Chapter 6

Exact relativistic Newtonian representation of gravitational static spacetime geometries

6.1 Introduction

The Schiff conjecture [Schiff, 1960] demands that any viable theory of gravity should respect Einstein equivalence principle which in turn implies that any feasible configuration of the gravitational field will correspond to a unique metric and the world lines of a test particle is the geodesics of that metric. Though no rigorous proof of the Schiff conjecture exists till now, the Eotvos-Dicke-Braginsky experiment and few other observations indicate the validity of the Schiff conjecture [Will, 1993, 2006]. General relativity, which is widely considered as the standard theory of gravitation, is the metric theory of gravitation; the theory was constructed based on space time geometry without any notion of potential which was the cornerstone in Newtonian picture or Newtonian like description of the physical universe. Baring one or two, all the viable alternative theories of gravitation are also based on spacetime metric (geometry).

The Newtonian description (through potential) of gravity is, however, mathematically much comprehensible, as well as simpler to understand the concerned phenomena physically in compare to geometrical (metric) description of gravity. As a result Newtonian potential continues to be used in most applications at far (weak) field static condition where Newtonian approximation is reasonably valid. At strong gravity, however, Newtonian description is no longer valid, even approximately. Nonetheless, for simpler understanding of the general relativistic (GR) behavior of a test particle under the influence of a non-gravitation potential in a curved background spacetime, efforts have been made to formulate or prescribe modified Newtonian like potentials to mimic GR gravitational features of few corresponding spacetime geometries, particularly in the strong field gravity. Although not intended to construct Newtonian analogues of gravitational spacetime geometries, they are mainly endeavoured to mathematically replicate few GR features approximately, to be mostly used to study inner relativistic dynamics of the accretion flow around spacetime geometries describing black holes (BHs)/compact objects; commonly called pseudo-Newtonian potentials (PNPs). These potentials, although not exactly Newtonian in nature, are notionally Newtonian in character, in a sense that the gravitational interaction can be treated through the form of a scalar potential and a corresponding force law. Since its inception by Paczyński and Witta [Paczynsky and Wiita, 1980] to mimic GR features of the spherically symmetric Schwarzschild geometry, many modified Newtonian like potentials have been proposed in the literature corresponding to different spacetime geometries. Nonetheless, most of these modified Newtonian like potentials which are either prescribed through an ad hoc proposition or formulated preserving certain specific procedures [Mukhopadhyay, 2002] lack the uniqueness to encompass most of the GR features simultaneously within a reasonable accuracy; a PNP corresponding to a given spacetime geometry is not unique. The modified potentials do not satisfy Poisson's equation, and can not be true Newtonian description of gravitational spacetimes (for detailed analysis of various PNPs and their use see [Ghosh and Mukhopadhyay, 2007, Ghosh et al., 2014, 2015, Sarkar et al., 2014, Bhattacharya et al., 2010]). Recently, a generic Paczyński-Wiita like PNP corresponding to any static spherical BH (in Einstein's gravity as well as in modified gravity theories) has been formulated [Faraoni et al., 2016]. In general, the modifications of Newtonian gravitational potential are done in such a manner that they are emphasized to correctly reproduce the location of the inner most stable circular orbit (ISCO) and the marginally bound orbit of the test particle motion in the strong field gravity, or at least approximately. More importantly they do not give the classical predictable GR effects or reproduce the classical experimental tests of general relativity like gravitational deflection of light, gravitational precession of test particles or gravitational time delay, even in the weak field regime which have been tested experimentally. These PNPs/modified Newtonian like potentials are best suited to be used or mostly restricted in the paradigm of accretion physics in the strong field regime, where the inner relativistic accretion phenomena is studied in a simple Newtonian framework by circumventing complex nonlinear tensorial equations of general relativity, especially complex GR gas dynamical equations, while the GR effects in the vicinity of the spacetime geometries are approximately captured through the corresponding PNPs. Recently, Wegg [Wegg, 2012] proposed couple of modified Newtonian like potentials to reproduce precessional effects in general relativity for orbits with large apoapsis, however, they are not quite effective in the vicinity of the Schwarzschild BH, in strong field gravity.

