J. Phys. Res. Edu., Vol. 2, March 2025

Phase plane of a non-canonical scalar field model and cosmological consequences

Agnidipto Bhattacharya[1] and Sujay Kr. Biswas[2]*

[1]*Department of Mathematics, Maulana Abul Kalam Azad University of Technology Haringhata Farm, West Bengal 741249.*

[2]*Department of Mathematics, University of North Bengal, Raja Ramohunpur, 734013, Darjeeling, India.*

This work deals with an investigation of phase plane analysis for a non-canonical scalar field in the background dynamics of spatially flat, homogeneous and isotropic Friedmann-Robertson-Walker (FRW) universe. Here, gravity is nonminimally coupled to scalar field having a real scalar field potential explicitly depending on cosmic time 't'. Nonlinear field equations are reduced to an autonomous system of ordinary differential equations by the suitable choice of dynamical variables. From the analysis of critical points, we have found the future decelerating phase of evolution of the universe.

I. INTRODUCTION

According to a number of astrophysical observations [1, 2], our universe is presently expanding at an accelerated rate, and there was a changeover from a deceleration phase to an acceleration phase in the recent past. Theoretically, such acceleration of cosmic evolution has been addressed by introducing some exotic type of fluid in Einstein's equations called Dark Energy (DE) which has huge negative pressure and is responsible for present acceleration.(Equation of State parameter $\omega < -\frac{1}{3}$). According to various sources of observations, it is believed that near about 96% of energy density of the total is coming from dark sector in which nearly 73% is DE and remaining 23% is Dark Matter, another mysterious object of the universe. The "Cosmological Constant" is the most straightforward of the several dark energy models that have been examined in the literature [3–7], but it has some theoretical problems like 'Cosmological Constant problem' or 'fine tuning' problem and the 'Coincidence problem' [8] . The cosmological constant problem can be solved by introducing dynamical dark energy models such as 'Quintessence' [9] based on the canonical scalar field and 'k-essence' [10] is an example of noncanonical scalar field model. The canonical scalar field plays an important role exhibiting the nature of DE due to its simple dynamics but cannot cross the phantom divide line ($\omega < -1$). The non-canonical scalar field [11] is a more general description of the canonical scalar field. The non-canonical scalar field's biggest advantage over the canonical scalar field is its ability to mitigate the coincidence problem without requiring fine-tuning.

 © 2025 Dept. of Physics, NBU

In this case, we examine a non-canonical scalar field whose potential term, 'V'is explicitly dependent on cosmic time, 't' Next, we use dynamical system analysis to investigate the dynamics of evolution.[[12–18]]. We transform the evolution equations into an independent system of ordinary differential equations because they are extremely non-linear. Next, we make first-order perturbations to the system near critical points. By analyzing the linearized Jacobian matrix's eigenvalues, the nature of crucial spots is examined. We expand upon the work of [19], which examines a non-canonical scalar field with particular scalar field potential selections in the context of dynamical system analysis. J. Dutta et al. [20] recently looked into non-canonical scalar fields with the means of dynamical analysis.

The organization part of the work is executed as follows: In the next section, we construct the autonomous system of ODEs from cosmological equations and the cosmological parameters with suitable dynamical variables. In section-III, we discuss about the critical points and the corresponding physical parameters, Eigen values of linearized Jacobian matrix and the phase plane analysis in detail. Section-IV consists of a short discussion about the possible future cosmological consequences of critical points and section-V concludes the work.

II. FORMATION OF AUTONOMOUS SYSTEM AND COSMOLOGICAL PARAMETERS

With the coupling parameter and scalar field, the action for a non-minimally coupled scalar field to gravity takes the following form

$$A = \int d^4x\sqrt{-g}[\frac{R}{2k^2} - \frac{1}{2}\lambda(\phi)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)] + \int d^4x\sqrt{-g}\mathcal{L}_m. \tag{1}$$

where the gravitational coupling constant is $k^2 = 8\pi G$, the coupling function of the scalar field ϕ is $\lambda(\phi)$, the scalar field potential is $V(\phi)$, and the matter Lagrangian density is $\mathcal{L}_m$.
When the action is changed relative to the metric in the spatially flat FRW model of the universe, the Friedmann and acceleration equations are obtained as follows

