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A Short Review on the Bosonization Duality in $2+1$D Chern-Simons-Matter Theories

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This review discusses Bose-Fermi dualities in $2+1$ dimensional Chern-Simons-matter theories, focusing on the conjecture that free (regular) fermions coupled to a Chern-Simons gauge field are level-rank dual to Chern-Simons gauged critical (Wilson-Fisher) bosons. In the bosonic theory, the sign of the mass deformation determines whether the system enters a Higgsed (condensed) or unHiggsed (uncondensed) phase. We review large-N computations of the thermal free energy in both phases of the bosonic theory and explore how these results align with their fermionic counterparts under the proposed duality. Notably, this mapping reveals a striking feature : in the unHiggsed phase, the fermions are dual to fundamental scalar excitations, while in the Higgsed phase, the same fermionic degrees of freedom map onto massive W-bosons.

I. INTRODUCTION

Dualities have been central to the development of our understanding of quantum field theory and statistical mechanics. Over the past few decades, many surprising connections have been discovered between theories that look very different on the surface but turn out to describe the same physics. These dualities often relate weakly coupled theories to strongly coupled ones, making it possible to learn about non-perturbative phenomena through a dual, more tractable description. In lower dimensions, where quantum fluctuations dominate and exact results are more accessible, dualities become especially powerful and often exact.

In recent years, a remarkable network of dualities has emerged in three-dimensional quantum field theories involving Chern-Simons gauge fields and matter couplings [1–4]. These dualities, sometimes referred to as "3D bosonization", generalize ideas familiar from two dimensions such as the duality between bosons and fermions [5, 6] to higher dimensions and richer settings. They have deep connections to both high-energy and condensed matter physics, appearing in studies of quantum Hall systems, topological phases, and strongly interacting gauge theories [7–11].

The goal of this review is to provide a pedagogical introduction to the subject of Chern-Simons-matter dualities, with a focus on the underlying physical ideas and the mathematical structure behind them. We will explore the different phases of these theories and the dualities that link them. In particular, we will take a closer look at one phase, explaining in detail how to compute physical quantities and how the duality shows up in that context.

This review is organized as follows. In the next section, we will take a step back and look at a few examples of dualities, starting with some from classical physics and then moving on to more modern ones in $1+1$ dimensional quantum field theories. This broader view will help set the stage for the main topic: dualities involving Chern-Simons gauge theories coupled to matter fields in $2+1$ dimensions, which we will explore in the following section.

II. A WEB OF DUALITIES

Before getting into Chern-Simons-matter dualities, it is useful to look at some familiar examples of dualities in physics. These dualities often relate theories that appear very different at first glance, but turn out to describe the same physical phenomena. One of the simplest examples is the electromagnetic duality in Maxwell theory in 3+1 dimensions. In vacuum (no charges or currents), Maxwell's equations take the form:

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 0, \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \\ \vec{\nabla}\cdot\vec{B} &= 0, \qquad \vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t}, \end{aligned} \tag{1}$$

where we used natural units (speed of light, $c = 1$). Notice that these equations remain the same under the exchange of electric and magnetic fields, $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$, reflecting a kind of self-duality of the field strength tensor $F_{\mu\nu}$ under Hodge duality, $F \to *F$. In more advanced settings, this idea gets extended to strong-weak dualities in non-abelian gauge theories, where coupling constants are inverted, and electrically charged particles are exchanged with magnetic monopoles.

Another famous example is the Kramers–Wannier duality in the 2D classical Ising model [12],

$$H = -J\sum_{\langle i,j\rangle} s_i s_j \ , \tag{2}$$

where $s_i = \pm 1$ is the spin at site i, J is the coupling constant and $\langle i,j\rangle$ indicates that the sum runs over all pairs of nearest-neighbor spins. It turns out that at low temperatures, the behavior of the Ising model is largely determined by its lowest energy states, whereas at high temperatures, the system is dominated by maximally disordered states. Interestingly, these two regimes are connected through a mathematical duality. Specifically, the partition function at inverse temperature β is related to the partition function at another temperature $\tilde{\beta}$ by the relation,

$$\sinh(2\tilde{\beta}J) = \frac{1}{\sinh(2\beta J)}. \tag{3}$$

This implies that the thermodynamics of the Ising model at one temperature fully determine those at the dual temperature. A particularly important consequence of this relation is that it reveals the critical temperature as the self-dual point—where the model is invariant under the transformation. Setting $\beta = \tilde{\beta} = \beta_c$ in equation (3), yields the well-known result for the critical temperature $k_B T_c = 2.269J$. This was one of the earliest signs that dualities can reveal deep structures like phase transitions without solving the model completely.

In 1+1D quantum field theories, dualities become even richer. Bosonization is a classic example, where a theory of free massless Dirac fermion is equivalent to a compact boson with periodicity $\phi \sim \phi + 2\pi$ [6],

$$\begin{aligned} S_{\text{f}} &= \int d^2x \, i\bar{\psi}\gamma^\mu \partial_\mu \psi \, , \\ S_{\text{b}} &= \int d^2x \frac{b^2}{2} \partial^\mu \phi \partial_\mu \phi \, , \end{aligned} \tag{4}$$

where γ^μ are the 2D gamma matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The dimensionless parameter b is the called the radius of the compact boson. This duality maps the fermionic vector current $j^\mu_{\text{f}} = \bar{\psi}\gamma^\mu\psi$ to the bosonic vector current $j^\mu_{\text{b}} = -\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi$, while the fermion mass term $\bar{\psi}\psi$ corresponds to $\cos\phi$ in the bosonic theory. Although the degrees of freedom look very different, the partition functions of the two theories agree, once the compactification radius of the boson is tuned to $b^2 = \frac{1}{4\pi}$. This duality is not just a mathematical trick—it provides a concrete mapping between fermionic and bosonic operators and has important applications in condensed matter physics.

A more involved example is the duality between the sine-Gordon model and the massive Thirring model [5]. The sine-Gordon theory describes a scalar field with action

$$S_{\text{SG}} = \int d^2x \left[\frac{1}{2}\partial^\mu \phi \partial_\mu \phi + \frac{a}{b^2}\Big(1 - \cos(b\phi)\Big) \right] , \tag{5}$$

where ϕ is a real scalar field, and a, b are constants. The massive Thirring model, on the other hand, describes a self-interacting Dirac fermion

$$S_{\text{T}} = \int d^2x \left[\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - g(\bar{\psi}\gamma^\mu\psi)^2 \right] . \tag{6}$$

These two theories are dual in the sense that the soliton (topological excitations) solutions of the sine-Gordon model correspond to the fermions of the Thirring model. The coupling constants are related by

$$\frac{4\pi}{b^2} = 1 + \frac{g}{\pi}, \tag{7}$$

and the operator mapping between the two theories reflects this identification. This duality provides a rare example of a strong-weak duality that can be checked explicitly, and it illustrates how topological objects in one theory become fundamental excitations in the dual.

In spacetime dimensions greater than two, establishing exact dualities is a highly non-trivial task and often requires the presence of supersymmetry to control quantum corrections and ensure non-perturbative consistency. Remarkably, however, in $2{+}1$D Chern-Simons gauge theories, exact dualities have been discovered even in the absence of supersymmetry [7]. These theories obey the *level-rank* duality and when we couple this theory to matter fields, they obey a bosonization duality in $2+1$ dimensions. These interesting things are the subjects to be discussed in the next section.

III. BOSE-FERMI DUALITY IN $2{+}1$D CHERN-SIMONS MATTER THEORIES

In order to understand dualities in Chern-Simons theories that include matter, it helps to first look at the simpler case of pure Chern-Simons (CS) theory, where no matter fields are present. One of the most important features of this theory is the level-rank duality, which we briefly describe below.

A. Level-Rank Duality in Pure Chern-Simons Theory

In quantum field theory, interactions are mediated by gauge bosons—quantum excitations associated with underlying gauge symmetries. A familiar example is electromagnetism, where the force is carried by a single massless spin-1 gauge boson: the photon. A natural generalization of electrodynamics is provided by Yang-Mills theory, in which the gauge field is extended from a single vector field to a matrix-valued, non-Abelian field taking values in the Lie algebra of a compact gauge group, typically $\mathrm{SU}(N)$. The action for the Yang-Mills theory in $3+1$ dimensions is given by

$$S_{\mathrm{YM}} = -\frac{1}{g_{\mathrm{YM}}^2}\int d^4x \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{8}$$

where the field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and g_{YM} denotes the Yang-Mills coupling constant. In contrast to the abelian case, the non-Abelian field strength contains a commutator term, leading to self-interactions among the gauge bosons. Yang-Mills theories form the foundation of the Standard Model of particle physics. Electroweak interactions are described by gauge groups such as $\mathrm{SU}(2) \times \mathrm{U}(1)$, while quantum chromodynamics (QCD), the theory of the strong force, is based on an $\mathrm{SU}(3)$ gauge symmetry.