In recent times, much accurate analogous potentials/modified Newtonian like potentials of corresponding GR geometries describing BHs/naked singularities have been derived from the conserved Hamiltonian of the test particle motion, comprising of a velocity dependent part of the particle motion along with the usual spherically symmetric part of the source and the other source dependent terms, and can be referred as some kind of Newtonian like analogous potentials (NAPs) [Tejeda and Rosswog, 2013, Sarkar et al., 2014, Ghosh et al., 2014, 2015]. Although many salient GR features can be reproduced with NAPs with precise/reasonable accuracy, however they fail to reproduce temporal effects like angular and epicyclic frequencies of test particle motion. Moreover they have been derived by assuming low energy limit of the test particle motion, and hence can not be treated as the correct relativistic Newtonian analogue of general relativity. Nonetheless, as compared to the existing PNPs/modified Newtonian like potentials, velocity dependent NAP of corresponding Schwarzschild geometry when used in astrophysical situations like in a simple accretion scenario or in tidal disruption of a star by a supermassive BH (SMBH), renders much better approximation to GR results [Tejeda and Rosswog, 2013]. Several other attempts have also been made to describe some form of Newtonian analogue of GR effects or to mimic GR effects with Newtonian dynamics ([Krasiński, 1980, Singh and Srivastava, 1987] and refernces therein).

In the present work, we propose to construct an *exact relativistic Newtonian representation* of a general class of gravitational spacetime geometries from a first principle approach. Such a construct should be essentially a relativistic Newtonian analogue described through a relativistic scalar potential and a corresponding force law, which then essentially demands that the corresponding modified Newtonian potential will give path equations of test particles exactly the same to those obtained from geodesic equations of the corresponding spacetime metric without any restriction on particle energy or the strength of gravitational field. Equivalently, it implies that if it can be demonstrated that any Newtonian analogous construct exactly reproduces the corresponding geodesic equations of motion of the corresponding spacetime geometry, then such a construct will essentially be an exact relativistic Newtonian analogous theory of gravitational spacetime geometries. The corresponding relativistic scalar potential which would be an exact relativistic Newtonian analogous potential, will reproduce all the relativistic features of the corresponding spacetime geometry in-toto including all the classical experimental tests of general relativity, while working in Newtonian framework.

In the next section, we construct our proposed relativistic Newtonian analogous theory. Although our construct would be generic in nature, however, we would mainly focus on a class of static spacetime geometries.

6.2 Formulation of exact relativistic Newtonian analogous potential for static spacetime geometries

Static spacetimes are among the simplest class of Lorentzian manifolds with a non-vanishing timelike irrotational Killing vector field K^{α}. We choose to represent a general class of gravitational static spacetimes in standard spherical geometry, given by

$$ds^2 = -f(r)^{\beta}\, c^2\, dt^2 + \frac{1}{f(r)^{\beta}}\, dr^2 + f(r)^{1-\beta} r^2 d\Omega^2\,, \tag{6.1}$$

where, $f(r)$ is the generic metric function and β is an arbitrary constant parameter. $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. For Schwarzschild geometry, $\beta = 1$ and $f(r) = 1 - 2GM/c^2 r$, where M is the gravitational mass of the source, and c is the usual speed of light. With $\beta = 1$, other static spacetime geometries that can be represented through Eq. (6.1) are ReissnerNordström solution, Schwarzschild-de Sitter/anti-de Sitter solution, Kiselev BH, Kehagias-Sfetsos solution, Ayón-Beato García solution, Weyl-conformal gravity, etc. With $\beta = \gamma\,(0 < \gamma \leq 1)$, $f(r) = 1 - 2GM/\gamma c^2 r$, it represents Janis-Newman-Winicor (JNW) solution describing naked singularity. As the effect of spacetime metrics can only be realized through the dynamics of the particle motion around the spacetime geometry, the corresponding relativistic analogous potential should contain the information of the velocity of the particle. As we intend to construct our relativistic Newtonian analogous theory from first principle approach, our relativistic Newtonian construct will then be described by an action $\mathcal{S}$ and a corresponding Lagrangian $\mathcal{L}$, comprising of the information of the test particle motion. We invent a most general three dimensional relativistic gravitational action per unit mass in analogous Newtonian framework, carrying the information of the metric function, defined by