$$3H^2 = \rho_m + \rho_\phi \tag{2}$$

$$\dot{H} = \frac{1}{2}(\rho_m + \rho_\phi + p_m + p_\phi) \tag{3}$$

where we suppose that $a(t)$ is the scale factor that depends on cosmic time "t," $k^2 = 8\pi G = c = 1$, and $H = \frac{\dot{a}}{a}$ is the Hubble parameter. The energy density and thermodynamic pressure of the non-canonical scalar field, which is non-minimally coupled to gravity, are as follows

$$\rho_\phi = \frac{1}{2}\lambda(\phi)\dot{\phi}^2 + V(\phi) \tag{4}$$

 © 2025 Dept. of Physics, NBU

$$p_\phi = \frac{1}{2}\lambda(\phi)\dot{\phi}^2 - V(\phi) \tag{5}$$

respectively, where $\lambda(\phi)$ represents the coupling in the scalar field; the scalar field behaves as quintessence when $\lambda(\phi) = 1$. The scalar field's equation of state parameter is

$$\omega_\phi = \frac{p_\phi}{\rho_\phi}. \tag{6}$$

Dark matter has a barotropic equation of state and is a perfect fluid

$$\omega_m = \frac{p_m}{\rho_m}, \tag{7}$$

where

$$\omega_m = \begin{cases} 0 & \text{dark matter behaves as dust fluid} \\ 1 & \text{dark matter exhibits radiation fluid behavior} \end{cases}$$

The energy conservation equations for DM and scalar field are given by

$$\dot{\rho_m} + 3H\rho_m(1 + \omega_m) = 0 \tag{8}$$

$$\dot{\rho_\phi} + 3H\rho_\phi(1 + \omega_\phi) = 0 \tag{9}$$

and the evolution equation for scalar field [from Eq. (9) using Eq. (4)] is given by

$$\frac{1}{2}\lambda'\dot{\phi}^2 + \lambda\ddot{\phi} + V' + 3H\lambda\dot{\phi} = 0 \tag{10}$$

Due to complicated nature of evolution [Eq. (2), Eq. (3), Eq. (8), and Eq. (10)], one cannot obtain the exact analytical solution, so to get a qualitative nature of solution, we develop an independent system of ODEs from which the phase plane image is derived. For this, we choose the dimensionless dynamical variables as follows:

$$x^2 = \frac{\lambda}{6}\frac{\dot{\phi}^2}{H^2}, \quad y^2 = \frac{V}{3H^2} \tag{11}$$

The Hubble scale is used to normalize them.

The autonomous system (after some algebraic calculations) is

$$\frac{dx}{dN} = -3x + \frac{3}{2}x\{1 + x^2 - y^2 + \omega_m(1 - x^2 - y^2)\} - \sqrt{\frac{3}{2}}y^2\frac{V'}{V\sqrt{\lambda}} \tag{12}$$

$$\frac{dy}{dN} = y\left[\sqrt{\frac{3}{2}}\frac{V'}{V\sqrt{\lambda}} + \frac{3}{2}\{1 + x^2 - y^2 + \omega_m(1 - x^2 - y^2)\}\right] \tag{13}$$

© 2025 Dept. of Physics, NBU

where the e-folding parameter is $N = ln(a)$.

It is possible to select the scalar field's potential function $V(\phi(t))$ and coupling $\lambda(\phi(t))$ so that $\frac{V'}{V\sqrt{\lambda}}$ is either constant or a function of x, y, or both. In the work, the authors have studied for $\frac{V'}{V\sqrt{\lambda}}$ is constant and is a function of x only. They have considered $V = V_0 a$, where scale factor $a(t)$ is expressed in terms of the scalar field ϕ and in a 2D autonomous system they obtained Einstein's empty universe only. Here, we shall extend the analysis. considering

$$\frac{V'}{V\sqrt{\lambda}} = -\sqrt{2}\frac{y}{x}$$

so that the potential takes the form

$$V = \frac{1}{(t + V_0)^2}$$

where t is the function of the scalar field ϕ. Then the autonomous system [Eq. (12) and Eq. (13)] reduces to

$$\frac{dx}{dN} = -3x + \frac{3}{2}x\{1 + x^2 - y^2 + \omega_m(1 - x^2 - y^2)\} + \frac{\sqrt{3}y^3}{x} \tag{14}$$

$$\frac{dy}{dN} = y\left[-\sqrt{3}y + \frac{3}{2}\{1 + x^2 - y^2 + \omega_m(1 - x^2 - y^2)\}\right] \tag{15}$$