While the Yang-Mills term governs the high-energy dynamics in $3{+}1$ dimensions, the situation is qualitatively different in $2{+}1$ dimensions. In lower dimensions, especially in the infrared (IR) limit, a topological term known as the Chern-Simons action becomes dominant. The pure Chern-Simons action is given by [13]

$$S_{\text{CS}} = \frac{k}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}\text{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right), \tag{9}$$

where k is the Chern-Simons level, and A_μ is a gauge field taking values in the adjoint representation of the gauge group $SU(N)$. Unlike the Yang-Mills term, the Chern-Simons action involves only a single derivative and does not depend on the spacetime metric, reflecting it's topological nature. In $2+1$ dimensions, this term dominates the low-energy behavior of the theory, effectively governing the infrared dynamics in the absence of matter.

The equations of motion that follow from the Chern-Simons action require the field strength to vanish: $F_{\mu\nu} = 0$. This means that any solution to the equations must be locally pure gauge (flat connections), taking the form $A_\mu = g^{-1}\partial_\mu g$ for some group-valued function g. As a result, pure Chern-Simons theory has no local, propagating degrees of freedom. Moreover, consistency under gauge transformations imposes a quantization condition: the level k must be an integer. This makes the coupling constant discrete and prevents it from being continuously renormalized.

Even though the theory is topological, it has a surprisingly rich structure. It is exactly solvable and exhibits what is known as *level-rank duality* [7] - an equivalence between a theory with gauge group $\text{SU}(N)$ at level k and one with gauge group $\text{SU}(k)$ at level N.

Although pure Chern-Simons theory has no local degrees of freedom, the theory becomes significantly richer when matter fields are introduced. In particular, Chern-Simons theories coupled to either bosonic or fermionic matter in the fundamental representation display a variety of interesting features, especially in the large-N limit where they become exactly solvable. One of the most remarkable phenomena to emerge in this setting is a strong-weak duality that relates two very different-looking theories: a Chern-Simons theory coupled to regular (free) fundamental fermions, and another coupled to critical (Wilson-Fischer) fundamental bosons. Despite having distinct ultraviolet (UV) descriptions, these theories flow to the same infrared (IR) physics. This equivalence, often referred to as a *Bose-Fermi duality*, offers a powerful window into the non-perturbative behavior of strongly coupled quantum field theories. In what follows, we explore these dualities in detail.

B. Bose-Fermi dualities in Chern-Simons-matter theories

In this review, we focus on two classes of Chern-Simons-matter theories that are central to the study of Bose-Fermi dualities in three dimensions. The first is the *regular fermion* (RF) theory, defined by the action

$$S_{\text{RF}}[\psi] = S_{\text{CS}} + \int d^3x\,\left(\bar\psi\gamma_\mu D^\mu\psi + m_F^{\text{reg}}\,\bar\psi\psi\right), \tag{10}$$

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where ψ is a Dirac fermion in the fundamental representation of the $SU(N_F)$ gauge group, and m_F^{reg} is the bare mass parameter for the regular fermion. The term S_{CS} denotes the pure Chern-Simons action introduced earlier. The second theory is the *critical boson* (CB) theory, defined by the action

$$S_{\text{CB}}[\phi, \sigma_B] = S_{\text{CS}} + \int d^3x \left(D_\mu \bar{\phi} D^\mu \phi + \sigma_B \left(\bar{\phi}\phi + \frac{N_B}{4\pi} m_B^{\text{cri}} \right) \right), \tag{11}$$

where ϕ is a complex scalar field in the fundamental representation of $SU(N_B)$ and m_B^{cri} is the bare mass parameter for the critical boson. σ_B is an auxiliary field introduced to enforce a constraint on the norm of the bosonic field. The presence of σ_B in the bosonic theory reflects the fact that we are working with the 'critical' or gauged Wilson-Fisher scalar theory, as opposed to the regular or gauged free scalar theory. Note that in both cases, the gauge dynamics are entirely governed by the Chern-Simons term S_{CS}, with no Yang-Mills kinetic term included.

The proposed duality relates an SU(N_F) Chern-Simons theory coupled to a regular fermion at level k_F with an SU(N_B) Chern-Simons theory coupled to a critical boson at level k_B. This correspondence holds under a level-rank duality given by,

$$N_B = |k_F| \, , \quad N_F = |k_B| \, . \tag{12}$$

In the large N limit, it is often more convenient to describe these theories using the so-called 'renormalized' levels κ_F, κ_B and associated 't Hooft couplings λ_F, λ_B defined by

$$\begin{aligned} \kappa_F &= \text{sgn}(k_F)\,(|k_F| + N_F)\,, & \kappa_B &= \text{sgn}(k_B)\,(|k_B| + N_B)\,, \\ \lambda_F &= \frac{N_F}{\kappa_F}, & \lambda_B &= \frac{N_B}{\kappa_B}\,. \end{aligned} \tag{13}$$

In terms of these variables, the duality map takes a particularly simple form:

$$\begin{gathered} \kappa_F = -\kappa_B, \quad \lambda_F = -\text{sgn}(\lambda_B) + \lambda_B \\ m_F^{\text{reg}} = -\lambda_B \, m_B^{\text{cri}}. \end{gathered} \tag{14}$$

Note that both λ_B and λ_F take values in the interval $[-1, 1]$. As $|\lambda_F|$ approaches zero—corresponding to the weak-coupling regime- $|\lambda_B|$ approaches one, indicating strong coupling. This highlights the fact that the duality exchanges strong and weak coupling regimes.

Over the past decade, substantial evidence has emerged supporting this duality, particularly in certain parametric regimes and in the large-N limit. Various physical observables have been computed on both sides of the duality to test and verify this correspondence. One major line of evidence comes from the precise matching of the thermal partition functions and the thermal free energies in the large-N limit, as demonstrated in a series of works [1–3]. Another strong piece of evidence comes from the comparison of scattering amplitudes [14, 15]. On one side of the duality, one computes the S-matrix for two-particle scattering in the fermionic theory; on the other side, the same is done for the bosonic theory. Remarkably, at large-N, the two results agree precisely. These results establish the duality between the massive, non-conformal versions of the two theories. When the fermion mass m_F^{reg} and the boson mass m_B^{cri} are tuned to zero, the theories become conformal. At this conformal point, one finds a matching of the spectrum of gauge-invariant operators in the two theories- an agreement that is nearly trivial at large N. More strikingly, the three-point correlation functions of single-trace operators, which carry nontrivial dependence on the coupling, also agree across the duality. There are infinitely many such operators on both sides, and the agreement holds for all of them [16–18]. This bosonization duality was first formulated in it's precise and complete form by Aharony in late 2015 [19].

However, there is a subtle and important issue in the study of dualities in such theories that was not fully addressed in earlier works. This concerns the behavior of the scalar sector in the critical boson theory, particularly the role of the Lagrange multiplier field σ_B, which imposes a constraint on the scalar field ϕ via its equation of motion

$$\bar{\phi}\phi = -\frac{N_B}{4\pi} m_B^{\text{cri}} \,. \tag{15}$$

This equation has no real solution when $m_B^{\text{cri}} > 0$. In this case, often referred to as the "unHiggsed phase" or "uncondensed phase," one proceeds by assuming that the scalar field has a vacuum at $\phi = 0$, and then checks the validity of this assumption. Specifically, one computes the quantum effective action for σ_B by integrating out the ϕ fields, and verifies that it admits a stable minimum, confirming that the vacuum is indeed located at $\phi = 0$.

In contrast, when $m_B^{\text{cri}} < 0$, the constraint equation becomes

$$\bar{\phi}\phi = \frac{N_B}{4\pi} |m_B^{\text{cri}}| \,, \tag{16}$$

which admits classical solutions. However, these solutions are not unique and related by $SU(N_B)$ gauge transformations. One can fix the gauge freedom by choosing unitary gauge, leading to a solution of the form

$$\phi^i = \delta^{iN_B} \sqrt{\frac{N_B}{4\pi} |m_B^{\text{cri}}|} \,, \tag{17}$$

which completely determines the scalar field configuration. In this so-called "Higgsed phase" or "condensed phase", the original $SU(N_B)$ gauge symmetry is spontaneously broken down to $SU(N_B - 1)$, and the scalar field ϕ becomes non-dynamical. As we will see, while ϕ is the dynamical degree of freedom in the unHiggsed bosonic phase, it is replaced by the W-bosons as the relevant excitations in the Higgsed bosonic phase.

