)e^{-i\lambda\tau}\right\}\right], \qquad \text{.... (3.74)}$$

where $q_0(\eta)$, $q_1(\eta)$ and $q_2(\eta)$ are the combination of primary and secondary velocity distribution and

$$q_0(\eta)=1-\frac{sinh(1-\eta)L_1}{sinhL_1}, \qquad \text{.... (3.75)}$$

$$q_1(\eta)=1-\frac{sinh(1-\eta)L_2}{sinhL_2}, \qquad \text{.... (3.76)}$$

$$q_2(\eta)=1-\frac{sinh(1-\eta)L_3}{sinhL_3}, \qquad \text{.... (3.77)}$$

$$\left.\begin{aligned} &L_1=\left(M^2+2iK\right)^{1/2},\ L_2=\left(M^2+2iK+i\lambda\right)^{1/2},\\ &L_3=\left(M^2+2iK-i\lambda\right)^{1/2},\\ &K=\frac{\Omega d^2}{\nu}\text{ is the rotation parameter,}\\ &\lambda=\frac{\omega d^2}{\nu}\text{ is frequency parameter,}\\ &M=B_0 d\left(\frac{\sigma}{\mu}\right)^{\frac{1}{2}}\text{ is the Hartman number.} \end{aligned}\right\} \qquad \text{.... (3.78)}$$

The initial and boundary condition are as follows

$$\left.\begin{aligned} \theta(0,X,\eta) &=1 \qquad \text{for } |X| \le \frac{1}{2}X_s\\ &=0 \qquad \text{for } |X| \rangle \frac{1}{2}X_s \end{aligned}\right\}, \qquad \text{.... (3.79)}$$

$$\theta(\tau,\infty,\eta)=0, \qquad \text{.... (3.80a)}$$

$$\frac{\partial\theta}{\partial\eta}(\tau,X,\pm1)=0. \qquad \text{.... (3.80b)}$$

3.10 SOLUTION

Following Gill and Sankarasubramanian [3], the solution of (3.73) subject to (3.79) and (3.80) is formulated as

$$\theta(\tau,X,\eta)=\sum_{k=0}^{\infty} f_k(\tau,\eta)\frac{\partial^k\theta_m}{\partial X^k}, \qquad \text{.... (3.81)}$$

where

$$\theta_m=\frac{1}{2}\int_{-1}^{1}\theta d\eta. \qquad \text{.... (3.82)}$$

Substituting (3.81) in (3.73) we get

$$\frac{\partial\theta_m}{\partial\tau}+\Psi_1(\eta,\tau)\frac{\partial\theta_m}{\partial X}-\frac{1}{Pe^2}\frac{\partial^2\theta_m}{\partial X^2}+\sum_{k=0}^{\infty}\left[\left(\frac{\partial f_k}{\partial\tau}-\frac{\partial^2 f_k}{\partial\eta^2}\right)\frac{\partial^k\theta_m}{\partial X^k}\right.$$

$$\left.+\Psi(\tau,\eta)f_k\frac{\partial^{k+1}\theta_m}{\partial X^{k+1}}-\frac{1}{Pe^2}f_k\frac{\partial^{k+2}\theta_m}{\partial X^{k+2}}+f_k\frac{\partial^{k+1}\theta_m}{\partial\tau\partial X^k}\right]=0. \qquad \text{.... (3.83)}$$

Integration of (3.73) gives upon using (3.82)

$$\frac{\partial\theta_m}{\partial\tau}=\frac{1}{Pe^2}\frac{\partial^2\theta_m}{\partial X^2}-\frac{1}{2}\int_{-1}^{1}\Psi(\tau,\eta)\frac{\partial\theta}{\partial X}d\eta. \qquad \text{.... (3.84)}$$

We assume that the process of distribution of θ_m is diffusive in nature right from time zero. Following Gill and Sankarasubramanian's approach, the generalized dispersion model with time dependent dispersion coefficients can be written as

$$\frac{\partial\theta_m}{\partial\tau}=\sum_{i=1}^{\infty}K_i(\tau)\frac{\partial^i\theta_m}{\partial X^i}, \qquad \text{.... (3.85)}$$

where

$$K_1(\tau)=-\frac{1}{2}\int_{-1}^{1}\Psi(\tau,\eta)f_0(\tau,\eta)d\eta, \qquad \text{.... (3.86a)}$$

$$K_2(\tau)=\frac{1}{Pe^2}-\frac{1}{2}\int_{-1}^{1}\Psi(\tau,\eta)f_1(\tau,\eta)d\eta, \qquad \text{.... (3.86b)}$$

$$\vdots$$

$$K_{i+2}(\tau)=-\frac{1}{2}\int_{-1}^{1}\Psi(\tau,\eta)f_{i+1}d\eta \qquad (i=1,2,3.............). \qquad \text{.... (3.86c)}$$