$$\mathcal{S} = \int \left[-\frac{e}{2} - \frac{c^4}{2e}\, f(r)^{\beta} + \frac{e}{2c^2} \left(\frac{\dot{r}^2}{f(r)^{2\beta}} + \frac{r^2\, \dot{\Omega}^2}{f(r)^{2\beta-1}} \right) \right] dt\,, \tag{6.2}$$

where the corresponding Lagrangian per unit mass is

$$\mathcal{L} = -\frac{e}{2} - \frac{c^4}{2e}\, f(r)^{\beta} + \frac{e}{2c^2} \left(\frac{\dot{r}^2}{f(r)^{2\beta}} + \frac{r^2\, \dot{\Omega}^2}{f(r)^{2\beta-1}} \right)\,. \tag{6.3}$$

Here e is an arbitrary constant having the dimension of energy, which we will determine from equations of motion. Overdots here always denote the derivative with respect to coordinate time 't'. We next obtain the geodesic equations of motion from the Euler-Lagrange equations in spherical geometry, which describe the complete behavior of the test particle dynamics in the presence of gravity defined by our action, given by

$$\begin{aligned} \ddot{r} \;&= -\tfrac{c^2\,\beta}{2}\, f(r)^{3\beta-1}\, \tfrac{\partial f}{\partial r}\, \tfrac{c^4}{\epsilon^2} \;+\; \tfrac{\gamma}{f(r)}\, \tfrac{\partial f}{\partial r}\, \dot{r}^2 \;+\; \Big[r\, f(r) \\ &\qquad - \tfrac{r^2}{2}(2\beta - 1)\tfrac{\partial f}{\partial r} \Big]\; \dot{\Omega}^2\,, \end{aligned} \tag{6.4}$$

$$\ddot{\phi} = -\frac{2\,\dot{r}\dot{\phi}}{r} \left(1 + \frac{1-2\beta}{2}\, \frac{r}{f}\, \frac{\partial f}{\partial r} \right) \;-\; 2\, \cot\theta\, \dot{\phi}\, \dot{\theta} \tag{6.5}$$

and

$$\ddot{\theta} = -\frac{2\,\dot{r}\dot{\theta}}{r} \left(1 + \frac{1-2\beta}{2}\, \frac{r}{f}\, \frac{\partial f}{\partial r} \right) \;+ \sin\theta\, \cos\theta\, \dot{\phi}^2\,, \tag{6.6}$$

respectively. Integrating the radial geodesic equation of motion given in Eqn.(6.4), and using the relation of specific angular momentum $\lambda = \frac{\epsilon}{c^2} \frac{r^2\, \dot{\Omega}}{f(r)^{2\beta-1}}$, we obtain the equation for $\dot{r}^2$ which uniquely describes the test particle dynamics, given by

$$\frac{e^2}{c^2}\frac{\dot{r}^2}{f(r)^{2\beta}} + \frac{c^2}{f(r)^{1-2\beta}}\frac{\lambda^2}{r^2} + f(r)^{\beta}\, c^4 = \mathbb{E}\,, \tag{6.7}$$

where $\mathbb{E}$ is the integration constant directly associated with the integrals of motion, in the presence of the gravity. In the low energy and test particle motion with small velocity in the Newtonian limit, the integral constant $\mathbb{E}$ can be easily identified as the square of the conserved specific energy of the test particle motion ($\mathcal{E}$), i.e., $\mathbb{E} = \mathcal{E}^2$, resembling the similar scenario in general relativity in the weak field limit. Equivalently, the equation for $\dot{r}^2$ can be obtained by computing the conserved specific Hamiltonian $\mathcal{H}$ of the test particle motion using Eqn. (6.3), given by

$$\frac{e^2}{c^2}\frac{\dot{r}^2}{f(r)^{2\beta}} + \frac{c^2}{f(r)^{1-2\beta}}\frac{\lambda^2}{r^2} + f(r)^{\beta}\, c^4 = e\,(2\mathcal{H} - e)\,. \tag{6.8}$$