A central objective of this review is to discuss the thermal free energy in both the unHiggsed and Higgsed phases of the bosonic theory [20] and to compare these results with those of the corresponding fermionic dual that exists in the literature. This comparison will provide us insight into how the duality extends across various phases.

We now briefly outline the procedure for computing the thermal free energy in these theories. It was shown in [3] that the partition function of the theories [Eq. (10) and Eq. (11)] on $S^2 \times S^1$ can be computed in a specific large N and high-temperature limit using a two-step procedure.

In the first step, one studies the theory on $\mathbb{R}^2 \times S^1$. The key ingredient in this setup is the holonomy U of the gauge field around the thermal circle S^1. Up to gauge transformations, the holonomy is completely characterized by its eigenvalues $e^{i\theta_j}$, where $j = 1, \ldots, N$, and the phases θ_j lie in the interval $(-\pi, \pi)$. In the large N limit, it is convenient to describe these eigenvalues using a continuous eigenvalue distribution function $\rho(\alpha)$, defined as

$$\rho(\alpha) = \frac{1}{N} \sum_{i=1}^{N} \delta(\alpha - \theta_i)\,. \tag{18}$$

At fixed holonomy U, one evaluates the path integral over all remaining fields. This defines the so-called free energy functional $v[U]$, or equivalently $v[\rho]$ in the large N limit, through the relation

$$e^{-\mathcal{V}_2 T^2 v[U]} = \int_{\mathbb{R}^2 \times S^1} [d\phi]\, e^{-S[\phi, U]}\,, \tag{19}$$

where $\mathcal{V}_2$ represents the volume of $\mathbb{R}^2$ and T stands for the temperature.

In the second step, the full partition function on $S^2 \times S^1$ is obtained by integrating over the holonomy U, using a Chern-Simons modified Haar measure $[dU]_{\text{CS}}$:

$$\mathcal{Z}_{S^2 \times S^1} = \int [dU]_{\text{CS}}\, e^{-\mathcal{V}_2 T^2 v[U]}\,. \tag{20}$$

In the large N limit, this matrix integral is evaluated using the saddle-point approximation.

It turns out that for the purpose of verifying the duality, it is sufficient to carry out only the first step of the procedure–namely, computing the functional $v[U]$, or more precisely $v[\rho]$, as defined in Eq. (19). We will therefore focus exclusively on this part of the analysis.

An explicit computation of an infinite sum of loop diagrams for the fermionic and bosonic fields yields the off-shell free energy functionals $v_F(|c_F|, \rho_F)$ and $v_B(|c_B|, \rho_B)$ for the two respective theories. It is important to note that these off-shell free energies depend on an additional auxiliary variable — $|c_F|$ or $|c_B|$, in addition to the holonomy eigenvalue distribution functions, ρ_F or ρ_B. The on-shell free energy functionals, as defined in Eq. (19), are functions only of the holonomy fields and are obtained by extremizing the off-shell functionals with respect to $|c_F|$ and $|c_B|$. The values of $|c_F|$ and $|c_B|$ at the extremum correspond to the thermal pole masses for the fermionic and bosonic theories, respectively, in units of temperature. In cases where the off-shell free energy admits multiple extrema, we select the extremum corresponding to the lowest free energy.

(i) Fermionic Phases

The off-shell free energy for the Chern-Simons gauged regular fermion theory is given by [20],

$$\begin{aligned} v_F(|c_F|, \rho_F) = \frac{N_F}{6\pi}\Bigg[&|c_F|^3 \frac{(\lambda_F - \mathrm{sgn}(X_F))}{\lambda_F} + \frac{3}{2\lambda_F}\hat{m}_F^{\mathrm{reg}} c_F^2 - \frac{(\hat{m}_F^{\mathrm{reg}})^3}{2\lambda_F(\lambda_F - \mathrm{sgn}(X_F))^2} \\ &- 3\int_{-\pi}^{\pi} \rho_F(\alpha)\, d\alpha \int_{|c_F|}^{\infty} dy\, y \left(\log(1+e^{-y-i\alpha}) + \log(1+e^{-y+i\alpha})\right)\Bigg] , \end{aligned} \tag{21}$$

where, $\hat{m}_F^{\mathrm{reg}}$ represents the fermion bare mass divided by temperature, while c_F is the thermal (or pole) mass of the fundamental fermionic excitations. The quantity X_F that appears in the free energy v_F is defined as,

$$X_F = 2\lambda_F \mathcal{C}(|c_F|) + \hat{m}_F^{\mathrm{reg}} , \tag{22}$$

where $\mathcal{C}(|c_F|)$ is given by,

$$\mathcal{C}(|c_F|) = \frac{1}{2}\int d\alpha\, \rho_F(\alpha) \left[\log\left(2\cosh\left(\frac{|c_F| + i\alpha}{2}\right)\right) + \log\left(2\cosh\left(\frac{|c_F| - i\alpha}{2}\right)\right)\right] . \tag{23}$$

The expression [Eq. (21)] represents the complete off-shell free energy of the CS gauged regular fermion theory. We will see that depending on the signs of λ_F and X_F, the theory falls into one of two distinct regimes. These two regimes correspond to the two different branches (unHiggsed and Higgsed) of the bosonic dual theory.

To determine the on-shell value of the parameter c_F, we extremize the free energy v_F as given in Eq. (21). This leads to the so-called *fermionic gap equation*, given by

$$|c_F| = \mathrm{sgn}(X_F)\left[\, 2\lambda_F \mathcal{C}(|c_F|) + \hat{m}_F^{\mathrm{reg}}\right] = |X_F| . \tag{24}$$

This gap equation admits solutions in two separate regimes. The first regime corresponds to the condition,

$$\mathrm{sgn}(\lambda_F)\mathrm{sgn}(X_F) \geq 0 . \tag{25}$$

The fermionic free energy, in this regime, simplifies to

$$v_F^{\text{unHiggsed}}(|c_F|,\rho_F) = \frac{N_F}{6\pi}\Bigg[|c_F|^3\frac{(|\lambda_F|-1)}{|\lambda_F|} + \frac{3}{2\lambda_F}\hat{m}_F^{\text{reg}}c_F^2 - \frac{(\hat{m}_F^{\text{reg}})^3}{2\lambda_F(|\lambda_F|-1)^2} - 3\int_{-\pi}^{\pi}\rho_F(\alpha)\,d\alpha\int_{|c_F|}^{\infty}dy\,y\left(\log(1+e^{-y-i\alpha})+\log(1+e^{-y+i\alpha})\right)\Bigg]\,. \tag{26}$$

This regime turns out to be the fermionic dual of the *unHiggsed* phase of the CS gauged critical boson theory.

The second regime corresponds to the condition,

$$\text{sgn}(\lambda_F)\text{sgn}(X_F) \leq 0\,. \tag{27}$$

In this case, the fermionic free energy becomes

$$v_F^{\text{Higgsed}}(|c_F|,\rho_F) = \frac{N_F}{6\pi}\Bigg[|c_F|^3\frac{(|\lambda_F|+1)}{|\lambda_F|} + \frac{3}{2\lambda_F}\hat{m}_F^{\text{reg}}c_F^2 - \frac{(\hat{m}_F^{\text{reg}})^3}{2\lambda_F(|\lambda_F|+1)^2} - 3\int_{-\pi}^{\pi}\rho_F(\alpha)\,d\alpha\int_{|c_F|}^{\infty}dy\,y\left(\log(1+e^{-y-i\alpha})+\log(1+e^{-y+i\alpha})\right)\Bigg]\,. \tag{28}$$

This regime corresponds to the fermionic dual of the *Higgsed* phase of the CS gauged critical boson theory.