Substituting (2.85) in (2.83) and equating the coefficients $\partial^k\theta_m/\partial X^k$, we obtain the following equations for $f_k(\tau,\eta)$:

$$\frac{\partial f_0}{\partial \tau}=\frac{\partial^2 f_0}{\partial \eta^2}, \qquad \text{.... (3.87a)}$$

$$\frac{\partial f_1}{\partial \tau}=\frac{\partial^2 f_1}{\partial \eta^2}-\left[\Psi(\tau,\eta)+K_1(\tau)\right]f_0, \qquad \text{.... (3.87b)}$$

$$\frac{\partial f_2}{\partial \tau}=\frac{\partial^2 f_2}{\partial \eta^2}+\left[\frac{1}{Pe^2}-K_2(\tau)\right]f_0-\left[\Psi(\tau,\eta)+K_1(\tau)\right]f_1, \qquad \text{.... (3.87c)}$$

$$\vdots$$

$$\frac{\partial f_k}{\partial \tau}=\frac{\partial^2 f_k}{\partial \eta^2}-\left[\Psi(\tau,\eta)+K_1(\tau)\right]f_{k-1}$$

$$+\left[\frac{1}{Pe^2}-K_2(\tau)\right]f_{k-2}-\sum_{i=3}^{k}K_i f_{k-i} \quad (k=3,4,5,\ldots\ldots\ldots\ldots). \qquad \text{.... (3.87d)}$$

Equations (3.79) and (3.82) give

$$\left.\begin{array}{lll}\theta_m(0,X)=1 & \text{for} & |X|\le \dfrac{X_s}{2}\\ \theta_m(0,X)=0 & \text{for} & |X|> \dfrac{X_s}{2}\end{array}\right\}, \qquad \text{.... (3.88a)}$$

$$\theta_m(\tau,\infty)=0. \qquad \text{.... (3.88b)}$$

Now from (3.81), the initial conditions for f_k can be taken as

$$f_0(0,\eta)=1\,, \qquad f_k(0,\eta)=0 \quad \text{for } k=1,2,3,\ldots\ldots\ldots\ldots . \qquad \text{.... (3.89)}$$

Similarly the boundary conditions for f_k are derived from (3.79), (3.80) and (3.81) as

$$\frac{\partial f_k}{\partial \eta}=0 \quad \text{at } \eta=\pm 1 \qquad (k=0,1,2,\ldots\ldots\ldots\ldots). \qquad \text{.... (3.90)}$$

Further Equations (3.81) and (3.82) are consistent if

$$\int_{-1}^{1} f_0 d\eta = 2 \quad \text{and} \quad \int_{-1}^{1} f_k d\eta = 0 \qquad \text{for } k = 1,2,3,\ldots\ldots. \qquad \ldots.\ (3.91)$$

With the initial and boundary conditions given by (3.89)–(3.91), we determine the functions $f_k(\tau,\ \eta)$ by solving the system of Equations (3.87a)–(3.87d) and then obtain the diffusion coefficients $K_i(\tau)$ from (3.86a)–(3.86c).

We next proceed to determine $K_1(\tau)$. It can be readily shown that the solution of (3.87a) subject to (3.89)–(3.91) is given by

$$f_0(\tau,\eta) = 1. \qquad \ldots.\ (3.92)$$

Substitution from (3.74) and (3.92) in (3.86a) then gives

$$K_1(\tau) = -\frac{1}{P}\left[2 + 2\varepsilon\cos\lambda\tau + \frac{1-\cosh 2\delta L_1}{\delta L_1 \sinh 2\delta L_1}\right.$$
$$\left. + \frac{\varepsilon}{\delta}\left(A\cos\lambda\tau - B\sin\lambda\tau\right)\right], \qquad \ldots.\ (3.93)$$

where

$$\left.\begin{aligned} P &= \frac{\sinh 2\delta L_1 - \sinh \delta L_1}{\sinh 2\delta L_1}, \\ A &= \frac{1}{2}\left\{\frac{1-\cosh 2\delta L_2}{L_2 \sinh 2\delta L_2} + \frac{1-\cosh 2\delta L_3}{L_3 \sinh 2\delta L_3}\right\}, \\ \mathrm{B} &= -\frac{i}{2}\left\{\frac{1-\cosh 2\delta L_2}{L_2 \sinh 2\delta L_2} - \frac{1-\cosh 2\delta L_3}{L_3 \sinh 2\delta L_3}\right\}. \end{aligned}\right\}. \qquad \ldots.\ (3.94)$$

With the help of Duhamel's theorem and using (3.74) and (3.93) respectively, we solve equation (3.87b). The result is as follows

$$f_1(\tau,\eta) = \frac{1}{\delta P L_1 \sinh 2\delta L_1}\left[\frac{\sinh(1-\eta)\delta L_1}{-\delta L_1} - \frac{(1-\cosh 2\delta L_1)}{4}\eta^2\right]$$

$$+\frac{\varepsilon}{2P\delta}\left[\frac{\sinh(1-\eta)\delta L_2}{\delta L_2{}^2\sinh 2\delta L_2}e^{i\lambda\tau}+\frac{\sinh(1-\eta)\delta L_3}{\delta L_3{}^2\sinh 2\delta L_3}e^{-i\lambda\tau}-(A\cos\lambda\tau-B\sin\lambda\tau)\right.$$

$$\left.\times\frac{\eta^2}{2}\right]-\frac{\varepsilon}{2P\delta}\left\{(2C+A)\cos\lambda\tau+(2\bar{D}-B)\sin\lambda\tau\right\}\eta+\frac{1}{2\delta PL_1\sinh 2\delta L_1}$$

$$\times\left[\frac{(\cosh 2\delta L_1-1)}{\delta^2 L_1{}^2}+\frac{1}{6}(1-\cosh 2\delta L_1)\right]-\frac{(1+\cosh 2\delta L_1)}{2\delta PL_1\sinh 2\delta L_1}\eta+\frac{\varepsilon}{2P\delta}\times$$