There is a mathematical singularity at $x = 0$ in Eq. (14). The following are the pertinent cosmological parameters expressed in terms of dynamical variables:

Equation of state parameter for DE: $\omega_\phi = \frac{x^2 - y^2}{x^2 + y^2}$

Matter's density parameter: $\Omega_m = 1 - x^2 - y^2$

Parameter of density for the scalar field DE: $\Omega_\phi = x^2 + y^2$

Parameter of the effective equation of state: $\omega_{eff} = x^2(1 - \omega_m) - y^2(1 + \omega_m) + \omega_m$

Deceleration parameter: $q = -1 + \frac{3}{2}\{1 + x^2 - y^2 + \omega_m(1 - x^2 - y^2)\}$

III. CRITICAL POINTS OF THE AUTONOMOUS SYSTEM AND PHASE SPACE ANALYSIS

The critical points of the above autonomous systems [Eq. (14)-Eq. (15)] are as follows:

A $(1, 0)$ B $(-1, 0)$

C $\left(\sqrt{\frac{\sqrt{13}-1}{6}}, \frac{\sqrt{13}-1}{2\sqrt{3}}\right) \approx (0.66, 0.75)$ and

D $\left(-\sqrt{\frac{\sqrt{13}-1}{6}}, \frac{\sqrt{13}-1}{2\sqrt{3}}\right) \approx (-0.66, 0.75)$

We have taken critical points C and D in the above form for mathematical simplicity only. Although there are other critical points $\left(\pm\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ for $\omega_m = 0$ and $\left(\pm 2\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right)$ for $\omega_m = \frac{1}{3}$ obtained from the autonomous

© 2025 Dept. of Physics, NBU

system, but they do not satisfy the energy conditions (i.e., out of phase boundary $0 \leq x^2 + y^2 \leq 1$). The value of Ω_ϕ is greater than 1. These points are discarded from our analysis

Table I displays the critical points and the physical parameters that correspond to them, when dark matter is taken as pressure-less dust matter ($\omega_m = 0$) and in Table II for radiation ($\omega_m = \frac{1}{3}$). The tables also display the linearized Jacobian matrix's eigenvalues appropriately.

Critical points	x	y	Ω_ϕ	Ω_m	ω_ϕ	ω_{eff}	q	eigen values
A	1	0	1	0	1	1	2	3,3
B	-1	0	1	0	1	1	2	3,3
C	0.66	0.75	0.9981	0.0019	-0.127	-0.1269	0.31	-0.39, -4.67
D	-0.66	0.75	0.9981	0.0019	-0.127	-0.1269	0.31	-0.39, -4.67

TABLE I. Critical points, corresponding physical parameters and Eigen values for dust ($\omega_m = 0$)

Critical points	x	y	Ω_ϕ	Ω_m	ω_ϕ	ω_{eff}	q	eigen values
A	1	0	1	0	1	1	2	2,3
B	-1	0	1	0	1	1	2	2,3
C	0.66	0.75	0.9981	0.0019	-0.127	-0.1263	0.30	-1.383, -4.664
D	-0.66	0.75	0.9981	0.0019	-0.127	-0.1263	0.30	-1.383, -4.664

TABLE II. Critical points, corresponding physical parameters and Eigen values for dust ($\omega_m = \frac{1}{3}$)