(ii) UnHiggsed bosonic phase

The off-shell free energy v_B in the unHiggsed bosonic phase can be obtained by expanding around the unHiggsed bosonic vacuum [20]. As discussed earlier, under certain conditions — typically when the critical boson mass m_B^{cri} is negative — this vacuum does not correspond to the true ground state of the bosonic theory. Therefore, the expression for v_B provided below is expected to be valid only within a specific range of parameters,

$$v_B^{\text{unHiggsed}}(|c_B|,\rho_B)| = \frac{N_B}{6\pi}\Bigg[\frac{3}{2}\hat{m}_B^{\text{cri}}c_B^2 - \frac{1}{2}\left(\hat{m}_B^{\text{cri}}\right)^3 - |c_B|^3 + 3\int_{-\pi}^{\pi}\rho_B(\alpha)d\alpha\int_{|c_B|}^{\infty}dy\,y\left(\log\left(1-e^{-y-i\alpha}\right)+\log\left(1-e^{-y+i\alpha}\right)\right)\Bigg]\,. \tag{29}$$

Here, $\hat{m}_B^{\text{cri}}$ denotes the critical boson bare mass divided by the temperature, while c_B is the thermal (or pole) mass of the fundamental bosonic excitations. Extremizing the free energy [Eq. (29)] on-shell leads to the *bosonic gap equation*, in the unHiggsed phase as,

$$2\mathcal{S}(|c_B|) = \hat{m}_B^{\text{cri}}\,, \tag{30}$$

where the function $\mathcal{S}(|c_B|)$ is defined as

$$\mathcal{S}(|c_B|) = \frac{1}{2}\int d\alpha\, \rho_B(\alpha)\left[\log\left(2\sinh\left(\frac{|c_B|+i\alpha}{2}\right)\right) + \log\left(2\sinh\left(\frac{|c_B|-i\alpha}{2}\right)\right)\right] . \tag{31}$$

It turns out that the gap [Eq. (30)] admits solutions only when the following condition is satisfied:

$$\mathrm{sgn}(\lambda_B)\mathrm{sgn}(X_B) \leq 0 , \tag{32}$$

where

$$X_B \equiv 2\lambda_B \mathcal{S} - \lambda_B \hat{m}_B^{\mathrm{cri}} - \mathrm{sgn}(\lambda_B)|c_B| . \tag{33}$$

The condition, given in Eq. (32) defines the parameter regime in which the bosonic free energy expression in Eq. (29) remains valid. It can be explicitly verified that, under the duality transformation specified by Eq. (14), and supplemented by the holonomy constraint

$$|\lambda_B|\rho_B(\alpha) + |\lambda_F|\rho_F(\pi-\alpha) = \frac{1}{2\pi} , \tag{34}$$

the bosonic gap equation in the Higgsed phase [Eq. (30)] precisely maps to the fermionic gap equation [Eq. (24)], provided the branch is chosen according to Eq. (25). This last relation connects the bosonic and fermionic holonomy eigenvalue distributions, $\rho_B(\alpha)$ and $\rho_F(\alpha)$, respectively. When Eq. (34) is satisfied, one can further show that [24]

$$\begin{aligned} \lambda_B\mathcal{S} &= \lambda_F\mathcal{C} - \frac{\mathrm{sgn}(\lambda_F)}{2}|c_F| , \\ \lambda_F\mathcal{C} &= \lambda_B\mathcal{S} - \frac{\mathrm{sgn}(\lambda_B)}{2}|c_B| . \end{aligned} \tag{35}$$

Furthermore, under the same duality, the bosonic free energy in Eq. (29) exactly maps to the fermionic free energy [Eq. (26)].

On the other hand, the fermionic gap equation also allows for solutions that satisfy the complementary condition [Eq. (27)]. The corresponding bosonic dual solutions lie outside the regime where the unHiggsed vacuum is valid. In this case, the bosonic free energy must be computed by expanding around the Higgsed vacuum, where W-bosons become the relevant degrees of freedom. We will explore this regime in more detail in the following subsection.

C. Higgsed bosonic phase

We now turn our attention in computing the thermal free energy in the Higgsed phase of the bosonic theory [20] and checking how it corresponds to its fermionic dual, as described in Eq. (28).

Consider the $SU(N_B)$ Chern-Simons gauged critical boson theory defined by the Euclidean action S_{E}:

$$\begin{aligned}
S_{\text{E}} &= S_{\text{CS}} + S_{\text{B}} \ , \\
S_{\text{CS}} &= \frac{i\kappa_B}{4\pi}\int d^3x \epsilon^{\mu\nu\rho}\,\text{Tr}\left(X_\mu \partial_\nu X_\rho - \frac{2i}{3}X_\mu X_\nu X_\rho\right) \ , \\
S_{\text{B}} &= \int d^3x \sqrt{\det g}\left(D_\mu \bar{\phi} D^\mu \phi + \sigma_B\left(\bar{\phi}\phi + \frac{N_B}{4\pi}m_B^{\text{cri}}\right)\right) \ ,
\end{aligned} \tag{36}$$

where the covariant derivative is defined as $D_\mu\phi = \partial_\mu\phi - iX_\mu\phi$. Here, we write X for the $SU(N_B)$ gauge field while we preserve A for the unbroken $SU(N_B-1)$ gauge field. The fields $X_\mu = X_\mu^a T^a$ are $N \times N$ Hermitian matrices where $T^a (a = 1, 2, \cdots, N_B^2 - 1)$ are the generators of the $SU(N_B)$ gauge group, normalized such that $\text{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$. The parameter κ_B denotes the 'renormalized' Chern-Simons level.

Here we are interested in the case, $m_B^{\text{cri}} < 0$, leading to

$$\bar{\phi}\phi = \frac{N_B}{4\pi}|m_B^{\text{cri}}| \equiv |\kappa_B| v^2 \,, \tag{37}$$

as was discussed in Eq. (16), where we define

$$v^2 = \frac{N_B}{4\pi|\kappa_B|}|m_B^{\text{cri}}| = \frac{|\lambda_B|}{4\pi}|m_B^{\text{cri}}| \,. \tag{38}$$

As explained earlier, by choosing the unitary gauge, we can fix the scalar field ϕ^i [see Eq. (17)], which spontaneously breaks the original $SU(N_B)$ gauge symmetry to $SU(N_B-1)$.

At first glance, this gauge fixing may appear to eliminate the scalar field entirely. However, the physical degrees of freedom do not disappear—they are instead redistributed due to the Higgs mechanism. In gauge theories with spontaneous symmetry breaking, the degrees of freedom of the matter fields are not lost but are absorbed by the gauge fields corresponding to the broken generators. In the present case, the gauge bosons which belonged to the full $SU(N_B)$ group but not to the unbroken $SU(N_B-1)$ subgroup acquire the missing degrees of freedom. These components become massive and propagate as W bosons. They inherit the dynamics that originally came from the scalar field. We now take a closer look at how this process works.

Let $(X_\mu)^i_j$ denote the ij^{th} component of the gauge field X_μ introduced in Eq. (36). The indices i and j run from 1 to N_B. For convenience, we single out the last index, $i = N_B$, and similarly for j, and denote the remaining indices by $a, b = 1, \ldots, N_B - 1$. With this notation, we define the following decomposition:

$$\begin{aligned}
(X_\mu)^a_b &= (A_\mu)^a_b - \frac{Z_\mu}{N_B-1}\delta^a_b, &\qquad (X_\mu)^a_{N_B} &= \frac{W^a_\mu}{\sqrt{\kappa_B}}, \\
(X_\mu)^{N_B}_b &= \frac{(\bar{W}_\mu)_b}{\sqrt{\kappa_B}}, &\qquad (X_\mu)^{N_B}_{N_B} &= Z_\mu \ .
\end{aligned} \tag{39}$$

Here, A_μ is a traceless matrix representing the gauge field for the unbroken $SU(N_B-1)$ subgroup. The fields W^a_μ and $(\bar{W}_\mu)_b$ are complex conjugates of each other, i.e., $(W^a_\mu)^* = (\bar{W}_\mu)_a$. The field W_μ transforms in the fundamental representation of the unbroken gauge group $SU(N_B-1)$, while $\bar{W}_\mu$ transforms in the antifundamental.

(i) Effective action

With the above decomposition, the Euclidean action becomes

$$S_{\mathrm{E}}[A,W,Z] = \frac{i\kappa_B}{4\pi}\int d^3x\,\mathrm{Tr}\left(AdA - \frac{2i}{3}AAA\right) + \frac{i}{4\pi}\int d^3x\,\left[2\bar{W}DW + \kappa_B Z dZ - 2iZ\bar{W}W\right] + \mathrm{sgn}(\kappa_B)\,v^2\int d^3x\sqrt{\det g}\,\left(\kappa_B Z_\mu Z^\mu + \bar{W}_\mu W^\mu\right)\ . \tag{40}$$

Here, $D_\mu = \partial_\mu - iA_\mu$ denotes the covariant derivative with respect to the unbroken gauge field A_μ. The exterior product ABC is a shorthand for $\epsilon^{\mu\nu\rho}A_\mu B_\nu C_\rho$.