$$\left[\frac{(\cosh 2\delta L_2-1)}{\delta^3 L_2{}^3\sinh 2\delta L_2}e^{i\lambda\tau}+\frac{(\cosh 2\delta L_3-1)}{\delta^2 L_3{}^3\sinh 2\delta L_3}e^{-i\lambda\tau}+\frac{1}{3}(A\cos\lambda\tau-B\sin\lambda\tau)\right]$$

$$+\sum_{n=1}^{\infty}\left[\frac{(1-\cosh 2\delta L_1)\cos\lambda_n}{\delta PL_1\sinh 2\delta L_1}\left(\frac{\delta L_1}{\lambda_n{}^2+\delta^2 L_1{}^2}+\frac{1}{\lambda_n{}^2}\right)e^{-\lambda_n{}^2\tau}-\frac{\varepsilon}{2}\frac{\cos\lambda_n}{P\delta}\times\right.$$

$$\left\{\frac{(\cosh 2\delta L_2-1)}{L_2{}^2\sinh 2\delta L_2}\frac{(i\lambda e^{i\lambda\tau}+\lambda_n{}^2e^{-\lambda_n{}^2\tau})}{(\lambda_n{}^2+\delta^2 L_2{}^2)(\lambda_n{}^2+i\lambda)}+\frac{(\cosh 2\delta L_3-1)}{L_3{}^2\sinh 2\delta L_3(\lambda_n{}^2+\delta^2 L_3{}^2)}\right.$$

$$\frac{(\lambda_n{}^2e^{-\lambda_n{}^2\tau}-i\lambda e^{-i\lambda\tau})}{(\lambda_n{}^2-i\lambda)}-\frac{2}{\lambda_n{}^2}\left(\frac{A(\lambda^2\cos\lambda\tau-\lambda_n{}^2\lambda\sin\lambda\tau)}{\lambda_n{}^4+\lambda^2}+\frac{A\lambda_n{}^4e^{-\lambda_n{}^2\tau}}{\lambda_n{}^4+\lambda^2}\right.$$

$$\left.\left.\left.-\frac{B(\lambda_n{}^2\lambda\cos\lambda\tau+\lambda^2\sin\lambda\tau-\lambda_n{}^2\lambda e^{-\lambda_n{}^2\tau})}{\lambda_n{}^4+\lambda^2}\right)\right\}\right], \qquad \ldots.\ (3.95)$$

where

$$\lambda_n=n\pi,$$

$$C=\frac{1}{2}\left\{\frac{\cosh 2\delta L_2}{L_2\sinh 2\delta L_2}+\frac{\cosh 2\delta L_3}{L_3\sinh 2\delta L_3}\right\},\qquad \bar{D}=-\frac{i}{2}\left\{\frac{\cosh 2\delta L_3}{L_3\sinh 2\delta L_3}-\frac{\cosh 2\delta L_2}{L_2\sinh 2\delta L_2}\right\}.$$

Substituting (3.74) and (3.95) in (3.86b), we obtain the diffusion coefficient $K_2(\tau)$ as follows

$$K_2(\tau)=\frac{1}{Pe^2}+S+D_2(\tau), \qquad \text{.... (3.96)}$$

where S and $D_2(\tau)$ represent the steady and time dependent part of the diffusion coefficient respectively.

The expression for S and $D_2(\tau)$ are given as

$$S=-\frac{1}{P\delta\{\sinh 2\delta L_1\text{-}\sinh\delta L_1\}}\left[-\frac{2\sinh^2\delta L_1}{\delta^2 L_1^3}-\frac{(1-\cosh 2\delta L_1)}{2L_1}\left\{\frac{1}{3}-\frac{1}{2\sinh 2\delta L_1}\right.\right.$$

$$\left.\times\left(\frac{2\sinh^2\delta L_1}{\delta L_1}+\frac{4\sinh^2\delta L_1}{L_1^3\delta^3}-\frac{4\cosh\delta L_1\sinh\delta L_1}{L_1^2\delta^2}\right)\right\}-\frac{\varepsilon^2\sinh 2\delta L_1}{8\delta^2\sinh 2\delta L_2\sinh 2\delta L_3}$$

$$\times\left(\frac{\sinh 2(L_2+L_3)\delta}{L_2+L_3}-\frac{\sinh 2(L_2-L_3)\delta}{L_2-L_3}\right)\left(\frac{1}{L_2^2}+\frac{1}{L_3^2}\right)+\frac{(1+\cosh 2\delta L_1)\cosh\delta M}{\delta L_1^2\sinh 2\delta L_1}$$

$$\times\left(\frac{\sinh\delta L_1}{L_1\delta}-\cosh\delta L_1\right)+\frac{(\cosh 2\delta L_1-1)}{L_1}\left(\frac{1}{\delta^2 L_1^2}-\frac{1}{6}\right)\left(1-\frac{\sinh^2\delta L_1}{\delta L_1\sinh 2\delta L_1}\right)-$$

$$\frac{\varepsilon^2\sinh 2\delta L_1}{2\delta^3\sinh 2\delta L_2\sinh 2\delta L_3}\left\{\frac{(\cosh 2\delta L_2-1)\sinh^2\delta L_3}{L_3 L_2^3}+\right.$$