Specific Hamiltonian $\mathcal{H}$ should be equivalent to specific energy $\mathcal{E}$. Now comparing Eqns. (6.7) and (6.8), one can find that the constant e is simply equivalent to the conserved specific energy $\mathcal{E}$ of the test particle motion. With this, the equation for $\dot{r}^2$ in (6.7) or in (6.8) and the geodesic equations in (6.4), (6.5) and (6.6) exactly resemble the corresponding equations for the spacetime metrics described by Eqn. (6.1). In other words, the action in Eqn. (6.2) or the corresponding Lagrangian in Eqn. (6.3) exactly reproduce the corresponding geodesic equations of motion for the spacetime geometries described by Eqn. (6.1). The potential correspond to this Lagrangian could then be easily computed using the most general expression of a relativistic Lagrangian per unit mass in the Newtonian framework which is given by

$$\mathcal{L} = -c^2\sqrt{1 - v^2/c^2} - V\,, \tag{6.9}$$

where $v = \sqrt{\dot{r}^2 + r^2\dot{\Omega}^2}$, the net velocity of the test particle, and V is the potential. The Lagrangian in Eqn. (9) would then be equivalent to the expression of Lagrangian in Eqn. (6.3). Equating Eqns. (6.9) and (6.3), we evaluate the potential function V which then would be the potential correspond to the Lagrangian in Eqn. (6.3), and hence can be unambiguously treated as the most generalized exact relativistic Newtonian analogous potential correspond to a class of static spacetime geometries described by Eqn. (6.1), given by

$$V \equiv V_{\rm GN} = \frac{c^4}{2e}\, f(r)^{\beta} - \frac{e}{2c^2}\left[\frac{1}{f(r)^{2\beta}}\,\left(\dot{r}^2 + f(r)\, r^2\, \dot{\Omega}^2\right) + \frac{2c^4}{e}\sqrt{1 - \frac{\dot{r}^2 + r^2\,\dot{\Omega}^2}{c^2}} - c^2\right], \tag{6.10}$$

where, subscript 'GN' symbolizes 'Geometric-Newtonian'. Thus $V_{\rm GN}$ in Eq. (6.10) which is a three dimensional relativistic scalar gravitational potential, is the correct relativistic generalization of Newtonian gravitational potential, and can be considered to be the most generalized Newtonian potential that is consistent with the features of corresponding spacetime geometries described by Eqn. (6.1). In a real astrophysical scenario, the constant 'e' in $V_{\rm GN}$ which is equivalent to the specific energy of the particle motion, will be evaluated from the asymptotic boundary condition. With appropriate metric function $f(r)$ for static spacetimes and the value of β, $V_{\rm GN}$ would furnish exact Newtonian analogous potential correspond to a class of static spacetime geometries as mentioned previously, after Eqn. (6.1). It is being interestingly found that in the low energy limit of the test particle motion ($\epsilon/c^2 \sim 1$) or equivalently in the limit $v^2/c^2 << 1$, $V_{\rm GN}$ reduces to the Newtonian like analogous potentials of the corresponding static GR geometries prescribed in [Tejeda and Rosswog, 2013, Sarkar et al., 2014, Ghosh et al., 2014, 2015]. Correspond to Schwarzschild metric, the exact relativistic Newtonian analogous potential is then given by

$$V_{\rm GN}\big|_{\rm SW} = \frac{c^4}{2e}\left(1-\frac{2GM}{c^2\,r}\right) - \frac{e}{2c^2}\left[\frac{r^2\,\dot{r}^2}{(r-2r_s)^2} + \frac{r^3\,\dot{\Omega}^2}{r-2r_s} + \frac{2c^4}{e}\sqrt{1-\frac{\dot{r}^2+r^2\,\dot{\Omega}^2}{c^2}} - c^2\right], \tag{6.11}$$

where, $r_s = GM/c^2$. In the usual low energy limit and in the far field approximation $(r >> r_s)$, $V_{\rm GN}\big|_{\rm SW}$ reduces to that in Newtonian gravity.