At leading order, the equation of motion for A_μ becomes $F_{\mu\nu}=0$, showing that this field does not have any propagating degrees of freedom. In contrast, the fields W_μ and Z_μ obey the linearized equations

$$\frac{i\epsilon^{\mu\nu\rho}}{4\pi}2\partial_\nu W_\rho + \mathrm{sgn}(\kappa_B)\,v^2\,W^\mu = 0, \qquad \frac{i\epsilon^{\mu\nu\rho}}{4\pi}2\partial_\nu Z_\rho + \mathrm{sgn}(\kappa_B)\,v^2\cdot 2Z^\mu = 0\ . \tag{41}$$

These equations describe massive vector fields. Using the expression for W_μ and working in momentum space, it is straightforward to verify that the W-boson is indeed a propagating degree of freedom with mass

$$m_W = 2\pi v^2\ . \tag{42}$$

Together, the W and Z fields account for $2(N_B-1)+1 = 2N_B-1$ real massive degrees of freedom, precisely matching those of the original scalar field after its modulus is fixed by the σ_B equation of motion. In other words, the scalar field's condensation does not eliminate these degrees of freedom but rather it effectively 'transmutes' them from a spin zero scalar into a spin ±1 vector modes.

Thus, the finite-temperature partition function takes the form

$$\mathcal{Z} = \int [dA\,dW\,dZ]\,e^{-S_{\mathrm{E}}[A,W,Z]}\ , \tag{43}$$

where the path integral is evaluated over the Euclidean space $\mathbb{R}^2\times S^1$.

To proceed with the computation, we need to fix the unbroken $SU(N_B-1)$ gauge invariance in the action [Eq. (40)]. We choose to work in the lightcone gauge $A_- = 0$, [25] where the cubic term in the Chern-Simons action vanishes, simplifying the action [Eq. (40)] considerably. The resulting form of the

Euclidean action is

$$\begin{aligned} S_{\rm E}[A,W,Z] = \frac{i}{4\pi}\int d^3x\, \text{Tr}\left(\kappa_B\, \epsilon^{\tilde\mu - \tilde\nu}\, A_{\tilde\mu}\partial_- A_{\tilde\nu}\right) \\ + \int d^3x\, \bar W_\mu \left(\tfrac{i}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu + \text{sgn}(\kappa_B)\, v^2\, g^{\mu\rho}\right) W_\rho \\ + \int d^3x\, Z_\mu \left(\tfrac{i\kappa_B}{4\pi}\epsilon^{\mu\nu\rho}\partial_\nu + |\kappa_B|\, v^2\, g^{\mu\rho}\right) Z_\rho \\ + \frac{1}{2\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\bar W_\rho\,(A_\mu - Z_\mu)\, W_\nu\,, \end{aligned} \tag{44}$$

where the indices $\tilde\mu, \tilde\nu$ run over $+, 3$.

It is convenient to work in Fourier space, where an arbitrary field $\psi(x)$ can be written as

$$\psi(x) = \int \frac{d^3q}{(2\pi)^3}\, e^{ix\cdot q}\psi(q), \tag{45}$$

with the integration measure denoted by d^3q. At finite temperature, our theory lives on the Euclidean manifold $\mathbb{R}^2 \times S^1$. On this space, the momentum measure along $\mathbb{R}^2$ is the usual $dp_1 dp_2$, while the momentum integral along the compact S^1 direction, which we denote dp_3, takes the form

$$\int \mathcal{D}p_3\, f(p_3) = \int_{-\pi}^{\pi} \rho_B(\alpha)\, d\alpha\, \frac{2\pi}{\beta} \sum_{n=-\infty}^{\infty} f\left(\frac{2\pi n + \alpha}{\beta}\right), \tag{46}$$

where $\rho_B(\alpha)$ is the holonomy eigenvalue distribution defined earlier, and $f(p_3)$ is any function of the Matsubara momentum p_3.

Using this, the action [Eq. (44)] can be rewritten in momentum space as

$$\begin{aligned} S_{\rm E} = \int \frac{d^3p}{(2\pi)^3}\, \text{Tr}\left(\frac{A_{\tilde\mu}(-p)K^{\tilde\mu\tilde\rho}(p)A_{\tilde\rho}(p)}{2} + J_A^{\tilde\mu}(-p)A_{\tilde\mu}(p)\right) \\ + \int \frac{d^3p}{(2\pi)^3}\left(\frac{Z_\mu(-p)K_Z^{\mu\rho}(p)Z_\rho(p)}{2} + J_Z^{\mu}(-p)Z_\mu(p)\right) \\ + \int \frac{d^3p}{(2\pi)^3}\, \bar W_\mu(-p)K_W^{\mu\rho}(p)W_\rho(p), \end{aligned} \tag{47}$$

where the kernels are defined as

$$\begin{aligned} K^{\tilde\mu\tilde\rho}(p) &= \frac{-\kappa_B}{2\pi}\epsilon^{\tilde\mu - \tilde\rho}p_-, \\ K_Z^{\mu\rho}(p) &= \frac{-\kappa_B}{4\pi}\epsilon^{\mu\nu\rho}p_\nu + |\kappa_B|v^2 g^{\mu\rho}, \\ K_W^{\mu\rho}(p) &= -\frac{1}{2\pi}\epsilon^{\mu\nu\rho}p_\nu + \text{sgn}(\kappa_B)v^2 g^{\mu\rho}, \end{aligned} \tag{48}$$

and the spin-1 currents

$$\begin{aligned} J_A^{\tilde\mu}(p) &= \frac{1}{2\pi}\epsilon^{\tilde\mu\nu\rho}\int \frac{d^3q}{(2\pi)^3} W_\nu(p-q)\bar W_\rho(q), \\ J_Z^{\mu}(p) &= -\text{Tr}\, J_A^{\mu}(p). \end{aligned} \tag{49}$$

Since the terms involving A_μ and Z_μ are quadratic, they can be integrated out, resulting in the following effective action for the W_μ and $\bar{W}_\mu$ fields

$$\begin{aligned} S_{\mathrm{E}}[W] = &\int \frac{d^3p}{(2\pi)^3}\, \bar{W}_{a,\mu}(-p) K_W^{\mu\rho}(p) W_\rho^a(p) \\ &- \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3}\,[\bar{W}_\alpha W_\beta](q,-p)\,\Lambda^{\alpha\beta\alpha'\beta'}(q-q',p)\,[\bar{W}_{\alpha'}W_{\beta'}](q',p), \end{aligned} \tag{50}$$

where the interaction kernel $\Lambda^{\alpha\beta\alpha'\beta'}$ is given by the sum of two terms

$$\Lambda^{\alpha\beta\alpha'\beta'}(q-q',p) = \Lambda_A^{\alpha\beta\alpha'\beta'}(q-q') + \Lambda_Z^{\alpha\beta\alpha'\beta'}(p), \tag{51}$$

with

$$\begin{aligned} \Lambda_A^{\alpha\beta\alpha'\beta'}(q-q') &= \frac{1}{(2\pi)^2}\epsilon^{\beta\alpha'\tilde{\mu}} K^{-1}_{\tilde{\mu}\tilde{\mu}'}(q-q')\epsilon^{\tilde{\mu}'\beta'\alpha}, \\ \Lambda_Z^{\alpha\beta\alpha'\beta'}(p) &= \frac{1}{(2\pi)^2}\epsilon^{\alpha\beta\mu} K^{-1}_{Z,\mu\mu'}(p)\epsilon^{\mu'\alpha'\beta'}. \end{aligned} \tag{52}$$

The inverse kernels in Eq. (52) are defined as

$$\begin{aligned} &K^{-1}_{\tilde{\mu}\tilde{\nu}}(p) = \frac{2\pi}{\kappa_B p_-}\epsilon_{\tilde{\mu}-\tilde{\nu}}, \quad K^{-1}_{\tilde{\mu}\tilde{\nu}}(p)K^{\tilde{\nu}\tilde{\rho}}(p) = \delta^{\tilde{\rho}}_{\tilde{\mu}}, \\ &K^{-1}_{Z,\mu\nu}(p) = \frac{-2\pi m_Z}{|\kappa_B|(p^2+m_Z^2)}\left(\delta_{\mu\nu} - \mathrm{sgn}(\kappa_B)\epsilon_{\mu\nu\rho}\frac{p^\rho}{m_Z} + \frac{p_\mu p_\nu}{m_Z^2}\right), \quad K^{-1}_{Z,\mu\nu}K_Z^{\nu\rho} = \delta^\rho_\mu\,. \end{aligned} \tag{53}$$

We used a shorthand notation $[\bar{W}_\alpha W_\beta]$ for the bilinear appearing in Eq. (50),

$$[\bar{W}_\alpha W_\beta](q,p) \equiv \bar{W}_\alpha\left(q+\frac{p}{2}\right) W_\beta\left(-q+\frac{p}{2}\right), \tag{54}$$

where p is the center-of-mass momentum of the bilinear field and q the relative momentum.