$$\left.\left.\frac{(\cosh 2\delta L_3-1)\sinh^2\delta L_2}{L_2 L_3^3}\right\}\right]. \qquad \text{.... (3.97)}$$

$$D_2(\tau)=-\frac{1}{\delta P^2 L_1\ \sinh 2\delta L_1}\left[\frac{\varepsilon e^{i\lambda\tau}}{2\delta L_1\sinh\delta L_2}\left(\frac{\sinh 2(L_2+L_1)\delta}{2(L_2+L_1)\delta}-\frac{\sinh 2(L_2-L_1)\delta}{2(L_2-L_1)\delta}\right)\right.$$

$$+\frac{\varepsilon e^{-i\lambda\tau}}{2\delta L_1\sinh\delta L_3}\left(\frac{\sinh 2(L_1+L_3)\delta}{2(L_1+L_3)\delta}-\frac{\sinh 2(L_1-L_3)\delta}{2(L_1-L_3)\delta}\right)+\varepsilon\cos\lambda\tau\left\{\frac{(1-\cosh 2\delta L_1)}{6}\right.$$

$$\left.-\frac{2\sinh^2\delta L_1}{\delta^2 L_1^2}\right\}+\frac{\varepsilon e^{i\lambda\tau}(1-\cosh 2\delta L_1)}{8\sinh\delta L_2}\left\{\frac{\sinh^2\delta L_2}{\delta L_2}+\frac{4\sinh^2\delta L_2}{\delta^3 L_2^3}-\frac{2\sinh 2\delta L_2}{L_2^2\delta^2}\right\}$$

$$\left.\left.+\frac{\varepsilon e^{-i\lambda\tau}(1-\cosh 2\delta L_1)}{8\sinh\delta L_3}\left\{\frac{4\sinh^2\delta L_3}{\delta^3 L_3^3}-\frac{2\sinh 2\delta L_3}{\delta^2 L_3^2}+\frac{\sinh^2\delta L_3}{\delta L_3}\right\}\right]\right]-\frac{\varepsilon}{2\delta P^2}\times$$

$$\left[\frac{2e^{i\lambda\tau}\sinh^2\delta L_2}{\delta^2 L_2^3\sinh 2\delta L_2}+\frac{2e^{-i\lambda\tau}\sinh^2\delta L_3}{\delta^2 L_3^3\sinh 2\delta L_3}-\frac{e^{i\lambda\tau}}{\delta L_2^2\sinh 2\delta L_1\sinh 2\delta L_2}\times\right.$$

$$\left\{\frac{\sinh 2(L_2+L_1)\delta}{2(L_2+L_1)\delta}-\frac{\sinh 2(L_2-L_1)\delta}{2(L_2-L_1)\delta}\right\}-\frac{e^{-i\lambda\tau}}{\delta L_3^2\sinh 2\delta L_1\sinh 2\delta L_3}$$

$$\left\{\frac{\sinh 2(L_3+L_1)\delta}{2(L_3+L_1)\delta}-\frac{\sinh 2(L_1-L_3)\delta}{2(L_1-L_3)\delta}\right\}-\frac{\varepsilon e^{2i\lambda\tau}}{2\delta L_2^2\sinh^2 2\delta L_2}\left\{\frac{\sinh 4\delta L_2}{4\delta L_2}-1\right\}$$

$$+\frac{2\varepsilon\cos\lambda\tau\sinh^2\delta L_3}{\delta^2 L_3^3\sinh 2\delta L_3}e^{-i\lambda\tau}-\left(A\cos\lambda\tau-B\sin\lambda\tau\right)\left\{\frac{2}{3}-\frac{1}{\sinh 2\delta L_1}\left(\frac{\sinh^2\delta L_1}{\delta L_1}\right.\right.$$

$$\left.\left.+\frac{2\sinh^2\delta L_1}{L_1^3\delta^3}-\frac{\sinh 2\delta L_1}{L_1^2\delta^2}\right)\right\}-\frac{\varepsilon e^{i\lambda\tau}}{2\sinh 2\delta L_2}\left\{\frac{2\sinh^2\delta L_2}{\delta L_2}+\frac{4\sinh^2\delta L_2}{L_2^3\delta^3}\right.$$

$$-\frac{2\sinh 2\delta L_2}{L_2^2\delta^2}\Bigg\}-\frac{\varepsilon e^{-i\lambda\tau}}{2\sinh 2\delta L_3}\Bigg\{\frac{2\sinh^2\delta L_3}{\delta L_3}+\frac{4\sinh^2\delta L_3}{L_3^3\delta^3}-\frac{2\sinh 2\delta L_3}{L_3^2\delta^2}\Bigg\}$$

$$+\frac{2}{3}\varepsilon\cos\lambda\tau\Bigg]-\frac{1}{P}\Bigg[\frac{(1+\cosh 2\delta L_1)}{2P\delta L_1\sinh 2\delta L_1}\Bigg\{\frac{\varepsilon e^{i\lambda\tau}}{2\sinh 2\delta L_2}\Bigg(\frac{\sinh 2\delta L_2}{L_2^2\delta^2}-\frac{2\cosh^2\delta L_2}{\delta L_2}\Bigg)$$

