Chapter 9

NOVEL SOLUTIONS: APPLICATIONS

9.1 Introduction

The origin of the arrow of time is still considered to be a puzzling problem in theoretical physics [47]. Another difficult problem is the $1/f$ noise [48], a footprint of complexity. The ubiquity of $1/f$-like noise in diverse natural and biological processes seems, in particular, to signal to certain key, but still not clearly understood, dynamical principles that might be at work, universally at the heart of any given dynamical process. Recently, there seems to have been an emerging urge in literature [4] for a new principle for understanding complex, intrinsically irreversible, processes in nature.

In this chapter, we explain how both the above problems might acquire a natural resolution in the class of such novel solutions. Both the problems were originally discussed in [27, 31, 32]. We also reinterpret, in the present framework, some new results on unimodal logistic map [51, 52] at the chaos threshold. At the end, we show how a hyperbolic type distribution arises naturally at the asymptotic late time ($t \to \infty$) limit even from a normally distributed variate.

9.2 Randomness

Is randomness a fundamental principle (law) controlling our life and all natural processes? Or, is it simply a projection of our limitations in comprehending such complex phenomena? Is there any well defined boundary separating simple and complex? These and similar related questions on the actual status of randomness are being vigorously investigated in the literature [4, 49]. The new class of discontinuous solutions sheds altogether a new light on the ontological status of randomness. (It is believed that randomness in quantum mechanics arises at a fundamental level. However, the Schrödinger equation, the governing equation of any quantal state, is purely deterministic and time symmetric. The random behaviour is ascribed only through an extraneous hypothesis of a 'collapsed state' at the level of measurement (see,[4]))

Let us note that equation (8.1) is the simplest ODE, t being an ordinary real variable. In the framework of the conventional analysis, one can not, in any way, expect, at such an elementary level, a random behaviour in its, so called unique (Picard's), solution. (The chaos and unpredictability could arise only in the presence of explicit nonlinearity in higher order ODEs.) However, the nontrivial scaling, along with the initial ansatz (8.2) and (8.8), reveals not only the self similarity of C^{2^n-1} solutions over scales η^{2^n}, but also exposes a subtle role of *decision making* and randomness in generating nontrivial late t behaviour of the solution.

A basic assumption in the framework of the standard calculus is that a real variable t changes by linear translation only. Further, t assumes (attains) every real number exactly. However, in every computational problem within a well specified error bar, a real number is determined only up to a finite degree of accuracy ϵ_0, say. Suppose, for example, in a computation, a real variable t is determined upto an accuracy of ± 0.01, so that t here effectively stands for the closure of set $t_\epsilon \equiv \{t \pm \epsilon,\ \epsilon < \epsilon_0 = 0.01\}$ with cardinality c of the continuum. We call ϵ an 'infinitesimally small' real number (variable). Now, any laboratory computational problem (experiment) is run only over a finite time span, and the influence of such infinitesimally small ϵ's, being insignificantly small, could in fact be disregarded. Consequently, the variable t could be written near 1, for instance, as $t_1 = 1 + \eta$, where η is an ordinary real variable close to 0, having 'exact' values as long as one disregards infinitesimal numbers ($< \epsilon_0$) due to *practical limitations.* At the level of mathematical analysis, such practical limitations being indicative only of natural (physical/biological) imperfections that should not jeopardy the existence of an abstract theory of sets, real number system, calculus and so on, shaping the logical framework for an exact and deterministic understanding of natural processes. That such an attempt would remain as an unfulfilled dream is not surprising in view of the new class of discontinuous solutions. As discussed in [30](see also [31, 32, 33]), the C^{2^n-1} solution shows that the real number set R should actually be identified with a nonstandard real number set $\mathbf{R}$ [35] so that every real number t *is a fat* real (hyperreal) number $\mathbf{t}$, which means that $t \equiv \mathbf{t}$. Accordingly, a real number t could not be represented simply by a structureless point, but in fact is embedded in a sea of irreducible fluctuations of infinitesimally small numbers $\mathbf{t} = tt_f$,