The interaction kernel $\Lambda^{\mu\nu\mu'\nu'}$ can be further simplified to give

$$\Lambda^{\mu\nu\mu'\nu'}(q-q',0) = \frac{1}{2\pi\kappa_B(q-q')_-}\left(\epsilon^{\nu\mu'\nu'}\delta^\mu_- - \epsilon^{\nu\mu'\mu}\delta^{\nu'}_-\right) - \frac{1}{2\pi|\kappa_B|m_Z}\left(\delta^{\mu\mu'}\delta^{\nu\nu'} - \delta^{\mu\nu'}\delta^{\nu\mu'}\right). \tag{55}$$

For reference, we list here some useful symmetry properties of this interaction kernel:

$$\begin{aligned} &\Lambda_A^{\mu\nu\mu'\nu'}(p) = -\Lambda_A^{\mu\nu\mu'\nu'}(-p) = -\Lambda_A^{\mu\mu'\nu\nu'}(p) = -\Lambda^{\nu'\nu\mu'\mu}(p)\,, \\ &\Lambda_Z^{\mu\nu\mu'\nu'}(0) = -\Lambda_Z^{\nu\mu\mu'\nu'}(0) = -\Lambda_Z^{\mu\nu\nu'\mu'}(0)\,. \end{aligned} \tag{56}$$

Integrating out the gauge fields A_μ and Z_μ gives rise to the determinants $\det_A$ and $\det_Z$, respectively. As a result, the final path integral to be evaluated is

$$\mathcal{Z} = \int [dW]\, e^{-S_{\mathrm{E}}[W]}\,\det{}_A\,\det{}_Z\;, \tag{57}$$

However, due to our choice of gauge $A_- = 0$, the determinant $\det_A$ vanishes. Moreover, the contribution from $\det_Z$ is suppressed at large N_B and can be neglected in a $1/N_B$ expansion.

The integral in Eq. (50) is quartic in the field W. However, to simplify the analysis we take advantage of the large-N_B limit and employ the Hubbard-Stratonovich trick. This method allows us to rewrite the quartic interaction in terms of auxiliary fields. Specifically, we introduce two bilocal, $SU(N_B - 1)$ singlet auxiliary fields—$\Sigma^{\mu\nu}(q,p)$ and $\alpha_{\mu\nu}(q,p)$—into the path integral using the identity

$$
\begin{aligned}
1 &= \int [d\alpha]\, \delta\left[\kappa_B \alpha_{\mu\nu}(q,p) + [\bar{W}_\mu W_\nu](q,p)\right] \\
&= \int [d\alpha][d\Sigma]\, \exp\left(\int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3}\, i\Sigma^{\nu\mu}(-q,-p) \left(\kappa_B \alpha_{\mu\nu}(q,p) + [\bar{W}_\mu W_\nu](q,p)\right)\right) .
\end{aligned}
\tag{58}
$$

Inserting the identity[Eq. (58)] into the path integral allows us to rewrite the action [Eq. (50)] as

$$
\begin{aligned}
\frac{S_E[\alpha,\Sigma,W]}{N_B} = &-\frac{i}{\lambda_B} \int \frac{\mathcal{D}^3p}{(2\pi)^3} \frac{\mathcal{D}^3q}{(2\pi)^3} \Sigma^{\nu\mu}(q,p)\alpha_{\mu\nu}(-q,-p) \\
&+ \frac{1}{N_B} \int \frac{\mathcal{D}^3q}{(2\pi)^3} \frac{\mathcal{D}^3p}{(2\pi)^3} \bar{W}_\mu(-q - \tfrac{p}{2})\, Q^{\mu\nu}(q,p)\, W_\nu(q - \tfrac{p}{2}) \\
&- \frac{1}{2\lambda_B} \int \frac{\mathcal{D}^3p}{(2\pi)^3} \frac{\mathcal{D}^3q}{(2\pi)^3} \frac{\mathcal{D}^3q'}{(2\pi)^3}\, \alpha_{\mu\nu}(q,-p)\, \kappa_B \Lambda^{\mu\nu\mu'\nu'}(q - q', p)\, \alpha_{\mu'\nu'}(q',p),
\end{aligned}
\tag{59}
$$

where

$$
Q^{\mu\nu}(q,p) = (2\pi)^3 \delta(p) K_W^{\mu\nu}(q) - i\Sigma^{\nu\mu}(q,p) \,. \tag{60}
$$

Finally, we express the effective action in a compact and convenient form as

$$
\begin{aligned}
\frac{S_E[\alpha,\Sigma,W]}{N_B} = &\frac{1}{N_B} \int \frac{\mathcal{D}^3q}{(2\pi)^3} \frac{\mathcal{D}^3p}{(2\pi)^3} \bar{W}_\mu(-q - \tfrac{p}{2})\, Q^{\mu\nu}(q,p)\, W_\nu(q - \tfrac{p}{2}) \\
&+ V[\alpha] - \frac{i}{\lambda_B} \int \frac{\mathcal{D}^3p}{(2\pi)^3} \frac{\mathcal{D}^3q}{(2\pi)^3} \Sigma^{\nu\mu}(q,p)\alpha_{\mu\nu}(-q,-p).
\end{aligned}
\tag{61}
$$

where the function $V[\alpha]$ encodes the interaction among the auxiliary fields and is defined as

$$
V[\alpha] = -\frac{1}{2\lambda_B} \int \frac{\mathcal{D}^3p}{(2\pi)^3} \frac{\mathcal{D}^3q}{(2\pi)^3} \frac{\mathcal{D}^3q'}{(2\pi)^3}\, \alpha_{\mu\nu}(q,-p)\, \kappa_B \Lambda^{\mu\nu\mu'\nu'}(q - q', p)\, \alpha_{\mu'\nu'}(q',p) \,. \tag{62}
$$

Since the effective action [Eq. (61)] is quadratic in the W-boson fields, they can be integrated out exactly. This results in an effective action for the bilocal auxiliary fields

$$
S_{\text{eff}}[\alpha,\Sigma] = N_B \left(-\frac{i}{\lambda_B}\, \Sigma \cdot \alpha + \log\det Q + V[\alpha] \right) . \tag{63}
$$

Since this effective action is proportional to N_B, the path integral over α and Σ can be evaluated, at leading order in $1/N_B$, using the saddle point approximation. We assume that the saddle point respects translational invariance, so the auxiliary fields take the form

$$
\begin{aligned}
\Sigma^{\mu\nu}(q,p) &= (2\pi)^3 \delta(p) \Sigma^{\mu\nu}(q) \,, \\
\alpha_{\mu\nu}(q,p) &= (2\pi)^3 \delta(p) \alpha_{\mu\nu}(q) \,.
\end{aligned}
\tag{64}
$$

With this assumption, the expression for $Q^{\mu\nu}(q,p)$ in Eq. (60) also simplifies to

$$Q(q,p) = (2\pi)^3\delta(p)Q(q)\ , \quad \text{with} \quad Q(q) = K_W(q) - i\Sigma^T(q)\ . \tag{65}$$

The Gaussian path integral over W is now straightforward, yielding the effective action

$$\frac{S_{\text{eff}}[\alpha,\Sigma]}{N_B\mathcal{V}_3} = V_0[\alpha] - \frac{i}{\lambda_B}\int\frac{\mathcal{D}^3q}{(2\pi)^3}\Sigma^{\nu\mu}(q)\alpha_{\mu\nu}(-q) + \int\frac{\mathcal{D}^3q}{(2\pi)^3}\log\det\left(K_W(q) - i\Sigma^T(q)\right)\ , \tag{66}$$

where $\mathcal{V}_3 = \mathcal{V}_2\beta$ is the spacetime volume. The term $V_0[\alpha]$ is obtained by setting the center-of-mass momentum $p = 0$ in the integrand of $V[\alpha]$ in and dividing by $\mathcal{V}_3$ [Eq. (62)]

$$V_0[\alpha] = -\frac{1}{2\lambda_B}\int\frac{\mathcal{D}^3q}{(2\pi)^3}\frac{\mathcal{D}^3q'}{(2\pi)^3}\,\alpha_{\mu\nu}(q)\,\kappa_B\Lambda^{\mu\nu\mu'\nu'}(q-q',0)\,\alpha_{\mu'\nu'}(q')\ . \tag{67}$$

Equation (66) is the final form of the effective action for the Chern-Simons gauged critical boson theory in the Higgsed phase.