$$+\frac{\varepsilon e^{-i\lambda\tau}}{2\sinh 2\delta L_3}\Bigg(\frac{\sinh 2\delta L_3}{L_3^2\delta^2}-\frac{2\cosh^2\delta L_3}{\delta L_3}\Bigg)\Bigg\}+\frac{\varepsilon}{2P\delta\sinh 2\delta L_1}\Bigg(\frac{\sinh 2\delta L_1}{L_1^2\delta^2}-$$

$$\frac{2\cosh^2\delta L_1}{L_1\delta}\Bigg)\{(2C+A)\cos\lambda\tau+(2\bar{D}-B)\sin\lambda\tau\}+\frac{\varepsilon^2}{2P\delta}\{(2C+A)\cos\lambda\tau$$

$$+(2\bar{D}-B)\sin\lambda\tau\}\Bigg\{\frac{e^{i\lambda\tau}}{2\sinh 2\delta L_2}\Bigg(\frac{\sinh 2\delta L_2}{L_2^2\delta^2}--\frac{2\cosh^2\delta L_2}{\delta L_2}\Bigg)+\frac{e^{-i\lambda\tau}}{2\sinh 2\delta L_3}$$

$$\times\Bigg(\frac{\sinh 2\delta L_3}{L_3^2\delta^2}-\frac{2\cosh^2\delta L_3}{\delta L_3}\Bigg)\Bigg\}\Bigg]+\frac{1}{P}\Bigg[\frac{\varepsilon(\cosh 2\delta L_1-1)}{2\delta P L_1\sinh 2\delta L_1}\Bigg(\frac{1}{\delta^2L_1^2}-\frac{1}{6}\Bigg)$$

$$\times\Bigg\{\frac{e^{i\lambda\tau}\sinh^2\delta L_2}{\delta L_2\sinh 2\delta L_2}+\frac{e^{-i\lambda\tau}\sinh^2\delta L_3}{\delta L_3\sinh 2\delta L_3}-2\cos\lambda\tau\Bigg\}-\frac{\varepsilon}{P\delta}\Bigg(1-\frac{\sinh^2\delta L_1}{\delta L_1\sinh 2\delta L_1}\Bigg)\times$$

$$\Bigg\{\frac{(\cosh 2\delta L_2-1)}{\delta^2L_2^3\sinh 2\delta L_2}e^{i\lambda\tau}+\frac{(\cosh 2\delta L_3-1)}{\delta^2L_3^3\sinh 2\delta L_3}e^{-i\lambda\tau}+\frac{1}{3}(A\cos\lambda\tau-B\sin\lambda\tau)\Bigg\}$$

$$+\frac{\varepsilon^2}{2P\delta}\left\{\frac{(\cosh 2\delta L_2-1)}{\delta^2 L_2{}^3\sinh 2\delta L_2}e^{i\lambda\tau}+\frac{(\cosh 2\delta L_3-1)}{\delta^2 L_3{}^3\sinh 2\delta L_3}e^{-i\lambda\tau}\right.$$

$$\left.\left.+\frac{1}{3}(A\cos\lambda\tau-B\sin\lambda\tau)\right\}\right\}\left\{\frac{\sinh^2\delta L_2}{\delta L_2\sinh 2\delta L_2}e^{i\lambda\tau}+\frac{\sinh^2\delta L_3}{\delta L_3\sinh 2\delta L_3}e^{-i\lambda\tau}\right.$$

$$\left.\left.\left.-2\cos\lambda\tau\right\}\right]\right]+\sum_{n=1}^{\infty}\frac{1}{P}\left[\frac{\cos\lambda_n(1-\cosh 2\delta L_1)}{\delta P L_1\sinh 2\delta L_1}\left(\frac{\delta L_1}{\lambda_n{}^2+\delta^2 L_1{}^2}+\frac{1}{\lambda_n^2}\right)e^{-\lambda_n{}^2\tau}\right.$$

$$-\frac{\varepsilon\cos\lambda_n}{2P\delta}\left\{\frac{(\cosh 2\delta L_2-1)(i\lambda e^{i\lambda\tau}+\lambda_n{}^2 e^{-\lambda_n{}^2\tau})}{L_2{}^2(\lambda_n{}^2+\delta^2 L_2{}^2)(\lambda_n{}^2+i\lambda)\sinh 2\delta L_2}+\frac{(\cosh 2\delta L_3-1)}{L_3{}^2(\lambda_n{}^2+\delta^2 L_3{}^2)}\right.$$

$$\times\frac{(\lambda_n{}^2 e^{-\lambda_n{}^2\tau}-i\lambda e^{-i\lambda\tau})}{(\lambda_n{}^2-i\lambda)\sinh 2\delta L_3}-\frac{2}{\lambda_n{}^2}\left(A\frac{\lambda^2\cos\lambda\tau-\lambda_n{}^2\lambda\sin\lambda\tau}{\lambda_n{}^4+\lambda^2}+A\frac{\lambda_n{}^4}{\lambda_n{}^4+\lambda^2}\times\right.$$

$$\left.\left.\left.e^{-\lambda_n{}^2\tau}-B\frac{\left(\lambda_n{}^2\lambda\cos\lambda\tau+\lambda^2\sin\lambda\tau\right)}{\lambda_n{}^4+\lambda^2}+B\frac{\lambda_n{}^2\lambda e^{-\lambda_n{}^2\tau}}{\lambda_n{}^4+\lambda^2}\right)\right\}\right]\times\left[\frac{L_1\delta\cos\lambda_n}{(\lambda_n{}^2+L_1{}^2\delta^2)}\right.$$