$t_f = 1 + \phi$ denoting *universal* random corrections from infinitesimals ϕ (c.f., equation(8.9)). The origin of randomness is obviously tied to the *freedom* of injecting an infinite sequence of arbitrary scaling parameters ϵ_n into the C^{2^n-1} solutions because of scale invariance of equation (8.1), introducing small scale uncertainty (indeterminacy) in the original variable. (Note that scaling at each stage introduces a degree of uncertainty in the original variable viz., $\delta\eta_1 = |\eta_1' - \eta_1| = \epsilon_1/\alpha_1$ and so on.) It is also clear that the set of infinitesimals $\mathbf{0} = \{\pm\phi\}$ has a Cantor set like structure viz., discrete, dust like points separated by voids of all possible sizes– and having the cardinality 2^c [31, 32]. Accordingly, an infinitesimal variable could change (within an infinitesimal neighbourhood of a point, say, 1) only by discrete *jumps* (inversions) of the form $t_- = t_+^{-\alpha}$, $t_\pm = 1 \pm \phi$, to cross the gulf of emptiness, length of jumps being arbitrary because of an arbitrary α. Note that, the value of a small real number η_0 is uncertain not only upto $O(\eta_0^2)$, but also because of the arbitrary parameter ϵ. Note also that the solution (8.8) proves explicitly that inversion is also a valid mode of change for a real variable, at least in an *infinitesimal* neighbourhood of an ordinary real number. We note further that two solutions τ_g and τ_s are indistinguishable for $t \sim O(1)$ and $\eta_0^2 << 1$. However, for a sufficiently large $t \sim O(\epsilon^{-1})(\equiv O(\eta_0^{-2}))$, the behaviours of two solutions would clearly be different. Finally, the order of discontinuity could be "controlled" by an application of an *intelligent decision* invoking a nonzero $\epsilon_n \neq 0$, $\epsilon_m = 0$, $m < n$ only at the nth level of iteration. This *freedom of decision making* could either be utilized at a pretty early stage of iterations, for instance, $n = 1$, say, making the system corresponding to the ODE (8.1) *fully intelligent*, or be *postponed* indefinitely ($n = \infty$) reproducing the standard Picard's solution for a material (non-intelligent) system [33]. We note that this randomness and potential intelligence being intrinsic properties of the solutions of equation (8.1) *are indelibly rooted to the real number system and hence could not simply be interpreted as due to some coarse graining effect* analogous to thermodynamics and statistical mechanics [4].

The new mathematical results explored here (and in[30, 31, 32, 33]) would likely to have some profound implications. With intelligence (and decision making) emerging as a fundamentally new ingredient (degree of freedom) from the mathematical analysis, the traditional framework of a physical theory viz., space, time, matter, energy (or in a relativistic theory, spacetime and energy) might, in future, be extended and replaced by a *truly dynamical* framework consisting of *intelligence, space, time, matter, energy* as envisioned in Ref.[4]. To explore the dynamic properties of the new solutions further, we now examine the origin of time's arrow in the following.

9.3 Reflection symmetry breaking and time

It is well known that *time* is *directed*, that is to say, we all have a sense of a forward moving time. The problem of time asymmetry [47] points to a fundamental dichotomy between the (Newtonian) 'time' in physics and mathematics and that of our (objective) experiences. The Newtonian time is non-directed. There is no way to distinguish between a space like variable x with a time variable t. Further, most of the fundamental equations of physics viz, the Newton's classical equation of motion, the General Relativistic field equations and the quantum mechanical Schrodinger equation, are time reversal symmetric. However, the existence of C^{2^n-1}solutions of equation (8.1) presents us with a new scenario! One is now obliged to re-examine the conventionally accepted standard notions under this new light. As mentioned already, we show here that time *does* indeed have an arrow, which is inherited, not only by all (physical /biological /social) dynamical systems, but it is also indelibly inscribed even to a real number. The concept of time thus turns out to be more fundamental compared to space and may even be considered at par with the real number system (and hence to the existence of intelligence as a fundamental entity)!

To see how a time sense is attached to C^{2^n} solutions, we recall first that the variable t is, in general, non-dynamical, and need not denote the (forward) flow of time. In fact, it simply behaves as a labeling parameter. Further, the inversion $t_- \leftrightarrow t_-^{-1} = t_+$ may at most be considered as a reversible random fluctuation between $t_\pm$ (for a given $\eta > 0$) with equal probability 1/2. In the usual treatment of ordinary calculus and classical dynamics, t is a non-random ordinary variable, and the above inversion reduces to the symmetry of equation (8.1) under reversal of sign (parity) $t \to -t$. The infinitely differentiable standard solution $\tau_s(t)$, written (in the notation of equation (8.2)) as $\tau_{s-} = t_-$, $\tau_{s+} = t_+$, $(\tau_s(1) = 1)$ is obviously symmetric under this reversible inversion $(\eta \to -\eta)$. The nontrivial solution $\tau(t)$ in equation (8.8), however, constitutes an explicit example where *this parity invariance is dynamically broken*, viz.; when the inversion is realized in an irreversible (one-sided, directed) sense.