(ii) Gap Equations

By varying the effective action [Eq. (66)] with respect to $\Sigma^{\mu\nu}(-q)$ and $\alpha_{\mu\nu}(-q)$, we obtain the corresponding *gap equations*. These equations take the following form:

$$\alpha_{\nu\mu}(q) = -\lambda_B(Q^{-1}(q))_{\nu\mu}, \tag{68}$$

$$\Sigma^{\nu\mu}(q) = -\frac{i}{4\pi}\int\frac{\mathcal{D}^3q'}{(2\pi)^3}\left(\epsilon^{\nu\mu'\nu'}\delta^{\mu}_{-} - \epsilon^{\nu\mu'\mu}\delta^{\nu'}_{-} - \epsilon^{\nu'\mu\nu}\delta^{\mu'}_{-} + \epsilon^{\nu'\mu\mu'}\delta^{\nu}_{-}\right)\frac{\alpha_{\mu'\nu'}(q')}{(q'+q)_-} \tag{69}$$

where $\alpha_{\mu\nu}$, $\Sigma^{\mu\nu}$ and $Q^{\mu\nu}$ enjoy the following symmetries at all orders in λ_B,

$$\begin{aligned}
\alpha^{\mu\nu}(q) &= \alpha^{\nu\mu}(-q), \quad \text{i.e.,} \quad \alpha(q) = \alpha^T(-q),\\
\Sigma^{\mu\nu}(q) &= \Sigma^{\nu\mu}(-q), \quad \text{i.e.,} \quad \Sigma(q) = \Sigma^T(-q),\\
Q^{\mu\nu}(q) &= Q^{\nu\mu}(-q), \quad \text{i.e.,} \quad Q(q) = Q^T(-q).
\end{aligned} \tag{70}$$

In this formulation, $\alpha_{\mu\nu}$ serves as the gauge-invariant propagator, while $\Sigma^{\mu\nu}$ plays the role of the self-energy.

From the gap Eq. (69), one can see that each component of $\Sigma^{\mu\nu}$ is independent of q_3 and depends only on q^+ and q^-. We choose to parametrize the non-zero components of $\Sigma^{\mu\nu}$ as

$$\begin{aligned}
\Sigma^{--}(q) &= \frac{1}{2\pi q_-^2}F_1(w),\\
\Sigma^{+-}(q) &= +\Sigma^{-+}(q) = \frac{1}{2\pi}F_2(w),\\
\Sigma^{3-}(q) &= -\Sigma^{-3}(q) = \frac{1}{2\pi q_-}F_3(w),\\
\Sigma^{3+}(q) &= -\Sigma^{+3}(q) = \frac{q_-}{2\pi}F_4(w),
\end{aligned} \tag{71}$$

where

$$w = q_s^2 = 2q_+q_- \; . \tag{72}$$

In terms of these new parameters F_1, F_2, F_3, F_4, the matrix $Q^{\mu\nu}$ becomes

$$Q^{\mu\nu}(q) = \frac{1}{2\pi}\begin{bmatrix} 0 & -i(F_2+im+q_3) & iq_-(1-F_4) \\ -i(F_2+im-q_3) & -\frac{i}{q_-^2}F_1(w) & -\frac{i}{q_-}(F_3+\frac{w}{2}) \\ -iq_-(1-F_4) & \frac{i}{q_-}(F_3+\frac{w}{2}) & m \end{bmatrix} , \tag{73}$$

where

$$m \equiv -\frac{\lambda_B m_B^{\rm cri}}{2} = \text{sgn}(\kappa_B)2\pi v^2 = |m_W| \; . \tag{74}$$

Note that m changes sign as κ_B changes sign.

$\det Q = 0$ gives the pole mass of the W-boson propagator, where we compute

$$\det Q = -\frac{m}{8\pi^3}\left(q^2 + M^2(w)\right) , \tag{75}$$

with

$$\begin{aligned} q^2 = w + q_3^2 \;\; &= \;\; 2q_+q_- + q_3^2 \\ M^2(w) = -(F_2+im)^2 - \frac{i}{m}F_1(1-F_4)^2 &- \frac{i}{m}(F_2+im)(w+2F_3)(1-F_4) - w \; . \end{aligned} \tag{76}$$

Using Eqs. (68), (71), and (73), we now rewrite the gap equations in terms of the four unknown single-variable functions $F_1, \ldots, F_4$. Since there are just four independent components of Σ, there will be just four integral equations [26]

$$\begin{aligned} F_1(w) &= -\frac{\lambda_B}{(2\pi)^2}\int_0^w \frac{dw'}{4\pi}\chi(w')\left(2F_1(1-F_4) + (F_2+im)(2F_3+w')\right) , \\ F_2(w) &= \frac{\lambda_B}{(2\pi)^2}\int_w^\infty \frac{dw'}{4\pi}\chi(w')(1-F_4)(F_2+im), \\ F_3(w) &= \frac{\lambda_B}{(2\pi)^2}\int_0^w \frac{dw'}{4\pi}\chi(w')\left((F_3+\tfrac{w'}{2})(1-F_4) - im(F_2+im)\right) , \\ F_4(w) &= \frac{\lambda_B}{(2\pi)^2}\int_w^\infty \frac{dw'}{4\pi}\chi(w')(1-F_4)^2. \end{aligned} \tag{77}$$

The function $\chi(w)$ appears as we 'integrate over q_3', which is actually taking a discrete summation,

$$\int \frac{\mathcal{D}q_3}{2\pi} f(q_3) = \int d\alpha \rho_B(\alpha)\beta^{-1}\sum_{n\in\mathbb{Z}} f\left(\frac{\alpha+2\pi n}{\beta}\right) . \tag{78}$$

$$\begin{aligned}\chi(z) &\equiv -\frac{(2\pi)^3}{m\beta}\int d\alpha\rho_B(\alpha)\sum_{n\in\mathbb{Z}}\frac{1}{(2\pi\frac{n}{\beta}+\frac{\alpha}{\beta})^2+(z+M^2(z))},\\ &= -\frac{2\pi^3}{m}\int d\alpha\rho_B(\alpha)\frac{1}{\sqrt{z+M^2(z)}}\left(\coth\left(\tfrac{\beta}{2}(\sqrt{z+M^2(z)}+i\tfrac{\alpha}{\beta})\right)+\coth\left(\tfrac{\beta}{2}(\sqrt{z+M^2(z)}-i\tfrac{\alpha}{\beta})\right)\right).\end{aligned} \tag{79}$$

The four classical coupled integral equations are highly non-linear and challenging, and solving them exactly seems almost impossible. However, using physical intuition, it is possible to find exact solutions to these equations. The key insight is that the zeros of $\det Q$ are physical, implying that $\det Q$ must be simple.

It turns out that the pole mass $M(w)$ is a constant, specifically $M(w) = M = \frac{c_B}{\beta}$. The solutions to the four coupled integral equations (77) are given by,

$$\begin{aligned}F_1(w) &= img(w)\left(M^2\left(g(w)-g(0)\right)-\frac{m^2}{3}\left(g(w)^3-g(0)^3\right)+wg(w)-\mathcal{I}(w)\right),\\ F_2(w) &= im\lambda_B\xi(w),\\ F_3(w) &= -\frac{w}{2}+\frac{1}{g(w)}\left(\frac{1}{2}\mathcal{I}(w)-\frac{m^2}{3}\left(g(w)^3-g(0)^3\right)\right),\\ F_4(w) &= 1-\frac{1}{1+\lambda_B\xi(w)},\end{aligned} \tag{80}$$

where

$$\begin{aligned}\xi(z) &= -\frac{1}{2(2\pi)^3}\int^z dw'\chi(w'),\\ &= \frac{1}{2m\beta}\int d\alpha\rho_B(\alpha)\Big[\log 2\sinh(\tfrac{\beta}{2}(\sqrt{z+M^2(z)}+i\tfrac{\alpha}{\beta}))+\log 2\sinh(\tfrac{\beta}{2}(\sqrt{z+M^2(z)}-i\tfrac{\alpha}{\beta}))\Big].\end{aligned} \tag{81}$$

and

$$\begin{aligned}g(w) &= 1+\lambda_B\,\xi(w),\\ \mathcal{I}(w) &= \int_0^w g(z)\,dz.\end{aligned} \tag{82}$$

The functions $F_1(w)$, $F_2(w)$, $F_3(w)$, and $F_4(w)$ satisfy the gap equations, provided that the constant $M = \frac{c_B}{\beta}$ satisfies the following equation:

$$(2c_B)^2 = \left(-\lambda_B\hat{m}_B^{\text{cri}}+2\lambda_B\mathcal{S}\right)^2 = \left(-|\lambda_B|\hat{m}_B^{\text{cri}}+2|\lambda_B|\mathcal{S}\right)^2\ , \tag{83}$$

where $\hat{m}_B^{\text{cri}} = \beta m_B^{\text{cri}}$ and $\mathcal{S}(c_B)$ is given by,

$$\mathcal{S}(c_B) = \frac{1}{2}\int d\alpha\,\rho_B(\alpha)\left[\log 2\sinh\left(\frac{\beta}{2}\left(M+i\frac{\alpha}{\beta}\right)\right)+\log 2\sinh\left(\frac{\beta}{2}\left(M-i\frac{\alpha}{\beta}\right)\right)\right]. \tag{84}$$

Equation (83) is the *gap equation* for the thermal mass of the W-bosons, and it matches precisely with the fermionic gap Eq. (24), provided the branch is chosen according to Eq. (27). This indicates that, under the duality, the pole mass of the W-boson corresponds to the pole mass of the fermionic excitations.