$$\times\frac{\cosh 2\delta L_1-1}{\sinh 2\delta L_1}+\frac{\varepsilon\delta L_2\cos\lambda_n(\cosh 2\delta L_2-1)}{2(\lambda_n{}^2+L_2{}^2\delta^2)\sinh 2\delta L_2}e^{i\lambda\tau}+\frac{\varepsilon\delta L_3\cos\lambda_n}{2(\lambda_n{}^2+L_3{}^2\delta^2)}$$

$$\left.\times\frac{(\cosh 2\delta L_3-1)}{\sinh 2\delta L_3}e^{-i\lambda\tau}\right]. \qquad \text{.... (3.98)}$$

3.11 RESULTS AND DISCUSSION

It is seen that an exact solution to the convective diffusion equation can be constructed in a rather simple way by using the series expansion proposed by Gill and Sankarasubramanian [2]. This exact solution involves a dispersion model that includes third order and higher order derivative of θ_m w.r.to X. Fortunately the coefficient $K_{i+2}(\tau)$ (i =1,2,3,4,.......) can be calculated from equation (3.86c) and are very small compared to $K_2(\tau)$. Therefore, to a good approximation, the higher order terms in equation (3.85) can be neglected in determining θ_m. Neglecting $K_3(\tau)$ and higher order dispersion coefficients equation (3.85) can be written as

$$\frac{\partial\theta_m}{\partial\tau} = K_1(\tau)\frac{\partial\theta_m}{\partial X} + K_2(\tau)\frac{\partial^2\theta_m}{\partial X^2}. \qquad \text{.... (3.99)}$$

The solution of equation (3.99) subject to (3.88a), (3.88 b) is obtained as

$$\theta_m = \frac{1}{2}\left[erf\,\frac{\frac{1}{2}X_s + X_1}{2T_0^{\frac{1}{2}}} + erf\,\frac{\frac{1}{2}X_s - X_1}{2T_0^{\frac{1}{2}}} \right],$$

where

$$X_1 = X + \int_0^\tau K_1(\eta)d\eta\ , \qquad T_0 = \int_0^\tau K_2(\eta)d\eta\,.$$

It reveals that the dimensionless solute concentration in oscillating hydro magnetic Couette flow in a rotating system mainly depends on Hartman number M and rotation parameter K. So we have computed dispersion coefficient $K_2(\tau)$ for several values of M and K. It is interesting to note that the dispersion of solute in unsteady flow arising out of rotation of entire system gives rise to dispersion coefficient, which consists of both steady and fluctuating part. We have computed S the steady part of dispersion coefficient for various values of M (Hartman no.) and K(rotation parameter) with Sc=1000 (Schmidt number) and plotted in Fig.-3.12. From Fig.-3.12. it is evident that S increases as M increases and attains maximum nearly at M=1 and then slowly decreases with increasing M. Fig.-3.12. also reveals that value of S decreases as K(rotation parameter) increases. Naturally, the effect of moderate or large values of K and M has less or no effect on the steady part S of dispersion coefficient .The fluctuating part due to τ of dispersion coefficient $K_2(\tau)$ is evaluated for different values of τ, M ,K for Sc=1000. Fig.-3.13. shows the plot of $D_2(\tau)$ with τ for several values of M. The plot of $D_2(\tau)$ is oscillating in nature and amplitude of oscillation is more in case of higher values of M. It is also seen that moderate or large values of M has more effect on fluctuating part than steady part. Fig.-3.14. shows the variation of $D_2(\tau)$ with τ for different values of K and amplitude of oscillation decreases as K increases. Clearly the effect of rotation parameter K on the fluctuating part of dispersion coefficient is much more than steady part. Present analysis reveals the striking results that the dispersion of solute in oscillating hydro magnetic Couette flow is affected by the unsteadiness of the flow due to rotation.