To state the above more precisely, let $P : Pt_\pm = t_\mp$ denote the reflection transformation near $t = 1$ ($P\eta = -\eta$ near $\eta = 0$). Clearly, equation (8.1) is parity symmetric. So is the standard solution $\tau_{s\pm} = t_\pm$ (since $P\tau_s = \tau_s$). However, the solution (8.8) breaks this discrete symmetry spontaneously: $\tau_-^P = P\tau_+ = t_-$, $\tau_+^P = P\tau_- = C\frac{1}{t_-}\frac{1}{t'_{1+}}\frac{1}{t'_{2+}}\ldots$, which is of course a solution of equation (8.1), but clearly differs from the original solution, $\tau_\pm^P \neq \tau_\pm$.

To see more clearly how this one-sided inversion is realized, let $t \to 1^-$ from the initial point $t \approx 0$. Then at *a* point in the infinitesimal neighbourhood of $t_- \lessapprox 1$, the solution τ_- carries (transfers) t_- instantaneously to t_+ by an inversion $\tau_- \approx 1/t_+$ (we disregard here the $O(\eta_0^2)$ and lower order self similar fluctuations) and subsequently the solution follows the (standard) path $\tau_+ = t_+$ in the small scale variable $\eta \gtrapprox 0$, as it is now free to follow the standard path till it grows to $O(\lessapprox 1)$, when second order transition to the next smaller scale variable by inversion becomes permissible: $\eta = 1/\eta_+, \eta_+ = 1 + \bar{\eta}, \bar{\eta} \gtrapprox 0$, and so on. Clearly, the generic pattern of (irreversible) (time asymmetric) evolution in $\tau(t)$ over smaller and smaller scales resembles more and more closely the infinite continued fraction of the golden mean: $\tau_-(t) = t$, $0 < t \uparrow\lessapprox 1, \to \tau_- = 1/(1+\eta)$, $0 < \eta \uparrow\lessapprox 1, \to \tau_- = 1/(1 + 1/(1+\bar{\eta}))$, $\bar{\eta} \gtrapprox 0$, and so, $\tau_-(t) \to \nu$, $\nu = (\sqrt{5}-1)/2$, the golden mean, as $t \to \infty$. Here, $t \uparrow$ means that t is an increasing variable. We note that the (macroscopic) variable t is *reversible* as long as $t \sim 1$. Subsequently, this parity symmetry is broken by a random inversion, leading the evolution irreversibly to

9.4 Logistic Maps

Let us now point out an interesting application in noninverible one dimension maps in nonlinear dynamics. There have been some new, previously unexposed, asymptotic scaling properties of the iterates of unimodal logistic maps at the edge of chaos [51, 52]. For definiteness, we consider here the dynamics of the sensitivity ξ_t to the initial conditions for large iteration time t, at the chaos threshold $\mu = \mu_\infty = 1.40115...$ of the map $f_\mu(x) = 1 - \mu|x|^2$, $-1 \leq x \leq 1$ (for notations see [51]). For a sufficiently large t, the ordinary exponential behaviour of sensitivity gives away to a power law behaviour, and is shown to have the $q-$exponential form given by

$$\xi_t = \exp_q(\lambda_q t) \equiv [1 + (1-q)\lambda_q t)]^{1/(1-q)}. \tag{9.1}$$

The standard exponential dependence $\xi_t = \exp(\lambda_1 t)$ is retrieved at the limit $q \to 1$. The system is said to be strongly insensitive (sensitive) to initial conditions if $\lambda_1 < 0$ ($\lambda_1 > 0$). The behaviour, however, gets altered at the edge of chaos. Using the Feigenbaum's RG doubling transformation $\hat{R}f(x) = \alpha f(f(x/\alpha))$ n times to the fixed point map $g(x)$ viz., $g(x) = \hat{R}^n g(x) \equiv \alpha^n g^{2^n}(x/\alpha^n)$, the values of the $q-$Lyapunov coefficient λ_q and q are determined to be $\lambda_q = \ln\alpha/\ln 2\,(>0)$, $q = 1 - \ln 2/\ln\alpha\,(< 1)$, α being one of the universal Feigenbaum constants, $\alpha = 2.50290....$ Consequently, the critical dynamics corresponds to *weak chaos.* Notice that the dynamics at the chaos threshold, being the most prominent and readily accessible to numerical experiments, among the critical points of a quadratic map, reveals a *universal* concerted behaviour, described by the fixed point solution of the RG doubling transformation. One, however, needs to explain the origin of weak chaos, i.e., the system being weakly *sensitive*, rather than being only weakly *insensitive*, to initial conditions, at the chaos threshold, when the system approaches the threshold from the left through period doubling route (for $\mu < \mu_\infty$).