From the autonomous system, we obtain a 2D phase plane (x-y plane). In Table-1 for dust matter we have obtained four critical points in the phase-plane. The critical points always exist in the phase plane. All points are hyperbolic in nature in the plane because they have non- zero eigenvalues. Due to their entire DE dominance (Ω_ϕ = 1) and the stiff fluid behavior of DE, points A (1,0) and B (-1,0) have the same nature in the phase plane. Since q = 2, there is always deceleration close to the points. Also, they behave as unstable node in the phase plane which are shown in Fig 1. when DM is taken as dust. The points A and B have the same nature when DM is taken as radiation like fluid. Figure 2 shows that the points A and B are unstable nodes in phase plane. The critical points C (0.66, 0.75) and D (-0.66, 0.75) are same in all respect. Since all of the eigenvalues are non-zero, they represent a hyperbolic mixture of DE and DM matter. Figure 1 shows that ω_m = 0 and Fig. 2 shows that ω_m = $\frac{1}{3}$. These nodes are stable in the phase-plane. Thus, the points show the universe's late-time evolution. Here, DE (Ω_ϕ = 0.9981) as non-canonical scalar field dominates over DM (Ω_m = 0.0019) in the evolution. But, the points are always decelerating in nature (see Table I and Table II). This is an interesting prediction through these points that future deceleration is possible in a non-canonical scalar field.

© 2025 Dept. of Physics, NBU

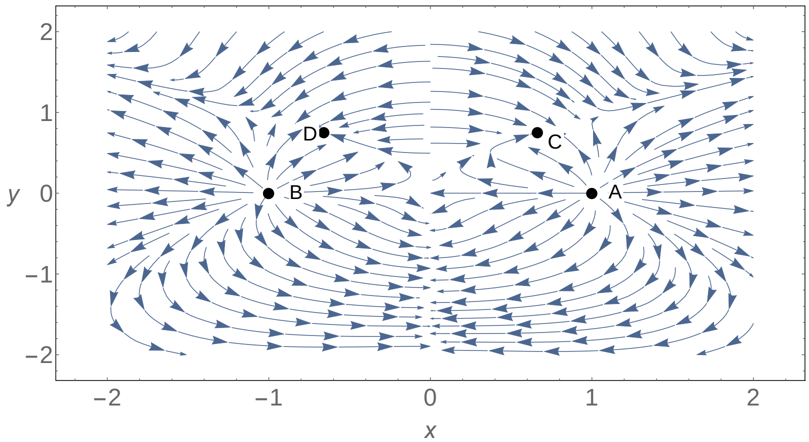

FIG. 1. 2D phase plane of autonomous system [Eq. (14) and Eq. (15)] for $\omega_m = 0$. A and B points are unstable nodes and the points C and D are stable sink.

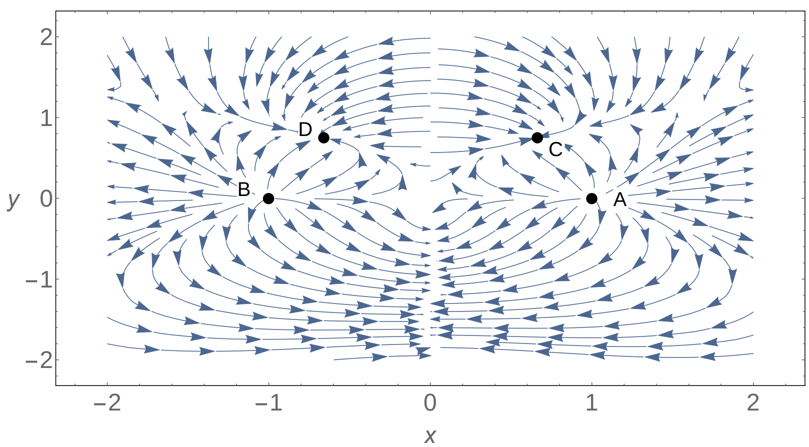

FIG. 2. 2D phase plane of autonomous system [Eq. (14) and Eq. (15)] for $\omega_m = \frac{1}{3}$. Here, points A and B are unstable points and C and D are stable points in phase plane.