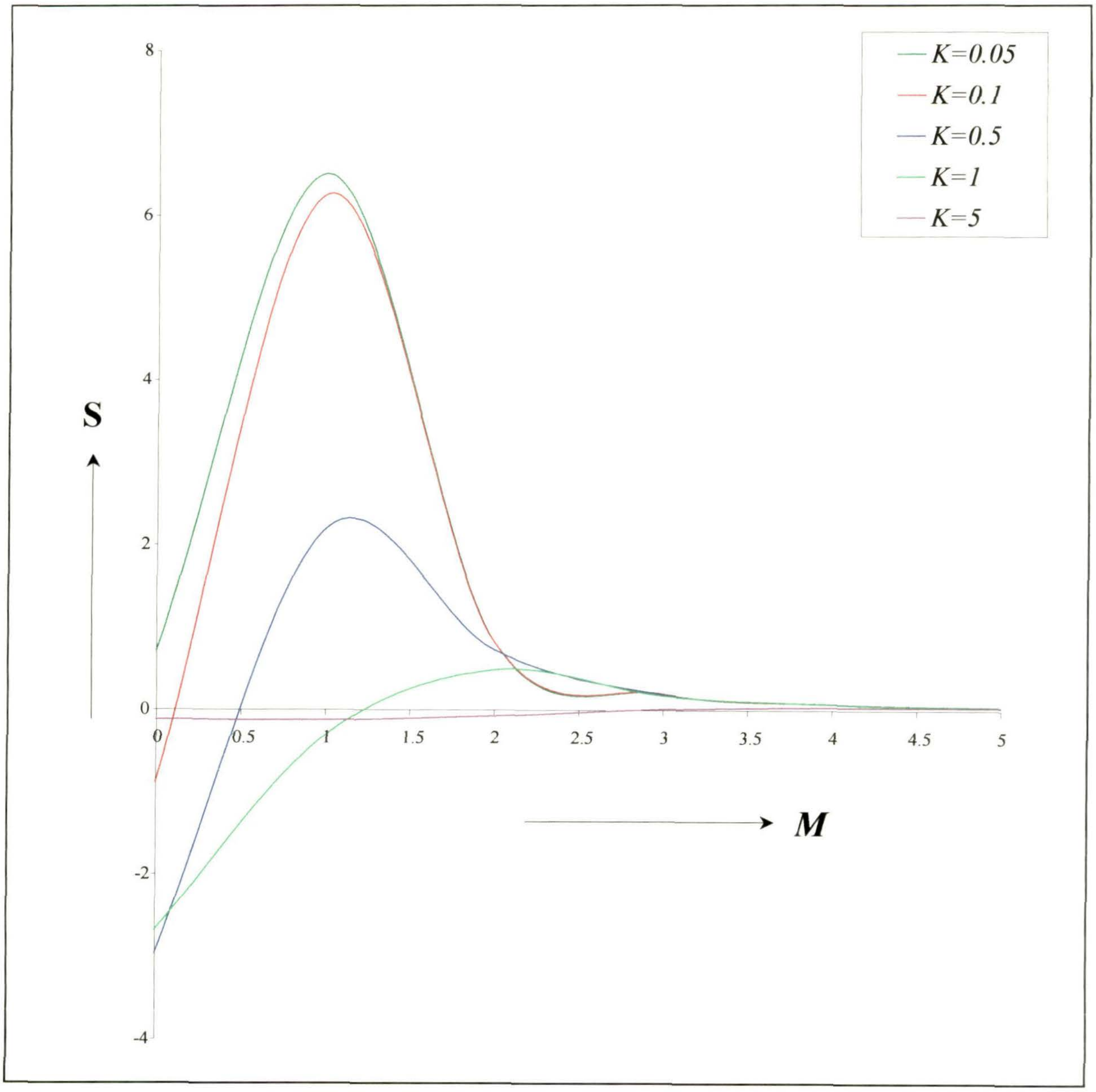

Fig **- 3.12 Plot of S against *M* for different values of *K*.**

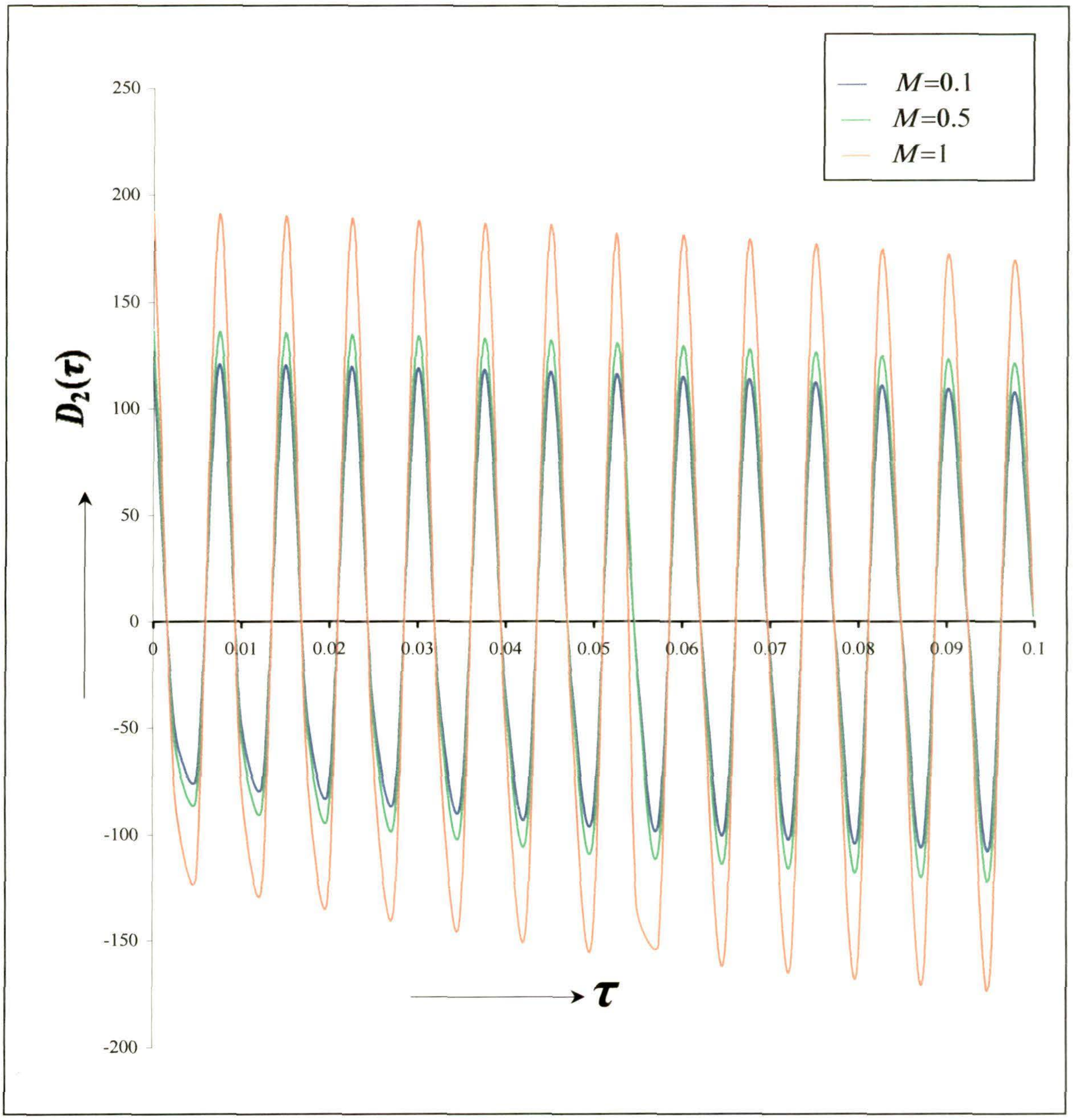

Fig. **3.13 Plot of $D_2(\tau)$ vs. τ for different values of M.**