To reinterpret above observations in the present context, we note first of all that the fixed point equation $\tilde{g}(x) = g(x)$ where $\tilde{g}(x) = \alpha g(g(x/\alpha))$, is a solution of equation (8.1), viz., $d\tilde{g}/dg = \tilde{g}/g$. Consequently, one expects that the critical dynamics of unimodal quadratic maps at the onset of chaos would be linked directly to our nontrivial solutions to equation (8.1). As an explicit example, let us investigate the origin of the q-exponential type power law dynamics from the ordinary exponential one

$$\frac{d\xi}{dt} = \lambda_1 \xi. \tag{9.2}$$

As the critical point is approached from left (say), the Lyapunov exponent $\lambda_1 \to 0^-$, and hence gets replaced, in our extended framework, by a 'dynamic' infinitesimal (ordinary zero is an equivalence class of infinitesimals) of the form $\lambda_1 \to -\epsilon\lambda_p\phi(t_1)$, where $\epsilon\,(> 0)$ is an infinitesimal ($\epsilon \neq 0$, ϵ^2 =O(0)) scaling parameter, $\phi = t_1\tau(t_1^{-1})$ ($t_1 = \epsilon t$), and $\lambda_p > 0$, $p = p(\epsilon)$ are two generalized constants (c.f., equation (8.11)), both of O(1). However, when the variation in the intrinsic time like variable ϕ becomes relevant at the scale $t \sim 1/\epsilon$, that of λ_p would be relevant only at a longer scale viz., $t \sim 1/\epsilon^2$ or more and could be considered as an ordinary constant. The reason for introducing the (third) generalized constant (again with a slower variation than ϕ) p will become clear below. At a critical point, Equation (9.2) now reduces to the RG -like equation

$$t_{1+}\frac{d\xi}{dt_{1+}} = \lambda_p \xi. \tag{9.3}$$

To explain the derivation of the above equation let us proceed in steps.

(i) The rescaled variable t_1, defined by $t_1 = \epsilon t$, is O(1) when $t \sim$ O($1/\epsilon$). Consequently, the critical dynamics would be revealed only at *a sufficiently long time scale* $t_{1-} = 1 - \eta \to 1^-$ i.e., $t \to (1/\epsilon)^-$. That means *the dynamics at the chaos threshold needs to be probed in an extended framework, viz., in the sense of a limit as the control parameter* μ *approaches* μ_∞ *from left (say) through period-doubling cascade* instead of simply replacing μ by μ_∞ in the map and then iterating. In any computational problem, this extended framework is automatically realized, because of the inherent finite bit (decimal) representation of a real number, such as μ_∞, exact value of which could only be approached recursively by increasing its accuracy. The infinitesimal ϵ (along with an O(1) variability as represented by $\lambda_p\phi(t_1)$) then simply corresponds to the infinite trailing bits in the finitely represented real number e.g., μ_∞.

(ii) The critical point equation (9.3) follows when one makes use of the relation $d\ln t_{1-} = -d\ln t_{1+}$, which is valid for infinitesimal η with O(η^2)=0 [33]. Clearly, the solution to the above equation is $\xi = [1 + p\lambda_p\eta]^{1/p}$ which corresponds exactly to the q exponential (9.1) provided we choose $q = 1 - p$, $p\lambda_p = 1$.

(iii) The q exponential solution is, however, valid not only in the 'infinitesimal' neighbourhood of $t_1 = 1$, but for arbitrarily large t_1, because of the scale invariance of equation (9.3). Indeed, writing $2^n t_1 = 1 + \tilde{t}$, the q exponential sensitivity takes the form $\xi = [1 + (1-q)\lambda_q \tilde{t}]^{1/(1-q)}$, $\tilde{t}$ being large.