(iii) Free Energy

Recall that the free energy functional v_B is obtained, in the saddle-point approximation, from the effective action for the auxiliary fields α and Σ [see Eq. (66)]:

$$\begin{aligned}\mathcal{V}_2 T^2 v_B(|c_B|, \rho_B) = S_{\text{eff}}[\alpha, \Sigma] = N_B \mathcal{V}_3 \bigg(V_0[\alpha] - \frac{i}{\lambda_B} \int \frac{\mathcal{D}^3 q}{(2\pi)^3} \Sigma^{\nu\mu}(q) \alpha_{\mu\nu}(-q) \\ + \int \frac{\mathcal{D}^3 q}{(2\pi)^3} \log\det\left(K_W(q) - i\Sigma^T(q)\right) \bigg) .\end{aligned} \tag{85}$$

The term $V_0[\alpha]$ in this expression is somewhat complicated, as it involves a double momentum integral, even when evaluated on translationally invariant solutions. However, on-shell (i.e., when the equations of motion are satisfied), this term can be simplified. Since $V_0[\alpha]$ is a homogeneous polynomial of degree 2 in α, we can use the equation of motion for α to write

$$V_0[\alpha] = \frac{1}{2} \frac{\delta V_0}{\delta \alpha} \cdot \alpha = \frac{1}{2} \frac{i}{\lambda_B} \Sigma \cdot \alpha \, . \tag{86}$$

Substituting this back into Eq. (85), the on-shell effective action takes a simpler form

$$S_{\text{eff}} = N_B \mathcal{V}_3 \bigg(- \frac{i}{2\lambda_B} \int \frac{\mathcal{D}^3 q}{(2\pi)^3} \Sigma^{\nu\mu}(q) \alpha_{\mu\nu}(-q) + \int \frac{\mathcal{D}^3 q}{(2\pi)^3} \log\det\left(K_W(q) - i\Sigma^T(q)\right) \bigg) . \tag{87}$$

From this expression, after a detailed and nontrivial computation [27], one can finally show that the free energy of the Chern-Simons gauged critical boson in the Higgsed phase is given by

$$\begin{aligned} v_B^{\text{Higgsed}}(|c_B|, \rho_B) = \frac{N_B}{6\pi} \bigg[3|\lambda_B||\hat{m}|\mathcal{S}^2 + 2|\lambda_B|^2 \mathcal{S}^3 - |c_B|^3 \\ + 3 \int_{-\pi}^{\pi} \rho_B(\alpha)\, d\alpha \int_{|c_B|}^{\infty} dy\, y \left(\log\left(1 - e^{-y - i\alpha}\right) + \log\left(1 - e^{-y + i\alpha}\right) \right) \bigg] . \end{aligned} \tag{88}$$

It is straightforward to check that the bosonic free energy in Eq. (88) exactly maps to the fermionic free energy [Eq. (28)] under the same duality transformation prescribed in the subsection .

IV. SUMMARY AND OUTLOOK

In this review, we explained several important dualities and their significance in understanding various physical phenomena. Our primary focus was on dualities in three-dimensional Chern-Simons-matter theories. In particular, we examined how to calculate the thermal free energy of the large N_B Chern-Simons gauged critical boson theory in its Higgsed phase, and found that our results align perfectly with the predictions from its proposed fermionic dual. Specifically, we demonstrated that the pole mass of the W boson corresponds to the pole mass of the fermionic excitations under duality. This suggests that, under duality, the elementary fermions— which map to scalar excitations in the unHiggsed phase—transform into W bosons in the Higgsed phase.

An important aspect of our theories is that the thermal free energy behaves as an analytic function of mass—a feature confirmed by our explicit large N_B computations, though in a rather unexpected way. In the bosonic theory, the free energy at finite temperature is computed differently depending on whether the theory is in its Higgsed or unHiggsed phase. These two calculations rely on distinct sets of degrees of freedom, are valid in separate regimes, and are dominated by very different saddle points. Despite these differences, the final expressions for the free energy in both phases turn out to be analytic continuations of one another. This outcome is somewhat surprising and suggests a deeper structure, which deserves further investigation.

The partition function of the scalar theory in the Higgsed phase on $S^2 \times S^1$ can be computed as an integral over holonomies, with the integrand given by the free energy $v_B[\rho]$ derived in this paper. From a physical standpoint, it would be valuable to study this integral more closely, especially in the presence of a finite chemical potential. In the large volume limit, the eigenvalue distribution at the saddle point is expected to approach a universal "tabletop" form [28]. However, at finite volume, deviations from this universal form are expected. These deviations can lead to a rich phase structure, featuring multiple interesting phase transitions, extending the analysis presented in [3].

As a continuation of our work [20], we later extended our analysis to other theories—specifically, the Chern-Simons gauged regular (free) boson and the Chern-Simons gauged critical (Gross–Neveu) fermion—in subsequent work [21]. We also carried out an in-depth study of the phase structure of Chern-Simons gauged $\mathcal{N} = 2$ supersymmetric theory, which includes a single fundamental fermion and boson [22]. Our findings revealed that, at zero temperature, the two dimensional phase diagram of this theory consists of four distinct topological phases separated by lines of first and second order phase transitions.

There are several interesting directions for future work. One promising avenue is the computation of three-point correlation functions of conserved currents with specific spins in the Higgsed phase of the Chern-Simons (CS) gauged critical boson theory. Comparing these results with those from the regular fermion theory would provide a direct test of the proposed Bose-Fermi duality. It would also be interesting to use the techniques developed in [20] to extend the S-matrix calculations of [14, 15] to the bosonic theory in the Higgsed phase. The goal would be to compare the resulting bosonic S-matrices with their fermionic counterparts, as predicted by duality. Another natural extension is to explore how these dualities behave in the presence of boundaries. Recent advances in boundary conformal field theory (BCFT) [23] provide useful tools for this analysis. Understanding how the duality is modified or preserved when a boundary is introduced could offer deeper insights into its robustness. It would also be fascinating to study entanglement entropy in CS-matter theories as a further test of the duality. Since entanglement entropy captures the number of degrees of freedom in a quantum system, computing it in both the CS gauged regular fermion and the CS gauged critical boson theories could serve as an independent and nontrivial check of the duality.

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[24] Equation (35) is the identity used in establishing the equivalence of the bosonic and fermionic gap equations.

[25] Our conventions are as follows:

$$x^{\pm} = \frac{x^1 \pm ix^2}{\sqrt{2}}, \quad p_{\mp} = \frac{p^1 \pm ip^2}{\sqrt{2}}, \quad A_{\mp} = \frac{A^1 \pm iA^2}{\sqrt{2}} \,. \tag{89}$$

In these coordinates, the non-vanishing components of the metric are $g_{+-} = g_{-+} = g_{33} = 1$, and the Levi-Civita symbol is defined as $\epsilon^{+-3} = \epsilon_{-+3} = -i$. The Kronecker delta is given by $\delta^{\mu}_{\nu} = 1$ if $\mu = \nu$ and 0 otherwise.

[26] For the detailed derivation of these equations, see [20].

[27] See [20] for the detailed derivation.

[28] This distribution is given by $\rho(\alpha) = 0$ for $|\alpha| > \pi|\lambda_B|$, and $\rho(\alpha) = \frac{1}{2\pi|\lambda_B|}$ for $|\alpha| < \pi|\lambda_B|$.