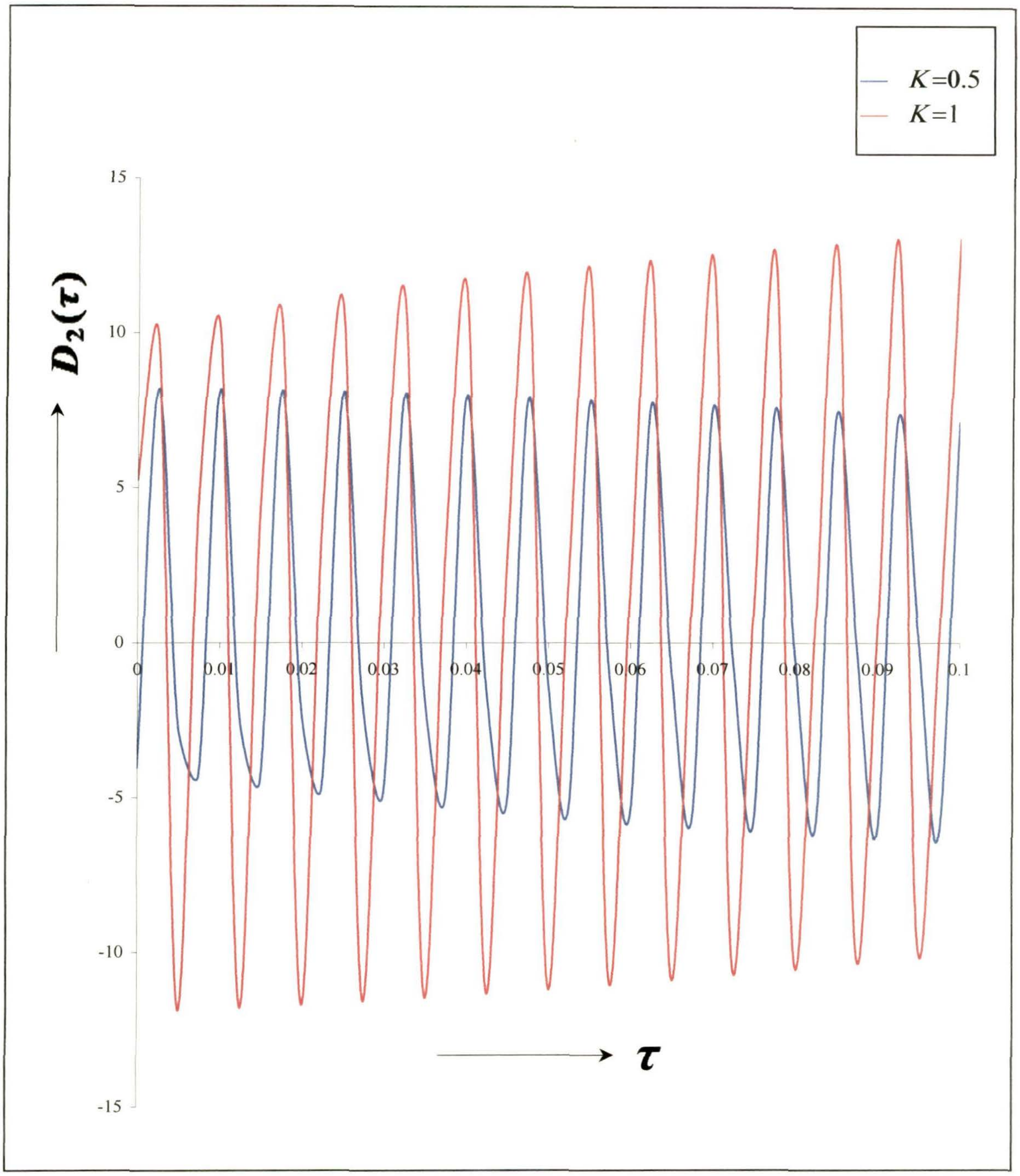

***Fig*. 3.14 Plot of $D_2(\tau)$ vs. τ for different value of K.**

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