IV. COSMOLOGICAL CONSEQUENCES OF CRITICAL POINTS

Autonomous system [Eq. (14) and Eq. (15)] has some critical points like A and B having similar nature from a cosmological standpoint. Though the points are completely dominated by the kinetic part of the non-canonical scalar filed, they are unstable (which is shown in Fig. 1 and Fig. 2, respectively). Additionally, the universe's expansion is always slowed down. Therefore, from a cosmological perspective, the points behave in a totally non-physical manner. However, the phase plane analysis predicts some intriguing nature from locations C and D. Throughout the progression, DE outperforms DM even though the points are a combination of both. The points are always stable in the phase plane (see Fig. 1 and Fig. 2). So, from the dynamical system perspective, the late time evolution of the universe can be realized by the points. The

© 2025 Dept. of Physics, NBU

expansion of the universe near the points are always decelerating. Therefore, in the case of a non-canonical scalar field, where the potential of the scalar field is as a function of time only, the points C and D attain a late time DE dominated decelerated universe.

V. CONCLUSION

By converting the evolution equations into dynamical equations through a suitable transformation of variables normalized over the Hubble scale, we have investigated the dynamical analysis of non-canonical scalar fields against the backdrop of the flat FRW model of the universe, where the scalar field's potential is taken as an explicit function of time. We have identified four critical points in the phase plane, two of which (A and B) are not physically relevant for this model since they yield unstable DE-dominated solutions with a decelerating nature. The critical points are then computed by setting the right portion of the autonomous system to zero. By analyzing the linearized Jacobian matrix's eigenvalues, we can determine the nature of critical points. Based on the phase plane analysis, we have two places, C and D, which are both combinations of DE and DM and are consistently stable yet show the decelerating universe. Therefore, when the potential is considered as an explicit function of cosmic time 't', one can forecast the universe's future slowing in the dynamics of non-canonical scalar fields based on our research.

ACKNOWLEDGEMENT

The authors are grateful to Dr. Rabi Kumar Dey and Maharaj Swami Kamalasthananda of Ramakrishna Mission Vivekananda Centenary College, Rahara, Kolkata-700118, where a significant volume of the present research was carried out.

* sujay.math@nbu.ac.in

[1] Adam G. Riess et al. AJ **116**, 1009 (1998).
[2] S. Perlmutter et al. ApJ **517**, 565 (1999).
[3] S. M. Carroll, W. H. Press, and E. L. Turner, Annual review of astronomy and astrophysics **30**, 499 (1992).
[4] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
[5] V. Sahni and A. Starobinski, Int. J. Mod. Phys. D, **9**, 373 (2000).
[6] P. J. E. Peebles and B. Ratra, The Astrophysical Journal **325**, L17 (1988).
[7] T. Padmanabhan, Physics Reports, **380**, 235 (2003).
[8] S. M. Carroll, Living reviews in relativity **4**, 1 (2001).

© 2025 Dept. of Physics, NBU

[9] G. K. Goswami, A. Pradhan, and A. K. Beesham, Modern Physics Letters A, **35**, 2050002 (2020).

[10] L. García, J. Tejeiro, and L. Castaneda, arXiv:1210.5259 (2012).

[11] W. Fang, H. Q. Lu, and Huang, ZG, Classical and Quantum Gravity **24**, 3799 (2007).

[12] E. J. Copeland, A. R. Liddle, and D. Wands, Physical Review D **57**, 4686 (1998).

[13] S. Carloni, J. A. Leach, S. Capozziello, and P. K. S. Dunsby, Classical and Quantum Gravity **25**, 035008 (2008).

[14] G. Leon, J. Saavedra, and E. N. Saridakis, Classical and Quantum Gravity, **30**, 135001 (2013).

[15] C. G. Böhmer, N. Tamanini, and M. Wright, Phys. Rev. D **91**, 123002 (2015).

[16] S. Bahamonde, C. G. Böhmer, S. Carloni, E. J. Copeland, W. Fang, and N. Tamanini, Physics Reports **775**, 1 (2018).

[17] S. K. Biswas and S. Chakraborty, International Journal of Modern Physics D **24**, 1550046 (2015).

[18] S. K. Biswas, and S. Chakraborty, General Relativity and Gravitation **47**, 22 (2015).

[19] N. Mahata and S. Chakraborty, General Relativity and Gravitation **46**, 1721 (2014).

[20] J. Dutta, W. Khyllep, and H. Zonunmawia, The European Physical Journal C **79**, 359 (2019).

© 2025 Dept. of Physics, NBU