(iv) The scale factors 2^n correspond to the times to determine the trajectory positions x_{2^n} of the logistic map with an initial position x_{in-} [51]. It follows therefore that the ratio of the sensitivities at times 2^n and 2^{n+1} viz., $\frac{\xi_{2^{n+1}}}{\xi_{2^n}} = (\frac{2^n t_1}{2^{n+1} t_1})^{1/p} = \alpha$ when p remains constant at $1/p = \ln\alpha/\ln 2$ upto a time $T \approx 2^N$, $N \approx |\ln\xi_{in}/\ln\alpha|$, ξ_{in} being the initial value of sensitivity, corresponding to the initial iterate $n = 0$.

(v) Indeed, to obtain the later estimate, we note that $\xi_{in}^p = t_1$, which follows from equation (9.3). For a non-zero (sufficiently large) n, it now follows that $\xi_n \equiv (2^n \xi_{in}^p)^{1/p} = \alpha^n$ which translates to $p = \frac{\ln 2}{\ln\alpha}(1 + \frac{\ln\xi_{in}}{n\ln\alpha})$. Consequently, a possible variation in p would be revealed only when $n \approx |\ln\xi_{in}/\ln\alpha|$, as claimed. Note that q exponential form in the neighbourhood of $t_1 = 1$ belongs to the class of solutions (8.12) provided the generalized constant p is given by $1/p = \ln f/\ln t_1 - 1$, $(dp/\ln t_1) = 0$.

The change in sensitivity from strongly insensitive case ($\lambda_1 < 0$) to weakly sensitive ($\lambda_q > 0$, $q < 1$) power law behaviour is thus explained as an effect of nontrivial infinitesimals and associated inversion $t_{1-}t_{1+} \approx 1$ in the infinitesimal neighbourhood of $t_1 = 1$. The origin of Feigenbaum's constant α (notice our use of $\xi_n = \alpha^n$ in the above derivation) in the present formalism along with other relevant issues will be considered elsewhere. Another interesting problem is to identify the golden mean number ν in the critical dynamics.

9.5 *1/f spectrum*

The relevance of higher derivative discontinuous solutions to the origin in $1/f$ noise problem have been discussed in detail in[[31, 32]]. We note here that in the extended framework of a dynamical theory, accommodating these solutions, any physical, variable t, say time, is replaced by $t^{1+\sigma}$, $\sigma = \ln(1+\phi)/\ln t$ (c.f., Sec.3) where σ typically is small $O(\epsilon)$ for any $\epsilon > 0$. The nonzero exponent σ introduces a small stochastic fluctuations over the ordinary (time) variable t. Clearly, these small scale stochastic fluctuations, existing purely in the real number system, would remain insignificant for any terrestrial (laboratory) *inanimate* system which persists over a moderate time scale, such as the motion of a (classical) particle under gravity. However, even for simple electrical circuits where the voltage fluctuation spectrum $S_V(f)$ is known to vary proportionally with the thermal fluctuation spectrum $S_T(f)$, the origin of $1/f$ noise as observed in [53] could be naturally ascribed to the C^{2^n-1} solutions of

$$c\frac{\mathrm{d}T}{\mathrm{d}t} = -g(T - T_0). \tag{9.4}$$

This equation describes the macroscopic (equilibrium) variations of the temperature (T) of a resistive system with heat capacity c, coupled through a thermal conductance g to a heat source at temperature T_0 [53]. According to the conventional knowledge one does not expect $1/f$ spectrum from such a simple, purely deterministic, linear uniscale system. One needs, in fact, to consider extraneous nonlinear effects from environment to explain the origin of the generic $1/f$ fluctuations.

However, according to the present analysis, even this simple system would behave stochastically because of small scale, intrinsic fluctuations in the time variable t. These scale free fluctuations could influence the late time behaviour of the system provided *the system is 'allowed' to survive* over a period $t >> 1/\epsilon$, $\epsilon = g/c$. Notice that ordinarily a system following purely equation (9.4) is assumed to relax to the equilibrium temperature T_0 after $t \approx 1/\epsilon$. The late time variability that is observed in any resistive system is then ascribed normally to the complex nature of the resistive medium and/or (nonlinear) interactions with environment [53], asking for an explicit modeling. The generic observation of $1/f$ fluctuations in metal films and semiconductors still eludes a universal explanation for its microscopic origin in the framework of conventional dynamical theories [56].