6.3 Discussion

Experimental verifications of classical predictable GR effects, namely gravitational deflection of light, perihelion shift and gravitational time delay, to a high degree of precision led general relativity as the theory of gravitation replacing Newtonian theory. Over a century general relativity has successfully withstood all the experimental tests conducted so far, the predictions of general relativity being confirmed in almost all observations and experiments till date, however, mostly in the weak field limit [Will, 1993, 2006] and most of the alternative theories of gravity are generally developed in such a manner that they reduce to general relativity in the weak field limit so that the theories automatically satisfy the classical solar system tests of gravitation. The recent announcement of the first direct observation and evidence of gravitational waves corresponding to the inspiral and merger of two black holes [Abbott et al., 2016] validates the general relativity in the strong field region leaving very little window for the alternative gravitational theories [Konoplya and Zhidenko, 2016]. Since begining, there were efforts to cheque whether the observations concerning gravity can be described by Newtonian like potential. In this regard various corrections have been offered to the Newtonian potential to reproduce the relativistic features of spacetime geometries (see introduction) but none of them can describe all the salient relativistic features simultaneously or make it consistent with all the classical experimental tests of gravity. The present work perhaps provides for the first time a method to construct a relativistic Newtonian analogous theory described through a relativistic scalar potential (and the corresponding force law) from a first principle approach for a class of static spacetime geometries, starting directly from a generic relativistic gravitational action in analogous Newtonian framework, that give exactly identical geodesic equations of motion to those of general relativity or of any viable alternative gravitational theory. Consequently, the analogous relativistic potential exactly reproduces the relativistic features of any static spacetime geometry (described by Eqn. (6.1)) in its entirety, including the classical experimental tests of gravity. In constructing this Newtonian analogous theory, no premises (axioms and principles) of geometric theory of gravitation has been used, the only information that has been used to construct the Newtonian analogous theory is the inclusion of the metric function of the corresponding spacetime geometries into the relativistic Newtonian gravitational action; the term that correlates relativistic Newtonian analogous theory with geometric spacetimesm. The Newtonian analogous construct is thus an exact relativistic Newtonian representation of a class of gravitational static spacetime geometries, and a true Newtonian description of gravitational spacetimes, and can be conservatively said that the corresponsing relativistic scalar potential is the correct relativistic generalization of Newtonian gravitational potential.

Being an exact relativistic Newtonian analogous construct, the corresponding relativistic scalar potential would be profoundly useful in analyzing wide range of complex astrophysical phenomena in strong field gravity. The analytical solution of n-body problem in metric theories remain very difficult, if not elusive. The Newtonian analogous potential should be quite useful in this regard. Most importantly, it is to represent gravitational interaction through a scalar potential by avoiding complex tensorial construct of relativistic gravitation, where by, relativistic astrophysical phenomena, especially in the strong field gravity can be studied accurately in a simple Newtonian framework using the exact Newtonian analogous potential, circumventing complex nonlinear tensorial equations of geometric gravitation. A very appropriate feasible scenario in this regard to use this relativistic potential is to accurately study complex relativistic accretion phenomena around massive compact objects like BH/neutron star/naked singularity or around any other possible spacetime geometry in the Newtonian framework, in strong field gravity. Other situations where Newtonian analogous potential may be useful include tidal disruption of a star by a SMBH, binary systems or in binary mergers or analyzing galactic dynamics around SMBH, and can even be tested in other applications of gravity as well. This generic Newtonian analogous potential should also be useful to observationally discriminate alternative theories of gravity among themselves and from general theory of relativity in the strong field gravity; consequently to test theory of relativistic gravitation in the strong field regime. There is an ongoing struggle to unify general relativity (or one of its viable alternative) and the other three fundamental forces namely the strong, weak, and electromagnetic forces. The metric theories of gravitation imply gravity as an effect of spacetime curvature. Gravity thus cannot be treated as a fundamental force and the unification of GR with the other fundamental forces thereby becomes difficult. It would be interesting to examine whether the relativistic Newtonian analogous potential can be useful in this regard.