In view of C^{2^n-1} solutions, we now have an extended framework to re-examine the above problem. The solution of equation (9.4) now have the form $T(t) - T_0 = t^{\sigma}e^{-\epsilon t}$, σ being a small fluctuating variable. As noted already, this random exponent would lead to small scale stochastic modulations over the (mean) macroscopic decay mode, as observed in physical systems. These small scale (power law) fluctuations would persists even far beyond the ordinary relaxation time. Accordingly, a time series of temperature fluctuations ($T_f(t) = (T(t) - T_0)e^{\tau}$) recorded over a period of a few decades (1 to 10^4, say), in the unit of the dimensionless time $\tau = \epsilon t$, would reveal a scale free $1/f$ type variability. For, the two point autocorrelation function of this intrinsic fluctuations has the form $C(t) =< T_f(t)T_f(0) >= c < T_f(t) >= c < t^{\sigma} >\approx ct^{<\sigma>}$, $T_f(0) = c$, where $< \sigma >$ is the expectation value of the random exponent and $c \sim$O(1) is the initial (background) noise in the system. The associated probability distribution would have a generic late t behaviour, resembling infinitely divisible Levi type distributions (see below). The corresponding power spectrum of this stochastic scale free fluctuation is given by $S(f) \sim 1/f^{1-<\sigma>}$. We note that an arbitrarily small nonzero σ is sufficient to generate a $1/f$-like fluctuation. In other words, intrinsically random, infinitesimal scales associated with the time variable t could act as a perennial source of small scale fluctuations leading to the universal low frequency $1/f$ spectrum. However, an accurate determination of the intensity of the fluctuations (viz, the constant of proportionality in the observed spectrum $S_V(f) \propto V^2/f^a$, $a \approx 1$) [53, 56] may require further work. The relevance of number theory to $1/f$ noise problem is also pointed out by Planat [57]. El Naschie suggested that the exponent β of the $1/f^\beta$ noise to semiconductors would be related to the golden mean in the framework of the E-infinity theory [58].

9.6 Hyperbolic distribution

In Ref.[31] we show that the scale free infinitesimal fluctuations follow a nongaussian, Bramewell-Holdsworth-Pinton (BHP) [59] distribution. Here we show how a hyperbolic, power law tail gets superposed generically in *any* distribution when the concerned random variate is assumed to leave in **R**. To see this it suffices to consider only a normally distributed variate, because by the central limit theorem a normal probability density acts as the attractor for any probability density with finite moments.

Let t be a zero mean normal variate, with unit standard deviation. The corresponding fat variate could be written as $\mathbf{t}^2 = t^2 + \phi$, $\phi = \epsilon(t)\ln t^2$, being a random infinitesimal satisfying equation (8.11) in the logarithmic variable $\ln t$ (c.f., Appendix). Consequently, the normal density function $\propto e^{-t^2/2}$ gets a generic power law tail $t^{-\epsilon}e^{-t^2/2}$. This generic power law tail in the present extended formalism should become important in the future studies on the statistics of rare events. We close with the remark that occasional detections of exceptional events in an experiment of, for instance, a normal variate over a prolonged period could be explained by this slowly varying tail. Note that the power law variability would become visible only in the asymptotic limits ($t \to \pm\infty$) because of an infinitesimal ϵ, so that ϕ remains vanishingly small in any laboratory experiments over a finite time scale.

9.7 Appendix

A continuously differentiable function $f(t)$ of a real variable could be defined as an integral of the ODE $\frac{\mathrm{d}x}{\mathrm{d}t} = f'(t)$. For the gaussian $e^{-t^2/2}$, the relevant equation is $\frac{\mathrm{d}x}{\mathrm{d}t} = -te^{-t^2/2}$, and hence the corresponding hyperreal (fat) extension is given by $\mathbf{t}^2 = t^2 + \epsilon(t)\ln t^2$. The extension of the linear variable t is given by $\mathbf{t} = t + \epsilon(t)\ln t$. The extended exponential $e^{\mathbf{t}} = t^{\epsilon}e^{t}$ would therefore have a slowly fluctuating power law tail. Note that the variable t gets the infinitesimal correction term in the logarithmic variable $\ln t$, when the infinitesimal ϵ satisfies the equation

$$\sigma\frac{\mathrm{d}\epsilon}{\mathrm{d}\sigma} = -\epsilon,\ \sigma = \ln t$$